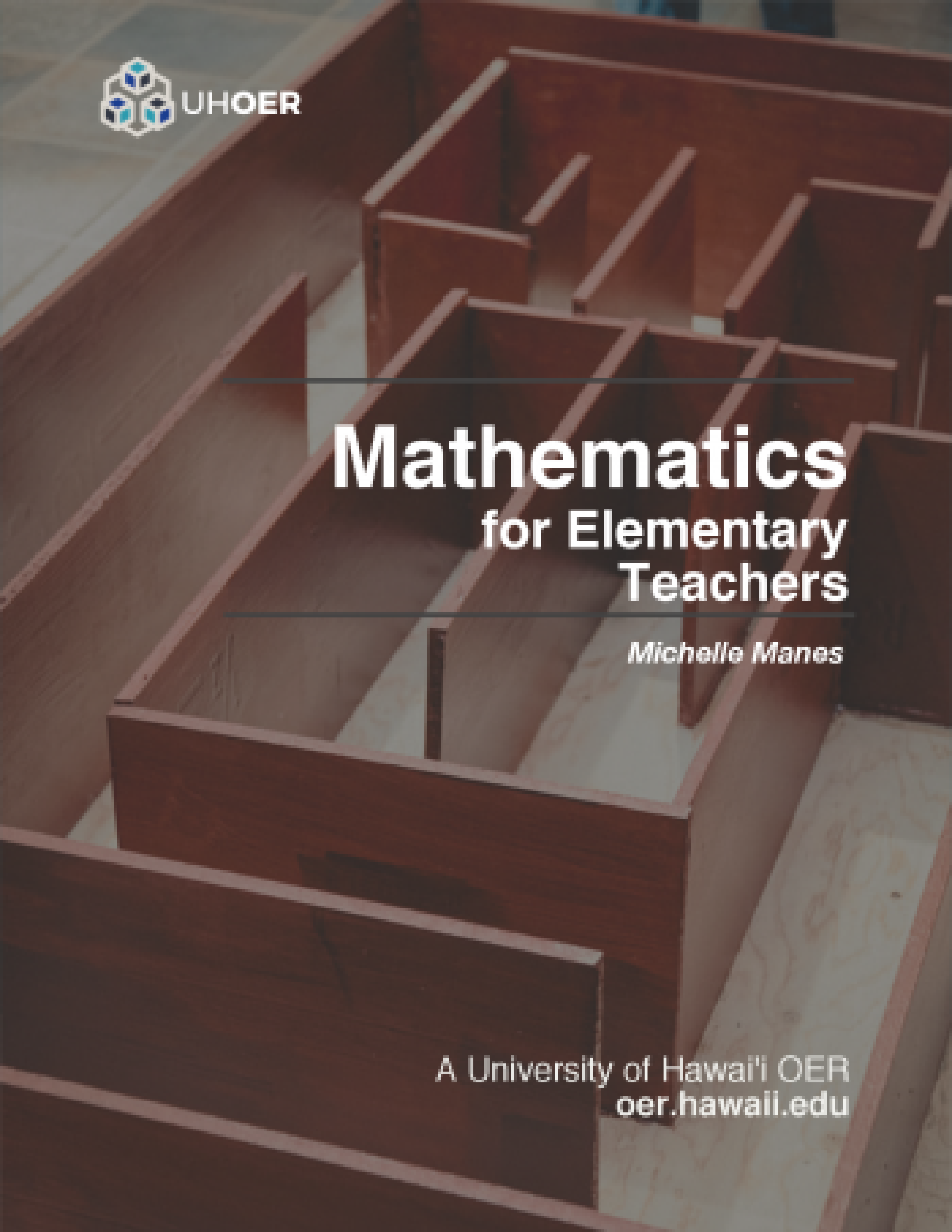

UHOER

Mathematics
for Elementary Teachers

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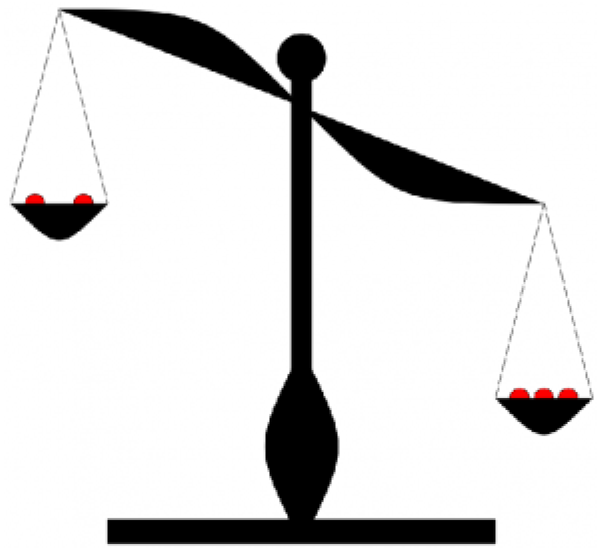


> Solving a problem for which you know there's an answer is like climbing a mountain with a guide, along a trail someone else has laid. In mathematics, the truth is somewhere out there in a place no one knows, beyond all the beaten paths.
>
> – Yoko Ogawa

Introduction

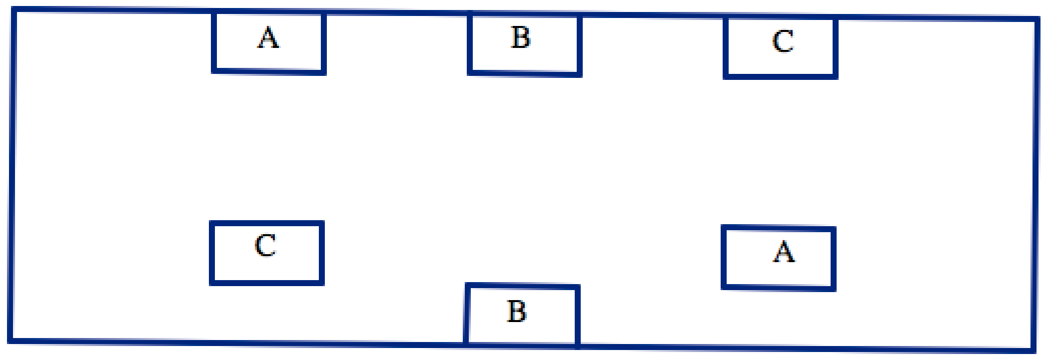
The Common Core State Standards for Mathematics (http://www.corestandards.org/Math/Practice) identify eight "Mathematical Practices" — the kinds of expertise that all teachers should try to foster in their students, but they go far beyond any particular piece of mathematics content. They describe what mathematics is really about, and why it is so valuable for students to master. The very first Mathematical Practice is:

> Make sense of problems and persevere in solving them. Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary.

This chapter will help you develop these very important mathematical skills, so that you will be better prepared to help your future students develop them. Let's start with solving a problem!

Problem 1 (ABC)

Draw curves connecting A to A, B to B, and C to C. Your curves cannot cross or even touch each other,they cannot cross through any of the lettered boxes, and they cannot go outside the large box or even touch it's sides.



