Teaching Mathematics at Secondary Level

Tony Gardiner

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About the author

Tony Gardiner, former Reader in Mathematics and Mathematics Education at the University of Birmingham, was responsible for the foundation of the United Kingdom Mathematics Trust in 1996, one of the UK's largest mathematics enrichment programs. Gardiner has contributed to many educational articles and internationally circulated educational pamphlets. As well as his involvement with mathematics education, Gardiner has also made contributions to the areas of infinite groups, finite groups, graph theory, and algebraic combinatorics. In the year 1994-1995, he received the Paul Erdös Award for his contributions to UK and international Mathematical Challenges and Olympiads. In 1997 Gardiner served as President of the Mathematical Association, and in 2011 was elected Education Secretary of the London Mathematical Society.

Introduction and summary

This extended essay started out as a modest attempt to offer some supporting structure for teachers struggling to implement a rather unhelpful National Curriculum. It then grew into a *Mathematical manifesto* that offers a broad view of secondary mathematics, which should interest both seasoned practitioners and those at the start of their teaching careers. **This is not a DIY manual on *how to teach*.** Instead we use the official requirements of the new National Curriculum in England as an opportunity:

- to clarify certain crucial features of elementary mathematics and how it is learned—features which all teachers need to consider *before* deciding 'How to teach'.

In other words, teachers will find here a survey of some of the mathematical background which schools need to bear in mind when choosing their approach, when thinking about long-term objectives, and when reflecting on (and trying to understand and improve) observed outcomes.

We leave others to draft recipes for translating the official curriculum into a scheme of work with the minimum of thought or reflection. This study is aimed at anyone who would like to think more deeply about the discipline of "elementary mathematics", so that whatever decisions they may take will be more soundly based. Feedback on earlier versions suggested that this analysis of secondary mathematics and its central principles should provide food for thought for anyone involved in school mathematics, whether as an aspiring teacher, or as an experienced professional—challenging us all to reflect upon what it is that makes secondary school mathematics educationally, culturally, and socially important.

The contents demand repeated reading, and should be weighed and digested *slowly*.

- The reader should begin with the very short Part I, which sets the scene.
- We suggest they should then work through Part II, which concentrates on the *Aims* etc. of the published curriculum, and on the general requirements in the section headed *Working mathematically*. But readers should not worry if some aspects remain unclear on a first reading.
- Ultimately all the sections are interlinked; but we expect the reader will then select sections in Part III (the listed *Subject content*) which are of most immediate interest—whether *Number*, or *Algebra*, or *Geometry and measures*, or *Probability and Statistics*—and extract whatever is found useful. Again, each section may bear repeated reading over a number of years, so do not be frustrated if at first some parts appear more immediately applicable than others.
- Part IV is a revised version of our "humane mathematics curriculum for all, written from a mathematical viewpoint". This is offered as a "sample" rather than as an ideal "model". It tries to avoid the *hubris* of some recent reforms and to show how more modest goals mesh together over time, and with each other. For example, we include stages intended to ensure that everyone should manage to learn their tables by the end of primary school, with reinforcement in lower secondary school (even if some pupils achieve fluency earlier); and though we emphasise the central role of fractions for everyone in secondary mathematics, we avoid their early introduction.

The reader is assumed to be an *active* reader. We repeatedly emphasise drawing, calculating, and making; but we have left these delights for the reader, who should always have pencil and paper to hand. In particular, problems and calculations included in the text should be tackled before reading on, and diagrams described in the text should be drawn.

The important messages are best understood in the context where they arise. However, we were advised to include a summary of some of the key messages at the outset. We therefore end this Introduction with a list of some of the most important messages that arise in the ensuing text, even though many of these messages cannot be easily summarised. Hence we also urge readers to construct their own list of key principles as they work through the main text.

- Key Stage 3 (lower secondary school, age 11–14) is a crucial transition stage, which needs concerted support (see Part I).
- We need to recognise that, if what is *learned* is to bear fruit in the medium term, whatever is *taught* needs to be analysed and taught within an organised *didactical framework*.
- What is taught also has to build on what is already known, so teachers need to exercise judgement about pupils' readiness to progress.
- Mathematics can be daunting; but everyone can make progress with perseverance. So it is important to pace the initial material to allow this message to register.
- Whenever possible one should exploit opportunities for pupils to calculate, to draw, to measure and to make things for themselves.
- Whenever possible, one should establish and check pupils' grasp of the inner structure of elementary mathematics through on-going class oral and mental work.
- Regularly extend routine oral and mental work to encourage an atmosphere in which thoughtful conjectures are expressed and tested, and where proof is increasingly valued.
- Actively develop pupils' powers of remembering. Gradually extend the range and scope of important results and methods that pupils understand and *know by heart*. Help them to see that having to work things out from scratch each time seriously restricts the kind of problems one can tackle and solve.
- Each theme must be given sufficient time and variety for pupils to achieve the kind of robust fluency, and the shift of focus that is needed for subsequent progression.

- Special and recurring attention needs to be paid to strengthening key themes (such as place value, fractions, structural arithmetic, simplification, ratio and proportion) in a suitably robust form.
- An effective programme must allow pupils to appreciate links and connections, and to gradually become aware of the way in which simple ideas from different mathematical domains relate to each other.
- Always look for alternatives to 'acceleration'. Aim for all pupils to achieve robust mastery in sufficient depth to maximise their preparation for subsequent progression. The easier a pupil finds a topic, or a group of topics, the more important it is for them to master that topic in serious depth before moving on.
- Use carefully designed sets of graded *exercises* that range from the very simple to the general, routinely exploring the more demanding 'indirect' variations, which are needed in many subsequent applications.
- Recognise the link between each *direct* operation or process (such as addition, or multiplying out brackets) and the corresponding *inverse* operation or process (such as subtraction, or factorising). Whilst fluency in the direct operation is essential, its main purpose is to serve as a foundation for solving the harder, and more important inverse problems. In particular, resist the temptation to break harder inverse problems into manageable (direct) steps.
- Routinely include simple *word problems* alongside technical exercises, so that pupils learn to identify and extract relevant information from short (two or three sentence) problems given in words.
- Regularly include short, non-routine *problems* (including two-step and multi-step problems), that cultivate pupils' willingness to face the unexpected, and to think how to link known techniques into effective solution chains.
- Routinely re-visit old material and replace old methods by more flexible, forward-looking alternatives. Distinguish clearly between *backward-looking* methods (that may deliver answers, but which hinder progression) and *forward-looking* methods (that may at first seem unnecessarily difficult, but which hold the key to future progression).

The final version owes much to many friends and colleagues, whose comments on successive drafts kept alive the vision of trying to write something of value in difficult times: I hope they will accept my profound thanks without my running the risk of trying to name them all. The London Mathematical Society provided essential support for this project over an extended period. But the book would never have seen the light of day without the endless encouragement and Herculean efforts of Alexandre Borovik.

I. Background: Why focus on Key Stage 3?

When designing a mathematics scheme of work for Key Stage 3, the obvious move would be to try to adapt the official programme of study.[1]

However:

- the programme of study incorporates some startling omissions of essential content that simply cannot be skipped (to give just two examples: there is no reference to the subtleties of teaching the arithmetic of negative numbers, or of combining negatives and 'minus signs' in algebra; nor is there any explicit mention of isosceles triangles, or of deriving and using their properties in other settings);
- many of the officially listed themes require careful interpretation in other ways;
- in the official programme of study the connections between topics are rarely elaborated; and
- the grouping and sequencing of, and the progression through, topics is far from clear.

In short, the programme of study needs to be supplemented and 'fleshed out' (and sometimes corrected). Moreover, unlike the programmes for Key Stage 1 and Key Stage 2,

> **the programme of study for Key Stage 3 has no year-by-year structure and no accompanying *Notes and guidance*.**

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[1] National curriculum in England: mathematics programmes of study, https://www.gov.uk/government/publications/national-curriculum-in-england-mathematics-programmes-of-study; https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/239058/SECONDARY_national_curriculum_-_Mathematics.pdf

The fact that we need to think more carefully about mathematics teaching at Key Stage 3 has been a theme of the Ofsted triennial reports on mathematics:

> *Mathematics: Understanding the score* (2008)[2]

and

> *Mathematics: Made to measure* (2012).[3]

These reports have not been as widely read as they deserved. Their analysis is unusually forthright for official documents, and provides a sobering starting point for any school seeking to review its mathematics provision at Key Stage 3. The reports summarise observations from hundreds of inspections—but they do so in an unusually constructive spirit. For example, having classified half of secondary maths lessons, and more than half of the schemes of work, as being either 'inadequate' or 'requiring improvement', Ofsted went out of their way to provide down-to-earth advice.[4]

This down-to-earth Ofsted DIY guide begins with a four-page table contrasting

- the general features of "good mathematics teaching"

with

- those of "mathematics teaching deemed to require improvement".

The Ofsted guide then presents a string of specific examples chosen to clarify the differences between 'weak' and 'more effective' mathematics teaching, and to challenge schools to reflect on, and to improve, their own teaching. Hence this collection of examples and advice should probably

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[2] http://webarchive.nationalarchives.gov.uk/20141124154759/http://www.ofsted.gov.uk/resources/mathematics-understanding-score

[3] https://www.gov.uk/government/publications/mathematics-made-to-measure

[4] http://webarchive.nationalarchives.gov.uk/20141124154759/http://www.ofsted.gov.uk/resources/mathematics-understanding-score-improving-practice-mathematics-secondary

be taken seriously by any school seeking to revise its published scheme of work for Key Stage 3.

Key Stage 3 mathematics teaching is important because it marks a transition from the more *informal* approach in primary schools to the formal, *more abstract* mathematics of Key Stage 4 and beyond. Hence those teaching Key Stage 3 classes need a clear picture of how the constituent parts of secondary mathematics interlock, and how Key Stage 3 work can best support progression—first progression to Key Stage 4, and then to Key Stage 5 (at ages 16-18). In this regard the 2012 report *Made to measure* highlights the uncomfortable fact that (p. 4):

> "More than 37,000 pupils who had attained Level 5 at primary school gained no better than grade C at GCSE in 2011. Our failure to stretch some of our most able pupils threatens the future supply of well-qualified mathematicians, scientists and engineers."

This illustrates the extent to which current provision at Key Stage 2 and Key Stage 3 fails to lay the necessary foundations for **subsequent** stages, and raises the question of how to improve provision at Key Stage 3. The question is especially relevant given that so many schools feel unable to allocate their strongest mathematics teachers to Key Stage 3 classes. So there is clearly a need to provide more detailed guidance for those who teach at this level.

The quality of existing support and guidance at school level is summarised in the key findings of the 2008 report *Understanding the score* (p. 6):

> "Schemes of work in secondary schools were frequently poor, and were inadequate to support recently qualified and non-specialist teachers."

The 'Executive Summary' (p. 4) noted:

> "Evidence suggests that strategies to improve test and examination performance, including 'booster' lessons, revision classes and extensive intervention, coupled with a heavy emphasis on 'teaching to the test', succeed in preparing pupils

> to gain the qualifications but **are not equipping them well enough mathematically for their futures**. It is of vital importance to shift from a narrow emphasis on disparate skills towards a focus on pupils' mathematical understanding. Teachers need encouragement to invest in such approaches to teaching." [emphasis added]

And the 'Recommendations' (p. 8) included:

> "Schools should [. . .]
>
> - enhance schemes of work to include guidance on teaching approaches and activities that promote pupils' understanding and build on their prior learning."

Pages 19–25 of the 2008 report provide useful additional details: Figure 4 on p. 19, and Figure 5 on p. 24 summarise the observed weaknesses in secondary schools, and the surrounding paragraphs make clear suggestions as to what needs attention.

The 2012 report *Made to measure* echoes, and reinforces the concerns expressed in the 2008 report:

p. 9:

> "Teaching was strongest in the Early Years Foundation Stage and upper Key Stage 2 and **markedly weakest in Key Stage 3**." [emphasis added]

p. 18:

> "Learning and progress [. . .] were **least effective in Key Stage 3**, where only 38% of lessons were good or better and 12% were inadequate" [emphasis added]

p. 19:

> "[. . .] Quick-fix approaches were particularly popular. Aggressive intervention programmes, regular practice of examination-style questions and extra provision, such as

> revision sessions and subscription to revision websites, **allowed pupils to perform better in examinations than their progress in lessons alone might suggest**.
>
> These tactics account for the rise in attainment at GCSE; this is not matched by better teaching, learning and progress in lessons, or by pupils' deeper understanding of mathematics. In almost every mathematics inspection, inspectors recommended improvements in teaching or curriculum planning, in most cases linked to improving pupils' understanding of mathematics or their ability to use and apply mathematics.
>
> [. . .] It remains a concern that secondary pupils seemed so readily to accept the view that learning mathematics is important but dull." [emphasis added]

The analysis in this book may be seen as an attempt to help schools respond to one of the main 'Recommendations' in the 2012 report (p. 10):

> "Schools should:
>
> - tackle in-school inconsistency of teaching, making more of it good or outstanding, so that every pupil receives a good mathematics education
> - increase the emphasis on problem solving across the mathematics curriculum
> - develop the expertise of staff:
>   - in choosing teaching approaches and activities that foster pupils' deeper understanding, including through the use of practical resources, visual images and information and communication technology
>   - in checking and probing pupils' understanding during the lesson, and adapting teaching accordingly
>   - in understanding the progression in strands of mathematics over time, so that they know the key knowledge and skills that underpin each stage of learning

> – ensuring policies and guidance are backed up by professional development for staff to aid consistency and effective implementation.”

The seriousness of the current situation summarised in these two reports, and the weaknesses in the published Key Stage 3 programme of study may explain why these notes and guidance grew into an ‘extended essay’, rather than being effectively distilled into a punchy DIY manual. Despite (or perhaps because of) this, we hope that all teachers (from those just beginning their careers, or those aiming to take responsibility as Head of Department, to the most experienced practitioners), and those who train teachers will find that what follows provides food for thought, and that schools will find what is presented here helpful in reviewing their current provision in lower secondary school.

II. The general advice in the Key Stage 3 programme of study

Schools will naturally try to implement and adapt the published programme as it stands. It is therefore important to decide

- when it is safe simply to copy what is listed;
- when the given list of topics needs to be reordered or supplemented in some way; and
- when there are strong mathematical reasons to *reinterpret* an official requirement (and to clarify in one's own mind why it needs to be reinterpreted).

Hence the remaining sections of this book are presented in the form of a line-by-line commentary (where comment seems needed) on the published programme. The present part, Part II, concentrates

- on the *Aims* etc. which appear on page 2 of the published programmes of study (Section 1 below), and
- on the broad expectations discussed in the section headed *Working mathematically* on pages 4 and 5 of the published programmes of study (Section 2 below).

1. Aims

1.1. [*Aims* p. 2]

> **Mathematics is an interconnected subject in which pupils need to be able to move fluently between [different] representations and mathematical ideas.**

Elementary mathematics derives its power from the way a simple idea sometimes has other interpretations, and from the way simple ideas from different domains can be *combined* to deliver more than one might expect. The published programme of study does not always make it easy to identify these connections and interactions. Hence it is important to consider how to sequence and to link the listed material in a way that clarifies and develops the interdependencies between topics and ideas.

For example, if we consider the most familiar idea of all—namely 'place value'—schools may recognise the need to reinforce:

- how the place value notation for integers works, and how it extends to decimals;
- that it does so in a way that links
  - the more familiar *positive* powers of 10 (tens, hundreds, thousands),
  - with $10^0 = 1$ (the 'units' or '1s' place), and
  - with *negative* powers of 10 (for places to the right of the decimal point);
- the fact that powers of 10 multiply together in a way that foreshadows the index laws for general powers;
- that the written algorithms of column arithmetic, which were developed in primary school for integers, extend naturally to decimals—giving plenty of opportunity to reinforce both the procedures themselves and *why they work*, and hence to strengthen pupils' sense of 'place value'.

Schools will benefit from identifying such recurring themes and important connections for themselves, and from organising the required Key Stage 3 content so that pupils come to appreciate these themes and connections. Some of these are very basic. The next ten bullet points indicate a few selected examples to illustrate the need

- to consider each of the requirements listed in the programme of study,
- to decide what links need to be explicitly mentioned, and
- where possible to include these in any scheme of work.

- The way work with *pure numbers* (that is, numbers like 1, 23, $\frac{4}{5}$, or -67.8, stripped of any units), and the arithmetic of integers and decimals, links to simple applications—where purely numerical calculations allow one to solve problems involving *measures*, and to make sense of, and solve, all sorts of 'word problems'.
- The way multiplication and division of decimals and fractions hold the key to routinely solving almost any problem involving rates, or percentages, or ratios, or proportion.
- The way blind calculation gives way to *simplification* and "structural arithmetic", which links naturally to effective calculation in algebra.
- The way "I'm thinking of a number ..." problems should at first be tackled without algebra (as 'inverse mental arithmetic'), but can later be formulated as a simple equation in one unknown, then routinely solved.
- The way any linear *equation* in one unknown x reduces to $ax + b = 0$, with solution $x = -\frac{b}{a}$; and any linear *inequality* in one unknown x reduces either to

  (i) $ax + b > 0$ (or $ax + b \geqslant 0$) with $a > 0$, having solution $x > -\frac{b}{a}$ (or $x \geqslant -\frac{b}{a}$)—i.e. a 'half-line'; or alternatively to

  (ii) $ax + b < 0$ (or $ax + b \leqslant 0$) with $a > 0$.

- The way any linear *equation* $y = mx + c$ in two unknowns x, y corresponds geometrically to the set of all points (x, y) on a straight line, that the line divides the plane into two 'half-planes', and that the solutions of the corresponding linear *inequality* ($y > mx + c$, or $y \geqslant mx + c$) correspond to the set of all points (x, y) in one of these two half-planes.

- The fact that two simultaneous linear equations can be solved exactly, and that the solution is the point of intersection of the two lines corresponding to the linear equations (provided the two lines meet).
- The way short and long division (combined with a little algebra) shows that fractions correspond precisely to terminating or recurring decimals.
- The way the basic property of parallel lines forces the sum of the angles in a triangle to be equal to the sum of the angles at a point on a straight line.
- The way the congruence criterion and the parallel criterion allow us to justify the standard ruler and compass constructions, and to prove the basic facts about areas (of parallelograms and triangles), which lead to a proof that in any right angled triangle the square on the hypotenuse is miraculously equal to the sum of the squares on the other two sides, which then links with coordinate geometry by allowing us to calculate *exactly* the distance between any two given points in 2D or in 3D.

1.2. [*Aims* p. 2]

Pupils should build on Key Stage 2

This is excellent advice—provided it is suitably interpreted. Key Stage 3 has to start out from pupils' experience at Key Stage 2. But this prior experience also needs to be revisited and developed in fresh ways if it is to be used as a reliable foundation for further work. In commenting on this principle, we consider one example in modest detail (1.2.1), then digress to make three important general points (1.2.2–1.2.4), indicate some further examples more briefly (1.2.5), and end with a gentle warning about the likely impact of the Key Stage 2 programme of study on Key Stage 3 (1.2.6).

1.2.1 Mental calculation work should not end with Key Stage 2. It should continue in Year 7, but should increasingly use what pupils know in a way that **exploits structure**, rather than calculating blindly.

- Pupils need to learn to be on the look-out for ways of extracting 10s and 100s in additions such as

  $$73 + 48 + 27 = \ldots;$$

  or in multiplications such as

  $$14 \times 45 = 7 \times (2 \times 5) \times 9 = 630,$$

  or

  $$75 \times 28 = 3 \times (25 \times 4) \times 7 = 2100.$$

- Decimal calculations (such as $7 \times 0.8 = \ldots$, and $12 \times 1.2 = \ldots$, and $0.7 \times 0.08 = \ldots$, and $1.2 \times 1.2 = \ldots$) should be routinely related to their familiar integer equivalents, exploiting opportunities to reflect on how multiplying and dividing by powers of 10 affects the decimal point.

- Common factors among a list of added terms should be seen as an opportunity to 'group' using the distributive law, as in

  $$17 \times 23 + 17 \times 7 = 17 \times (23 + 7) = 17 \times 30,$$

  rather than to calculate the left hand side blindly. In general, common factors among terms which are to be added or subtracted, multiplied or divided, should be seen as an opportunity to simplify and to cancel.

- Lots of simple work involving fractions should include (a) switching to common denominators (by scaling up both numerator and denominator) in order to simplify the arithmetic, and (b) moving in the opposite direction when using cancellation to simplify fractions.

Written calculation with integers also needs to be strengthened and extended to decimals—but we shall have more to say on this in Section 1.2.5 below.

1.2.2 In Part I we saw clear evidence (in the two Ofsted reports) of the unfortunate consequences when a Key Stage seeks to maximise performance on immediately impending assessments, and forgets

> that our primary responsibility is always **to prepare pupils for the Key Stages that follow.**

Pressure to "achieve" in the short-term often encourages pupils to become dependent on (and teachers to allow) 'backward-looking' methods that deliver answers in easy cases, but which sooner or later become an obstacle to progress. Hence any internal scheme of work needs to make a clear distinction between

(a) **backward-looking methods** that get answers in the short-term, but which trap pupils in old ways of working (as with finger counting, or idiosyncratic calculation methods, or reducing multiplication to repeated addition, or modelling questions about fractions in terms of pizzas—all of which may have transitional value, but which are known to block later progress if they become too strongly embedded), and

(b) **forward-looking methods**, that may seem unnecessary if the perceived goal is merely to get answers to simple problems at a given stage, but which are important because of the way they reflect the inner structure of elementary mathematics, and are often essential for *progress at the next stage*.

It is not easy for a mere listing of curriculum *content* to capture this crucial distinction. An effective *primary* school is one whose pupils are taught in such a way that allows them to flourish at Key Stages 3 and 4. Similarly, effective teaching at Key Stage 3 prepares the ground for, and leads to solid achievement at Key Stage 4 and beyond. Insofar as the revised programme of study incorporates this idea, it tends to do so in ways that are not immediately apparent, so we shall occasionally comment on how Key Stage 3 material impacts on mathematics at Key Stage 4 and beyond.

1.2.3 The previous subsection drew attention to the distinction between *backward-looking* and *forward-looking* methods. Another important distinction is that between

- a **direct** operation (such as addition, or multiplication, or evaluating powers, or multiplying out brackets), and
- the associated **inverse** operation (such as subtraction, or division, or identifying roots, or factorising).

The distinction may be easier to appreciate if we consider a strictly artificial example—namely the "24 game". Four numbers are given, and each is to be used once. These four numbers may be combined using any three operations chosen from the four rules (with brackets as required), with the goal being to "make 24".

If one is given the starting numbers "3, 3, 4, 4", then one scarcely notices the distinction between

- a 'direct' calculation (such as "Work out $(3 \times 4) + (3 \times 4) = \ldots$"), and
- the 'inverse' challenge of having to "invent for oneself a way to make 24" (let's try "$(3 + 3) + (4 \times 4) = 22$"— not quite; or "$(3 \times 3) + (4 \times 4) = 25$"—nearly; or "$(3 \times 4) + (3 \times 4) = \ldots$").

When faced with the inverse challenge to "make 24 using 3, 3, 4, 4", it is almost as easy to dream up a combination that works as it is to evaluate the expression once it has been invented. But

- evaluating the answer of a given sum is a *direct*, or mechanical, process, whereas
- juggling possibilities to come up with a calculation which produces the required answer of "24" is an *inverse* operation, which is far from mechanical (even if in this case it is rather easy).

The distinction between *direct* and *inverse* operations becomes slightly clearer if the given numbers are "3, 3, 5, 5". Here the *inverse* task of coming up with a sum that delivers the required answer of "24" is significantly harder. The relevant tools are the direct processes of arithmetic—except that it is not clear which to use, so one has to scan what one knows, and select approaches which seem to be the most promising. It is precisely this willingness to juggle intelligently with numbers, and to think flexibly with simple ideas that is needed in many everyday applications. But once one is told what calculation to carry out, then the *direct* calculation "$(5 \times 5) - (3 \div 3)$" is entirely routine.

This distinction between the *direct* operation (which is straightforward, and which requires only that one should implement a given calculation to check that the answer is equal to "24"), and the *inverse* operation (which is much harder, and which here requires us to invent a sum that has the required answer "24"), becomes markedly more clear if one is given the starting numbers 3, 3, 6, 6, and is left to find a way to "make 24" (or if one is given the starting numbers 3, 3, 7, 7; or 3, 3, 8, 8).

To sum up: the reasons why this distinction is important are that

- almost every mathematical technique one learns comes initially in a *direct*, or mechanical, form, but leads naturally to *inverse* problems (as addition leads naturally to subtraction);
- *inverse* problems are usually much more demanding than their *direct* cousins;
- mastery of the *inverse* form depends on a prior robust mastery of the *direct* form;
- **but in the long run, it is the *inverse* operation which is generally more important**.

Those who complain that pupils, or school leavers, cannot "use" what they are supposed to know, often fail to notice that what pupils have been taught (and what has been assessed) has usually focused on *direct* procedures, whereas what is required is the ability to think more flexibly when faced with some kind of *inverse* problem. Inverse problems often come in different forms, or variations something that has been a focal point of the recent teacher exchanges with Shanghai, where the idea of "exercises with variation" has emerged as a recurring didactical theme

Given this, one might expect formal assessments to include a strong focus on ensuring mastery of the many *inverse* operations and the ability to solve the standard inverse problems in elementary school mathematics. In reality, *inverse* processes have been neglected, or (worse) have been distorted by providing ready-made intermediate stepping stones that reduce every *inverse* problem to a sequence of *direct* steps. Why is this?

Direct operations are relatively easy to teach, and to assess. The associated *inverse* operations may be more important, but they are **harder to assess**. *Inverse* problems are more demanding, and cannot be reduced to deterministic methods. So they give rise to **low scores**, and they do so in a way that is hard to predict. This makes them distinctly awkward for those who devise test items within a target-driven and test-driven culture, where the assessors may be contractually obliged to return predictable results, and to avoid low scores. Hence, if such problems are set at all, they are usually adapted in some way to make pupil performance more predictable (for example, by breaking down the unpredictable *inverse* problem into a more manageable sequence of steps—each of which is essentially a routine *direct* task).

Teachers need to recognise the importance of such problems for pupils' subsequent progress, and then devote sufficient time to them for pupils to achieve a degree of mastery. But it would obviously help if assessments regularly required, and rewarded, such mastery!

1.2.4 The bald listing of content in the official programme of study is rather dry and formal—focusing on "what" rather than "how". In one sense, this emphasis is healthy. But it ignores the essential interplay between **content** and **didactics**.

Procedural fluency is rightly stressed. But this emphasis is too often repeated in isolation—as though a robust grasp of place value (for example) will emerge spontaneously as a result of banging on about fluency in specified procedures. It won't. So something more is needed. If it is to serve as a useful guide, a content list or programme of study needs to be constructed in a way that indicates, and supports, a clear underlying "didactical architecture". In contrast, the given programme of study routinely misses the opportunity to convey key central principles (such as the contrast between *backward-looking* and *forward-looking* methods, or between *direct* and *inverse* operations), and important details (such as the key didactical stages which can lead from:

(a) talking about "half a pint" or "half an hour" in Year 2, to competence with the arithmetic of fractions in Year 9; or

(b) from meeting negative quantities for the first time in Key Stage 2, to calculating freely with negative numbers, and simplifying expressions which combine subtraction and minus signs in algebra at Key Stage 3/4).

1.2.5 In early Key Stage 3 we need to reinforce Key Stage 2 work on the familiar written arithmetical procedures for *integers* in order to extend them to more serious long multiplication, to division, and to decimals. Column arithmetic for integers provides an excellent opportunity to cement number bonds and multiplication tables. It also develops the ability to carry out a sequence of simple steps completely reliably. The extension of these procedures to decimals provides fresh opportunities to address 'place value'. At the simplest level, pupils need to understand why it is essential to align the units and tens "places" when carrying out column addition and subtraction of integers, so that the requirement to align the decimal points when adding and subtracting decimals is recognised as being essential (see example 1.2.2C "$42.65 + 5.748 = \ldots$" in Part III). The *logic* of short and long multiplication will also need to be clarified before these procedures are extended to decimals. Integer arithmetic (including mental arithmetic in both direct and inverse forms with all variations) also needs to be in good shape before we extend integer arithmetic to fractions.

The extension of long multiplication and division to decimals may need to be slightly delayed. When they are addressed, pupils need first to know how the decimal point behaves under multiplication and division by powers of 10, so that they can understand how this allows multiplication and division of *decimals* to be transformed into *integer* multiplication and division.

Short and long division develop *the inverse* of multiplication, in that they require pupils to use what they know about multiplication in a flexible way. When asked to divide 17 onto 918, the initial *inverse* question:

> "How many times does 17 go into 91? And what is the remainder?"

requires greater mental agility than the two *direct* questions:

> "What is 17×5?", and "What is $91 - 85$?".

