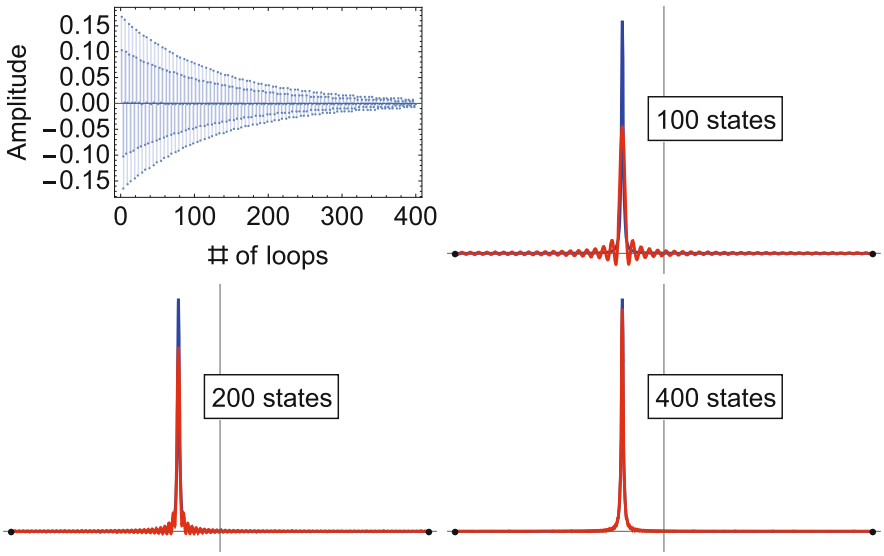


Fig. 6.6 Constructing the position state from the middle picture in Figure 6.3. We need over 400 energy states to build this one state! Notice the position state in blue is noticeably taller than the reconstructed states even when we use the first 200 energy states. The position state is still a bit taller even with 400 energy states!

Now that we have explored the concept of superposition, let's build a more realistic position state, similar to the top right graph in Fig. 6.3, from the energy states. Figure 6.6 shows the mathematical results needed to construct that position state from the energy basis set. As you can see, we needed at least twice as many energy states to get a good approximation for a position state. Despite needing many more energy states to accurately reconstruct a single position state, the result remains the same: if the quantum mechanical system were in that position state and we measured its energy, we could get many different answers with probabilities given by the squared amplitudes.

The Uncertainty Principle Part 2
Measuring an observable puts the system into a single state for that observable and in a superposition of states for a second observable. If a second observable shares states with the basis set of the first observable, the value of that second observable is already determined. We can measure both quantities as many times as we'd like, and we will never change the state of the system. We will always get the same answer as the first time we measured.

(continued)

If the two observables have different basis sets, knowing the value of one observable does not inform you of the value of the other. These observables are incompatible. Measuring one observable places the system in a superposition of states for the other observable where none of the amplitudes are 1. Therefore, the state of the second observable after measurement is probabilistic.

States and superposition of states are core concepts in quantum mechanics. Let's summarize the entire chapter using two generalized observables α and β.

- Both α and β have states associated with them, denoted as $\psi_{\alpha,1}$, $\psi_{\alpha,2}$, ... and $\psi_{\beta,1}$, $\psi_{\beta,2}$, ...
- The set of states that are needed to describe all possible values for α is called the basis set of α. The set of states that are needed to describe all possible values for β is called the basis set of β.
- Each state has a measurable value associated with it. If the system is in state $\psi_{\alpha,7}$ and we measure α, we will get the value associated with that state, α_7. We will never measure, for example, α_5.
- If α and β are incompatible, they have different basis sets. If they are compatible, they share a basis set.
- A state for α can be written as a superposition of states for β, and vise-versa. For example,

$$\psi_{\alpha,7} = A_{\beta,1}\psi_{\beta,1} + A_{\beta,2}\psi_{\beta,2} + A_{\beta,3}\psi_{\beta,3} + ... \tag{6.3}$$

  or

$$\psi_{\beta,137} = A_{\alpha,1}\psi_{\alpha,1} + A_{\alpha,2}\psi_{\alpha,3} + A_{\alpha,3}\psi_{\alpha,3} + ... \tag{6.4}$$

- The square of the amplitudes in the above equations tell us the probability, upon measurement, we will find the system in that state with the value associated with that state. For example, if the system was in state $\psi_{\beta,137}$ with value β_{137} and we measured α, the probability the system is now in a particular state of α, say state $\psi_{\alpha,3}$ with value α_3, is the amplitude squared, $A^2_{\alpha,3}$.
- Suppose we measured β and got β_{137}. Next, let's say we measure α and get α_7. If α and β are incompatible and we then re-measure β, it is highly likely we will get something other than β_{137}.

6.4 The Energy Basis Set for a Quantum Harmonic Oscillator

The quantum harmonic oscillator is one of the first quantum systems learners encounter in a quantum mechanics class, and it is a very practical system to study. It can be used to model many physical systems including molecular vibrations, see Fig. 6.7. Consider two atoms connected by chemical bonds to form a single molecule. A classical model for this system is to have the two atoms connected by a spring. If the two atoms are not moving and the spring is not stretched or compressed, the system would just sit there at rest and not vibrate. However, if the spring is stretched slightly, it tries to pull the atoms back closer together. As the atoms move closer, the spring passes its equilibrium length and compresses. Once compressed, the spring tries to push the atoms apart until it stretches past equilibrium again, and the process repeats. This is an oscillation. In a quantum mechanical version, this system will have discrete energy levels, meaning it will have specific energy states. If the spring from this classical analogy were behaving quantum mechanically, the spring would only oscillate at specific frequencies.

We can use the Schrödinger equation to find the energy states and their energies for the quantum harmonic oscillator. The energy states with the four smallest energies for the quantum harmonic oscillator are shown in Fig. 6.8. Notice the similarities with the standing waves on a string example in Sect. 6.1. Although the shape of the states look different, the lowest energy state has 1 loop, the second lowest energy state has 2 loops, etc. If we measure the vibrational energy of a molecule modeled by the quantum harmonic oscillator, we would find the system in one of the energy states such as those shown in Fig. 6.8. Each state has a defined, discrete energy. All of the same bullet points that summarized Sect. 6.3 are still true! Observables that are compatible with energy, for example the frequency of vibration, have the same basis set, allowing us to measure all

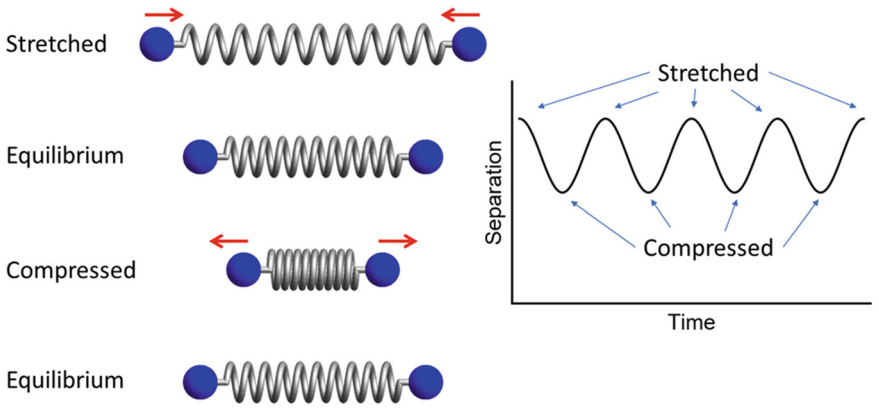


Fig. 6.7 Left: Two atoms connected by a spring. This is a model for two atoms connected by molecular bonds. Right: The separation of the atoms as a function of time

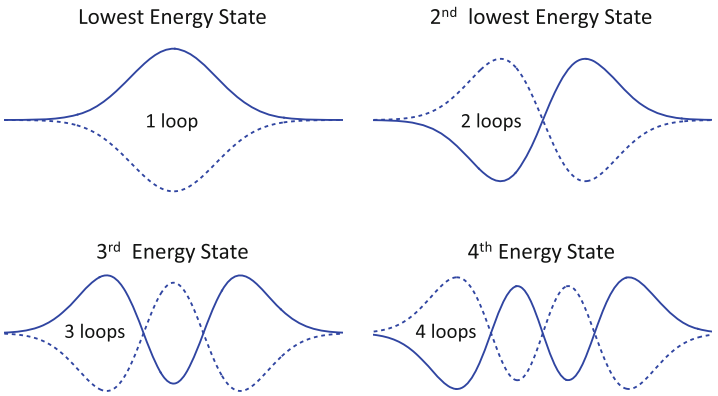


Fig. 6.8 The energy states with the four smallest energies for the quantum harmonic oscillator. Each of these energy states also corresponds to a specific energy or frequency of vibration of the atoms in the molecule

compatible observables repeatedly without disturbing the outcome of measuring other compatible observables. Observables that are incompatible with energy have different basis sets. If we measure energy, the system will be in a state from the energy basis set and a superposition of states for the incompatible observable's basis set. The amplitudes in the superposition formula allow us to calculate the probability of what we will measure for the incompatible observable.

6.5 The Uncertainty Principle Part 3

Reminder Definition

- **Reduced Planck's constant (h-bar):** $\hbar = \frac{h}{2\pi} = 1.054 \times 10^{-34}\,\text{kg} \cdot \text{m}^2/\text{s};$

The most common explanation of the Uncertainty Principle is about intrinsic limitations to the precision with which pairs of certain physical properties of a particle can be simultaneously known. Let's connect that idea with what we talked about in this chapter. Suppose a quantum system is in the rectangular state shown in Fig. 6.4a. This is neither an energy state, a position state, nor a momentum state. However, we now know that we can write this state as a superposition of states from any observable's basis set.

For momentum, we would write[5]

$$\psi = A_{p,1}\psi_{p,1} + A_{p,2}\psi_{p,2} + A_{p,3}\psi_{p,3} + ... = \sum_{i=1}^{\infty} A_{p,i}\psi_{p,i}$$

where $\psi_{p,i}$ are the states making up the momentum basis set. Similarly, we could write the state as a superposition of states from the position basis set:

$$\psi = A_{x,1}\psi_{x,1} + A_{x,2}\psi_{x,2} + A_{x,3}\psi_{x,3} + ... = \sum_{i=1}^{\infty} A_{x,i}\psi_{x,i}$$

where $\psi_{x,i}$ are the states making up the position basis set. If the system was in the state shown in Fig. 6.4a, and we measured momentum, we would get a *range* of possible values. More specifically, we have a $A_{p,1}^2$ chance of measuring p_1, a $A_{p,2}^2$ chance of measuring p_2, and so on. Equivalently, we would have some average value of momentum. For example, if we had a 100% chance of measuring p_4, the range (or spread) of possible measurements for momentum is 0 while the average value would be p_4. Let's call the range Δp and the average value we measure $\langle p \rangle$. Similarly, if we instead had measured position, we would get a range of possible position measurements, which we will call Δx, and an average position measurement, which we will call $\langle x \rangle$.

If two observables A and B are compatible, there is a scenario where $(\Delta A)(\Delta B) = 0$. That scenario would be when the system is in a state of A, which also happens to be a state of B since they are compatible. In that scenario, there is no range of measurements for either A or B: $\Delta A = 0$ and $\Delta B = 0$. However, if the system was not in a state of A and B, there would be a range of possible measurements for both observables. Therefore,

$$(\Delta A)(\Delta B) \geq 0 \qquad (6.5)$$

The uncertainty principle for two compatible observables.

This equation covers all possible scenarios for two compatible observables.

Now let's explore incompatible observables, for example momentum and position. Since momentum and position are incompatible, then there is no situation where $(\Delta x)(\Delta p) = 0$. If we did the math to determine the relationship, we would find that there is a minimum possible value to this product. For these two observables, the minimum possible value is

$$(\Delta x)(\Delta p) \geq \frac{\hbar}{2} \qquad (6.6)$$

[5] For the advanced reader, the next two equations should technically be integrals since momentum and position have a continuous range of possible values, unlike energy that has discrete values.

No matter what state you consider, there will always be a range of possible measurements for at least one of the observables. You might be asking yourself the question, "But wait, if the system was in a position state, wouldn't $\Delta x = 0$ and thus $(\Delta x)(\Delta p) = 0 \times (\Delta p) = 0$?" If the system was in a position state, the range of possible measurements for momentum would be ∞. So, we would have $(0)(\infty)$, which is undefined. However, if we were to carefully take the limit as $\Delta x \to 0$ and $\Delta p \to \infty$, the product is still greater than $\hbar/2$.

For every pair of incompatible observables, the product of the ranges is greater than some minimum value. The result is easier to read for some pairs of incompatible observables than others. For example, Eq. 6.6 is fairly straightforward: the range of possible measurements of position multiplied by the range of possible measurements of momentum is always greater than or equal to the constant $\hbar/2$. Other times, the uncertainty principle is harder to interpret. The uncertainty principle for position and energy is

$$(\Delta x)(\Delta E) \geq \frac{\hbar}{2m}|\langle p \rangle| \tag{6.7}$$

This equation is a bit harder to read. I would interpret it as follows: for a given state, the product of the spread of possible position measurements and the spread of possible energy measurements will always be greater than a constant times the magnitude of the average value of possible momentum measurements. This is a formula that you must be careful with. There are scenarios where $|\langle p \rangle| = 0$, but that does not mean that there is a scenario where position and energy are compatible. This just says that the product of their ranges must be greater than or equal to 0.

6.6 Looking Forward

This chapter solidifies some concepts we explored in the first part of this book while adding some more. Some observables are compatible (i.e., they share a basis set and subsequent measurements do not disrupt the values of the observables) and some observables are incompatible (i.e., they do not share a basis set and measuring one observable puts the system in a superposition of states for the other observable). The next questions to think about are

- What observables are compatible with energy?
- What observables are incompatible with energy?

We will explore these questions in the next few chapters.

A reflection from Will: This was a really hard chapter for me to write![a] There are so many fun things to explore in quantum mechanics that I had a difficult time narrowing it down to just a few important concepts. So, please take this chapter as an initial introduction. If you choose to go on and learn more about quantum mechanics (and I hope you do!), you can use this chapter as a base to build more understanding. There are many fascinating things to explore like quantum entanglement, Bell's Inequality, Ehrenfest's theorem, and the no-cloning theorem, just to name a few. The uncertainty principle also has much more to explore and learn. As with all things worth learning in life,

questions + repetition + critical thinking = mastery

[a] And I learned a lot by doing so.

Problems

6.1 In your own words, summarize the Uncertainty Principle.

6.2 Suppose a quantum system has two incompatible observables. We now know two important principles:

1. A state for one of the observables can be constructed from a superposition of states from the basis set of the other observable.
2. The probability that, upon measurement of the other observable, we find the system in a particular state can be calculated by squaring the amplitude of that state.
   (a) What would we get if we squared all the amplitudes and added them up? In other words, what is:

$$\sum_{i=1}^{\infty} A_i^2 = ? \tag{6.8}$$

   Explain.
   (b) Now suppose there is a quantum system with two compatible observables. Do the two principles listed in the paragraph above still hold true for two compatible observables? Why or why not?

6.3 For the quantum harmonic oscillator, are position and energy compatible observables? Explain.

6.4 Suppose we had a quantum mechanical system in the energy state represented by the second state in Fig. 6.2 (the energy state with two loops).

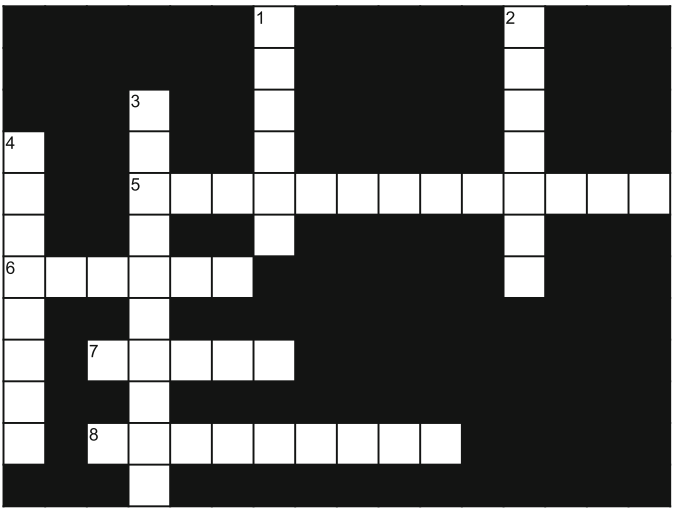
(a) Draw that energy state on a piece of paper. Reconstruct that energy state by drawing position states (Fig. 6.3) so that when you add them together you get that energy state.
(b) Now you want to measure position. Where are you most likely to find the quantum mechanical particle?
(c) Let's put the system back into that energy state. What is the probability, upon measurement of position, that the particle is found exactly in the middle? Why?

6.5 (The Measurement Game) Suppose we have three observables: A, B, and C. A and B are compatible observables while C is incompatible with both A and B. Below is a list of measurements that happen chronologically. If you will measure a specific value, state that value you will measure. For example, if you know that you are going to measure B_4, state, "I will measure B_4 with 100% probability." If you will not measure a specific value, state, "I can measure a variety of things." and then make one up, for example, "I measured A and got A_{17}."

You measure A and get A_4.
Now you measure B. What can/will you measure?
Now you measure A. What can/will you measure?
Now you measure C. What can/will you measure?
Now you measure A. What can/will you measure?
Now you measure C. What can/will you measure?
Now you measure B. What can/will you measure?
Now you measure A. What can/will you measure?
Now you measure B. What can/will you measure?
Now you measure A. What can/will you measure?
Now you measure A. What can/will you measure?
Now you measure C. What can/will you measure?

6.6 Write Eq. 6.1 using bra-ket notation.

6.7 Just for fun, here is a crossword with some terms from this chapter. For each word, write a clue.



Angular Momentum

7

Abstract

In this chapter, we explore angular momentum, a key concept in quantum mechanics and atomic physics. We cover the quantization of angular momentum, the role of quantum numbers, and their impact on atomic states. We will learn about the different types of angular momentum, the states representing these types of angular momentum, and how angular momenta are added together in quantum mechanics. The chapter also examines the compatibility of different observables, such as the magnitude and components of angular momentum, and the impact of measurements on quantum states.

Learning Goals

By the end of this chapter, you should be able to understand:

- that angular momentum is a vector that has both magnitude and direction.
- that quantum mechanics restricts angular momentum to certain discrete magnitudes and directions.
- that there are multiple angular momentum vectors in an atom, and understanding how these vectors add together impacts the energy of a particular state.
- that quantum numbers are key tools for understanding angular momentum in quantum mechanics.
- the concept of compatible and incompatible observables in the context of angular momentum measurements.

W. Raven, *Atomic Physics for Everyone*,
https://doi.org/10.1007/978-3-031-69507-0_7

7.1 Definitions

Note There are ***a lot*** of new definitions in this section. We will talk about and try to understand all of them, but don't worry too much about memorizing everything right now. We will use some of the definitions continually throughout the rest of the book. In future discussions, you should come back to these definitions as needed. We are all learners, and learning takes repetition.☺

Momentum This is the same definition from Chap. 6: a property of an object that is moving. For a classical object like a baseball, the formula for momentum is $p = mv$, where p is the object's momentum, m is the object's mass, and v is the object's velocity. An object's momentum will change if something acts on the object from the outside. The larger an object's momentum, the harder it is to stop. The unit of momentum is $\mathrm{kg\,m/s}$.

Force Below are two definitions of force. The first is the most common definition. However, it is only true if the object's mass is not changing, which is a very common scenario. The second is the real definition of force. The unit of force is $\mathrm{kg\,m/s^2}$, which we also call a newton (N).

Definition #1 An external interaction that causes an object to accelerate (i.e., change velocity). Imagine pushing a block up a ramp. You are pushing on the block and causing its velocity to change. That push is an external interaction (force) on the block. Any single force acting on an object will cause the object to accelerate (change velocity). There can be multiple forces acting on an object. Sometimes those forces cancel each other out. If there are multiple forces that perfectly balance one another so the net force is 0, then the object will not accelerate. However, if the net force is not 0, it will accelerate. This is summarized by Newton's 2nd Law: "The sum of all forces acting on an object causes an object with mass to accelerate."

Definition #2 An external interaction that causes an object's momentum to change. Newton's 2nd Law for momentum: "The sum of all forces acting on an object causes an object's momentum to change."

Torque A measure of how effective a force is in causing an object to rotate. The two things that matter for torque are the size of the force that causes the rotation and how far from the axis of rotation that force is being applied. Imagine you have a bicycle wheel. If you exert a force horizontally at the edge of the wheel, the wheel starts to spin. Since that force caused a rotation, there is a torque. Applying that same force on the axle or pushing on the tire directly towards the axle does not cause any rotation, so in those two scenarios there is no torque. The unit for torque is $\mathrm{Nm} = \mathrm{kg\,m^2/s^2}$. Interestingly, torque and energy share the same unit: $\mathrm{kg\,m^2/s^2}$. However, torque is not energy; they are incredibly different!! To avoid confusion,

we tend to not write the unit $\mathrm{kg\,m^2/s^2}$ for either energy or torque. We use joules (J) for energy and newton meters (Nm) for torque. Torque will be important in this chapter.

Angular Momentum A property of an object that is rotating. It is the thing that changes when a torque is applied to the object. There is a parallel between the ideas of force and momentum, and torque and angular momentum. Momentum is a property of an object that is moving along a straight line, and it changes when a force is applied to the object. The larger an object's momentum, the harder it is to stop the object from moving (a bigger force is needed). In the same way, angular momentum is a property of an object that is rotating, and the object will start to rotate faster or slower when a torque is applied (a bigger torque makes the change happen faster). The unit of angular momentum is $\mathrm{kg\,m^2/s}$. Angular momentum is ***extremely*** important in quantum mechanics and atomic physics.

Reduced Planck's Constant (h-Bar) $\hbar = \frac{h}{2\pi} = 1.054 \times 10^{-34}\,\mathrm{kg\,m^2/s}$.

▶ **Important Comment** Both Planck's constant and the reduced Planck's constant have units of angular momentum, $\mathrm{kg\,m^2/s}$, which can also be written as joules times seconds, Js. The reduced Planck's constant will be used a lot in this chapter. You will see why we call it a fundamental constant.

Quantum Number In quantum mechanics, a lot of different properties are quantized, i.e., can only have certain values. We already know that the energy levels of an atom are quantized. Other properties, like angular momentum, are also quantized. When something is quantized, a quantum number is usually associated with that property. A quantum number is an integer or half-integer. Knowing a quantum number will allow someone to calculate some property of an atom. Quantum numbers have no units.

As an example, we will soon talk about the electronic orbital angular momentum quantum number (that is a mouthful!). That quantum number is represented by a script ℓ. ℓ is a quantum number that can be zero or a positive integer: 0, 1, 2, 3, etc. ℓ is a discrete number because, according to quantum mechanics, the size or magnitude of the orbital angular momentum of electron is found to have only certain, discrete values. If I tell you that a state in an atom has $\ell = 2$, we can calculate the magnitude of the electron's orbital angular momentum. $\ell = 2$ is not the actual size of the orbital angular momentum, but the quantum number can be used to calculate it from the formula $\sqrt{\ell(\ell+1)}\hbar$. Notice that everything in that formula is either the quantum number ℓ or the reduced Planck's constant. In a way, knowing a quantum number is very similar to how spectroscopists use $\mathrm{cm^{-1}}$ as an energy unit. $\mathrm{cm^{-1}}$ is not the correct unit for energy, but we can multiply it by fundamental constants to get the actual energy. The same is true with quantum numbers. Knowing a quantum number is equivalent to knowing a specific property. You might have to

do some math, but the only other parameters in the equation should be fundamental constants. So, being told $\ell = 2$ is the same thing as being told that the magnitude of the electron's orbital angular momentum is $\sqrt{6}\hbar = 2.582 \times 10^{-34}$ Js. We will soon find that we need many quantum numbers to describe a particular state in an atom.

7.2 Angular Momentum

Electrons, protons, and neutrons all have angular momentum. A classical analogy is thinking about the moon orbiting the earth.[1] The moon has orbital angular momentum because it is orbiting the earth. The moon is also spinning on its own axis, so it also has "spin" angular momentum. So, the moon has two types of angular momentum: orbital and spin. Likewise, an electron in an atom can have orbital angular momentum and spin, which is sometimes called intrinsic angular momentum. Interestingly, an electron always has intrinsic angular momentum, but it does not always have orbital angular momentum. One of the craziest things about quantum mechanics is that we don't have a great analogy to think about the orbital angular momentum or spin of the electron. The electron is not orbiting around the nucleus or spinning on its axis like the moon, but it has the same properties as if it were. The electron, as far as we can tell, has no size! So, how can it be spinning? Imagine how confusing that must have been when physicists were first trying to understand the electron.[2] It has all the properties one would expect for a ball spinning on its axis, but it is not a ball and it is not spinning! Understanding electron spin is still a wonderful mystery.

Since there are two types of angular momentum, we could ask the question, "What is the total electronic angular momentum of the electron in an atom?" The total electronic angular momentum is not a simple sum of orbital angular momentum and spin. In other words, you cannot just add the two angular momenta together like $2 + 3 = 5$. Angular momentum is something called a vector, which is something with magnitude (i.e., size or length) and direction. To get the total electronic angular momentum you have to add the orbital angular momentum and spin together in their vector forms. As a classical analogy, imagine you walk 30 m due north, see Fig. 7.1. That is a vector because it has a magnitude (30 m) and a direction (due north). Next, you walk 40 m due west. Once again, that is a vector. If you were to add the two vectors together, you would be asking the question, "If I restarted my journey but

[1] Remember that in quantum mechanics, an electron is not like the moon orbiting the earth but more like a wave that is surrounding the nucleus. We are just using the moon and earth as an analogy to introduce angular momentum.

[2] There are, at least, two things in physics that really boggle my mind. The first is spin. The second is something called the fine structure constant, which you should Google if you want your mind blown!

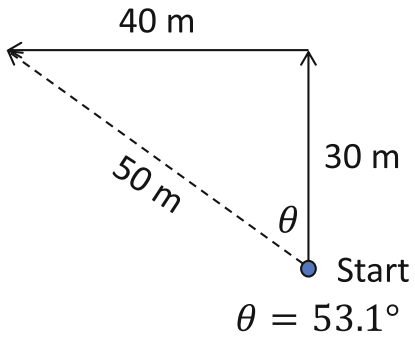


Fig. 7.1 An example of adding two vectors together

took the shortest path, how far and in what direction would I walk?" The answer to this particular scenario is 50 m at $\tan^{-1}\left(\frac{40\,\text{m}}{30\,\text{m}}\right) = 53.1°$ West of North.

The Super Important Take Home Message
Adding two vectors together produces a third vector. The size and direction of the third vector depends upon the sizes and directions of the first two vectors.

Let's get back to talking about angular momentum in a quantum system. An electron in an atom has a total angular momentum. That total angular momentum of the electron comes from the vector addition of its orbital angular momentum and its spin. There are quantum numbers associated with all three of these angular momentum vectors (orbital, spin, and total). The rest of this chapter is going to be focused on understanding angular momentum in a quantum system. We will find that both the size and direction of an angular momentum vector are quantized (i.e., can only have certain values like the size of an orbital angular momentum vector is $\sqrt{\ell(\ell+1)}\hbar$ where ℓ is a positive integer or zero). We will start by exploring the orbital angular momentum of a single electron in an atom and then build up the complexity by exploring (and adding) more and more angular momentum vectors to the system.

7.3 Orbital Angular Momentum of a Single Electron

The magnitude of the orbital angular momentum vector for a single electron in an atom is represented by the quantum number ℓ. The direction (or orientation) is also quantized and is represented by the quantum number m_ℓ. Figure 7.2a) shows an example of the three possible orientations of an orbital angular momentum vector represented by $\ell = 1$. Each of these vectors has the same magnitude (they all have $\ell = 1$), but they all point in different directions (they all have a different

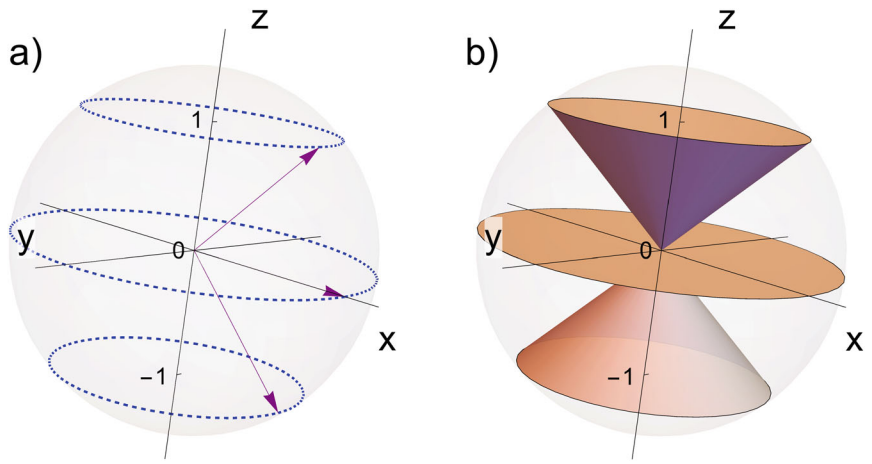


Fig. 7.2 Two popular ways to think about angular momentum states for a quantum mechanical system, like an electron in an atom. (**a**) A vector representation of the three possible vector orientations for an electron with orbital angular momentum quantum number $\ell = 1$. (**b**) The same information using cones instead of a vector and a dotted circle

m_ℓ quantum number). While ℓ tells us the magnitude of the vector, m_ℓ tells us how much of the vector points along the z-axis; this is called the z-component or z-projection. Figure 7.2a) has three values of m_ℓ: -1, 0, and +1. The dashed blue circles are meant to indicate that the angular momentum vector points anywhere on that circle. Because the angular momentum vector can point anywhere on the circle, some people like to think of angular momentum as a cone, see Fig. 7.2b).[3] These cones are also referred to as angular momentum states, which we will make more sense as we work our way through Chaps. 7 and 8.

Orientation of the angular momentum vector, height of the cone, or component/projection along the z-axis are all valid ways to think about m_ℓ. After solving the math for quantum mechanics, we find the component of orbital angular momentum along the z-axis to be $m_\ell\hbar$. For the upside-down cone in Fig. 7.2b), the orbital angular momentum vector is represented by the two quantum numbers $\ell = 1$ and $m_\ell = -1$, or the size of the orbital angular momentum is $\sqrt{1(1+1)}\hbar = \sqrt{2}\hbar$ and the amount of orbital angular momentum along the z-axis (or height of the cone) is $-\hbar$. The flat cone has the same size ($\ell = 1$), but it has no component along the z-axis ($m_\ell = 0$). Finally the upright cone has a component along the z-axis represented by $m_\ell = +1$. Since m_ℓ is the amount pointing along the z-axis, that number is also restricted by the value of ℓ. For example, if $\ell = 1$, m_ℓ cannot be 2. If it was, the amount pointing along the z-axis would be larger than the total magnitude!

[3] I, personally, like using cones. But, as long as you understand the concepts, it doesn't matter which version you use.

Experimentalists can measure these values. If an electron in an atom has orbital angular momentum represented by $\ell = 1$, the experimentalist will measure the size to be $\sqrt{2}\hbar$ every single time. However, if they measured the z-component, they could find $+\hbar$, 0, or $-\hbar$. But wait, couldn't they measure the x-component? What makes the z-component quantized while the x- and y- components just have to be somewhere along the dashed blue circle? On that note, how does an electron even decide which direction is z?

The general answer to all those questions is that we (the physicists) just decided to call one of the directions z. After we randomly pick a direction and call it z, we measure the amount along that axis. When we do that, we will only find those three values: $+\hbar$, 0, or $-\hbar$. We could certainly measure the x-component and get a value. In fact, if we measured the x-component we would get three possible values: $+\hbar$, 0, or $-\hbar$. The pictures in Fig. 7.2 would look very similar except the circles/cones would be aligned along the x-axis. Let's call those components $m_{\ell,x} = 1$, $m_{\ell,x} = 0$, and $m_{\ell,x} = -1$. The three possible values for the z-component form a basis set of size three. The three possible values for the x-component also form a basis set of size three, but those states making up that basis set are different than the state making up the basis set for the z-component. More importantly, those two basis sets are incompatible with each other.

Reminder of Important Definitions

- **Incompatible observables:** If two observables cannot be precisely measured at the same time, they are called incompatible. Measuring one observable changes the system, making subsequent measurements of the other observable unpredictable.
- **Compatible observables:** If two observables can be precisely measured at the same time, they are called compatible. Measuring one observable does not change the system in a way that affects the measurement of the other observable. Remeasuring the first observable will return the same value as initially measured.

Important
In the following paragraphs, all of the states (ψ) are for angular momentum. I am going to use ψ_z to represent the z-component of the angular momentum state, ψ_y for the y-component, and ψ_x for the x-component.

Suppose our system is in a state $\ell = 1$, $m_\ell = 1$. This is the upright cone in Fig. 7.2b). Since the x- and z- components are incompatible, we don't know what we will measure along the x-direction. We know there will be three possible outcomes for

the amount of angular momentum along the x-axis ($+\hbar$, 0, or $-\hbar$), but we don't know which value we will find until we measure it. This concept is what is being represented by the blue circles or the cones.

Mathematically, we can also think about this using the principle of superposition. Each set of three cones forms a basis set. One basis set is the three cones representing orbital angular momentum along z and other is the three cones along x. A state from one basis set can be constructed from a superposition of the other basis set. For example,

$$\psi_z(\ell = 1, m_\ell = 1) = c_{x,1}\psi_{x,1} + c_{x,0}\psi_{x,0} + c_{x,-1}\psi_{x,-1}, \tag{7.1}$$

where $\psi_z(\ell = 1, m_\ell = 1)$ is the state representing that upright cone (orbital angular momentum along z), $\psi_{x,1}$, $\psi_{x,0}$, $\psi_{x,-1}$ are the three states making up the x-component of the orbital angular momentum basis set, and $c^2_{x,1}$, $c^2_{x,0}$, and $c^2_{x,-1}$ are the probabilities that, upon measurement of the x-component of angular momentum, the system has x-component values $m_{\ell,x} = 1$, $m_{\ell,x} = 0$, or $m_{\ell,x} = -1$ (remember the actual value of the x-component of orbital angular momentum is $m_{\ell,x}\hbar$). We could repeat the same argument with the y-component. In fact, all three components are incompatible with each other. To summarize this, here are the three basis sets:

$$\begin{aligned} x &: \psi_{x,1}, \psi_{x,0}, \psi_{x,-1} \\ y &: \psi_{y,1}, \psi_{y,0}, \psi_{y,-1} \\ z &: \psi_{z,1}, \psi_{z,0}, \psi_{z,-1} \end{aligned} \tag{7.2}$$

Each state represents a cone along a particular axis. We are certainly allowed to measure any component of angular momentum we want. Upon measurement, we will be in one of those states. However, since they are all incompatible observables, measuring one of the components puts the system in a superposition of the states for the other obervables. Interestingly, we have two separate ways to construct the state $\psi_z(\ell = 1, m_\ell = 1)$:

$$\begin{aligned} \psi_z(\ell = 1, m_\ell = 1) &= c_{x,1}\psi_{x,1} + c_{x,0}\psi_{x,0} + c_{x,-1}\psi_{x,-1} \\ \psi_z(\ell = 1, m_\ell = 1) &= c_{y,1}\psi_{y,1} + c_{y,0}\psi_{y,0} + c_{y,-1}\psi_{y,-1} \end{aligned} \tag{7.3}$$

In a way, this makes sense. If the quantum mechanical system is in the state $\psi_z(\ell = 1, m_\ell = 1)$ and we measure the x-component of angular momentum, we will get $m_{\ell,x} = 1$ with probability $c^2_{x,1}$, $m_{\ell,x} = 0$ with probability $c^2_{x,0}$, or $m_{\ell,x} = -1$ with probability $c^2_{x,-1}$. On the other hand, we could have decided to measure the y-component at which point we would need to construct the state from the y-basis set to predict what we would measure if we measured the y-component. I should point out that we can calculate those amplitudes using quantum mechanics. For this particular system, $c_{x,1} = 1/2$, $c_{x,0} = -1/\sqrt{2}$, and $c_{x,-1} = 1/2$.