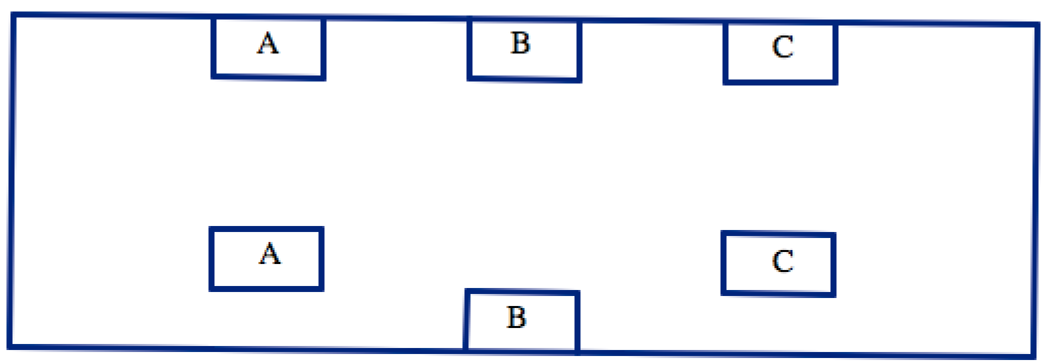
Think / Pair / Share

After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it).

- What did you try?
- What makes this problem difficult?
- Can you change the problem slightly so that it would be easier to solve?

Problem Solving Strategy 1 (Wishful Thinking). *Do you wish something in the problem was different? Would it then be easier to solve the problem?*

For example, what if ABC problem had a picture like this:



Can you solve this case and use it to help you solve the original case? Think about moving the boxes around once the lines are already drawn.

Here is one possible solution.

Problem or Exercise?

The main activity of mathematics is solving problems. However, what most people experience in most mathematics classrooms is practice exercises. An exercise is different from a problem.

In a **problem**, you probably don't know at first how to approach solving it. You don't know what mathematical ideas might be used in the solution. Part of solving a problem is understanding what is being asked, and knowing what a solution should look like. Problems often involve false starts, making mistakes, and lots of scratch paper!

In an **exercise**, you are often practicing a skill. You may have seen a teacher demonstrate a technique, or you may have read a worked example in the book. You then practice on very similar assignments, with the goal of mastering that skill.

Note: What is a problem for some people may be an exercise for other people who have more background knowledge! For a young student just learning addition, this might be a problem:

Fill in the blank to make a true statement: $\_\_\_\_ + 4 = 7$.

But for you, that is an exercise!

Both problems and exercises are important in mathematics learning. But we should never forget that the ultimate goal is to develop more and better skills (through exercises) so that we can solve harder and more interesting problems.

Learning math is a bit like learning to play a sport. You can practice a lot of skills:

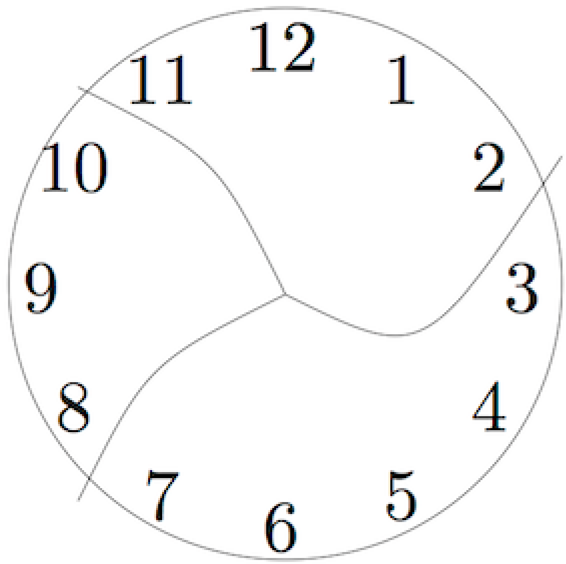
- hitting hundreds of forehands in tennis so that you can place them in a particular spot in the court,
- breaking down strokes into the component pieces in swimming so that each part of the stroke is more efficient,
- keeping control of the ball while making quick turns in soccer,
- shooting free throws in basketball,
- catching high fly balls in baseball,
- and so on.

But the point of the sport is to play the game. You practice the skills so that you are better at playing the game. In mathematics, solving problems is playing the game!

On Your Own

For each question below, decide if it is a *problem* or an *exercise*. (You do not need to solve the problems! Just decide which category it fits for you.) After you have labeled each one, compare your answers with a partner.

1. This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers.(Note: **Consecutive numbers** are whole numbers that appear one after the other, such as 1, 2, 3, 4 or 13, 14, 15.)



Can you break another clock into a different number of pieces so that the sums are consecutive numbers? Assume that each piece has at least two numbers and that no number is damaged (e.g. 12 isn't split into two digits 1 and 2).

2. A soccer coach began the year with a $500 budget. By the end of December, the coach spent $450. How much money in the budget was not spent?

3. What is the product of 4,500 and 27?

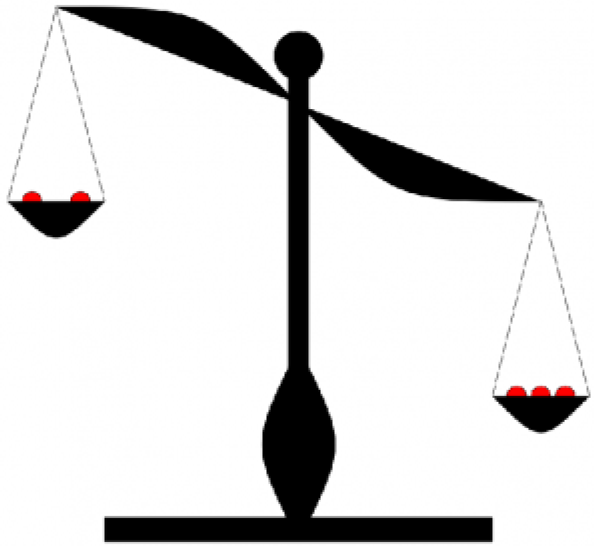
4. Arrange the digits 1–6 into a "difference triangle" where each number in the row below is the difference of the two numbers above it.

5. Simplify the following expression:

$$\frac{2 + 2(5^3 - 4^2)^5 - 2^2}{2(5^3 - 4^2)}.$$

6. What is the sum of $\frac{5}{2}$ and $\frac{3}{13}$?

7. You have eight coins and a balance scale. The coins look alike, but one of them is a counterfeit. The counterfeit coin is lighter than the others. You may only use the balance scale two times. How can you find the counterfeit coin?



8. How many squares, of any possible size, are on a standard 8 × 8 chess board?

9. What number is 3 more than half of 20?

10. Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1's are separated by one digit, the 2's are separated by two digits, the 3's by three digits, and the 4's by four digits.

Problem Solving Strategies

Think back to the first problem in this chapter, the ABC Problem. What did you do to solve it? Even if you did not figure it out completely by yourself, you probably worked towards a solution and figured out some things that *did not* work.

Unlike exercises, there is never a simple recipe for solving a problem. You can get better and better at solving problems, both by building up your background knowledge and by simply practicing. As you solve more problems (and learn how other people solved them), you learn strategies and techniques that can be useful. But no single strategy works every time.