Short and long division also require pupils to string together *a chain of steps*, each of which is accessible, but where the whole chain has to be implemented 100% reliably for the process as a whole to succeed. And the power of the process becomes apparent when one discovers how it extends naturally to allow division of decimals. Later the division process helps to establish the remarkable connection between fractions and decimals.

Some pupils will benefit from the challenge of tackling (or extending their prior facility with) serious long division. This topic is listed in Key Stage 2 *for all pupils*. It is unclear what effect this may have; but we may well find that serious long division is appropriate for only around half of the cohort, even at Key Stage 3.

1.2.6 In exhorting teachers at Key Stage 3 to "build on Key Stage 2" it is only fair to mention that the Key Stage 2 programme of study may prove problematic in some respects. A preliminary indication of the extent of this difficulty may be gleaned from an earlier paper.[5] In particular

- a significant amount of material has been included at Key Stage 2 in a way that is likely to prove premature; and
- some of the listed topics which are entirely appropriate in Year 5 and 6 have been specified rather poorly.

Hence one can anticipate that many pupils entering Key Stage 3 will have at best a superficial grasp of some of the listed content from Key Stage 2.

Among the listed topics that are inappropriate and unnecessary in Year 6, many are implicit in the early Key Stage 3 programme of study, so could be safely delayed until Year 7. Some primary schools may recognise this and concentrate on more age-appropriate material—leaving other content to be treated more effectively at Key Stage 3. But many schools will go by the book and will try to cover whatever is listed—with predictable consequences. For both groups, this problematic material will need to be revisited at Key Stage 3 in order to establish a secure platform for progression. Examples of topics which may have been 'covered' at Key Stage 2, but which will need serious attention in Years 7 and 8 include:

- the extension of place value to decimals;

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[5] http://education.lms.ac.uk/wp-content/uploads/2012/02/DMG_4_no_3_2013.pdf

- the arithmetic of decimals;
- work with measures—especially compound measures;
- the arithmetic of fractions;
- ratio and proportion;
- the use of negative numbers;
- work with coordinates in all four quadrants;
- simple algebra.

1.3. [*Aims* p. 2]

> **Decisions about progression should be based on the security of pupils' understanding and their readiness to progress to the next stage.**

Secondary schools will need to know how this excellent principle of "readiness to progress" has been handled at Key Stage 2. We give just one example of many.

There is a general welcome for the requirement that pupils should learn (i.e. know, and be able to use) their tables. But there is unanimity that this will not be achieved by the end of Year 4 as specified in the official programme of study, and that a more realistic objective may be to expect most pupils to achieve this by the end of Year 5 or Year 6. Hence material listed in Year 5 and Year 6 that depends on '**prior** mastery of tables' will not be accessible at the expected stage, so will prove unrealistic at that level. (For example, until tables are secure, one is limited in what one can achieve in factorising integers, finding HCFs, working with prime numbers, with short division and long division, with squares and cubes, with equivalent fractions and with cancellation.)

If primary schools feel obliged to try to teach inappropriately ambitious material purely because it is officially listed, this will lead to problems that are entirely avoidable. Thus secondary schools may have to encourage their feeder primary schools to trust their professional judgement in such matters, and to recognise those aspects of the Year 6 programme where work should remain 'preparatory', with a serious treatment being delayed until Year 7.

Some of the material that is listed in Key Stage 2 seems inappropriate at that level—partly because we know that it is hard to teach it well even at Key Stage 3. For example, it may make sense at Key Stage 2 to use symbols to summarise familiar formulae: such as re-writing the verbal equation

"(area of a rectangle) = (length times breadth)" as "$A = l \times b$".

However, it would be premature to expect most primary pupils to learn more serious *elementary algebra* (and most primary teachers are in no position to teach it effectively). And while there is every reason to engage pupils at Key Stage 2 in tackling "I'm thinking of a number ..." problems, they are best addressed at that age by using '*inverse* mental arithmetic': that is, where the missing number is discovered by using intelligent, flexible, *inverse* mental arithmetic, rather than by prematurely trying to formulate such problems algebraically as *equations* (as suggested by the official Year 6 programme listed under *Algebra*).

Even where secondary schools liaise effectively with most of their feeder primaries, they should think carefully—as part of ensuring "readiness to progress"—how to consolidate key ideas and techniques from Key Stage 2 in early Key Stage 3, and should be prepared to clear up misunderstandings that may have arisen as a result of material having been introduced prematurely.

A key application of this crucial principle of "readiness to progress" arises because the Key Stage 3 programme of study is now an explicit part of the GCSE specification. Hence decisions about progress through the Key Stage 3 curriculum are bound up with *decisions about future GCSE entry*. The Key Stage 4 programme of study states explicitly:

> **Together the mathematical content set out in the Key Stage 3 and Key Stage 4 programmes of study covers the full range of material contained in the GCSE Mathematics qualification. Wherever it is appropriate, given pupils' security of understanding and readiness to progress, pupils should be taught the full content set out in this programme of study.**

In its understated way this both presents a challenge to teach *as much of the listed material as possible* to as many pupils as possible, and at the same time leaves considerable scope for teachers to use their professional experience to decide *where this aspiration may not be "appropriate"*.

Those pupils who should progress comfortably to GCSE Higher tier may be able to swallow the complete Key Stage 3 programme **by the end of Year 9**. But those who may land up taking Foundation tier GCSE will often benefit from proceeding **more slowly** through Key Stage 3 in order to establish a solid foundation for those parts of the Key Stage 4 programme which they might subsequently manage to cover, and perhaps master. In other words, schools would seem to be free to interpret the Key Stage 3 programme *as part of GCSE*, and to allow some material to spill over into Year 10 where this seems appropriate. Those pupils heading for Foundation tier are far more likely to achieve mastery of some of this material if they are allowed to proceed more steadily (e.g. taking four years rather than three), than if they are forced to cover the material prematurely, and then have to repeat it.

1.4. [*Aims* p. 2]

> **Pupils who grasp concepts rapidly should be challenged through being offered rich and sophisticated problems before any acceleration through new content in preparation for Key Stage 4. Those who are not sufficiently fluent**

> **should consolidate their understanding, including through additional practice, before moving on.**

The second sentence reinforces the comments made at the end of 1.3 above.

The first sentence advises against acceleration. It also highlights the fact that each listed topic can be treated on *many levels*, and states the important general principle that those who grasp a basic concept should be faced **with more challenging variations on the same material** before they move ahead. This is an extension of the idea of "readiness to progress": namely that

> **before allowing pupils to progress to more advanced topics, we should routinely expect a much deeper understanding on the part of those who might one day proceed further.**

At present we routinely let down large numbers of pupils by failing to establish a sufficiently robust mastery of important basic ideas. For example, the very first item under *Number* (*Subject content* p. 5: see Part III, section 1) states that pupils should

> **understand and use place value for decimals, measures and integers of any size.**

Other requirements under the sub-heading *Number* relate to calculating with fractions, working with percentages, and simple algebra. But the evidence is that, even when teaching such basic material we in England have expected far too little—including from our more able pupils. Consider the following items, given to Year 9 pupils in around 50 different countries as part of the major international comparison TIMSS 2011:[6]

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[6] http://timss.bc.edu/timss2011/

1.4A Which fraction is equivalent to 0.125?

A: $\frac{125}{100}$ B: $\frac{125}{1000}$ C: $\frac{125}{10000}$ D: $\frac{125}{100000}$

1.4B Which number is equal to $\frac{3}{5}$?

A: 0.8 B: 0.6 C: 0.53 D: 0.35

1.4C $\frac{4}{100} + \frac{3}{1000} =$

A: 0.043 B: 0.1043 C: 0.403 D: 0.43

1.4D The fractions $\frac{4}{14}$ and $\frac{\dots}{21}$ are equivalent. What is the value of …?

A: 6 B: 7 C: 11 D: 14

1.4E Which of these number sentences is true?

A: $\frac{3}{10}$ of 50 = 50% of 3 B: 3% of 50 = 6% of 100

C: $50 \div 30 = 30 \div 50$ D: $\frac{3}{10} \times 50 = \frac{5}{10} \times 30$

1.4F Which shows a correct method for finding $\frac{1}{3} - \frac{1}{4}$?

A: $\frac{1-1}{4-3}$ B: $\frac{1}{4-3}$ C: $\frac{3-4}{3 \times 4}$ D: $\frac{4-3}{3 \times 4}$

1.4G Write $3\frac{5}{6}$ in decimal form rounded to 2 decimal places.

1.4H Simplify the expression

$$\frac{3x}{8} + \frac{x}{4} + \frac{x}{2}.$$

Show your work.

Success rates are never easy to interpret. But it seems sensible to compare the success rates for Year 9 pupils in England with those in Russia, in Hungary, in the USA, and in Australia rather than with countries from the Far East (for the released items and the corresponding results, see http://timss.bc.edu/timss2011/international-released-items.html). We note that:

- in Russia, children start school only at age 7, and in Hungary at age 6;
- the primary curriculum in Russia may include the idea of fractional parts, and the link with decimals, but calculation with fractions would seem to begin only in secondary school;
- tasks 1.4A–1.4F are multiple-choice questions with just four options, and some of the options could never be obtained as a result of making a mistake (which suggests that the English success rates for 1.4A–1.4C are already embarrassing).

1.4A Russia 86%, USA 76%, Hungary 74%, Australia 67%, England 62%;

1.4B Russia 84%, USA 83%, Australia 70%, Hungary 67%, England 59%;

1.4C Russia 83%, Australia 68%, Hungary 63%, USA 63%, England 57%;

1.4D Russia 62%, USA 55%, Hungary 49%, Australia 45%, England 43%;

1.4E Russia 58%, Hungary 53%, Australia 36%, USA 36%, England 33%;

1.4F Russia 63%, Australia 34%, Hungary 33%, USA 29%, England 28%;

1.4G Russia 39%, Australia 31%, Hungary 29%, USA 29%, England 24%;

1.4H Russia 35%, Hungary 34%, USA 19%, Australia 14%, England 9%.

The implication of these comparisons would seem to be that we in England

- are failing to achieve basic competence even for our more able pupils,
- that we routinely allow (or even encourage) pupils to move on to some "higher level" before basic material has been properly understood, and
- that we need to *slow down* and routinely use **slightly harder** and more varied problems to probe and strengthen pupils' understanding before they move on in this way.

This inference was supported by the recent ICCAMS study which set a sample of 15 year olds in English schools problems that had been used in a similar study in the late 1970s. We give just two examples:

> **1.4J** On the motorway my car can go 41.8 miles on each gallon of petrol. How many miles can I expect to travel on 8.37 gallons?
>
> [Six calculations involving 41.8 and 8.37 were given, and the relevant calculation was to be 'circled', not implemented.]

30 years ago **54%** of 14 year olds managed to circle 8.37 × 41.8; now **only 33%** manage this.

> **1.4K** Six tenths written as a decimal is 0.6. How would you write eleven tenths as a decimal?

30 years ago **36%** managed to write 1.1; now **just 16%** of 14 year olds respond correctly.

The message would seem to be clear. We need to do much more work with the **most basic** material to ensure that pupils grasp the relevant concepts. *The last thing our more able pupils need is to be accelerated.* They need to slow down, and to strengthen their understanding by tackling **harder**, and more varied, problems involving the same material as their peers. In particular, notwithstanding the wording of the requirement at the start of Section 1.4, able pupils may need challenges that are surprisingly basic, before they are confronted with material that is "rich and sophisticated".

The need to replace a philosophy of premature "acceleration" by a strategy of deepening and strengthening was strongly argued in the recent ACME report *Raising the bar*.[7] Any mathematics department which appreciates the importance of avoiding acceleration, but which anticipates being challenged by parents, or by senior management, will find valuable support in this report.

Ministerial advice regarding early GCSE entry has recently changed to reflect the same position. This change in official policy is partly based on overwhelming evidence. The instructive paper,[8] which was prepared by the Department for Education for the House of Commons Select Committee, contains some astonishing statistics that should also help to convince sceptical parents and management that acceleration incurs substantial human and resource costs with no evident benefits. Indeed, those who take GCSE early rarely benefit as a result.

This recent shift in policy is in line with the longstanding professional consensus, which was first stated in the analysis and the recommendations of the old report *Acceleration or enrichment*? (2000).[9]

2. Working mathematically

This section of the official programme of study contains eighteen bullet points under three headings: *Develop fluency*, *Reason mathematically*, and *Solve problems*. Many of these bullet points appear relatively unproblematic. Hence we restrict our remarks to those requirements that invite comment.

2.1. [*Develop fluency*, p. 4]

The list of themes referred to in the bullet points under this sub-heading in the official programme of study needs to be further supplemented: e.g. at

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[7] http://www.acme-uk.org/media/10498/raisingthebar.pdf

[8] http://www.parliament.uk/documents/commons-committees/Education/MemoSelectCommitteeGCSEMultipleEntryFinal.pdf

[9] http://education.lms.ac.uk/wp-content/uploads/2012/02/Acceleration_or_Enrichment_15Aug12.pdf

present there is no mention of measures, or of ratio and proportion, or of word problems, or of geometry.

In recent years those who decided what a typical pupil in England should be expected to learn have downplayed the importance of *memorisation*, and of *fluency*. Yet there are all sorts of reasons why we need to learn certain things **by heart**, and in general to achieve much higher levels of fluency. The word "fluency" is not quite the same as raw speed; but fluency, and the related notions of "learning by heart" and "automaticity", are useful indicators of understanding and mastery.

Memory contributes significantly to what we are, and to what we can do. We need to be completely on top of that limited collection of *basic facts* and techniques in terms of which most elementary mathematics can be understood. But we need to memorise far more than this. For example, when tackling an unfamiliar problem, one must be able

- to consider and choose between possible approaches and to compare the alternative intermediate steps in order to assess what seems to be the most promising strategy; and
- to achieve this, the possible steps or techniques need to be robustly internalised and immediately accessible.

Where a pupil struggles to use an idea, or fails to implement a learned procedure quickly and reliably, one can infer either that *the ingredient steps* need to be strengthened, or that more time needs to be devoted to *integrating these steps* into an effective method (or both).

When faced with routine inverse problems (such as "simplify $\frac{36}{54}$"; or "factorise $x^4 - 7x^2 + 1$"; or "make 24 with $3, 3, 5, 5$; or with $3, 3, 6, 6$; or with $3, 3, 7, 7$; or with $3, 3, 8, 8$"), one cannot begin unless the relevant direct facts are immediately to hand. Only then do we have a chance of recognising the relevance of those direct facts.

- We need immediate recognition that $36 = 4 \times 9$ and $54 = 6 \times 9$ in order to "simplify $\dfrac{36}{54} = \dfrac{4}{6} = \dfrac{2}{3}$".
- Given "3, 3, 5, 5 to make 24" we need to notice immediately that "5×5" is "close to 24", and then that "$3 \div 3$" makes up the difference.

- Later (at Key Stage 4 or beyond), unless the identity

  $$(a^2 - b^2) = (a - b)(a + b)$$

  is second nature, we are most unlikely to notice that

  $$x^4 - 7x^2 + 1 = (x^2 + 1)^2 - 9x^2 = (x^2 - 3x + 1)(x^2 + 3x + 1).$$

That is, we need to memorise enough to enable us to respond flexibly.

What you don't know by heart, and so can't access instantly, you can't use.

This observation applies not only to facts (such as $36 = 4 \times 9$, and $5 \times 5 = 25$), but also to *procedures*. That is, we need to attain **fluency** in handling a wide range of arithmetical, algebraic, trigonometric and geometrical procedures, so that each new procedure can eventually be exercised *automatically*, quickly, and accurately. Once this level of **automaticity** is achieved, the brain is free to focus on those more demanding aspects of a problem that require genuine thought (such as trying to see whether $x^4 - 7x^2 + 1$ can be written as a difference of two squares).

2.1.1 [*Develop fluency* p. 4]:

> - **consolidate their numerical and mathematical capability from Key Stage 2 and extend their understanding of the number system and place value to include decimals, fractions, powers and roots**
> - **select and use appropriate calculation strategies to solve increasingly complex problems**

2.1.1.1 Consolidating Key Stage 2 work, and choosing and using appropriate calculation strategies should start immediately in Year 7. In particular, **mental** work should continue, but should move beyond idiosyncratic methods (which may have been quite rightly encouraged at some stage, but which should then have moved on to more efficient methods) towards *structural arithmetic* in preparation for algebra. (The meaning of "structural arithmetic" is explained briefly in Subsection 2.1.1.2.)

There should also be a continuing thread of *word problems*, through which pupils learn to extract information from given text and to

> "select and use appropriate calculation strategies to solve ... problems".

(What is meant by "word problems" is outlined in Section 2.3 *Solve problems*, and in particular in Subsection 2.3.3.)

Particular attention should be paid

- to pupils' facility in working with decimals as an extension of earlier work with integers, including a robust grasp of
  (a) the "transition across boundaries" (from 0.9 to 1.0, or from 1.19 to 1.20, or from 2.99 to 3.0, etc.),
  (b) multiplying by a suitable power of 10 to change decimals into integers and conversely,
  (c) translating decimals into fractions and vice versa,
  (d) adding and subtracting fractions and decimals (see the ICCAMS and TIMSS examples 1.4A–1.4K above, and example 1.2.2C in Part III);
- to consolidating long multiplication and short division, and *simple* long division for integers;
- to extending the standard written arithmetical procedures for *integers to decimals* (column addition and subtraction, short and long multiplication and short division)—using these procedures to reinforce the idea of place value, and to solve word problems and other problems involving measures;
- linking division to quotients, or fractions, so that pupils understand how decimal division can be effected by multiplying both divisor and dividend by a suitable power of 10 to change the divisor into an integer.

2.1.1.2 We end this subsection by explaining briefly what we refer to as *structural arithmetic*. One feature of mathematics teaching at all levels is the need to re-visit topics and methods which have been previously learned, in order to think about familiar things in new ways. As long as one avoids simply repeating what was done before, much may be gained from time spent revising and strengthening vaguely familiar ideas, language, and methods—even when the material has already been well taught. Where pupils failed to grasp a topic at the first encounter, subsequent re-visiting and revision is essential if they are to progress; and those pupils who appeared to understand things the first time round can always benefit from re-visiting basic material *in the right spirit*.

The 2003, 2007, and 2011 results from TIMSS (a 4-yearly study of school mathematics in different countries) revealed a significant improvement in *average* success rates among **Year 5** pupils in England when tackling internationally designed test items. The natural response was to see this as constituting resounding support for the extensive efforts that had gone into the early *Numeracy Strategy*. But closer inspection (for example, of those problems where English pupils performed less well) suggested that these improved average scores

- derived mainly from success on relatively simple tasks, where correct answers could be obtained using "backward-looking" methods, and that
- pupils in Year 5 struggled with precisely the material that is most relevant to subsequent progress at Key Stage 3.

This impression was reinforced by the fact that **the apparent improvement in average Year 5 scores was not reflected in any corresponding improvement at Year 9** (even though the 2007 Year 9 sample was from exactly the same cohort as the 2003 Year 5 sample; and the 2011 Year 9 sample was from exactly the same cohort as the 2007 Year 5 sample). If this analysis is correct, then we clearly need to focus our mathematics teaching rather differently, so that our approach to the content being taught in Years 5–8 *actively prepares the ground* for the way elementary mathematics will develop subsequently.

In particular, at the interface between Key Stage 2 and Key Stage 3 the approach to mental calculation needs to move beyond methods designed solely to "get the answer". As the range of numbers in calculations expands (to include arbitrarily large integers, decimals, fractions, and surds), most of the expressions one could conceivably be asked to calculate are so messy that they cannot be easily evaluated or simplified. Something similar occurs in algebra when, during Key Stage 3 and Key Stage 4, the possible algebraic structure of the expressions to be manipulated gets progressively more complicated. Attention then shifts away from working with "expressions in general" and concentrates on expressions whose "structure" allows them to be evaluated or simplified. Progress in mathematics then depends more and more on learning to use the *algebraic rules* which sometimes allow one to *simplify* unexpectedly. Hence from Key Stage 2 onwards, calculation should begin to move beyond bare hands evaluation, and should concentrate on developing

- flexibility in looking for ways to exploit *place value* (as in $73 + 48 + 27 = \dots$, or $17.18 + 7460 + 22.82 = \dots$), and
- an awareness of the algebraic *structure* lurking just beneath the surface of so many numerical or symbolical expressions [as in $3 \times 17 + 7 \times 17 = \dots$, or $\frac{6 \times 15}{10} = \dots$, or $16 \times 17 - 3 \times 34 = \dots$, or $6(a - b) + 3(2b - a) = \dots$].

This habit of looking for, and then exploiting, **algebraic** structure in **numerical** work is what we call *structural arithmetic*.

2.1.2 [*Develop fluency* p. 4]:

> **use algebra to generalise the structure of arithmetic, including to formulate mathematical relationships**

Elementary algebra does not really "*generalise* the structure of arithmetic" as suggested in the above official requirement: algebra **copies** the structure of arithmetic *exactly* (that is, the four rules, together with the commutative laws, the associative laws, and the distributive law) and *applies it to a new "mixed universe"* of symbols (or letters) and numbers. Thus it is not the *structure* that is generalised, but the *universe* to which the old structure is applied.

This new domain of "elementary algebra" has several distinct aspects, or sub-domains, each of which sheds a slightly different light upon the subject.

Some of these sub-domains are more natural for beginners than others. The four most obvious ones—in approximate order of sophistication—are *formulae, equations, expressions,* and *identities.*

- *Formulae.* Here letters are used in place of familiar entities (e.g. $A = l \times b$ for the area A of a rectangle of length l and breadth b; or $C = 2\pi r$ for the circumference C of a circle of radius r). In each such formula, the letters can take different numerical values. The simplest formulae (such as $C = 2\pi r$) are rather like the simplest calculations that we meet at Key Stages 1 and 2, in that they tell us how the value of one entity (the circumference C) can be calculated once we know the values of certain others (the radius r).
- *Equations.* The first *equations* one meets involve a single letter (often denoted by "x"). This letter is usually referred to as the "unknown". An equation can be interpreted as a constraint which some unknown number "x" has to satisfy. Later one meets equations, and even pairs of equations, linking two or more "unknowns" (or "variables"). In all cases the strategy is the same: namely to transform the equations *using the rules of algebra* in a way that pins down the "unknown number (or numbers)" more precisely than was apparent in the original equation (or equations).
- *Expressions.* Given a formula, such as $C = 2\pi r$, we very soon want to move the letters around. For example:
  - Suppose we use string to measure the circumference C of a tall cylindrical lamp post and want to calculate the radius r (a length which we cannot measure directly). We then need to re-write the formula as $r = \frac{C}{2\pi}$ so that we can calculate r as soon as we know the value of the circumference C.
  - Or we may want to "expand" $(x+4)(x+2)$ or $(x+3)^2$; or to rewrite the quadratic equation "$x^2 + 6x + 8 = 0$" by "factorising" the LHS to get "$(x+4)(x+2) = 0$", or by "completing the square" to get "$(x+3)^2 - 1 = 0$".

In all these settings we need to know how to work with *expressions* made up of letters, and to transform them "as if the letters stood for numbers" (since this is exactly what the letters represent).

The fourth subdomain of elementary algebra—*identities*—is not mentioned explicitly in the Key Stage 3 programme of study. But it has already arisen in the previous bullet point; and it is highlighted by the later Key Stage 3 requirement (see Part III, section 2.4 below)

> "to simplify and manipulate algebraic expressions **to maintain equivalence**" [emphasis added].

Hence this subdomain is bound to arise in Key Stage 3, even if it is more evident at Key Stage 4 and beyond.

- *Identities*: In primary arithmetic the = sign at first tends to connect some required calculation such as "$13 + 29$" (on the left hand side) with the answer "42" (on the right hand side): $13 + 29 = 42$. But the = sign then broadens its meaning and is used to connect *any two numerically equivalent expressions*—such as "$13 + 29 = 6 \times 7$", or "$6^2 - 1 = 5 \times 7$", or "$\frac{28}{42} = \frac{10}{15}$". Something similar arises in the algebra of expressions, where pupils first learn that, given a jumble of symbols on the left hand side, one is expected to *simplify* it in some way and set it "equal" to something a bit like an "answer" (on the right hand side). For example one might be given an expression such as

$$\left(\frac{x}{x-1} - \frac{x+1}{x}\right)^{-1}$$

  and be expected to rewrite it as "$= x^2 - x$" (or as "$= x(x-1)$"). However one later broadens this use of the equals sign so that "=" simply links two expressions that are "algebraically equivalent"—that is, where one side can be transformed into the other side via the rules of algebra. Any such equation that links two expressions that are algebraically equivalent is called an *identity*.

Pupils become aware of these four subdomains gradually, generally starting with formulae, then equations and expressions. The goal throughout should be to establish two main principles:

- letters are essentially placeholders for numbers, and so are subject only to the laws of arithmetic (or algebra);
- in formulae and equations, the letters can take any values that are consistent with the constraint expressed by the formula or equation; in an expression or identity the only constraints are the laws of arithmetic—so the x in "$\frac{1}{x}$" cannot be set $= 0$, but otherwise the letters can be replaced by any values whatsoever, as long as different instances of the same letter are given the same value.

2.1.3 [*Develop fluency* p. 4]:

> **substitute values into expressions, rearrange and simplify expressions, and solve equations**

2.1.3.1 The requirement in 2.1.3 reinforces the immediately preceding bullet point. In an equation, the letters are constrained, so can only take **particular** (as yet unknown) values.

- In contrast, the letters in an algebraic *expression* are only required to satisfy the rules of arithmetic (or of algebra), *so can be replaced by **any** numbers whatsoever* (provided they are not clearly "forbidden values"—such as those that would make a denominator equal to zero).

Many pupils never grasp this fact, and so move letters around without realising that they are little more than placeholders for numbers, and must be treated as such. Pupils need more experience of **substituting given numerical values for the letters in an expression**, in order to internalise the idea that a letter can be given any value *provided all occurrences of the same letter are given the same value*. The act of substituting and evaluating also provides opportunities

- to exercise mental arithmetic, and
- to check that standard algebraic notation (juxtaposition as multiplication, brackets, powers, the fraction bar notation, priority of operations, etc.) is translated correctly when calculating with numbers.

Moreover, evaluating expressions in this way begins to convey the key idea that

- each choice of *inputs* gives rise to a single, determined *output* value for the expression.

That is, such expressions provide the simplest examples of what we will later call a *function* (of its component variables).

2.1.3.2 The expression "rearrange and simplify" in the quote at the start of 2.1.3 gives a slightly misleading impression. Algebra almost never involves "rearranging" for the sake of it: one "rearranges" the terms of a compound expression for a reason—and that reason is almost always **to simplify** in some way. We are formally allowed to rearrange, or to manipulate, expressions in any way that respects the rules of algebra; but in practice we ignore almost all rearrangements, and focus on those which seem likely to lead to a more manageable, or "simpler", result. Hence "rearrange and simplify" might have been better expressed as

> "rearrange *in order* to simplify".

In any event it is clear that pupils need more exercises (and *class discussion*) to help them learn what kinds of outputs are mathematically "simpler" (such as "fully cancelled" expressions, or those in "fully factorised" form), and to understand when and why the simpler forms are to be preferred.

2.1.3.3 The requirement at the start of Subsection 2.1.3 ends with three innocent-looking words: "and solve equations". In mathematics the expression "solve equations" strictly means "solve **exactly**"—by algebraic methods. We delay further comment on exactly what this means until Subsection 2.2.2.2. However, once this basic notion is understood, it can be modified, or re-interpreted in other fruitful ways.

The first such reinterpretation is to interpret the equations and the solving process *geometrically*. This reinterpretation does not help in the solution process itself, but it gives rise to interesting applications; it also provides a valuable alternative way of thinking about what is going on.

A different variation on the idea of "solving an equation" arises when we have no obvious way of finding an *exact* solution. It is then worth looking for effective ways of "getting close to" the elusive exact solution—that is, to find an *approximate* solution. The standard way to do this is to devise a process which allows us to "creep up on" a solution by generating a sequence which approaches the exact solution ever more closely. And the preferred kind of process is one which always takes the output from the previous step and operates on it in the same way to get the next approximation. This kind of repetition of a single process is called "iteration"—an idea which appears unannounced in the GCSE specification.[10] There, in item 20 under the heading *Number*, we read:

> **find approximate solutions to equations numerically using iteration;**

and in item 16 of *Ratio, proportion, and rates of change* we read:

> **work with general iterative processes**.