Ok, so all three components are incompatible with each other, but what about compatibility with the magnitude of the angular momentum vector? Interestingly, the magnitude of the angular momentum vector is a compatible observable with all three components of angular momentum! At first, this might seem a little weird; how can all the components be incompatible with each other yet each component is compatible with the magnitude? But, in a way, this also makes physical sense. The length of the vector is the same whether the cones point along the x-direction, the y-direction, or the z-direction. Measuring a component of angular momentum does not change the length, but it does change the orientation. For this system, measuring the magnitude of angular momentum will always return $\sqrt{2}\hbar$ (or $\ell = 1$). It doesn't matter if we are in a state for the x-component, y-component, or z-component, $\ell = 1$.

The rest of the chapter will be spent making the system more complex and complete. To do this, we will need to pick one component of angular momentum to represent the system. Since the x-, y-, and z- components are just rotations of each other, it doesn't really matter which one we use to represent direction. By tradition, we pick the z-component.

In the end, the above discussion can be summarized as follows: for $\ell = 1$, there are three possible projections for an axis and they all look the same ($+\hbar$, 0, and $-\hbar$). So, we need to pick some axis to represent the idea that there are three orientations for the orbital angular momentum vector.

A lot has happened in this section, so let's summarize:

Summary

- Orbital angular momentum is quantized in two ways: the magnitude and the projection on an axis.
- The magnitude is represented by the quantum number ℓ, which is used to calculate the magnitude using the formula $\sqrt{\ell(\ell+1)}\hbar$.
- The component of the orbital angular momentum along the z-axis is represented by the quantum number m_ℓ, which is used to calculate the component along the z-axis using the formula $m_\ell\hbar$.
- The choice of the z-axis is random, but we need an axis to represent the fact that there are only certain directions the vector can point along that axis.
- We can still measure along other directions, but the measurements along those directions are incompatible with measurements along the z-direction. We use cones to remind us that, while we know the z-component of angular moment, the x- and y-components are incompatible with the z-component.

7.4 The Magnitude and Projection of Angular Momentum

The magnitude of an orbital angular momentum vector is given by the formula $\sqrt{\ell(\ell+1)}\hbar$. The amount of the vector pointing along the z-axis (or the height of the cone) is $m_\ell\hbar$. Notice that the formulas contain only quantum numbers and the reduced Planck's constant. If you hear someone say, "Angular momentum comes in units of $\hbar$", this is what they mean. Being told that a quantum mechanical system has quantum numbers $\ell = 4$ and $m_\ell = -2$ tells us all we need to know about the orbital angular momentum of the system: the magnitude is $\sqrt{4(4+1)}\hbar = \sqrt{20}\hbar$ and the z-component is $-2\hbar$ (the cone is pointed in the negative z-direction with a height of $2\hbar$).

There are also mathematical restrictions on the quantum numbers themselves. ℓ can be either zero (no orbital angular momentum) or a positive integer (it has orbital angular momentum). The component along the z-axis must be smaller than the magnitude of the orbital angular momentum, so m_ℓ is restricted to be between -ℓ and +ℓ in integer steps.

Example #1 An electron in an atom has orbital angular momentum represented by the quantum number $\ell = 3$. There are 7 possible orientations (cone heights) of that vector represented by m_ℓ= -3, -2, -1, 0, 1, 2, 3, see Fig. 7.3. If we measure the magnitude of the orbital angular momentum, we will measure $\sqrt{3(3+1)}\hbar = \sqrt{12}\hbar$. If we measure the z-components, we will measure either $-3\hbar$, $-2\hbar$, $-\hbar$, 0, $\hbar$, $2\hbar$, or, $3\hbar$.

Example #2 An electron in an atom has orbital angular represented by $\ell = 2$. There are 5 possible orientations of that vector represented by $m_\ell = -2$, -1, 0, 1, and 2.

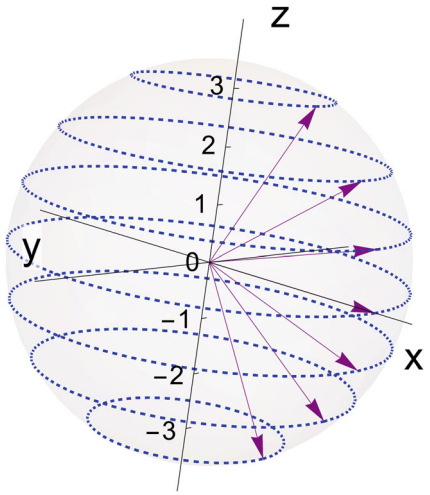


Fig. 7.3 A vector representation of the seven possible vector orientations for an electron with orbital angular momentum quantum number $\ell = 3$

The form of $\sqrt{\ell(\ell+1)}\hbar$ and $m_\ell\hbar$ are the same for other types of angular momentum. All we have to do is replace the quantum numbers representing orbital angular momentum with the quantum numbers that represent the type of angular momentum we are interested in. In the next few sections, we are going to expand our discussion to talk about electron spin, the total electronic angular momentum (orbital + spin), the total orbital angular momentum of multiple electrons in an atom, and more. Each of the above examples of angular momentum will have two quantum numbers associated with it, one for magnitude and one for the z-component (cone height). For example, electron spin will have quantum numbers s and m_s. The formulas for electron spin are $\sqrt{s(s+1)}\hbar$ and $m_s\hbar$. While the quantum numbers for orbital angular momentum are integers, we will find that the other types of angular momentum quantum numbers can be either integers or half-integers. But they all use the same two formulas and tell us the same information, magnitude and z-component (cone height).

7.5 Adding Angular Momentum Vectors Together

Imagine you had 2 electrons in your atom. Each of those electrons has orbital angular momentum. We want to ask the question, "What is the *total* orbital angular momentum of the system?" To answer that question, we need to add together two quantum mechanical vectors. As an example, suppose we have an electron with the magnitude of the orbital angular momentum vector represented by $\ell_A = 1$ (Fig. 7.2 with 3 possible orientations) and a second electron with $\ell_B = 3$ (Fig. 7.3 with 7 possible orientations). We want to add these two vectors together as in Fig. 7.1, but we have to be thoughtful for two big reasons: (1) the vectors are really more like "vector cones" and (2) there might be incompatible observables we have to think about.

While the math is surprisingly complicated, the result is quite nice:

$$L = (\ell_A + \ell_B), (\ell_A + \ell_B - 1), (\ell_A + \ell_B - 2), \ldots, |\ell_A - \ell_B| \tag{7.4}$$

where L is the quantum number for the composite system. For each value of L, there are $2L+1$ possible projections represented by the quantum number m_L. This might be, understandably, confusing. So, let's unpack this statement using our example of $\ell_A = 1$ and $\ell_B = 3$.

If we add together two quantum mechanical vectors whose magnitudes are represented by quantum numbers ℓ_A and ℓ_B, the total angular momentum of the system will be represented by new quantum numbers L and m_L. The math shows there are multiple possible magnitudes one can get when adding two angular momentum vectors together. However, the new cones for L and m_L follow the same rules that we have been working with: L tells us the magnitude of the angular momentum for the system and m_L tells us the component (or height of the cone). From the math, the largest possible magnitude of L is given by $\ell_A + \ell_B$, which is $L = 4$ in the thought experiment we are working through. The smallest possible magnitude is $L = |\ell_A - \ell_B|$, or $L = 2$ for the example we are working on. To

find all possible values of L, we make a list that starts with the smallest value and continually add 1 until we reach the largest value. In this case L can equal 2, 3, or 4.

To put it another way, there are three different orientations for ℓ_A and seven different orientations for ℓ_B. Depending on the orientations, adding those two angular momentum vectors results in different possible magnitudes. Working through all possible combinations results in three possible magnitudes. For each possible magnitude, there are different orientations (cone heights). In this example, the individual orbital angular momentum vectors can add together to make $L = 2$ (5 possible cones with the same length vector $\sqrt{2(2+1)}\hbar = \sqrt{6}\hbar$), $L = 3$ (7 possible cones with the same length vector $\sqrt{3(3+1)}\hbar = \sqrt{12}\hbar$), and $L = 4$ (9 possible cones with the same length vector $\sqrt{4(4+1)}\hbar = \sqrt{20}\hbar$).

After all the math, there are $5+7+9 = 21$ possible cones. This makes sense since the original vectors had 3 and 7 possible orientations, and $3 \times 7 = 21$. However, I want to caution you in that the new cones are not constructed simply by taking one cone chosen from $\ell_A = 1$ and one cone from $\ell_B = 3$ and adding them together. This is because the z-components of the individual electrons are incompatible with the z-component of the composite system. So, the new cones, which we will call the basis set for the composite system, are a superposition of all the cones from the individual electrons, which we call the individual electron basis set. The complicated math, which we skip in this book, tells us the amplitude or amount that we need from each of the individual electron basis set to construct the new composite system basis set. We will explore this more in Sect. 7.5.1.

Another example: Suppose $\ell_A = 3$ and $\ell_B = 5$. The largest value for L is $3 + 5 = 8$, and the smallest value is $|3 - 5| = 2$. Therefore, L can have values of 2, 3, 4, 5, 6, 7, or 8. For each value of L, there will be different orientations represented by the quantum number m_L. If $L = 3$, there are 7 possible vector cones represented by $m_L = 3, 2, 1, 0, -1, -2$, and -3.

Quick problem (the answers are in the footnotes[a]):
Suppose $\ell_A = 1$ and $\ell_B = 2$.

(a) How many projections are there for $\ell_A = 1$?
(b) How many projections are there for $\ell_B = 2$?
(c) How many possible cones can we add together from part (a) and part (b)? In other words, what is the size of the individual electron basis set?
(d) What values of L are possible?
(e) For each value of L, how many projections are there?
(f) Add up all the possible projections from part (e).

The answer to (c) should match the answer in (f).

[a](a) 3; (b) 5; (c) $3 \times 5 = 15$; (d) 1, 2, or 3; (e) 3, 5, or 7; (f) 15.

7.5.1 Compatible or Incompatible?

For just these two electrons, we have a bunch of quantum numbers floating around. Each quantum number corresponds to something we can measure. For these two electrons, we have six quantum numbers: ℓ_A, $m_{\ell,A}$, ℓ_B, $m_{\ell,B}$, L, and m_L. Which of these represent compatible observables? Is there something we can measure that will disrupt the value of another observable? As learners of quantum mechanics, this is important to keep track of.

All of the magnitudes are compatible with everything. If we know $\ell_A = 1$, $\ell_B = 3$, and $L = 2$, we can measure any of the observables represented by the six quantum numbers without disrupting future outcomes of magnitude measurements. However, the composite z-component m_L is incompatible with the z-components of the individual electrons, $m_{\ell,A}$ and $m_{\ell,B}$. Therefore, we can know either the z-component of the system or the z-components of the individual electrons, but not all three at the same time. The z-components of the two individual electrons are compatible with each other. So, if we know the z-component of the system and want to know what we might measure if we were to measure the z-component of the individual electrons, we have to write the state of the composite system as a superposition of states from the individual electron basis states. The concept remains the same; we would just have to do the math to find the amplitudes.

Generalization and Summary
The above discussion is true when adding *any* two quantum mechanical angular momentum vectors together. Suppose we have two angular momentum vectors **A** and **B**. Their magnitudes are represented by quantum numbers A and B, while their z-components are represented by m_A and m_B. We want to add **A** and **B** together to get **C**. The possible values for the quantum number representing the magnitude of **C** are:

$$C = (A + B), (A + B - 1), \ldots, |A - B| \tag{7.5}$$

For each value of C, $m_C = -C, \ldots, C$ in integer steps. The compatibility of the observables is as follows:

- The magnitudes are always compatible with everything.
- The z-components of the individual angular momenta are compatible with each other.
- The z-component of the summed angular momentum is incompatible with the z-components of the individual angular momenta.

7.6 Other Types of Angular Momentum

An electron, whether it is in an atom or free, always has spin. The quantum numbers for spin are s and m_s for the magnitude and orientation (cone height), respectively. For an electron, $s = 1/2$ always, so the possible orientations are always $m_s = -1/2$ or $m_s = +1/2$. From tradition, we call $m_s = +1/2$ "spin up" and $m_s = -1/2$ "spin down".

If the electron is in an atom, it can also have orbital angular momentum. We can add orbital angular momentum and spin together and ask the question, "What is the total angular momentum of that electron?" This new total angular momentum of the electron is represented by the quantum numbers j and m_j. Using our new rule (Eq. 7.5), the possible magnitudes of the total are represented by $j = l+s, \ldots, |l-s|$ in integer steps. Thus, if $\ell = 1$ and $s = 1/2$, j can be $3/2$ or $1/2$. As before, m_ℓ can be any value between $-\ell$ and $+\ell$ in integer steps, m_s can be any value between $-s$ and $+s$ in integer steps, and m_j can be any value between $-j$ and $+j$ in integer steps. Knowing the quantum numbers j, ℓ, and s tells us the magnitude of the angular momentum vectors. Knowing m_j, m_ℓ, and m_s tells us the orientation of each of their respective angular momentum vectors. However, m_j is incompatible with m_s and m_ℓ.

Here is a table summarizing all of the current types of angular momenta for a single electron in an atom (QN means quantum number):

Type	QN	Rule	Formula
Orbital	ℓ	Zero or positive integer	Magnitude: $\sqrt{\ell(\ell+1)}\hbar$
	m_ℓ	$-\ell$ to $+\ell$ in integer steps	Cone height: $m_\ell\hbar$
Spin	s	For a single electron, $s = 1/2$	Magnitude: $\sqrt{s(s+1)}\hbar = \sqrt{3/4}\hbar$
	m_s	$-1/2$ and $+1/2$	Cone height: $m_s\hbar$
Total electronic	j	Zero, positive integer, or half-integer	Magnitude: $\sqrt{j(j+1)}\hbar$
	m_j	$-j$ to $+j$ in integer steps	Cone height: $m_j\hbar$

Example An electron in an atom has quantum numbers $\ell = 3$ and $s = 1/2$.

(a) What is the magnitude of the electronic orbital angular momentum?
Solution: The magnitude is $\sqrt{\ell(\ell+1)}\hbar = \sqrt{12}\hbar \approx 3.46\hbar$

(b) What is the magnitude of the electron's spin?
Solution: The magnitude is $\sqrt{s(s+1)}\hbar = \sqrt{\frac{3}{4}}\hbar \approx 0.87\hbar$

(c) What are the possible magnitudes of the total electronic angular momentum?
Solution: The possible values for j are $j = (\ell+s), \ldots, |\ell-s|$ in integer steps. The largest value is $3+\frac{1}{2} = \frac{7}{2}$ and the smallest value is $|3-\frac{1}{2}| = \frac{5}{2}$. So, $j = \frac{7}{2}$ or $j = \frac{5}{2}$.

When $j = 7/2$, the magnitude is $\sqrt{j(j+1)}\hbar = \sqrt{\frac{7}{2}\frac{9}{2}}\hbar = \sqrt{\frac{63}{4}}\hbar \approx 3.97\hbar$.
When $j = 5/2$, the magnitude is $\sqrt{j(j+1)}\hbar = \sqrt{\frac{5}{2}\frac{7}{2}}\hbar = \sqrt{\frac{35}{4}}\hbar \approx 2.96\hbar$.

We now have all the building blocks we need to add together as many angular momentum vectors as we need. We first add two together to come up with the new possible quantum numbers and projections for this composite system. Next, we treat the composite system as its own individual angular momentum and add a third angular momentum. Then keep going until you have accounted for all electrons in your atom. The magnitudes of each angular momenta will be compatible with everything. The z-component of any summed angular momentum will be incompatible with the two individual angular momenta.

In practice, we don't go through all this effort. As we will see in Chap. 8, there are shortcuts we can take and databases that tell us all we need to know. For atoms with more than 1 electron, we do not keep track of subscripts (i.e., electron 1, electron 2, etc.), we simply ask the question "how much orbital, spin, and total electronic angular momentum do all of the electrons have?" The formulas and ideas are identical, but we use capital letters when we ask how much angular momentum the electrons as a whole have:

Type	QN	Rule	Formula
Orbital	L	Zero or positive integer	Magnitude: $\sqrt{L(L+1)}\hbar$
	m_L	$-L$ to $+L$ in integer steps	Cone height: $m_L\hbar$
Spin	S	Zero, positive integer, or half-integer	Magnitude: $\sqrt{S(S+1)}\hbar$
	m_S	$-S$ to $+S$ in integer steps	Cone height: $m_S\hbar$
Total electronic	J	Zero, positive integer, or half-integer	Magnitude: $\sqrt{J(J+1)}\hbar$
	m_J	$-J$ to $+J$ in integer steps	Cone height: $m_J\hbar$

All of the magnitudes are compatible with everything. However, m_J is incompatible with m_L and m_S.

Something Super Annoying, But Unfortunately Super Important Historically, we assign letters to orbital angular momentum quantum numbers. A system with no orbital angular momentum has the quantum number $L = 0$ (or $\ell = 0$ for a single electron) is given the letter designation S (or s for a single electron). But, S and s are also the quantum numbers for spin. They are both angular momenta, but very different types. Technically the quantum numbers are italicized and the letter designations are not, but that is not a rule everyone follows. Below is a table with quantum numbers, the letter designation, and double uses with other types of angular momentum:

Orbital quantum number	Letter designation	Other uses
$L = 0$ ($\ell = 0$)	S (s)	S and s are also used for spin
$L = 1$ ($\ell = 1$)	P (p)	
$L = 2$ ($\ell = 2$)	D (d)	
$L = 3$ ($\ell = 3$)	F (f)	F is also used for nuclear angular momentum
$L = 4$ ($\ell = 4$)	G (g)	

I admit it is extremely annoying to have orbital angular momentum represented by letters that also mean different types of angular momentum, but it is unfortunately the way of spectroscopy. It takes some getting used to.

The Nucleus The nucleus can also have angular momentum. Traditionally, this is called nuclear spin, but that is a bit of a misnomer. The angular momentum of the nucleus comes from the spin of the protons and neutrons as well as any orbital angular momentum of those nucleons. So, nuclear spin should really be called total nuclear angular momentum but no one uses that phrase. Nuclear spin has the quantum numbers I and m_I. Nuclear spin (I) can be added to the total electronic angular momentum (J) find the total angular momentum of the atom. The total angular momentum is represented by the quantum numbers F and m_F, where F is found from J and I using Eq. 7.5:

$$F = (J + I), (J + I - 1), \ldots, |J - I| \tag{7.6}$$

As always, m_F is incompatible with m_J and m_I.

So, if an atom has nuclear spin, our table gets a bit longer:

Type	QN	Rule	Formula
Orbital	L	Zero or positive integer	Magnitude: $\sqrt{L(L+1)}\hbar$
	m_L	$-L$ to $+L$ in integer steps	Cone height: $m_L\hbar$
Spin	S	Zero, positive integer or half-integer	Magnitude: $\sqrt{S(S+1)}\hbar$
	m_S	$-S$ to $+S$ in integer steps	Cone height: $m_S\hbar$
Total electronic	J	Zero, positive integer, or half-integer	Magnitude: $\sqrt{J(J+1)}\hbar$
	m_J	$-J$ to $+J$ in integer steps	Cone height: $m_J\hbar$
Nuclear spin	I	Zero, positive integer, or half-integer	Magnitude: $\sqrt{I(I+1)}\hbar$
	m_I	$-I$ to $+I$ in integer steps	Cone height: $m_I\hbar$
Total Atomic	F	Zero, positive integer, or half-integer	Magnitude: $\sqrt{F(F+1)}\hbar$
	m_F	$-F$ to $+F$ in integer steps	Cone height: $m_F\hbar$

We will explore nuclear spin more in Chap. 9.

The Photon The photon has intrinsic angular momentum as well. Photon spin is the quantum mechanical description of light polarization, and it has a quantum number of 1. In Chap. 1, we talked about how light can be linearly polarized, circularly polarized, or elliptically polarized. That statement is for a laser beam, which is composed of many photons. A single photon is circularly polarized. To help visualize this, imagine a photon traveling straight towards you. The photon would be rotating either clockwise or counterclockwise. To be clear, a photon is not actually rotating just like the electron is not actually spinning like a top. However, both the electron and the photon behave and interact with other particles as if they are spinning or rotating. If it is spinning counterclockwise, the height of the cone is $+\hbar$ (the z-component to photon spin with a quantum number of +1). If the photon is spinning clockwise, the height of the cone is $-\hbar$.[4]

Circularly polarized light just means that all the photons are rotating in the same direction. A laser beam composed of photons coming towards you that are all rotating clockwise is said to have left (or left-handed) circularly polarized light. A laser beam composed of photons that are all rotating counterclockwise is said to have right (or right-handed) circularly polarized light. To get linearly polarized light, you need to have equal amounts of photons spinning clockwise and counterclockwise. Elliptically polarized light has an imbalance between the number of clockwise and counterclockwise photons.

7.7 A Bit More on Compatible and Incompatible Observables

Every quantum number in every table in Sect. 7.6 is something we can measure. Consider the hydrogen atom, which has a single electron. For now, we will ignore the fact that the nucleus of a hydrogen atom has angular momentum (this is the topic of Chap. 9 so we will revisit compatible and incompatible observables again in that chapter). Here is a list of things we can measure: ℓ, m_ℓ, s, m_s, j, and m_j.

Technically, we could also measure L, m_L, S, m_S, J, and m_J, but, since the hydrogen atom has only a single electron, measuring, for example, the total orbital angular momentum of all the electrons is the same as measuring the orbital angular momentum of the single electron in the system.

We want to ask the question, which of these observables is compatible with energy? In other words, if the hydrogen atom was in an energy state, what other properties could we measure and still leave the electron in that same energy state? The answer is:

- Compatible with energy: ℓ, s, j, and m_j
- Incompatible with energy: m_ℓ and m_s

That means if the system was in an energy state and we measured the z-component of the electron's orbital angular momentum, the system is now in a superposition

[4] Because the photon has no mass, the math, interestingly, forbids a z-component of 0.

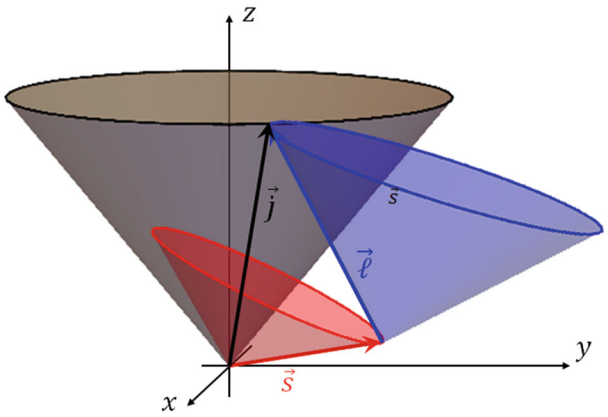


Fig. 7.4 An example of adding together a spin cone and a orbital angular momentum cone to get the total electronic angular momentum cone. Notice that the z-component of both the spin and orbital angular momentum cones is not the same value at all points of the cone tops. This means the z-component does not have a specific, discrete value, which means that both the z-component of spin and the z-component of orbital angular momentum are incompatible with the z-component of j

of energy states. But what makes m_ℓ and m_s incompatible with m_j and energy? While it isn't obvious why m_ℓ and m_s are incompatible with the energy state, we can visualize why they are incompatible with m_j. Figure 7.4 visualizes the incompatibility. This figure shows the addition of two angular momentum cones. Since m_j is compatible with the energy state, we will define the z-axis to be along the cone for j. I want to point out a few things. First, notice that the cones for both ℓ and s are tilted with respect to the z-axis. This means that the height of those two cones (i.e., the result of a measurement of m_ℓ or m_s) are no longer perfect projections onto the z-axis. In other words, if we measured the z-component of spin with respect to this z-axis, we will be in a superposition of the m_j basis set. Compare this to the measurement of the z-component of j. The cone for j has a rim that is constant along the z-axis, so we will always measure the same value of m_j. Since m_j is compatible with the energy state, measuring m_ℓ or m_s would also put the system into a superposition of energy states.

The second thing I would like to point out is that the length of the both s and ℓ are constants. If we were to measure the slant height of either cone, we will always get the same answer. In other words, both s and ℓ are compatible with j and energy.

To clarify which observables are compatible with the energy states, I'll reintroduce the quantum number for energy, n, from Chap. 6. We use the quantum numbers representing those observables that are compatible with energy in a ket: $|n\ \ell\ s\ j\ m_j\rangle$. We will discuss these quantum numbers and their relationships to electron shells in more detail in Chap. 8. If the electron in the hydrogen atom is in an energy state represented by one of these kets, we can measure any of those observables and still be in that same energy state. If we measured m_ℓ or m_s, the system would be in a superposition of energy states.

Let's solidify this idea with a couple of examples. We will do Example #1 together, and then you should do Example #2.

Example #1 How many states will there be for the electron in a hydrogen atom that has $n = 2$, $\ell = 1$, and $s = 1/2$?

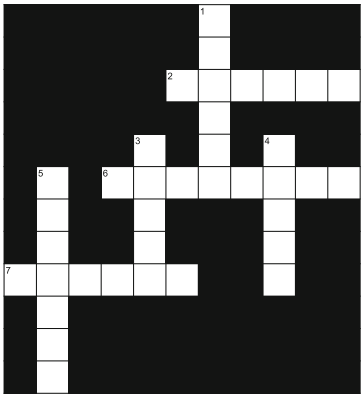
To start this problem, let's first calculate what values of j are possible. j can range from $\ell + s$ to $|\ell - s|$ in integer steps, so j can be $1/2$ or $3/2$. If $j = 1/2$, then m_j can be $1/2$ or $-1/2$. If $j = 3/2$, then m_j can be 3/2, 1/2, -1/2, or -3/2. Therefore, there are 6 energy states with $n = 2$, $\ell = 1$, and $s = 1/2$. Using the notation $|n\ \ell\ s\ j\ m_j\rangle$, these states are: $|2\ 1\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\rangle$, $|2\ 1\ \frac{1}{2}\ \frac{1}{2}\ \text{-}\frac{1}{2}\rangle$, $|2\ 1\ \frac{1}{2}\ \frac{3}{2}\ \frac{3}{2}\rangle$, $|2\ 1\ \frac{1}{2}\ \frac{3}{2}\ \frac{1}{2}\rangle$, $|2\ 1\ \frac{1}{2}\ \frac{3}{2}\ \text{-}\frac{1}{2}\rangle$, and $|2\ 1\ \frac{1}{2}\ \frac{3}{2}\ \text{-}\frac{3}{2}\rangle$.

The above math tells us that there is an energy state in the hydrogen atom that is represented by, for example, the quantum numbers $|2\ 1\ \frac{1}{2}\ \frac{3}{2}\ \text{-}\frac{3}{2}\rangle$. If the hydrogen atom is in this state, it has a defined magnitude of orbital angular momentum (in this case $\sqrt{\ell(\ell+1)}\hbar = \sqrt{1(1+1)}\hbar \approx 1.41\hbar$), a defined magnitude for spin, a defined magnitude for the total electronic angular momentum, and a defined z-component for the total electronic angular momentum. That state does not have a well defined z-component of orbital angular momentum nor a well defined z-component of spin. We can measure any observable compatible with energy and the system will stay in the $|2\ 1\ \frac{1}{2}\ \frac{3}{2}\ \text{-}\frac{3}{2}\rangle$ state. However, if we measure m_ℓ or m_s, the system will be in a superposition of energy states.

Example #2 The electron in the hydrogen is in the state $|2\ 1\ \frac{1}{2}\ \frac{3}{2}\ \text{-}\frac{1}{2}\rangle$.

(a) What ***will*** we measure for the electron's orbital angular momentum, the electron's spin, the electron's total angular momentum, and the electron's z-component of the electron's total angular momentum?
(b) If we measured the z-component of the electron's orbital angular momentum, what might we get?

The solutions are below the crossword.



Across:

2 ___ - ___ notation is a common way to show all the quantum numbers that are compatible with energy.

6 Applying a force changes this

7 ℓ is an example of a quantum ____

Down:

1 Apply this to change an object's angular momentum.

3 Apply this to change an object's momentum.

4 The cone with height $+\hbar$ in Figure 7.2 is an example of an angular momentum ____

5 Applying a torque changes an object's ____ momentum.

Solution

(a) $\ell = 1$: We will measure $\sqrt{1(1+1)}\hbar \approx 1.41\hbar$ with 100% probability
$s = 1/2$: We will measure $\sqrt{\frac{1}{2}(\frac{1}{2}+1)}\hbar \approx 0.87\hbar$ with 100% probability
$j = 3/2$: We will measure $\sqrt{\frac{3}{2}(\frac{3}{2}+1)}\hbar \approx 1.94\hbar$ with 100% probability
$m_j = -1/2$: We will measure $-\frac{1}{2}\hbar$ with 100% probability
(b) We would measure either $-\hbar$, 0, or $\hbar$ with different probabilities. We would need advanced math to calculate those probabilities, so I will just emphasize that we will not measure a definite value for the z-component of the electron's orbital angular momentum. After the measurement, the system will no longer be in the state $|2\ 1\ \frac{1}{2}\ \frac{3}{2}\ -\frac{1}{2}\rangle$, but a superposition of energy states. If we now went back and measured j, we might get $j = 1/2$ or $j = 3/2$ with hard to calculate probabilities.

Problems

7.1 Angular momentum seems to be a pretty important concept in quantum mechanics and atomic physics! In your own words, describe a classical (i.e., not quantum) system that has angular momentum and a classical system that does not. Speculate on a few ways the classical system with angular momentum might behave differently if it were a quantum system.

7.2 An electron in an atom has the quantum numbers $\ell = 0$ and $s = 1/2$.

(a) What are the magnitudes and projection along the z-axis for ℓ?
(b) What are the magnitudes and projection along the z-axis for s?
(c) What are the possible magnitudes and projections along the z-axis for j?

7.3 An atom has two electrons. One electron has $\ell = 1$ and the other has $\ell = 0$.

(a) Find all possible values of L.
(b) Find all possible values of S.
(c) Find all possible values of J.

Hint: There will be a different set of J values for each combination of L and S. For example, if your answer to part (a) was $L = 2$ or $L = 1$ (I hope you didn't get these numbers, because they aren't correct) and your answer to part (b) was $S = 0$ or $S = 1$, then there would be 4 sets of answers to part (c). Word your answer similar to:

"For $L = 2$ and $S = 0$, J can be ..."
"For $L = 2$ and $S = 1$, J can be ..."

"For $L = 1$ and $S = 0$, J can be ..."
"For $L = 1$ and $S = 1$, J can be ..."

7.4 An atom has two electrons. One electron has $\ell = 1$ and the other has $\ell = 2$.

(a) Find all possible values of L.
(b) Find all possible values of S.
(c) Find all possible values of J.

7.5 The atom in Problem 3 has a nuclear spin of $I = 3$. Find all possible values of F.
Hint: There will be a different set of F values for each J value.

Electronic Structure and Atomic Notation

8

Abstract

In this chapter, we explore the fundamental principles governing the electronic structure of atoms and the notation used to describe it. We begin by examining energy level spacings across various elements, highlighting how the number of electrons influences the complexity of these levels. The Coulomb interaction and electron shell filling patterns are introduced as key factors determining atomic state energies. Through detailed discussions on electronic configurations and term symbols, we illustrate how angular momentum, both in individual electrons and in atoms as a whole, impacts energy states. We also discuss fine structure splitting and hyperfine structure splitting. By the end of this chapter, readers will have a comprehensive understanding of the labels used by atomic physicists for electronic levels in atoms. Additionally, we differentiate between fermions and bosons, emphasizing their roles and significance in atomic and nuclear physics.

Learning Goals

By the end of this chapter, you should be able to understand:

- the Coulomb interaction and its role in determining the energy levels within an atom.
- describe how electrons fill shells and subshells according to quantum mechanical principles.
- the concept of angular momentum in the context of atomic physics and how it affects atomic energy states.

W. Raven, *Atomic Physics for Everyone*,
https://doi.org/10.1007/978-3-031-69507-0_8

- the influence of Coulomb interactions, electron shell filling, and angular momentum on the energy of atomic states.
- interpret and describe electronic configurations and term symbols to understand the angular momentum of individual electrons and the overall atom.
- explain the difference between fine structure splitting and hyperfine structure splitting.
- distinguish between fermions and bosons and explain their significance in atomic and nuclear physics.

8.1 Energy Level Spacings

Figure 8.1 shows the energy levels for hydrogen, helium, lithium, and europium. As a reminder, hydrogen has 1 electron, helium has 2, lithium has 3, and europium has 63. Notice how different the energy levels are! Hydrogen's first excited state is over 80,000 cm^{-1} above the ground state, while helium's first excited state is around

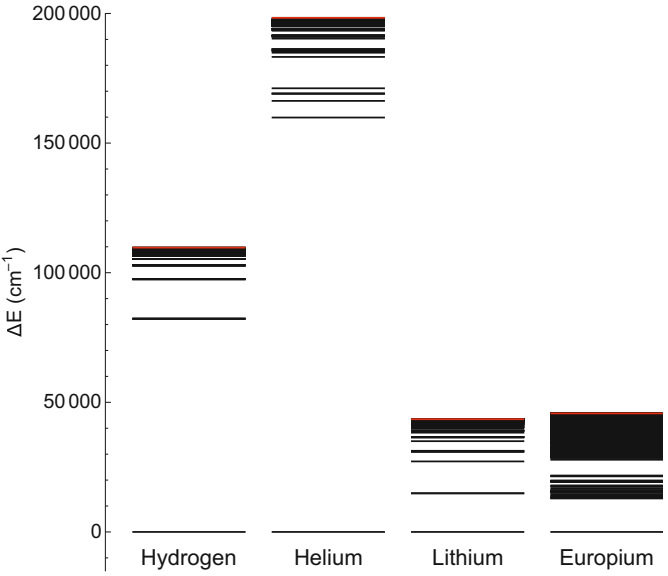


Fig. 8.1 The energy levels of hydrogen, helium, lithium, and europium. The red line at the top of each element is the ionization threshold, which is the energy required to remove an electron from the atom

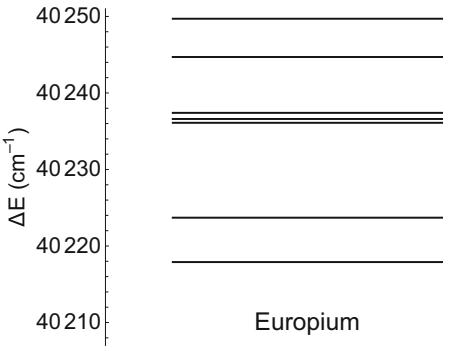


Fig. 8.2 A zoom in on some of the denser states in Europium. Even though there are many states, they are still discrete

160,000 cm^{-1} above the ground state![1] The hydrogen states seem to get closer and closer to each other as the energy increases, but helium seems to "clump" a bit more. And look at Europium; there seems to be a big gap between the ground state before a really dense set of energy levels, a gap, and then even more! The red line on each element is the energy needed to rip an electron from the atom. We call this the **ionization threshold**.

Even though the energy levels are so densely packed, they are still discrete. The energy levels looking like a solid band is just an artifact of making a picture with many, many discrete energy levels (the europium diagram has 500 levels in it). Figure 8.2 is a zoom in of the Europium energy levels from 40,210 cm^{-1} to 40,250 cm^{-1}, which is right in the middle of one of the dense patches. As you can see, these 7 levels are very close together, but still discrete. Plotting all 500 levels together really highlights the groupings.

Guiding Question
Why are the energy levels so different for each element?

Give it some thought and then read on.

[1] Remember that the ground states of hydrogen, helium, lithium, and europium do not have 0 energy. Each element's ground state has an energy like the lowest mode of a standing wave has energy. The actual ground state energy of hydrogen is very different than that of helium or lithium. But, we always do spectroscopy with respect to the element's ground state energy. Figure 8.1 is a nice learning tool, but it can be misleading by giving the impression that, for example, the first excited state of lithium has a similar energy to the first excited state of europium. They do not!! The energy difference compared to their ground state is similar. A subtle, but important difference. ☺

8.2 The Coulomb Interaction and Electron Shells

Definitions

- **Coulomb force:** The force between two charged particles, described by $F = k_e \frac{q_1 q_2}{r^2}$, where F is the force, $k_e = 8.987 \times 10^9\ \mathrm{N \cdot m^2/C^2}$ is Coulomb's constant, q_1 and q_2 are the charges, and r is the distance between them. It represents the attraction or repulsion between the charged particles.
- **Coulomb interaction:** The interaction between two charged particles due to their electric charges. It is governed by the Coulomb force and is fundamental in understanding the behavior of charged particles.