Pólya's *How to Solve It*

George Pólya was a great champion in the field of *teaching* effective problem solving skills. He was born in Hungary in 1887, received his Ph.D. at the University of Budapest, and was a professor at Stanford University (among other universities). He wrote many mathematical papers along with three books, most famously, "How to Solve it." Pólya died at the age 98 in 1985.[1]

George Pólya, circa 1973

In 1945, Pólya published the short book *How to Solve It*, which gave a four-step method for solving mathematical problems:

1. Image of Pólya by Thane Plambeck from Palo Alto, California (Flickr) [CC BY 2.0 (http://creativecommons.org/licenses/by/2.0)], via Wikimedia Commons

1. First, you have to understand the problem.
2. After understanding, then make a plan.
3. Carry out the plan.
4. Look back on your work. How could it be better?

This is all well and good, but how do you actually do these steps?!?! Steps 1. and 2. are particularly mysterious! How do you "make a plan?" That is where you need some tools in your toolbox, and some experience to draw upon.

Much has been written since 1945 to explain these steps in more detail, but the truth is that they are more art than science. This is where math becomes a creative endeavor (and where it becomes so much fun). We will articulate some useful problem solving strategies, but no such list will ever be complete. This is really just a start to help you on your way. The best way to become a skilled problem solver is to learn the background material well, and then to solve a lot of problems!

We have already seen one problem solving strategy, which we call "Wishful Thinking." Do not be afraid to change the problem! Ask yourself "what if" questions:

- What if the picture was different?
- What if the numbers were simpler?
- What if I just made up some numbers?

You need to be sure to go back to the original problem at the end, but wishful thinking can be a powerful strategy for getting started.

This brings us to the most important problem solving strategy of all:

Problem Solving Strategy 2 (Try Something!). *If you are really trying to solve a problem, the whole point is that you do not know what to do right out of the starting gate. You need to just try something! Put pencil to paper (or stylus to screen or chalk to board or whatever!) and try something. This is often an important step in understanding the problem; just mess around with it a bit to understand the situation and figure out what is going on.*

And equally important: If what you tried first does not work, try something else! Play around with the problem until you have a feel for what is going on.

Problem 2 (Payback)

Last week, Alex borrowed money from several of his friends. He finally got paid at work, so he brought cash to school to pay back his debts. First he saw Brianna, and he gave her 1/4 of the money he had brought to school. Then Alex saw Chris and gave him 1/3 of what he had left after paying Brianna. Finally, Alex saw David and gave him 1/2 of what he had remaining. Who got the most money from Alex?

Think/Pair/Share

After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it). What did you try? What did you figure out about the problem?

This problem lends itself to two particular strategies. Did you try either of these as you worked on the problem? If not, read about the strategy and then try it out before watching the solution.

Problem Solving Strategy 3 (Draw a Picture). *Some problems are obviously about a geometric situation, and it is clear you want to draw a picture and mark down all of the given information before you try to solve it. But even for a problem that is not geometric, like this one, thinking visually can help! Can you represent something in the situation by a picture?*

Draw a square to represent all of Alex's money. Then shade 1/4 of the square — that's what he gave away to Brianna. How can the picture help you finish the problem?

After you have worked on the problem yourself using this strategy (or if you are completely stuck), you can watch someone else's solution.

Problem Solving Strategy 4 (Make Up Numbers). *Part of what makes this problem difficult is that it is about money, but there are no numbers given. That means the numbers must not be important. So just make them up!*

You can work forwards: Assume Alex had some specific amount of money when he showed up at school, say \$100. Then figure out how much he gives to each person. Or you can work backwards: suppose he has some specific amount left at the end, like \$10. Since he gave Chris half of what he had left, that means he had \$20 before running into Chris. Now, work backwards and figure out how much each person got.

Watch the solution only after you tried this strategy for yourself.

If you use the "Make Up Numbers" strategy, it is really important to remember what the original problem was asking! You do not want to answer something like "Everyone got \$10." That is not true in the original problem; that is an artifact of the numbers you made up. So after you work everything out, be sure to re-read the problem and answer what was asked!

Problem 3 (Squares on a Chess Board)

How many squares, of any possible size, are on a 8×8 chess board? (The answer is not 64… It's a lot bigger!)

Remember Pólya's first step is to understand the problem. If you are not sure what is being asked, or why the answer is not just 64, be sure to ask someone!

Think / Pair / Share

After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it). What did you try? What did you figure out about the problem, even if you have not solved it completely?

It is clear that you want to draw a picture for this problem, but even with the picture it can be hard to know if you have found the correct answer. The numbers get big, and it can be hard to keep track of your work. Your goal at the end is to be absolutely positive that you found the right answer. You should never ask the teacher, "Is this right?" Instead, you should declare, "Here's my answer, and here is why I know it is correct!"

Problem Solving Strategy 5 (Try a Simpler Problem). *Pólya suggested this strategy: "If you can't solve a problem, then there is an easier problem you can solve: find it." He also said: "If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem?" In this case, an 8 × 8 chess board is pretty big. Can you solve the problem for smaller boards? Like 1 × 1? 2 × 2? 3 × 3?*

Of course the ultimate goal is to solve the original problem. But working with smaller boards might give you some insight and help you devise your plan (that is Pólya's step (2)).

Problem Solving Strategy 6 (Work Systematically). *If you are working on simpler problems, it is useful to keep track of what you have figured out and what changes as the problem gets more complicated.*

For example, in this problem you might keep track of how many 1 × 1 squares are on each board, how many 2 × 2 squares on are each board, how many 3 × 3 squares are on each board, and so on. You could keep track of the information in a table:

size of board	# of 1 × 1 squares	# of 2 × 2 squares	# of 3 × 3 squares	# of 4 × 4 squares	…
1 by 1	1	0	0	0	
2 by 2	4	1	0	0	
3 by 3	9	4	1	0	
…					

Problem Solving Strategy 7 (Use Manipulatives to Help You Investigate). *Sometimes even drawing a picture may not be enough to help you investigate a problem. Having actual materials that you move around can sometimes help a lot!*

For example, in this problem it can be difficult to keep track of which squares you have already counted. You might want to cut out 1 × 1 squares, 2 × 2 squares, 3 × 3 squares, and so on. You can actually move the smaller squares across the chess board in a systematic way, making sure that you count everything once and do not count anything twice.

Problem Solving Strategy 8 (Look for and Explain Patterns). *Sometimes the numbers in a problem are so big, there is no way you will actually count everything up by hand. For example, if the problem in this section were about a 100 × 100 chess board, you would not want to go through counting all the squares by hand! It would be much more appealing to find a pattern in the smaller boards and then extend that pattern to solve the problem for a 100 × 100 chess board just with a calculation.*

Think / Pair / Share

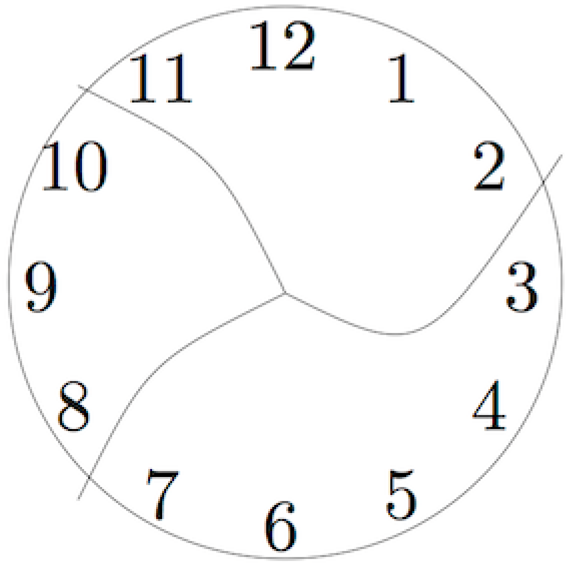
If you have not done so already, extend the table above all the way to an 8 × 8 chess board, filling in all the rows and columns. Use your table to find the total number of squares in an 8 × 8 chess board. Then:

- Describe all of the patterns you see in the table.
- Can you explain and justify any of the patterns you see? How can you be sure they will continue?
- What calculation would you do to find the total number of squares on a 100 × 100 chess board?