There is no officially required preparatory work at Key Stage 3. However, when finding approximate points of intersection of two straight lines, or of the graph of a curve and the x-axis, it may make sense to alert pupils to the desirability of having a deterministic numerical (rather than graphical) process that finds such solutions to any required degree of accuracy. At the same time one can prepare the ground as part of work with sequences, by exploring the behaviour of standard sequences that "converge" (such as $x_n = \left(\frac{1}{10}\right)^n$, or $x_n = \left(\frac{1}{2}\right)^n$, or $x_n = 1 - \left(-\frac{1}{2}\right)^n$, or $x_n = \frac{n-1}{n}$), and others that "diverge" (such as $x_n = 10^n$, or $x_n = 2^n$).

The geometrical interpretation of the meaning of "solve" arises because algebraic equations correspond to geometrical curves or surfaces. This important link between algebra and geometry was forged by Descartes (1596-1650), who in 1637 showed that solving an equation corresponds to

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[10] https://www.gov.uk/government/publications/gcse-mathematics-subject-content-and-assessment-objectives

- looking for points on a curve or surface where some expression takes a particular value (as with contour lines on an Ordinance Survey map); or
- looking for points where a curve or surface intersects a line (such as the x-axis), or a plane.

Hence the solutions of an equation, or of a system of equations, can be thought of as the coordinates of some point or points where two or more curves, or surfaces, meet. This is a powerful idea which can help to explain, for example,

- why some quadratic equations, such as $x^2 + 1 = 0$, have no solutions (because the curve $y = x^2 + 1$ never crosses the x-axis—that is, the line "$y = 0$"),
- why other quadratic equations, such as $x^2 = 0$, have just one solution (because the curve $y = x^2$ *touches* the x axis "$y = 0$"), and
- why many quadratic equations, such as $x^2 - 1 = 0$, have exactly two solutions (because the curve $y = x^2 - 1$ cuts the x-axis "$y = 0$" in two points: $(-1, 0)$ and $(1, 0)$).

The geometrical interpretation makes it possible for pupils to engage the *hand* and the *eye* to draw the relevant curves and to find the approximate coordinates of the points which correspond to solutions of the given equation(s)—a process that can help the *brain* to make sense of the **exact** algebraic solution process, which might otherwise remain a purely abstract idea. Without the insights provided by this geometrical interpretation, pupils can all too easily misapply the rules of algebra—even with such simple examples as:

- $x = x^2$ (where thoughtless cancellation can easily lead one to lose the solution $x = 0$). If we interpret solutions of this equation as the two points $(0, 0)$ and $(1, 1)$ where the familiar curves $y = x$ and $y = x^2$ cross, this can illustrate the error, can underline the importance of only cancelling factors which are never zero, and can help to reinforce the reasons for the standard algebraic method (when dealing with quadratics and higher powers) of "taking everything to one side and factorising".

The same idea applies to less familiar equations such as

- $x = 2x^4$ (or $x = 3x^5$) where one can again consider the *two* points where the curves $y = x$ and $y = 2x^4$ cross (or the *three* points where the curves $y = x$ and $y = 3x^5$ cross).

Later one can apply the same idea to $\cos x = 1$, to $\sin x = \frac{1}{2}$, or to $x = \tan x$, to see that each equation has *infinitely many* solutions.

Sketching the lines or curves corresponding to two equations can allow one to find approximate solutions by estimating the coordinates of the points where the lines or curves intersect. This kind of geometrical visualisation is didactically and *psychologically* invaluable. But it is not a logical, or mathematical way of actually "solving the equation"—any more than the unknown length of the hypotenuse of a right angled triangle with legs of lengths 3 and 4 (or a and b) can be mathematically calculated by drawing an approximate 3 by 4 rectangle (or an a by b rectangle) and then *measuring* the diagonal.

2.2. Reason mathematically

2.2.1 [*Reason mathematically* p. 4]:

extend and formalise their knowledge of ratio and proportion

The words "ratio" and "proportion" are here used correctly! But they are so often used incorrectly that we go into considerable detail (here and in Part III, Section 1.9) to explain the background that is needed if pupils are to "formalise their knowledge of ratio and proportion". We should perhaps stress that our comments throughout are designed to provide food-for-thought for teachers, and are not intended to constitute a teaching sequence for pupils.

Elementary mathematics comes into its own (and needs to be seriously *taught*!) as soon as we take the step from addition to *multiplication*. Ratios are the quintessential "multiplicative relations", and work with ratios links naturally to work with fractions.

The basic "knowledge of ratio and proportion" which all pupils need to build on is relatively familiar and accessible to all: so all can make some progress. And this matters, because the topic is important, and has many applications. However, the step that leads from a "common sense" view to its mathematical analysis is more delicate; and though the art of teaching consists in finding ways to make such things easier to digest, one should not underestimate the challenge in this case.

The initial stage is purely numerical.

> A position is advertised at £8.30 per hour (including specified breaks).
>
> If my weekly schedule counts as 25 hours, then I expect to earn
>
> $$25 \times £8.30 = £207.50.$$

Here the given data includes the "**unit cost**" of "earnings per hour", and the calculation reduces to a single multiplication. Despite the disturbingly low success rate for problem 1.4J above, this kind of multiplication can be made accessible to almost everyone. So it should be possible (even if it takes time and care) to extend this idea to problems where the "unit cost" has to be extracted first, before it can be used to find the required answer:

> If **my** schedule counts as 25 hours per week and **I** earn £207.50, what would **you** expect to earn if **your** schedule counted as 30 hours?

All that is needed is to insert an extra reverse step, before repeating essentially the same calculation:

if 25 hours ⟶ earns £207.50,
then **1 hour** ⟶ earns £ …
so 30 hours ⟶ should earn 30 × £8.30 = £249

In other words, all that is needed (once one has a template to organise one's thoughts and calculations) is to carry out **two** multiplications ("$\times\frac{1}{25}$" and

"×30") instead of one multiplication. This two-step process, where the **unit cost** is extracted first, is often referred to as "the **unitary method**".

The general situation of which the above is an example arises whenever two quantities, in this case

"hours worked" and "pay received (in £)"

vary together in such a way that, whenever we have *two linked pairs* of quantities, such as:

25 hours *corresponds to* £207.50,

and

30 hours *corresponds to* £249,

then the ratio between the two quantities of the first kind

25 : 30

is equal to the ratio between the two quantities of the second kind

207.50 : 249

This "equality of ratios" is called a **proportion**. We also say "the two quantities—hours worked, and pay received—vary in proportion to each other". (Slightly confusingly, this is sometimes referred to as "*direct* proportion"—to underline the contrast with "*inverse* proportion", where the two quantities x, y vary in such a way that the first quantity x varies in proportion to the *inverse* $\frac{1}{y}$ of the second quantity).

The "equality of ratios" can be re-written as an equality of *fractions*:

$$\frac{25}{30} = \frac{207.5}{249}$$

This is all very well, but each time we choose a different "linked pair of quantities" we get two new ratios. The new ratios are again equal, but they are *different from* the previous two ratios that were equal. **However**, if we rewrite the fraction equation in the form

$$\frac{25}{207.5} = \frac{30}{249}$$

then something remarkable happens: we obtain a quotient which is **always the same**—and which is called the **constant of proportionality**.

Teachers and schools will no doubt have their own ways of simplifying this idea. But there is no escaping from the need to prepare the ground by doing sufficient prior work with *word problems*, with multiplication and division, with *fractions*, and with *ratios*.

The teacher is like a midwife—using their own higher knowledge to coax ideas into pupils' minds. But to do this effectively, the teacher (like the midwife) needs to see the bigger picture—even if they then choose to suppress some of the details. *To generate a* ***ratio***, all that is needed is *a single class of pairwise* ***comparable*** *magnitudes* (that is, a class of magnitudes where any two given entities can be 'compared', so that we can decide which is the larger). The simplest examples of such a class of "comparable magnitudes" are the set of positive rational numbers, and the set of positive real numbers. In the context of ratios, real numbers generally arise as the set of possible numerical *measures* of some set of objects (relative to some chosen unit). *Numerical* ratios are easier to handle (replacing the class of *objects* by their *measures*). But ratios are not necessarily numerical. They arise naturally in mathematics whenever one has a class of "comparable entities" (such as line segments, or 2D shapes): we do not have to turn everything into numbers by measuring (for a very simple example, see Part III, Section 1.9.2).

The example above illustrates how a *proportion* arises whenever two **different** classes of entities are linked in a special (but very common) way. For example, suppose that one class consists of

"quantities of petrol"

and the other class consists of

"amounts of money in £".

If 1 litre of petrol	costs £1.50,
then we expect 2 litres	to cost £3 (= 2 × £1.50)

That is, for any two purchases from the same outlet at the same time, the ***quantities*** *purchased* (*in litres*)

are in the same ratio as

the ***amounts*** *paid* (*in £*).

If I buy a litres of petrol and pay £c,
and you buy b litres of petrol and pay £d,
then the ratio $a : b$ is equal to the ratio $c : d$.

The equality

$$\boldsymbol{a : b = c : d}$$

is what we call a *proportion*.

Note that since a, b, c, d are magnitudes, with a, b of one kind and c, d of another kind, then $a : b$ is a perfectly well-defined ratio; but **"$a : c$" makes no sense**, because a and c are not comparable magnitudes. One cannot have a ratio between a quantity of fluid and an amount of money. **However**, if we replace the different quantities and amounts by their numerical *measures*, then the equality of ratios "$a : b = c : d$" can be written as an *equation between fractions*, which can then be treated purely numerically (or algebraically), to give an equality of quotients, or fractions:

$$\frac{a}{b} = \frac{c}{d} \qquad (*)$$

The two quotients in equation $(*)$ are always equal, but can take any positive value. You could consider buying

$b = 2a$ litres of petrol and pay $d = 2c$ pounds,

and the quotients would then both take the value $\frac{1}{2}$. Or you could buy

$b = \frac{1}{2}a$ litres of petrol and pay $d = \frac{1}{2}c$ pounds,

and the quotients would then both take the value 2.

However, if we now treat the equation $(*)$ purely algebraically, then we can rewrite it in the form

$$\frac{c}{a} = \frac{d}{b}$$

This equation looks very similar to equation $(*)$, but it is completely different. The two sides do not represent ratios, but specify the *constant of proportionality* (relative to the two chosen units: litres and pounds (£)).

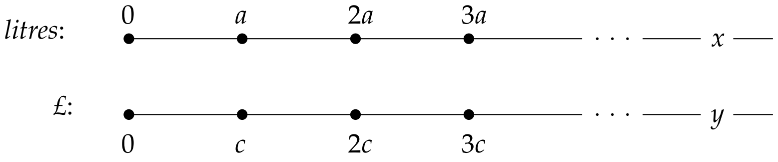
That is, once we choose units and give numerical values a and c to the basic pair of corresponding magnitudes—one from one class and one from the other

$$\boldsymbol{a}\ \textbf{litres} \longrightarrow \textbf{cost £}\boldsymbol{c}$$

the value of the quotient $\frac{c}{a}$ is a **constant**, the **constant of proportionality**. That is, it has the same value as the corresponding quotient $\frac{d}{b}$ *for any other pair* of corresponding magnitudes b, d (one from one class and one from the other).

This is the simplest, and perhaps the most valuable, application of school mathematics—to life, to science and to mathematics itself. It applies whenever two quantities are related so that if one quantity doubles, or triples, so does the other: that is, where the numerical measures a, c or b, d of the two quantities have a **constant ratio**. Two quantities that vary in such a way as to preserve a constant ratio between their values are said to be "in **proportion**".

The fact that "$\frac{c}{a}$ is a constant" means that the *number lines* corresponding to the two families of measures "line up" in such a way that one scale is simply a multiple ($\times\frac{c}{a}$) of the other:



If we imagine a linked pair (x, y) of unknown variables—where "x litres costs £y"—then these variables are connected by the linear equation

$$y = \left(\frac{c}{a}\right) x.$$

Any particular proportion problem that pupils may be required to solve is likely to involve just two pairs (a, c) and (b, d),

- where a and b come from one class of magnitudes, and c and d come from the other class.

In a typical proportion problem, three of the four values are given and the fourth is to be found. Hence one pair is completely known, and we take this as our "base", or "reference pair":

$$\textbf{\textit{a} litres} \xrightarrow{\qquad\qquad} \textbf{cost £\textit{c}}$$

One of the *other* two values b, d is "to be found". So the four ingredients can be thought of as the corners of a rectangular array, where three of the values are known and the fourth is to be calculated:

$$\begin{array}{l}
\text{If} \quad \textbf{\textit{a} litres} \xrightarrow{\qquad\qquad} \textbf{cost £\textit{c}} \\
\text{then } b \text{ litres} \xrightarrow{\qquad\qquad} \text{cost £??}
\end{array}$$

Alternatively, the missing value may be the one in the bottom left corner:

$$\begin{array}{l}
\text{If} \quad \textbf{\textit{a} litres} \xrightarrow{\qquad\qquad} \textbf{cost £\textit{c}} \\
\text{then ?? litres} \xrightarrow{\qquad\qquad} \text{cost £}d
\end{array}$$

This standard way of representing the four pieces of information in a proportion—with three known values and one generally unknown—is referred to here as the *rectangular template* for displaying *proportion* problems. We will revise and extend this example in Part III (p. 137ff).

2.2.2 [*Reason mathematically* p. 4]:

- **make and test conjectures about patterns and relationships; look for proofs or counterexamples**
- **begin to reason deductively in geometry, number and algebra, including using geometrical constructions**

Learning to distinguish between a plausible guess and a provable fact should be part of school mathematics from the earliest years. In Key Stage 3 this distinction takes on a new importance—but the requirement stated in 2.2.2 is difficult to interpret because the logical framework within which such deduction is to take place remains undeclared (e.g. for Euclidean geometry).

2.2.2.1 The problems begin already with the requirement to "reason deductively in number" when making sense of simple *numerical patterns*. At present the patterns pupils meet are often chosen in a way that misleads everyone into thinking that

patterns that ***seem*** *genuine,* ***always are*** *genuine.*

This makes it hard for pupils to discover the need for **proof**.

Consider, for example, the first 17 terms of what should be a familiar endless sequence:

$$2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384,$$
$$32768, 65536, 131072, \ldots$$

These are the successive *powers* of 2. Pupils can extend the sequence as far as they need simply by repeatedly multiplying by 2.

Now consider the two sequences that arise naturally from this sequence of "powers of 2" by looking at the two "ends" of each term of this sequence:

first the succession of *units* digits:

$$2, 4, 8, 6, \qquad 2, 4, 8, 6, \qquad 2, 4, 8, 6, \qquad 2, 4, 6, 8, \qquad 2, \ldots$$

then the succession of *leading* digits:

$$2, 4, 8, 1, 3, 6, 1, 2, 5, 1, \qquad 2, 4, 8, 1, 3, 6, 1, 2, \ldots .$$

As one continues to extend the original sequence of powers of 2, it is hard not to notice that both these sequences of digits **seem** to *recur*.

But do they really? And if they do, are these two conjectures really similar?

It is relatively easy to **prove** that **the first sequence "2, 4, 8, 6, 2, 4, 8, 6, ..." really does recur**. For we know that when we carry out the short multiplication, multiplying by 2 each time,

- each ***new*** *units digit* arises *from multiplying the* ***previous*** *units digit* by 2.

  So each time we reach a units digit of **2**, we notice that

- the units digit of the next term is **4** (since "2 × **2** ends in **4**");
- then "2 × **4** ends in **8**";
- then "2 × **8** ends in **6**";
- then "2 × **6** ends in **2**"—and the sequence "$\mathbf{2, 4, 8, 6}$" starts to repeat.

However, **the second sequence**

$$\mathbf{2, 4, 8, 1, 3, 6, 1, 2, 5, 1, \qquad 2, 4, 8, 1, \ldots}$$

is different. There is no obvious reason why the **leading** digits should recur as they seem to do.

> *Somehow pupils need to learn that what looks like a pattern may not be a pattern at all!*

So we have to insist that, in the absence of an acceptable proof, no pattern is simply "believed"—no matter how persistent it may seem to be.

2.2.2.2 The requirement to "reason deductively in **algebra**" is more interesting—and is explored surprisingly rarely. Proof in *algebra* has to be based on combining

- use of the commutative and associative laws of addition and multiplication, and the distributive law, to simplify *expressions*,

together with

- the idea that one is allowed to operate on the two sides of any equals sign in the same way without destroying the equality.

The most obvious example at Key Stage 3 and Key Stage 4 (about which the programme of study remains stubbornly silent) is the proof that

$$(-1) \times (-1) = 1.$$

There are all sorts of heuristic arguments that can be used to "justify" this crucial mathematical fact. One of the more plausible explanations is to consider *dieting* and *weight loss*.

- If I consistently *put on* 1kg per month, then in 3 months time, I will be (3×1)kg *heavier* than now; and

3 months *ago*, i.e. "**in $-$3 months time**", my weight was $[(-3) \times 1]$kg more than now.

- If I consistently *lose* 1 kg per month (that is, if I "put on (-1)kg per month"), then I know that

  in 3 months time, my weight will be 3kg less than it is now; and

  ($*$) 3 months ago, my weight would have been **3kg more** than it is now.

  If we try to express these observations arithmetically we see that

  in 3 months time, my weight will be $[3 \times (-1)]$kg more than it is now; whereas

  ($**$) 3 months ago, my weight must have been $[(-3) \times (-1)]$kg **more** than it is now.

  Taken together ($*$) and ($**$) seem to suggest that: $(-\mathbf{3}) \times (-\mathbf{1}) = \mathbf{3}$.

Such linguistic plausibility is fine at Key Stage 3. But at some stage in Key Stage 4, those who may move on to A level need to know that the fact has a simple mathematical basis. All we need to use is that:

(i) *multiplying by* 1 changes nothing: $a \times 1 = a$ for all a;

(ii) *adding* 0 changes nothing: $a + 0 = a$ for all a;

(iii) the distributive law holds.

It then follows that

- $$\begin{aligned} & a = 1 \times a \\ \therefore\quad & a = (1+0) \times a \\ \therefore\quad & \boldsymbol{a} + 0 = 1 \times a + 0 \times a = \boldsymbol{a} + 0 \times a \\ \therefore\quad & 0 = 0 \times a \text{ for all } a \end{aligned}$$

- $$\begin{aligned} \therefore 0 &= 0 \times (-1) \quad (\text{putting } a = -1 \text{ in ``} 0 \times a = 0 \text{ for all } a\text{''}) \\ &= (1 + (-1)) \times (-1) \\ &= 1 \times (-1) + (-1) \times (-1) \\ &= (-1) + [(-1) \times (-1)] \\ \therefore\quad & (-\mathbf{1}) \times (-\mathbf{1}) = \mathbf{1}.\ \textbf{QED} \end{aligned}$$

The proof given here is for teachers, and is based on the fact that

(i) is the defining property of the multiplicative unit "1", and

(ii) is the corresponding defining property of the additive identity "0".

Some readers may judge that *for pupils* the initial step—the fact that "$0 \times a = 0$ for all a"—is so familiar that the first bullet point is best suppressed.

A quite different fact that is often confused with the above is the fact that "subtracting a negative is the same as adding":

$$a - (-x) = a + x.$$

However tempting it may seem, little is gained by summarising this and the result proved above as "two minuses make a plus". The two results are in fact rather different: in the above equation there is no multiplication in sight. Moreover, the symbol "$-x$" should not be referred to as "a negative", since its value depends on the value of x itself: it is simply the "additive inverse of x"; that is, "$-x$" is the "negative of x", or that number which cancels out "x" under addition and produces 0.

Claim $a - (-x) = a + x$ for all a, x

Proof

(i) $a + (-x) + x = a + 0 = a$

Now subtract x from both sides:

$$\therefore a + (-x) = a - x$$

(ii) $a - (-x) + (-x) = a$

Use part (i) to replace "$+(-x)$" by "$-x$":

$$\therefore a - (-x) - x = a$$

Now add x to both sides:

$$\therefore a - (-x) = a + x. \quad \textbf{QED}$$

At Key Stage 3 schools will need to develop their own ways of achieving fluency in using such algebraic rules—for they are far from obvious! If the proof is illustrated numerically, one must first establish part (i), so that it can be used in part (ii); and it is important to give three or four examples—e.g. replacing a and x first by 1 and 2, then by 1 and -2, then by -1 and 2, and finally by -1 and -2.

A rather different opportunity for pupils to "reason deductively in algebra" arises in the solution of equations. We pointed out in Subsection 2.1.3 that "to solve equations" really means to solve **exactly**—by algebraic methods. A given equation in a single unknown "x" has an imagined (but unknown) set of "solutions", or possible values for the unknown "x". The art of solving equations algebraically is a process which exploits exactly two kinds of moves.

- The first kind of move allows us to replace any constituent expression on either side of the equation by another expression which is *algebraically equivalent* to it. Because "algebraically equivalent" expressions are equal **for all values of** x, this kind of move is *reversible*, so **exactly the same values** of the unknown "x" satisfy the new equation as satisfied the old equation.
- The second kind of move is to subject both sides of the equation to the same operation.

  - **If** this operation is *reversible* (such as adding or subtracting the same thing from both sides, or multiplying or dividing both sides by a given expression *that is never equal to zero*, or cubing both sides), **then** we can again be sure that **exactly the same values** of the unknown "x" satisfy the new equation as satisfied the old equation.
  - However, we are also free to subject both sides of the equation to an operation which is not reversible, such as squaring both sides of the equation. In this case we can only be sure that

    > *any value of "x" which satisfied the original equation*
    > *will also satisfy the new equation.*

    That is, any solution of the original equation is also a solution of the new equation, so we can be sure that *we have not* ***lost*** *any solutions*. However, we may have **gained** some new solutions which did not satisfy the original equation. For example,

    > if "$A = B$", then we can square both sides to get the new equation "$A^2 = B^2$";

  but the change may introduce new solutions, since $A^2 = B^2$ includes the possibility that $A = -B$, which is quite different from the original equation.

A third domain where pupils should learn to "reason deductively in algebra" arises with *inequalities*. In many ways equations are rather rare. In mathematics and in life *inequalities* are much more common. For example, a business never quite 'breaks even': it either makes a surplus, or it makes a loss. And for a production line to keep running, one can never order the *exact* amount of material that is required: one has to slightly over-order to make sure that the supply of what is needed *never runs out* (and one would like to do so in such a way that waste is reduced to a minimum). This means that real problems are often formulated in terms of *inequalities*.

Much of what holds for *equations* translates to *inequalities*.

- The solution of a **linear** *equation* in one unknown "x" is a single point on the x-axis; and the solution of the corresponding **linear** *inequality* consists of all values *on one side* of this point (a "half-line").

- The possible solutions of a **linear** *equation* in two variables x, y correspond to the set of all points (x, y) on a line, which divides the plane into two "half-planes"; and the solutions of the corresponding **linear** *inequality* in two variables x, y consists of all points *on one side* of the line—that is, in one of the two half-planes.

The algebraic rules for "solving inequalities" are very similar to the rules for solving equations. For example, one is allowed to add the same to both sides of an inequality, or to multiply both sides of a given inequality by a *positive* quantity. But there is a twist: a *negative* multiplier *reverses* the inequality!

The extent to which inequalities are neglected in England is clear from one of the 2011 TIMSS Year 9 items:

2.2.2.2A "Solve the inequality: $9x - 6 < 4x + 4$".

We can transform the given inequality by collecting terms (or more correctly, by "adding $6 - 4x$ to both sides") to get $5x < 10$.

We can then multiply both sides by the positive multiplier $\frac{1}{5}$ to obtain "$x < 2$".

The percentage of correct responses to this problem from a representative sample of 15 year olds in more than 50 countries was not encouraging, and included:[11]

2.2.2.2A Korea 60% Russia 46% Hungary 38% USA 21%
Australia 8% England 3%

This suggests rather starkly that our approach to deduction and calculation in algebra needs to change in order to establish a clear connection between the familiar processes used in solving *equations* and those required to solve *inequalities* (which are listed in the Key Stage 3 programme in the third bullet point of "Algebra", and which feature in the GCSE mathematics subject content list, so certainly warrant preliminary work at this level—even if a more formal treatment can be delayed until Key Stage 4).

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[11] http://timss.bc.edu/timss2011/international-released-items.html

2.2.2.3 The requirement to "begin to reason deductively in **geometry**" and to include "geometrical constructions" is in some ways easier to achieve. But it is in other ways more delicate.

Geometry at Key Stage 1 and 2 is predominantly experiential and descriptive. However, once the basic repertoire of shapes and language has been established, one can begin to organise the subject matter at Key Stage 3 into a logical, or deductive, hierarchy. For example:

- once one knows that angles at a point P on a straight line add to 180°,
- one can **prove** that, whenever two lines cross at a point P, any pair of *vertically opposite* angles A and A' at P are necessarily equal:

  [**Proof:** Let B be the angle "between" the two vertically opposite angles A and A'. Then $A + B$ is the straight angle on one line, and $B + A'$ is the straight angle on the other line.
  $\therefore A + B = B + A'$, so $A = A'$. **QED**]

This proof only depends on the assumption (which pupils and teachers alike accept without even noticing) that:

> **all 'straight angles'** (at possibly different points on two straight lines) **are equal**.

Plane geometry deals with imagined *points* and *lines*. Two points determine exactly one line, and two lines which are not parallel meet in exactly one point. This much should be clear—though it needs to be reinforced in Year 7 through appropriate drawing exercises.

More importantly, the methods and language of geometry require us to make a clear distinction between

- the *line* AB (which passes through the two points A, B, and which extends forever in both directions)

and

- the *line* ***segment*** AB (that starts at A, runs to B, and then stops), which in the UK is usually also written as AB.

For example, the sides of triangle ABC are not the "lines" AB, BC, CA, but rather the line *segments* AB, BC, CA. Geometrical experience prior to Year 8 needs to ensure that these ideas can be taken for granted without drawing explicit attention to them. However, to go further, one needs a clear framework within which the basic results of Euclidean geometry can be derived. And since such a framework remains largely implicit (or even hidden) in the programme of study, it may help if we give (here and in Part III, Section 3) a brief outline of the necessary background.

The whole of geometry in 2D and in 3D rests on one key idea, which needs to be cultivated at Key Stage 2, and strengthened at Key Stage 3 through drawing, and through making and examining standard structures. This is the discovery that **triangles** hold the key to the construction and analysis of more complicated shapes. Every integer can be factorised as a product of prime numbers, and this factorisation tells us important things about the original number, even though most of the details of this factorisation cannot be seen when one first looks at the starting number. In much the same way important properties of complicated geometrical configurations can be analysed in terms of their constituent *triangles*, even though these triangles may not be immediately apparent in the initial configuration. (This strategy of reducing geometrical reasoning in general to reasoning about *triangles* is also related to the fact that the rigidity of structures in engineering—such as cranes, or roof trusses, or bridges, or the Wembley arch—often comes down to the way triangles are built in to their design.)

Mathematics succeeds by translating sense impressions and language, or sounds, into symbols which allow *exact calculation*. "The sum of three consecutive even integers" makes perfect sense in English, but the words alone suggest nothing special. However, as soon as we translate the words into symbols and write this as

$$2n + (2n + 2) + (2n + 4) = 6n + 6$$

it is clear that the result is always a multiple of 6. The same is true in geometry. The English words "triangle" or "quadrilateral" may conjure up a visual impression in the mind's eye of an imagined shape. But one cannot calculate with such an impression. If we wish to refer to a particular triangle or quadrilateral, we may point it out; and others may notice things about the indicated shape; but they cannot talk, or reason deductively about a triangle or quadrilateral which has been indicated in this way. Just as "consecutive even integers" were given names in accordance with the rules of algebra, so **a triangle or quadrilateral has to be given a name** in accordance with certain conventions before we can begin to calculate with it.

Labelling conventions have to communicate reliably between individuals, and so are chosen to reflect the underlying geometric structure. For example, a polygon is a collection of line segments, where successive pairs meet at a shared vertex. Hence *the sequence in which the vertices are labelled matters*. A quadrilateral $ABCD$ has to be labelled **in cyclic order**, where the edges are the successive line segments, or edges, that make up the quadrilateral: AB and BC (meeting at B), BC and CD (meeting at C), CD and DA (meeting at D), and DA and AB (meeting at A). Just as the neglect of grammar and spelling makes it impossible for pupils to organise and to express their thoughts, and hence to be understood, so it is an indication of the anarchy in English school geometry that standard geometric conventions are routinely flouted without the serious consequences being recognised.