We use the phrase Coulomb interaction when we are thinking about general interactions between charged particles, such as how these interactions can affect energy levels. We use the phrase Coulomb force when we are specifically thinking about the force one charged particle exerts on the other.

There are many interactions that happen inside the atom. Each of these interactions affects where the discrete energy levels end up. If we, as physicists, understand all of these interactions we should be able to calculate properties of the atom like the energy of a particular state, the natural linewidth of each transition, and what would happen if we put the atoms in a particular situation such as inside an electric field, a magnetic field, or if we put a bunch of these atoms together so strongly compressed under gravity that the gravitational energy is converted into heat and eventually the atoms fuse together.[2] While there are many interactions happening in the atom, the two biggest contributions that determine the energy of an atomic state are the Coulomb interaction and how electrons fill up "shells."

The **Coulomb interaction** is how we describe the interaction between charged particles. For the hydrogen atom, there is only 1 Coulomb interaction. That interaction is between the single electron, which has negative charge, and the single proton, which has positive charge. In fact, this interaction is so simple we can exactly solve the Schrödinger equation to find the effect this interaction has on the energy levels. The helium atom has two electrons and two protons. The two protons are inside the nucleus and, while there is technically a Coulomb interaction between the two protons, that interaction is overwhelmed by the strong nuclear force, which is a topic in Chap. 10. Therefore, we can think of the nucleus as a single particle with a charge of $+2$. The two electrons on the other hand are not confined tightly together like the two protons in the nucleus. For the helium atom, there are effectively 3 Coulomb interactions: one between the two electrons, one between one of the electrons and the nucleus, and one between the other electron and the nucleus. The increased number of interactions causes most of the dramatic difference between the energy levels of hydrogen and helium.

[2] That was a really complicated way of saying "a star."

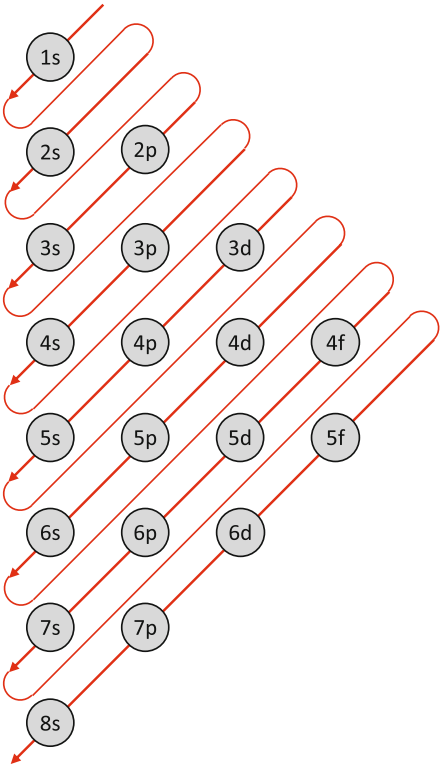


Fig. 8.3 How electrons fill shells. This diagram is often called Madelung energy ordering rule, named after the German physicist Erwin Madelung. The top row is the $n = 1$ shell and has one subshell. The second row is the $n = 2$ shell and has two subshells. The third row is the $n = 3$ shell and has three subshells, and so on

Lithium has 3 electrons and europium has 63. Like helium, we can think about the nucleus of lithium as 1 particle with a charge of +3. Even though europium has 63 protons in the nucleus, we can still think of the nucleus as 1 particle with a charge of +63.[3] Now we have a lot of interactions. Each electron has a Coulomb interaction with every other electron and each electron has a Coulomb interaction with the nucleus. One of the homework problems for this chapter will have you count the number of Coulomb interactions for the first few atoms in the periodic table.

The second major contribution comes from something called **electron shells**, which are made up of **electron subshells**. The first shell is called 1, and it contains 1 subshell labelled 1s, see Fig. 8.3. The second shell is 2 and has two shells. The third shell is 3 and has 3 subshells and so on. In Fig. 8.3, we stop showing all possible subshells on shell 5. There is a 5g subshell, 6f, 6g, 6h, etc. that is not shown. In quantum mechanics and atomic physics, the shell number $(1,\ 2,\ 3,\ \ldots)$

[3] The neutrons in the nucleus also matter, but not as much as the proton charge. The effect the neutrons have on the energy states is a topic in Chap. 10.

is a quantum number called the principal quantum number, represented by the letter n. This is the same quantum number n explored in Chap. 6.

The electron subshells come about because an electron in an atom can have orbital angular momentum and spin. The letter accompanying the principal quantum number is the orbital angular momentum quantum number for the electron in that subshell. The subshells use the historical letter designations discussed in Chap. 7. As a reminder, if an electron has the label s, it has $\ell = 0$. If it has a label p, it has $\ell = 1$, and so on. For example, 3p tells us that $n = 3$ and $\ell = 1$. From solving the Schrödinger equation, we find that n is the upper limit on possible values for ℓ. As a reminder, ℓ is a zero or a positive integer. From the math, we find $\ell_{\max} = n - 1$. For example, if $n = 3$, then $\ell = 0$, 1, or 2. That means the $n = 3$ shell has three subshells: 3s, 3p, and 3d. There is no 3f subshell since $\ell = n$, which is not allowed.

An s-subshell can hold 2 electrons in total, one will have spin up ($m_s = +1/2$) and the other spin down ($m_s = -1/2$). A p-subshell can hold 6 electrons in total: 3 spin up and 3 spin down. A d-subshell can hold 10 electrons (5 up and 5 down), an f-subshell can hold 14, a g-subshell can hold 18, etc. The subshells fill in a specific order roughly shown in Fig. 8.3.

The first subshell to fill is the 1s subshell, which can hold 2 electrons. After the 1s subshell is filled, electrons start to fill the 2s subshell, which also holds 2 electrons. Once the 2s subshell is filled, electrons start to fill the 2p subshell, which holds 6 electrons. Next is 3s followed by 3p, 4s, 3d, 4p, etc.

Why Can a p-Shell Hold 6 Electrons? An electron with the label p means that $\ell = 1$. There are three possible orientations for this angular momentum vector: $m_\ell = -1$, 0, and 1. In addition, the electron can be either spin up or spin down. That means there are 6 orientations for an electron to have $\ell = 1$. Using the notation (m_ℓ, m_s), the possible orientations for a p-subshell are:

$$\left(1, +\tfrac{1}{2}\right) \quad \left(1, -\tfrac{1}{2}\right) \quad \left(0, +\tfrac{1}{2}\right) \quad \left(0, -\tfrac{1}{2}\right) \quad \left(-1, +\tfrac{1}{2}\right) \quad \left(-1, -\tfrac{1}{2}\right)$$

An electron with the label d means that $\ell = 2$. There are 5 possible orientations for this vector (m_ℓ = -2,-1,0,1,and 2). Including spin up and spin down for each m_ℓ gives 10 possible orientations. Using the notation (m_ℓ, m_s), the possible orientations for a d-subshell are:

$$\begin{matrix} \left(2, +\tfrac{1}{2}\right) & \left(1, +\tfrac{1}{2}\right) & \left(0, +\tfrac{1}{2}\right) & \left(-1, +\tfrac{1}{2}\right) & \left(-2, +\tfrac{1}{2}\right) \\ \left(2, -\tfrac{1}{2}\right) & \left(1, -\tfrac{1}{2}\right) & \left(0, -\tfrac{1}{2}\right) & \left(-1, -\tfrac{1}{2}\right) & \left(-2, -\tfrac{1}{2}\right) \end{matrix}$$

Figure 8.4 are Bohr model pictures of hydrogen, helium, and lithium in their lowest energy state (the ground state). Remember that electrons are actually more like waves, but I like to draw them as little balls to explore this concept. Hydrogen has 1 electron so the 1s subshell is half filled. Helium has 2 electrons that fully fill

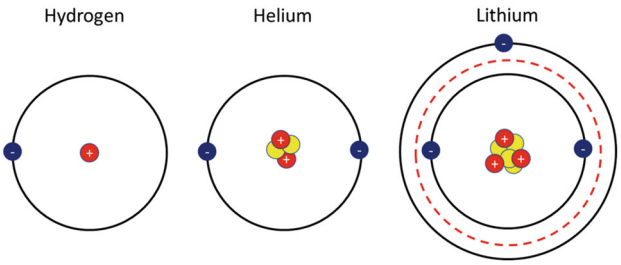


Fig. 8.4 A Bohr model picture of hydrogen, helium, and lithium. Electrons will first fill the 1s subshell. Hydrogen half fills the 1s subshell, while Helium fills the 1s subshell completely. Lithium has 3 electrons that completely fill the 1s subshell leaving a single electron in the 2s subshell

this 1s subshell. If we move up the periodic table to lithium (3 electrons), 2 electrons fill the 1s shell while the last electron is in the 2s shell.

So, why does this affect the energy levels in the atom? The answer is every time we fill a shell, it is like creating a new nucleus with a reduced charge. For example, let's look at a picture of lithium in Fig. 8.4. I added a red dashed line between the 1s and 2s shells. Inside this red dashed line, the effective charge is +1 (2 electrons and 3 protons). For all it knows, that 1 electron sitting in the 2s subshell is all by itself interacting with a nucleus with charge +1. This is just like the hydrogen atom with a few key differences. The first is the nucleus of lithium has a lot more mass than the nucleus of hydrogen. The second is that the outermost electron in lithium is farther away from the nucleus. The reduced Coulomb force means the energy levels for lithium are not as widely spaced compared to hydrogen. This is why the hydrogen atom and the lithium atom energy levels look so similar, but the lithium energy levels are closer together, see Fig. 8.5. All of the atoms in the first column of the periodic table (hydrogen, lithium, potassium, rubidium, cesium, and francium)

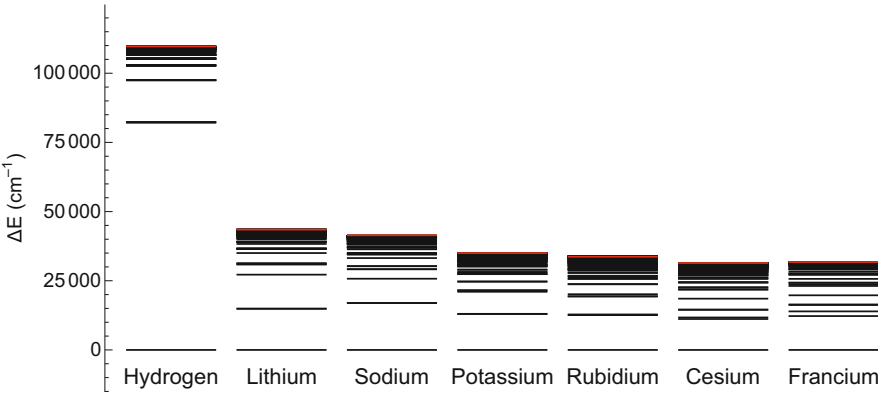


Fig. 8.5 The energy levels for the first column of the periodic table. Each of these elements has fully closed shells except for a single electron in an s-shell

have 1 electron sitting outside a closed shell. Therefore, the energy levels of these atoms are very similar to hydrogen.

We write the **electron configuration** with superscripts that tell us how many electrons are in a subshell; if there is no superscript present, there is only 1 electron in that subshell. For example, the electronic configuration for the ground state of helium is $1s^2$; there are 2 electrons in the 1s subshell. If we excite the atom so that 1 electron moves to the 3d subshell, the electronic configuration is 1s3d; 1 electron is in the 1s subshell and 1 is in the 3d subshell.

As we move through the periodic table, the atoms gain more and more electrons filling up shells according to Fig. 8.3. I should note that while the chart does a really good job, particularly for lighter atoms, it messes up sometimes for heavier atoms. For example, the chart predicts that the ground state of silver (47 electrons) should have a filled 5s subshell and 9 electrons in the 4d subshell:

$$1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 5s^2 4d^9$$

However, the actual electronic configuration has a completely filled 4d subshell and a single electron in the 5s subshell:

$$1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 4d^{10} 5s$$

As physicists, we ask, "Why?" Why would silver decide that filling the 4d subshell is better than following that chart and keeping the 5s subshell filled? The short answer is that nature tends to try and get to a configuration with the lowest overall energy.

A General Rule of Nature
A system always seeks to reach a state of lower energy.

As an analogy, think about a ball in a bowl. The ball will want to hang out at the bottom of the bowl, which is the spot of lowest energy. Because there are so many electrons in silver and so many interactions, the electron configuration that fills the 4d shell has less energy than if the 5s shell was filled.

Quick quiz:
Considering the silver atom, how many electrons are in the 3d subshell? The answer is in the footnotes.[4]

[4] 10

Here is the electron configuration for the ground state of europium, which has 63 electrons:

$$1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 4d^{10} 5s^2 5p^6 6s^2 4f^7$$

From this electronic configuration we can see which shells are filled. After the 6s subshell fills up, the 4f subshell starts to fill until we run out of electrons. There are 7 electrons in the unfilled 4f subshell. Combined with the Coulomb interactions, the end result is a pretty complicated energy level diagram, see Figs. 8.1 and 8.2. This is why I'm glad to be an experimentalist. I don't have to try to calculate these levels, just measure them ☺.

Summary
The dominant interactions in atoms that determine the energy level locations are the Coulomb interaction and the way electrons fill shells.

8.3 Term Symbols

Reminder Definition
- **Torque:** A measure of how effective a force is in causing an object to rotate. Applying torque changes the angular momentum of an object. There is energy associated with that torque.

We spent almost all of Chap. 7 studying angular momentum and emphasizing the orientation of the vectors. The theory of electricity and magnetism reveals an important fact: because electrons and the nucleus have charge, and the electrons have angular momentum, everything in the atom exerts a torque on everything else.

A Brief Aside
The Coulomb interaction is used to describe the interaction between charged particles, resulting in an electric field between the charges. This approach ignores the motion associated with their angular momentum. When we include angular momentum, it is more appropriate to use the electromagnetic interaction because moving charges create magnetic fields. Therefore, the interaction between moving charged particles involves both electric and magnetic forces, which is what the electromagnetic force describes.

Definitions

- **Electromagnetic force:** A fundamental force encompassing both electric and magnetic forces, including the Coulomb force and magnetic forces due to moving charges.
- **Electromagnetic interaction:** The interaction between charged particles due to both electric and magnetic forces, including interactions involving stationary charges (electrostatics) and moving charges.

The magnetic forces from the moving charges cause the internal torques. The magnitude of this internal torque depends on the orientation and size of the angular momentum vectors. In short, both the size and the orientation of the angular momentum vectors impact the energy of a state. That is so important, I'm going to give the sentence its own red box.

Important
Both the size and the orientation of the angular momentum vectors impact the energy of a state.

If we had a magic wand to change the orientation of an angular momentum vector, the energy of that state would change slightly. Now we can add a third property that affects where the energy levels end up: the Coulomb interaction, electron shells, and angular momentum.

The total orbital angular momentum of all the electrons, the total spin of all of the electrons, and total angular momentum of all of the electrons affects the energy of a state. Therefore, we need to include that information in the description of a state. By tradition, we combine these quantum numbers into a "term symbol", which looks like this:

$$^{2S+1}L_J \tag{8.1}$$

where S is the quantum number representing the total spin of all the electrons, L is the quantum number for the total orbital angular momentum of all the electrons, and J is the quantum number for the total angular momentum of all of the electrons. For example, suppose an energy level has a term symbol $^3D_2$. That tells us that the spin of the electrons has quantum number $S = 1$ ($2S + 1 = 3$), the orbital angular momentum of the electrons has quantum number $L = 2$ (represented by the letter D), and the total electronic angular momentum of the electrons has quantum number $J = 2$. We can use those numbers to calculate the magnitude of angular momentum.

Table 8.1 The first 6 states of atomic oxygen. The last column is the frequency of a laser needed if we were to try to excite the atom from the ground state to that state

Electron configuration	Term symbol	Energy (cm^{-1})	f (Hz)
$1s^2 2s^2 2p^4$	$^3P_2$	0	
$1s^2 2s^2 2p^4$	$^3P_1$	158.265	4.74×10^{12}
$1s^2 2s^2 2p^4$	$^3P_0$	226.977	6.80×10^{12}
$1s^2 2s^2 2p^4$	$^1D_2$	15,867.862	4.76×10^{14}
$1s^2 2s^2 2p^4$	$^1S_0$	33,792.583	1.01×10^{15}
$1s^2 2s^2 2p^3 3s$	$^5S_2$	73,768.200	2.21×10^{15}

Two Important Reminders

1. The term symbol is important not just because it tells us three quantum numbers for the electrons in an atom as a whole, but also because those three quantum numbers affect the energy of a state.
2. The observables represented by S, L, and J are all compatible with energy.

The shift of an energy level due to electrons having angular momentum is called fine structure splitting.[5] The nucleus can also have angular momentum (often called nuclear spin), which is the topic of Chap. 9. The shift in the energy of a state due to the nucleus having angular momentum is called hyperfine structure splitting.[6]

Definitions

- **Fine structure splitting:** The shift in the energy of a state due to the electrons having orbital angular momentum and spin.
- **Hyperfine structure splitting:** The shift in the energy of a state due to the nucleus having angular momentum.

Table 8.1 shows the six lowest energy states of atomic oxygen (8 electrons, 8 protons, and 8 neutrons). This particular type of oxygen is known as oxygen-16. Oxygen-16 has no nuclear angular momentum, so the nucleus does not play a role in the following discussion. Notice that the first five energy states all have the same electronic configuration but different term symbols. The next few paragraphs are all about adding lots and lots of angular momentum vectors together. We are going to start with the ground state and then work our way through the table.

[5] This is sometimes called fine splitting.

[6] This is sometimes called hyperfine splitting.

Reminder of Eq. 7.5
Suppose we have two quantum mechanical angular momenta vectors whose magnitudes are represented by quantum numbers L and S. Depending on the orientation of the two vectors, adding them together produces a new quantum mechanical angular momentum vector that can have different magnitudes, represented by the quantum numbers

$$J = (L + S), \ldots, |L - S|$$
in integer steps

For example, if $L = 3$ and $S = 1$, then J can be 4, 3, or 2. If $J = 4$, then there are 9 possible orientations of this new angular momentum vector represented by the quantum number $m_J = 4, 3, 2, 1, 0, -1, -2, -3$, or -4. Similarly, if $J = 2$, then there are 5 possible orientations of this new angular momentum vector represented by the quantum number $m_J = 2, 1, 0, -1$, or -2.

The ground state of oxygen has the designation $1s^2 2s^2 2p^4\ ^3P_2$. This designation tells us that both the 1s and 2s subshells are completely filled, and we can essentially ignore them. The 2p subshell is partially filled with 4 electrons (the 2p subshell can hold up to 6 electrons). From the electronic configuration, we know the orbital angular momentum of each electron. The 4 electrons in the 2p subshell each have $\ell = 1$, or each electron in this subshell has orbital angular momentum with size $\sqrt{\ell(\ell+1)}\hbar = \sqrt{2}\hbar$.

Each of the electrons in the 2p subshell have the same magnitude of orbital angular momentum ($\ell = 1$), but have different orientations. If we were to add all four of those vectors together, we will get a new vector that represents the orbital angular momentum of all four electrons as a composite system. For the ground state, the orbital angular momentum of the composite system has a magnitude represented by quantum number $L = 1$ (P in the term symbol).

Those same four electrons also have individual spin vectors that add up to $S = 1$ ($2S + 1 = 3$). Again, that spin vector represents the total spin of all four electrons as a composite system. Since the orbital angular momentum vector for all four electrons has a size and orientation and the spin vector for all four electrons has a size and orientation, we can ask the question, "What are the possible magnitudes for the total electronic angular momentum for the composite system?" The answer is $J = 2$, 1, or 0. The orientations that produce a vector whose magnitude is represented by $J = 2$ has the lowest energy, so that is the ground state.

Important Foreshadowing Statement
The total electronic angular momentum for the ground state ($J = 2$) has 5 possible orientations that, in the absence of nuclear spin, all have the same energy. In other words, there are 5 states that all have the same energy. When there are states that have the same energy, we call those states **degenerate**.

Now let's move on to the next energy level. This level has an energy that is $158.265\,\text{cm}^{-1}$ larger than the ground state. The only difference in the designation for the ground state and this state is that J in the term symbol ($J = 2$ for the ground state and $J = 1$ for this state). The orbital angular momentum for all four electrons and the spin for all four electrons still add together, but the result has a different total electronic angular momentum for the composite system. The orientation of all the electrons to produce this new composite vector has a different internal torque, so that orientation has a different energy than the $J = 2$ ground state.

The first 5 energy levels of atomic oxygen have the same electron configuration: $1s^2 2s^2 2p^4$. To determine how the angular momenta of the last 4 electrons combine, we look at the term symbols. As you can see from the table, there are multiple ways this can happen, each resulting in a different energy level. In a hypothetical world without the internal torque, all 5 of these levels would be degenerate. However, due to the internal interactions, these levels split into the 5 distinct levels observed experimentally. This phenomenon is known as fine structure splitting.[7]

While the nucleus of oxygen-16 has no angular momentum, a different nucleus might. That nuclear angular momentum, whose magnitude is represented by the quantum number I, exerts an additional internal torque that, once again, shifts the energy of the states. This additional internal torque "breaks" the degeneracy of the fine structure states. For example, oxygen-17 has nuclear spin $I = 5/2$ (6 possible orientations). The ground state has a total electronic angular momentum of $J = 2$ (5 possible orientations). Adding together vector J and vector I produces a new angular momentum vector whose magnitude is represented by the quantum number F with possible values that range from $J + I$ to $|J - I|$ in integer steps, or in this case $F = 9/2$, $7/2$, $5/2$, $3/2$, and $1/2$. So the ground state of oxygen-17 further splits into six hyperfine levels, each with a different energy. From the "important foreshadowing statement," the orientation of a particular F vector will not change the energy. If $F = 3/2$, there are four orientations represented by $m_F = 3/2$, $1/2$, $-1/2$ and -1. All four of these states are degenerate. We will explore hyperfine structure and how the nucleus affects our energy levels further in Chap. 9.

[7] For completeness, Einstein's theory of special relativity, which deals with the physics of fast-moving objects, also shifts the energy of a state and is included as part of fine structure splitting, but we will not explore that here. If this book has inspired you to pursue further studies in quantum mechanics ☺, you will learn about perturbation theory and the effects of electron velocity.

Common Question
If $J = 2$, there are 5 states with different orientations that have the same energy. Can we do anything to "split" those final orientations so they have different energies? Yes! The internal torque doesn't do it, but we can apply a torque from the outside to "break the degeneracy". Applying an external torque using an external magnetic field is called the Zeeman Effect, named after the Dutch physicist Pieter Zeeman. Applying an external torque by applying an electric field is called the Stark Effect, named after the German physicist Johannes Stark. Both of these effects "break" the degeneracy of those states.

Summary
The orientation and size of angular momentum impact the energy of a state. For an atom with no nuclear spin, a state can be described by the individual electron configurations, which tells us the angular momentum of each individual electron, and the term symbol, which tells us about the orientation of the angular momentum vectors of the electrons as a whole.

Super Short Summary
The orientation and size of the angular momentum vectors matter.

8.4 Connecting Angular Momentum to Orbitals

You may have learned about orbitals in high school chemistry. Orbitals are visual representations of different energy states for an electron in a hydrogen atom. Figures 8.6 and 8.7 show some orbital pictures for different energy states in hydrogen. These are analogous to the shaking energy modes we studied in Chap. 1 (Fig. 1.7) and Chap. 6 (Fig. 6.2). There are 3 numbers on each plot given in bra-ket notation. The first number is the principal quantum number (also called the shell number). The second is the orbital angular momentum quantum number, and the last is the projection of the orbital angular momentum quantum number along the z-axis (orientation). For example, $|4\ 3\ -1\rangle$ means $n = 4$, $\ell = 3$, and $m_\ell = -1$. You can think of all of these orbitals as different standing waves of the electron. The "loops" are a bit harder to see in 3-dimensional space, so in Fig. 8.6 we plot a few different orbitals in 3D and in Fig. 8.7 we plot a few cross sectional views. The reason they are so hard to visualize is that we only have 3 dimensions to view 4 dimensions of information.

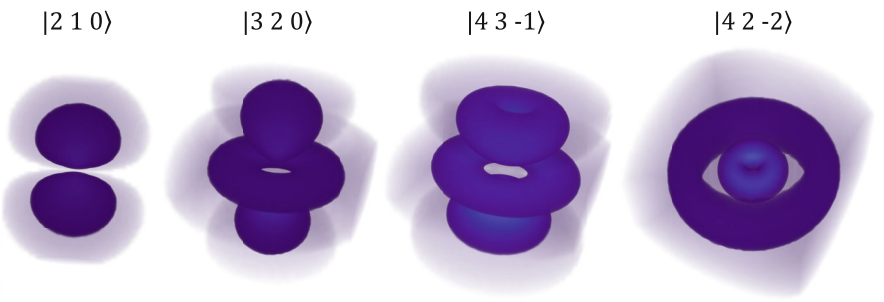


Fig. 8.6 3D illustrations of different electron orbitals in a hydrogen atom. The darker the shade the larger the amplitude of the standing wave

For example, if you look at Fig. 6.2 you'll see that we need two dimensions to see the energy state for 1 dimension of shaking energy. The vertical axis shows the amplitude of the energy state while the horizontal shows position in 1 dimension. An example of a 2 dimensional standing wave would be a drum head vibrating. To visualize a 2 dimensional standing wave, we need a 3 dimensional plot: 1 dimension for the amplitude and 2 for the position dimensions. We run into a problem for

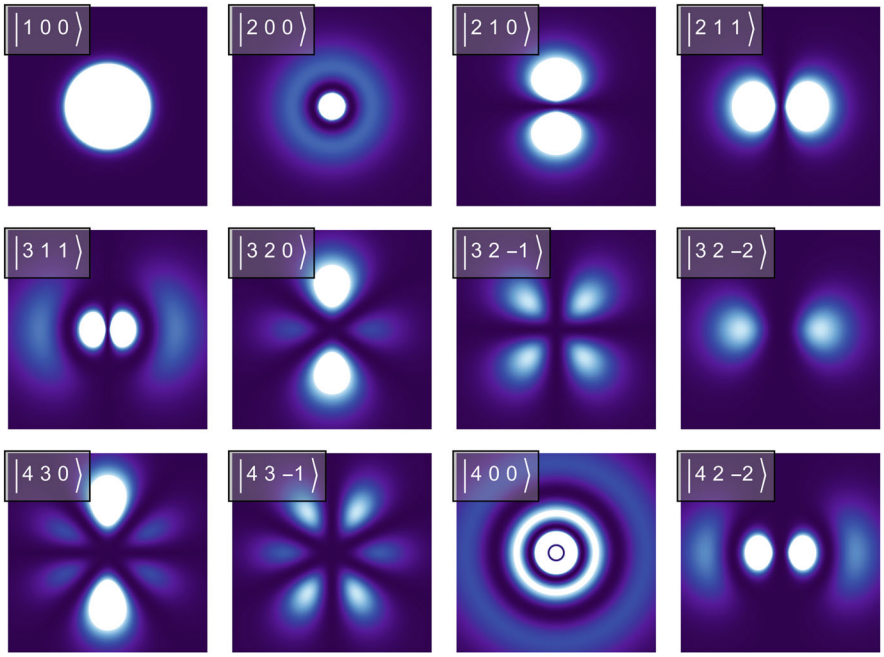


Fig. 8.7 A cross sectional view of different electron orbitals in the Hydrogen atom. The closer the color is to white, the larger the amplitude of the standing wave

3 dimensional energy states. We now need to plot 3 dimensions of position, but we also need to visualize the amplitude. Figures 8.6 and 8.7 attempt to show the amplitude by shading.

It is important to point out that these orbitals are not discrete regions of space (for example an orbital is not like a hollow ping pong ball or a uniformly filled sphere). It is more like a foam ball where the density of the foam ball is not uniform but varying in space. If we were to measure the position of the electron, it would be most probable to find the electron at a place where the energy state is most dense. As an analogy, if we were to measure the position of the quantum particle for a system in an energy state seen in Fig. 6.2, the most probable place to find the quantum particle would be where the amplitude is largest.

8.5 Fermions and Bosons

In atomic and nuclear physics, particles are categorized into two major groups: **fermions** and **bosons**. Fermions are particles with half-integer spin quantum numbers (e.g., 1/2, 3/2, etc.). Examples of fermions are electrons, protons, and neutrons, all of which have a spin quantum number of 1/2. Bosons are particles with integer spin quantum numbers (e.g., 0, 1, etc.). So far, we have encountered only one boson, the photon, which has a spin quantum number of 1.

This is important because two *identical* fermions cannot "hang out" in the same space while two identical bosons can. This is the essence of the **Pauli Exclusion Principle**: two fermions cannot simultaneously occupy the same quantum state; that is, no two fermions can have the same set of quantum numbers within a quantum system. As an example, consider the two electrons, both of which are fermions, in the atomic ground state of a helium atom, which has the electronic configuration $1s^2$. Let's write out the quantum numbers for the two electrons using bra-ket notation with the quantum numbers $|n\ \ell\ m_\ell\ s\ m_s\rangle$. The two electrons have quantum numbers:

$$\begin{aligned} \text{First Electron:}\ & |1\ 0\ 0\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle \\ \text{Second Electron:}\ & |1\ 0\ 0\ \tfrac{1}{2}\ \text{-}\tfrac{1}{2}\rangle \end{aligned}$$

Notice that the two electrons have a different set of quantum numbers. If they did have the same quantum numbers, the math from quantum mechanics shows that the two electron wavefunctions would cancel each other out.[8] This is a very bad thing. If the individual electron wavefunctions added together to produce no wavefunction, neither electron would exist. Therefore, the two electrons must have different quantum numbers. This is what is meant by the phrase, "two *identical* fermions cannot hang out in the same space". They must have a different set of quantum numbers.

[8] More specifically, the wavefunction for the two-electron system is "antisymmetric" and would be zero everywhere if the electrons had identical quantum numbers.

The Pauli Exclusion Principle is part of why electron shells and subshells exist. Every electron in the atom *must* have a different set of quantum numbers. Interestingly, bosons don't have this problem. Two bosons can have the same set of quantum numbers and not destructively interfere each other out of existence. So, two identical bosons can hang out in the same space. In fact, the two bosons can constructively interfere with each other.

There can also be composite fermions and bosons. For example, helium-4 is a system that acts as a composite boson. Helium-4 contains 2 protons, 2 neutrons, and 2 electrons. All of these particles are fermions, but they can pair up to behave like bosons. Other examples of composite bosons include Cooper pairs[9] (important for superconductors) and Bose–Einstein condensates[10]. Helium-3, which has 2 protons, 2 electrons, and 1 neutron, is a composite fermion.

Problems

8.1 As discussed in the chapter, hydrogen has 1 Coulomb interaction and helium has 3. Assuming the nucleus is one big particle with a positive charge, how many Coulomb interactions do lithium, beryllium, and boron have?

Challenge: What is the general formula to calculate the number of Coulomb interactions for an atom with N electrons, assuming the nucleus is one big particle with positive charge?

8.2 The electron configuration for hydrogen is 1s. There is 1 electron in the first s subshell. The electron configuration for helium is $1s^2$. That means there are 2 electrons in the first s subshell, which completely fills the shell. Lithium has 3 electrons, so the electron configuration is $1s^2 2s$. Notice since the first shell is filled, we begin to fill the second shell. Write out the electron configurations for

(a) beryllium (4 electrons)
(b) boron (5 electrons)
(c) carbon (6 electrons)
(d) nitrogen (7 electrons)
(e) oxygen (8 electrons)
(f) fluorine (9 electrons)
(g) neon (10 electrons)
(h) sodium (11 electrons).

[9] Also known as Bardeen–Cooper–Schrieffer pairs named after American physicists John Bardeen, Leon Cooper, and John Schrieffer.

[10] Named after Indian physicist Satyendra Nath Bose and German born physicist Albert Einstein.

Note: The electron configurations are something that you can easily find on the internet. You can also find the answers in Appendix B. Don't search for the answer before you try yourself first.

8.3 For each of the following term symbols, what is the magnitude of the total electronic spin, the total electronic orbital angular momentum, and the total electronic angular momentum?

(a) $^3P_2$
(b) $^3P_0$
(c) $^1S_0$
(d) $^1F_2$

8.4 Appendix B has a list of all of the elements with their ground state electronic configurations. Look through the list and find all the atoms that will have an energy level structure similar to hydrogen (i.e. all the shells are filled except for a single electron in the last s subshell). All of these elements will have energy level diagrams that look similar to hydrogen.

8.5 If we ionized (removed 1 electron) from beryllium, its electronic structure would look just like lithium, see Fig. 8.8. While the energy level spacings are proportionally similar, lithium is more compact. Why?

8.6 In Sect. 8.3 when exploring oxygen, we had the sentence, "This designation tells us that both the 1s and 2s subshells are completely filled, and we can essentially ignore them." Let's explore this sentence a bit more.

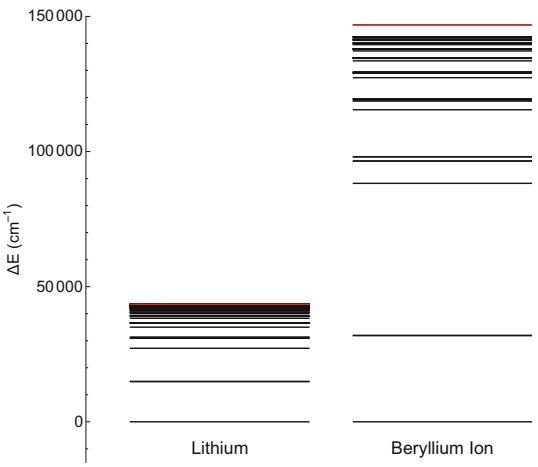


Fig. 8.8 Energy levels for a neutral lithium atom and a singly ionized beryllium atom

(a) Write the term symbol for the two electrons that fill the 1s subshell. Explain the reasoning you used to determine the term symbol. The answer (without explanation) is in the footnotes.[11]
(b) Write the term symbol for the two electrons that fill the 2s subshell. Explain your reasoning.
(c) Write the term symbol for the four electrons with the electronic configuration $1s^2 2s^2$. This is the same as the ground state of a beryllium atom, which has 4 electrons. Explain your reasoning.
(d) Write the term symbol for the ground state of neon. Neon has 10 electrons, so the 2p subshell is completely filled. Explain your reasoning.
(e) Write the term symbol for the ground state of lithium, which has 3 electrons. Explain your reasoning.
(f) Write the term symbol for the ground state of cesium, which has all subshells filled except for the last electron that is in the 6s subshell. Explain your reasoning.
(g) Boron has all subshells filled except for the last electron that is in the 2p subshell. For that last electron, there are two possible values for J. What are they?
(h) For part (g), the term symbol with the lower value of J corresponds to the lower energy and thus the ground state. Write the term symbol for the ground state of boron.

8.7

(a) Make a sketch of a saturated absorption spectroscopy spectrum where the lower state has $J = 0$ and the upper state has $J = 1$. The transition frequency is $f_r = 652,000,000\,\text{MHz}$ and the natural linewidth is 1 MHz.

Next, we put the atoms in an external magnetic field. The magnetic field shifts the energy levels according to the formula $\Delta E = \left(1\frac{\text{GHz}}{\text{T}}\right) h\, m_J\, B_{\text{ext}}$, where B_{ext} is the external magnetic field and the unit T stands for Tesla (the unit for magnetic field).
(b) We place the atoms in a uniform magnetic field with $B_{\text{ext}} = 0.1\,\text{T}$. Make a sketch of a saturated absorption spectroscopy spectrum where the lower state has $J = 0$ and the upper state has $J = 1$. Don't worry about the relative amplitudes.
(c) Optional Challenge: Next, we place the atoms in a magnetic field described by the function $B_{\text{ext}} = \left(0.001\frac{\text{T}}{\text{cm}}\right) z$. Make a sketch of the three $J = 1$ states as a function of z. Use frequency units instead of Joules for the vertical axis (this is equivalent to using a laser to excite the atoms).