(We will come back to this question soon. So if you are not sure right now how to explain and justify the patterns you found, that is OK.)

Problem 4 (Broken Clock)

This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers. (**Consecutive numbers** are whole numbers that appear one after the other, such as 1, 2, 3, 4 or 13, 14, 15.)



Can you break another clock into a different number of pieces so that the sums are consecutive numbers? Assume that each piece has at least two numbers and that no number is damaged (e.g. 12 isn't split into two digits 1 and 2.)

Remember that your first step is to understand the problem. Work out what is going on here. What are the sums of the numbers on each piece? Are they consecutive?

Think / Pair / Share

After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it). What did you try? What progress have you made?

Problem Solving Strategy 9 (Find the Math, Remove the Context). *Sometimes the problem has a lot of details in it that are unimportant, or at least unimportant for getting started. The goal is to find the underlying math problem, then come back to the original question and see if you can solve it using the math.*

In this case, worrying about the clock and exactly how the pieces break is less important than worrying about finding consecutive numbers that sum to the correct total. Ask yourself:

- What is the sum of all the numbers on the clock's face?
- Can I find two consecutive numbers that give the correct sum? Or four consecutive numbers? Or some

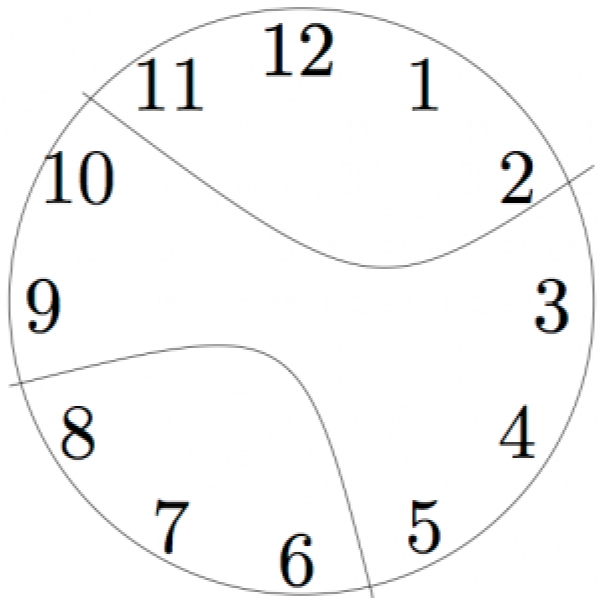
other amount?

- How do I know when I am done? When should I stop looking?

Of course, solving the question about consecutive numbers is not the same as solving the original problem. You have to go back and see if the clock can actually break apart so that each piece gives you one of those consecutive numbers. Maybe you can solve the math problem, but it does not translate into solving the clock problem.

Problem Solving Strategy 10 (Check Your Assumptions). *When solving problems, it is easy to limit your thinking by adding extra assumptions that are not in the problem. Be sure you ask yourself: Am I constraining my thinking too much?*

In the clock problem, because the first solution has the clock broken radially (all three pieces meet at the center, so it looks like slicing a pie), many people assume that is how the clock must break. But the problem does not require the clock to break radially. It might break into pieces like this:



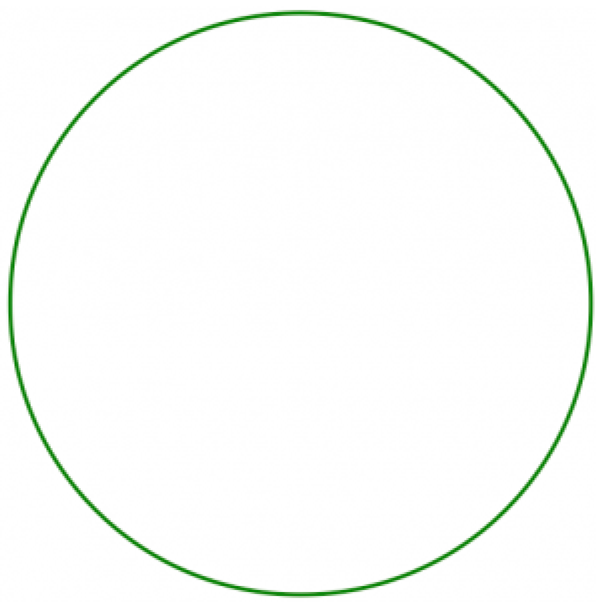
Were you assuming the clock would break in a specific way? Try to solve the problem now, if you have not already.

Beware of Patterns!

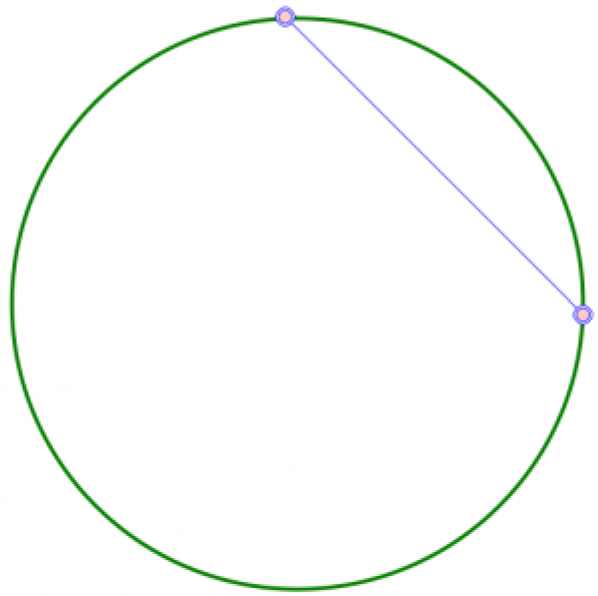
The “Look for Patterns” strategy can be particularly appealing, but you have to be careful! Do not forget the “and Explain” part of the strategy. Not all patterns are obvious, and not all of them will continue.

Problem 5 (Dots on a Circle)

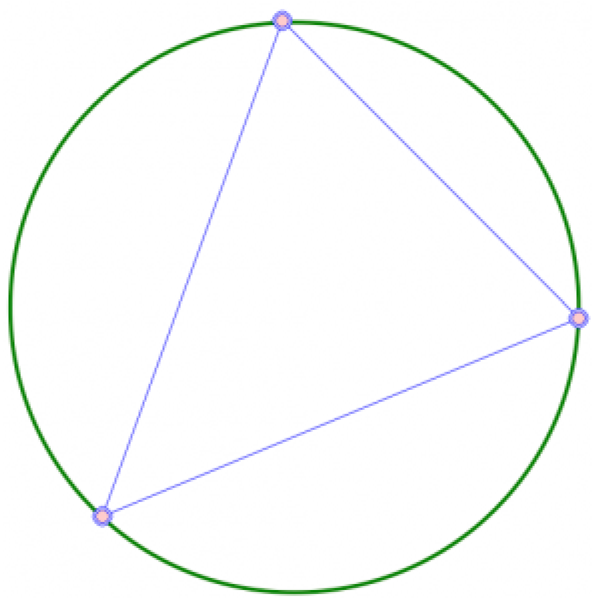
Start with a circle.