There is another oversight which may prove harder for some to swallow. The reader is invited to imagine (and to draw, and to label) two adjacent unit squares—$ABCF$, $FCDE$. The squares $ABCF$ and $FCDE$ are clearly different, but very much alike. But we would not usually quibble if I referred to the first square as $ABCF$ and **you** referred to it as $BCFA$ (or even $BAFC$—but **not** $AFBC$). However, $ABCF$ and $BCFA$ are in some sense *different*—whether they are different squares or just different *labellings* need not be decided immediately. The difficulty may be clearer if one considers the two rectangles $ABDE$ and $DBAE$: much of the time one may loosely think of these as "different ways of referring to the same rectangle". But life is much easier if one views them as *different*—though closely related. This becomes clear as soon as one tries to sharpen the feeling that the two rectangles are "the same", or "congruent"; for then the "sameness" one is trying to capture requires one to match them up in a way that essentially *changes* the labelling of the second rectangle, since "AB" (the first side mentioned in $ABDE$) is a short side, whereas "DB" (the first side mentioned in $DBAE$) is a long side. It does not matter whether the matching up leads one to think of the second rectangle as $ABDE$, or $BAED$, or $DEAB$, or $EDBA$; but it becomes silly to insist on calling it $DBAE$ while also insisting that it is "the same as $ABDE$". Even if we do not strictly insist on such precision all the time, each time we do some kind of "calculation" with a triangle, or a quadrilateral, we find that the **order** matters (as well as the sequential labelling of the vertices).

So there is a clear sense in which, whenever push comes to shove, a "triangle" is not just a three-cornered shape: it is a *labelled, or ordered, triple* ABC, where **the order matters**. (If one only knows the three vertices, but not the order, then this corresponds to several *different* triangles: $\triangle ABC$, $\triangle BCA$, $\triangle CAB$, $\triangle BAC, \ldots$.)

Each triangle involves *six* different pieces of data:

- the three side lengths: $AB = c$, $BC = a$, $CA = b$, and
- the three angles: $\angle ABC$ (often abbreviated as "B"), $\angle BCA$ (abbreviated as "C"), and $\angle CAB$ (abbreviated as "A").

There are three basic organising principles on which deductive reasoning in geometry is based. Two of these principles (relating to *congruence* and to *parallels*) belong naturally to Key Stage 3; the third organising principle (the *similarity criterion*) belongs slightly later—perhaps in Year 9 or Year 10.

The first organising principle is the *congruence criterion*. This underlines the central role played by triangles, and should arise naturally as a formal summary of pupils' extensive experience from drawing, where they should discover that:

> one does not need to know *everything* about a triangle in order to specify it *uniquely*.

The "congruence criterion" then summarises the information that is needed to specify a triangle uniquely.

Two (ordered) triangles $\triangle ABC$ and $\triangle DEF$ are *congruent* if the (ordered) correspondence

$$A \longleftrightarrow D, \quad B \longleftrightarrow E, \quad C \longleftrightarrow F$$

matches up each of the six ingredients of triangle $\triangle ABC$ with those of triangle $\triangle DEF$ in such a way that

- all three *corresponding* pairs of line segments are equal: $AB = DE$, $BC = EF$, $CA = FD$, and
- all three *corresponding* pairs of angles are equal: $A = D$, $B = E$, $C = F$.

We write this as: $\triangle ABC \equiv \triangle DEF$ (which we read as "Triangle ABC is congruent to triangle DEF").

"Congruence of triangles" only makes sense between **ordered** triangles. And it can help pupils to see more clearly which vertex of the first triangle corresponds to which in the second triangle, and which side of the first triangle corresponds to which in the second triangle, if pupils initially write:

$$\begin{array}{rl} & \triangle ABC \\ \equiv & \triangle DEF \end{array}$$

since this lines up

- corresponding vertices (with A directly above D, B directly above E, C directly above F), and
- corresponding sides (with AB directly above DE, BC directly above EF, CA directly above FD).

Pupils' experience of drawing should then reveal that, in order to guarantee congruence

> "just three pieces of data suffice",

provided we avoid the two triples that don't suffice! Hence they need to understand

- that any of SSS, SAS, ASA determine the triangle uniquely,
- that "AAA" determines the shape, but says nothing about the scale, or size, of the triangle;

- that the appropriately named "ASS" criterion is different, in that it may give rise to two possible triangles (for example, a triangle with $\angle B = 30^\circ$ and $BC = \sqrt{3}$, $CA = 1$ could have either $AB = 2$ with $\angle A$ acute, or $AB = 1$ with $\angle A$ obtuse).

The *congruence criterion* summarises the first of these three bullet points:

- triangles $\triangle ABC$ and $\triangle DEF$ are congruent (by SSS) if $AB = DE$, $BC = EF$, and $CA = FD$;
- triangles $\triangle ABC$ and $\triangle DEF$ are congruent (by SAS) if $AB = DE$, $\angle BAC = \angle EDF$, and $AC = DF$;
- triangles $\triangle ABC$ and $\triangle DEF$ are congruent (by ASA) if $\angle BAC = \angle EDF$, $AB = DE$, and $\angle ABC = \angle DEF$.

The RHS congruence criterion is not part of this basic congruence criterion, so does not really belong at this stage. It arises as the degenerate instance of the failed ASS criterion (where the angle "A" in "ASS" is a *right angle*, and so is neither acute nor obtuse). The fact that RHS guarantees congruence follows somewhat later (once we have proved *Pythagoras' Theorem*, since knowing two sides and a right angle then determines the third side. So RHS is a special case of SSS).

SSS, SAS, and ASA congruence allow one to prove such results as:

- The two diagonals of a square $ABCD$ are equal

  [**Proof** The two triangles $\triangle ABC$, and $\triangle BCD$ are congruent by SAS:

  $$\begin{aligned}&\triangle ABC\\ \equiv\ &\triangle BCD \quad \text{(since } AB = BC,\ \angle B = \angle C, \text{ and } BC = CD).\end{aligned}$$

  Hence AC (in $\triangle ABC$) $= BD$ (in $\triangle BCD$). **QED**].

- The base angles of an isosceles triangle are equal:

  [**Proof 1** Suppose $AB = AC$. Construct the midpoint M of the base BC. Then, by SSS,

$$\begin{aligned}&\triangle AMB\\ \equiv\ &\triangle AMC \quad (\text{since } AM = AM,\ MB = MC\ (M \text{ is the midpoint of } BC),\\ &\qquad\qquad \text{and } AB = AC\ (\text{given})).\end{aligned}$$

Hence $\angle ABM$ (in $\triangle AMB$) $= \angle ACM$ (in $\triangle AMC$). **QED**].

It is worth pondering on a different proof of this result, which exploits the fact that $\triangle ABC$ and $\triangle ACB$ are different triangles.

[**Proof 2** Suppose $AB = AC$. Then the two different ordered triangles $\triangle BAC$, and $\triangle CAB$ are congruent by SAS:

$$\begin{aligned}&\triangle BAC\\ \equiv\ &\triangle CAB \quad (\text{since } BA = CA,\ \angle A = \angle A, \text{ and } AC = AB).\end{aligned}$$

Hence $\angle B$ (in $\triangle BAC$) $= \angle C$ (in $\triangle CAB$). **QED**].

- Any triangle with equal base angles is isosceles:

[**Proof** Suppose $\angle ABC = \angle ACB$. Then the two different ordered triangles $\triangle ABC$, and $\triangle ACB$ are congruent by ASA:

$$\begin{aligned}&\triangle ABC\\ \equiv\ &\triangle ACB \quad (\text{since } \angle ABC = \angle ACB,\ BC = CB, \text{ and } \angle BCA = \angle CBA).\end{aligned}$$

Hence AB (in $\triangle ABC$) $= AC$ (in $\triangle ACB$). **QED**].

- In an isosceles triangle, the bisector of the apex angle, the median to the base, and the perpendicular to the base are all the same.

Isosceles triangles constitute one of the simplest and most fruitful sources of geometrical deduction. For example, in a circle any chord AB forms an isosceles triangle OAB with the centre O, so isosceles triangles allow one to deduce all sorts of properties of circles (the so-called "circle theorems").

The congruence criterion is also needed to prove that the basic ruler and compass constructions do what they claim to do:

- to bisect a given angle,

- to bisect a given line segment,
- to construct a perpendicular to a given line from a given point, and
- to construct a line parallel to a given line through a given point.

For example:

- ***To bisect a given angle*** $\angle \boldsymbol{BAC}$. Let the circle with centre A and passing through B meet the half line AC at the point B'. Let the two circles—one with centre B and passing through A, the other with centre B' and passing through A— meet again at D. Then $\boldsymbol{AD}$ ***bisects*** $\angle \boldsymbol{BAC}$.

  [**Proof** We show that $\triangle BAD \equiv \triangle B'AD$ (by the SSS congruence criterion), since
  $\boldsymbol{BA} = \boldsymbol{B'A}$ (both are radii of the same circle with centre A).
  $\boldsymbol{AD}$ (in $\triangle ADB$) $=$ $\boldsymbol{AD}$ (in $\triangle ADB'$) (the one segment is part of both triangles)
  $DB = AB$ (both are radii of the same circle with centre B)
  $AB = AB'$ (both are radii of the same circle with centre A)
  $AB' = DB'$ (both are radii of the same circle with centre B')
  $\therefore \boldsymbol{DB} = \boldsymbol{DB'}$.
  Hence $\angle DAB$ (in $\triangle BAD$) $=$ $\angle DAB'$ (in $\triangle B'AD$), so DA bisects $\angle BAC$. **QED**]

The second organising principle in geometry is the criterion for two lines in the plane to be *parallel*.

Given any two lines in the plane, a *transversal* is a third line that cuts both of the two given lines. The *parallel criterion* declares that:

- two lines are *parallel* precisely when the *alternate angles* (or the *corresponding angles*) created by a transversal are equal.

This is a rather subtle criterion, but one which can be made thoroughly plausible. It immediately allows one to prove:

Claim The angles in any triangle $\triangle ABC$ add to 180° (i.e. a "straight angle").

Proof Construct the line XAY through vertex A which is parallel to BC (where X, B both lie on the same side of the line AC).
$\therefore \angle XAB = \angle CBA = \angle B$ (alternate angles)
and $\angle YAC = \angle BCA = \angle C$ (alternate angles)

$$\begin{aligned}\therefore \angle A + \angle B + \angle C &= \angle A + \angle XAB + \angle YAC \\ &= \angle XAY (\text{a straight angle at } A \text{ on the line } XY). \ \textbf{QED}\end{aligned}$$

And this in turn provides access to hundreds of wonderful (non-obvious, multi-step) problems involving *angle chasing*. This term is a shorthand for any activity in which a 2D configuration is specified, with the sizes of certain angles given, from which the sizes of other angles are to be logically determined (by reasoning and calculation, not by measuring). If the required angle were the third angle in a triangle whose other two angles were given, then the required angle could be immediately deduced. In general the size of the required angles may not be immediately deducible, but may force one to first calculate certain intermediate results. That is, *angle-chasing* refers to a restricted (geometrical) class of problems that are *multi-step*, and that are also deductive exercises in using the basic angle properties (angles at a point, angles on a straight line, vertically opposite angles, angles in a triangle—and later alternate angles). See for example, *Extension mathematics Book Alpha* by Tony Gardiner (Oxford 2007), Sections T9 and E2, and *Book Beta* Sections T17, C11 and E4.

If we combine the *parallel criterion* with the *congruence criterion*, we can prove the basic facts about parallelograms, and derive the fundamental fact that the area of a triangle is equal to

$$\frac{1}{2}(\text{base} \times \text{height}).$$

This then allows us to prove *Pythagoras' Theorem*.

The congruence criterion and the parallel criterion allow one to transfer *exact* relations (such as *equality* of line segments or of angles) from one place to another. The third organising principle of secondary school Euclidean geometry, the *similarity criterion*, goes beyond this world of *exact equality* to allow one to deal with ratios, scaling, and enlargement. The introduction of this criterion is probably best delayed until the basic consequences of congruence and parallelism have been fully explored, and until pupils are sufficiently confident in working with ratio.

As with congruence, similarity in general is formulated in terms of "similarity of *triangles*". *The similarity criterion* summarises the minimum requirement for two given triangles to be "similar". Two (ordered) triangles $\triangle ABC$ and $\triangle DEF$ are *similar* (which we write as $\triangle ABC \sim \triangle DEF$) if

- corresponding angles are equal:

$$\angle A = \angle D, \quad \angle B = \angle E, \quad \angle C = \angle F,$$

and

- corresponding sides are proportional:

$$AB : DE = BC : EF = CA : FD.$$

The *similarity criterion* may be thought of as a substitute for the (evidently false) "AAA congruence criterion", in that it states that each of the above bullet points **implies the other**:

> if corresponding angles are equal:
>
> $$\angle A = \angle D, \quad \angle B = \angle E, \quad \angle C = \angle F,$$
>
> then corresponding sides are proportional:
>
> $$AB : DE = BC : EF = CA : FD;$$

and

> if corresponding sides are proportional:
>
> $$AB : DE = BC : EF = CA : FD,$$
>
> then corresponding angles are equal:
>
> $$\angle A = \angle D, \quad \angle B = \angle E, \quad \angle C = \angle F.$$

Special cases of this can be proved using the *exact* relation of congruence. For example, one can prove the *Midpoint Theorem*, which says that:

> if in $\triangle ABC$, M is the midpoint of AB and N is the midpoint of AC,
> then MN is parallel to BC and $BC : MN = 2 : 1$.

That is $\triangle ABC \sim \triangle AMN$, with the corresponding scale factor $AB : AM = AC : AN = BC : MN = 2 : 1$.

Some further detail concerning geometry may be found in Part III, Section 3.

2.2.2.4 The requirements listed at the start of Section 2.2.2 suggest that during Key Stage 3 pupils should

> "make and test conjectures"

and

> "begin to reason deductively".

This should be interpreted as part of the (unstated) requirement that pupils should at all times expect the methods of elementary mathematics to *make sense*. But there are different kinds of "sense making": some involve *inference*; some involve plausibility arguments; and some are rooted in *deduction*. The requirement for pupils to "reason deductively" means that they need to be clear

- when they are experimenting or conjecturing, and when they are working "deductively";

and also

- when they are working in rough, and when they are writing for others to read.

That is, they need some way of demonstrating (to themselves and to others) which mode they are in at any given time. For pupils who are ready for the formal procedures of elementary mathematics at Key Stage 3, the calculations and methods should be increasingly justified in ways that are *exact* and deductive (rather than approximate, inferential, or based on the authority of the teacher). The essence of elementary mathematics at secondary level incorporates the twin facts

- that its domain is restricted and abstract, and
- that within this limited domain, the knowledge it delivers is **certain**—that is, *objective*, rather than subjective (or approximate, or based only on experience, or conjecture, or convention).

For such pupils calculations and solutions need to be increasingly presented in a way that constitutes a **proof** that the answer to the original problem is undeniably what emerges at the end of the calculation or solution. And this is best conveyed by laying out calculations and deductions **line-by-line**,

- with the given information, and any symbols representing "unknowns" declared at the outset,
- with each fresh step on a new line (and any explanation given alongside),

and

- with the final answer clearly displayed at the end.

The sequence of successive steps can then be grasped as a single *chain of reasoning*, in which each step follows clearly from those which went before. This logical structure is equally applicable

- to simple calculations,
- to the solution of an angle-chasing problem,
- to setting up and solving an equation,
- to proving that two algebraic (or trigonometric) expressions are identical,
- to a ruler and compass construction,
- to a **proof** (such as, that the angle in a semicircle is a right angle), or

- to the way pupils present their solutions to set problems.

It is hard to convey this style consistently in a discursive text such as this one. But it can be seen in the way we present short proofs. And it is also visible elsewhere—such as in the solution at the end of Section 2.3.5 below.

2.3. Solve problems

[*Solve problems* p. 5]:

> – **develop their mathematical knowledge, in part through solving problems and evaluating their outcomes, including multi-step problems**
>
> – **develop their use of formal mathematical knowledge to interpret and solve problems, including in financial mathematics**
>
> – **begin to model situations mathematically and express the results using a range of formal mathematical representations**
>
> – **select appropriate concepts, methods and techniques to apply to unfamiliar and non-routine problems**

These four bullet points are clearly meant to encourage pupils and teachers to see school mathematics as more than endless practise with dry-as-dust formal technique. But beyond this admirable aspiration, it is far from clear what exactly is being advocated. We base our commentary on three questions.

- What is meant by a “problem”, rather than (say) an “exercise”?
- What does it mean to “solve problems”?
- And why are “multi-step” problems important?

2.3.1 We begin by clarifying the distinction between “exercises” and “problems”.

An **exercise** is a task, or a collection of tasks that provide *routine practice* in some technique or combination of techniques. The techniques being exercised will have been explicitly taught, so the meaning of each task should be clear. Each sequence of exercises is designed to cultivate fluency in using the relevant techniques, and all that is required of pupils is that they implement the procedures more-or-less as they were taught in order to produce an answer. The overall goal of such a sequence of exercises is merely to establish mastery of the relevant technique in a suitably robust form. In particular, a well-designed set of exercises should help to avoid, or to eliminate, standard misconceptions and errors.

Exercises are not meant to be particularly exciting, or especially stimulating. But they can give pupils a quiet sense of satisfaction. Without a regular diet of suitable *exercises*, ranging from the simple to the suitably complex (including standard variations), pupils are likely to lack the repertoire of basic techniques they need in order to make sense of mildly more challenging tasks (as the examples 1.4A-1.4K above show). In other words,

> *exercises* are the bread-and-potatoes of the mathematics curriculum.

Pupils in England clearly need more (carefully prepared) "bread-and-potatoes" exercises than they currently get. However, bread and potatoes alone do not constitute a healthy diet. Pupils also need more challenging activities both to whet their mathematical appetites, and to cultivate an inner willingness to tackle, and to persist with, simple but unfamiliar (or "non-routine") **problems**. A *problem* is any task which we do not immediately recognise as being of a familiar type, and for which we therefore know no standard solution method. Hence, when faced with a *problem*, we may at first have no clear idea how to begin.

The first point to recognise is that a task does not have to be all that unfamiliar before it becomes a *problem* rather than an *exercise*! In the absence of an explicit problem solving culture, an exercise may appear to the pupil to be a *problem* simply because its solution method has not been mentioned for a week or so, or because it is worded in a way which fails to announce its connection with recent work. The second point is that the distinction between a *problem* and an *exercise* is not quite as clear-cut as we have made it look, and is to some extent time- and pupil-dependent. For example, an "I'm thinking of a number" *problem* from Year 5 or Year 6 should by Year 8 be seen to be a mere *exercise* in setting up and solving a simple equation.

Most useful techniques involve a *chain* of simple steps, and the technique as a whole is only an effective tool if *the complete chain* can be carried out **entirely reliably**—a requirement which may only be achieved after extensive practice. Examples include: any of the standard written algorithms; the process of turning a fraction into a decimal; the sequence of steps required to add or subtract two fractions, or to solve an equation or inequality, or to multiply out and simplify an algebraic expression. Hence each set of *exercises* should include tasks that force pupils to think a little more flexibly, and that require them to string simple steps together in a reliable way. Too many sets of *exercises* get stuck at the level of "one piece jigsaws"—with one-step routines being practised in isolation, ignoring key variations. Pupils need to learn from their everyday experience that the whole purpose of achieving fluency in routine bread-and-potatoes *exercises* is for them to learn to marshal these techniques to solve more demanding **multi-step** *exercises*, and more interesting, if mildly unsettling, *problems*.

2.3.2 This distinction between *exercises* and *problems* affects how we choose to introduce each new topic or technique. Should we concentrate on relatively simple examples that minimise pupil difficulties, and which seem likely to guarantee a quick pay-off? Or should we—when working with the whole class—move quickly on to examples that provide a significant challenge, and so require pupils from the outset to grapple with (carefully chosen) tasks of a more demanding nature?

How challenging one can safely be will depend on the pupils. But experience from those who observe lessons in other countries suggests that the English preference for concentrating the initial worked examples on easy cases **increases the extent of subsequent failure**. Easy initial examples lead to cheap apparent success; but this initial pupil success may be based on pupils' own inferred methods that appear to work in easy cases, but which are flawed in some way; or on backward-looking methods, that seem (to the pupil) to work in simple instances, but which do not extend to the general case. So we need to consider the benefits of starting each new topic with a harder "class problem" that brings out the full complexity of the method that we want pupils to master, and then to follow this up with *exercises* that may start simply, but which oblige pupils to think flexibly from the outset, and to handle standard variations including inverse problems.

2.3.3 The last 30 years have witnessed a consistent concern about pupils' ability to "use" the elementary mathematics they are supposed to know. Previous versions of the mathematics National Curriculum displayed an admirable determination to incorporate "Using and applying" within teaching and assessment. But such determination is not enough. The experience of the last 25 years in England is more useful as a guide to what does **not** work than to what does work. Much effort has been expended in trying to do better—but with limited effect. In particular, ambitious attempts to coerce change—using extended investigations, coursework, and "modelling"—have mostly served to demonstrate what should **not** be officially required at this level.

Somewhere along the line we seem to have lost sight of simple **word problems**. *Word problems* typically consist of two or three short sentences, from which pupils are required

- to extract the intended meaning and any required information,
- to identify what needs to be done,

and then

- to carry it out, and interpret the answer in the context of the problem.

Everyday uses of elementary mathematics tend to come in some variation of this form. Yet the simplest exercises, which might be solved routinely if they were presented *without words*, become powerful discriminators when given this gentle packaging. The need for pupils to read and extract the relevant data from two or three English sentences may appear routine—but it is a skill that has to be learned the hard way, and that constitutes the initial stepping-stone *en route* to the ultimate solution of almost any problem. This simple format can be tweaked to cover the standard variations of the underlying task (e.g. so that it appears both in *direct* and in the various *indirect* forms).

During Key Stage 1 *word problems* are important because they reflect the fundamental links between

- the world of mathematical ideas and mathematical reasoning,

and

- the world of language.

Indeed, for young children, the *logic* of mathematics is inextricably bound up with the *grammar* of language.

At later stages *word problems* continue to serve as an invaluable way of linking the increasingly abstract world of mathematics and the world where its ideas can be applied. That is, they constitute the simplest exercises and problems in any programme that seeks to ensure that elementary mathematics can be used.

The suggestion that improving mathematical literacy depends on rediscovering the world of carefully structured word problems is both more ambitious and more modest than what has been attempted in recent English reforms.

- It is *more* ambitious in that the evidence from other countries shows just how much more we might achieve were we to incorporate a *permanent thread* of such focused material from the earliest years.

- It is more modest in that it explicitly encourages *more focused* (and hence more manageable) tasks—short problems with a clearly specified beginning and end, but with the path from one to the other left for the solver to devise. Such problems have "closed" beginnings and "closed" ends, but are **open-middled**. Almost any mental arithmetic problem, or word problem, might serve as an example. Suppose we ask:

  > "*I pack peaches in* 51 *boxes with* 16 *peaches in each box.*
  >
  > **How many boxes would I use if each box contained just** 12 **peaches?**"

  What is given and what is required is "closed"—i.e. specified uniquely. But the mode of solution is left entirely open:

- some pupils might calculate the total number of peaches and then divide by 12;
- one would prefer to see a more structural version of this representing the total number of peaches as "51×16" without evaluating, and the required number of boxes as $\frac{51 \times 16}{12}$ before cancelling

$$\frac{17 \times (3 \times 4) \times 4}{12} = 17 \times 4;$$

- others might notice that $3 \times 16 = 4 \times 12$, and look for the number x satisfying "$x : 51 = 4 : 3$";
- while some might remove 4 peaches from each of the 51 boxes and group the 4s in groups of 3×4 to get 17 additional boxes.

2.3.4 Pupils need a regular diet of problems and activities designed to strengthen the link between elementary mathematics on the one hand and its application to simple problems from the wider world on the other. *Word problems* are only a beginning.

Some have advocated using "real-world" problems. But though these may have a superficial appeal, their educational utility is limited. Problems which support the move towards using and applying beyond the limited world of *word problems* need to be very carefully constructed, so that the real context truly reflects the mathematical processes pupils are expected to use as part of their solution. (Problems which have to be carefully designed in this way are sometimes called "realistic".)

The related claim that technology allows pupils to work with "real-world problems" and with "real (or 'dirty') data" becomes important once the underlying ideas have been grasped. However, for relative beginners the claim too often ignores the distracting effect of the *noise* which is created by "real" contexts, by "real" data, and by the non-mathematical interface that so easily prevents pupils from grasping the underlying mathematical message.

2.3.5 The official programme of study makes repeated reference to the need to solve **multi-step** problems. A *multi-step* problem is like a challenge to cross a stream that is *too wide to straddle with a single jump*, so that the prospective solver is obliged to look for stepping-stones—intermediate points which reduce the otherwise inaccessible challenge of crossing from one bank (what is given) to the other (the completed solution) to *a chain of individually manageable step*s. In elementary mathematics, this art has to be learned the hard way. It should not be seen as optional, or as a matter of taste. It is central to what elementary mathematics is about, and to how it is used.

One might think that—given the original emphasis on *Using and applying*—this goal has been an integral part of the National Curriculum since its inception. But that is not quite true—for we have too often confused

- "solving problems", and tackling "multi-step" problems

with

- *real-world* problems, and *extended* tasks.

The limitations of "real-world" problems were outlined in the previous Section 2.3.4. An *extended* task allows pupils considerable freedom, and can be beneficial precisely because the outcomes lie to some extent outside the teacher's control. However, this lack of predictability and control means that extended tasks are **not** an effective way for most pupils to *learn* the art of solving *multi-step* problems. For most teachers, this art is much more effectively addressed through **short**, easily stated problems in a specific domain (such as number, or counting, or algebra, or Euclidean geometry), where

- what is given and what is required are both clear,
- but the route from one to the other requires pupils to identify one or more intermediate stepping-stones (that is, they are "open-middled")—as with
  - solving a simple number puzzle, or
  - interpreting and solving word problems, or
  - proving a slightly surprising algebraic identity, or
  - angle-chasing (where a more-or-less complicated figure is described and has to be drawn, with some angles given and some sides declared to be equal, and certain other angles are to be found—using the basic

repertoire of angles on a straight line, vertically opposite angles, angles in a triangle, and base angles of an isosceles triangle), or

- proving two line segments or two angles are equal, or that two triangles are congruent (where the method of proof is not immediately apparent).

The steps in the solution to a multi-step problem are like the separate links in a chain. And the difficulty of such problems arises from the need to select and to link up the constituent steps into a single logical chain. Suppose pupils are faced with:

> **Question:** "I'm thinking of a two-digit number $N < 100$, which is divisible by three times the sum of its digits? How many such numbers are there?"

In Year 7 pupils may see no alternative to guessing, or to testing each "two digit number" in turn. But by Year 9 one would like some to respond to the trigger in the question

> "three times the sum of its digits"

by gradually noticing some of the hidden stepping stones.

Steps toward a solution

1. The number has to be a multiple of 3 ("divisible by **three times** the sum of its digits").
2. Hence the sum of its digits must be a multiple of 3 (standard divisibility test).
3. But then the number is divisible by 9 ("divisible by three times a multiple of 3").
4. And so the sum of its digits must be a multiple of 9 (standard divisibility test).
5. So the number is divisible by 27 ("divisible by three times a multiple of 9").
6. So we only have to check 27, 54, and 81. **QED**

The sequencing of the steps, and the connections between the steps, are part of the solution. In short, *basic routines become useful only insofar as sufficient time is devoted to making sure they can be linked together to solve more interesting (multi-step) problems.*

2.3.6 Expecting pupils to select and to coordinate simple routines to *create* a chain of steps in order to solve simple multi-step problems should be part of mathematics teaching for all pupils. In contrast, recent efforts to improve the effectiveness of mathematics instruction in England have concentrated on:

- the teacher, textbook author, or examiner *breaking up* each complex procedure into easy steps, and then concentrating on teaching and assessing the easy steps, or atomic outcomes (one-piece jigsaws),
- monitoring centrally whether these atomic outcomes can be performed in *isolation*, and
- ignoring the fact that we have neglected the most demanding skill of all—namely that of *integrating* the separate steps into an effective *multi-step procedure*.

The evidence from international studies confirms what should have been obvious: this reductionist process of de-constructing elementary mathematics into atomic parts, combined with central monitoring that rewards partial success, has distorted the way pupils and teachers perceive elementary mathematics in a most unfortunate way. Improved problem solving and more effective mathematics teaching depend on enhancing the skill of the teacher. In contrast, the policy of focusing on *targets* and *testing*, and our misplaced dependence on crude measures of "pupils' progress", have tended to undermine the authority, the professional judgement, and the perceived long-term responsibility of the teacher.

Solving problems is hard. Any system that uses targets and testing to exert pressure on schools soon discovers the awkward facts that assessment items that require pupils to link two or more steps

- have a high failure rate, and
- generate pupil responses whose profile is at odds with the contractual demands placed on those who design centrally administered tests.