[11] $^1S_0$.

Hyperfine Structure

9

Abstract

In this chapter, we explore hyperfine structure, which occurs in atoms with a nucleus that has angular momentum. We will discuss the theoretical framework, including key equations for calculating hyperfine energy shifts, and apply this knowledge through practical examples involving cesium, europium, and oxygen. This chapter aims to provide a comprehensive understanding of hyperfine structure and its significance in atomic physics.

Learning Goals

By the end of this chapter, you should be able to understand:

- why hyperfine structure exists and identify the conditions under which hyperfine splitting occurs.
- the difference between fine structure and hyperfine structure.
- how to interpret spectroscopic data to extract hyperfine constants.

9.1 Hyperfine Structure

The nucleus contains protons and neutrons, each of which has intrinsic angular momentum. In addition to their intrinsic spins, nucleons (protons and neutrons) can also have orbital angular momentum due to their motion within the nucleus. When discussing the angular momentum of the nucleus as a whole, we primarily refer to the combination of the intrinsic spins of the nucleons and their orbital angular momentum. This combined angular momentum should be called the "total nuclear angular momentum," but it is commonly referred to as "**nuclear spin**." Despite the

W. Raven, *Atomic Physics for Everyone*,
https://doi.org/10.1007/978-3-031-69507-0_9

terminology, neither the nucleus nor the protons and neutrons in the nucleus are literally spinning or orbiting in the classical sense. Nuclei with both an even number of protons and an even number of neutrons tend to have zero nuclear spin because the individual spins pair up (one spin-up and one spin-down) and cancel each other out. Nuclei with an odd number of protons and/or neutrons generally have a non-zero nuclear spin.

When an atom has a non-zero nuclear spin, the nucleus interacts with the magnetic fields produced by the electrons. More specifically, because the nucleus has both charge and angular momentum (I), and the electrons have both charge and angular momentum (J), there is an additional internal torque between the nucleus and the electrons. This interaction leads to the splitting of atomic energy levels into closely spaced sub-levels known as hyperfine levels, analogous to the interaction between an electron's spin and its orbital angular momentum, which results in fine structure splitting. I want to point out that atomic physicists use **hyperfine levels** and hyperfine states interchangeably. We tend to use hyperfine levels when thinking about energy splittings and hyperfine states when using quantum numbers, but the two terms mean the same thing.

Table 9.1 is a copy of a table from Chap. 7 that summarizes all of the angular momentum vectors and angular momentum quantum numbers for the system of electrons and the atom as a whole. Let's look at some examples. Oxygen has three stable isotopes: oxygen-16 (99.76% of all oxygen on earth is oxygen-16), oxygen-17 ($\sim$0.04%), and oxygen-18 ($\sim$0.20%). Isotopes are elements with the same number of protons but different numbers of neutrons. Oxygen-16 has 8 electrons, 8 protons, and 8 neutrons. Oxygen-17 has 8 electrons, 8 protons, and 9 neutrons. Oxygen-18 has 8 electrons, 8 protons, and 10 neutrons. Since each isotope has a different number of neutrons, the transition frequencies are slightly different. This small shift, called an isotope shift, will be discussed in Chap. 10.

Table 9.1 This table summaries all of the angular momentum quantum numbers

Type	QN	Rule	Formula
Orbital	L	Zero or positive integer	Magnitude: $\sqrt{L(L+1)}\hbar$
	m_L	$-L$ to $+L$ in integer steps	Cone height: $m_L\hbar$
Spin	S	Zero, positive integer, or half-integer	Magnitude: $\sqrt{S(S+1)}\hbar$
	m_S	$-S$ to $+S$ in integer steps	Cone height: $m_S\hbar$
Total electronic	J	Zero, positive integer, or half-integer	Magnitude: $\sqrt{J(J+1)}\hbar$
	m_J	$-J$ to $+J$ in integer steps	Cone height: $m_J\hbar$
Nuclear spin	I	Zero, positive integer, or half-integer	Magnitude: $\sqrt{I(I+1)}\hbar$
	m_I	$-I$ to $+I$ in integer steps	Cone height: $m_I\hbar$
Total atomic	F	Zero, positive integer, or half-integer	Magnitude: $\sqrt{F(F+1)}\hbar$
	m_F	$-F$ to $+F$ in integer steps	Cone height: $m_F\hbar$

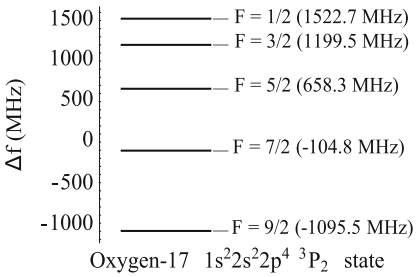


Fig. 9.1 The hyperfine structure of the ground state of oxygen-17. The hyperfine levels are shown with respect to the center of gravity of the ground state

Important Reminders

- We can represent the energy difference between two states using energy units, wavelength units, or frequency units. For hyperfine structure, frequency units are the most convenient unit because the energy spacing between hyperfine levels is small.
- When adding two angular momentum vectors together, both the magnitude and orientation of the individual angular momentum vectors will affect the magnitude and orientation of the resulting vector. Each possible magnitude of the composite angular momentum vector will have an effect on the energy of a state.
- **Center of gravity:** The energy of a state if there was no nuclear spin.

The nuclear spin quantum number for oxygen-17 is $I = 5/2$. The nuclear spin quantum number for both oxygen-16 and oxygen-18 is $I = 0$. Therefore oxygen-17 will have hyperfine structure while the other two isotopes do not. All isotopes of oxygen have a ground state term symbol of $^3P_2$, or $S = 1$ ($2S + 1 = 3$), $L = 1$, and $J = 2$. However, the ground state of oxygen-17 looks different compared to the other two isotopes, see Fig. 9.1. If oxygen-17 had no nuclear spin, it would have a single level precisely at 0. Nuclear spin "splits" this single level into 5 hyperfine levels. For example, the level labeled $F = 7/2$ has a slightly smaller energy than the center of gravity while the $F = 5/2$ state has a higher energy. Both oxygen-16 and oxygen-18 have no nuclear spin, so they have a single ground state that would be labeled 0 energy and have no F quantum number designation.

Adding together nuclear spin, represented by the quantum number I, and the total electronic angular momentum, represented by the quantum number J, results in a new angular momentum vector that we call the total angular momentum of the atom, represented by the quantum number F. Hyperfine structure generally has a much smaller energy splitting compared to fine structure. Like every angular momentum vector we have encountered, the F vector can also point in different orientations. For example, an $F = 3/2$ state has four possible orientations described by the quantum

numbers m_F =3/2, 1/2, −1/2, and −3/2. All four of those orientations have the same energy.

Compatibility with Energy We often care about what observables are compatible with energy. At the end of Chap. 7, we used the ket $|n\ \ell\ s\ j\ m_j\rangle$ to represent the states of the hydrogen atom (one electron). All the different quantum numbers in this ket represent observables that are compatible with energy.

Important Reminder
The observables represented by m_ℓ and m_s are not compatible energy.

Let's update our ket with information we have learned from this chapter. When we add together nuclear spin and total electronic angular momentum, the cones representing those angular momenta are tilted with respect to the total atomic angular momentum (F). Just like when we added orbital and electron spin, the cone heights of I and J, which are represented by the quantum numbers m_I and m_J, are no longer compatible with energy. Therefore, our new ket is $|n\ \ell\ s\ j\ I\ F\ m_F\rangle$.

$$\begin{array}{ll} |n\ \ell\ s\ j\ m_j\rangle & \text{Energy state with no nuclear spin} \\ |n\ \ell\ s\ j\ I\ F\ m_F\rangle & \text{Energy state with nuclear spin} \end{array} \tag{9.1}$$

The nucleus of a hydrogen atom has a nuclear spin represented by $I = 1/2$. The atomic ground state of hydrogen has the electronic configuration 1s, or $s = 1/2$, $\ell = 0$, and $j = 1/2$. Because the nucleus has spin, there will be two hyperfine levels represented by $F = 0$ ($m_F = 0$) and $F = 1$ ($m_F = 1$, 0, and −1). Therefore, the four hyperfine levels for the ground state have kets:

$$\begin{array}{l} |1\ 0\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ 1\ 1\rangle \\ |1\ 0\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ 1\ 0\rangle \\ |1\ 0\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ 1\ \text{-}1\rangle \\ |1\ 0\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ 0\ 0\rangle \end{array} \tag{9.2}$$

The three hyperfine levels with $F = 1$ are degenerate and have the same energy. The hyperfine level with $F = 0$ has a different energy.

So, the ground state of hydrogen has two hyperfine levels represented by the quantum numbers $F = 1$ and $F = 0$. For something a bit more complicated, let's look at the ground state of europium. Europium has 7 electrons in its last, unfilled subshell. All isotopes of europium have a ground state term symbol $^8S_{7/2}$, or $S = 7/2$ ($2S+1 = 8$), $L = 0$, and $J = 7/2$. If europium had no nuclear spin, that would

be the end of the story. We would define the ground state:

$$1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 4d^{10} 5s^2 5p^6 6s^2 4f^7\ {}^8S_{7/2} \tag{9.3}$$

to have 0 energy. However, both stable isotopes of europium have nuclear spin: europium-151 (63 electrons, 63 protons, 88 neutrons, and $I = 5/2$) and europium-153 (63 electrons, 63 protons, 90 neutrons, and $I = 5/2$). Other isotopes of europium will have different nuclear spin. For example, europium-152, which is radioactive with a half-life of 13.5 years, has a nuclear spin quantum number of $I = 3$. Each of these europium isotopes have hyperfine levels.

Summary
If an isotope has no nuclear spin, there would be 1 ground state. If it does have nuclear spin, there are multiple ground states. This is because the angular momentum from the nucleus exerts a torque on the electrons, which results in a small splitting and shift of the state's energy.

There is a single exception to the above summary that we will discuss more in Sect. 9.2. If the state has no total electronic angular momentum ($J = 0$), there is still only a single state; the single state does not split into multiple hyperfine levels. However, that single state will still have an F quantum number as a label.

9.2 Math

We can find the possible values of F by using the largest to smallest in integer steps rule. The rule is:

$$F = (I + J), \ldots, |I - J| \quad \text{in integer steps.} \tag{9.4}$$

Let's do a few examples.

Example 1 Hydrogen-1 has a nuclear spin quantum number of $I = 1/2$ and the ground state has a total electronic angular momentum quantum number of $J = 1/2$ (also $j = 1/2$ since hydrogen has a single electron). Therefore, the possible F values for the ground state range from $1/2 + 1/2 = 1$ to $|1/2 - 1/2| = 0$ in integer steps. Thus, there are 2 hyperfine levels for the ground state of hydrogen-1.

Example 2 Oxygen-17 has a nuclear spin quantum number of $I = 5/2$ and the ground state has a total electronic angular momentum quantum number of $J = 2$. Therefore, the possible F values for the ground state range from $5/2 + 2 = 9/2$ to $|5/2 - 2| = 1/2$ in integer steps. Thus, there are 5 hyperfine levels for the ground

state of oxygen-17. Those values, which are shown in Fig. 9.1, are $F = 9/2$, 7/2, 5/2, 3/2, and 1/2.

Example 3 Let's consider the two stable isotopes of europium: europium-151 and europium-153. Both stable isotopes of europium have the same nuclear spin quantum number of $I = 5/2$. The ground state has a total electronic angular momentum quantum number of $J = 7/2$. Therefore, the possible values for F range from $5/2 + 7/2 = 6$ to $|5/2 - 7/2| = 1$ in integer steps, resulting in F =6, 5, 4, 3, 2, or 1.

Quick Quiz

1. The radioactive isotope europium-152 has a nuclear spin quantum number of $I = 3$. How many hyperfine levels will the ground state have, and what are their F quantum numbers?
2. Beryllium-9 has a nuclear spin quantum number of $I = 3/2$ with a ground state total electronic angular momentum quantum number of $J = 0$. How many hyperfine levels will the ground state have, and what are their F quantum numbers? The answers are below the maze shown in Fig. 9.2.

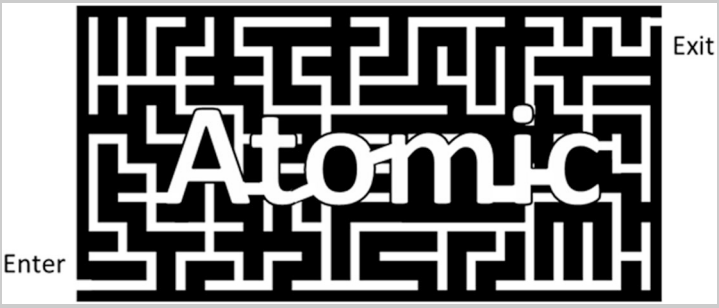


Fig. 9.2 A fun maze to separate the quiz from the answers

The answers are:

1. 7 levels: F =13/2, 11/2, 9/2, 7/2, 5/2, 3/2, or 1/2
2. 1 level: F =3/2.

Notice that beryllium-9 still has a single ground state even though the nucleus has spin. This is because $(I + J) = |I - J|$, so the range of possible F values is 3/2 to 3/2. This always happens when $J = 0$. However, the nucleus still has angular momentum, so we include $F = 3/2$ in the description of the state.

Using quantum mechanics, we can derive the energy splitting of a hyperfine level with respect to the center of gravity:

$$\begin{gathered}\Delta E = \frac{1}{2}KA + \frac{\frac{3}{2}K(K+1) - 2I(I+1)J(J+1)}{2I(2I-1)2J(2J-1)}B \\ K = F(F+1) - I(I+1) - J(J+1) \\ A = 0 \text{ unless both } I > 0 \text{ and } J > 0 \\ B = 0 \text{ unless both } I > 1/2 \text{ and } J > 1/2\end{gathered} \tag{9.5}$$

where A is called the magnetic dipole hyperfine constant and B is called the electric quadrupole hyperfine constant. Notice that everything else in the above equation apart from A and B is a quantum number. Theorists can calculate the hyperfine constants A and B while experimentalists measure them. When we perform spectroscopy on an atom with hyperfine structure, we can measure the energy spacing between all of the hyperfine levels and use the above equation to back out experimental values of A and B. Some hyperfine constants were measured many years ago while others have yet to be measured. For example, the ground state hyperfine constants for europium-151 and europium-153 were measured for the first-time way back in 1960 by P.G.H. Sandars and G.K. Woodgate and published in the journal *Proceedings of the Royal Society A*.[1] Sandars and Woodgate found that the magnetic dipole hyperfine constant and the electric quadrupole hyperfine constant for the ground state of europium-151 is $A = -20.0523 \pm 0.0002$ MHz and $B = -0.7012 \pm 0.0035$ MHz. To convert those numbers into energy, just plug the hyperfine constants into the above equation and multiply the result by Planck's constant, h. If someone already measured those numbers, we can use those as a starting point for our fitting algorithms. If not, we have to determine them ourselves.

For every state in an atom, the electrons have different quantum numbers. States with higher n tend to be farther from the nucleus while the angular momentum quantum numbers represent different orbitals. Therefore, for an atom with nuclear spin, every state in that atom will have different hyperfine constants resulting in a different hyperfine splitting. Even the same state in two different isotopes that happen to have the same nuclear spin will have different hyperfine constants because the nuclei of the two isotopes are slightly different.

Summary
The "splitting" of the center of gravity into hyperfine levels is described by Eq. 9.5. The magnetic dipole hyperfine constant A is zero unless both $I > 0$ and $J > 0$. The magnetic quadrupole hyperfine constant B is zero unless both $I > 1/2$ and $J > 1/2$.

[1] How cool of a journal name is that?! See reference [2] for the full citation.

One Final Thing The magnetic dipole term (the term with A) in Eq. 9.5 tends to be larger than the electric quadrupole term (the term with B). For example, the $F = 6$ ground hyperfine state of europium-151 has a splitting $\Delta E = (-175.458\,\text{MHz}) + (-0.175\,\text{MHz}) = -175.633\,\text{MHz}$. The first term in parentheses is from the magnetic dipole term and the second is from electric quadrupole term. There are additional terms to Eq. 9.5, but they are very small compared to the electric quadrupole term. The next term in the formula is the magnetic octupole term, which contains quantum numbers and the magnetic octupole hyperfine constant C. This term is generally unnecessary unless you have exceptionally good data. The magnetic octupole constant is zero unless both $I > 1$ and $J > 1$.

9.3 Transition Frequencies

Suppose we have an atom with a nuclear spin quantum number of $I = 3/2$. To help distinguish between the lower and upper states, we will use primes on the quantum numbers for the excited states. In this example, the lower state has a total angular momentum quantum number of $J = 1/2$ and the upper state has $J' = 1/2$. Our goal is to write an equation for the transition frequency between two hyperfine levels. We first need to find the possible values for F, which can range from $3/2 + 1/2 = 2$ to $|3/2 - 1/2| = 1$ in integer steps, giving $F = 1$ and $F = 2$. Since $J' = 1/2$ as well, the possible values for F' are $F' = 1$ and $F' = 2$, see Fig. 9.3. In this hypothetical example, the $F = 1$ hyperfine level has a smaller energy than the $F = 2$ hyperfine level while the order is reversed in the excited state; the ordering of the quantum number all depends upon the interaction with the nucleus.

Next, we want to find the transition frequency from the $F = 2$ hyperfine level to the $F' = 1$ hyperfine level. Using Eq. 9.5, we can calculate the hyperfine energy splitting. We will use "LS" for lower state and "US" for upper state. Note that since $J = 1/2$ and $J' = 1/2$, both $B_{\text{LS}} = 0$ and $B_{\text{US}} = 0$. Evaluating Eq. 9.5 with the given quantum numbers, we find $\Delta E_{\text{LS},F=2} = \frac{3}{4}A_{\text{LS}}$ and $\Delta E_{\text{US},F'=1} = -\frac{5}{4}A_{\text{US}}$. According to Fig. 9.3, the $F = 2$ state has more energy than the center of gravity for the lower level, meaning $\Delta E_{\text{LS},F=2} > 0$. Since $\Delta E_{\text{LS},F=2} = \frac{3}{4}A_{\text{LS}}$, we also learn that $A_{\text{LS}} > 0$. For the upper state, the hyperfine energy splitting is also positive.

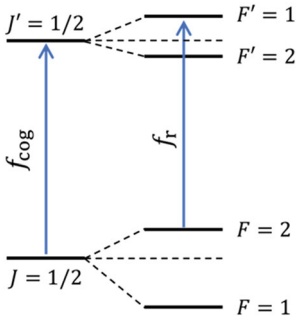


Fig. 9.3 A simple Grotrian diagram for a made-up atom

For this state, the formula is $\Delta E_{\mathrm{US},F'=1} = -\frac{5}{4}A_{\mathrm{US}}$, which implies that $A_{\mathrm{US}} < 0$ in order to make $\Delta E_{\mathrm{US},F'=1} > 0$.

Quick Quiz
Suppose an electron in our made up atom is in the $F = 2$ lower hyperfine level. What is the formula to calculate the transition frequency from the $F = 2$ lower hyperfine level to the $F' = 1$ upper hyperfine level? Your answer should contain the center of gravity frequency f_{cog} and the two magnetic dipole hyperfine constants A_{LS} and A_{US}. The answer is in the footnotes.[2]

The general formula to calculate the transition frequency between two hyperfine levels is

$$f_r = f_{\mathrm{cog}} - \Delta E_{\mathrm{LS}}(A_{\mathrm{LS}}, B_{\mathrm{LS}}) + \Delta E_{\mathrm{US}}(A_{\mathrm{US}}, B_{\mathrm{US}}). \tag{9.6}$$

For many learners, the signs of the shifts can be confusing, so let's explore the signs with our toy example. Let's start with the minus sign in front of $\Delta E_{\mathrm{LS}}(A_{\mathrm{LS}}, B_{\mathrm{LS}})$. According to Fig. 9.3, the $F = 2$ hyperfine state reduces the transition frequency compared to f_{cog}. Since $\Delta E_{\mathrm{LS}}(A_{\mathrm{LS}}, B_{\mathrm{LS}}) > 0$ for that state, the minus sign makes sense. What about if we wanted the transition frequency from the $F = 1$ lower hyperfine level? For that state, $\Delta E_{\mathrm{LS}}(A_{\mathrm{LS}}, B_{\mathrm{LS}}) < 0$, so that minus sign turns into a plus sign, increasing the frequency needed compared to f_{cog}. This is a subtle but important point, so take some time to convince yourself the transition frequency from any lower hyperfine level to any upper hyperfine level is given by Eq. 9.6. Using this same logic, convince yourself that the plus sign in front of $\Delta E_{\mathrm{US}}(A_{\mathrm{US}}, B_{\mathrm{US}})$ makes sense.

9.4 Example with Cesium-133

Figure 9.4 shows a spectrum from a paper we published in 2018,[3] in which we performed saturated absorption spectroscopy on cesium-133. Cesium-133 has a nuclear spin quantum number of $I = 7/2$. The first five shells of cesium are completely filled, leaving a single electron in the 6s subshell. The ground state has the term symbol $^2\mathrm{S}_{1/2}$ while the excited state has a term symbol $^2\mathrm{P}_{3/2}$. So, the ground state has two hyperfine levels, $F = 3$ and $F = 4$, while the excited state has four hyperfine levels, $F' = 2$, 3, 4, and 5. The $F = 3$ and $F = 4$ ground hyperfine levels are well separated from each other (~9.192 GHz), which is much larger than

[2] $f_{F=2\to F'=1} = f_{\mathrm{cog}} - \frac{3}{4}A_{\mathrm{LS}} - \frac{5}{4}A_{\mathrm{US}}$.
[3] Reference [1].

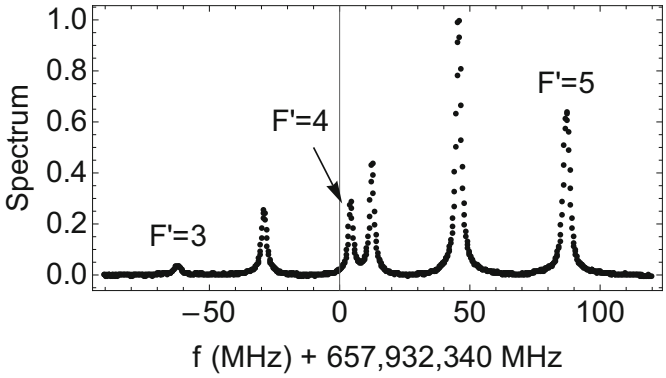


Fig. 9.4 Experimental spectroscopic results of the 6s $^2S_{1/2}$ $F = 4 \rightarrow$ 7p $^2P_{3/2}$ transition in neutral cesium-133. This is an experimental result from my research group, see reference [1]. A simplified Grotrian diagram for the transition can be found in Fig. 5.12

the Doppler width for transitions from either state. Therefore, we will not have any Λ crossovers despite having two lower states.

We performed spectroscopy from the $F = 4$ ground hyperfine level, which has an energy approximately 4.021 GHz above the center of gravity for the ground state. Given The Rule (Eq. 3.15) $\Delta F = -1$, 0, or 1 with the exception $F = 0 \not\rightarrow F = 0$, we can excite an atom from the ground state hyperfine level with $F = 4$ to the excited state hyperfine levels with $F' = 3$, 4, or 5.[4] Each of these three real transitions has a Lorentzian lineshape. We also have three V crossovers. For a review of crossovers, see Chap. 5, Sect. 5.2. The first crossover comes from the two real transitions $F = 4 \rightarrow F' = 3$ and $F = 4 \rightarrow F' = 4$. We often write this crossover in shorthand form: $F = 4 \rightarrow F' = 3/4$. The other two crossovers are the $F = 4 \rightarrow F' = 4/5$ and $F = 4 \rightarrow F' = 3/5$ crossovers. Despite having 6 spectroscopic features, the frequencies of all 6 features depend on only five parameters: the two hyperfine constants for the ground state, the two hyperfine constants for the excited state, and the transition frequency between the center of gravity for the ground state and the center of gravity for the excited state, see Eq. 9.6. I should point out that the width and amplitude of each peak are also free parameters, but we didn't really care about those. Our main goal was to measure the center of gravity frequency and the hyperfine constants. Having 5 parameters to determine the position of all spectroscopic features is typical. The experimental data from our group on a transition in europium-151 had up to 77 spectral features in that spectrum![5] Even still, the center of each feature is determined by only 5 free parameters.

[4] A full list of the rules that need to be satisfied for an electron to transition between two atomic states is given in Appendix C.

[5] See Reference [4] for the full citation.

Let's apply Eq. 9.6 using numbers from cesium-133. The spectrum in Fig. 9.4 is from the $J = 1/2$, $F = 4$ ground state. The excited state has $J' = 3/2$ and $F' = 3$, 4, and 5.[6] Let's find ΔE for all of the features.

9.4.1 Real Transitions

There are three real transitions in the spectrum. They are

1. $F = 4 \rightarrow F' = 3$
2. $F = 4 \rightarrow F' = 4$
3. $F = 4 \rightarrow F' = 5$

Let's calculate the shift from the center of gravity for the 7p $^2P_{3/2}$ state for each real transition. Since these are shifts for the excited state, we will use the primed values in Eq. 9.5. We will also need Eq. 9.6 to find the transition frequency between two hyperfine levels. Since the hyperfine splitting for the ground state is so large, we will perform our spectroscopy with respect to the $F = 4$ ground hyperfine level. For ease of reference, here are those equations:

$$\begin{aligned} \Delta E &= \tfrac{1}{2} K A_{\text{US}} + \frac{\frac{3}{2}K(K+1) - 2I(I+1)J'(J'+1)}{2I(2I-1)2J'(2J'-1)} B_{\text{US}} \\ K &= F'(F'+1) - I(I+1) - J'(J'+1) \end{aligned} \tag{9.7}$$

$$f_r = f_{F=4 \rightarrow \text{cog}} + \Delta E_{\text{US}}(A_{\text{US}}, B_{\text{US}})$$

where $f_{F=4 \rightarrow \text{cog}} = f_{\text{cog}} - \Delta E_{\text{LS,F=4}}$ is the frequency of light needed to go from the $F = 4$ ground hyperfine level to the center of gravity of the 7p $^2P_{3/2}$ state. I also replaced the quantum numbers with primed values and the hyperfine constants with "US" subscripts.

Let's do the math for the $F = 4 \rightarrow F' = 3$ real transition. First, let's find K:

$$\begin{aligned} K &= F'(F'+1) - I(I+1) - J'(J'+1) \\ &= 3(3+1) - \tfrac{7}{2}(\tfrac{7}{2}+1) - \tfrac{3}{2}(\tfrac{3}{2}+1) \\ &= -\tfrac{15}{2} \end{aligned}$$

Now we can find the hyperfine splitting for the $F' = 3$ state from the center of gravity for the 7p $^2P_{3/2}$ state.

[6] There is also an $F' = 2$ excited hyperfine level, but we cannot excite an atom from the $F = 4$ ground hyperfine level to this state.

Table 9.2 The energy shifts for the hyperfine levels with respect to the center of gravity for the $7p\ ^2P_{3/2}$ state in cesium-133. All of the real transitions in this example are from the $6s\ ^2S_{1/2}\ F = 4$ ground hyperfine level

Transitions	$K = F'(F'+1) - I(I+1) - J'(J'+1)$	ΔE
$F = 4 \rightarrow F' = 3$	$3(3+1) - \frac{7}{2}(\frac{7}{2}+1) - \frac{3}{2}(\frac{3}{2}+1) = -\frac{15}{2}$	$-\frac{15}{4}A_{\mathrm{US}} - \frac{5}{28}B_{\mathrm{US}}$
$F = 4 \rightarrow F' = 4$	$4(4+1) - \frac{7}{2}(\frac{7}{2}+1) - \frac{3}{2}(\frac{3}{2}+1) = +\frac{1}{2}$	$+\frac{1}{4}A_{\mathrm{US}} - \frac{13}{28}B_{\mathrm{US}}$
$F = 4 \rightarrow F' = 5$	$5(5+1) - \frac{7}{2}(\frac{7}{2}+1) - \frac{3}{2}(\frac{3}{2}+1) = +\frac{21}{2}$	$+\frac{21}{4}A_{\mathrm{US}} + \frac{1}{4}B_{\mathrm{US}}$

$$
\begin{aligned}
\Delta E &= \tfrac{1}{2}KA_{\mathrm{US}} + \frac{\frac{3}{2}K(K+1)-2I(I+1)J'(J'+1)}{2I(2I-1)2J'(2J'-1)}B_{\mathrm{US}} \\
&= \tfrac{1}{2}\left(-\tfrac{15}{2}\right)A_{\mathrm{US}} + \frac{\frac{3}{2}\left(-\frac{15}{2}\right)\left(\left(-\frac{15}{2}\right)+1\right)-2\left(\frac{7}{2}\right)\left(\frac{7}{2}+1\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}+1\right)}{2\left(\frac{7}{2}\right)\left(2\left(\frac{7}{2}\right)-1\right)2\left(\frac{3}{2}\right)\left(2\left(\frac{3}{2}\right)-1\right)}B_{\mathrm{US}} \\
&= -\tfrac{15}{4}A_{\mathrm{US}} + \frac{\left(-\frac{45}{4}\right)\left(-\frac{13}{2}\right)-7\left(\frac{9}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{7(6)(3)(2)}B_{\mathrm{US}} \\
&= -\tfrac{15}{4}A_{\mathrm{US}} + \frac{\left(\frac{585}{8}\right)-\left(\frac{945}{8}\right)}{252}B_{\mathrm{US}} \\
&= -\tfrac{15}{4}A_{\mathrm{US}} + \frac{\left(-\frac{360}{8}\right)}{252}B_{\mathrm{US}} \\
&= -\tfrac{15}{4}A_{\mathrm{US}} + \left(\frac{-45}{252}\right)B_{\mathrm{US}} \\
&= -\tfrac{15}{4}A_{\mathrm{US}} - \tfrac{5}{28}B_{\mathrm{US}}
\end{aligned}
$$

Table 9.2 shows the results of the math. The hyperfine constants for the $7p\ ^2P_{3/2}$ state are kept as unknowns that we find while fitting the data. If there weren't any crossovers, we would be done. Our fitting function would be the sum of three Lorentzian functions[7] whose centers are taken from the above table:

$$
\begin{aligned}
g(f) = &\frac{C_1}{1+\frac{\left(f-(f_{F=4\rightarrow\mathrm{cog}}-\frac{15}{4}A_{\mathrm{US}}-\frac{5}{28}B_{\mathrm{US}})\right)^2}{\gamma_1^2}} + \frac{C_2}{1+\frac{\left(f-(f_{F=4\rightarrow\mathrm{cog}}+\frac{1}{4}A_{\mathrm{US}}-\frac{13}{28}B_{\mathrm{US}})\right)^2}{\gamma_2^2}} \\
&+\frac{C_3}{1+\frac{\left(f-(f_{F=4\rightarrow\mathrm{cog}}+\frac{21}{4}A_{\mathrm{US}}+\frac{1}{4}B_{\mathrm{US}})\right)^2}{\gamma_3^2}}.
\end{aligned}
\tag{9.8}
$$

where $f_{F=4\rightarrow\mathrm{cog}}$ is the frequency of light needed to go from the $F = 4$ ground hyperfine level to the center of gravity of the $7p\ ^2P_{3/2}$ state. To avoid accidentally mixing up the magnetic dipole hyperfine constant of the $7p\ ^2P_{3/2}$ state and the amplitude of the Lorentzian functions, I changed the variable for amplitude to C.

[7] There are many subtleties that we are glossing over here. Spectra sometimes have an offset, a sloped offset, or a Gaussian pedestal for various reasons. When you fit data, sometimes you have to add something to your fit function. However, you should always have a reason for why you are adding something to your fit function.

If we wanted to write the transition frequencies in terms of the actual center of gravity frequency, we would replace $f_{F=4\rightarrow \text{cog}}$ with $f_{\text{cog}} - \Delta E_{LS,F=4}$.

9.4.2 Crossover Transitions

For this spectrum, there are three V crossovers: $F = 4 \rightarrow F' = 3/4$, $F = 4 \rightarrow F' = 4/5$, and $F = 4 \rightarrow F' = 3/5$. The center of the spectral feature due to the $F = 4 \rightarrow F' = 3/4$ crossover is found using the same procedures we learned about in Chap. 5, which is adding the two real transitions and dividing by 2:

$$\begin{aligned}\Delta E_{F'=3/4} = \frac{\Delta E_{F'=3} + \Delta E_{F'=4}}{2} &= \frac{\left(-\frac{15}{4}A_{\text{US}} - \frac{5}{28}B_{\text{US}}\right) + \left(\frac{1}{4}A_{\text{US}} - \frac{13}{28}B_{\text{US}}\right)}{2} \\ &= -\frac{7}{4}A_{\text{US}} - \frac{9}{28}B_{\text{US}}\end{aligned} \tag{9.9}$$

The same procedure can be done for the other two crossovers. In the end, the fit function is a sum of 6 Lorentzian functions. From the fit, we find A_{US}, B_{US}, and $f_{F=4\rightarrow \text{cog}}$, all with uncertainty. You will get to practice this in a homework problem.

9.5 Optional: Amplitudes

A common question I am asked is if there is a way to calculate the amplitudes of the Lorentzian functions. Finding the absolute amplitude is possible, but quite difficult. However, if the spectral features aren't overlapping (we often use the phrase "well-separated features"), we can find the relative amplitudes for the real transitions. It uses a mathematical symbol you may not have seen before, but you should think of it as a placeholder for a lot of hidden algebra. Not many people do the hidden algebra themselves—that's what Mathematica is for—but if you search the internet for "Wigner 6-j," you can find all the math behind the symbol. The scaled amplitudes of the real transitions are given by the following formula:

$$I_r = (2F+1)(2F'+1)\begin{Bmatrix} J & I & F \\ F' & 1 & J' \end{Bmatrix}^2, \tag{9.10}$$

where the primed quantum numbers are for the excited state. That last symbol is called the Wigner 6-j symbol (it is squared in the above equation). Also, the second element of the second row is the number 1 (lots of folks accidentally read that as the nuclear spin quantum number I). In Mathematica, you would type SixJSymbol{J,I,F},{Fp,1,Jp}^{2} (Mathematica doesn't allow J' or F' as variable names, so I replaced them with Jp and Fp). There are also online calculators you can find by searching the internet for "Wigner 6-j symbol calculator". Notice this formula has no units. We use this formula to take ratios to find relative amplitudes.

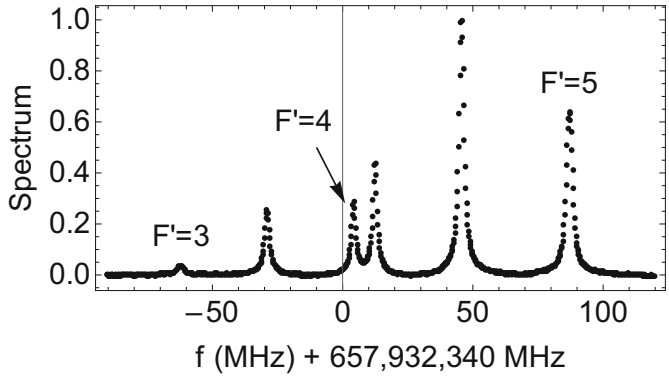


Fig. 9.5 Experimental spectroscopic results of the 6s $^2S_{1/2}$ $F = 4 \rightarrow$ 7p $^2P_{3/2}$ transition in neutral cesium-133. This is an experimental result from my research group. A simplified Grotrian diagram for the transition can be found in Fig. 5.12

Let's find I_r for all three real transitions in the cesium-133 example we have been studying. To make things a bit easier, I included a copy of Fig. 9.4 on this page. As a reminder, the ground state has quantum numbers $J = 1/2$ and $F = 4$, the excited state has $J' = 3/2$ and $F' = 3,\ 4,$ and 5, and the nuclear spin is $I = 7/2$.

$$1.\ F = 4 \rightarrow F' = 3:\ I_r = (2(4)+1)(2(3)+1)\begin{Bmatrix} \frac{1}{2} & \frac{7}{2} & 4 \\ 3 & 1 & \frac{3}{2} \end{Bmatrix}^2 = \frac{7}{16} = 0.4375$$

$$2.\ F = 4 \rightarrow F' = 4:\ I_r = (2(4)+1)(2(4)+1)\begin{Bmatrix} \frac{1}{2} & \frac{7}{2} & 4 \\ 4 & 1 & \frac{3}{2} \end{Bmatrix}^2 = \frac{21}{16} = 1.3125$$

$$3.\ F = 4 \rightarrow F' = 5:\ I_r = (2(4)+1)(2(4)+1)\begin{Bmatrix} \frac{1}{2} & \frac{7}{2} & 4 \\ 5 & 1 & \frac{3}{2} \end{Bmatrix}^2 = \frac{11}{4} = 2.75$$

From this math we see that the $F = 4 \rightarrow F' = 5$ is the largest transition. Using this largest transition as the reference, we can compare the size of the other two transitions to it. The $F = 4 \rightarrow F' = 4$ transition, which is the peak around 5 MHz in Fig. 9.5, is $\frac{21/16}{11/4} = \frac{21}{44} = 0.47$ times smaller than the $F = 4 \rightarrow F' = 5$ transition. Finally, the $F = 4 \rightarrow F' = 3$ transition is $\frac{7/16}{11/4} = \frac{7}{44} = 0.16$ times smaller than the $F = 4 \rightarrow F' = 5$ transition. If you look at the amplitudes of the real transitions, you'll see that these estimates are pretty close to what was measured experimentally.