If I put two dots on the circle and connect them, the line divides the circle into two pieces.



If I put three dots on the circle and connect each pair of dots, the lines divides the circle into four pieces.



Suppose you put one hundred dots on a circle and connect each pair of dots, meaning every dot is connected to 99 other dots. How many pieces will you get? Lines may cross each other, but assume the points are chosen so that three or more lines never meet at a single point.

Think / Pair / Share

After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it). What strategies did you try? What did you figure out? What questions do you still have?

The natural way to work on this problem is to use smaller numbers of dots and look for a pattern, right? If you have not already, try it. How many pieces when you have four dots? Five dots? How would you describe the pattern?

Now try six dots. You will want to draw a big circle and space out the six dots to make your counting easier. Then carefully count up how many pieces you get. It is probably a good idea to work with a partner so you can check each other's work. Make sure you count every piece once and do not count any piece twice. How can you be sure that you do that?

Were you surprised? For the first several steps, it seems to be the case that when you add a dot you double the number of pieces. But that would mean that for six dots, you should get 32 pieces, and you only get 30 or 31, depending on how the dots are arranged. No matter what you do, you cannot get 32 pieces. The pattern simply does not hold up.

Mathematicians love looking for patterns and finding them. We get excited by patterns. But we are also very skeptical of patterns! If we cannot explain why a pattern would occur, then we are not willing to just believe it.

For example, if my number pattern starts out: 2, 4, 8, … I can find lots of ways to continue the pattern, each of which makes sense in some contexts. Here are some possibilities:

- 2, 4, 8, 2, 4, 8, 2, 4, 8, 2, 4, 8, …

This is a a repeating pattern, cycling through the numbers 2, 4, 8 and then starting over with 2.

- 2, 4, 8, 32, 256, 8192, …

To get the next number, multiply the previous two numbers together.

- 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, …
- 2, 4, 8, 14, 22, 32, 44, 58, 74 …

Think / Pair / Share

- For the last two patterns above, describe in words how the number sequence is being created.
- Find at least two other ways to continue the sequence 2, 4, 8, . . . that looks different from all the ones you have seen so far. Write your rule in words, and write the next five terms of the number sequence.

So how can you be sure your pattern fits the problem? You have to tie them together! Remember the "Squares on a Chess Board" problem? You might have noticed a pattern like this one:

If the chess board has 5 squares on a side, then there are

- $5 \times 5 = 25$ squares of size 1×1.
- $4 \times 4 = 16$ squares of size 2×2.
- $3 \times 3 = 9$ squares of side 3×3.
- $2 \times 2 = 4$ squares of size 4×4.
- $1 \times 1 = 1$ squares of size 5×5.

So there are a total of

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

squares on a 5×5 chess board. You can probably guess how to continue the pattern to any size board, but how can you be absolutely sure the pattern continues in this way? What if this is like "Dots on a Circle," and the obvious pattern breaks down after a few steps? You have to tie the pattern to the problem, so that it is clear why the pattern must continue in that way.

The first step in explaining a pattern is writing it down clearly. This brings us to another problem solving strategy.

Problem Solving Strategy 11 (Use a Variable!). *One of the most powerful tools we have is the use of a variable. If you find yourself doing calculations on things like "the number of squares," or "the number of dots," give those quantities a name! They become much easier to work with.*

Think/Pair/Share

For now, just work on describing the pattern with variables.

- Stick with a 5×5 chess board for now, and consider a small square of size $k \times k$. Describe the pattern: How many squares of size $k \times k$ fit on a chess board of size 5×5?
- What if the chess board is bigger? Based on the pattern above, how many squares of size $k \times k$ should fit on a chess board of size 10×10?
- What if you do not know how big the chess board is? Based on the pattern above, how many squares of size $k \times k$ should fit on a chess board of size $n \times n$?

Now comes the tough part: explaining the pattern. Let us focus on an 8×8 board. Since it measures 8 squares on each side, we can see that we get $8 \times 8 = 64$ squares of size 1×1. And since there is just a single board, we get just one square of size 8×8. But what about all the sizes in-between?

Think/Pair/Share

Using the Chess Board video in the previous chapter as a model, work with a partner to carefully explain why the number of 3×3 squares will be $6 \cdot 6 = 36$, and why the number of 4×4 squares will be $5 \cdot 5 = 25$.

There are many different explanations other than what is found in the video. Try to find your own explanation.

Here is what a final justification might look like (*watch the Chess Board video as a concrete example of this solution*):

Solution (Chess Board Pattern). Let n be the side of the chess board and let k be the side of the square. If the square is going to fit on the chess board at all, it must be true that $k \leq n$. Otherwise, the square is too big.

If I put the $k \times k$ square in the upper left corner of the chess board, it takes up k spaces across and there are $(n - k)$ spaces to the right of it. So I can slide the $k \times k$ square to the right $(n - k)$ times, until it hits the top right corner of the chess board. The square is in $(n - k + 1)$ different positions, counting the starting position.

If I move the $k \times k$ square back to the upper left corner, I can shift it down one row and repeat the whole process again. Since there are $(n - k)$ rows below the square, I can shift it down $(n - k)$ times until it hits the bottom row. This makes $(n - k + 1)$ total rows that the square moves across, counting the top row.

So, there are $(n - k + 1)$ rows with $(n - k + 1)$ squares in each row. That makes $(n - k + 1)^2$ total squares.

Thus, the solution is the sum of $(n - k + 1)^2$ for all $k \leq n$. In symbols:

$$\text{number of squares on an } n \times n \text{ board} = \sum_{k=1}^{n} (n - k + 1)^2.$$

Once we are sure the pattern continues, we can use it to solve the problem. So go ahead!

- How many squares on a 10×10 chess board?
- What calculation would you do to solve that problem for a 100×100 chess board?

There is a number pattern that describes the number of pieces you get from the "Dots on a Circle" problem. If you want to solve the problem, go for it! Think about all of your problem solving strategies. But be sure that when you find a pattern, you can explain *why it is the right pattern for this problem*, and not just another pattern that seems to work but might not continue.