Such problems are therefore deemed unsuitable, and the tests tend to concentrate on more manageable *one-step* routines (or break down longer questions into a pre-ordained sequence of one-step "subroutines"). As long as teachers are judged on test outcomes, and as long as unfamiliar, multi-step problems are largely excluded from the official tests, teachers will continue to conclude that "in the (short-term) interests of their pupils" they dare not waste time developing the only thing that matters in the long run—namely:

> *to provide their pupils with the skills and attitudes they need for the* ***next*** *phase.*

In short, England has adopted an "improvement strategy" that guarantees neglect of the delicate art of solving multi-step problems, and that is therefore self-defeating. Central prescription, and political pressure to demonstrate relentless year-on-year improvement, have resulted in a national didactical blind spot, with curriculum objectives and assessment—and hence teaching—becoming atomised, so that pupils are only expected to handle "one piece jigsaws". Exams have routinely broken down each problem into a succession of easy steps—in order to minimise the risk of failure, and to ease "follow through marking" for the examiner. Teachers have then concluded that the delicate art of *interlinking* simple steps can be safely ignored. And we have all pretended that

- candidates who can implement (most of) the constituent steps separately
- have thereby achieved mastery of the *integrated* technique.

This is a delusion. The individual steps may be a starting point; but the power and challenge of elementary mathematics lies in learning *how simple ideas can be combined* to solve problems that would otherwise be beyond our powers. That is, the essence of the discipline lies not so much in the techniques themselves as in the *connections* between its ideas and methods. Hence the curriculum (and, where possible, its assessment) need to cultivate the ability to tackle *multi-step* problems without them being artificially broken down into steps.

A curriculum or syllabus can specify the individual techniques, or steps; but this is futile if one then forgets that it is the **linking** of the material which determines whether it can be effectively used to solve problems. This interlinking is an elusive property, *which depends entirely on the way the material is* **taught**: that is, it depends on the teacher. So we need a system in which teachers are free (nay, in which teachers feel professionally obliged) to value this activity in their classrooms, even though its value will only become apparent at *subsequent* stages—after their pupils have moved on to other classes.

2.3.7 Two further issues warrant comment before we move on to consider the listed *Subject content* requirements in Part III. The first is the matter of *exactness* and *approximation*, and the second is the repeated reference to "financial mathematics".

Mathematics used to be known as "the *exact* science". Mathematical objects sometimes have their roots in the world of human experience; but they become *mathematical* only when the underlying ideas are abstracted from these roots. Unlike disciplines that work with real data or objects, mathematics studies a world of idealised, *mental* objects. For example,

- *numbers* have their roots in experience;
- but they soon become "mental objects" with exact properties, and are manipulated in the mind.

In much the same way, a sheet of A4 paper, or a wooden door, may serve as a suggestive model for a rectangle, but

- a *mathematical rectangle* is a perfect mental object, whose diagonals are *exactly* equal—their length being given *exactly* (in terms of the sides) by *Pythagoras' Theorem*.

The mathematical universe consists of *imagined* objects, which are precisely defined, and hence uniquely knowable. In particular, mathematics, or "the art of exact calculation", belongs to a completely different conceptual universe from the practical world in which one might

> "draw a scale diagram of a rectangle and *measure* the approximate length of the diagonal".

Helping pupils to appreciate the difference between these two universes, and to see the advantages—even for the most practical of purposes—of engaging with the *exact* world of mathematics, should constitute a key (though often unstated) goal of any curriculum.

The process of developing internal methods of calculating with these exact *mental objects* (whether numbers, symbols, shapes, or functions) is much the same today as it ever was—and is rooted in mental work and written hand calculation. Once these ideas are suitably embedded in the mind, calculators and other tools have much to offer: but initially, the learning process proceeds more naturally without such distractions.

The fact that the world of mathematics operates on *ideal objects* allows its ideas, its notation, its methods of calculation, and its processes of logical deduction to be *exact*. This guarantees that the answers and conclusions produced in mathematics are as reliable as the information that was fed into the relevant calculation or deduction. The importance of this aspect of elementary mathematics has been considerably blurred in recent years—for example, by inappropriate and premature dependence on calculators, by reduced emphasis on the need to attain mastery of the art of exact calculation, and by the way "valuing children's own reasons" has been misconstrued.

In contrast to the *exact* mental universe of mathematics, the world of experience, of measurements, and of ideas is inescapably "fuzzy". It should be a goal of any curriculum to convey implicitly this key distinction between the *exact* world of mathematics, and the approximate world where mathematics is used and applied.

Mathematical *exactness* is quite different from *precision*. The very idea of "precision" recognises that, outside mathematics, all measurements incorporate a degree of error, and so are **approximate**. In contrast, *exactness* in mathematics allows no scope whatsoever for error; indeed, in an *exact* calculation an error of any kind undermines the validity of the whole process. Mathematical methods can be applied to values which are only known *approximately*; but the "exact answer" which mathematics then provides indicates the exact *range of values* within which the actual answer must lie. To achieve this, we first need to know

- the maximum extent of potential error in the given data, and

- how these potential errors accumulate when one carries out exact calculations with numbers that are only known *up to this level of accuracy*.

For pupils to master the art of approximating arithmetical calculations in integers, they first need to master the art of *exact* calculation. Only then can they use their knowledge of exactness as a fulcrum for thinking precisely about more elusive *approximation*, or *estimation*. (This matter is explored further in Part III, at the end of Section 1.7.) And when they come to analyse the errors introduced by such approximations, they will find that this is done via the exact calculations of elementary algebra. Thus, even when seeking to transcend the inherent exactness of arithmetic by cultivating the art of making *estimates*, there is no escape from the maxim:

> *Mathematics is the science of exact calculation.*

Finally, while it is perfectly fair to require that pupils be required to

> "develop their use of formal mathematical knowledge to interpret and solve problems",

this challenge applies to problems of many different kinds. So there is no possible excuse for adding the words "including financial mathematics" in the second bullet point of 2.3. There is no such subject area as "financial mathematics" at Key Stage 3; so its explicit inclusion can only reflect an enforced response to improper political lobbying. Some material relating to financial matters will inevitably be included (e.g. as an application of percentage increases and decreases, and of iterated powers as a model for the returns on long term investment or the accumulation of debt). But the precise words are no more worthy of special mention in a national curriculum than many other examples.

III. The listed subject content for Key Stage 3

In Part III we examine the detail of the listed *Subject content*. To comment on each bullet point in turn would tend to reinforce the fragmentation that arises when a curriculum is reduced to a mere content list. So we have tried instead to group the bullet points in a way that allows us to identify common threads and underlying themes, and to indicate some of the linking that may be needed.

1. Number (and ratio and proportion)

1.1. [Subject content: *Number* pp. 5–6]

- **understand and use place value for decimals, measures and integers of any size**
- **order positive and negative integers, decimals and fractions; use the number line as a model for ordering of the real numbers; use the symbols** $=, \neq, <, >, \leqslant, \geqslant$
- **use standard units of mass, length, time, money and other measures, including with decimal quantities**
- **round numbers and measures to an appropriate degree of accuracy [for example, to a number of decimal places or significant figures]**
- [*Algebra* p. 6] **work with coordinates in all four quadrants**

At Key Stage 3 basic number work acts as an essential bridge, reaching back to Key Stage 2, and looking ahead to the more subtle multiplicative methods of Key Stage 3—with 'structural arithmetic' serving as a template for elementary algebra.

Within this context, the five requirements listed in 1.1 constitute a very simple beginning, since they focus on the size of numbers, and do not yet address *arithmetic*. But it would be unwise to assume that these ideas will therefore not require consolidation and strengthening. Consider these two released items[12] from TIMSS 2011 which were set to pupils in Year 5.

1.1A In which number does the 8 have the value 800?

A 1,468   B 2,587   C 3,809   D 8,634

1.1B Which number is 100 more than 5,432?

A 6,432   B 5,532   C 5,442   D 5,433

These are very basic questions; and the answer to each question is given as one of four options. One should therefore expect almost all pupils to answer correctly. But the results suggest that we in England may expect less than comparable countries (some of whom start school significantly later than we do). We have included here the results from Flemish Belgium (who took part in TIMSS 2011 at Year 5, but not at Year 9).

1.1A Russia 90%, USA 87%, Flem Bel 87%, Australia 75%, England 68%, Hungary 66%

1.1B Flem Bel 84%, Russia 82%, USA 80%, Australia 73%, England 73%, Hungary 73%

Moreover, the examples 1.4A, 1.4B, 1.4C, 1.4D, 1.4G, 1.4K in Part II above suggest that this weakness needs to be (and is often not) addressed between Year 5 and Year 9.

Given the fourth requirement listed at the start of 1.1 we include an additional item from TIMSS 2011 for pupils in Year 9:

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[12] http://timss.bc.edu/timss2011/international-released-items.html

1.1C Write $3\frac{5}{6}$ in decimal form rounded to two decimal places.

Here one expects significantly lower scores—but the English success rate is nevertheless disappointing:

1.1C Russia 39%, Australia 31%, Hungary 29%,
USA 29%, (Ave. 25%), England 24%,

The second bullet point at the start of 1.1 refers to "the number line". At Key Stages 1 and 2 the number line provides a valuable image which allows the different forms of "number" to be seen as part of a *single number system*. Moving along the number line also provides a useful physical model for skip-counting and for addition and subtraction—including with negative numbers (though it is less helpful with multiplication and division). But during Key Stage 3 the number line gradually loses its separate existence and becomes identified with the x-axis (and y-axis) in a coordinate system. The ordering of real numbers is then needed on both axes to locate points in the plane, where pupils need to learn to work comfortably with coordinates "in all four quadrants".

At Key Stage 3 the family of real numbers extends to include not only decimals and fractions, but also *negative numbers*, and later *surds*. A lot of work is needed to ensure that negative numbers and their arithmetic become a natural part of pupils' mental universe of mathematics. For example:

- locating "-3" and "-2.5" on the number line, or x-axis, helps to underline the ordering (e.g. $-3 < -2.5 < -2$);
- common sense may suggest that "measures" and "quantities" have to be *positive*, but pupils need to learn to interpret *negative* quantities in practical situations, so that, for example, "-3 hours' from now" is routinely interpreted as "3 hours *ago*".

The inequality symbols mentioned in the second requirement listed at the start of 1.1 may appear unproblematic. We see $2 < 3$ as being entirely natural; and $-3 < 2$ may seem only marginally less obvious (though it still needs to become second nature). However $-3 < -2$ is nowhere near as obvious as one might think, and has clearly not been well handled in the past.[13]

There seem to be few TIMSS 2011 released items on ordering numbers. But one Year 5 item suggests a need for further work on ordering fractions.

> **1.1D** Which of these fractions is larger than $\frac{1}{2}$?
>
> A $\frac{3}{5}$ B $\frac{3}{6}$ C $\frac{3}{8}$ D $\frac{3}{10}$
>
> **1.1D** USA 62%, Russia 62%, Flem Bel 58%, Australia 54%, England 50%, Hungary 48%

Each such set of responses needs to be assessed on its own merits—bearing in mind that there are many hidden details that make the raw data hard to interpret reliably. For example, as far as one can tell, the primary curriculum in Russia does not seem to include explicit work on fractions or their arithmetic; but the *idea* of a fraction is clearly addressed in some preliminary way. The success rates in other countries are therefore merely guides as to what might reasonably be expected. The success rate for English pupils in example 1.1D is in fact just above the "international average"; but this "average" is skewed by many countries whose education systems are much less well developed. So it makes sense to focus any comparison on systems that are more naturally comparable with England.

In helping pupils make sense of "$<$" and "$\leqslant$", we need to be aware that these are *relations*, which are true if used for certain *pairs* of real numbers, and are false for other pairs. The truth of "$2 < 3$" and "$2 \leqslant 3$" may seem obvious. But it can be harder for pupils to accept that "$2 \leqslant 2$" is equally true.

In many countries, the list of standard symbols in the second bullet point at the start of 1.1 would include a symbol (usually $\approx$) to stand for "approximately equal to". It is perfectly natural to stretch the use of "$=$" to include

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[13] http://www.manchestereveningnews.co.uk/news/greater-manchester-news/cool-cash-card-confusion-1009701

> "$2\pi = 6.28$ (2 d.p.)", or "$\sqrt{2} = 1.4$ (2 s.f.)", or "$\sin 60^\circ = 0.866$ (3 d.p.)".

But given the requirement to use symbols "correctly", and to work with *rounding, estimates* and *approximations*, it is worth introducing a special symbol "$\approx$", and using it consistently whenever one is "actively approximating", as in:

$$35,941 \times 273 \approx 33,333 \times 300 \approx 10,000,000 = 1 \times 10^7.$$

These matters are addressed in more detail in Section 1.7 below.

The reference to "measures" in the first, second, and fourth requirements must include *compound measures*. A "compound" measure arises when two basic measures are combined: *area* is a compound measure, where length is multiplied by length, measured in "cm^2" (say); *speed* arises when length is divided by time, and is measured in "metres per second" or "miles per hour"; *density* arises when mass is divided by volume, and is measured in "grams per cubic centimetre". Other compound measures include "rates of pay", "fuel consumption", and "unit prices". One might think that compound units will be familiar from Key Stage 2 (even if only implicitly), because any problem which involves "measures" and "multiplication" inevitably involves *compound* measures:

> **Question** "I travel at 60 mph for 4.5 hours. What distance do I cover?"
>
> **Answer** $4.5 \times 60 = 270$ miles
>
> **Question** "My car consumes 8 litres of petrol per 100km. How much fuel is needed to drive 170 kilometres?"
>
> **Answer** $8 \times 1.7 = 13.6$ litres.

Yet compound measures are not explicitly mentioned in the Key Stage 2 programme of study! So those teaching at Key Stage 3 must anticipate that time may be needed to ensure that pupils can work comfortably with compound measures.

We end by mentioning one topic that can contribute much to pupils' understanding of place value, but which has dropped out of the official curriculum. That is, to engage in numerical work in *other bases*. We particularly recommend work in *base* 2, in *base* 9, and in *base* 11. *Base* 2 lies behind the 0–1 of all electronic devices; but it has other pedagogical advantages (such as allowing a row of seated pupils to emulate the sequence of digits representing a number, and to enact a human numerical "counter", with each pupil standing for "1" and sitting for "0"). *Base* 9 and *base* 11 are closer to the familiar *base* 10; and it can be highly instructive for pupils to extend the standard written algorithms by inventing and working with a new symbol for the "digit 10"—say "*X*"—when working in *base* 11. They can also discover the thought-provoking fact that

> in *base* 11 a number is "divisible by ten"
>
> precisely when "the sum of the digits is divisible by ten",

which matches the base 10 rule for divisibility by 9 (see Section 1.4.4 below). For more confident pupils it can be highly instructive to extend the notation for integers to "decimals" in these other bases, and to realise that whether a fraction has a terminating "decimal" depends on the *base*, not on the fraction itself.

1.2. [Subject content: *Number* p. 5]

- **use the four operations, including formal written methods, applied to integers, decimals, proper and improper fractions, and mixed numbers, all both positive and negative**
- **use conventional notation for the priority of operations, including brackets, powers, roots and reciprocals**
- **recognise and use relationships between operations including inverse operations**

The final paragraph of Section 1.1 above illustrates how difficult it is to separate the notation for *place value* from arithmetic, or work with *operations* (the four rules, powers, etc.), which is the focus of the present section.

1.2.1 Throughout the official Key Stage 3 programme of study there is an unfortunate silence concerning *mental* and *oral* work with numbers. The increase in the variety of forms in which "numbers" are encountered (positive integers, fractions, terminating and recurring decimals, negative numbers, surds, etc.) *increases* the need for such oral work at this level.

- Work with integers needs to be continually exercised, and extended to negatives.
- The same mental procedures need to be actively extended to work with decimals.
- Work with integers needs to be extended rather differently to support work with fractions.
- The "algebraic" conventions (for powers, for fractions, for brackets, for priority of operations, and for roots) need to be exercised fluently and automatically *with numerical expressions*, so that they are clearly understood **before** these conventions are extended to symbols.

As the examples 1.4A–1.4K in Part II indicate, such mental work has clearly been undervalued in English secondary schools for some decades, with significant consequences for pupils' subsequent progression. Here we can only illustrate what is needed on the simplest level, where pupils should be routinely expected to evaluate mentally such expressions as:

> $1.2 + 0.8$, $2(14.3 - 3.8)$, 17×0.9, 1.2×80, $1.08 \div 1.2$, $1.7 \times 13 + 0.3 \times 13$, $(0.8)^2$, $(0.4)^3$, $(1.2)^2$, $(0.12)^2$, $\sqrt{2.25}$, $\sqrt{1.96}$, $\sqrt{6.25}$, $\sqrt{16}$, $(\sqrt{2})^3$, $\sqrt{27}$, $\sqrt{100}$, $\sqrt{1000}$, $\sqrt[3]{27}$, $\sqrt[3]{64}$, $0.625 + \frac{3}{5}$, $\frac{4}{100} + \frac{35}{10000}$ as a decimal, $\frac{3}{10}$ of 40% of $50 \div 60$, $\frac{1}{3} - \frac{1}{4}$, $3\frac{5}{6}$ as a decimal.

In addition to mastering simple calculation, mental and oral work is perhaps even **more** important, and even less common, in **thinking about** calculation, and numerical relations. This is indicated by the following three released items[14] for Year 5 pupils from TIMSS 2001.

> **1.2.1A** $\square$ stands for the number of pencils Pete had. Kim gave Pete 3 more pencils. How many pencils does Pete now have?
>
> A $3 \div \square$ B $\square + 3$ C $\square - 3$ D $3 \times \square$
>
> **1.2.1B** $4 \times \square = 28$. What number goes in the box to make this sentence true?
>
> **1.2.1C** $3 + 8 = \square + 6$. What number goes in the box to make this number sentence true?

In all three cases English success rates are around, or below the international average.

> **1.2.1A** Russia 91%, Flem Bel 85%, USA 83%, Hungary 82%, Australia 79%, England 75%
>
> **1.2.1B** Russia 95%, Flem Bel 94%, Hungary 91%, USA 87%, England 82%, Australia 77%
>
> **1.2.1C** Russia 80%, Hungary 50%, Flem Bel 49%, USA 47%, Australia 33%, England 29%

1.2.2 The standard *written* algorithms need further attention at Key Stage 3 to secure their reliability for integers. More confident pupils can avoid mere repetition by concentrating on *inverse problems* to test their understanding (the meaning of "inverse problems" was explained in Part II, Section 1.2.3). We offer two more released items from TIMSS 2011 for Year 5 pupils as evidence that there will still be plenty to do in Year 7.

> **1.2.2A** $5631 + 286 = \ldots$
>
> **1.2.2B** $23 \times 19 = \ldots$

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[14] http://timss.bc.edu/timss2011/international-released-items.html

Some will find the English success rates acceptable. But these are exercises one should expect almost all pupils to get right—as the results from other countries tend to confirm. In all cases the English performance is either below or just above the "international average".

> **1.2.2A** Russia 89%, USA 84%, Hungary 77%,
> England 67%, Flem Bel 66%, Australia 57%
>
> **1.2.2B** Russia 74%, USA 59%, Hungary 40%,
> England 37%, Flem Bel 26%, Australia 11%

Schools who actively seek to strengthen arithmetic in Year 7 and who need harder "inverse" problems for pupils whose arithmetic is strong, could do worse than to include lots of "missing digit" problems (for example, see Tony Gardiner, *Extension Mathematics Book Alpha* p. 46, p. 61, p. 74, p. 125).

These written procedures then need to be extended to decimals. And the simplest calculations with *decimals* (such as 71.6×2.8, or $271.6 \div 2.8$) demonstrate that this extension to decimals needs the corresponding integer procedures to routinely handle *multi-digit inputs* (at the very least 716×28, and $2716 \div 28$). In the released TIMSS 2011 items at Year 9, decimal arithmetic mostly arises in context. But the following item tends to reinforce the suggestion that we currently expect too little.

> **1.2.2C** $42.65 + 5.748 = \ldots$
>
> **1.2.2C** Russia 90%, USA 89%, Hungary 88%,
> Australia 82%, England 79%

1.2.3 At this level, calculation with fractions becomes increasingly pervasive (solving simple numerical problems involving multiplication; understanding how the standard written algorithms of column arithmetic for integers extend to those for decimals; rearranging equations and simplifying expressions; using percentages; working with ratio and proportion). And something clearly needs to change if many more pupils are to learn to calculate reliably and confidently with fractions: examples 1.4A–1.4K in Part II above suggest that we currently fail to lay the most basic foundations. Rather than offer a trite summary here, we postpone discussion of fractions until Section 1.6 below— where, as a tentative contribution to the re-thinking that is needed, we outline some of the relevant background.

1.2.4 All three of the official requirements listed at the start of 1.2 include the word "use"; but the intended scope of the word is left unexplained. The official intention here may be restricted to *technical* usage, rather than to "applications". But we take the opportunity to explore what it means for pupils to be able to use what they have learned.

The last 35 years have witnessed a stream of complaints that those leaving school cannot "use" what they have been certified as "knowing". This suggests that everyone may have misunderstood what is required if a learned technique is to become available for use.

The ability to use the mathematics one knows

- includes its use within other parts of mathematics; and
- extends to simple applications, or word problems (see Section 2.3.3 in Part II for an explanation of what is meant by *word problems*).

In both domains, pupils' inability to "use what they know" often has the same cause, and stems from

- the fact that a typical technique is first learned as a deterministic *direct* procedure,
- whereas applications frequently require a flexibility in using the procedure in the spirit of the corresponding *inverse* process (the distinction between *direct* and *inverse* is explained in Part II, Section 1.2.3).

In other words, pupils' difficulties often reflect our failure to recognise the gulf between

- fluency in the underlying easy *direct* skill, and
- what is needed to work flexibly with this direct skill, and to handle the related *inverse problems*, or variations, which is what is generally needed for most applications.

Mathematics teaching and assessment have focused too strongly on the easy *direct* skills, and have often overlooked the fact that fluency, flexibility, and "use" require that far more attention be given to simple *inverse* problems. A pupil may know how to

- "find 75% of £120"

yet fail to relate this *direct* operation to *inverse* variations, such as

- "A price of £90 is raised to £120. What percentage increase is this?", or
- "Calculate the original price if I got 25% off and paid £90".

For each direct process, we need to allow far more time to develop the flexibility that is needed if pupils are to use the process effectively to solve related *indirect* problems.

1.2.5 The distinction in the previous subsection is illustrated in its simplest form by the third requirement listed at the start of 1.2. Once one moves into Key Stage 3, the key to arithmetic (and later to algebra) lies in *simplification*. One no longer applies brute force to calculate with each expression as it is given. Instead one looks first for ways of *simplifying*. And the key to simplification lies in looking for

> "complexifications that cancel each other out",

that is, for hidden instances of operations cancelled out by their inverses. For example, when faced with the question:

> "How many weeks are there in 5040^2 seconds?"

one would like pupils to set up the relevant equations

$$\begin{aligned} 5040^2 \text{ seconds} &= \frac{5040 \times 5040}{60} \text{ minutes} \\ &= \frac{5040 \times 5040}{60 \times 60} \text{ hours} \\ &= \frac{5040 \times 5040}{60 \times 60 \times 24} \text{ days} \\ &= \frac{5040 \times 5040}{60 \times 60 \times 24 \times 7} \text{ weeks} \end{aligned}$$

without evaluating $5040^2 = \ldots$, and without carrying out long divisions (or using a calculator), and then to look for ways of *cancelling*.

When dealing with algebraic expressions:

- It is permissible (but usually silly) to split up a single term and to spread the parts around to change a given expression into one that looks much more complicated; it is more helpful to reverse such "complexifications" by "collecting up" similar-looking terms to produce a more compact expression, which is then much easier to comprehend at a glance.
- It is equally permissible (and usually equally silly) to multiply the numerator and denominator of a given (numerical or algebraic) fraction by the same non-zero expression, and then to multiply out to make a new rational expression that appears more complicated than the original; but it is generally more sensible to factorise, to identify (non-zero) common factors, and to cancel in order to simplify.

That is,

- operations come in linked "direct-inverse" pairs which cancel each other out (addition-subtraction; multiplication-division; powers-roots; multiplying out and factorising; etc.).

Simplification is essentially the art of spotting such combinations, and cancelling them out.

This key algebraic art needs to be exercised and mastered first within *arithmetic*—so that *numerical expressions* are no longer "blindly evaluated", but are routinely simplified, using what we have called *structural arithmetic* (see Part II, Section 2.1.1)—so that one routinely notices: that

$$28 + 186 + 72 = (28 + 72) + 186 = 286;$$

or that

$$\frac{36}{54} = \frac{4 \times 9}{6 \times 9} = \frac{4}{6} = \frac{2}{3}.$$

One is then in a position to be pleasantly surprised by equivalences that are less obvious (such as that $\sqrt{3 + 2\sqrt{2}} = \sqrt{(1 + \sqrt{2})^2} = 1 + \sqrt{2}$).

1.2.6 The first of the requirements listed at the start of 1.2 refers to "proper and improper fractions" and to "mixed fractions". The expressions "proper fraction" and "improper fraction" make sense in Key Stage 2, but they are no longer really appropriate at Key Stage 3.

Fractions are introduced in Key Stage 1 and Key Stage 2 as parts of a whole, and so are automatically *less than* 1; hence, **at that stage**, when one comes to refer to fractions that are *greater than* 1, it makes sense to call them "improper". But the distinction is not a mathematical distinction; it arises because of the way fractions are introduced.

From Key Stage 3 onwards all fractions, whether greater than 1 or less than 1, should be treated in the same way, as the quotient of two integers $\frac{p}{q}$, with $q > 0$. Hence the use of words like "proper" and "improper" should be left behind (along with such language as "timesing").

Similarly, though it may sometimes be appropriate to present an answer in "mixed" form (say as $3\frac{5}{6}$), the expression "mixed number" is out of place in secondary mathematics.

1.3. [Subject content: *Number* p. 6]

- **use a calculator and other technologies to calculate results accurately and then interpret them appropriately**

"Calculators and other technologies" were first advocated at secondary level some 40 or more years ago. Yet we still do not seem to have forged a consensus as to when their use is "appropriate", and when not.

The opening *Aims* (see page 2 of the National Curriculum programmes of study for Key Stage 3) include the sensible warning that calculators, etc.

> "**should not** be used as **a substitute** for good written and mental arithmetic" [emphasis added].

However, this sound advice still needs to be interpreted. And the positive guidance as to when calculator use is "appropriate" is only slightly more helpful. The general advice offered at the beginning of the programmes of study for Key Stages 1 and 2, on pages 3 and 4,[15] says that calculators should only be introduced

> "to support pupils' conceptual understanding and exploration of more complex number problems, **if written and mental arithmetic** are secure" [emphasis added].

The dilemma highlighted by this advice refers to *integer* arithmetic in *primary* schools. But the same dilemma recurs throughout Key Stage 3—with *decimal arithmetic*, with *fractions*, with *surds*, and so on. Secure calculation by hand and in the head is a crucial ingredient of *the way beginners internalise meaning, structures, and procedures*. So in each case the above instruction would seem to imply that

- pupils should achieve conceptual understanding and mental and written fluency before routinely using a calculator,
- but that once a suitable level of fluency has been achieved, one can safely delegate "more complex number problems" to the calculator, and exploit the power of the calculator to extend conceptual understanding into new realms (see the example at the end of this section).

The introduction to the programmes of study for Key Stage 1 and 2 and for Key Stage 3 both state that

> "In both primary and secondary schools, teachers should use their judgement about when ICT tools should be used."

But the wider community remains confused. The judgement in the previous paragraph (that "secure calculation is an important part of the way beginners internalise meaning") would seem to be a reasonable summary of views in many other countries. But teachers in England will know that the mathematics education community here remains divided. Hence teachers must be prepared to develop and to use their own judgement as they are exhorted to do.

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[15] https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/335158/PRIMARY_national_curriculum_-_Mathematics_220714.pdf

To illustrate the divide, we give just two recent examples. The first is a report published by the *Joint Mathematical Council*[16] and a riposte.[17] The second is a debate between a strong advocate of "computer based mathematics" in schools and an agnostic:[18] (see "Technology and maths") .

Technology is clearly seen as "sexy" by politicians and by enthusiasts. And its evident potential should certainly be explored. But it is not easy for ordinary teachers to see beyond the rhetoric in order to discern

- whether we have already discovered some magic "royal road" to elementary mathematics, that removes the need for beginners to master the art of hand calculation; or
- whether those who currently advocate increased use of technology by beginners are getting ahead of themselves, and are misleading the rest of us as to what is currently in pupils' interests.

Whatever may be the eventual impact of technology on the learning of mathematics, the present evidence from international studies (illustrated by examples 1.4A–1.4K in Part II) would seem to be that we in England have tended to delegate calculation to the calculator or computer *far too easily*. Instead of using technology to achieve *more*, we have used it as a convenient *alternative* to achieving meaning and mastery. That is, we have failed to heed the exhortation of the official programme of study, and have allowed technology to be "**used as a substitute for**" pupils' understanding of written and mental arithmetic.

Computation by hand, or in the head, has too often been repudiated as if it were merely outmoded drudgery, or some puritanical hangover. But the importance of calculation at all levels stems from the role played by mental and written procedures in the subtle process of human **sense-making**. So we should perhaps hesitate before discarding it until such time as we are sure that we have other ways of establishing the kind of meaning that will allow pupils to use elementary mathematics with confidence.