While it is possible to calculate the amplitudes of the crossovers, it is much more complicated. One reason is that the number of atoms with a particular velocity, which creates the crossovers, depends on the temperature of your vapor cell. However, we at least have a fairly straightforward way of finding the relative amplitudes of the real transitions.

9.6 Example with Oxygen-16

Oxygen-16 has no nuclear spin, so there is no hyperfine structure. Shortest section ever!

9.7 Example with Oxygen-17

Oxygen-17 has a nuclear spin quantum number of $I = 5/2$, so there will be hyperfine structure. Below is a table of the seven lowest energy states for atomic oxygen.

Electron configuration	Term symbol	Energy (cm^{-1})	f (Hz)
$1s^2 2s^2 2p^4$	$^3P_2$	0	
$1s^2 2s^2 2p^4$	$^3P_1$	158.265	4.74×10^{12}
$1s^2 2s^2 2p^4$	$^3P_0$	226.977	6.80×10^{12}
$1s^2 2s^2 2p^4$	$^1D_2$	15,867.862	4.76×10^{14}
$1s^2 2s^2 2p^4$	$^1S_0$	33,792.583	1.01×10^{15}
$1s^2 2s^2 2p^3 3s$	$^5S_2$	73,768.200	2.21×10^{15}
$1s^2 2s^2 2p^3 3s$	$^3S_1$	76,794.978	2.30×10^{15}

For oxygen isotopes with no nuclear spin, such as oxygen-16, this table provides all the information about their atomic states. There is a single ground state with electronic configuration and term symbol $1s^2 2s^2 2p^4\ ^3P_2$. However, due to hyperfine structure, the table does not tell the whole story for oxygen-17. Let's examine the ground state of oxygen-17. For this state, the total electronic angular momentum quantum number is $J = 2$ and the nuclear spin quantum number is $I = 5/2$. The possible values for the total atomic angular momentum quantum number are:

$$F = \left(2 + \frac{5}{2}\right), \ldots, \left|2 - \frac{5}{2}\right| = \frac{9}{2}, \frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2} \tag{9.11}$$

We can also calculate the energy splitting using the hyperfine splitting formula. For convenience, here is the hyperfine splitting formula:

$$\begin{gathered} \Delta E = \tfrac{1}{2} K A + \frac{\frac{3}{2} K(K+1) - 2I(I+1)J(J+1)}{2I(2I-1)2J(2J-1)} B \\ K = F(F+1) - I(I+1) - J(J+1) \\ A = 0 \text{ unless both } I > 0 \text{ and } J > 0 \\ B = 0 \text{ unless both } I > 1/2 \text{ and } J > 1/2. \end{gathered} \tag{9.12}$$

The first thing to notice is that for the $^3P_2$ ground state, both $I > 1/2$ and $J > 1/2$, so we will need both the magnetic dipole term (the term with A) and the electric

quadrupole term (the term with B). For example, the hyperfine splitting for the $F = 9/2$ hyperfine level is:

$$K = F(F+1) - I(I+1) - J(J+1) = \frac{9}{2}\left(\frac{11}{2}\right) - \frac{5}{2}\left(\frac{7}{2}\right) - 2(3) = 10$$

$$\Delta E_{F=9/2} = 5A_{\text{gs}} + \frac{1}{4}B_{\text{gs}} \tag{9.13}$$

where A_{gs} and B_{gs} are the hyperfine constants for the $^3\text{P}_2$ ground state. Each energy level in the above table will have different hyperfine constants.

Next, let's explore the $^3\text{P}_0$ state, which has an energy of 226.977 cm^{-1} above the ground state and a total electronic angular momentum quantum number of $J = 0$. Using the rule to find F, we find a single possible value: $F = (J+I), \ldots, |J-I| = (0+I), \ldots, |0-I| = I \rightarrow F = I = 5/2$. This only happens when a state has quantum number $J = 0$. With the constraint $F = I$, we find

$$\begin{aligned} K &= F(F+1) - I(I+1) - J(J+1) \\ &= I(I+1) - I(I+1) - 0(0+1) \\ &= 0 \\ \rightarrow \Delta E &= 0 \end{aligned} \tag{9.14}$$

Even though the nucleus has angular momentum, there is no energy shift when $J = 0$. This hyperfine level has the same energy as the center of gravity. However, we will still label this state with the F quantum number: $1s^2 2s^2 2p^4\ ^3\text{P}_0\ F = \frac{5}{2}$. You will explore the hyperfine structure of oxygen-17 more in Problem 9.3.

Problems

9.1 Table 9.2 shows the energy shifts for the hyperfine levels with respect to the center of gravity for the 7p $^2\text{P}_{3/2}$ state in cesium-133. All of the real transitions in this problem are from the 6s $^2\text{S}_{1/2}$ $F = 4$ ground hyperfine level.

(a) In Sect. 9.4.1, we found ΔE for the $F = 4 \rightarrow F' = 3$ real transition. Confirm ΔE for the other two real transitions.
(b) Find ΔE for the three V crossovers, similar to what we did in Sect. 9.4.2.

9.2 Below is table for four states in europium.[8] The first row represents the ground state, and the three subsequent rows are excited states. Europium has two stable isotopes, europium-151 and europium-153.

[8] Ground state citation: Reference [2]
[8] $^8\text{P}_{5/2}$ and $^8\text{P}_{7/2}$ citation: Reference [3]
[8] $^8\text{P}_{9/2}$ citation: Reference [4].

Electron configuration	A_{151} (MHz)	B_{151} (MHz)	A_{153} (MHz)	B_{153} (MHz)
$4f^7 6s^2\ {}^8S_{7/2}$	−20.0523±0.0002	−0.7012±0.0035	−8.8532±0.0002	−1.7852±0.0035
$4f^7 6s6p\ {}^8P_{5/2}$	−157.01±0.03	74.5±0.4	−69.43±0.14	191±2.6
$4f^7 6s6p\ {}^8P_{7/2}$	−218.66±0.04	−293.4±0.8	−97.15±0.13	−750±3
$4f^7 6s6p\ {}^8P_{9/2}$	−228.84±0.02	226.9±0.5	−101.87±0.06	575.4±1.5

(a) Select either the 151 or 153 isotope and one of the three excited states from the table above. Use the largest to smallest in integer steps rule to find the possible F values for the ground state and your chosen excited state.
(b) Using the hyperfine splitting equation, calculate the energy splitting for the ground state. Don't worry about the uncertainties.
(c) Using the hyperfine splitting equation, calculate the energy splitting for the excited state. Don't worry about the uncertainties.
(d) Determine the transition frequency from a single ground hyperfine level (your choice) to a single excited state hyperfine level (your choice). Report your answer in MHz. Your final answer should include f_{cog} in it.
Note: f_{cog} is different for the two isotopes. This small difference is called an isotope shift, which we will explore in Chap. 10.

9.3 Oxygen-17 has a nuclear spin quantum number of $I = 5/2$, resulting in hyperfine structure. Sect. 9.7 contains a table that lists the seven lowest energy states.

(a) Find the possible F quantum numbers for all seven states. Hint: There are only three calculations here!
(b) For the $1s^2 2s^2 2p^4\ {}^3P_1$ state, calculate the hyperfine energy shift ΔE for each hyperfine level in terms of the hyperfine constants. Label your hyperfine constants A_{3P1} and B_{3P1}.
(c) Suppose you were to excite an oxygen atom from the $1s^2 2s^2 2p^4\ {}^3P_1$ $F = 7/2$ state to the $1s^2 2s^2 2p^3 3s\ {}^3S_1$ $F' = 5/2$ state. What is the transition frequency? Your answer should look something like $f = f_{\text{cog}} + \#A_{3P1} + \#B_{3P1} + \#A_{3S1} + \#B_{3S1}$.
(d) Suppose you performed saturated absorption spectroscopy across all hyperfine levels for the $1s^2 2s^2 2p^4\ {}^3P_1 \rightarrow 1s^2 2s^2 2p^3 3s\ {}^3S_1$ transition. Write the fit function for this spectrum, assuming we have no crossovers.
(e) Challenge: Include crossovers!
(f) Optional: Find the relative intensities of all the real hyperfine transitions for the spectrum in part (d). The relative intensities should be with respect to the largest amplitude transition.

9.4 Consider the $3s\ {}^2S_{1/2} \rightarrow 3p\ {}^2P^{\circ}_{3/2}$ transition in sodium-23. The little circle on the excited state term symbol is the parity of the state, a topic we don't cover

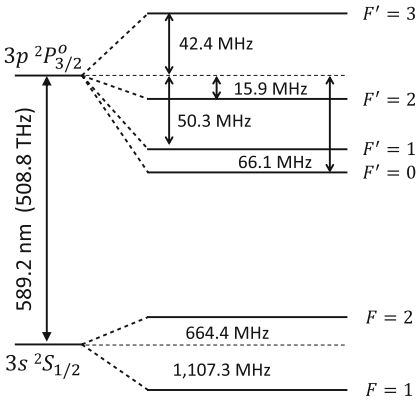


Fig. 9.6 A Grotrian diagram for a transition in sodium-23

in this book. Parity is an advanced topic that is easy to say in words, but hard to understand.[9] Fig. 9.6 is a Grotrian diagram that shows the hyperfine structure of both states. We will call the hyperfine constants for the ground state A_{2S} and B_{2S} and the hyperfine constants for the excited state A_{2P} and B_{2P}.

(a) What is the nuclear spin quantum number for sodium-23?
(b) What is B_{2S}?
(c) Write the formula for the transition frequency between the 3s $^2\mathrm{S}_{1/2}$ $F = 1$ state and the 3p $^2\mathrm{P}_{3/2}$ $F' = 2$ state. Leave the hyperfine constants as A_{2S}, A_{2P}, and B_{2P}. You will find their values in parts (d) and (e).
(d) Determine the hyperfine constants for the 3s $^2\mathrm{S}_{1/2}$ state.
(e) Determine the hyperfine constants for the 3p $^2\mathrm{P}_{3/2}$ state.
(f) The center of gravity frequency is $f_{\mathrm{cog}} = 508,848,717.1\,\mathrm{MHz}$. Determine the numerical value for the transition frequency for the 3s $^2\mathrm{S}_{1/2}$ $F = 1 \rightarrow$ 3p $^2\mathrm{P}_{3/2}$ $F' = 2$ transition.
(g) Optional: Find the relative amplitudes of the real transitions between the 3s $^2\mathrm{S}_{1/2}$ $F = 2$ state and the allowed excited states.

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[9] You should, of course, do an internet search for "parity physics" if you'd like to learn more.

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Isotope Shifts, Radioactive Decay, and the Nuclear Forces

10

Abstract

In this chapter, we explore the nucleus, focusing on how the number of neutrons in a nucleus influences transition frequencies and the stability of atoms. We begin by examining the effect of neutrons on transition frequencies, then shift into detailed discussions of the nuclear forces and principles governing atomic stability and radioactive decay. Various modes of radioactive decay, including neutron and proton emission, α decay, spontaneous fission, β^- decay, β^+ decay, and electron capture, are investigated, with an emphasis on the conditions under which each occurs. Additionally, we explore the nuclear shell model to understand the energetics behind different types of decays and the stability of isotopes.

Learning Goals

By the end of this chapter, you should be able to understand:

- that the number of neutrons in the nucleus can
  - cause a small shift in the spectrum,
  - make an isotope unstable.
- that unstable atoms undergo radioactive decay characterized by a time called the half-life.
- that a system always seeks to reach its lowest energy state.
- that the daughter system of a decay has lower energy than the parent system.
- that the table of isotopes is similar to the periodic table but includes all isotopes.
- that the strong nuclear force is the force that holds the nucleus together.

W. Raven, *Atomic Physics for Everyone*,
https://doi.org/10.1007/978-3-031-69507-0_10

- that the weak nuclear force is the force that facilitates the annihilation and creation of particles during radioactive decay.
- the nuclear shell model and the energy arguments behind radioactive decay.

10.1 Isotope Shifts

Definitions:
- **Mass number:** The number of neutrons plus protons in an element, represented by the variable A.
- **Nucleon:** A general word for either a neutron or a proton.
- **Isotope shift:** The change in the resonance frequency of a transition between two states in an atom caused by changing the number of neutrons in the nucleus.

Different isotopes of a particular element have different numbers of neutrons. For example, europium-151 has 63 protons and 88 neutrons. The '151' in the name of the isotope indicates the total number of protons and neutrons in the nucleus: $63 + 88 = 151$. Europium-153 has 63 protons and 90 neutrons: $63 + 90 = 153$. The element name europium is just a placeholder for 'the atom with 63 protons.' To display all the information at once, we often write an isotope as follows:

$$^{A}_{Z}\mathrm{X}_{N} \tag{10.1}$$

where Z is the proton number, N is the neutron number, $A = Z + N$ is the mass number, and X is the element symbol. See Appendix B for a full list of element symbols. For example, europium-153 would be written as $^{153}_{63}\mathrm{Eu}_{90}$.

Quick Quiz
Use Appendices A or B to find the proton number for the following atoms:

1. How many neutrons does oxygen-17 have?
2. How many neutrons does hydrogen-2 have?
3. How many neutrons does beryllium-9 have?

The answers are in the footnotes.[1]

[1] $A = Z + N \rightarrow N = A - Z$:

1. Oxygen-17 has 8 protons, so $17 - 8 = 9$ neutrons
2. Hydrogen-2 has 1 proton, so $2 - 1 = 1$ neutron
3. Beryllium-9 has 4 protons, so $9 - 4 = 5$ neutrons

The number of neutrons in the nucleus slightly affects the energy of the states in an atom. Neutrons affect the energy of the states in two ways: they change the mass of the nucleus and slightly alter the charge distribution within the nucleus (how the protons are distributed in the nucleus). For a given isotope, the energy of each state in the atom will shift a different amount because each state has a different electron configuration and term symbol (i.e. different angular momentum and different distance from the nucleus). To study how changing the number of neutrons affects the energy of a state, we typically select the same transition and measure how the transition frequency changes as a function of neutron number. Since the energy of each state in an atom shifts a different amount, the transition frequency between those two states also changes. That shift is called the isotope shift.

Table 10.1 shows experimental results from my research group for three transitions for two different isotopes of europium. The transitions are from the $4f^7 6s^2\ {}^8S_{7/2}$ ground state to the $4f^7({}^8S^\circ)6s6p({}^1P^\circ)\ {}^8P_J$ states, where $J = 5/2$, $7/2$, or $9/2$. In the above electronic configurations, I did not write out the closed subshells (e.g., $1s^2 2s^2 \ldots$). Notice that adding two neutrons to the nucleus decreased the transition frequency for all three transitions. Because the two isotopes have a different number of neutrons, all of the states in europium-151 have a different energy than the states in europium-153. The difference in energy between the ground states of the two isotopes is different from the difference in energy between, say, the $J = 5/2$ excited states. Therefore, the transition frequency from the ground state to the $J = 5/2$ excited state is slightly different for the two isotopes; this is the isotope shift for this transition. Similarly, the difference in energy between the $J = 7/2$ excited states for the two isotopes is different from the difference in energy between the $J = 5/2$ excited states, so the isotope shift for this transition is different from the isotope shift for the $J = 7/2$ state. In the end, every transition in the atom will have a different isotope shift.

▶ **Important Comment** Isotope shifts are measured with respect to the center of gravity of a state. This is because each isotope for an element can have a different nuclear spin. Studying the effects of neutron number is much more convenient after removing hyperfine splitting.

Table 10.1 The isotope shifts for three transitions in the europium atom. All of the transitions are from the atomic ground state. The labels for the excited state column are the total electronic angular momentum quantum number J for the excited state: $4f^7({}^8S^\circ)6s6p({}^1P^\circ){}^8P_J$. f_r is the frequency for the transition between the center of gravity of the ground state and the center of gravity for an excited state. The numbers come from references [3] and [4]

Excited state	f_r (MHz) for ${}^{151}_{63}Eu_{88}$	f_r (MHz) for ${}^{153}_{63}Eu_{90}$ (MHz)	Isotope Shift (MHz)
$J = 9/2$	652,389,757.16±0.34	652,386,593.2±0.5	3163.8±0.6
$J = 7/2$	647,708,930.6±0.6	647,705,958.4±2.6	2972.8±0.5
$J = 5/2$	642,894,493.3±0.4	642,891,693.3±0.9	2799.54±0.20

> We accomplish this using Eq. 9.5. When you read studies of isotope shifts from scientific papers, the numbers you are reading have already accounted for hyperfine structure.

While experimentalists measure the frequency difference between isotopes, mathematically we break down the isotope shift into three expressions: the normal mass shift $\delta f_{\mathrm{NMS}}^{AA'}$, the specific mass shift $\delta f_{\mathrm{SMS}}^{AA'}$, and the field shift $\delta f_{\mathrm{FS}}^{AA'}$. The formula for the isotope shift is:

$$\delta f^{AA'} = \delta f_{\mathrm{NMS}}^{AA'} + \delta f_{\mathrm{SMS}}^{AA'} + \delta f_{\mathrm{FS}}^{AA'}, \tag{10.2}$$

where A is the mass number for one isotope and A' is the mass number of the other isotope. Both the normal mass shift and the specific mass shift come about because the two isotopes have different masses. The field shift comes about because the nuclei for the two isotopes have a slightly different size. The different sizes change the charge distribution (the distribution of protons) inside the nucleus and this leads to a small shift in the energy levels.

The easiest way to think about normal mass shift is a classical analogy. The moon orbiting the earth implies that the moon is orbiting about the center of the earth and the earth is not orbiting around the moon. This is a bit of a misnomer. The moon and the earth are actually orbiting around a point somewhere along a line between the center of the earth and the center of the moon. Because the earth has so much more mass than the moon, that point happens to be very close to the center of the earth. If the moon and the earth had the same mass, they would both be orbiting about a point directly between them. That point is called the center of mass. Analogously, the electron is not orbiting around the center of the nucleus. Instead, the nucleus and the electron are orbiting around the center of mass point. That point moves if we change the mass of the nucleus. Accounting for this shift in the center of mass point is the normal mass shift. The specific mass shift is present in atoms with more than 1 electron. It comes about because the electrons are moving and interacting with one another. Changing the mass of the nucleus has a small effect on those interactions.

Out of the 3 contributions, only the normal mass shift has a simple formula:

$$\delta f_{\mathrm{NMS}}^{AA'} = \frac{m_e}{m_p} \frac{A' - A}{AA'} f_r, \tag{10.3}$$

where A' is the mass number of an isotope (for example the 151 in europium-151), A is the mass number of a different isotope (for example the 153 in europium-153), $\delta f_{\mathrm{NMS}}^{AA'}$ is the isotope shift of the isotope with mass number A' with respect to the isotope with mass number A due to the normal mass shift, m_e is the mass of an electron, m_p is the mass of a proton, and f_r is the transition frequency from the

center of gravity from the lower energy state to the center of gravity of the higher energy state.[2]

Example
Rubidium, an element with 37 protons, has two naturally occurring isotopes: rubidium-85 and rubidium-87.[a] There is a transition from the ground state to an excited state with a center of gravity transition frequency of 377.107 THz. What is the normal mass shift between the two isotopes?

For this example, let's make $A' = 87$ and $A = 85$:

$$\begin{aligned}\delta f_{\text{NMS}}^{AA'} &= \frac{m_e}{m_p}\frac{A'-A}{AA'} f_r \\ &= \frac{9.11\times10^{-31}\,\text{kg}}{1.67\times10^{-27}\,\text{kg}}\frac{87-85}{(85)(87)}(377.107\times10^{12}\,\text{Hz}) \\ &= 55.6\,\text{MHz}\end{aligned} \tag{10.4}$$

Since we made $A' = 87$ and $A = 85$, a positive result means that rubidium-87 would have a larger transition frequency, at least before we account for the specific mass shift or the field shift.

[a] On earth, about 72% of rubidium is rubidium-85 and about 28% is rubidium-87.

The experimental value of the isotope shift between rubidium-87 and rubidium-85 for this transition is (77.583 ± 0.012) MHz, see reference [1]. We found the normal mass contribution was 55.6 MHz. The remaining shift comes from the other two contributions that are, unfortunately, difficult to calculate. However, we can try to extract information about the nucleus by measuring a large number of isotopes for a particular element.

To visually compare different isotopes, we can make a plot of the isotope shifts for a particular transition as a function of neutron number. This plot is called a King plot[3] and is a nice way to visualize how changing the number of neutrons in a nucleus affects the spectrum. Figure 10.1 shows an example of experimentally measured isotope shifts for a transition in the krypton atom that is excited using a laser near 760 nm. It was collected by Keim et al. in 1995, see reference [2]. All of

[2] If you are paying close attention, you might ask the question, "Does f_r refer to the transition frequency for the isotope with mass number A or A'?" The answer is neither. f_r is a calculated transition frequency assuming a nucleus with infinite mass. However, in practice, you can use f_r for either isotope or the calculated value for a nucleus with infinite mass. We can do this because isotope shifts are usually on the order of MHz to a few GHz, which is typically more than 100,000 times smaller than f_r. The formula to calculate the transition frequency assuming a nucleus with infinite mass is $\frac{m_e+M}{M} f_r$, where M is the mass of a nucleus for a particle isotope and f_r is the transition frequency for that isotope. That fraction is very close to 1.

[3] Named after the British physicist William H. King.

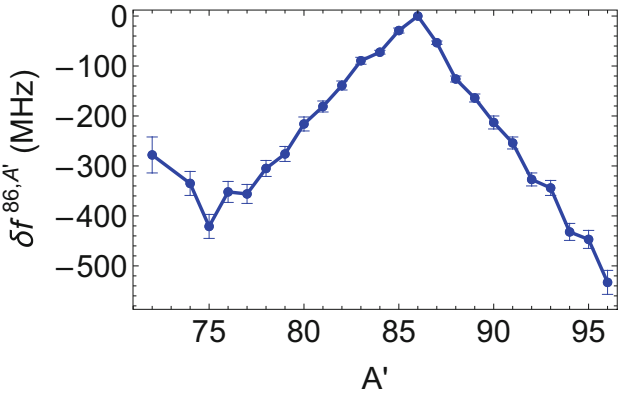

Fig. 10.1 The isotope shift of a transition in krypton near 760 nm using data from reference [2]. The isotope shifts are measured with respect to the isotope with mass number $A = 86$

the isotope shifts are measured with respect to krypton-86. Notice there is a "kink" in the graph at $A' = 86$. As physicists, we want to know why! If we truly understand the system, we should be able to calculate the isotope shifts using Eq. 10.2 and predict such a graph. The reason for the "kink" at $A' = 86$ is that nucleons fill shells in the nucleus just like electrons fill shells. We discuss nuclear shells in Sect. 10.6. There is a nuclear shell for neutrons that fills at 50 neutrons. Since krypton has 36 protons, this neutron shell fills at $A' = 86$. The "kink" at $A' = 75$ is not from a shell filling. That "kink" is thought to come from the nucleus itself beginning to deform due to lack of neutrons. There is so much one could ask and explore with this data. What questions would you ask? The wonderful thing about physics is that if we understand the system, we can interpret the data. If there is something we don't understand, this sort of data gives us a starting point to explore and learn more.

10.2 Radioactive Decay

Inside the nucleus are neutrons and protons. Experimental evidence shows that if the neutron-to-proton ratio isn't "correct," the nucleus becomes unstable and will undergo radioactive decay. Radioactive decay is often written in the form of an equation. The left hand side of the equation is the unstable isotope. We will call this the parent system. The right hand side shows all of the decay particles. We will call this the daughter system.[4] The equation looks something like this:

[4] Historically, parent and daughter are the names used to describe radioactive decay. The terminology also shows up in biology during cell division (parent cell and daughter cells). Sometimes the daughter system is called the decay system.

$$\text{parent} \rightarrow \text{daughter} \tag{10.5}$$

There are a number of decays that can happen:

Neutron Emission The nucleus will eject a neutron. An example of an isotope that undergoes neutron emission is helium-5 (2 protons and 3 neutrons), which decays to helium-4 (2 protons and 2 neutrons) and a neutron:

$$^{5}_{2}\mathrm{He}_3 \rightarrow {}^{4}_{2}\mathrm{He}_2 + \mathrm{n}$$

In the above equation, helium-5 is called the parent nucleus and helium-4 is called the daughter nucleus. The general formula for neutron emission is:

$$^{A}_{Z}\mathrm{X}_N \rightarrow {}^{A}_{Z}\mathrm{X}_{N-1} + \mathrm{n} \tag{10.6}$$

Proton Emission The nucleus will eject a proton. An example of an isotope that undergoes proton emission is scandium-39 (21 protons and 18 neutrons), which decays to calcium-38 (20 protons and 18 neutrons) and a proton:

$$^{39}_{21}\mathrm{Sc}_{18} \rightarrow {}^{38}_{20}\mathrm{Ca}_{18} + \mathrm{p}$$

In the above equation, scandium-39 is called the parent nucleus and calcium-38 is called the daughter nucleus. The general formula for proton emission is:

$$^{A}_{Z}\mathrm{X}_N \rightarrow {}^{A}_{Z-1}\mathrm{X}_N + \mathrm{p} \tag{10.7}$$

β^- Decay (Beta Minus Decay) A neutron will transform into a proton, an electron, and a particle called an "anti-electron neutrino" (which has the label $\bar{\nu}_e$). We haven't talked about neutrinos, but you can find more information about them with a quick internet search. For our purposes, neutrinos are very light particles with no electric charge. Both the electron and the anti-electron neutrino are ejected while the proton remains in the nucleus. The electron that is ejected from the nucleus usually has a ton of energy and is called a β^- particle. An example of an isotope that undergoes β^- decay is carbon-14:

$$^{14}_{6}\mathrm{C}_8 \rightarrow {}^{14}_{7}\mathrm{N}_7 + \beta^- + \bar{\nu}_e$$

In the above equation, carbon-14 (6 protons and 8 neutrons) is called the parent nucleus and nitrogen-14 (7 protons and 7 neutrons) is called the daughter nucleus. The general formula for β^- decay is:

$$^{A}_{Z}\mathrm{X}_N \rightarrow {}^{A}_{Z+1}\mathrm{X}_{N-1} + \beta^- + \bar{\nu}_e \tag{10.8}$$

β^+ Decay (Beta Plus Decay or Positron Decay) A proton will transform into a neutron, a positron,[5] and an electron neutrino (which has the label ν_e). Both the positron and the electron neutrino are ejected while the neutron remains in the nucleus. Like the electron in β^- decay, the positron has a ton of energy and is called a β^+ particle. An example of an isotope that undergoes β^+ decay is potassium-40, which is found in bananas and decays to argon-40:

$$^{40}_{19}\mathrm{K}_{21} \rightarrow {}^{40}_{18}\mathrm{Ar}_{22} + \beta^+ + \nu_e$$

The general formula for β^+ decay is:

$$^{A}_{Z}\mathrm{X}_{N} \rightarrow {}^{A}_{Z-1}\mathrm{X}_{N+1} + \beta^+ + \nu_e \tag{10.9}$$

Electron Capture The nucleus will steal an electron from the atom. That electron and a proton in the nucleus transform into a neutron and an electron neutrino. The electron neutrino is ejected while the neutron stays in the nucleus. An example of an isotope that undergoes electron capture is beryllium-7, which decays to lithium-7:

$$^{7}_{4}\mathrm{Be}_{3} + e \rightarrow {}^{7}_{3}\mathrm{Li}_{4} + \nu_e$$

The general formula for electron capture is:

$$^{A}_{Z}\mathrm{X}_{N} + e \rightarrow {}^{A}_{Z-1}\mathrm{X}_{N+1} + \nu_e \tag{10.10}$$

α Decay (Alpha Decay) Heavy nuclei can emit a cluster of two protons and two neutrons known as an α particle. This usually happens when a pair of protons is far enough away from the other protons that the Coulomb force from all the other protons is larger than the strong nuclear force (discussed in Sect. 10.4), which keeps those protons attached to the nucleus. As we will see in Sect. 10.6, an α particle is a super-stable combination of protons and neutrons. An example of an isotope that undergoes α decay is radon-222, which is a gas that leaks into some basements and decays to polonium-218:

$$^{222}_{86}\mathrm{Rn}_{136} \rightarrow {}^{218}_{84}\mathrm{Po}_{134} + \alpha$$

The general formula for α decay is:

$$^{A}_{Z}\mathrm{X}_{N} \rightarrow {}^{A-4}_{Z-2}\mathrm{X}_{N-2} + \alpha \tag{10.11}$$

Spontaneous Fission While most heavy nuclei undergo α decay, superheavy atoms can emit nuclei of other elements. For example, californium-252 (98 protons and 154 neutrons) can break apart into xenon-140 (54 protons and 86 neutrons),

[5] A positron is the antimatter partner to the electron. Antimatter is a topic in Chap. 11.

ruthenium-108 (44 protons and 64 neutrons), and 4 neutrons (notice that $140 + 108 + 4 = 252$). This is called spontaneous fission:

$$^{252}_{98}\text{Cf}_{154} \rightarrow {}^{140}_{54}\text{Xe}_{86} + {}^{108}_{44}\text{Ru}_{64} + 4\text{n}$$

In the above equation, both xenon-140 and ruthenium-108 are considered daughters.

If the balance between neutrons and protons deviates too far from a stable equilibrium, the nucleus will undergo radioactive decay. Some isotopes can undergo two or more forms of decay. For example, francium-220 undergoes α decay 99.65% of the time and β^- decay for the remaining 0.35%. Radium-226 almost always undergoes α decay, but 3.2×10^{-9}% of the time it will undergo spontaneous fission by emitting a carbon-14 nucleus. Exploring the physics behind the balance between neutrons and protons is the topic of Sect. 10.6.

Will's Rant
It is important to understand that during β^- decay, β^+ decay, and electron capture, particles cease to exist and new particles come into existence. Some books imply, for example, that a neutron is composed of a proton, an electron, and an anti-electron neutrino, as if you could look inside the neutron and find those three particles squished together. This is incorrect and drives me a little crazy, hence the rant ☺. The neutron is actually composed of three smaller particles called quarks, a topic in Chap. 11. What is important here is that during β^- decay, the neutron ceases to exist. The neutron is present one moment and gone the next. The proton, electron, and anti-electron neutrino did not exist before the decay; these particles are created during the decay. In contrast, no new particles are created or destroyed in neutron emission, proton emission, or α decay.

There is a general rule about whether or not an isotope will undergo radioactive decay. In fact, this rule is more general than radioactive decay. As a general rule of nature, things try to get to the place of lowest energy.

A General Rule of Nature
A system always seeks to reach a state of lower energy.

As an analogy, think about a ball that is sitting halfway up a bowl. When released, the ball will try to get to the lowest point in the bowl and, with friction present, will eventually settle at this lowest point. The reason the ball settled at the lowest point is because this is the place of lowest energy.

Consider the radioactive decay of carbon-14:

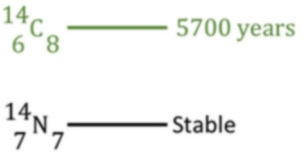

Fig. 10.2 An example of how to think about why carbon-14 decays to nitrogen-14

$$^{14}_{6}C_8 \rightarrow {}^{14}_{7}N_7 + \beta^- + \bar{\nu}_e$$

The daughter nucleus, nitrogen-14, has lower energy than the parent, carbon-14. The decay happens because carbon-14 is a higher energy state than nitrogen-14, see Fig. 10.2. The higher energy state has a lifetime of about 5700 years. After that characteristic time, it will decay to nitrogen-14. We will discuss the mechanisms behind radioactive decay in Sects. 10.4, 10.5 and 10.6.

All unstable isotopes have a characteristic time for decay, known as the half-life, represented by the parameter $t_{1/2}$. Radioactive decay is a random process, but we can state probabilities for whether an isotope has decayed. Half-life is defined as the time it takes for there to be a 50% probability that the isotope has decayed. If the radioactive sample is large enough, the half-life can also be thought of as the time it takes for half of the parent nuclei to decay to the daughter nucleus. The number of remaining parents as a function of time is given by the formula:

$$N(t) = N_0\left(\frac{1}{2}\right)^{\frac{t}{t_{1/2}}}, \tag{10.12}$$

where N_0 is the number of radioactive atoms we start with, t is the time elapsed, and $t_{1/2}$ is the half-life. For example, rubidium-84 has a half-life of $t_{1/2} = 32.82$ days. If we had a sample of 1,000,000,000 rubidium-84 atoms, half would decay after 32.82 days leaving about 500,000,000 rubidium-84 atoms. Of those remaining atoms 500,000,000 rubidium-84 atoms, half of those will decay in the next 32.82 days. This continues until there are no radioactive atoms remaining, see Table 10.2.

Table 10.2 How radioactive rubidium-84 atoms decay over time

$t_{1/2}$	Days	$N(t)$
0	0	1,000,000,000
1	32.82	500,000,000
2	65.64	250,000,000
3	98.46	125,000,000
4	131.28	62,500,000
5	164.10	31,250,000
6	196.92	15,625,000
7	229.74	7,812,500
8	262.56	3,906,250
9	295.38	1,953,125

Extra Fun
Carbon-14 is a radioactive isotope of carbon with a half-life of about 5700 years. It is created in the atmosphere when high energy neutrons from space, also known as cosmic rays, slam into the stable isotope nitrogen-14. The high energy neutron basically crashes into a proton in the nitrogen-14 nucleus, pushing the proton out while remaining behind. This radioactive carbon-14 attaches to an oxygen molecule to form radioactive carbon dioxide. Since there is a source (cosmic rays hitting nitrogen-14) and a sink (radioactive decay), the atmosphere reaches an equilibrium between the carbon-12 (the stable isotope) and carbon-14 forms of carbon dioxide. This equilibrium results in a constant isotopic abundance. If you collected a bunch of carbon dioxide from the air, about 1 molecule in every 10^{12} is radioactive. Living organisms in contact with the atmosphere breathe in, absorb, or ingest this radioactive carbon dioxide, thereby reaching equilibrium with the atmosphere. If the organism ceases interaction with the atmosphere,[6] the isotopic abundance decreases (the radioactive carbon decays without being replaced). If sometime later you dig up the formally living being and you know the starting isotopic abundance, you can quickly calculate how long that being has been removed from the atmosphere. This process is known as radiocarbon dating. There are other types of radioactive dating including radioargon dating and radiokrypton dating.

[6] RIP ☹.

10.3 The Table of isotopes

The table of isotopes, also known as the table of nuclides, is similar to the periodic table, but includes all known elements and isotopes. Figure 10.3 is a screenshot of the table of isotopes from a website maintained by the International Atomic Energy Agency (IAEA).[7] Figure 10.4 is a zoom in for the lighter elements. The vertical axis represents the number of protons in the isotope, while the horizontal axis represents the number of neutrons in the isotope. The entire row with 2 protons consists of helium isotopes; the entire row with 63 protons are europium isotopes. The black

[7] I consider the Live Chart of Nuclides website one of three essential tools for an atomic and nuclear physicist: https://www-nds.iaea.org/relnsd/vcharthtml/VChartHTML.html. The other two are the IAEA app for your phone, known as 'Isotope Browser', and the NIST spectral database, https://www.nist.gov/pml/atomic-spectra-database.

boxes indicate stable isotopes, and the colors representing the various types of radioactive decay are shown as insets in both Figs. 10.3 and 10.4. The interactive table is a lot of fun to play with.