Problem Bank

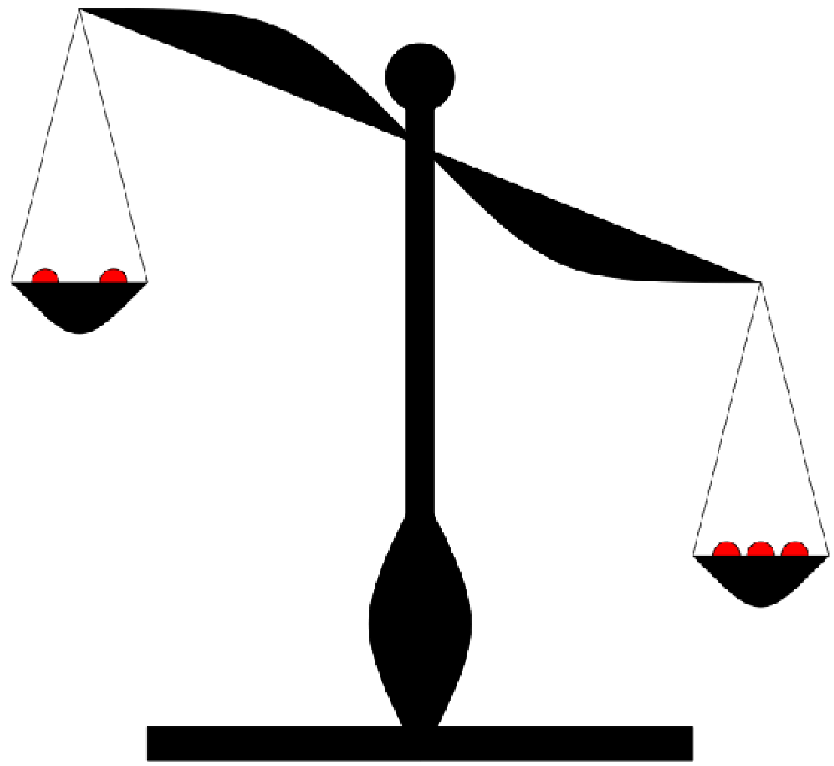
You have several problem solving strategies to work with. Here are the ones we have described so far (and you probably came up with even more of your own strategies as you worked on problems).

1. Wishful Thinking.
2. Try Something!
3. Draw a Picture.
4. Make Up Numbers.
5. Try a Simpler Problem.
6. Work Systematically.
7. Use Manipulatives to Help you Investigate.
8. Look for and Explain Patterns.
9. Find the Math, Remove the Context.
10. Check Your Assumptions.
11. Use a Variable.

Try your hand at some of these problems, keeping these strategies in mind. If you are stuck on a problem, come back to this list and ask yourself which of the strategies might help you make some progress.

Problem 6

You have eight coins and a balance scale. The coins look alike, but one of them is a counterfeit. The counterfeit coin is lighter than the others. You may only use the balance scale two times. How can you find the counterfeit coin?



Problem 7

You have five coins, no two of which weigh the same. In seven weighings on a balance scale, can you put the coins in order from lightest to heaviest? That is, can you determine which coin is the lightest, next lightest, . . . , heaviest.

Problem 8

You have ten bags of coins. Nine of the bags contain good coins weighing one ounce each. One bag contains counterfeit coins weighing 1.1 ounces each. You have a regular (digital) scale, not a balance scale. The scale is correct to one-tenth of an ounce. In one weighing, can you determine which bag contains the bad coins?

Problem 9

Suppose you have a balance scale. You have three different weights, and you are able to weigh every whole number from 1 gram to 13 grams using just those three weights. What are the three weights?

Problem 10

There are a bunch of coins on a table in front of you. Your friend tells you how many of the coins are heads-up. You are blindfolded and cannot see a thing, but you can move the coins around, and you can flip them over. However, you cannot tell just by feeling them if the coins are showing heads or tails. Your job: separate the coins into two piles so that the same number of heads are showing in each pile.

Problem 11

The digital root of a number is the number obtained by repeatedly adding the digits of the number. If the answer is not a one-digit number, add those digits. Continue until a one-digit sum is reached. This one digit is the digital root of the number.

For example, the digital root of 98 is 8, since 9 + 8 = 17 and 1 + 7 = 8.

Record the digital roots of the first 30 integers and find as many patterns as you can. Can you explain any of the patterns?

Problem 12

If this lattice were continued, what number would be directly to the right of 98? How can you be sure you're right?

	3		6		9		12		…
1	2	4	5	7	8	10	11	13	…

Problem 13

Arrange the digits 0 through 9 so that the first digit is divisible by 1, the first two digits are divisible by 2, the first three digits are divisible by 3, and continuing until you have the first 9 digits divisible by 9 and the whole 10-digit number divisible by 10.

Problem 14

There are 25 students and one teacher in class. After an exam, everyone high-fives everyone else to celebrate how well they did. How many high- fives were there?

Problem 15

In cleaning out your old desk, you find a whole bunch of 3¢ and 7¢ stamps. Can you make exactly 11¢ of postage? Can you make exactly 19¢ of postage? What is the largest amount of postage you cannot make?

Problem 16

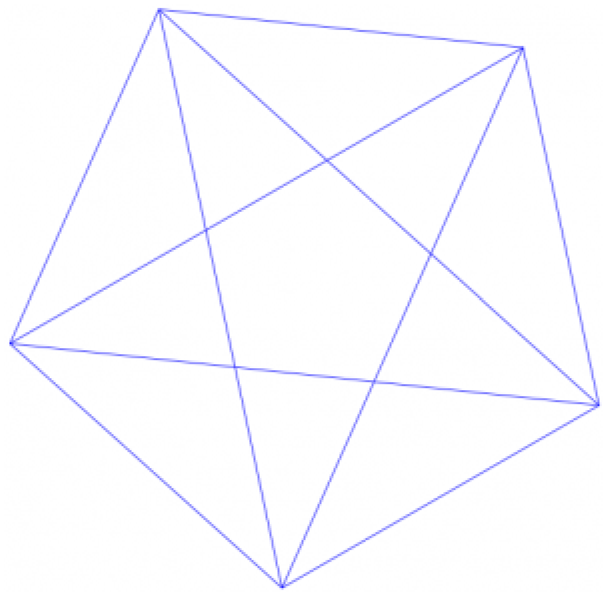
Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1's are separated by one digit, the 2's are separated by two digits, the 3's by three digits, and the 4's by four digits.

Problem 17

Kami has ten pockets and 44 dollar bills. She wants to have a different amount of money in each pocket. Can she do it?

Problem 18

How many triangles of all possible sizes and shapes are in this picture?



Problem 19

Arrange the digits 1–6 into a "difference triangle" where each number in the row below is the difference of the two numbers above it.

Example: Below is a difference triangle, but it does not work because it uses 1 twice and does not have a 6:

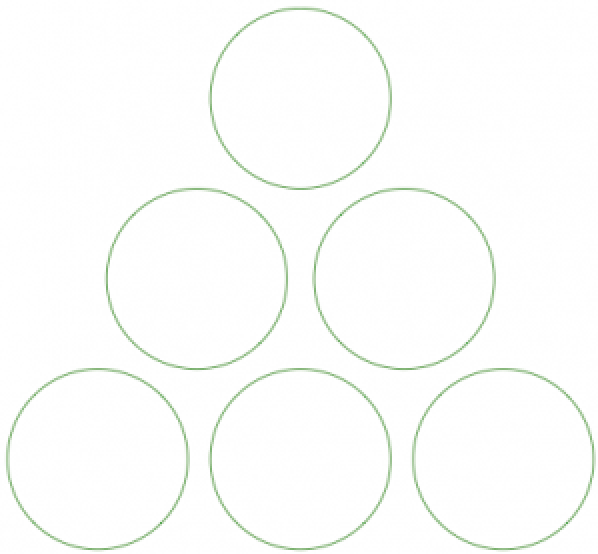
4		5		3
	1		2	
		1		

Problem 20

Certain pipes are sold in lengths of 6 inches, 8 inches, and 10 inches. How many different lengths can you form by attaching three sections of pipe together?

Problem 21

Place the digits 1, 2, 3, 4, 5, 6 in the circles so that the sum on each side of the triangle is 12. Each circle gets one digit, and each digit is used exactly once.