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[16] http://www.jmc.org.uk/documents/JMC_Report_Digital_Technologies_2011.pdf

[17] http://education.lms.ac.uk/wp-content/uploads/2012/02/Gardiner_on_JMC.pdf

[18] http://www.cambridgeassessment.org.uk/news/playlist/view/maths-podcasts/

The requirement that pupils should "use calculators and other technologies"

> "to calculate results **accurately** and then interpret them appropriately" [emphasis added]

needs to be interpreted with care. A calculator certainly allows us all to work with messier numerical data than we could otherwise manage. But for most calculations, a calculator is the opposite of "accurate": its value lies in the fact that it is "quick and dirty", and produces an answer which is a very good **approximation**, but which may not be exact. The ubiquity of calculators, and their ease of use makes it important for pupils to develop their own internal sense of number so that they can use calculators intelligently, interpret the approximate answers which they produce, and use these tools to extend their own powers of analysis.

To give an example from within elementary mathematics (having one eye on the next subsection), one might invite more able pupils in Year 8 or Year 9 to work (initially *without a calculator*) to address these three questions:

(a) Find a prime number which is one less than a square.

(b) Find another such prime number. And another.

(c) How many such prime numbers are there?

Different teachers will exploit the proposed task in different ways. Pupils must first access whatever internal register of squares they have, and then reinforce and extend their internal list to generate:

$$(1^2-1=\mathbf{0},)\ 2^2-1=\mathbf{3},\ 3^2-1=\mathbf{8},\ 4^2-1=\mathbf{15},\ 5^2-1=\mathbf{24},$$
$$6^2-1=\mathbf{35},\ 7^2-1=\mathbf{48},\ 8^2-1=\mathbf{63},\ 9^2-1=\mathbf{80},\ 10^2-1=\mathbf{99},$$
$$11^2-1=\mathbf{120},\ 12^2-1=\mathbf{143},\ 13^2-1=\mathbf{\ldots}\ ,\ldots$$

They must then decide which of these numbers are prime. The associated "noise" (of having first to think about squares, then to subtract 1) makes this more awkward than simply asking pupils to test given integers to see whether they are prime. So one can anticipate some surprising mistakes. For example: though 8, 15, 24, 35, 48 are unlikely to be labelled as primes, the surrounding "noise" means that part (b) may well lead to 63 and 143 being proposed as candidate primes.

There are challenges here for pupils on many levels. A calculator may at first be used simply to extend the list of squares. If so, then 168, 195, 224, 255, 288, 360, 440 are unlikely to be proposed as primes; but 399 and 483 might well be, and 323 will almost certainly feature.

However, once the proposed candidates 63 ($= 7 \times 9$), 143 ($= 11 \times 13$), and 323 ($= \cdots \times \ldots$) have been seen to fail, one would like pupils to think rather than just press buttons and guess. A mixture of patience and prodding should allow them to discover the apparent pattern

$$8 = 2 \times 4,$$
$$15 = 3 \times 5,$$
$$24 = 4 \times 6, \text{ etc.},$$

and they can then to use the distributive law to multiply out

$$(n-1)(n+1) = n(n+1) - 1(n+1) = n^2 + n - n - 1 = n^2 - 1,$$

and to discover

- the advantages of thinking and working with **symbols** ("$n^2 - 1$")
- rather than with words ("one less than a square").

1.4. [Subject content: *Number* p. 5]

- **use the concepts and vocabulary of prime numbers, factors (or divisors), multiples, common factors, common multiples, highest common factor, lowest common multiple, prime factorisation, including using product notation and the unique factorisation property**
- **use integer powers and associated real roots (square, cube and higher), recognise powers of 2, 3, 4, 5 and distinguish**

between exact representations of roots and their decimal approximations

This collection of topics related to integer arithmetic deserves to be taken more seriously than has perhaps traditionally been the case at secondary level. The following released item from TIMSS 2011 for pupils in Year 9 suggests that work on primes and factors from primary school is often not followed up.

> **1.4A** Which of these shows how 36 can be expressed as a product of prime factors?
>
> A 6×6 B 4×9 C $4 \times 3 \times 3$ D $2 \times 2 \times 3 \times 3$
>
> **1.4A** Hungary 69%, Russia 68%, USA 64%,
> England 51%, Australia 45%

Bare hands integer arithmetic may suffice for pupils to find HCFs (to cancel fractions), and LCMs (to add or subtract fractions by writing both with a common denominator). But if the official requirements are interpreted coherently, then the listed ideas constitute a valuable "Key Stage 3 introduction to *Number theory*", a subject which is increasingly important in a world dominated by "calculators and other technologies".

1.4.1 The second listed requirement in 1.4 "use integer powers" is perhaps the simplest starting point. Pupils should recognise and work with

> squares: $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, $7^2 = 49$, $8^2 = 64$, $9^2 = 81$, $10^2 = 100$, $11^2 = 121$, $12^2 = 144, \ldots$; and
>
> cubes: $1^3 = 1$, $2^3 = 8$, $3^3 = 27$, $4^3 = 64$, $5^3 = 125$, $6^3 = 216, \ldots$, $10^3 = 1000$.

They should also recognise the powers of 10 in exponent form and know the corresponding values:

> powers of 10: 10, $10^2 = 100$, $10^3 = 1000$, $10^4 = 10000$, $10^5 = 100000$, $10^6 = 1000000$, etc.

And they should work with and recognise powers of small integers, such as:

powers of 2: $2, 2^2 = 4, 2^3 = 8, 2^4 = 16, 2^5 = 32, 2^6 = 64, 2^7 = 128, 2^8 = 256, 2^9 = 512, 2^{10} = 1024$

powers of 3: $3, 3^2 = 9, 3^3 = 27, 3^4 = 81, 3^5 = 243$

powers of 4: $4, 4^2 = 16, 4^3 = 64, 4^4 = 256, 4^5 = 1024$

powers of 5: $5, 5^2 = 25, 5^3 = 125, 5^4 = 625$.

Squaring is a "unary operation" or function (in that the *output* n^2 is uniquely determined by a single *input*). Once sufficiently many squares are known, they can be exploited to interpret the *exact* meaning of the *inverse* unary operation, that is the square root function $\sqrt{}$ where

$\sqrt{n}$ denotes "the **positive** number whose square is equal to n".

Notice that, since $\sqrt{}$ is to be a *function*, $\sqrt{4}$ must denote a *unique* value—namely the positive number whose square is equal to 4: i.e. 2. In contrast, the quadratic equation "$x^2 = 4$" has **two** solutions, which are $\pm\sqrt{4}$.

Later, appropriate groups of pupils can help to formulate and **prove**:

Claim If $a^2 = b^2$, then $a = \pm b$.

Proof Suppose $a^2 = b^2$.

$\therefore a^2 - b^2 = 0$

$\therefore (a - b)(a + b) = 0$

$\therefore a - b = 0$, or $a + b = 0$, so $a = \pm b$. **QED**

This shows that there is just one positive number whose square has a given positive value.

Provided n is a perfect square, pupils can find the *exact* value of $\sqrt{n}$: for small squares:

$$\sqrt{4} = 2, \quad \sqrt{9} = 3, \quad \sqrt{16} = 4, \quad \sqrt{25} = 5;$$

and for larger squares:

$$\sqrt{81} = 9, \quad \sqrt{100} = 10, \quad \sqrt{121} = 11, \quad \sqrt{256} = 16.$$

They may be encouraged to notice that

$$\sqrt{4 \times 9} = \sqrt{36} = 6 = 2 \times 3 = \sqrt{4} \times \sqrt{9},$$

and that

$$\sqrt{9 \times 9} = 9 = \sqrt{9} \times \sqrt{9}.$$

They can then use this as a short cut to find the square root of larger squares such as $\sqrt{16 \times 25}$.

> [Later they can prove that:
>
> **Claim** $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ whenever a and b are positive:
>
> **Proof** $\sqrt{a} \times \sqrt{b}$ is clearly positive (since $\sqrt{a}$ and $\sqrt{b}$ are both positive).
>
> And $(\sqrt{a} \times \sqrt{b})^2 = (\sqrt{a})^2 \times (\sqrt{b})^2 = ab$
>
> $\therefore \sqrt{a} \times \sqrt{b} = \sqrt{ab}$. **QED]**

And once sufficiently many *cubes* are known, pupils can find $\sqrt[3]{n}$ when n is a perfect cube:

$$\sqrt[3]{8} = 2,\ \sqrt[3]{27} = 3,\ \sqrt[3]{64} = 4,\ \sqrt[3]{1000} = 10.$$

With help they may notice that

$$\sqrt[3]{8 \times 27} = \sqrt[3]{216} = 6 = 2 \times 3 = \sqrt[3]{8} \times \sqrt[3]{27}.$$

This basic repertoire of calculations using powers and roots can then develop in two very different directions—one focusing on *calculation*, and the other on *structure*.

1.4.2 Further calculation The notation $\sqrt{n}$ and $\sqrt[3]{n}$ for square roots and cube roots has many features in common with the notation for fractions.

Some fractions, like $\frac{8}{2} = 4$, or $\frac{1}{4} = 0.25$, stand for familiar numbers, and can be *exactly* evaluated. But most fractions one can write down (such as $\frac{1}{6} \approx 0.167$) do not stand for any otherwise familiar number, and cannot be evaluated exactly. The value of the fraction notation is that it provides a way of writing *exact expressions* for "ideas of numbers", which we often have no other way of writing exactly, such as

> "that number—six identical copies of which add up to 1".

Similarly, the functions $\sqrt{}$ and $\sqrt[3]{}$ allow us to write *exact* expressions for numbers, most of which cannot be evaluated exactly as decimals, or in any other way. We know that $\sqrt{4} = 2$. But what number is represented by $\sqrt{2}$? Or by $\sqrt{3}$? Or by $\sqrt{300}$? Or by $\sqrt{0.3}$? Or by $\sqrt{\frac{1}{3}}$?

Before we worry about the square root of fractions or decimals, there is plenty of work to be done to establish the meaning and the arithmetical rules for working with *surds*: that is numbers of the form $\sqrt{n}$ when n is an integer. For example, we need to ensure

- that $\sqrt{10}$ is understood formally to be "the (positive) number whose square is 10";
- that since 10 lies between 9 and 16, $\sqrt{10}$ is seen to be slightly bigger than $\sqrt{9} = 3$ (and a lot less than $\sqrt{16} = 4$);
- that pupils compare the side length of a square of area 10 square units, with that for a square of area 9, and one of area 16; and
- that they later compare the length of a *diagonal* of a 1 by 3 rectangle

(a) with the length (= 3) of the longest side, and

(b) with the length (= 4) of the route round the perimeter of the rectangle from one corner to the opposite corner.

These ideas can later be taken further. *Pythagoras' Theorem* shows that an isosceles right angled triangle with legs of length 1 has a hypotenuse of length exactly $\sqrt{2}$. The hypotenuse is clearly longer than each of the two legs; and the *triangle inequality* shows that the hypotenuse is less than the sum of the two shorter sides. So we know that $1 < \sqrt{2} < 2$. But to pin down the value of $\sqrt{2}$ more accurately requires us to use a little of what we know about integer squares:

$$14^2 = 196 < 200 < 225 = 15^2$$
$$\therefore 14 = \sqrt{196} < \sqrt{200} < \sqrt{225} = 15$$
$$\therefore 14 < \sqrt{100 \times 2} = \sqrt{100} \times \sqrt{2} < 15$$
$$\therefore 14 < 10\sqrt{2} < 15$$

$\therefore 1.4 < \sqrt{2} < 1.5$

[In short: $1.4^2 = 1.96 < 2$, and $1.5^2 = 2.25 > 2$.]

Similarly, *Pythagoras' Theorem* shows that an equilateral triangle of side 2 has height exactly $\sqrt{3}$, and that this height is less than the hypotenuse, so $\sqrt{3} < 2$; and the triangle inequality shows that $1 + \sqrt{3} > 2$. Hence $1 < \sqrt{3} < 2$. But to pin down the value $\sqrt{3}$ more accurately we have to use what we know about integer powers to find reasonable estimates:

$17^2 = 289 < 300 < 324 = 18^2$

$\therefore 17 = \sqrt{289} < \sqrt{300} < \sqrt{324} = 18$

$\therefore 17 < \sqrt{100 \times 3} = \sqrt{100} \times \sqrt{3} < 18$

$\therefore 17 < 10\sqrt{3} < 18$

$\therefore 1.7 < \sqrt{3} < 1.8$

[In short: $1.7^2 = 2.89 < 3$, and $1.8^2 = 3.24 > 3$.]

In the same way one can use what pupils know about perfect cubes to ensure

- that $\sqrt[3]{10}$ is interpreted as "the number whose cube is equal to 10";
- that this number is seen to be slightly bigger than $\sqrt[3]{8} = 2$ and considerably smaller than $\sqrt[3]{27} = 3$;
- that pupils compare an imagined cube of volume 10 cubic units with a smaller cube of volume 8 and a larger cube of volume 27 cubic units—noting and understanding how a modest increase in the edge length leads to a cube with *three times* the volume!

1.4.3 Structure: the index laws The *structural* (or *algebraic*) theme related to powers prepares the ground for the *index laws*. The index laws are not explicitly mentioned within the Key Stage 3 programme of study, but there are several reasons why they need to be squarely addressed at this level.

One reason is that, as we shall see in Section 1.5, zeroth and negative powers are needed to represent real numbers *in standard form*; and the way we define these powers only really makes sense if we think in terms of the advantages of "preserving the index laws".

A more basic reason is for pupils to understand why

> when we multiply a digit in the 10^m place (or column) by a digit in the 10^n place (or column), the answer belongs in the 10^{m+n} column.

For this to make sense, pupils already need to know in their bones how products of powers work: for example, that

$$10^2 \times 10^5 = (10 \times 10) \times (10 \times 10 \times 10 \times 10 \times 10) = 10^{2+5}, \text{ and}$$
$$2^2 \times 2^5 = (2 \times 2) \times (2 \times 2 \times 2 \times 2 \times 2) = 2^{2+5}.$$

Once pupils

- think of the *place value* of positions, or columns, in terms of the exponent of the "power of 10", rather than verbally as "units, tens, hundreds, etc.", and
- realise that "when we *multiply* powers, we *add* exponents",

it becomes natural to think of the **unit** as $\mathbf{10^0 = 1}$.

> The rightmost place when representing an integer
> then corresponds to the "(units digit) $\times 10^0$".

The fact that $10^0 = 1$ then fits in with the way powers multiply (since we want $10^1 \times 10^0 = 10^{(1+0)} = 10$).

Once the units column (*just to the* **left** of the decimal point) is associated with 10^0, it becomes plausible that the place *immediately to the* **right** of the decimal point might correspond to "10^{-1}". And the idea that "when we *multiply* powers, we *add* exponents" also helps to explain why we take "10^{-1}" to equal $\frac{1}{10}$ (since we want: $10^1 \times 10^{-1} = 10^{1+(-1)} = 10^0 = 1 = 10 \times \frac{1}{10}$).

1.4.4 Introduction to number theory It is easy to compare, and to add, two fractions with the **same** denominator; but it is not at all obvious how to compare, or to add, two fractions with **different** denominators m, n. However, as soon as we change each fraction to one that is equivalent to it, and which has denominator "$LCM(m,n)$", comparison is again immediate, and addition, subtraction and division can be carried out easily. Hence LCMs come into their own as soon as we wish to compare, or to add, subtract, or divide two fractions with different denominators m and n. In general HCFs and LCMs feature whenever a problem requires us to switch to a common unit that works for both m and n (whether a multiple of each, or a submultiple—or factor—of each).

The HCF and LCM of two given integers m, n are easy to find in a primitive way.

> **HCF**: Each of the given integers m, n has a finite number of *factors*, and these can be listed; the two lists can then be scanned to find the *"highest"*, or largest, factor *in both lists*.
> **LCM**: The LCM of the given integers m, n can be found by making a list of (positive) multiples of each number ($2m$, $3m$, $4m, \ldots$; and $2n$, $3n$, $4n, \ldots$) and looking for the *"least"* multiple that occurs *in both lists*.

These primitive approaches are easy to implement, but are slightly unwieldy. Moreover, they do not immediately suggest, or explain why it is always true that:

$$HCF(m,n) \times LCM(m,n) = m \times n.$$

For suitable groups of pupils it is worth making sure that this result is discovered, or at least noticed, and if possible proved.

[**Proof** Let $HCF(m,n) = h$.

$\therefore m = h \times m'$ and $n = h \times n'$, where m' and n' have no common factors.

$\therefore m' \times n = m' \times (h \times n') = (m' \times h) \times n' = m \times n'$ is a multiple of m and of n, so is a common multiple of both m and n.

The fact that it is the LCM follows from the important fact that every common multiple of both m and n is also a multiple of their LCM. (So if there were a *smaller* common multiple of m and n, say k, then it would have to be a proper factor of $m' \times h \times n'$ and the quotient would be a factor of both m' and n'.)

$\therefore HCF(m,n) \times LCM(m,n) = h \times (m' \times n) = (h \times m') \times n = m \times n.$ **QED**]

The observation that $LCM(m,n)$ is a factor of every common multiple of m and n is not hard, but cannot easily be proved at this level. However, it can be established as a "fact of experience" by listing the common multiples of suitable pairs, such as:

2 and 3: 6, 12, 18, 24, ...
6 and 8: 24, 48, 72, 96, ...
6 and 14: 42, 84, 126, ...
30 and 42: 210, 420, 630, ...

And the fact that

$$HCF(m,n) \times LCM(m,n) = mn$$

can be re-explained later when one is in a position to look at HCFs and LCMs in terms of the prime factorisations of the two integers m and n.

The Key Stage 3 requirements relating to prime numbers and prime factorisation extend what is expected at Key Stage 2. There we find that pupils (in Year 5) are supposed to

- "know and use the vocabulary of prime numbers, prime factors and composite numbers"
- "establish whether a number up to 100 is prime and recall prime numbers up to 19", and
- "recognise and use square numbers and cube numbers and the notation for squared (^{2}) and cubed (^{3})".

Although we have been told that "Key Stage 3 should build on Key Stage 2", it may be wise to revisit, and to reinforce, these ideas in Year 7 before ploughing ahead (especially with regard to the third bullet point, which seems unnecessarily premature). A sensible initial goal at Key Stage 3 is

- to get to know the twenty five prime numbers up to 100

by implementing the *Sieve of Eratosthenes* (Greek, 3rd century BC).

- Write out the integers 1–100 in ten columns. Cross out 1 (as 1 is **not** a prime).

- Circle the first uncrossed integer (the prime 2) and cross out all its larger multiples.
- Circle the first uncrossed integer (the prime 3) and cross out all its larger multiples.
- Circle the first uncrossed integer (the prime 5) and cross out all its larger multiples.
- Circle the first uncrossed integer (the prime 7) and cross out all its larger multiples.

Then check that all of the remaining uncrossed integers

$$11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$$

are in fact primes. (The reason why should be revisited later when the "square root test" has been understood—see later in this section.)

As part of this exercise one would like pupils to learn that, although unfamiliar integers sometimes "smell like a prime", this may be simply because (like 51, or 91, or 323) they are not routinely encountered in the multiplication tables. Pupils will later need to develop a systematic way of testing any three-digit integer to see whether it is prime (the "square root test").

The programme of study includes "prime factorisation" as an explicitly declared goal. So it is important to explain why we do not count "1" as a prime number (and to make it clear that this has nothing to do with enforcing an arbitrary definition of a "prime" as an integer with "*exactly two factors*"). Pupils should understand (from their own extensive experience of factorising integers: see below) that

- *prime numbers* are the "multiplicative atoms" for integers.

Hence we can break up any given integer as the product of its constituent prime factors. Once we grasp this important property of prime numbers, it should be clear that "1 is different", e.g.

$$1 = 1 \times 1 = 1 \times 1 \times 1 = \dots,$$

and

$$2 = 2 \times 1 = 2 \times 1 \times 1 = 2 \times 1 \times 1 \times 1 = \ldots.$$

So "1" is not such a constituent atom, and it would simply get in the way if we made the mistake of calling it a prime.

Some thought is needed when choosing a systematic procedure for "factorising integers". "Factor trees" may have a place for beginners, but it is worth thinking carefully why they are best left behind when we come to Key Stage 3 (along with oblongs, timesing, improper fractions, and mixed numbers). The most suitable systematic algorithm for achieving prime factorisation of a given integer is to carry out successive short divisions—upside down:

"Write **2310** as a product of prime powers."

2 is clearly a factor of 2310: $\begin{array}{rr} 2 & \underline{|2310} \\ & 1155 \end{array}$

$\therefore 2310 = 2 \times 1155$

3 is clearly a factor of 1155: $\begin{array}{rr} 3 & \underline{|1155} \\ & 385 \end{array}$

$\therefore 2310 = 2 \times 1155 = 2 \times 3 \times 385$

5 is clearly a factor of 385: $\begin{array}{rr} 5 & \underline{|385} \\ & 77 \end{array}$

$\therefore 2310 = 2 \times 3 \times 5 \times 77$

7 is clearly a factor of 77: $\begin{array}{rr} 7 & \underline{|77} \\ & 11 \end{array}$

$\therefore 2310 = 2 \times 3 \times 5 \times 7 \times 11$

If we apply a slightly compressed version of the same procedure to less carefully chosen starting integers—such as 1234, or 12345, or 123456, or 4321, or 54321, or 654321, then we quickly discover the need for an efficient way of deciding whether "large" integers are prime.

1234: 2 is clearly a factor:

$$\begin{array}{r|r} 2 & 1234 \\ \hline & 617 \end{array}$$

$\therefore 1234 = 2 \times 617$. **But is 617 prime?**

12345: 5 is clearly a factor:

$$\begin{array}{r|r} 5 & 12345 \\ \hline 3 & 2469 \\ \hline & 823 \end{array}$$

$\therefore 12345 = 3 \times 5 \times 823$. **But is 823 prime?**

123456: 2 is clearly a factor:

$$\begin{array}{r|r} 2 & 123456 \\ \hline 2 & 61728 \\ \hline 2 & 30864 \\ \hline 2 & 15432 \\ \hline 2 & 7716 \\ \hline 2 & 3858 \\ \hline 3 & 1929 \\ \hline & 643 \end{array}$$

$\therefore 123456 = 2^6 \times 3 \times 643$. **But is 643 prime?**

These unanswered questions lead naturally to the square root test for deciding whether a given integer is prime:

Square root test: Suppose that 643 is not prime.

Then 643 factorises—say as $643 = a \times b$, where a, b are both "proper factors" (i.e. $a, b > 1$) We may choose a to be the smaller of the two proper factors: so $1 < a \leqslant b$.

Then

$$\begin{aligned} 643 &= a \times b \\ &\geqslant a \times a \quad (\text{since } b \geqslant a) \end{aligned}$$

$\therefore \sqrt{643} \geqslant \sqrt{a \times a} = a$, so the smaller factor $a \leqslant \sqrt{643} < 26$.

Hence **to test whether 643 is prime, we only need to test for factors up to 25**.

The first few short divisions can be done in the head:

> 2 is clearly not a factor of 643;
>
> 3 is not a factor (the simple 'divisibility tests' are discussed below);
>
> (4 cannot be a factor—or else 2 would have been a factor);
>
> 5 is clearly not a factor;
>
> (6 cannot be a factor or else 2 and 3 would have been factors);
>
> 7 is not a factor;
>
> (8 cannot be a factor or 2 would have been a factor; similarly 9 and 10 cannot be factors);
>
> 11 is not a factor; and so on.

The reasons why we do not have to check 4, 6, 8, 9, 10, . . . show that we only have to check for possible **prime** factors up to $\sqrt{643}$—that is up to 23. And once the easy short divisions have been checked, it makes perfect sense to use a calculator to test for larger possible prime factors (say beyond 7, or 11). Moreover calculator use makes the power and speed of the method even more evident:

> $643 \div 13 = 49.46\ldots;$
>
> $643 \div 17 = 37.82\ldots;$
>
> $643 \div 19 = 33.84\ldots;$
>
> $643 \div 23 = 27.95\ldots.$
>
> $\therefore$ **643 is prime**

Pupils can now look back at the "sieve of Eratosthenes" for the integers 1–100 and understand *why it stopped at multiples of* 7:

> **Proof** Any **non-prime** $\leqslant 100$ must have a **prime factor** $\leqslant \sqrt{\mathbf{100}} = 10$.
>
> That is, every **non-prime** $\leqslant 100$ is a multiple of 2, or of 3, or of 5, or of 7. **QED**

Armed with this method, they can then complete a "*sieve of Eratosthenes*" to find all prime numbers **up to 500** (by following the same procedure—circling the first uncrossed number and crossing out all higher multiples—for primes up to $\sqrt{500} = 22.36\ldots$—that is up to 19). Hence, in order to extend the list from 100 to 500 we only need to carry out **four extra steps**, to eliminate multiples of 11, of 13, of 17, and of 19.

The fact that every positive integer can be factorised *in just one way* as a product of prime powers cannot be proved at this level. Instead the uniqueness of prime factorisation emerges as a "fact of experience": the factorisation procedure above churns out the prime factorisation each time, and the subtle question as to its uniqueness is unlikely to arise.

There is plenty of mileage in exploiting prime factorisation. For example:

- to recognise squares as precisely those integers whose prime factorisation only involves primes to *even powers*
- to recognise cubes as precisely those integers whose prime factorisation only involves primes raised to powers that are all multiples of 3
- to see how $HCF(m,n)$ is just the product of those prime powers that occur both in the prime factorisation of m and in the prime factorisation of n, and hence to re-prove

$$HCF(m,n) \times LCM(m,n) = m \times n.$$

Divisibility tests are not explicitly mentioned in the Key Stage 3 programme of study. However, the requirements to understand place value (Section 1.1) and to test for factors (Section 1.4) should highlight the need to discuss these excellent examples of *structural arithmetic*.

The fact that **multiples of 10** are precisely the integers having "units digit = 0" is an evident consequence of *place value*: for example

$$\begin{aligned} 3210 &= 3000 + 200 + 10 \\ &= 300 \times 10 + 20 \times 10 + 1 \times 10 \\ &= 321 \times 10 \end{aligned}$$

Any integer N can therefore be decomposed as "a multiple of 10" plus its "units digit". The first of these two terms "a multiple of 10" is also "a multiple of 2" (because $10k = (2 \times 5)k = 2 \times (5k)$).

> $\therefore$ An integer N is a **multiple of 2** precisely when its units digit is a multiple of 2.

That is, when it ends in 0, 2, 4, 6, or 8. (Be prepared to have to insist that "$0 = 0 \times 2$" is a multiple of 2, and so is *even*.)

Similarly, any multiple of 10 is also a "multiple of 5" (because $10k = (5 \times 2)k = 5 \times (2k)$).

> $\therefore$ an integer is a **multiple of 5** precisely when its units digit is a multiple of 5.

That is, when it ends in 0, or 5.

The same idea shows that multiples of 100 are precisely the integers having "both tens and units digits = 0".

Any integer N can be decomposed as "a multiple of 100" plus the number formed by its tens and units digits. The multiple of 100 is also a "multiple of 4" (because $100k = (4 \times 25)k = 4 \times (25k)$).

> $\therefore$ N is a **multiple of 4** precisely when "the number formed by its *last two digits* is a multiple of 4".

Multiples of 1000 are precisely the integers having hundreds, tens and units digits = 0.

Any multiple of 1000 is also a "multiple of 8" (because $1000k = (8 \times 125)k = 8 \times (125k)$); so an integer is a **multiple of 8** precisely when "the number formed by its *last three digits* is a multiple of 8".

This shows how the rules for spotting multiples of 2, or 4, or 5, or 8, or 10 derive from our *place value* system for writing numbers.

The divisibility tests for multiples of 3, and of 9 depend on the *place value* system in a more interesting way, which obliges us to think about the *algebraic* structure of the place value system. The key here lies in the fact that

> $10 - 1 = \ldots, 100 - 1 = \ldots, 1000 - 1 = \ldots$ etc. are all multiples of 9.

Later this can be seen as a special case of the beautiful factorisation

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1).$$

Hence any integer such as 12345, can be deconstructed into

$$\begin{aligned}\mathbf{12345} &= 1 \times 10000 + 2 \times 1000 + 3 \times 100 + 4 \times 10 + 5\\ &= 1 \times (9999+1) + 2 \times (999+1) + 3 \times (99+1) + 4 \times (9+1) + 5\\ &= (1 \times 9999 + 2 \times 999 + 3 \times 99 + 4 \times 9) + (\mathbf{1} + \mathbf{2} + \mathbf{3} + \mathbf{4} + \mathbf{5})\end{aligned}$$

The first bracket is clearly a multiple of 9—and so is also a multiple of 3.