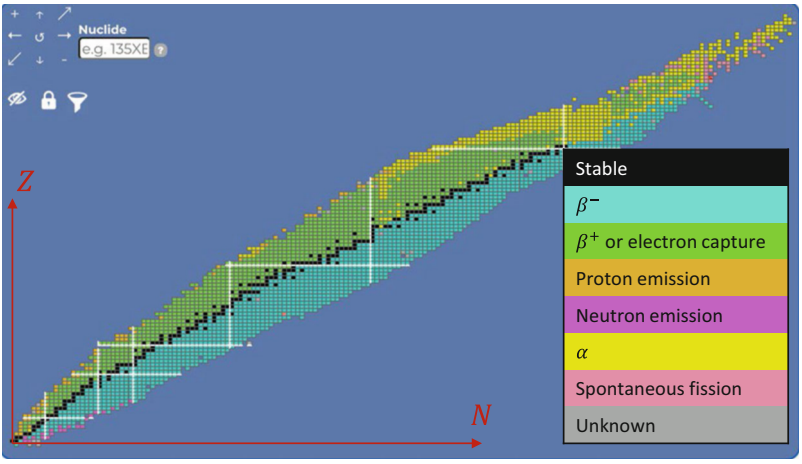

Fig. 10.3 A screenshot of the entire table of isotopes from the Live Chart of Nuclides maintained by the IAEA: https://www-nds.iaea.org/relnsd/vcharthtml/VChartHTML.html Note: I added the axes and the legend. Some isotopes have more than one type of decay. The legend shows the dominate decay

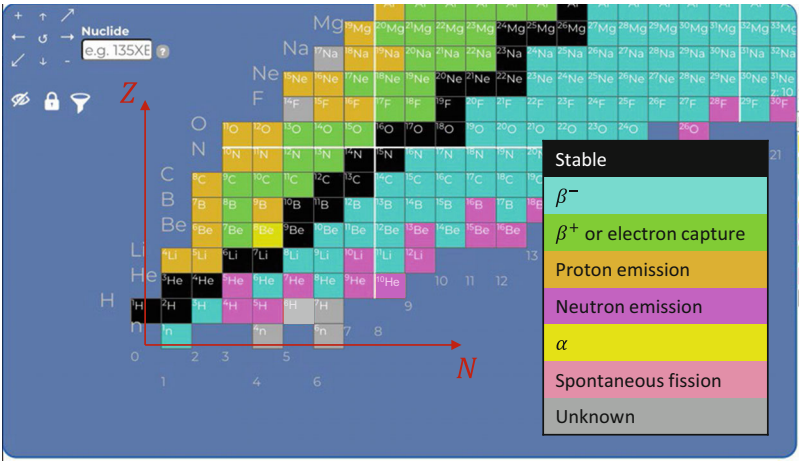

Fig. 10.4 A zoom in screenshot of the lighter elements from the table of isotopes. Note: I added the axes and the legend

10.4 The Strong Nuclear Force

Important Reminder
When two particles are interacting, there is a force between them. For example, an electron and a proton are attracted to each other because they have opposite charges. That attractive force is called the Coulomb force, which is also known as the electrostatic force.

We learned in Sect. 10.2 that an imbalance in the ratio of protons to neutrons results in radioactive decay. If the balance is "good", the atom is stable. If there is an imbalance, there will exist a daughter system that has a lower energy than the parent system, see Fig. 10.2 for an example. To fully explore the nucleus and atomic stability, we will first study the strong nuclear force (this section). The strong nuclear force is an attractive force that keeps the nucleons in the nucleus. In Sect. 10.5 we will explore the weak nuclear force, which is the force that facilitates β^- decay, β^+ decay, and electron capture. Basically, if a process involves the annihilation and creation of particles, the weak nuclear force is involved. Finally, we will tie everything together in Sect. 10.6 using energy principles to explore why some isotopes are stable and some are not.

The helium-4 nucleus has 2 protons and 2 neutrons. Since both protons have positive charge, they repel each other due to the Coulomb force. The protons in helium-4 are separated by about 1×10^{-15} m. This incredibly small distance is called a femtometer, which is given the unit fm: $1 \times 10^{-15}\,\text{m} = 1\,\text{fm}$. The repulsive Coulomb force between the two protons has a magnitude of about 230 newtons (about 50 pounds of force). Considering each proton has a mass of only 1.67×10^{-27} kg, this is huge!! If we took a free proton and acted on it with 230 Newtons of force, the proton would accelerate at $1 \times 10^{29}\,\text{m/s}^2$!!!! For comparison, freefall on earth is only about $9.8\,\text{m/s}^2$. The relative comparison is so enormous, I have absolutely no idea how to convey how large that repulsive force is.

However, the protons inside a helium-4 nucleus do not fly apart.[8] Therefore, there must be some other attractive force that is larger than the repulsive Coulomb force to stop the protons from being pushed out of the nucleus. An intuitive guess is that maybe gravity is keeping the protons in the nucleus. After all, gravity is the attractive force that we interact with everyday. Remarkably, gravity is an extremely weak force. You will explore just how weak gravity is in Problem 10.6. In addition, neutrons have no charge, so they don't feel any Coulomb force. What is keeping neutrons in the nucleus?

[8] This is a good thing!

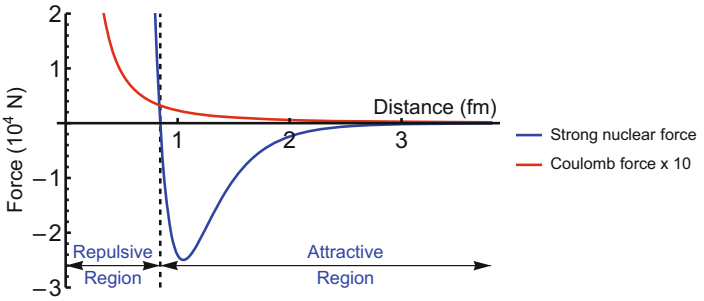

Fig. 10.5 An illustrative plot showing the forces between nucleons inside the nucleus. The strong nuclear force is attractive (negative force) above about 0.85 fm but falls off very quickly. The Coulomb force between two protons, which is increased by a factor of 10 for visibility, is always repulsive (positive force)

The force keeping the nucleons "glued" to one another is the strong nuclear force.[9] The strong nuclear force acts on both neutrons and protons, and it can be either an attractive or repulsive force. At a distance of 1 femtometer, the strong nuclear force is attractive and approximately 100 times stronger than the repulsive Coulomb force. Figure 10.5 shows a graph of both the Coulomb force and the strong nuclear force.[10] At short distances, both the strong nuclear force and the Coulomb force are repulsive. Somewhere around 0.85 fm, the nucleons start to be attracted by the strong nuclear force. The really important thing to take from this graph is that at around 1 fm, the strong nuclear force is more attractive than the Coulomb force is repulsive. The total force is the sum of the two, which for small distances, is attractive. Outside of about 5 fm, the strong nuclear force is basically 0 and the Coulomb force will dominate, pushing the protons away from each other.

Figure 10.5 is also useful for understanding why we cannot have a stable atom with more than about 200 nucleons. A proton near the edge of a large nucleus is far away from the nucleons that are on the other side of the nucleus. Because the strong nuclear force goes to zero so quickly, that proton does not feel any attractive strong nuclear force from the distant nucleons. It does, however, still feel the repulsive

[9] In Chap. 11, we are going to talk about quarks, which are the subparticles that make up protons and neutrons. The strong nuclear force is also what keeps the quarks glued to one another inside the nucleon. However, the strong nuclear force extends outside of a nucleon to interact with other nucleons. The force that extends out of the nucleon is often called the "strong residual force". Technically, it is still the same strong nuclear force, but there are subtle and important differences for how it behaves inside a nucleon and how it behaves between nucleons. Figure 10.5 is an illustrative plot for the strong residual force between nucleons.

[10] This strong nuclear force plot is calculated from the Reid potential, see Reference [3], which was developed and named after American physicist Roderick V Reid Jr. This is a popular model for the strong nuclear force first used in 1968.

Coulomb force from all the other protons. Consequently, this proton near the edge will be repelled away. This is why the periodic table does not have any stable elements above lead-208. The next atom on the periodic table is bismuth. Bismuth has no stable isotopes, although bismuth-209 is super long-lived with a half-life of 2×10^{19} years.

An Analogy
The Coulomb force is a long-range force. Two protons will weakly repel each other even if they are 10 meters apart. The strong nuclear force is similar to a contact force such as two pieces of Velcro. When the two pieces of Velcro are touching, they stick together really well. The moment they are no longer touching, the force between the two is zero.

So, why do we need neutrons for a stable nucleus? Neutrons add additional attractive strong nuclear force to the nucleus without adding any additional repulsive Coulomb forces. A nucleus with only 2 protons, which would be helium-2, is extremely unstable. The Coulomb force, while small compared to the strong nuclear force, is still large enough to push the two protons apart. So, the decay of helium 2 would be two protons flying away in different directions. Adding a neutron to the nucleus adds additional strong nuclear force (attractive) without adding any additional repulsive Coulomb forces. Both helium-3 (2 protons and 1 neutron) and helium-4 (2 protons and 2 neutrons) are stable. Helium-5 (2 protons and 3 neutrons) is unstable. But why is helium-5 unstable? An extra neutron should just add additional attractive strong nuclear force with no additional repulsive Coulomb force. In the end, it comes down to the energy analogy: there is a daughter system with lower energy than helium-5. We will explore this more in Sect. 10.6.

Strong Nuclear Force Summary
Inside the nucleus, the strong nuclear force is the mechanism that keeps nucleons in the nucleus. It is a short-range force. Neutrons add additional attractive strong nuclear force without adding any additional repulsive Coulomb force. If there are too many protons compared to neutrons or the nucleus is too large, proton emission or α decay may occur.

Fun Fact
If the strong nuclear force were only 2% larger, helium-2 could undergo β^- decay to hydrogen-2 (also known as deuterium or heavy hydrogen) instead of breaking apart into 2 protons.[11]

[11] See reference [4].

10.5 The Weak Nuclear Force

The strong nuclear force holds nucleons in the nucleus. If the balance between protons and neutrons is not correct, the nucleus will be unstable and undergo radioactive decay (more on this in Sect. 10.6). We have hinted at possible mechanisms behind proton emission and α decay (the Coulomb force overwhelms the strong nuclear force), but nothing we have talked about so far explains the forces behind the annihilation or creation of particles (i.e., β^-, β^+, and electron capture; see the beginning of Sect. 10.2 for a recap of the types of radioactive decays). To explore those decays, we need to understand the weak nuclear force.

Important
If the radioactive decay process involves the annihilation and creation of particles, the weak nuclear force is involved.

Let's start by reviewing a few important concepts. The Coulomb force acts only on charged particles. It is a long-range force and can be either an attractive force (opposite charges) or a repulsive force (same charges). The strong nuclear force acts on protons and neutrons but not on electrons. It is a short-range force with a range on the order of a few femtometers and is, for our purposes, an attractive force.

The weak nuclear force, sometimes referred to simply as "the weak force," acts on electrons, protons, and neutrons. Like the Coulomb force, it can be either attractive or repulsive. However, unlike the Coulomb force, it is an *extremely* short-range force. While the strong nuclear force has a range on the order of a few femtometers, the weak nuclear force has a range on the order of 1×10^{-18} m, or about 0.1% of the diameter of a proton. This distance is called an attometer (am): $1 \times 10^{-18}\,\text{m} = 1\,\text{am}$.

As you can probably guess from the name, the weak nuclear force is weak compared to the strong nuclear force and the Coulomb force. Consider two protons about 1 fm apart. The weak nuclear force is about 1 million times weaker than the strong nuclear force. For stable atoms, the strong nuclear force dominates the weak

nuclear force, but for atoms with a few too many neutrons or protons, the weak nuclear force can play an important role.

To explore the importance of the weak nuclear force, let's discuss the neutron. A neutron outside of a nucleus, called a free neutron, will undergo radioactive decay with a half-life of about 10.2 minutes according to the equation:

$$n \rightarrow p + \beta^- + \bar{\nu}_e, \tag{10.13}$$

where n is the neutron that decays into a proton p, a β^- particle (a high energy electron), and an anti-electron neutrino $\bar{\nu}_e$.

A Reminder of an Important Rule and an Analogy from Sect. 10.2

A General Rule of Nature A system always seeks to reach a state of lower energy.

The Analogy Think about a ball that is sitting half way up a bowl. When released, the ball will try to get to the lowest point in the bowl and, with friction present, will eventually settle at this lowest point. The reason the ball settled at the lowest point is because this is the place of lowest energy. Both gravity and friction were the mechanisms (forces) that got the ball to settle at the bottom of the bowl. Gravity pulled the ball down, and friction dissipated the extra energy into heat. The ball went from a higher energy state to a lower energy state.

The proton, β^- particle, and anti-electron neutrino (i.e., the daughter system) have lower energy than the free neutron (the parent system). Something had to facilitate this system going from higher energy to lower energy. The force behind this interaction is the weak nuclear force. So, according to the general rule of nature, a free neutron "wants" to get to a place of lower energy, and the weak nuclear force is the force that helps it get there.

A free neutron will undergo radioactive decay facilitated by the weak nuclear force. For a neutron inside a nucleus, there is also the strong nuclear force from other nucleons interacting with the neutron. The question to ask is, "Is there a place of lower energy for the system to go?" If the answer is yes, the weak nuclear force will eventually get the system to a lower energy. If the answer is no, then the weak force has no place to push the system. So, the weak nuclear force will cause the free neutron to cease existing and create a proton, β^- particle, and anti-electron neutrino. In contrast, there is no place of lower energy for, for example, a helium-4 nucleus. So, the helium-4 nucleus is stable.

If a neutron in a helium-4 nucleus were to undergo β^- decay, the equation would be:

$$^{4}_{2}\mathrm{He}_{2} \rightarrow {}^{4}_{3}\mathrm{Li}_{1} + \beta^{-} + \bar{\nu}_{e}$$

This does not happen!

The system on the right of this equation (the lithium-4 atom, the β^- particle, and the anti-electron neutrino) has a higher energy than the helium-4 atom. Therefore, this process does not occur.

Current Summary
As a general rule of thumb:

- If there are way too many neutrons in the nucleus, a neutron will escape (more on this in Sect. 10.6) to get the system to a point of lower energy.
- If there are a few too many neutrons in the nucleus, the weak nuclear force will facilitate β^- decay to get the system to a point of lower energy.
- If there are way too many protons in the nucleus, the Coulomb force will push a proton out of the nucleus to get the system to a point of lower energy.

A free proton is stable[9] because a free proton is already the system of lowest energy. However, a proton can decay inside the nucleus. How?!? The answer always goes back to the energy argument: a system wants to get to the place of lowest energy. If a nucleus has a few too many protons, the weak nuclear force will find a way to that place of lower energy. In this case, a proton can transform into a neutron, a positron (more on positrons in Chap. 11), and an electron neutrino. This is called β^+ decay. As an example, consider the following decay for oxygen-15:

$$^{15}_{8}\mathrm{O}_{7} \rightarrow {}^{15}_{7}\mathrm{N}_{8} + \beta^{+} + \nu_{e} \tag{10.14}$$

This decay, which has a half-life of about 2 minutes, is used in Positron Emission Tomography (PET) scans to measure blood flow and oxygen metabolism.[10] A proton in the oxygen-15 nucleus transforms into a neutron, a β^+ particle (positron), and an electron neutrino. The right-hand side of that equation is lower energy than the left-hand side, so the weak nuclear force will make this decay happen.

The daughter particle, nitrogen-15, is the lowest energy state for this system, so it does not undergo any further type of decay. Therefore, neither of the following occurs:

[9] As far as we know.

[10] Due to its short half-life, it is made in an accelerator on site (usually in the basement of the hospital) and brought immediately to the patient.

$$^{15}_{7}\mathrm{N}_8 \rightarrow {}^{15}_{6}\mathrm{C}_9 + \beta^+ + \nu_e$$
$$^{15}_{7}\mathrm{N}_8 \rightarrow {}^{15}_{8}\mathrm{O}_7 + \beta^- + \bar{\nu}_e$$
Neither of these happen!

Both of these processes have daughter systems with higher energy than the parent system, so they will never occur.

There is one more process in which the weak nuclear force plays an important role. For some atoms, the electrons "orbiting" the nucleus can come too close. If that happens **and** there is a daughter system with a lower energy, that electron and a proton in the nucleus will transform into a neutron and an electron neutrino. This is called electron capture. The weak nuclear force, once again, is the force that is facilitating this process. I want to emphasize that the proton and electron do not combine to form a neutron. The proton and electron literally stop existing, and a neutron and an electron neutrino start existing (see Will's rant in Sect. 10.2). An atom that undergoes electron capture decay is the beryllium-7 atom, which has a half-life of about 53 days. The equation to describe beryllium-7 decay is:

$$^{7}_{4}\mathrm{Be}_3 + e \rightarrow {}^{7}_{3}\mathrm{Li}_4 + \nu_e, \tag{10.15}$$

where $^{7}_{3}\mathrm{Li}_4$ is the daughter lithium-7 atom. Once again, this process can only occur because the daughter system has lower energy than the parent system.

Summary for the Weak Nuclear Force
If particles are going to be annihilated and created to get the system to a place of lower energy, the weak nuclear force is the force that facilitates the process.

Current Summary
As a general rule of thumb:

- If there are way too many neutrons in the nucleus, a neutron will escape (more on this in Sect. 10.6) to get the system to a point of lower energy.
- If there are a few too many neutrons in the nucleus, the weak nuclear force will facilitate β^- decay to get the system to a point of lower energy.
- If there are a few too many protons in the nucleus, the weak nuclear force will facilitate either β^+ decay or electron capture to get the system to a point of lower energy.
- If there are way too many protons in the nucleus, the Coulomb force will push a proton out of the nucleus to get the system to a point of lower energy.

10.6 The Nuclear Shell Model and Energetics

When we talked about the strong nuclear force, we had an unanswered question: "Why is a nucleus with too many neutrons unstable?" After all, neutrons only add an attractive force to the system. There was an explanation about the existence of a lower energy state, but that explanation, at least to me, felt unsupported. In this section, we will explore what happens when more and more nucleons are added to a nucleus. We will also see why, from an energy viewpoint, different types of decays occur and why stable lighter atoms have about equal numbers of protons and neutrons while stable heavier atoms have more neutrons than protons, see the black boxes in Fig. 10.3. To do so, we will rely on the general rule of nature and the Pauli Exclusion Principle, see Chap. 8, Sect. 8.5 for a review of the Pauli Exclusion Principle.

Important Reminder

Pauli Exclusion Principle Two fermions cannot simultaneously occupy the same quantum state; that is, no two fermions can have the same set of quantum numbers within a quantum system. Electrons, protons, and neutrons are all fermions with spin quantum number 1/2.

In Chap. 8, we learned that electrons fill shells. The 1s subshell can hold two electrons. Using the ket notation $|n\ \ell\ m_\ell\ s\ m_s\rangle$, the two electrons have quantum numbers $|1\ 0\ 0\ \frac{1}{2}\ \frac{1}{2}\rangle$ and $|1\ 0\ 0\ \frac{1}{2}\ \text{-}\frac{1}{2}\rangle$. Notice they have different quantum numbers as required by the Pauli Exclusion Principle. All six electrons in a filled 2p subshell also have different quantum numbers. The 1s electrons are in a lower energy state than the 2p electrons, but the Pauli Exclusion Principle prevents any more electrons from being in the 1s subshell. The dominant interaction between the electrons and the nucleus is the electromagnetic force, so this force determines how many

Table 10.3 Electron configuration for shells and subshells

Shell	Subshell	Subshell max electrons	Shell max electrons
1	1s	2	2
2	2s	2	8
	2p	6	
3	3s	2	18
	3p	6	
	3d	10	
4	4s	2	32
	4p	6	
	4d	10	
	4f	14	

subshells there are for a particular shell. Table 10.3 shows the maximum number of electrons that can fit into each subshell and shell. I want to emphasize that every electron has a unique set of quantum numbers. Neon has a total of ten electrons to completely fill the $n = 1$ and $n = 2$ shells. Each of those ten electrons has a unique set of quantum numbers. As a reminder, electrons do not have to completely fill a shell before starting to fill the next shell (see Fig. 8.3). For example, potassium has a single electron in the 4s subshell with no electrons in the 3d subshell. This ordering is determined by the electromagnetic force.

Protons and neutrons are also fermions with a spin quantum number of $1/2$. Therefore, they also fill nuclear shells and subshells. Since protons and neutrons are different particles, they fill individual nuclear shells; protons fill the proton shells and neutrons fill the neutron shells. Isotopes with filled nuclear shells and subshells tend to be more stable. This stability occurs at so-called "magic numbers." The magic numbers for protons indicate how many protons are in the nucleus when this extra stability occurs. The magic numbers for protons are 2, 8, 20, 28, 50, 82, and 114. The magic numbers for neutrons are 2, 8, 20, 28, 50, 82, and 126. Oxygen-16 (8 protons and 8 neutrons) is called doubly magic because it has both a filled proton shell and a filled neutron shell. Oxygen-16 is an extra stable nucleus.

The configuration of the nuclear shells is very different from the electron shells because the nuclear shells are determined by both the strong nuclear force and the electromagnetic force. However, every proton (or neutron) in the nucleus has a unique set of quantum numbers and fills the proton (or neutron) shells. The last magic number for protons (114) is smaller than the last magic number for neutrons (126). This is because protons have an additional Coulomb force acting on them, and that additional force causes a different structure for the shells. I want to emphasize that these are not electron shells! Protons and neutrons do not fill up their nuclear shells according to the Madelung rule (see Fig. 8.3).

Let's use this as a starting point to understand nuclear stability. Figure 10.6 shows a planetary model for protons and neutrons filling their respective shells. As a reminder, the planetary model is not correct. Protons and neutrons are quantum mechanical particles that have wave-like properties. Nonetheless, this nuclear shell model can help us understand the stability of the nucleus.

The number of nucleons that can occupy a subshell is indicated on the left. The lowest energy proton shell can hold two protons (one spin up and one spin down) and the lowest energy neutron shell can hold two neutrons. This is the first magic number, 2. The next shell can hold a total of six nucleons distributed among two subshells. The sum of all protons (or neutrons for the neutron shells) that completely fill the first two shells is the second magic number, $2 + 6 = 8$. The third grouping of subshells holds a total of twelve nucleons, a magic number of $2 + 6 + 12 = 20$, and so on. Notice that as the energy increases, the neutron shells are lower in energy than the proton shells. This is because the Coulomb force acts on the protons, but not on the neutrons, and also because neutrons have a slightly larger mass than protons. It should be noted that the energy spacings are not correct; they are greatly simplified for us to explore concepts. There are also higher energy shells that are not shown in these diagrams.

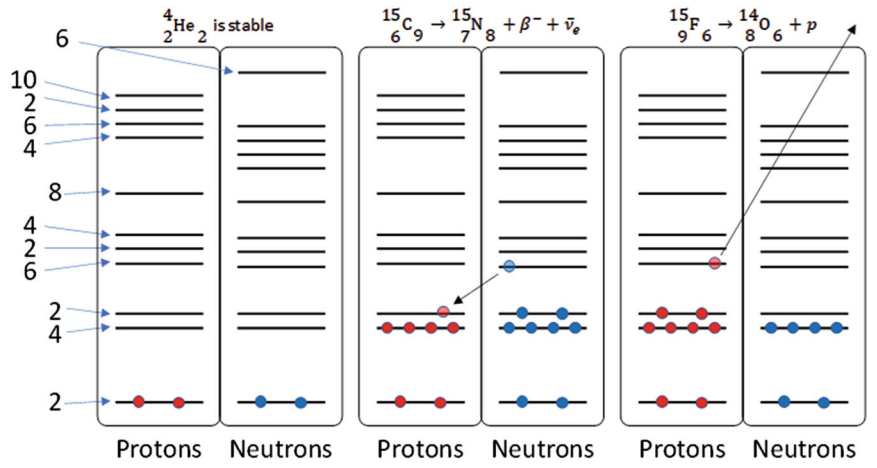

Fig. 10.6 The nuclear shells for protons and neutrons. Starting from the left, helium-4 is a stable isotope. The system is in its lowest energy state. The middle diagram is an example of $\beta-$ decay. The right diagram is an example of proton emission

Starting from the left diagram in Fig. 10.6, we see that helium-4 is a stable isotope. In fact, this is a super-stable, doubly magic nucleus because it perfectly fills the lowest nuclear shell for both the protons and the neutrons. There isn't any place of lower energy for any of the nucleons. The helium-4 nucleus is also the α particle from α decay. The middle diagram is an example of β^- decay. The system has a place of lower energy if the highest-energy neutron transforms into a proton (plus a β^- particle and anti-electron neutrino). One could ask the question, "Why is this β^- decay and not neutron emission?" The answer always come back to energy. For this scenario, it is energetically more favorable for the weak nuclear force to facilitate β^- decay than the neutron to just leave the nucleus. The right diagram is an example of proton emission. The highest-energy proton has a place to go to bring the system to a point of lower energy. It can leave the nucleus or undergo β^+ decay. In this scenario, the Coulomb force makes proton emission energetically more favorable than β^+ decay, so that is what happens. Finally, the left diagram in Fig. 10.7 shows an example of β^+ decay. In this scenario, it is energetically more favorable for the weak nuclear force to facilitate the decay of scandium-41 to calcium-41. And this is the essence of stable versus unstable nuclei. If there is a state of lower energy, the system will try to get there. Sometimes the mechanism is proton emission, neutron emission, α decay, or spontaneous fission. Other times, the weak nuclear force facilitates the annihilation and creation of particles. In the end, if there is a state of lower energy, the nucleus will decay to this lower state.

There is another interesting concept I'd like to explore with you. For smaller mass, stable nuclei, there are roughly equal numbers of neutrons and protons. This can be seen by the black boxes (stable nuclei) in Fig. 10.3, which make a linear line with a slope of one for smaller mass nuclei. This is because, for smaller masses, the nuclear energy levels for the protons and neutrons are about the same. This

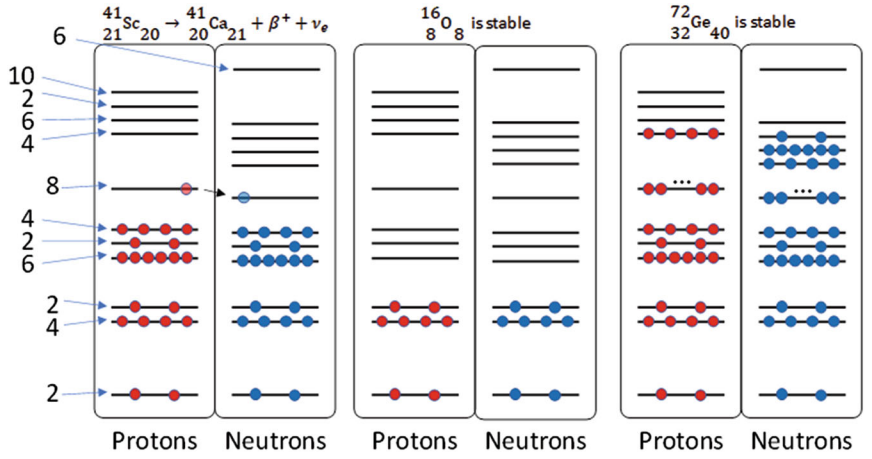

Fig. 10.7 Starting from the left, scandium-41 is an example of β^+ decay. The middle diagram and right diagram are examples of stable nuclei. Oxygen-16 is stable and has 8 protons and 8 neutrons. Germanium-72 is also stable, but it needs 40 neutrons to balance the energy of the 32 protons. The three dots on the subshell that can hold 8 nucleons are indicating the shell is filled; I couldn't fit 8 nucleons on the line

is shown in the left diagram in Fig. 10.6 for helium-4 and the middle diagram in Fig. 10.7 for oxygen-16. However, as more protons are packed into the nucleus, the Coulomb force becomes larger and larger, increasing the nuclear energy levels for the protons. As such, more neutrons are needed to balance the energy of the protons and neutrons. This is an equivalent way of saying more neutrons are needed to provide a larger strong nuclear force to overcome this increase in the Coulomb force repulsion. An example of a heavier nucleus is shown in the right diagram of Fig. 10.7. Germanium-72 needs eight more neutrons than protons to create a stable balance between the total proton energy and the total neutron energy. In fact, germanium-70, -72, -73, and -74 are all stable.[11] Notice how the diagrams for all the stable nuclei have approximately equal energy for the highest-energy neutrons and protons while the unstable nuclei do not.

If the nucleus gets too big, there is always a state of lower energy. The largest mass nucleus that is stable is lead-208. Lead-208 has a proton number of 82 and a neutron number of 126, which are both magic numbers. There are also isotopes that have two or more decay paths to states of lower energy. This is more likely for larger mass nuclei. For example, bismuth-212 can undergo both α decay (36% of the time) and β^- decay (64% of the time). For the heaviest of atoms, the dominant decay is not the annihilation and creation of particles, but particles simply leaving the nucleus to reach a lower energy state, see Fig. 10.3 or the online table of isotopes.

The nucleus is an amazing and wonderful system to study. It is such a rich system to explore and is needed to understand the world of the super small. Atomic

[11] Germanium-71 decays via electron capture, which is facilitated by the weak nuclear force.

physicists tend to use lasers to excite electrons to higher energy atomic states. We care about the nucleus, but usually in the context of how the structure of the nucleus affects electronic energy levels, for example, isotope shifts discussed in Sect. 10.1.

Nuclear physicists conduct similar experiments on nuclei with the goal of understanding the internal forces at play. Just like electrons, protons and neutrons can be excited to higher energy states. Compared to the Coulomb force and the mass of the electron, the strong nuclear force and the mass of the nucleons are much larger! This results in nuclear excited states with much higher energies than atomic energy levels. For the most part, we can't use lasers to excite nucleons to higher energy states. Thorium-229 is the one exception. The lowest energy nuclear excited state of thorium-229 requires a laser with a wavelength $\lambda = 150\,\text{nm}$, or a frequency of $2.00 \times 10^{15}\,\text{Hz}$. That laser isn't easy to make or use, but it is possible! The next isotope with the lowest lying nuclear excited state is protactinium-234, which would require a laser with $\lambda = 16.8\,\text{nm}$ ☹. However, nuclear physicists are clever and use other methods to study nuclear excited states. For example, nuclear physicists can use accelerators and high-energy collisions to excite a nucleus and then measure the energy of the high-energy photons that are emitted. They also use radioactive decay. For this technique, some parent systems decay to excited nuclear levels that then decay via high-energy photons. If you find the nucleus as fascinating as I do, nuclear physics might be the field for you!

Problems

10.1 Consider the transition in the beryllium atom $1s^2s^2\ {}^1S_0 \rightarrow 1s^2s2p\ {}^1P_1$. The resonance frequency for this transition is $f_r = 1,276,080,100\,\text{MHz}$.

(a) Calculate the normal mass shift as a function of mass number for $A' = 7$ to $A' = 12$ with respect to the stable isotope beryllium-9.
(b) Make a (modified) King plot of the normal mass shift versus A'.

10.2 Write the decay equation for the following radioactive decays:

- β^- decay:
  - Potassium-40 ($^{40}_{19}K_{21}$): Potassium-40 is the largest source of natural radioactivity in animals, including humans. About 89% of all potassium-40 decay is β^- decay.
  - Rubidium-87 ($^{87}_{37}Rb_{50}$): Rubidium-87 is used in rubidium-strontium dating, a radiometric dating technique used to determine the age of rocks and minerals.
- β^+ decay:
  - Sodium-22 ($^{22}_{11}Na_{11}$): Sodium-22 is used as a calibration source for positron emission tomography (PET) scans.
  - Carbon-11 ($^{11}_{6}C_5$): Carbon-11 is used in PET scans to detect sites of prostate cancer.

- Electron capture:
  - Potassium-40 ($^{40}_{19}\mathrm{K}_{21}$): Potassium-40 is used in potassium-argon dating. About 11% of all potassium-40 decay is β^- decay.
  - Beryllium-7 ($^{7}_{4}\mathrm{Be}_{3}$): Beryllium-7 is used in cosmogenic isotope studies to understand solar activity and atmospheric processes. It is also the lightest element to undergo electron capture.
- Alpha decay:
  - Uranium-238 ($^{238}_{92}\mathrm{U}_{146}$): Uranium-238 produces about 40% of the radioactive heat produced in the earth.
  - Thorium-232 ($^{232}_{90}\mathrm{Th}_{142}$): Thorium-232 is used in thorium reactors and in dating geological formations through thorium-lead dating.

10.3 (Carbon Dating) The current atmospheric isotopic abundance is $\frac{\text{carbon-14}}{\text{carbon-12}} = 10^{-12}$.

(a) Using the table of isotopes or online resources, find the half-life of carbon-14.
(b) A mummy was recently found in what was the ancient city of Memphis. The egyptologist who led the expedition sends a sample of the mummy to a radiocarbon dating specialist for carbon dating. The results came back as isotopic abundance, and not a date. The isotopic abundance of the sample is $\frac{\text{carbon-14}}{\text{carbon=12}} = 5.8 \times 10^{-13}$. Assuming the atmospheric isotopic abundance stays constant in time,[12] estimate how long ago the individual who is now a mummy died?

10.4 Locate rubidium-84 in the table of isotopes. Given a sample of 1 billion rubidium-84 atoms, determine the composition of atoms remaining after 100 days. Hint: Rubidium-84 decays into both krypton-84 and strontium-84.

10.5 A decay chain is a series of radioactive decays that shows the sequential process of decays. It is also known as a "radioactive cascade." For example, oxygen-20 undergoes β^- decay to fluorine-20, which then undergoes β^- decay to neon-20, which is stable. So, the decay chain is:

$$^{20}_{8}\mathrm{O}_{12} \xrightarrow{\beta^-} {}^{20}_{9}\mathrm{F}_{11} \xrightarrow{\beta^-} {}^{20}_{10}\mathrm{Ne}_{10}$$

Construct a decay chain for thorium-232, considering only the dominant decay mode at each step. When reaching bismuth-212, follow the decay path to thallium-208. The decay chain should terminate at lead-208.

[12] It is not constant, but we have ways to calibrate how the ratio has changed in times as well as geographical variations.

10.6 Newton's law of universal gravitation is a model of the gravitational force between two objects with mass. Mathematically, the force between two objects with mass m_1 and m_2 is:

$$F_G = \frac{Gm_1m_2}{r^2}, \tag{10.16}$$

where $G = 6.674 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}$ is a constant of nature, known as the gravitational constant, and r is the distance between the two objects. The gravitational force between two objects is always attractive.

Coulomb's law is a model of the force between two objects with charge. Mathematically, the force between two objects with charge q_1 and q_2 is:

$$F_C = \frac{kq_1q_2}{r^2}, \tag{10.17}$$

where $k = 8.988 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}$ is a constant of nature, known as the Coulomb constant, r is the distance between the two objects, and the unit C stands for Coulomb. A proton has a charge of $+1.602 \times 10^{-19}$ C and an electron has the same magnitude charge but opposite sign, -1.602×10^{-19} C.

(a) Consider two protons in a nucleus. How much larger is the Coulomb force compared to the gravitational force?
Hint: Take the ratio of the two forces first. The distance between the two protons will cancel out.

(b) For an object with mass m_1 at the surface of the earth, Newton's law of universal gravitation is:

$$F_G = \frac{Gm_1m_E}{R_E^2}, \tag{10.18}$$

where $m_E = 5.972 \times 10^{24}$ kg is the mass of the earth, $R_E = 6.378 \times 10^6$ m is the radius of the earth. This formula can be simplified to:

$$F_G = m_1 g, \tag{10.19}$$

where $g = \frac{Gm_E}{R_E^2}$. What is the numerical value of g? The units should simplify to m/s^2.
Hint: A newton N is the same thing as $\frac{\text{kg m}}{\text{s}^2}$.