Problem 22

Find a way to cut a circular pizza into 11 pieces using just four straight cuts.

Careful Use of Language in Mathematics

This section might seem like a bit of a sidetrack from the idea of problem solving, but in fact it is not. Mathematics is a social endeavor. We do not just solve problems and then put them aside. Problem solving has (at least) three components:

1. Solving the problem. This involves a lot of scratch paper and careful thinking.
2. Convincing yourself that your solution is complete and correct. This involves a lot of self-check and asking yourself questions.
3. Convincing someone else that your solution is complete and correct. This usually involves writing the problem up carefully or explaining your work in a presentation.

If you are not able to do that last step, then you have not really solved the problem. We will talk more about how to write up a solution soon. Before we do that, we have to think about how mathematicians use language (which is, it turns out, a bit different from how language is used in the rest of life).

Mathematical Statements

Definition

A mathematical statement is a complete sentence that is either true or false, but not both at once.

So a "statement" in mathematics cannot be a question, a command, or a matter of opinion. It is a complete, grammatically correct sentence (with a subject, verb, and usually an object). It is important that the statement is either true or false, though you may not know which! (Part of the work of a mathematician is figuring out which sentences are true and which are false.)

Think / Pair / Share

For each English sentence below, decide if it is a mathematical statement or not. If it is, is the statement true or false (or are you unsure)? If it is not a mathematical statement, in what way does it fail?

1. Blue is the prettiest color.
2. 60 is an even number.
3. Is your dog friendly?
4. Honolulu is the capital of Hawaii.
5. This sentence is false.
6. All roses are red.
7. UH Manoa is the best college in the world.
8. 1/2 = 2/4.
9. Go to bed.
10. There are a total of 204 squares on an 8 × 8 chess board.

Now write three mathematical statements and three English sentences that fail to be mathematical statements.

Notice that “1/2 = 2/4” is a perfectly good mathematical statement. It does not look like an English sentence, but read it out loud. The subject is “1/2.” The verb is “equals.” And the object is “2/4.” This is a very good test when you write mathematics: try to read it out loud. Even the equations should read naturally, like English sentences.

Statement (5) is different from the others. It is called a **paradox**: a statement that is self-contradictory. If it is true, then we conclude that it is false. (Why?) If it is false, then we conclude that it is true. (Why?) Paradoxes are no good as mathematical statements, because it cannot be true and it cannot be false.

And / or

Consider this sentence:

> After work, I will go to the beach, or I will do my grocery shopping.

In everyday English, that probably means that if I go to the beach, I will not go shopping. I will do one or the other, but not both activities. This is called an “exclusive or.”

We can usually tell from context whether a speaker means “either one or the other or both,” or whether he means “either one or the other but not both.” (Some people use the awkward phrase “and/or” to describe the first option.)

Remember that in mathematical communication, though, we have to be very precise. We cannot rely on context or assumptions about what is implied or understood.

Definition

In mathematics, the word "or" always means "one or the other or both."

The word "and" always means "both are true."

Think / Pair / Share

For each sentence below:

- Decide if the choice $x = 3$ makes the statement true or false.
- Choose a different value of that makes the statement true (or say why that is not possible).
- Choose a different value of that makes the statement false (or say why that is not possible).

1. x is odd or x is even.
2. x is odd and x is even.
3. x is prime or x is odd.
4. $x > 5$ or $x < 5$.
5. $x > 5$ and $x < 5$.
6. $x + 1 = 7$ or $x - 1 = 7$.
7. $x \cdot 1 = x$ or $x \cdot 0 = x$.
8. $x \cdot 1 = x$ and $x \cdot 0 = x$.
9. $x \cdot 1 = x$ or $x \cdot 0 = 0$.

Quantifiers

Problem 23 (All About the Benjamins)

You are handed an envelope filled with money, and you are told "Every bill in this envelope is a $100 bill."

- What would convince you beyond any doubt that the sentence is true? How could you convince someone else that the sentence is true?

- What would convince you beyond any doubt that the sentence is false? How could you convince someone else that the sentence is false?

Suppose you were given a different sentence: “There is a $100 bill in this envelope.”

- What would convince you beyond any doubt that the sentence is true? How could you convince someone else that the sentence is true?
- What would convince you beyond any doubt that the sentence is false? How could you convince someone else that the sentence is false?

Think / Pair / Share

What is the difference between the two sentences? How does that difference affect your method to decide if the statement is true or false?

Some mathematical statements have this form:

- “Every time…”
- “For all numbers. . . ”
- “For every choice. . . ”
- “It’s always true that. . . ”

These are *universal* statements. Such statements claim that something is always true, no matter what.

- To prove a universal statement is false, you must find an example where it fails. This is called a **counterexample** to the statement.
- To prove a universal statement is true, you must either check every single case, or you must find a logical reason why it would be true. (Sometimes the first option is impossible, because there might be infinitely many cases to check. You would never finish!)

Some mathematical statements have this form:

- “Sometimes…”
- “There is some number. . . ”
- “For some choice. . . ”
- “At least once…”

These are *existential* statements. Such statements claim there is some example where the statement is true, but it may not always be true.

- To prove an existential statement is true, you may just find the example where it works.
- To prove an existential statement is false, you must either show it fails in every single case, or you must find a logical reason why it cannot be true. (Sometimes the first option is impossible!)

Think / Pair / Share

For each statement below, do the following:

- Decide if it is a universal statement or an existential statement. (This can be tricky because in some statements the quantifier is "hidden" in the meaning of the words.)
- Decide if the statement is true or false, and do your best to justify your decision.

1. Every odd number is prime.
2. Every prime number is odd.
3. For all positive numbers x, $x^3 > x$.
4. There is some number x such that $x^3 = x$.
5. The points (1, 1), (2, 1), and (3, 0) all lie on the same line.
6. Addition (of real numbers) is commutative.
7. Division (of real numbers) is commutative.

Look back over your work. you will probably find that some of your arguments are sound and convincing while others are less so. In some cases you may "know" the answer but be unable to justify it. That is okay for now! Divide your answers into four categories:

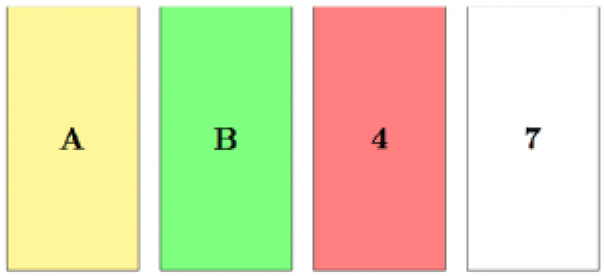
1. I am confident that the justification I gave is good.
2. I am not confident in the justification I gave.
3. I am confident that the justification I gave is not good, or I could not give a justification.
4. I could not decide if the statement was true or false.

Conditional Statements

Problem 24 (Card Logic)

You have a deck of cards where each card has a letter on one side and a number on the other side. Your friend claims: “If a card has a vowel on one side, then it has an even number on the other side.”

These cards are on a table.



Which cards must you flip over to be certain that your friend is telling the truth?

Think / Pair / Share

After you have thought about the problem on your own for a while, discuss your ideas with a partner. Do you agree on which cards you must check? Try to come to agreement on an answer you both believe.