Hence, for 12345 to be **a multiple of 3** the second bracket—that is, its **digit-sum** "$1+2+3+4+5$"—must be a multiple of 3 (which it is!).

And for 12345 to be **a multiple of 9**, the second bracket—that is, its **digit-sum** "1+2+3+4+5"—must be a multiple of 9 (which it is not). This yields a simple (and intriguing) test for divisibility by 3 and by 9.

The test for divisibility by 6 is mildly different: an integer is divisible by 6 precisely when it is divisible both by 2 **and** by 3. Similarly, an integer is divisible by 12 precisely when it is divisible both by 4 **and** by 3. Here it is important that $HCF(3,4) = 1$. (Notice that 18 is a multiple of 6 and of 9; but 18 is **not** a multiple of $6 \times 9 = 54$, because $HCF(6,9) \neq 1$.)

Divisibility by $11 = 10 + 1$ depends on a simple variation of the reasoning for divisibility by $9 = 10 - 1$. The key here lies in the fact that

$$\begin{aligned}10 + 1 &= 11,\\ 100 - 1 &= 99,\\ 1000 + 1 &= 1001,\\ 10000 - 1 &= 9999,\end{aligned}$$

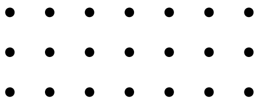
etc. are all multiples of 11.

An interesting consequence of the prime factorisation of an integer is that it allows an easy way of **counting the number of factors** which the integer has *without listing them all first*. The idea depends on "the product rule for counting" which is needed at Key Stage 3—but is not explicitly mentioned. However, it is optimistically hinted at rather vaguely in the Year 6 programme of study under

> "*Algebra*: – enumerate possibilities of combinations of two variables".

And the product rule is explicitly required at Key Stage 4.

The simplest version of the *product rule* tells us that the number of dots in a rectangular array is equal to "the number of dots in each row times the number of rows".



Instead of counting the dots individually, we note that there are 3 rows, each with 7 dots, so the total number of dots is "$7 + 7 + 7 = \mathbf{3} \times \mathbf{7}$".

A similar situation arises whenever we are effectively counting "ordered pairs". When we roll two dice, one red and one blue, each outcome can be listed systematically as an *ordered pair*:

> (red score, blue score).

The key observation is that each possible first coordinate has the same fixed number of possible second coordinates, so the total number of outcomes can be counted very easily.

> There are 6 possible red scores;
> and each red score can occur with each of the 6 possible blue scores;
> so there are
> $$6 + 6 + 6 + 6 + 6 + 6 = \mathbf{6} \times \mathbf{6}$$
> possible ordered pairs, or outcomes for rolling the two dice.

In the same way, if we want to count the possible factors of $12 = 2^2 \times 3$, then each factor must have the form $2^a \times 3^b$ with $a = 0$, 1, or 2, and $b = 0$, or 1. So

> there are 3 possible choices for a;
> and for each choice of a there are 2 choices for b. $\therefore$ **3 × 2 possible factors**:
>
> $2^0 \times 3^0 = 1, 2^0 \times 3^1 = 3, 2^1 \times 3^0 = 2, 2^1 \times 3^1 = 6, 2^2 \times 3^0 = 4, 2^2 \times 3^1 = 12.$

1.5. [Subject content: *Number* p. 5]

- **understand and use place value for decimals, measures and integers of any size**
- **interpret and compare numbers in standard form $A \times 10^n$, $1 \leqslant A < 10$, where n is a positive or negative integer of zero**

The two requirements in 1.5 are closely intertwined—even if the second bullet point seems slightly premature from a purely mathematical viewpoint. (*Standard form* may have been included at this level to support the requirements of science teaching. Yet there is no mention of "standard form" in the Key Stage 3 science programme of study—unless the numerical significance of the "pH scale" as

> "the decimal logarithm of the reciprocal of the hydrogen ion activity in a solution"

is to be explained in detail, or the value of "Newton's gravitational constant" is to be pulled out of a hat as "$\approx 6.67 \times 10^{-11} N \cdot (m/kg)^2$".)

The sequence of topics related to the requirements in 1.5 would seem to include:

- understanding and working with positive integer powers
- recognising that multiplication of powers of 10 corresponds to "adding exponents" (i.e. the index laws)
- understanding that defining "$10^0 = 1$" is consistent with the place value notation for integers (so that the tens column is in some sense the 1st column, and the units column is the "zeroth" column), and that this

definition of 10^0 preserves the index laws for multiplication ($10^3 \times 10^0 = 10^{(3+0)} = 10^3$)

- understanding that defining "10^{-n}" to be equal to the reciprocal of 10^n then allows us to interpret the decimal places to the right of the decimal point in the same way (as the "$(-1)^{\text{th}}$" column, the "$(-2)^{\text{th}}$ column", the "$(-3)^{\text{th}}$ column", the "$(-4)^{\text{th}}$ column", and so on), and that this also respects the index laws
- learning to write any integer with $n+1$ digits as a decimal A ($1 \leqslant A < 10$) multiplied by 10^n (by moving the decimal point n places to the left to follow the leading digit), and learning to translate numbers which are given in standard form back into their more familiar guise
- extending this notation to numbers which are less than 1, so that it can be used for all positive real numbers
- learning to compare numbers given in standard form
- learning to interpret the conventions associated with rounding, where numbers are specified to so many "significant figures", or to so many "decimal places"
- learning how to multiply and divide, and to add and subtract, numbers given in standard form (bearing in mind the specified levels of accuracy).

Experience with different groups of pupils will determine which parts of this sequence are better delayed until Year 10 (or even Year 11). For example, some pupils may be able to compare relatively simple examples of numbers given in standard form, but will need to revisit and extend the idea in Years 10 and 11. However, the final bullet point in the sequence seems much too demanding at this stage, since it involves the interaction between standard form and rounding, or approximation. (Numbers given in standard form are almost never exact. So arithmetic with numbers given in standard form needs to be linked with an understanding of numerical data being "accurate to so many decimal places", and with the use of "significant figures".)

The first few bullet points in the above sequence were incorporated in our comments on powers in Section 1.4. On one level, in order to understand that

> 3.1×10 is equal to "31",

it is enough to know that

> $3.1 \times 10 = (3 + 0.1) \times 10$, and that
>
> 0.1 is equal to $\frac{1}{10}$ (that is, that the "1" in the first decimal place corresponds to "tenths").

However, the *general* procedure for interpreting standard form makes much more sense once it is clear that the digit that is k places to the right of the decimal point corresponds to a multiple of 10^{-k}, so that multiplying by a suitable power of 10 simply "moves the decimal point" that number of steps *to the right* (or keeps the decimal point fixed and moves the digits the same number of steps to *the left*).

The same ideas are worth addressing because they are needed to understand

- the way **division** by a decimal can be transformed into division by an integer (by multiplying both the divisor and the dividend by a suitable power of 10), and
- the way **multiplication** of decimals can be transformed into a three-step process
  - first *multiplying* by a suitable power of 10 to transform the calculation into a familiar multiplication of integers,
  - then carrying out the multiplication of integers,
  - then *dividing* by the same power of 10 (that is, re-positioning the decimal point in the answer) to find the required answer.

Hence it may well be possible to convey something of the meaning of the standard form notation before the end of Key Stage 3—at least for those who are likely to need it elsewhere. But, in the spirit of the declared *Aims* of the mathematics programme of study, we urge mathematics teachers to avoid simply presenting standard form as an uncomprehended formalism. Instead we hope schools will lay the necessary foundations in Year 7 and 8 (through exercises that expand and then simplify powers such as

$$10^2 \times 10^5 = (10 \times 10) \times (10 \times 10 \times 10 \times 10 \times 10) = 10^{2+5},$$

linking this to an understanding of long multiplication), so that some modest version of the notation can be properly understood in Year 9 say. (The index laws offer a rare opportunity for pupils to experience at first hand the way meanings and definitions are extended in mathematics, though this opportunity is generally missed. For a systematic development at this level see *Extension mathematics Book Gamma* (Oxford 2007), Sections T14, C24, C31, C38.)

However, before launching into standard form, it would be good if pupils understood why it is often helpful to think in terms of "powers of 10", and why we focus on the exponent (or "baby logs") when dealing with very large or very small quantities or measurements. An easily available point of entry would be to watch the classic short movie *Powers of 10*, made many years ago by the Eames brothers.[19] (The film invites repeat viewing, stopping from time to time to discuss what is being shown.)

One everyday instance, where we focus on the exponent (or the logarithm) rather than the number itself, arises with the *Richter scale* for measuring the strength of earthquakes. This may already be familiar to some pupils. Here an **increase of 1** in the measurement used on the Richter scale corresponds to an earthquake which is **10 times more powerful**, and an **increase of 2** corresponds to an earthquake which is **100 times more powerful**. Other instances where such "log-scales" are used include the measure for the brightness of stars, and the pH scale.

1.6. [Subject content: *Number* p. 5]

- **use the four operations [...] applied to [...] fractions**
- **work interchangeably with terminating decimals and their corresponding fractions (such as 3.5 and $\frac{7}{2}$, or 0.375 and $\frac{3}{8}$)**
- **define percentage as 'number of parts per hundred', interpret percentages and percentage changes as a fraction**

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[19] http://www.eamesoffice.com/the-work/powers-of-ten/

or a decimal, interpret these multiplicatively, express one quantity as a percentage of another, compare two quantities using percentages, and work with percentages greater than 100%

- **interpret fractions and percentages as operators**
- [*Ratio, proportion and rates of change* p. 7] **solve problems involving percentage change: including percentage increase, decrease and original value problems; and simple interest in financial mathematics**

As the last listed item here indicates, the boundary between this section and Section 1.9 below (on ratio and proportion) is blurred—so the two need to be considered together. The first listed requirement concerning calculation with fractions was also considered briefly in Section 1.2. However, since achieving fluency in calculating with fractions should be a central goal of Key Stage 3, this deserves to be addressed here in greater detail than was possible as part of Section 1.2.

1.6.1 Fractions as a unifying idea The central importance of calculation with fractions for all pupils only becomes apparent in late Key Stage 3 and early Key Stage 4. Before that pupils learn to work with division (sharing and grouping), parts of a whole, decimals, fractions, ratios, percentages, proportion, scale factors—first numerically and then within algebra. But at some stage pupils ideally discover that all of these apparently different ideas and procedures reduce to "calculation with fractions".

1.6.2 Prerequisites and follow-up When preparing to address the **arithmetic** of *fractions* in early Key Stage 3, the first move should be a check that the necessary prerequisites from *integer* arithmetic are firmly in place. These include: complete arithmetical fluency with integers; and flexibility in identifying common multiples (in order to switch to common denominators), and in identifying common factors (in order to simplify by cancelling).

The subsequent developments summarised below constitute a considerable challenge. But such examples as 1.4C, 1.4F, and 1.4H in Part II suggest rather clearly that the arithmetic of fractions needs to be given more time than has been usual in recent years. In particular, fraction work should be routinely included as part of solving equations, solving word problems, finding equations of straight lines through given points, and within other applications during the ensuing 2–3 years (where it has often been artificially avoided by restricting to problems with small integer solutions).

1.6.3 Fractions as operators and percentages The fourth requirement listed at the start of 1.6 reads as though pupils start out with a clear understanding of "fractions as numbers", and then need to interpret these "numbers" as "operators". This is potentially misleading.

Fractions are initially introduced (in Key Stages 1 and 2) as "parts of a whole"—that is, as [implicit] "operators". At that stage pupils have no conception of fractions *as numbers*, such as $\frac{1}{2}$ or $\frac{3}{4}$, but work only with "parts of an understood whole".

At some point these "parts of a whole", such as "half a pint" or "three quarters of a cake", have to give birth to the *numbers* $\frac{1}{2}$ and $\frac{3}{4}$. Exactly how this shift from working with "parts of a given whole" to "fractions as numbers" is supposed to be made is never clarified in the Key Stage 2 programme of study. So we may anticipate that many pupils entering Key Stage 3 will still think of fractions only as operators (so the word "fraction" will immediately conjure up the idea of "a fraction of" some whole).

The third, fourth and fifth requirements listed in 1.6 refer to percentages. The key here is to recognise that all work with percentages should eventually reduce to a particular instance of work with fractions (sometimes in decimal form). That is, "percentages" should eventually be no longer seen as a separate topic, and fractions (and their arithmetic) should become the unifying theme. We make three further comments on percentages.

First, once the transition from "fractions as operators" to "fractions as numbers" has been firmly established, pupils need to re-interpret fractions as "operators" once again, in order to implement the standard applications efficiently—so that, for example, a "20% increase" is naturally calculated by **multiplying by 1.2**, rather than by calculating 20% and adding.

The second comment on percentages has already been made in Section 1.2.3 of Part II, and in Section 1.2.4 above, but bears repetition in the context of percentages. Mathematics teaching and assessment too often focus on the easy *direct* skills, and overlook the fact that fluency, flexibility, and "use" generally require that far more attention needs to be given to simple *inverse* problems. A pupil may know how to

- "find 75% of (i.e. three quarters of) £120"

yet fail to relate this *direct* operation to the different *inverse* variations, such as

- "A price of £90 is raised to £120. What percentage increase is this? And what percentage decrease would then be required to revert to the original price?", or
- "Calculate the original price if I got 25% off and paid £120".

Pupils need to spend time tackling a suitable variety of problems on percentages ("including percentage increase, decrease and original value problems") in order to appreciate both the underlying direct process, and the slightly counterintuitive aspects of percentages that tend to arise only in connection with *indirect* variations.

The final comment is slightly awkward. It has become common in England to require pupils to treat "50%" as if it were a number equal to "$\frac{1}{2}$". This is not only false, but thoroughly confusing (and shows that textbook authors, editors, and examiners have themselves failed to distinguish between *numbers* and *operators*). The *number* "$\frac{1}{2}$" sits midway between 0 and 1. In contrast "50%" on its own has no more meaning than the "f" in $f(x)$: it is an operator, and gives rise to a quantity or value only when it is given a "whole" (or an "x") to act upon. "50% of" is another way of writing "$\frac{50}{100}$ of", which is in turn another way of writing "$\frac{1}{2}$ of". But this is an *operator*, and is not the same as the *number* "$\frac{1}{2}$". In particular, the arithmetic of fractions only applies to *numbers*: there is no similar way (at this level) of making sense of "adding and dividing operators".

1.6.4 The background to fraction arithmetic We noted in Section 1.6.1 that, by the age of 15 or so, it should be clear that large tracts of secondary mathematics come down to "fraction arithmetic". So we end Section 1.6 first with a uniform description of the mathematical *background* which underpins the **arithmetic of fractions**, and then look more closely at the link between fractions and decimals. This is not intended to be a "teaching sequence": its goal is to emphasise certain features of the arithmetic of fractions whose spirit needs to be incorporated into, and reflected in any teaching sequence which schools may adopt.

When introducing positive **integers**, we work at first in some detail with "copies of a concrete object" (such as sweets). Later we shift attention to the number "1" as a kind of abstract "*universal object*", which can itself be replicated (like the sweets, but more exactly, and wholly in the mind). Thus positive integers arise when we **replicate**, or take *multiples* of the unit 1:

$2 = 1 + 1$;

$3 = 1 + 1 + 1$; and so on.

In general, we may replicate the unit "1" n times to obtain

$$n = 1 + 1 + \cdots + 1.$$

All the facts of integer arithmetic follow from this "replication of the unit".

In a similar way, when introducing fractions, we begin by working in some detail with *concrete* objects and consider "parts of some given whole". That is, fractions are initially introduced as "parts of a whole", where the meaning depends on the particular "whole": in other words, the fractions are "fractions of" something, or operators. Before too long, we need to introduce the fundamental idea that if we take the number "1" to be the whole, and think of fractions as parts of this *universal object* "1", we obtain "fractions as numbers". That is, the unit "1" can be subdivided into n **equal** parts, each of which is equal to the **unit fraction** $\frac{1}{n}$. This opens the door to a uniform treatment of fractions—including working with fractions that are bigger than 1: the fraction $\frac{m}{n}$ can be made by taking m copies of this "unit fractional part" $\frac{1}{n}$.

To repeat this explicitly:

Integers were constructed by **multiplying** (or *replicating*) the unit to obtain "multiples of the unit 1":

$$n = 1 + 1 + 1 + \cdots + 1 \quad (n \text{ terms}).$$

Fractions as numbers arise as

> "that part of 1" that emerges when we treat the unit "1" as our "whole", and apply the fraction as an operator to it.

The *unit fraction* $\frac{1}{n}$ is obtained by **dividing** the unit, taking $\frac{1}{n}$ to be "a *submultiple* of the unit 1"—namely that "part" of which exactly n copies make 1:

$$1 = \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} \quad (n \text{ terms}).$$

Thus $\frac{1}{2}$ is precisely that number of which 2 identical copies make 1:

$$1 = \frac{1}{2} + \frac{1}{2};$$

$\frac{1}{3}$ is precisely that number of which 3 identical copies make 1:

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3};$$

$\frac{1}{4}$ is precisely that number of which 4 identical copies make 1:

$$1 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4};$$

and so on.

In the end, this is what every justification for calculation with fractions comes down to.

- The fraction $\frac{1}{q}$ is defined as above: namely that number of which q copies make 1.

  In the spirit of arithmetical *division*, this is interpreted as the result of dividing the unit 1 into q parts, and then taking one part. In other words, $\frac{1}{q}$ is the answer to the question

“$1 \div q = \ldots$?”.

- The fraction $\frac{p}{q}$ is then defined to be $p \times \frac{1}{q}$ (that is,

  $$\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}$$

  with exactly p terms).

  In the spirit of *division* of given quantities, this can then be **proved** to be equal to the result of dividing p units (or wholes) into q identical parts and then taking one of the q parts (which is easiest to see by dividing **each** of the p units into q equal parts [each part being equal to $\frac{1}{q}$], and selecting 1 of these parts from each of the p different units, to give $p \times \frac{1}{q}$). In other words, $\frac{p}{q}$ is defined to be $p \times \frac{1}{q}$, but turns out to be equal to the answer to the question

  “$p \div q = \ldots$?”.

- We know that $\frac{1}{nq}$ is the number of which nq identical copies make 1:

  $$1 = \frac{1}{nq} + \frac{1}{nq} + \frac{1}{nq} + \frac{1}{nq} + \frac{1}{nq} + \frac{1}{nq} + \frac{1}{nq} + \cdots + \frac{1}{nq} \quad (nq \text{ terms})$$

  Since there are exactly $n \times q$ terms on the RHS, we can bracket them into q successive groups with n terms in each bracket:

  $$\begin{aligned} 1 \quad = \quad & \left(\frac{1}{nq} + \frac{1}{nq} + \cdots + \frac{1}{nq}\right) + \left(\frac{1}{nq} + \frac{1}{nq} + \cdots + \frac{1}{nq}\right) + \cdots \\ & + \left(\frac{1}{nq} + \frac{1}{nq} + \cdots + \frac{1}{nq}\right) \end{aligned}$$

  There are now q equal brackets on the RHS, so (by the definition of $\frac{1}{q}$),

  - each bracket must be exactly equal to $\frac{1}{q}$;

- and each bracket contains n terms equal to $\frac{1}{nq}$, so each bracket is also equal to $n \times \frac{1}{nq}$, which is precisely what we call $\frac{n}{nq}$.

$$\therefore \frac{n}{nq} = \frac{1}{q}$$

An entirely similar argument shows that

$$\frac{np}{nq} = \frac{p}{q},$$

so we can replace any given fraction by another fraction *equivalent* to it by "cancelling", or by multiplying numerator and denominator by the same integer n.

- Addition and subtraction of fractions needs to be linked to reality by combining fractional parts of a *fixed* object.

- Any two fractions $\frac{a}{q}$ and $\frac{b}{q}$ with the *same* denominator can also be added or subtracted by remembering what they represent—namely $a \times \frac{1}{q}$ (that is,

$$\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}$$

with a terms) and $b \times \frac{1}{q}$ (that is,

$$\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}$$

with b terms), so that

  - their sum is

$$(a+b) \times \frac{1}{q} = \frac{a+b}{q}$$

(that is,

$$\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}$$

with $a + b$ terms), and

- their difference is

  $$(a - b) \times \frac{1}{q} = \frac{a - b}{q}$$

  (that is,

  $$\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}$$

  with $a - b$ terms).

- Any two fractions $\frac{a}{n}$ and $\frac{b}{q}$ with *different* denominators can be **added** or **subtracted** by first transforming them both into equivalent fractions with the same denominator

  $$\frac{aq}{nq}\left(= \frac{a}{n}\right), \quad \text{and} \quad \frac{nb}{nq}\left(= \frac{b}{q}\right)$$

  so that

  - their sum is

    $$(aq + nb) \times \frac{1}{nq} = \frac{aq + nb}{nq}$$

    (that is,

    $$\frac{1}{nq} + \frac{1}{nq} + \cdots + \frac{1}{nq}$$

    with $aq + nb$ terms), and

  - their difference is

    $$(aq - nb) \times \frac{1}{n}q = \frac{aq - nb}{nq}$$

    (that is,

    $$\frac{1}{nq} + \frac{1}{nq} + \cdots + \frac{1}{nq}$$

    with $aq - nb$ terms).

- *Division* of fraction x by fraction y needs to be linked to reality by discovering that both forms of division give the same answer:
  - "How many times does y go into x?" (or "How many times can I subtract y from x?"), and
  - "What do we multiply y by to get x?"
- We can formally divide any fraction $\frac{a}{q}$ by one with the same denominator, say $\frac{b}{q}$, by remembering what they represent—namely $a \times \frac{1}{q} = \frac{a}{q}$ and $b \times \frac{1}{q} = \frac{b}{q}$, so that we can switch to the equivalent fraction by multiplying both numerator and denominator by "q" to see that the quotient is $\frac{a}{b}$.
- To formally divide any fraction $\frac{a}{n}$ by one with a *different* denominator $\frac{b}{q}$, we first change them both to equivalent fractions

  $$x = \frac{aq}{nq} \quad \left(= \frac{a}{n}\right), \quad \text{and} \quad y = \frac{nb}{nq} \quad \left(= \frac{b}{q}\right)$$

  with the *same* denominator, and we can then evaluate the quotient by switching to an equivalent quotient by multiplying numerator and denominator by "nq" to see that the quotient is $\frac{aq}{nb}$.
- To *multiply* two *unit* fractions $\frac{1}{n}$ and $\frac{1}{q}$ we return to their definitions as submultiples of 1, and think about the product

  $$1 \times 1 = \left(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right) \times \left(\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}\right)$$

  [n terms in the 1st bracket, q terms in the 2nd].

  When we multiply out the two brackets we obtain nq equal terms, each equal to

  $$\left(\frac{1}{n}\right) \times \left(\frac{1}{q}\right),$$

whose sum is 1. But that is precisely the definition of the unit fraction "$\frac{1}{nq}$".

$\therefore \left(\frac{1}{n}\right) \times \left(\frac{1}{q}\right) = \frac{1}{nq}$.

When we multiply two general fractions $\frac{a}{n}$ and $\frac{b}{q}$ we can write each fraction out as:

$$\frac{a}{n} = a \times \frac{1}{n} = \left(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right) \quad (a \text{ terms})$$

$$\frac{b}{q} = b \times \frac{1}{q} = \left(\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}\right) \quad (b \text{ terms})$$

and then multiply out the two brackets 'long hand' to get

$$\frac{a}{n} \times \frac{b}{q} = \left(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right) \times \left(\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}\right)$$

[a terms in 1st bracket, b terms in the 2nd] where the RHS gives rise to exactly ab separate terms, each equal to

$$\left(\frac{1}{n}\right) \times \left(\frac{1}{q}\right) = \frac{1}{nq}.$$

$\therefore \frac{a}{n} \times \frac{b}{q} = ab \times \frac{1}{nq} = \frac{ab}{nq}$.

1.6.5 Fractions and terminating decimals

Pupils need exercises that clarify three features of terminating decimals.

The first is to use place value to interpret each decimal as a sum. Just as

$$375 = 3 \times 100 + 7 \times 10 + 5,$$

so place value tells us that 0.375 means precisely the sum

$$3 \times \frac{1}{10} + 7 \times \frac{1}{100} + 5 \times \frac{1}{1000}.$$

The second is to rewrite the constituent parts (from the separate "places") with a common power of 10 as denominator (here "1000") to obtain:

$$\frac{3}{10}+\frac{7}{100}+\frac{5}{1000}=\frac{300}{1000}+\frac{70}{1000}+\frac{5}{1000}=\frac{\mathbf{375}}{\mathbf{1000}} \qquad (*)$$

In other words, pupils need to connect the *definition* of place value (which breaks up the number into a sum of several parts—tenths, hundredths, thousandths, etc.) with the alternative reading of 0.375 as $\frac{375}{1000}$.

The third feature is more subtle, namely to realise precisely which fractions correspond to *terminating* decimals, and which correspond to *endless* decimals.

- If a fraction is given with denominator a power of 10, then it is easy to write it as a terminating decimal, in exactly the same way that equation $(*)$ tells us that

$$\frac{375}{1000}=0.375. \qquad (**)$$

- But what do we know about $\frac{3}{8}$ and $\frac{3}{18}$ that should tell us *in advance* that the first has a terminating decimal, but the second does not?

The key lies in the previous paragraph (and properties of prime factorisation which were addressed in Section 1.4).

Suppose we are given some unfamiliar fraction.

- The first move is to *cancel* any common factors between the numerator and the denominator which may mislead us.

For example, we know that the decimal for $\frac{1}{2}=0.5$, and so it terminates. But if we were faced instead by $\frac{3}{6}$, we might be misled by knowing that the decimal for $\frac{1}{6}$ does not terminate. This first move of "cancelling" puts the given fraction into its "standard form", or "lowest terms", $\frac{p}{q}$, where p, q have no common factors (other than 1): $HCF(p,q)=1$.

We have seen that a fraction whose denominator is equal to a power of 10 can always be written as a terminating decimal (as in equation $(**)$). Pupils need to extend this to see that

- if a given fraction $\frac{p}{q}$ **can be re-written** in a form with denominator equal to a power of 10 (in the same way that $\frac{3}{8} = \frac{375}{1000}$),

  then it will be equal to a terminating decimal.

That is, given a fraction $\frac{p}{q}$, we need to know when it can be rewritten as an equivalent fraction $\frac{np}{nq}$ which has denominator a power of 10.

> If nq is a power of 10 for some multiplier n,
>
> then **the denominator** q of the given fraction must be **a factor of some power of 10**.
>
> Now $10 = 2 \times 5$, so a power of 10, such as $10^m = (2 \times 5)^m$, has the form $2^m \times 5^m$.
>
> And any factor of $2^m \times 5^m$ must have the form $2^a \times 5^b$ for some $a, b \leqslant m$.
>
> $\therefore$ If the fraction $\frac{p}{q}$ has a terminating decimal, then the denominator q must have the form $2^a \times 5^b$: that is, a power of 2 times a power of 5.

- Conversely suppose we are given any fraction with denominator q of the form $2^a \times 5^b$.

  If $a \geqslant b$, then we can multiply by $n = 5^{a-b}$ to make $nq = 2^a \times 5^a = 10^a$; and if $b > a$, then we can multiply by $n = 2^{b-a}$ to make $nq = 2^b \times 5^b = 10^b$.

  $\therefore$ Any fraction with denominator of the form $2^a \times 5^b$ has a terminating decimal.

Hence whether a given fraction $\frac{p}{q}$ (where p, q have no common factors) has a **terminating** decimal or not depends entirely on the prime factorisation of the denominator q:

$$q = 1, 2, 4, 5, 8, 10, 16, 20, 25, 32, \ldots$$

all lead to terminating decimals, but

$$q = 3, 6, 7, 9, 11, 12, 13, 14, 15, 17, 18, 19, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31, \ldots$$

never do.

1.6.6 Fractions and recurring decimals

Section 1.6.5 shows that:

> every fraction $\frac{p}{q}$, where p, q have no common factors and q is **not** of the form $2^a \times 5^b$ has a decimal that does **not** terminate, and so must go on for ever.

In fact every such fraction has a decimal that "recurs": that is, its decimal consists of

> an initial sequence of digits (which can be of any finite length), followed by a "block of digits" that simply repeats over and over again *for ever*.

The most familiar examples are

$$\frac{1}{3} = 0.3333333\ldots$$

which recurs from the beginning with a repeating block "3" of length 1;

$$\frac{1}{11} = 0.090909\ldots$$

which recurs from the beginning with repeating block "09" of length 2;

$$\frac{1}{6} = 0.1666666\ldots$$

which recurs from the 2nd place with repeating block "6" of length 1;

$$\frac{1}{7} = 0.1428571428\ldots$$

which recurs from the start with repeating block "142857" of length 6.

The converse is also true, in that

> every decimal which recurs in this fashion is the decimal of some fraction.