(c) Calculate the force of gravity between you and the earth. The conversion between pounds and kilograms is 1 kg = 2.205 lbs.

(d) Find the ratio of the Coulomb force of two protons separated by 1 fm to your answer in part (c). For context, you are much, much more massive than a proton!
(e) The magnitude of the strong nuclear force between two nucleons separated by 1 fm is about 24,000 N. Find the ratio of this strong nuclear force compared to your answer in part (c).

10.7 (Advanced Math Problem: Connecting Half-Life to Lifetime) Equation 10.12 is the formula most everyone uses for radioactive decay. However, it can be useful to convert this equation to one of exponential decay: $N(t) = N_0 e^{-t/\tau}$, where τ is the lifetime. This is the same form we used to model the probability that an electron in an excited state decays to a lower energy state, see Sect. 3.3!

(a) Carbon-14 has a half-life of 5700 years. What is the lifetime?
(b) Make a graph of both equations. If your answer to part (a) is correct, the two functions should graph identically.

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The Standard Model of Particle Physics

11

Abstract

In this chapter, we explore the fascinating world of particle physics and the Standard Model of Particle Physics. We discuss the constraints of the Schrödinger equation and the necessity of quantum field theory, introducing key concepts like antimatter, vacuum fluctuations, and Feynman diagrams. The chapter details the fundamental particles and forces in the Standard Model and highlights unresolved questions, such as dark matter, dark energy, and the integration of gravity. Additionally, we examine the role of virtual particles and the impact of vacuum fluctuations on our understanding of particle interactions. We aim to provide a comprehensive overview of the current state of particle physics and the exciting challenges that lie ahead.

Learning Goals

By the end of this chapter, you should be able to understand:

- that quantum mechanics does a great job of describing the world of the super small, but it is not perfect.
- that the Standard Model of Particle Physics is a conceptual model of the world of the super small; the mathematical framework for it is called Quantum Field Theory.
- that the Standard Model of Particle Physics is not complete. There are problems with it, but we aren't entirely sure how to make a more complete model. There are lots of ideas, but so far, no one has been able to experimentally confirm a better model.

W. Raven, *Atomic Physics for Everyone*,
https://doi.org/10.1007/978-3-031-69507-0_11

11.1 Problems with Quantum Mechanics

We spent a lot of this book conceptually thinking about atoms. Many of the conceptual ideas come from quantum mechanics, which is mathematically described by the Schrödinger equation. Whenever you work with a model, it is important to know the limitations of your model. For example, you can't use classical physics, like Newton's second law, to describe the world of the super small. For things that are moving very fast,[1] we need Einstein's theory of special relativity. The goal of fundamental science is to develop a model that accurately describes nature. Since we now have a good conceptual background on the world of the super small, let's discuss the limitations of the quantum mechanical model.

- **Things moving fast:** The Schrödinger equation is only for slow-moving-particles. However, physicists have incorporated the ideas and mathematical models of special relativity into the Schrödinger equation. But none of these did a complete job. If you are interested in this topic, you should search the internet for "the Dirac equation" and "Klein–Gordon equation." Just a heads-up, the Wikipedia pages for these two equations are extremely mathematical. For completeness, folks also use something called perturbation theory that helps approximate the effects of fast-moving particles. It can do a really good job, but, in the end, perturbation theory is just an approximation.
- **Radioactive decay:** The mathematical model behind quantum mechanics, the Schrödinger equation, conserves particle number.[2] This will be one of the first things you prove if you take a quantum mechanics class.[3] Particle conservation means if a particle exists at one point in time, it has always existed and will always exist. We know this is not true. We create and destroy particles all the time! This is the whole purpose of a particle accelerator. In addition, we already talked about radioactive decay. For example, during β^- decay, a neutron stops existing and a proton, electron, and anti-electron neutrino come into existence. That means we went from 1 neutron to 0 neutrons. Particle number is definitely not conserved.
- **Excited state decay:** An atom that has an electron in an excited state will decay to a lower energy state and emit a photon. This process is random and mathematically modeled by an exponential decay (this was talked about back in Chap. 3 if you need a refresher). However, to get an atom to decay, it needs to be acted upon from the outside. Think about how we need force (or torque) to change an object's momentum (or angular momentum). But an excited state atom seems to decay all by itself with no external interactions.
- **Quarks**: Physicists discovered that a proton is composed of three particles known as quarks. While we will talk more about quarks in Sect. 11.5, the really

[1] By very fast, we mean something like 10% of the speed of light or faster.

[2] This is also called conservation of probability.

[3] At least you should. It is super important to always know the limitations of your model.

important thing here is that the mass of the three individual quarks is only about 1% of the mass of the proton. Think about that for a minute. Imagine I hand you 3 steel balls. You weigh each one individually and determine two of the balls have a mass of 2.2 grams and one of the balls has a mass of 4.7 grams. Now you put all 3 steel balls on the scale. A logical guess would say the total mass should be 2.2 grams + 2.2 grams + 4.7 grams = 9.1 grams. Instead, you find that the mass is 938 grams. Replace grams with a much smaller unit for mass and you have what actually happens ...

- **The speed of causality:** Imagine you had two charged particles that are separated by a few meters. Next you nudge one of the charged particles. According to the Schrödinger equation (and also classical electromagnetic theory), the other particle reacts instantaneously. You could repeat this experiment with one of the charged particles on earth and the other on Mars and get the same result. The two charged particles feel a force from the other charged particle and if you move one particle, the other will instantaneously feel a force that pushes or pulls on it. This violates special relativity. Special relativity says the fastest anything can move, including information about where a particle is located, is 299,792,458 m/s. This speed is known as the speed of light or the speed of causality. If the two charged particles were 299,792,458 meters apart and you nudged one of them, the other particle shouldn't know about that for an entire second.
- **Spectroscopy:** Simply put, the Schrödinger equation gets pretty close to calculating the correct energy for atomic states, but it is still wrong.

11.2 The Uncertainty Principle Part 4

In Chap. 6, we introduced the uncertainty principle. To summarize, the uncertainty principle says that if two observables are incompatible, we cannot simultaneously know the values of both observables. More specifically, measuring one observable puts the system in a superposition of the basis set for the other incompatible observable, so we will have a probabilistic result for the value of the second observable. The most famous uncertainty principle, which we talked about in Sect. 6.5, is the Heisenberg Uncertainty Principle named after the German physicist Werner Heisenberg. It is

$$(\Delta x)(\Delta p) \geq \frac{\hbar}{2} \tag{11.1}$$

where Δx is the uncertainty (or range of possible measurements) in the position of the particle and Δp is the uncertainty (or range of possible measurements) in the momentum of the particle. The greater than or equal sign is important as well. For some systems, the product of the two uncertainties is close to $\hbar/2$ while other systems are not. For example, the hydrogen atom with the electron in the ground state has a range of possible position values that can be calculated to be $\Delta x = \frac{\sqrt{3}}{2} a_0$, where $a_0 = 5.29 \times 10^{-11}$ m is a constant called the Bohr radius, named after Danish

physicist Niels Bohr. The range of measurements for momentum can be calculated as well, $\Delta p = \frac{\hbar}{a_0}$. The product of the two is $\frac{\sqrt{3}}{2}\hbar$. Notice this is larger than the minimum value of $\frac{\hbar}{2}$.

In quantum mechanics, every pair of incompatible observables has an uncertainty principle. An important uncertainty principle for this discussion is the energy-time uncertainty principle, which is mathematically written as

$$(\Delta E)(\Delta t) \geq \frac{\hbar}{2}. \tag{11.2}$$

This is a very useful uncertainty principle! It says that a state that only exists for a short time cannot have a definite energy. This is why an excited state in an atom has a natural linewidth (Chap. 3). The lifetime of an excited state tells us how long, on average, an electron stays in that state, see Fig. 3.6 and Eq. 3.7. Some electrons move to a lower energy state quickly, while some hang around for a while. This uncertainty principle helps us understand why the lifetime is connected to the natural linewidth of a state, see Eq. 3.8 and Fig. 3.8. In fact, the summary at the end of Chap. 3.3 is stating in words this uncertainty principle!

Let's work with Eq. 3.8. For ease, here is the equation again:

$$\tau = \frac{1}{\Gamma}. \tag{11.3}$$

τ is the lifetime of an excited state, but this is a characteristic time, see the graph in Fig. 11.1. Some states decay quickly ($t < \tau$) while others decay more slowly ($t > \tau$). There is a range or uncertainty in the time it takes for that state to decay. So, let's call the uncertainty in time τ: $\Delta t = \tau$. Now let's think about the energy of the excited state, see the energy level diagram in Fig. 11.1. The energy of the state is centered at the resonance frequency, but there is a width to that resonance. We call that width the natural linewidth. In energy units, the width is $\hbar\Gamma$. This is the range or uncertainty of the excited state: $\Delta E = \hbar\Gamma$. Let's multiply the two uncertainties

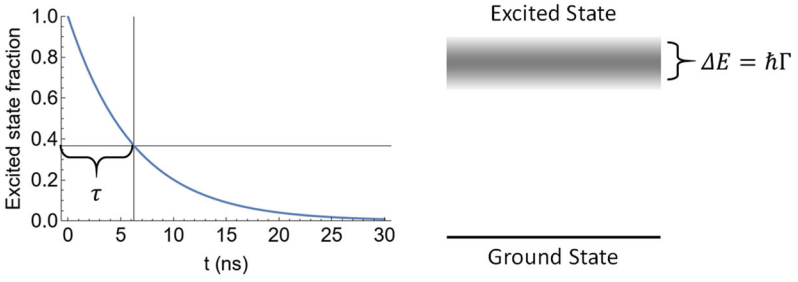

Fig. 11.1 An electron in an excited state decays with a characteristic lifetime τ. That lifetime is related to the linewidth of the excited state by Eq. 11.3. If τ is small, Γ is large. Conversely, if τ is large, Γ is small

together and compare it to the energy-time uncertainty principle, Eq. 11.2:

$$(\Delta E)(\Delta t) = \hbar\ \Gamma\tau \text{ (for an electron in an excited state)} \tag{11.4}$$

But, Eq. 11.3 tells us that $\Gamma\tau = 1$, so we find

$$(\Delta E)(\Delta t) = \hbar \text{ (for an electron in an excited state)} \tag{11.5}$$

This is really neat! We now know why a spectral feature has a width and an excited state has a lifetime: time and energy are incompatible observables. While that was fun, you might be thinking, "Why is this in Chap. 11, a chapter devoted to the issues and problems in quantum mechanics?" We will need it in Sect. 11.4.2 to understand something called vacuum fluctuations. But first, we need to discuss one more concept: **antimatter**.

11.3 Antimatter

The Famous Equation
In special relativity, Einstein showed us that energy and mass are the same thing:

$$E = mc^2$$

This equation is for a particle at rest. If we could somehow convert a proton, which has a mass of $1.6726219 \times 10^{-27}$ kg, to energy, we would have:

$$E = mc^2 = 938.27 \times 10^6 \text{ eV} = 938.27 \text{ MeV}$$

That is a lot of energy![a] This equation also provides us with a brand new unit for mass. The mass of the proton is $938.27\ \text{MeV}/c^2$.

[a]Converting mass to energy is the basic idea behind nuclear reactors.

In nature, there are a number of conservation laws that seem to always be true. For example, conservation of energy tells us that for an isolated system (i.e., nothing comes in or out), the total energy of the system is a constant value. Other conserved quantities include charge and angular momentum. As we will discuss in the next section, atom number is not a conserved quantity. For example, we can, and have, created particles from high-energy photons. This process is called **pair production**. To accomplish this, we send a high-energy photon towards a nucleus. The nucleus "nudges" the photon a little bit and, if the photon has enough energy, it transforms

into an electron and a positron.[4] Before the transformation, the photon existed while the electron and positron did not. After the transformation, the photon no longer exists.

A positron, which we introduced at various times throughout the book, is known as the antimatter partner to the electron. It has the same mass and spin as the electron, but it has opposite (positive) charge. Conservation laws are really helpful at telling us the properties of the positron. Since the photon has spin 1 and the electron has a spin of 1/2, conservation of spin tells us the positron also has a spin of 1/2. A photon has no charge. Likewise, the charges of the electron and positron are the same magnitude, but with opposite signs. This is called conservation of charge. Since the photon has no electric charge, if the photon is to be converted into particles, those particles must together have no charge. There are other differences between the electron and positron as well, but we aren't going to go through all of them.[5] If a positron collides with an electron, the two particles "annihilate" each other back into photons. Before the annihilation, the electron and positron existed while the photons did not. After the annihilation, the electron and positron no longer exist.

Other examples of pair production include creating a muon and an antimuon (see Sect. 11.5), a proton and an antiproton, and a neutron and an antineutron. For this book, the exact details of antimatter don't really matter. What matters is that these antimatter particles can exist, and we have created a lot of antimatter in accelerators or through studying β^+ decay. If an antimatter particle collides with its matter partner, they will annihilate each other creating photons.[6]

Fun Fact

The nuclei of an atom can also have excited states similar to the electrons in the atom. These nuclear excited states are much higher energy compared to electron excited states. Almost all of these excited states emit a high energy photon called an x-ray or a gamma ray to transition back to the nuclear ground state. However, there are nuclei that emit matter and antimatter pairs to transition back to the nuclear ground state! As an example, oxygen-16 has an excited state about 6050×10^3 eV above the nuclear ground state.[b] When

(continued)

[4] The nucleus is the external interaction needed to start the process. As a general rule, you always need something acting on the system to initiate such a process.

[5] If you'd like to explore more differences, search the internet for parity. Parity is a concept that is easy to say but hard to understand. Electrons and positrons have opposite parity. Fun fact: we used to think parity was a conserved quantity. However, in 1956, Chinese American physicist Tsung-Dao Lee and Chinese physicist Chen-Ning Yang proposed a theory that the weak force does not conserve parity. This idea was experimentally confirmed by Chinese American physicist Chien-Shiung Wu and her collaborators.

[6] For completeness, other particles besides photons can also be created during annihilation.

an oxygen-16 nucleus is in this excited state, it would violate conservation of angular momentum if it did decay to the nuclear ground state by emitting a photon. So, instead the nucleon gets rid of the energy by emitting an electron and positron in order for the nucleus to transition back to the nuclear ground state.

[b]For comparison, a 450 nm wavelength photon has an energy of 2.75 eV. So, this nuclear excited state has over 2 million times more energy!

11.4 Going from Quantum Mechanics to Quantum Field Theory

11.4.1 Remove the Conservation of Particle Number Constraint

Reminder
Quantum mechanics conserves particle number. However, unstable atoms decay, and we create and destroy particles all the time at particle accelerators.

To fix this incompatibility with nature (i.e., the theory says no, but experiment says yes), physicists had to remove the constraint of conservation of particle number from their models. Particle number is clearly not conserved in nature, so we should try to make a model that doesn't conserve particle number. However, this causes a really interesting problem. The Schrödinger equation models an individual particle. What do we do if the particle doesn't always exist? The solution was to form a mathematical model that doesn't try to model a particle, but something called a field. But what is a field? The easiest way to understand a field is with a few examples. Figure 11.2 is a temperature field of the minimum temperatures across the Nordic countries Norway, Sweden, and Finland from January 27, 1999. The region had an unusually cold day.[7] For reference, $-50°\,\mathrm{C} = -58°\,\mathrm{F}$ and $0°\,\mathrm{C} = +32°\,\mathrm{F}$. This type of field is known as a scalar field. Every point on the map has a temperature. If I told you a coordinate, you could tell me the temperature at that coordinate. In other words, this scalar field is really some function $T(x, y)$. If you knew the function, you could quickly find the temperature for any coordinate values of x and y.

Figure 11.3 is another type of field known as a vector field. This is a plot that shows the speed of wind in a simulated tornado. This is, again, a function, but the answer that you get from this function is a vector. It tells you both the speed of the

[7] Det här är för kallt!

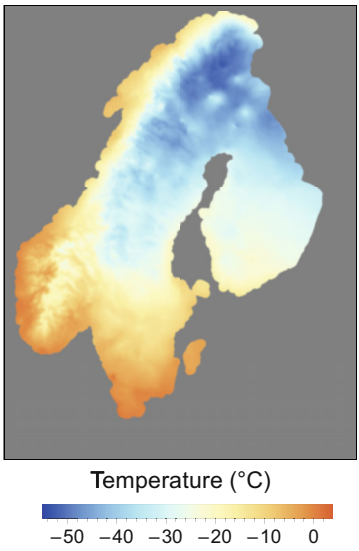

Fig. 11.2 A scalar field showing the minimum temperatures across the Nordic countries Norway, Sweden, and Finland on January 27, 1999. The data used to make this plot was taken from the Copernicus Climate Change Service, Climate Data Store, see Reference [1]

wind and the direction. Other vector fields you might have heard about are electric fields and magnetic fields.

Instead of following around a particle, quantum field theory keeps track of a field. Particles are "excitations" of the fields. As an analogy, let's think about the standing waves on a quantum mechanical string, see Fig. 6.2. The background field in this

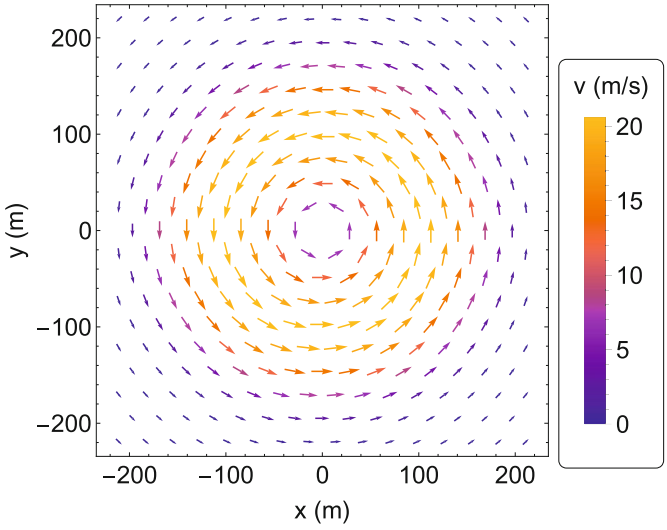

Fig. 11.3 A (simulated) vector field showing the wind speed of a tornado that has a diameter of about 300 m and a top speed of about 20 m/s. This type of plot gives us both direction (the flow is counterclockwise) and the speed of the air

analogy would be the state with a single loop. The particle would be an excitation of this field, or the state with two loops. A quantum particle can go into the field (in our analogy, the system went from two loops to one) or come out of it (one loop to two). Consider a free neutron moving through space. We know from Chap. 10 that a free neutron is unstable and will decay into a proton, an electron, and an anti-electron neutrino with a half-life of about 11 minutes. From a mathematical viewpoint, that neutron is described as moving through a quantum field. When the neutron decays, the neutron goes into the field and a proton, electron, and anti-electron neutrino come out of the field.[8]

Important Summary
Removing the constraint on conservation of particle number forced physicists to develop a model that doesn't rely on a particle always existing. Instead, quantum field theory models fields where particles can go into or out of these fields.

11.4.2 Vacuum Fluctuations

Vacuum fluctuations are temporary changes in the energy of a point in space due to the energy-time uncertainty principle, Eq. 11.2. Removing the constraint on conservation of particle number combined with the energy-time uncertainty principle, results in bizarre behavior. Quantum field theory predicts that particles are continually popping out of and returning into the field. Interestingly, this behavior is not forced; it naturally comes from removing conservation of particle number and including the energy-time uncertainty principle in the model! As a rough analogy, quantum field theory says these particles are excitations of fields similar to exciting a standing wave from one mode to another. This excited state will not last forever, but decay back into the field after a characteristic time Δt.

Think about that for a moment. According to quantum field theory, there is no such thing as empty space. Space is constantly being filled with particles that pop out of and decay back into fields. This might disturb you a bit. Einstein's famous equation says that $E = mc^2$. Reading this equation, we would say that mass and energy are the same thing just like laser frequency, wavelength, and photon energy are all the same thing.

So, doesn't this violate conservation of energy? Particles that have mass are popping out of the field. Mass is energy, so energy that didn't exist now exists. However, the energy-time uncertainty principle actually allows nature to seemingly violate conservation of energy, but only for a very short amount of time. The more

[8] To be completely correct, only a quark inside the neutron goes into the field and another quark comes out, but we haven't talked about quarks yet.

massive the particle that pops into existence (a big ΔE), the quicker that particle must return to the field (a small Δt). Removing conservation of particle number combined with the energy-time uncertainty principle allows this odd behavior to happen. It is important to reiterate that this behavior is a natural consequence and not something forced into the math. The temporarily existing particles are known as **virtual particles**.

These fluctuations can be used to explain why an electron in an excited state decays back to the ground state. These virtual particles, when they exist, push on the atom from the outside. This is the external interaction that causes the electron to fall back to the ground state. Remarkably, theorists can use these vacuum fluctuations to calculate the lifetime of an excited state, and their math correctly predicts the lifetime of excited states.

These virtual particles also explain why the mass of a proton or neutron is so much heavier than the three quarks that make them up. Inside the proton (or neutron), the three quarks are bound tightly together by the strong nuclear force, mediated by virtual particles known as gluons. We will discuss quarks and gluons more in Sect. 11.5. Because of this strong binding, quantum field theory predicts that virtual particles are continuously generated and annihilated within the field. The time-averaged mass of the proton thus includes not only the mass of the three quarks but also the contributions from these virtual particles. Ready to have your mind blown: 99% of the mass of a proton (or neutron) comes from particles popping into and out of the field. That means that 99% of your mass has now gone back into the field and been replaced with new particles from the field.[9]

I recognize that this all probably seems very, very odd. But, every experiment seems to confirm that these surprising predictions are true.

11.4.3 Speed of Causality and Feynman Diagrams

Einstein's theory of special relativity put a speed limit on the universe. Nothing, including information, can travel faster than the speed of light. However, the Schrödinger equation has instantaneous information transfer. How do we solve this problem of instantaneous information transfer?

The answer is **force carriers**. These force carriers are particles that temporarily come from the field to transmit information from one particle to another. These ideas are nicely illustrated in Feynman diagrams, which are named after the American physicist Richard Feynman.

[9] For completeness, there are other ways to think about this extra mass. Some physicists prefer to use something called binding energy or simply "fluctuations of the field" to account for the extra mass. All of these ideas are mathematically correct. I have found that virtual particles are a very accessible way for first-time learners to think about the world of the super small. They will also be helpful when we get to force carriers, which are virtual particles, in the next section.

How to read a Feynman diagram:

- Time goes from the bottom to the top.[c] The horizontal axis looks like it should describe a spatial coordinate, but it doesn't. The horizontal axis doesn't really mean anything.
- Particles are represented using straight lines with arrows on them. Matter has arrows that point in the same direction as time (up) while antimatter has arrows that point in the opposite direction as time (down).
- Force carriers are wavy lines.
- Particle creation and annihilation occur at the vertices.

[c]Some physicists like to rotate their Feynman diagrams so that time goes left to right instead of bottom to top.

A Feynman diagram is not just a wonderful way to visualize a quantum mechanical process. Each diagram is actually a visual representation of a complicated mathematical equation from quantum field theory. An example Feynman diagram can be seen in Fig. 11.4. In this Feynman diagram, we have two electrons that exchange a "virtual photon," which can exist for a short amount of time because of the energy-time uncertainty principle. That virtual photon comes from the field and does not exist for very long before returning to the field.

The virtual photon transmits information, including the electromagnetic force, from one electron to the other electron. This is how quantum field theory describes the Coulomb interaction! When two electrons are close to each other, the virtual photons that mediate the electromagnetic force between them can exist for a short time due to the energy-time uncertainty principle. Since the virtual photons don't have to exist for a very long time, they can have higher energy, resulting in a strong repulsive interaction between the like charges. As the two electrons move further

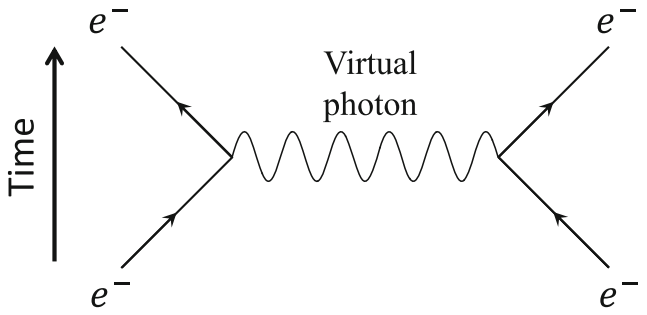

Fig. 11.4 An example of a Feynman diagram. This diagram illustrates a possible interaction between two electrons

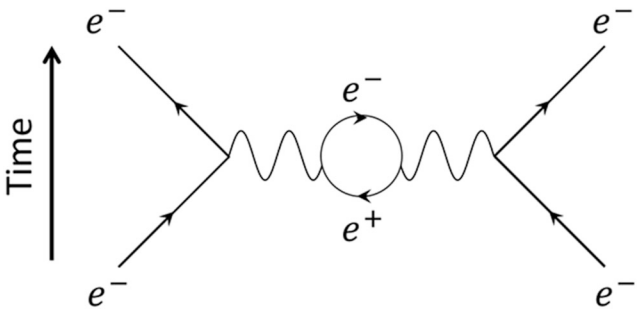

Fig. 11.5 A Feynman diagram depicting how a virtual photon can momentarily split into an electron-positron pair while being exchanged between two electrons

apart, the virtual photons need to exist for a longer time to mediate the interaction, which means they have less energy. Consequently, the force between the two charged particles decreases with distance. This is why, according to quantum field theory, the repulsive or attractive force between two charged particles diminishes as the distance between them increases.

Other things can happen as well. For example, Fig. 11.5 describes how a virtual photon can split into an electron-positron pair when traveling between the two charged particles. This doesn't happen very often compared to the simple exchange of a virtual photon (the electron-positron pair have much more energy, so the energy-time uncertainty principle makes it harder for this to happen), but when the electron-position pair pop into existence from the field, they have a real impact on the system.

Consider an electron in an atom. That electron is constantly interacting with the protons in the nucleus via virtual photons. Every once in a while, that virtual photon breaks apart into an electron-positron pair similar to Fig. 11.5 (just replace one of the electrons with a proton). The effect of the electron-positron pair is that the transition frequency between any two states is shifted by just a little bit, but this small shift is still big enough for us to experimentally measure!

There are other things besides virtual photons and electron-positron pairs that can happen, and theorists calculate the effect that all of these different scenarios will have on an atom. Adding up all of the different possible scenarios produces a single overall shift to the transition frequency. Experimentalists then go measure the transition frequency.

Quantum field theory uses all these ideas to make predictions about the atom. Remarkably, these predictions have almost always been confirmed by experiment. In fact, the theory has been confirmed so many times that if there is a disagreement between theory and experiment, we all assume someone made an error.

There are two major subfields to quantum field theory: quantum electrodynamics and quantum chromodynamics. Quantum electrodynamics (QED) is a mathematical framework that describes interactions between charged particles and light (including virtual photons). Quantum chromodynamics (QCD) is a mathematical framework

that describes the interactions inside the nucleus including the interactions of the quarks inside the proton and neutron.

11.4.4 One More Thing

You may have heard some form of the expression, "Particle accelerators make particles." This is a true statement, but we now have the background to understand how they make new particles. According to quantum field theory, we have virtual particles popping into existence from the field and hanging out for a short amount of time before returning to the field.

Important Reminder
Einstein showed us that energy and mass are the same thing

$$E = mc^2$$

Let's take two protons and give them a ton of kinetic energy. Next, we are going to smash the two protons together. If the kinetic energy of the protons is larger than the energy equivalent of the particle's mass coming from the field ($E = mc^2$), the production of that particle does not violate conservation of energy. In this case, the energy-time uncertainty principle is not the limiting factor, allowing the particle to exist as a real particle. This is exactly what particle accelerators do. Almost all of the particles created in accelerators are unstable and undergo radioactive decay, but they exist long enough for us to determine important properties of the particles, such as mass, charge, and spin.

11.5 The Standard Model of Particle Physics

Quantum field theory is currently the best mathematical model that we have to describe the world of the super small. The conceptual model is called the Standard Model of Particle Physics, or just the Standard Model for short, see Fig. 11.6.

A Little History Before the 1960s, the world of particle physics was really confusing. Physicists at particle accelerators were creating hundreds of new particles. It was a little overwhelming; could there really be hundreds upon hundreds of elementary particles? There were so many new particles that in 1956, an American physicist named Robert Oppenheimer coined the term "subnuclear zoo."

In 1964, American physicists Murray Gell-Mann and George Zweig independently realized that the whole crazy picture could be simplified to just a few elementary particles: quarks. Despite particle accelerators seemingly finding new

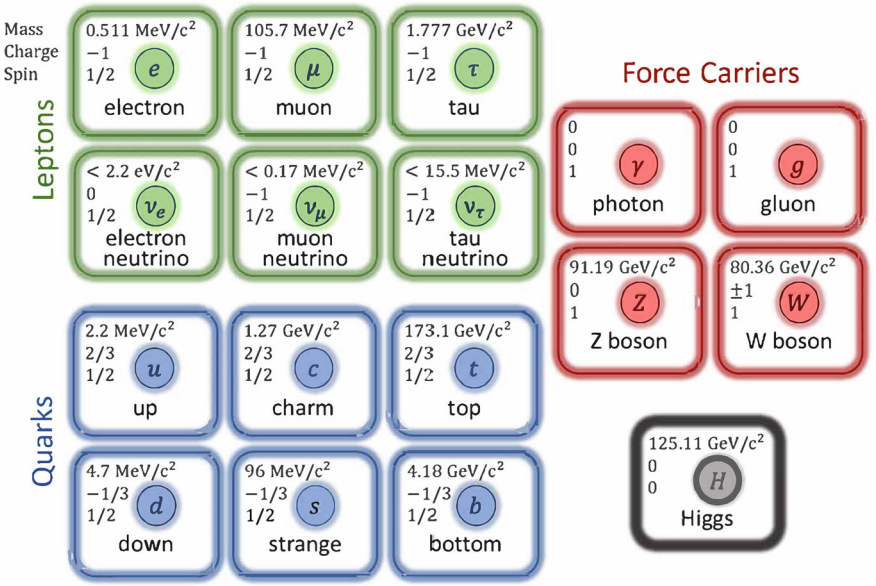

Fig. 11.6 The standard model of particle physics

particles almost every week, what they figured out was that all of the particles were made from twelve elementary particles and their antimatter counterparts. Six of these are now called the up quark, the down quark, the charm quark, the strange quark, the top quark, and the bottom quark,[10] and all of their antimatter counterparts.

At the moment, our best conceptual model of the world of the super small says that everything in the universe is made up of the elementary particles shown in Fig. 11.6. There are four major groupings in the Standard Model.

- **Leptons** are elementary particles that do not experience the strong force and include electrons, muons, taus, neutrinos, and all their antimatter counterparts. They are colored green in Fig. 11.6.
- **Quarks** are elementary particles that experience all fundamental forces, including the strong force. Each quark has an antimatter counterpart. Any particle made from quarks, including protons and neutrons, is called a hadron. They are colored blue in Fig. 11.6.
- **Force Carriers** are particles that mediate the fundamental forces. They include the photon (electromagnetic force), W and Z bosons (weak force), gluons (strong force), and the hypothetical graviton (gravity). They are colored red in Fig. 11.6.
- **Higgs Boson** is an elementary particle associated with the Higgs field, which gives mass to other particles. It is colored black in Fig. 11.6.

[10] Some of these names are odd, but it was the 1960s and 1970s . . . .

Protons are composed of 2 up quarks and 1 down quark. A neutron is made up of 1 up quark and 2 down quarks. Helium-4, which has 2 protons, 2 neutrons, and 2 electrons, is actually made up of 6 up quarks, 6 down quarks, and 2 electrons. All of the particles from the subnuclear zoo were made from the six quarks and their antimatter partners. For example, one of the many particles from the zoo is known as the charmed sigma particle, which is actually composed of 2 up quarks and 1 charm quark. Another particle from the zoo is the pion, which is composed of 1 up quark and 1 anti-down quark (the antimatter version of the down quark).

The four elementary particles colored red are the force carriers. Even though the gluon is listed once, there are actually 8 different types of gluons. All 8 of them are carriers for the strong force, which transmit information between quarks. The photon is the force carrier that transmits information between charged particles. Finally, the Z boson and the two W bosons are force carriers for the weak nuclear force, which is the force responsible for β^- decay, β^+ decay, and electron capture. Also, notice how massive the W and Z bosons are. This is why the weak force is such a short range force. The uncertainty principle tells us that those force carriers simply can't exist for very long making their range very, very short.

We can now be more precise in what happens during β^- decay, see Fig. 11.7. During this process, a down quark in the neutron is transformed into an up quark, resulting in the neutron becoming a proton. This transformation is facilitated by the emission of a W^- boson, which quickly decays into an electron and an anti-electron neutrino. The proton remains in the nucleus, while the electron and anti-electron neutrino are emitted from the atom.

Once again, I admit that all of this seems a little ... out there. But everything we have talked about so far really just naturally falls out of the math when we remove or add the restrictions that we talked about in Sect. 11.1. More importantly, the math behind this conceptual model seems to do an amazing job both modeling and predicting the world of the super small.

The final fundamental particle in the Standard Model is the Higgs boson, named after British physicist Peter Higgs. Simply put, the Higgs field interacts with all particles that have mass. Particles with mass are constantly interacting with the

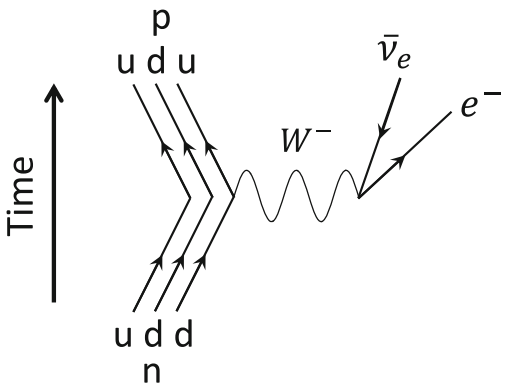

Fig. 11.7 A Feynman diagram showing how a neutron decays into a proton. A neutron is composed of 2 down quarks and 1 up quark. One of the down quarks is transformed into an up quark. This transformation is facilitated by the emission of a W^- boson, which quickly decays into an electron and an anti-electron neutrino

Higgs field through the Higgs boson. This interaction is, according to the Standard Model, the thing that gives the particles mass. This was the last particle to be discovered by particle physicists. The Higgs boson itself was theoretically predicted back in 1964 and finally created in a particle accelerator in 2012. Peter Higgs and Belgian physicist François Englert won the Nobel Prize for the Higgs boson in 2013.

11.6 So, What's Next?

This is exactly the question physicists are always asking! We want to understand nature. To do so, we develop models and test the models to make sure they accurately describe nature. There are different ways to test the Standard Model, and Part 1 of this book discusses how atomic and nuclear physicists use spectroscopy as a tool to test these models. The Standard Model of Particle Physics, which is mathematically described by quantum field theory, is our current best model for the world of the super small. The theory has been tested over and over again, and it has succeeded almost every single time. However, we know the theory is not complete. There are plenty of things in the universe that are not included in the Standard Model. That means that we might be able to create a more complete theory. Here is a list of some of the phenomena that are not in the Standard Model.

Gravity Believe it or not, quantum field theory does not include gravity. Einstein's theory of general relativity is our best model of gravity. General relativity has predicted fascinating phenomena such as gravitational lensing (the theoretically predicted and experimentally measured phenomenon of light bending around massive astronomical objects like large stars and galaxies) and gravitational waves (the theoretically predicted and experimentally measured phenomenon that creates "ripples" in space when two black holes merge; quite literally space is compressed just a little bit and this compression wave ripples out to be detected on earth). Like quantum field theory, general relativity is an incredibly successful theory. Interestingly, the two theories are incompatible with one another. There are a number of reasons why, but one of the big reasons is that general relativity requires "spacetime" to be smooth and continuous while quantum field theory discretizes the fields. In other words, quantum says no to smooth and continuous while general relativity says no to discrete and bumpy. Quantum field theory also needs a force carrier to transmit gravitational information between two objects with mass, which we call a graviton. However, we have yet to create and measure a graviton in the lab.