The proofs of these statements are discussed briefly in Section 1.8 below.

1.7. [Subject content: *Number* p. 6]

- **use approximation through rounding to estimate answers and calculate possible resulting errors expressed using inequality notation $a < x \leqslant b$**

In mathematics we calculate with *exact* "mental objects". But when mathematics is *applied*, the numbers often come from the real world. Discrete data from the real world (e.g. small counting numbers) can sometimes be "exact"; but most measurements are reliable only to a certain degree of accuracy. The approximate character of certain measurements is reflected in the "rounding conventions". When a digit is known to be, or is to be taken as being, just beyond the known or required limits of accuracy, the "rounding conventions" mean that

> "a digit of 5 or more is rounded **up**, and everything else is rounded **down**".

Hence a decimal like 37.45293 would be written as

> "37.45 to 2 decimal places", or "37.5 to 1 decimal place".

Conversely, if we are given a measurement "$x = 37.5$ to 1 decimal place", then all we know is that the "true" value of x lies somewhere *in an interval*: $37.45 \leqslant x < 37.55$. (The inequality given in the official requirement listed at the start of 1.7 should probably have been written as "$a \leqslant x < b$" to fit in with this convention.)

If the initial data is only known to a certain degree of accuracy, then any calculation with that data is approximate from the outset. Even when our data and our calculations are "theoretically exact", approximations may arise when exact terms (such as "$\sin 45^\circ$" or "$\sqrt{2}$") are "evaluated" at some point using a decimal approximation. All approximations affect the accuracy of the final result; so pupils need to understand how potential errors "accumulate" as a result of calculation, so that they can tell exactly how inaccurate the final result could be.

When *adding* or *subtracting* approximate numbers, *the errors in the data* **add up**. Given two lengths of 2.15cm and 1.75cm—each correct to within 0.05cm—their calculated difference of 0.40cm is only correct to within 0.1cm, so could actually be as low as 0.30cm or as high as 0.50cm. And if we were to add four lengths, each of which was accurate to within 0.05cm, then the result would only be accurate to within 0.2cm either way (so we would only know that the answer lies in an interval of length 0.4cm).

When *multiplying* or *dividing* the story is a more complicated. For example, the area of a rectangle whose dimensions are given as "15cm by 12cm", where each measurement is accurate to within an error of 0.1cm, is equal to $15 \times 12\text{cm}^2$, or 180cm^2, but only **to within 2.7cm^2**. And if we know that a rectangle has area 180cm^2 accurate to within 5cm^2, and that its length is 20cm accurate to within 0.1cm, then its width may be as small as small as $(175 \div 20.1)\text{cm} \approx 8.7\text{cm}$ (to 1 d.p.), or as large as $(185 \div 19.9)\text{cm} \approx 9.3\text{cm}$ (to 1 d.p.).

The art of making **estimates**, or *approximate* calculations, is more subtle than is often thought. It depends on:

- robust fluency in *exact* calculation, together with a "feeling for calculation" that is willing to think flexibly about the effect of any errors,
- a willingness to change *global* units intelligently (replacing the given units by larger or smaller "blocks" so as to make the eventual calculation more manageable), and
- an ability to make sensible *local* approximations (to find the approximate size of one of these new 'blocks' and to estimate the number of "blocks").

Consider first approximating an exact arithmetical calculation, such as 35941×273.

- We need the kind of flexibility that can think of this as 35941 "blocks" of 273, and combine this with a clear understanding of how the *exact* calculation would proceed using the given units—with 35941 copies of a collection of size 273.

- Instead of 35941 blocks (each of size 273) we then may see the advantage of interpreting the number of blocks as “slightly **more** than $\mathbf{33\frac{1}{3}}$ *thousand*”, and compensate the block size of 273 by thinking of it as “slightly **less** than **3** *hundreds*”.
- This then suggests that the required answer is “approximately 100 hundred thousands”, or 10 million.

By increasing one factor in the product and decreasing the other we managed to produce an answer that is fairly close to the actual value (9 811 893) of the product. But the method used gave us no clue as to whether we had overestimated or underestimated, or what our maximum error might be. To get such assurance we would have to approximate *consistently*—perhaps to work out

first an overestimate such as $36000 \times 300 = 10800000$,

then an underestimate such as $35000 \times 250 = 8750000$.

Similarly, in seeking to estimate the size of a large crowd, one may divide the whole into a number of blocks of more-or-less the same size, count (or estimate) the number in a given section of the crowd relatively accurately (for example, by counting the number of rows and the number in each row), and then multiply the answer by the number of blocks. A striking historical example of this approach to estimation occurs in Herodotus, *The Histories*, Book 7:

> “As nobody has left a record, I cannot state the precise numbers provided by each separate nation [towards the Persian army that Xerxes was leading against the Greeks in around 480BC], but the grand total, excluding the naval contingent, turned out to be 1 700 000. The counting was done by first packing ten thousand men as close together as they could stand and then drawing a circle around them on the ground; they were then dismissed and a fence, about navel-high, was constructed round the circle; finally the other troops were marched into the area thus enclosed and dismissed in their turn, until the whole army had been counted.”

1.8. [Subject content: *Number* p. 6]

- **appreciate the infinite nature of the sets of integers, real and rational numbers.**

There is an awkward clash between the precise, procedural language which is appropriate for specifying the ideas and processes of a school curriculum and this highly unusual and rather woolly "requirement". Indeed, it remains unclear how it survived the extended editing process.

The underlying idea would be fine as part of an *internal* curriculum—for in some sense, the whole of elementary mathematics is the story of "how we tame infinity". But to include such a requirement in a *national* curriculum (especially in such a curiously worded form) runs the risk that some examiner may decide that they are obliged to invent some way of "assessing" each year whether it has been addressed!

1.8.1 Mathematics begins when we move beyond the particular to the general. Every culture develops some way of referring to the *size* of small collections of objects (one, two, three, ...), and to the *ordering* of its objects (first, second, third, ...). They either develop some semi-systematic counting process or adapt that of some neighbouring culture for their own use. Some cultures go further and invent an "arithmetic" based on their counting process; but this is almost always done without worrying whether their form of counting could "go on for ever".

For example, the numeration system of the Egyptians, and that of the Babylonians, were both semi-systematic. But both were restricted by the need to invent specific new symbols each time they wanted to refer to larger numbers; so it is unclear whether they appreciated "the infinite nature of the set of integers". In contrast, there is something truly remarkable about the ease with which our Hindu-Arabic numeral system combines the ten digits 0-9 and the idea of "place value" to convey the **idea** that counting

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, \ldots, 98, 99, 100, 101, \ldots$$

can **continue for ever**, even though we soon run out of linguistic ways of "naming" the numbers whose numerals we can all write down so easily.

Despite their mathematical sophistication, the Greeks had no such systematic notation—a lack which may have forced them to develop their astonishingly modern approach to handling infinity and infinite processes. But it also meant that *Archimedes* had to go to considerable lengths to demonstrate (in his little book, *The Sand Reckoner*) that "the number of grains of sand in the universe is finite". This he did by repeatedly changing units in order to estimate a finite upper bound (around 8×10^{63}) based on constructing a large *power* of a number called (in Greek) a "myriad myriad"—in much the same way as Herodotus reported (Section 1.7 above) that the Persians counted the number of soldiers in Xerxes' army as a *multiple* of ten thousand.

Our numeral system avoids the inevitable finiteness of number *names*, and focuses instead on a **numeral** system based on place value, which allows us to write numbers *without giving them names*. It then seems clear that, using only the digits 0–9, our *written numerals* for counting numbers could go on for ever. (The truth is more delicate. In our numeral system we "deduce" the endlessness of the sequence of counting numbers by first *assuming* that the sequence of possible "places"—the units, tens, hundred, thousands, etc.—goes on for ever! However, this is unlikely to disturb anyone.)

In some sense, that is all there is to it. The counting numbers are the same as the positive integers, so the integers—both positive and negative—are also infinite (that is, "more than just finite"). The integers are precisely the "rational numbers with denominator 1", so the set of rational numbers is even bigger—and hence infinite. And the real numbers include all the rational numbers—so the set of all real numbers is also infinite.

1.8.2 Sequences: What else is there to be said at secondary level? Endless sequences use the counting numbers to label the successive *terms* of a sequence. The squares, the cubes, the powers of 2—all go on for ever, and all get larger and larger without bound (though some "grow" faster than others):

$$0^2 = 0,\ 1^2 = 1,\ 2^2 = 4,\ 3^2 = 9,\ 4^2 = 16,\ \ldots, n^2,\ \ldots;$$

$$0^3 = 0,\ 1^3 = 1,\ 2^3 = 8,\ 3^3 = 27\ ,4^3 = 64,\ldots,n^3,\ \ldots;$$

$$2^0 = 1,\ 2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,\ \ldots,2^n,\ \ldots.$$

Some sequences eventually stop. Others go on for ever, with one such term for each positive integer n. It is hard to see that there is much to make a fuss about.

However, there are two clear candidates at this level, which show that indeed there is indeed something interesting here, to which one might draw attention—at least for suitable groups of pupils. The first concerns prime numbers; the second concerns the way we can be sure that fractions are precisely the real numbers whose decimals either terminate or recur.

1.8.3 Prime numbers: Prime numbers are the multiplicative building blocks for integers.

> There are 4 prime numbers up to 10; 25 up to 100; and 168 up to 1000.
>
> That is: prime numbers make up 40% of integers up to 10; 25% up to 100; 16.8% up to 1000.

It is thus apparent that prime numbers seem to be "thinning out" as one goes up. So one might ask:

> Do the prime numbers eventually "peter out"? Or do they go on for ever?

There is no indication that anyone considered such a question before the Greeks (4th century BC), who **proved** that

> "the prime numbers are more than any assigned multitude".

That is, that *the prime numbers go on for ever*. Euclid's original proof is highly memorable and has impressed many a young mind—but it is often misrepresented. We give it here in a form that is both close to the original, and in the spirit of modern constructive mathematics.

> We know a few prime numbers, so we can clearly pick one to start with—say p_1. (We could choose $p_1 = 2$, but we do not have to.)

> We then set
>
> $$N_1 = p_1 + 1$$
>
> to be "1 more than the product of all the primes in our list so far".
>
> $\therefore$ p_1 is not a factor of N_1 (since it leaves remainder $= 1$); so **the smallest prime factor** p_2 of N_1 is a **new** prime number.
>
> Then set
>
> $$N_2 = p_1 \times p_2 + 1$$
>
> to be "1 more than the product of all the primes in our list so far".
>
> $\therefore$ Neither p_1 nor p_2 is a factor of N_2 (since both leave remainder $= 1$); so the smallest prime factor p_3 of N_2 is a **new** prime number.
>
> Then set
>
> $$N_3 = p_1 \times p_2 \times p_3 + 1$$
>
> to be "1 more than the product of all primes in our list so far".
>
> $\therefore$ None of p_1, p_2, p_3 is a factor of N_3 (since all leave remainder $= 1$); so the smallest prime factor p_4 of N_3 is a **new** prime number.
>
> And so it goes on, for ever. **QED**

That is, once the list gets started, no matter how many primes we have listed so far, we have a bomb-proof way of finding a **new** prime.

> Suppose we start with $p_1 = 2$, then $N_1 = 3$ is prime, so $p_2 = 3$;
>
> then $N_2 = 7$ is prime, so $p_3 = 7$;

> then $N_3 = 43$ is prime, so $p_4 = 43$.

It is important **not** to stop at this point, but to complete the next three stages in order to understand how the process really works.

> Work out
> $$N_4 = 2 \times 3 \times 7 \times 43 + 1,$$
> and hence find its **smallest** prime factor $p_5 = \dots$.
>
> Then work out
> $$N_5 = 2 \times 3 \times 7 \times 43 \times p_5 + 1,$$
> and hence find its **smallest** prime factor $p_6 = \dots$.
>
> Then work out
> $$N_6 = 2 \times 3 \times 7 \times 43 \times p_5 \times p_6 + 1,$$
> and hence find its **smallest** prime factor p_7 .

It is also worth starting with various different "initial primes" p_1 to see how this affects the sequence which is generated each time.

Those who took up our earlier suggestion (Section 1.3) of challenging pupils to

> "Find a prime number which is one less than a square. Find another. And another."

might also like to use the similar-sounding, but actually very different challenge:

> "(a) Find a prime number which is one **more** than a square
>
> (b) Find another such prime. And another."

If one tries this, then it quickly becomes clear that, except for the very first such prime $1^2 + 1 = 2$, one can restrict to looking for *odd* primes, and these must be "one more than an **even** square". Among the list of numbers that are "one more than an even square",

$$2^2 + 1,\ 8^2 + 1,\ 12^2 + 1,\ 18^2 + 1,\ 22^2 + 1,\ 28^2 + 1, \dots$$

are all multiples of 5.

If we eliminate these multiples of 5, we are left with a long list of candidate primes, starting:

(2,) 5, 17, 37, 101, 197, 257, 401, 577, 677, 901, …

Almost all of these 11 "*candidate* primes" turn out to be *genuine* primes (only one of those listed is not). This raises the question:

> Are there infinitely many prime numbers of the form "$n^2 + 1$"?
>
> Or does the list eventually peter out?

This is perhaps the simplest question one can pose at this level to which the answer is not yet known.

1.8.4 Recurring decimals: One other place where infinity features at Key Stage 3 and needs to be handled properly is the way the normal division process is extended to compute recurring decimals for fractions. We have seen that when we divide an integer p by another integer q, the process *terminates* precisely when the fraction $\frac{p}{q}$ is equivalent to a decimal fraction (one with denominator 10^n for some n)—as with

$$\frac{3}{24} = \frac{125}{1000},$$

or

$$\frac{5}{16} = \frac{3125}{10000},$$

and that this occurs whenever the fully simplified fraction has a denominator of the form $2^a \times 5^b$.

In all other cases, the division process continues indefinitely. For example, when one carries out the division for $\frac{1}{7}$, the output seems to recur: 0.14285714…. All too often pupils are left with the impression that

> the **output** to the division process "recurs" **because it seems to recur**.

This is like believing that the "leading digits" of the sequence of powers of 2 recur **because they look as though** they recur:

$$2, 4, 8, 1, 3, 6, 1, 2, 5, 1; 2, 4, 8, 1, \ldots .$$

The fact that the division of p by q recurs follows **not** from the apparent **output**, but from the **pattern of remainders**.

- The decimal for $\frac{p}{q}$ *terminates* precisely when at some point we obtain a remainder of 0.
- So if the decimal does **not** terminate, then the only possible remainders are

  $$1, 2, 3, \ldots, q-1.$$

  Hence, within at most q steps, we will always get a remainder r that we have seen before; and this remainder r becomes $10r$ in the next decimal place *as it did on the first occurrence of the remainder r*, so from then on the process simply repeats whatever happened after the previous occurrence of the remainder r.

For example, when calculating the decimal for $\frac{1}{7}$ we divide 7 into 1.000000 . . . .

- **Forget about the output**, or the "answer", and **concentrate on the remainders**.
- The process begins with a remainder of "1", then "3", then "2", then "6", then "4" then "5", then "1" (the first repeat)—which becomes "10" in the next column, as it did at the first stage when the initial "1" became "10 tenths".
- The process must then repeat from here on (giving the answer

  $$0.14285714285714\ldots,$$

  with the block "142857" repeating for ever).

The converse claim—namely that

> every number x whose decimal recurs is the decimal of some fraction

can be appreciated at this level (say Year 9 or Year 10) via the procedure for turning any such decimal back into a fraction. For example:

$$\begin{aligned}
&\text{Suppose } x = 0.37255555\ldots \text{ (for ever)}\\
&\text{Then } 10x = 3.72555555\ldots \text{ (for ever)}\\
&\therefore 9x = 10x - x\\
&\quad\;\; = 3.353 = \tfrac{3353}{1000}\\
&\therefore x = \tfrac{3353}{9000}.
\end{aligned}$$

$$\begin{aligned}
&\text{Suppose } x = 0.72525252525\ldots \text{ (for ever)}\\
&\text{Then } 100x = 72.5252525252\ldots \text{ (for ever)}\\
&\therefore 99x = 100x - x\\
&\quad\;\; = 71.8 = \tfrac{718}{10}\\
&\therefore x = \tfrac{718}{990} = \tfrac{359}{495}.
\end{aligned}$$

1.9. [Subject content: *Ratio, proportion and rates of change* p. 7]

- **change freely between related standard units (for example time, length, area, volume/capacity, mass)**
- **use scale factors, scale diagrams and maps**
- **express one quantity as a fraction of another, where the fraction is less than 1 and [where the fraction is] greater than 1**
- **use ratio notation, including reduction to simplest form**
- **divide a given quantity into two parts in a given part:part or part:whole ratio; express the division of a quantity into two parts as a ratio**
- **understand that a multiplicative relationship between two quantities can be expressed as a ratio or a fraction**
- **relate the language of ratios and the associated calculations to the arithmetic of fractions and to linear functions**

- **solve problems involving percentage change, including: percentage increase, decrease and original value problems and simple interest in financial mathematics**
- **solve problems involving direct and inverse proportion, including graphical and algebraic representations**
- **use compound units such as speed, unit pricing and density to solve problems**

1.9.1 This is a mixed bag of requirements linked to multiplication, ratio and proportion, and scaling—and hence, ultimately to the application of fractions.

- The first two listed requirements (the ability to switch "between related units", and to work with "scale factors, scale diagrams and maps") clearly involve "multiplying factors" and an application of ratios.
- We have already noted the relative neglect of compound units. So the last listed requirement in 1.9 should be interpreted in the light of comments already made in Section 1.1 above and in Part II, Section 1.2.
- Percentage and percentage change has already arisen in 1.6, but reappears here for good reason.
- The requirements for pupils to "express one quantity as a fraction of another" and to "divide a given quantity into two parts" underline the connections between the work required here and work involving fractions (see Sections 1.6.1–1.6.3 above).

1.9.2 We repeat and expand some of the ideas touched upon in Part II, Section 2.2.1. Elementary mathematics comes into its own (and needs to be seriously *taught*!) as soon as we take the step from addition to *multiplication*. Ratios are the quintessential "multiplicative relations", and work with ratios links naturally to work with fractions.

All that is needed to generate a ***ratio*** *is a single class of* ***comparable*** *magnitudes*—that is, a class of magnitudes

- where any two given entities can be "compared", so that we can decide which is the larger, and
- where one can also subtract the smaller from the larger, with the "difference" being another entity from the same class (as, for example, with line segments).

That is, one needs to be able to implement a version of the *Euclidean algorithm*.

The simplest example of a class of "comparable magnitudes" is the set of positive *real numbers*. In the context of ratios, real numbers normally arise as the set of *numerical measures* of some set of objects (relative to some chosen unit). Such *numerical* ratios are easy to handle (with the class of *objects* being replaced by their *measures*); but ratios also arise naturally in mathematics between comparable entities (such as line segments, or 2D shapes) without turning everything into numbers by 'measuring'.

For example, 3cm and 2cm are in the ratio "3 : 2". But we also have the same ratio between the *two line segments*, say AB and CD, that were measured as being of lengths 3cm and 2cm. Even if we do not know their exact lengths, there is often a natural way to be sure that "half of the second segment fits *exactly* three times into the first segment". For example, if we draw a circle with centre O passing through the point X, extend the radius XO to meet the circle again at Y, and construct the mid-point M of the segment OY, then we can be sure (without measuring) that

$$XM : XO = 3 : 2.$$

1.9.3 The rest of our comments in this section revisit and extend our previous remarks in Part II, Section 2.2.1. What follows explores further the background to *ratio and proportion*, which is the key idea that underlies most of the (rather vaguely worded) requirements listed at the start of Section 1.9. We repeat our earlier comment: this outline is intended for teachers, and is not a teaching sequence for pupils.

The word *proportion* has a colloquial usage, which is unfortunately copied in many mathematics texts and classrooms. People speak about "a proportion of the class", meaning exactly the same as "a fraction of the class". **This**

has nothing to do with mathematical "proportion". Sloppy language is neither helpful nor harmless: it confuses pupils, teachers, textbook authors, and examiners alike. In general, technical words are best used correctly and with care in the mathematics classroom (as is normal in many other countries). Because the underlying mathematics may not be second nature, it seems simplest to repeat the basic framework from Part II, Section 2.2.1, while adding a little more detail.

Given the notion of a class of comparable magnitudes, or quantities, a (mathematical) proportion arises when two different classes of entities are linked in a special (but very common) way. For example, suppose that one class consists of

"quantities of petrol"

and the other class consists of

"amounts of money in £".

If 1 litre of petrol costs £1.50,
then we expect 2 litres to cost £3 (= 2 × £1.50)

That is, for any two purchases from the same outlet at the same time,
the ***quantities*** *purchased* (*in litres*)

are in the same ratio as

*the **amounts** paid* (*in £*).

If I buy a litres of petrol and pay £c,
and you buy b litres of petrol and pay £d,
then the ratio $a : b$ is equal to the ratio $c : d$.

The equality

$$\boldsymbol{a : b = c : d}$$

is what we call a *proportion*.

Note that since a, b, c, d are magnitudes, with a, b of one kind and c, d of another kind, then "$a : b$" is a perfectly well-defined ratio; but "$a : c$" makes no sense, because a and c are not "comparable magnitudes". One can have a ratio $a : b$ between two quantities of petrol both measured in litres (or a ratio $c : d$ between two amounts of money—both measured in £); but one cannot have a ratio between a quantity of fluid and an amount of money.

However, something miraculous occurs if we replace the different quantities and amounts by their numerical measures. The equality of ratios

$$a : b = c : d$$

can then be written as an *equation between fractions*, which can be treated purely algebraically. That is, *if we replace each ratio by the quotient of the corresponding measures* we get an equality of quotients, or fractions:

$$\frac{a}{b} = \frac{c}{d} \tag{$*$}$$

The two quotients in equation $(*)$ are always equal, but can take *any positive value*. For example, we could buy

$b = 2a$ litres of petrol and pay $d = 2c$ pounds,

and the quotients would both take the value $\frac{1}{2}$. Or we could buy

$b = \frac{1}{2}a$ litres of petrol and pay $d = \frac{1}{2}c$ pounds,

and the quotients would both take the value 2.

The equation $(*)$ *between fractions* can be treated purely numerically (or algebraically) and can be rearranged to give

$$\frac{c}{a} = \frac{d}{b}.$$

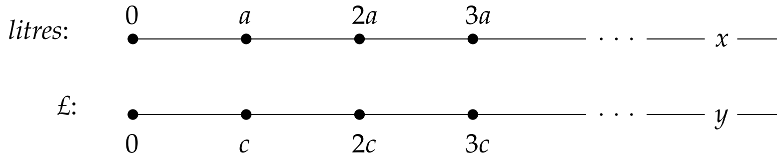
This equation looks very similar to equation $(*)$, but *it is completely different*. The two sides do not represent ratios, but specify the **constant of proportionality** (relative to the two chosen units: litres and pounds (£)). That is, once we choose units and give numerical values a and c to the basic pair of corresponding magnitudes—one from one class and one from the other

$$\boldsymbol{a} \quad \textbf{litres} \xrightarrow{\qquad\qquad} \textbf{cost £}\boldsymbol{c}$$

the value of the quotient $\frac{c}{a}$ is a **constant**: that is, it is the same as the value of the corresponding quotient $\frac{d}{b}$ *for any other pair* of corresponding magnitudes b, d (one from one class and one from the other). The purely numerical quotient $\frac{c}{a}$ can now be interpreted as the "multiplying factor" that links the two classes of related magnitudes.

This is the simplest, and perhaps the most valuable, application of school mathematics—to life, to science and to mathematics itself. It applies whenever two quantities are related so that if one quantity doubles, or triples, so does the other: that is, where the numerical measures a, c or b, d of the two quantities have a **constant ratio**. Two quantities that vary in such a way as to preserve a constant ratio between their values are said to be "in **proportion**".

The fact that "$\frac{c}{a}$ is a constant" means that the *number lines* corresponding to the two families of measures "line up" in such a way that one scale is simply a multiple ($\times\frac{c}{a}$) of the other:



If we imagine a linked pair (x, y) of unknown variables—where

$$x \quad \text{litres} \longrightarrow \quad \text{cost } £y$$

then these linked variables are related by the *linear equation*

$$y = \left(\frac{c}{a}\right) x.$$

Eventually (in late Key Stage 3 or Key Stage 4) one may want as many pupils as possible to appreciate this global picture, and to be able to

> "formulate proportional relations algebraically"

as is required in the quote at the start of Section 2.2.1 in Part II. However, this is unnecessary, and probably inappropriate for beginners, who first need to learn how to solve the various standard problems involving proportion.

Any particular proportion problem that pupils may be required to solve is likely to involve just two pairs (a, c) and (b, d),

> where a and b come from one class of magnitudes,

and c and d come from the other class.

In a typical proportion problem, three of the four values are known and the fourth is "to be found". This explains why the approach to solving this kind of problem is referred to in old texts as "the rule of three". Hence one pair is completely known, and we take this as our "base", or reference pair

$$\boldsymbol{a}\ \textbf{litres} \longrightarrow \textbf{cost £}\boldsymbol{c}$$

One of the *other* two values is to be found. So the four ingredients can be thought of as the corners of a rectangular array, where three of the values are known and the fourth is to be calculated, so we either have the unknown value in the bottom right corner:

$$\begin{array}{ll} \text{If} & \boldsymbol{a}\ \textbf{litres} \longrightarrow \textbf{cost £}\boldsymbol{c} \\ \text{then} & b \text{ litres} \longrightarrow \text{cost £??} \end{array}$$

or the missing value may be located bottom left:

$$\begin{array}{ll} \text{If} & \boldsymbol{a}\ \textbf{litres} \longrightarrow \textbf{cost £}\boldsymbol{c} \\ \text{then} & \text{?? litres} \longrightarrow \text{cost £}d \end{array}$$

This standard way of representing the four pieces of information in a *proportion*—with three known values and one generally unknown—is referred to here as the *rectangular template* for displaying *proportion* problems.

To repeat the earlier derivation, if we know the corresponding values a and c in our "reference pair"

$$\boldsymbol{a} \longrightarrow \boldsymbol{c}$$

then two unknown amounts x and y, which correspond to each other, provide the third and fourth vertices of our "rectangular template"

$$x \longrightarrow y$$

and so satisfy

$$x : a = y : c$$

If the two magnitudes of the first kind x, a, and the two magnitudes of the second kind y, c are replaced by their measures, then the proportion can be written as

$$\frac{x}{a} = \frac{y}{c}$$

and this can be rearranged to express the relationship between the two unknown values x and y as

$$y = \left(\frac{c}{a}\right) x$$

with multiplying factor $\frac{c}{a}$. If we are given the value of x, we can calculate the value of

$$y = \frac{c}{a} \times x;$$

and if we are given the value of y, then we can calculate the value of

$$x = \frac{a}{c} \times y.$$

For example:

if $a =$ £100 **is worth the same as** **$c = \$150$**

then

$x =$ £200 will be worth exactly $y = \frac{c}{a} \times x = \$\ldots.$

And

if $a = 1kg$ **is the same as** **$c = 2.205lbs$**

then

$x = \frac{a}{c} \times y = \ldots kg$ is the same as $y = 5lbs.$

Earlier we showed how the "number lines" corresponding to the two families of magnitudes in a proportion problem can be lined up to form what is sometimes called a "double number line". We have since seen how the simpler "rectangular template" picks out two data points (a and b) on the first number line, and two points (c and d) on the second number line, and have suggested that this is sufficient for the beginner to solve problems. (To link the two representations one has to imagine that the double number lines run *vertically*, with a and b chosen from the left hand line and c and d chosen from the right hand line.)

We typically know one pair of corresponding values, such as that

£100 is worth \$150;

and we want to know either:

"If I have £$x = $ £768 how many **\$*y*** can I expect in exchange?"

or

"How many **£*x*** should I expect in exchange for $y = \$1152$?".

Pupils who become sufficiently confident may solve the first kind of *proportion* question directly—and in one of two ways:

(i) extract the ratio $\frac{b}{a}$ from two of the known quantities of one kind (e.g. $\frac{768}{100}$ in the above example), and apply it to the third known quantity c of the other kind, to find the unknown required value

$$y = c \times \frac{b}{a}$$

($150 \times \frac{768}{100} = \ldots$ in the above example); or

(ii) identify the *constant of proportionality* $\frac{c}{a}$ ($= \frac{150}{100}$ in our example) derived from two known corresponding quantities of different kinds, and apply it to the third known quantity b to find the unknown value

$$y = \frac{c}{a} \times b$$

($\frac{150}{100} \times 768 = \ldots$ in the above example).

However, for most students, the **unitary method** provides an essential stepping stone *en route* to this general method—a stepping stone which one can return to in any setting to re-explain, or to reinforce, the logic of the general method.

Given three of the four relevant values,