Where Is All the Antimatter? Quantum field theory predicts there should be about equal parts matter and antimatter. For example, Fig. 11.8 shows a photon that has enough energy to create particles (we have done this at accelerators!). The particles that are created are always a matter/antimatter pair. However, when we look out in the universe, we only see matter. So, where is all the antimatter? We don't know. This problem is called baryon asymmetry. This isn't a bad thing. If the universe was equal parts matter and antimatter, they would have annihilated each other. So,

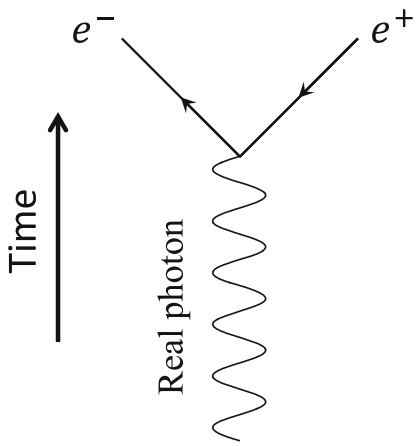

Fig. 11.8 Everything we have done in the lab produces equal amounts of matter and antimatter, like this photon making an electron and a positron

in a way, this is one experimental measurement that the Standard Model failed to predict. The Standard Model predicts equal amounts of matter and antimatter, while observations show a universe dominated by matter.

Why Do Neutrinos Have Mass? This one is short. The Standard Model says that neutrinos don't have mass. However, they do. This is why I use the phrase "The theory has been tested over and over again, and it has succeeded almost every single time." The Standard Model says neutrinos don't have mass, but we have found that they do.

What Is Dark Matter and Dark Energy? This one is also short. From astronomical observations, the universe is thought to be about 5% atoms, 26% dark matter, and 69% dark energy. However, we don't even know what dark matter or dark energy is . . . so it is a little hard to include them in a model. Still, it is kinda weird that our most successful model ever tested (quantum field theory) doesn't know how to deal with 95% of the universe.

Even though quantum field theory has been incredibly successful, there is still a lot to learn! For physicists, this is the best thing ever. There have been some incredibly interesting and ingenious ideas to make the Standard Model more complete. Some of these ideas include string theory, supersymmetry, and loop quantum gravity. However, no one has come up with any way to definitively test if any of these ideas are correct. But this is what makes everything so exciting! We still have more to learn, more to understand, and more to explore. To learn more and push our understanding always onward, we use the most important equation in all of physics:

$$\text{questions} + \text{repetition} + \text{critical thinking} = \text{mastery}$$

Problems

11.1 Explain why the Schrödinger equation is not suitable for describing particles moving at speeds close to the speed of light.

11.2 Describe the process of pair production and explain why it violates the conservation of particle number in quantum mechanics.

11.3 In this chapter, we talked about temperature as an example of a scalar field and the wind velocity of a tornado as an example of a vector field.

(a) What is another example of a scalar field?
(b) What is another example of a vector field?

11.4 In Sect. 11.3, we stated the mass of a proton can be written as $938.27\,\text{MeV}/c^2$.

(a) Show this to be true.
Hint: $1\,\text{MeV} = 1.602176 \times 10^{-13}\,\text{J}$
(b) The mass of an electron is $9.109384 \times 10^{-31}\,\text{kg}$. Write the mass in units of MeV/c^2.
(c) What is the mass of the charm quark in kilograms?

11.5 Draw a Feynman diagram for β^+ decay. The force carrier will be a W^+ boson.

11.6 Draw a Feynman diagram for electron capture. The force carrier will be a W^+ boson.

11.7 Explain how vacuum fluctuations can cause an electron in an excited state to decay back to its ground state.

11.8 An accelerator accelerates two protons to very high energy. The two protons travel in opposite directions with the same speed and collide. The goal is to create a top quark and an anti-top quark.

(a) What is the minimum energy each proton must have to make a top quark-antiquark pair?
(b) The relationship between the energy of a particle and its velocity comes from special relativity. The formula is

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Using your answer from part (a), what is the speed of a proton? Express your answer in units of c. For example, an incorrect answer is $v = 0.92c$, or 92% the speed of light.

Fun Fact The Tevatron was a particle accelerator at Fermi National Accelerator Laboratory (Fermilab), located in Batavia, Illinois, near Chicago. It accelerated protons and antiprotons to energies of 980 GeV, producing proton-antiproton collisions with energies of up to 1.96 TeV. Sadly, the Tevatron shut down in 2011 because the Large Hadron Collider (LHC) at CERN, which became operational in 2008, surpassed the Tevatron in terms of energy levels. The LHC can collide protons at energies of 7 TeV per beam (14 TeV in total). CERN is located in Geneva, Switzerland, and it is currently the world's largest particle accelerator. CERN is an acronym for the French name "Conseil Européen pour la Recherche Nucléaire," which translates to "The European Organization for Nuclear Research."

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The Periodic Table

A

W. Raven, *Atomic Physics for Everyone*,
https://doi.org/10.1007/978-3-031-69507-0

A Table of the Elements

B

Below is a list of all of the known elements sorted by the number of protons in the nucleus (Z). The last column lists the mass number (A) for all of the stable isotopes. For example, the stable isotope magnesium-25 has 12 protons and 13 neutrons for a total of 25 nucleons.

Z	Name	Symbol	Ground state electron configuration	Mass number for stable isotopes
1	Hydrogen	H	$1s\ ^2S_{1/2}$	1, 2
2	Helium	He	$1s^2\ ^1S_0$	3, 4
3	Lithium	Li	$[He]2s\ ^2S_{1/2}$	6, 7
4	Beryllium	Be	$[He]2s^2\ ^1S_0$	9
5	Boron	B	$[He]2s^22p^1\ ^2P_{1/2}$	10, 11
6	Carbon	C	$[He]2s^22p^2\ ^3P_0$	12, 13
7	Nitrogen	N	$[He]2s^22p^3\ ^4S_{3/2}$	14, 15
8	Oxygen	O	$[He]2s^22p^4\ ^3P_2$	16, 17, 18
9	Fluorine	F	$[He]2s^22p^5\ ^2P_{3/2}$	19
10	Neon	Ne	$[He]2s^22p^6\ ^1S_0$	20, 21, 22
11	Sodium	Na	$[Ne]3s^1\ ^2S_{1/2}$	23
12	Magnesium	Mg	$[Ne]3s^2\ ^1S_0$	24, 25, 26
13	Aluminium	Al	$[Ne]3s^23p^1\ ^2P_{1/2}$	27
14	Silicon	Si	$[Ne]3s^23p^2\ ^3P_0$	28, 29, 30
15	Phosphorus	P	$[Ne]3s^23p^3\ ^4S_{3/2}$	31
16	Sulfur	S	$[Ne]3s^23p^4\ ^3P_2$	32, 33, 34, 36
17	Chlorine	Cl	$[Ne]3s^23p^5\ ^2P_{3/2}$	35, 37
18	Argon	Ar	$[Ne]3s^23p^6\ ^1S_0$	36, 38, 40
19	Potassium	K	$[Ar]4s^1\ ^2S_{1/2}$	39, 40, 41
20	Calcium	Ca	$[Ar]4s^2\ ^1S_0$	40, 42, 43, 44, 46, 48

(continued)

W. Raven, *Atomic Physics for Everyone*,
https://doi.org/10.1007/978-3-031-69507-0

Z	Name	Symbol	Ground state electron configuration	Stable isotopes
21	Scandium	Sc	$[Ar]3d^1 4s^2\ ^2D_{3/2}$	45
22	Titanium	Ti	$[Ar]3d^2 4s^2\ ^3F_2$	46, 47, 48, 49, 50
23	Vanadium	V	$[Ar]3d^3 4s^2\ ^4F_{3/2}$	50, 51
24	Chromium	Cr	$[Ar]3d^5 4s^1\ ^7S_3$	50, 52, 53, 54
25	Manganese	Mn	$[Ar]3d^5 4s^2\ ^6S_{5/2}$	55
26	Iron	Fe	$[Ar]3d^6 4s^2\ ^5D_4$	54, 56, 57, 58
27	Cobalt	Co	$[Ar]3d^7 4s^2\ ^4F_{9/2}$	59
28	Nickel	Ni	$[Ar]3d^8 4s^2\ ^3F_4$	58, 60, 61, 62, 64
29	Copper	Cu	$[Ar]3d^{10} 4s^1\ ^2S_{1/2}$	63, 65
30	Zinc	Zn	$[Ar]3d^{10} 4s^2\ ^1S_0$	64, 66, 67, 68, 70
31	Gallium	Ga	$[Ar]3d^{10} 4s^2 4p^1\ ^2P_{1/2}$	69, 71
32	Germanium	Ge	$[Ar]3d^{10} 4s^2 4p^2\ ^3P_0$	70, 72, 73, 74, 76
33	Arsenic	As	$[Ar]3d^{10} 4s^2 4p^3\ ^4S_{3/2}$	75
34	Selenium	Se	$[Ar]3d^{10} 4s^2 4p^4\ ^3P_2$	74, 76, 77, 78, 80, 82
35	Bromine	Br	$[Ar]3d^{10} 4s^2 4p^5\ ^2P_{3/2}$	79, 81
36	Krypton	Kr	$[Ar]3d^{10} 4s^2 4p^6\ ^1S_0$	78, 80, 82, 83, 84, 86
37	Rubidium	Rb	$[Kr]5s^1\ ^2S_{1/2}$	85, 87
38	Strontium	Sr	$[Kr]5s^2\ ^1S_0$	84, 86, 87, 88
39	Yttrium	Y	$[Kr]4d^1 5s^2\ ^2D_{3/2}$	89
40	Zirconium	Zr	$[Kr]4d^2 5s^2\ ^3F_2$	90, 91, 92, 94, 96
41	Niobium	Nb	$[Kr]4d^4 5s^1\ ^6D_{1/2}$	93
42	Molybdenum	Mo	$[Kr]4d^5 5s^1\ ^7S_3$	92, 94, 95, 96, 97, 98
43	Technetium	Tc	$[Kr]4d^5 5s^2\ ^6S_{5/2}$	N/A
44	Ruthenium	Ru	$[Kr]4d^7 5s^1\ ^5F_5$	96, 98, 99, 100, 101, 102, 104
45	Rhodium	Rh	$[Kr]4d^8 5s^1\ ^4F_{9/2}$	103
46	Palladium	Pd	$[Kr]4d^{10}\ ^1S_0$	102, 104, 105, 106, 108, 110
47	Silver	Ag	$[Kr]4d^{10} 5s^1\ ^2S_{1/2}$	107, 109
48	Cadmium	Cd	$[Kr]4d^{10} 5s^2\ ^1S_0$	106, 108, 110, 111, 112, 113, 114, 116
49	Indium	In	$[Kr]4d^{10} 5s^2 5p^1\ ^2P_{1/2}$	113, 115
50	Tin	Sn	$[Kr]4d^{10} 5s^2 5p^2\ ^3P_0$	112, 114, 115, 116, 117, 118, 119, 120, 122, 124
51	Antimony	Sb	$[Kr]4d^{10} 5s^2 5p^3\ ^4S_{3/2}$	121, 123
52	Tellurium	Te	$[Kr]4d^{10} 5s^2 5p^4\ ^3P_2$	120, 122, 123, 124, 125, 126, 128, 130
53	Iodine	I	$[Kr]4d^{10} 5s^2 5p^5\ ^2P_{3/2}$	127
54	Xenon	Xe	$[Kr]4d^{10} 5s^2 5p^6\ ^1S_0$	124, 126, 128, 129, 130, 131, 132, 134, 136
55	Cesium	Cs	$[Xe]6s^1\ ^2S_{1/2}$	133

(continued)

Z	Name	Symbol	Ground state electron configuration	Stable isotopes
56	Barium	Ba	$[Xe]6s^2\ ^1S_0$	130, 132, 134, 135, 136, 137, 138
57	Lanthanum	La	$[Xe]5d^16s^2\ ^2D_{3/2}$	138, 139
58	Cerium	Ce	$[Xe]4f^15d^16s^2\ ^1G_4$	136, 138, 140, 142
59	Praseodymium	Pr	$[Xe]4f^36s^2\ ^4I_{9/2}$	141
60	Neodymium	Nd	$[Xe]4f^46s^2\ ^5I_4$	142, 143, 144, 145, 146, 148, 150
61	Promethium	Pm	$[Xe]4f^56s^2\ ^6H_{5/2}$	N/A
62	Samarium	Sm	$[Xe]4f^66s^2\ ^7F_0$	144, 147, 148, 149, 150, 152, 154
63	Europium	Eu	$[Xe]4f^76s^2\ ^8S_{7/2}$	151, 153
64	Gadolinium	Gd	$[Xe]4f^75d^16s^2\ ^9D_2$	152, 154, 155, 156, 157, 158, 160
65	Terbium	Tb	$[Xe]4f^96s^2\ ^6H_{15/2}$	159
66	Dysprosium	Dy	$[Xe]4f^{10}6s^2\ ^5I_8$	156, 158, 160, 161, 162, 163, 164
67	Holmium	Ho	$[Xe]4f^{11}6s^2\ ^4I_{15/2}$	165
68	Erbium	Er	$[Xe]4f^{12}6s^2\ ^3H_6$	162, 164, 166, 167, 168, 170
69	Thulium	Tm	$[Xe]4f^{13}6s^2\ ^2F_{7/2}$	169
70	Ytterbium	Yb	$[Xe]4f^{14}6s^2\ ^1S_0$	168, 170, 171, 172, 173, 174, 176
71	Lutetium	Lu	$[Xe]4f^{14}5d^16s^2\ ^2D_{3/2}$	175, 176
72	Hafnium	Hf	$[Xe]4f^{14}5d^26s^2\ ^3F_2$	174, 176, 177, 178, 179, 180
73	Tantalum	Ta	$[Xe]4f^{14}5d^36s^2\ ^4F_{3/2}$	180, 181
74	Tungsten	W	$[Xe]4f^{14}5d^46s^2\ ^5D_0$	182, 183, 184, 186
75	Rhenium	Re	$[Xe]4f^{14}5d^56s^2\ ^6S_{5/2}$	185, 187
76	Osmium	Os	$[Xe]4f^{14}5d^66s^2\ ^5D_4$	184, 186, 187, 188, 189, 190, 192
77	Iridium	Ir	$[Xe]4f^{14}5d^76s^2\ ^4F_{9/2}$	191, 193
78	Platinum	Pt	$[Xe]4f^{14}5d^96s^1\ ^3D_3$	190, 192, 194, 195, 196, 198
79	Gold	Au	$[Xe]4f^{14}5d^{10}6s^1\ ^2S_{1/2}$	197
80	Mercury	Hg	$[Xe]4f^{14}5d^{10}6s^2\ ^1S_0$	196, 198, 199, 200, 201, 202, 204
81	Thallium	Tl	$[Xe]4f^{14}5d^{10}6s^26p^1\ ^2P_{1/2}$	203, 205
82	Lead	Pb	$[Xe]4f^{14}5d^{10}6s^26p^2\ ^3P_0$	204, 206, 207, 208
83	Bismuth	Bi	$[Xe]4f^{14}5d^{10}6s^26p^3\ ^4S_{3/2}$	209
84	Polonium	Po	$[Xe]4f^{14}5d^{10}6s^26p^4\ ^3P_2$	N/A
85	Astatine	At	$[Xe]4f^{14}5d^{10}6s^26p^5\ ^2P_{3/2}$	N/A

(continued)

Z	Name	Symbol	Ground state electron configuration	Stable isotopes
86	Radon	Rn	$[Xe]4f^{14}5d^{10}6s^26p^6\ ^1S_0$	N/A
87	Francium	Fr	$[Rn]7s^1\ ^2S_{1/2}$	N/A
88	Radium	Ra	$[Rn]7s^2\ ^1S_0$	N/A
89	Actinium	Ac	$[Rn]6d^17s^2\ ^2D_{3/2}$	N/A
90	Thorium	Th	$[Rn]6d^27s^2\ ^3F_2$	232
91	Protactinium	Pa	$[Rn]5f^26d^17s^2\ ^4K_{11/2}$	231
92	Uranium	U	$[Rn]5f^36d^17s^2\ ^5L_6$	234, 235, 238
93	Neptunium	Np	$[Rn]5f^46d^17s^2\ ^6L_{11/2}$	N/A
94	Plutonium	Pu	$[Rn]5f^67s^2\ ^7F_0$	N/A
95	Americium	Am	$[Rn]5f^77s^2\ ^8S_{7/2}$	N/A
96	Curium	Cm	$[Rn]5f^76d^17s^2\ ^9D_2$	N/A
97	Berkelium	Bk	$[Rn]5f^97s^2\ ^6H_{15/2}$	N/A
98	Californium	Cf	$[Rn]5f^{10}7s^2\ ^5I_8$	N/A
99	Einsteinium	Es	$[Rn]5f^{11}7s^2\ ^4I_{15/2}$	N/A
100	Fermium	Fm	$[Rn]5f^{12}7s^2\ ^3H_6$	N/A
101	Mendelevium	Md	$[Rn]5f^{13}7s^2\ ^2F_{7/2}$	N/A
102	Nobelium	No	$[Rn]5f^{14}7s^2\ ^1S_0$	N/A
103	Lawrencium	Lr	$[Rn]5f^{14}7s^27p^1\ ^2P_{1/2}$	N/A
104	Rutherfordium	Rf	$[Rn]5f^{14}6d^27s^2\ ^3F_2$	N/A
105	Dubnium	Db	$[Rn]5f^{14}6d^37s^2\ ^4F_{3/2}$	N/A
106	Seaborgium	Sg	$[Rn]5f^{14}6d^47s^2\ ^5D_0$	N/A
107	Bohrium	Bh	$[Rn]5f^{14}6d^57s^2\ ^6S_{5/2}$	N/A
108	Hassium	Hs	$[Rn]5f^{14}6d^67s^2\ ^5D_4$	N/A
109	Meitnerium	Mt	$[Rn]5f^{14}6d^77s^2\ ^4F_{9/2}$	N/A
110	Darmstadtium	Ds	$[Rn]5f^{14}6d^97s^1\ ^3D_3$	N/A
111	Roentgenium	Rg	$[Rn]5f^{14}6d^{10}7s^1\ ^2S_{1/2}$	N/A
112	Copernicium	Cn	$[Rn]5f^{14}6d^{10}7s^2\ ^1S_0$	N/A
113	Nihonium	Nh	$[Rn]5f^{14}6d^{10}7s^27p^1\ ^2P_{1/2}$	N/A
114	Flerovium	Fl	$[Rn]5f^{14}6d^{10}7s^27p^2\ ^3P_0$	N/A
115	Moscovium	Mc	$[Rn]5f^{14}6d^{10}7s^27p^3\ ^4S_{3/2}$	N/A
116	Livermorium	Lv	$[Rn]5f^{14}6d^{10}7s^27p^4\ ^3P_2$	N/A
117	Tennessine	Ts	$[Rn]5f^{14}6d^{10}7s^27p^5\ ^2P_{3/2}$	N/A
118	Oganesson	Og	$[Rn]5f^{14}6d^{10}7s^27p^6\ ^1S_0$	N/A

Transition Rules: C

The following transition rules (also known as selection rules) must be satisfied for an electron to transition between two atomic states:

General rules for all atoms:

- $\Delta L = \pm 1$: The electronic orbital angular momentum quantum number must change by 1.
- $\Delta S = 0$: The spin quantum number must remain unchanged.
- $\Delta J = -1,\ 0,\ +1; \quad J = 0 \not\rightarrow J = 0$: The total electronic angular momentum quantum number must change by 1, 0, or -1. However, transitions between $J = 0 \rightarrow J = 0$ are forbidden.
- Parity must change: A transition must occur between states of opposite parity. Parity is indicated by a circle in the term symbol. For example, a state with odd parity has a small circle, like ${}^1\mathrm{P}_1^\circ$. A state with even parity does not have a circle, like ${}^1\mathrm{S}_0$.

For atoms with no nuclear spin:

- $\Delta m_J = -1, 0, +1$: The z-component (projection) of the total electronic angular momentum (m_J) can change by 1, 0, or -1. However, if $\Delta J = 0$, the transition between $m_J = 0 \rightarrow m_J = 0$ is forbidden.

For atoms with nuclear spin:

- $\Delta F = -1,\ 0,\ +1; \quad F = 0 \not\rightarrow F = 0$: The total atomic angular momentum quantum number including nuclear spin (F) must change by 1, 0, or -1. However, transitions between $F = 0 \rightarrow F = 0$ are forbidden.

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- $\Delta m_F = -1,\ 0,\ +1$: The z-component (projection) of the total atomic angular momentum including nuclear spin (m_F) can change by 1, 0, or -1. However, if $\Delta F = 0$, the transition between $m_F = 0 \rightarrow m_F = 0$ is forbidden.

Glossary

Absorption plot: A plot showing the percentage or fraction of photons absorbed from a laser beam as it passes through a vapor cell, plotted as a function of laser frequency. A dip in a transmission plot corresponds to a peak in the absorption plot.

Angular Momentum: A property of a rotating object that changes when a torque is applied. It is the rotational analog of linear momentum.

Antimatter: Particles that have the same mass as their matter counterparts but opposite charge, as well as opposite parity and spin projection. When an antimatter particle collides with its matter counterpart, they annihilate each other, often creating photons.

Atomic Physics: A field of physics that seeks to understand how the components of an atom interact and behave.

Basis set: The complete set of states for a particular observable in a given system.

Boson: a particle with integer spin. Bosons do not follow the Pauli Exclusion Principle, meaning multiple bosons can occupy the same quantum state.

Bra-ket notation: A common notation used to represent states in quantum mechanics. A "ket" represents a particular state and is written as $|\text{state label}\rangle$. A "bra," which is not used in this book but is common in advanced quantum mechanics, is written as $\langle\text{state label}|$.

Center of gravity: The energy of an atomic state assuming the nucleus has no angular momentum.

Compatible observables: Observables that can be precisely measured simultaneously without affecting each other's measurement. Measuring one observable does not alter the system in a way that impacts the measurement of the other.

Coulomb force: The force between two charged particles, given by $F = k_e \frac{q_1 q_2}{r^2}$, where F is the force, $k_e = 8.987 \times 10^9\ \text{N} \cdot \text{m}^2/\text{C}^2$ is Coulomb's constant, q_1 and q_2 are the charges, and r is the distance between them. It describes the attraction or repulsion between charged particles.

Coulomb interaction: The interaction between two charged particles due to their electric charges, governed by the Coulomb force. It is fundamental in understanding the behavior of charged particles in electrostatic situations

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Degenerate or Degeneracy: When multiple states in a quantum system have the same energy.

Detuning: The difference between the laser frequency and the resonance frequency. When using linear frequency units, the variable representing detuning is the lowercase Greek letter delta: $\delta = f - f_r$, where f is the laser frequency and f_r is the resonance frequency. When using angular frequency units, the variable representing detuning is the uppercase Greek letter delta: $\Delta = \omega - \omega_r$, where $\omega = 2\pi f$ is the frequency of the laser in angular frequency units and $\omega_r = 2\pi f_r$ is the resonance frequency in angular frequency units.

Diffraction: The bending of waves around the corners of an obstacle.

Dispersive element: An optical element that spatially separates light into its spectral components.

Doppler broadening: The widening of a spectral feature caused by the thermal motion of atoms.

Doppler effect: The change in frequency of sound, light, or other waves as the source and observer move toward or away from each other.

Doppler shift: The change in the frequency of light observed by an atom due to the Doppler effect.

Doppler profile: A spectral feature broadened due to the thermal motion of atoms.

Doppler width: The full width at half maximum (FWHM) of a Doppler-broadened spectral feature. The variable Δf_{FWHM} represents the Doppler width.

Electromagnetic force: A fundamental force encompassing both electric and magnetic forces, including the Coulomb force and magnetic forces due to moving charges.

Electromagnetic interaction: The interaction between charged particles due to both electric and magnetic forces, including interactions involving stationary charges (electrostatics) and moving charges.

Electron: A fundamental particle found in an atom outside of the nucleus. An electron has a negative charge of -1.602×10^{-19} C, a mass of 9.109×10^{-31} kg, and no measurable radius. Unlike protons or neutrons, electrons are not composed of smaller particles.

Electron Configuration: The arrangement of electrons in an atom or molecule within designated orbitals around the nucleus. Electron configurations are typically represented using notation that indicates the energy levels, subshells, and the number of electrons in each subshell. For example, the electron configuration of the ground state of carbon, which has six electrons, is $1s^2 2s^2 2p^2$.

Electron Shells: The different energy levels around the nucleus of an atom where electrons are likely to be found. These shells are defined by the principal quantum number n, with each shell corresponding to a specific energy level. Electrons in lower-numbered shells are closer to the nucleus, while those in higher-numbered shells are farther away. Each shell can hold a limited number of electrons, determined by the formula $2n^2$. The arrangement of electrons in these shells follows the rules of quantum mechanics and significantly influences the chemical properties of the atom.

Electron Subshells: Subdivisions of electron shells within an atom, characterized by the quantum number ℓ. Each subshell corresponds to a specific value of ℓ and is labeled using the letters s, p, d, f, etc., corresponding to $\ell = 0, 1, 2, 3$, and so on. Subshells define the shape of the electron's wavefunction and can hold a specific number of electrons: the s subshell can hold 2 electrons, the p subshell can hold 6, the d subshell can hold 10, and the f subshell can hold 14. The arrangement of electrons within these subshells follows the Pauli exclusion principle.

The Equipartition Theorem: A theorem in thermodynamics stating that the energy of a system is equally distributed among all its degrees of freedom.

Fermion: a particle with half-integer spin. Fermions follow the Pauli Exclusion Principle, meaning no two fermions can occupy the same quantum state.

Fine structure splitting: The splitting of atomic energy levels due to the interaction between an electron's orbital angular momentum and spin.

Force:

If the object's mass is constant: An external interaction that causes an object to accelerate (i.e., change velocity). If multiple forces act on an object, the net force causes the object to accelerate. This is Newton's 2nd Law for an object with constant mass: "The sum of all forces acting on an object causes the object to accelerate." This is the most common definition of a force but is a special case of the general definition.

If the object's mass is changing: An external interaction that causes an object's momentum to change. Newton's 2nd Law for momentum states: "The sum of all forces acting on an object causes its momentum to change." This is the general definition of Newton's 2nd law.

The variable F represents force.

Force carriers: Virtual particles that mediate the fundamental forces of nature in quantum field theory. They are bosons and are responsible for transmitting forces between particles. The primary force carriers and the forces they mediate are:

- **Photon**: The force carrier of the electromagnetic force. Photons are massless and mediate interactions between charged particles.
- **W and Z Bosons**: The force carriers of the weak nuclear force. These bosons are massive and are responsible for processes such as beta decay in radioactive materials.
- **Gluons**: The force carriers of the strong nuclear force. Gluons are massless and mediate the force that holds quarks together within protons and neutrons.
- **Graviton** (hypothetical): The proposed force carrier of gravity in quantum gravity theories. Gravitons are hypothesized to be massless and mediate the gravitational force, although they have not been experimentally observed.

Frequency: The number of oscillations per second. The variable f represents frequency. The unit for frequency is 1/second, called Hertz (Hz). Other common units for frequency include megahertz ($1\,\text{MHz} = 1 \times 10^6$ Hz), gigahertz ($1\,\text{GHz} = 1 \times 10^9$ Hz), and terahertz ($1\,\text{THz} = 1 \times 10^{12}$ Hz).

Higgs Boson: An elementary particle associated with the Higgs field, responsible for giving mass to other particles.

Hyperfine level: When a nucleus has angular momentum, an atomic state splits into multiple closely spaced states known as hyperfine levels.

Hyperfine structure splitting: The splitting of energy levels in an atom due to nuclear spin.

Incompatible observables: Observables that cannot be precisely measured simultaneously. Measuring one observable alters the system, making subsequent measurements of the other unpredictable.

Intensity: The intensity of a laser beam is the power of the laser divided by the cross sectional area of the laser beam, denoted as $I = P/A$.

Ion: An atom with a net charge due to an unequal number of protons and electrons. An ion with more protons than electrons is called a positive ion, while an ion with more electrons than protons is called a negative ion.

Ionization Threshold: The minimum energy required to remove an electron from an atom or molecule, thereby ionizing it. Each atom has a unique ionization threshold. Most atoms have multiple ionization thresholds depending on the angular momentum and energy state of the remaining electrons.

Isotope shift: The change in the resonance frequency of a transition between two states in an atom due to a change in the number of neutrons in the nucleus. The variable for isotope shift is $\delta f^{AA'}$, where A is the mass number for one isotope and A' is the mass number of the other isotope.

Kinetic energy: The energy of motion. The variable K represents kinetic energy.

Leptons: Elementary particles that do not experience the strong force, including electrons, muons, taus, neutrinos, and their antimatter counterparts.

Mass number: The total number of protons and neutrons in an atomic nucleus, represented by the variable A.

Molecule: A group of two or more atoms bonded together by chemical bonds.

Momentum: A property of a moving object. For a classical object, momentum is given by $p = mv$, where p is the object's momentum, m is the object's mass, and v is the object's velocity. An object's momentum changes if acted upon by an external force. The unit of momentum is kgm/s.

Natural linewidth: The minimum possible full width at half maximum (FWHM) of a spectral feature. The natural linewidth is a property of a transition, with each transition in an atom having a unique natural linewidth. The variable for natural linewidth is the lowercase Greek letter gamma γ when using linear frequency units and the uppercase Greek letter gamma Γ when using angular frequency units.

Neutral atom: An atom with no net charge, having an equal number of electrons and protons.

Neutron: A particle found inside the nucleus of an atom. A neutron has no charge, a mass of 1.675×10^{-27} kg, and a radius of 0.8×10^{-15} m $= 0.8$ fm. A neutron is composed of three quarks: 1 up quark and 2 down quarks. A free neutron is unstable and undergoes β^- decay with a half life of about 11 minutes.

Neutron number: The total number of neutrons in an atomic nucleus, represented by the variable N.

Nucleon: A term referring to either a neutron or a proton.

Nuclear spin: The angular momentum of atomic nuclei, arising from the angular momentum of protons and neutrons.

Observable: A physical quantity that can be measured experimentally, such as energy, position, momentum, and angular momentum.

Pair Production: The creation of a particle and its antiparticle from a photon. Examples include the creation of an electron and a positron, a muon and an antimuon, a proton and an antiproton, and a neutron and an antineutron.

Pauli Exclusion Principle: The principle stating that two fermions cannot simultaneously occupy the same quantum state; no two fermions can have the same set of quantum numbers within a quantum system.

Period: The time it takes for a wave to complete one full oscillation. The variable T represents period, and the unit is seconds. Frequency and period are related by the formula $T = 1/f$.

Photon: A fundamental particle of light, also known as a quantum of electromagnetic radiation. Photons have no mass, travel at the speed of light in a vacuum, and carry energy proportional to their frequency. The energy E of a photon is given by the equation $E = hf$, where h is Planck's constant and f is the frequency of the light. Photons exhibit both wave-like and particle-like properties, a phenomenon known as wave-particle duality.

Physics: Physics is a branch of science that seeks to understand the universe. It explores physical concepts and often explains them using the language of mathematics.

Planck's constant: A fundamental constant in quantum mechanics, with a value of $h = 6.626 \times 10^{-34}$ Js. The reduced Planck's constant is $\hbar = \frac{h}{2\pi} = 1.054 \times 10^{-34}$ Js. Planck's constant has units of angular momentum.

Probe beam: One of two laser beams used in saturated absorption spectroscopy. The probe beam has less power than the pump beam. In saturated absorption spectroscopy, the transmission of the probe beam is measured.

Proton: A particle found inside the nucleus of an atom. A proton has a positive charge of $+1.602 \times 10^{-19}$C, a mass of 1.672×10^{-27} kg, and a radius of 0.84×10^{-15}m $= 0.84$fm. A proton is composed of three quarks: two up quarks and one down quark. A free proton is stable and does not undergo radioactive decay.

Proton number: The total number of protons in an atomic nucleus, represented by the variable Z.

Pump beam: One of two laser beams used in saturated absorption spectroscopy. The pump beam has more power than the probe beam.

Quantum mechanics: A fundamental theory of physics that describes the behavior of atoms, subatomic particles, and molecules.

Quantum number: An integer (positive, negative, or 0) or half-integer used to represent a state of an observable, emphasizing the discrete nature of quantum mechanics. Quantum numbers have no units.

Quarks: Elementary particles that experience all fundamental forces, including the strong force. Each quark has an antimatter counterpart. Particles made from quarks, including protons and neutrons, are called hadrons.

Reduced Planck's constant (h-bar): Planck's constant divided by 2π: $\hbar = \frac{h}{2\pi} = 1.054 \times 10^{-34}\,\mathrm{J\,s}$.

Refraction: The redirection of a wave as it passes from one medium to another (e.g., air to water).

Resonance: When an atom is excited by a photon from one state to another, it "goes through resonance." This is similar to playing the trumpet, where blowing correctly into the trumpet excites a standing wave in the pipe to create a note. Similarly, when an atom is excited, the electron transitions from one standing wave mode to another. Atomic physicists use "excitation" and "resonance" interchangeably.

Resonance frequency: The frequency at which a laser excites an atom from one state to another. The variable f_r represents resonance frequency.

Saturated absorption plot: A plot representing the difference in the transmission of the probe beam with the pump beam on versus off. This technique removes Doppler broadening, leaving only the spectral features that would exist if the atoms were stationary.

Saturated absorption spectroscopy: A spectroscopic technique using a probe beam and a pump beam to produce spectral features from atoms with $v_\parallel = 0$.

Saturation intensity: The intensity of light required for an on-resonance laser to have 25% of atoms in a sample in the excited state at any given time. The variable I_s represents the saturation intensity.

Saturation parameter: The ratio of the laser intensity to the saturation intensity. The variable s represents the saturation parameter.

Scattering rate: The number of photons per second an atom absorbs and re-emits. The variable for scattering rate is $r_\gamma(\delta, s)$ when using linear frequency units and $r_\Gamma(\Delta, s)$ when using angular frequency units.

Schrödinger equation: The fundamental equation in quantum mechanics that describes how the quantum state of a physical system evolves over time. Using this equation, we can determine the wavefunctions and corresponding energy levels of the system.

Spectral component: "White" light is composed of many different wavelengths of light. Even light from the sun, which appears yellow, consists of various wavelengths. A single wavelength within a broader spectrum of light is called a spectral component.

Spectral feature: The lineshape on an absorption or transmission plot arising from the emission or absorption of a photon with energy corresponding to the difference between initial and final states of a transition.

Spectrometer: An instrument that separates and measures the amount of each spectral component.

Spectroscopy: A subfield of atomic physics that studies atoms through their interactions with light.

Standard Model of Particle Physics: A comprehensive theory of particle physics that describes the fundamental particles and their interactions, extending beyond quantum mechanics. It is one of the most successful theories to date.

Superposition: A concept in mathematics and physics stating that if multiple effects act on a system, the overall effect is the sum of the individual effects.

Torque: A measure of the effectiveness of a force in causing an object to rotate. The variable that represents torque is the lowercase Greek letter tau, τ.

Transmission plot: A plot showing the percentage or fraction of photons that pass through a vapor cell as a function of laser frequency.

$v_\parallel$: The velocity component of an atom in the direction of the laser beam. In the Doppler shift formula, $v_\parallel$ is negative if the atom is traveling towards the laser and positive if it is traveling away from the laser.

Virtual particles: Temporary fluctuations in quantum fields that arise from the energy-time uncertainty principle. They can mediate forces, like virtual photons in electromagnetic interactions, or appear as vacuum fluctuations, but they are not directly observable.

Wave interference: A phenomenon occurring when two or more waves overlap, resulting in a wave with greater (constructive interference) or lower (destructive interference) amplitude. Interference effects can be observed with all types of waves, including light waves, water waves, and electron waves.

Waist of a laser: The half-width of the intensity profile where the intensity is 13.5% of the maximum intensity, as shown in Fig. 3.7. The variable w represents waist.

Wavelength: The distance between two consecutive "like" points on a wave, such as two adjacent wave peaks. The variable that represents wavelength is the lowercase Greek letter lambda, λ.

Wave-particle duality: The concept that particles, like electrons and photons, exhibit both wave-like and particle-like behavior depending on the experiment, meaning they can act as waves in some situations and as particles in others.

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