Here is another very similar problem, yet people seem to have an easier time solving this one:

Problem 25 (IDs at a Party)

You are in charge of a party where there are young people. Some are drinking alcohol, others soft drinks. Some are old enough to drink alcohol legally, others are under age. You are responsible for ensuring that the drinking laws are not broken, so you have asked each person to put his or her photo ID on the table. At one table, there are four young people:

- One person has a can of beer, another has a bottle of Coke, but their IDs happen to be face down so you cannot see their ages.
- You can, however, see the IDs of the other two people. One is under the drinking age, the other is above it. They both have fizzy clear drinks in glasses, and you are not sure if they are drinking soda water or gin and tonic.

Which IDs and/or drinks do you need to check to make sure that no one is breaking the law?

Think / Pair / Share

After you have thought about the problem on your own for a while, discuss your ideas with a partner. Do you agree on which cards you must check? Compare these two problems. Which question is easier and why?

Definition

A **conditional statement** can be written in the form

If <u>some statement</u> then <u>some statement.</u>

Where the first statement is the **hypothesis** and the second statement is the **conclusion**.

Think / Pair / Share

These are each conditional statements, though they are not all stated in "if/then" form. Identify the hypothesis of each statement. (You may want to rewrite the sentence as an equivalent "if/then" statement.)

1. If the tomatoes are red, then they are ready to eat.
   The tomatoes are red. / The tomatoes are ready to eat.
2. An integer n is even if it is a multiple of 2.
   n is even. / n is a multiple of 2.
3. If n is odd, then n is prime.
   n is odd. / n is prime.
4. The team wins when JJ plays.
   The team wins. / JJ plays.

Remember that a mathematical statement must have a definite truth value. It is either true or false, with no gray area (even though we may not be sure which is the case). How can you tell if a conditional statement is true or false? Surely, it depends on whether the hypothesis and the conclusion are true or false. But how, exactly, can you decide?

The key is to think of a conditional statement like a promise, and ask yourself: under what condition(s) will I have broken my promise?

Examples

Example 1. Here is a conditional statement:

> If I win the lottery, then I'll give each of my students $1,000.

There are four things that can happen:

- **True hypothesis, true conclusion:** I do win the lottery, and I do give everyone in class $1,000. I kept my promise, so the conditional statement is TRUE.
- **True hypothesis, false conclusion:** I do win the lottery, but I decide not to give everyone in class $1,000. I broke my promise, so the conditional statement is FALSE.
- **False hypothesis, true conclusion:** I do not win the lottery, but I am exceedingly generous, so I go ahead and give everyone in class $1,000. I did not break my promise! (Do you see why?) So the conditional statement is TRUE.
- **False hypothesis, false conclusion:** I do not win the lottery, so I do not give everyone in class $1,000. I did not break my promise! (Do you see why?) So the conditional statement is TRUE.

What can we conclude from this? **A conditional statement is false only when the hypothesis is true and the conclusion is false.** In every other instance, the promise (as it were) has not been broken. If a mathematical statement is not false, it must be true.

Example 2. Here is another conditional statement:

> If you live in Honolulu, then you live in Hawaii.

Is this statement true or false? It seems like it should depend on who the pronoun "you" refers to, and whether that person lives in Honolulu or not. Let us think it through:

- Sookim lives in Honolulu, so the hypothesis is true. Since Honolulu is in Hawaii, she does live in Hawaii. The statement is true about Sookim, since both the hypothesis and conclusion are true.
- DeeDee lives in Los Angeles. The statement is true about DeeDee since the hypothesis is false.

So in fact it does not matter! The statement is true either way. The right way to understand such a statement is as a universal statement: "Everyone who lives in Honolulu lives in Hawaii."

This statement is true, and here is how you might justify it: "Pick a random person who lives in Honolulu. That person lives in Hawaii (since Honolulu is in Hawaii), so the statement is true for that person. I do not need to consider people who do not live in Honolulu. The statement is automatically true for those people, because the hypothesis is false!"

Example 3. How do we show a (universal) conditional statement is false?

You need to give a specific instance where the hypothesis is true and the conclusion is false. For example:

> If you are a good swimmer, then you are a good surfer.

Do you know someone for whom the hypothesis is true (that person is a good swimmer) but the conclusion is false (the person is not a good surfer)? Then the statement is false!

Think / Pair / Share

For each conditional statement, decide if it is true or false. Justify your answer.

1. If $2 \times 2 = 4$ then $1 + 1 = 3$.
2. If $2 \times 2 = 5$ then $1 + 1 = 3$.
3. If $\pi > 3$ then all odd numbers are prime.
4. If $\pi < 3$ then all odd numbers are prime.
5. If a number has a 4 in the one's place, then the number is even.
6. If a number is even, then the number has a 4 in the one's place.
7. If the product of two numbers is 0, then one of the numbers is 0.
8. If the sum of two numbers is 0, then one of the numbers is 0.

Think / Pair / Share (Two truths and a lie)

On your own, come up with two conditional statements that are true and one that is false. Share your three statements with a partner, but do not say which are true and which is false. See if your partner can figure it out!

Explaining Your Work

At its heart, mathematics is a social endeavor. Even if you work on problems all by yourself, you have not really solved the problem until you have explained your work to someone else, and they sign off on it. Professional mathematicians write journal articles, books, and grant proposals. Teachers explain mathematical ideas to their students both in writing and orally. Explaining your work is really an essential part of the problem-solving process, and probably should have been Pólya's step 5.

Writing in mathematics is different from writing poetry or an English paper. The goal of mathematical writing is not florid description, but clarity. If your reader does not understand, then you have not done a good job. Here are some tips for good mathematical writing.

Do Not Turn in Scratch Work: When you are solving problems and not exercises, you are going to have a lot of false starts. You are going to try a lot of things that do not work. You are going to make a lot of mistakes. You are going to use scratch paper. At some point (hopefully!) you will scribble down an idea that actually solves the problem. Hooray! That paper is not what you want to turn in or share with the world. Take that idea, and write it up carefully, neatly, and clearly. (The rest of these tips apply to that write-up.)

(Re)state the Problem: Do not assume your reader knows what problem you are solving. (Even if it is the teacher who assigned the problem!) If the problem has a very long description, you can summarize it. You do not have rewrite it word-for-word or give all of the details, but make sure the question is clear.

Clearly Give the Answer: It is not a bad idea to state the answer right up front, then show the work to justify your answer. That way, the reader knows what you are trying to justify as they read. It makes their job much easier, and making the reader's job easier should be one of your primary goals! In any case, the answer should be clearly stated somewhere in the writeup, and it should be easy to find.

Be Correct: Of course, everyone makes mistakes as they are working on a problem. But we are talking about after you have solved the problem, when you are writing up your solution to share with someone else. The best writing in the world cannot save a wrong approach and a wrong answer. Check your work carefully. Ask someone else to read your solution with a critical eye.

Justify Your Answer: You cannot simply give an answer and expect your reader to "take your word for it." You have to explain how you know your answer is correct. This means "showing your work," explaining your reasoning, and justifying what you say. You need to answer the question, "How do you know your answer is right?"

Be Concise: There is no bonus prize for writing a lot in math class. Think clearly and write clearly. If you find yourself going on and on, stop, think about what you really want to say, and start over.

Use Variables and Equations: An equation can be much easier to read and understand than a long paragraph of text describing a calculation. Mathematical writing often has a lot fewer words (and a lot more equations) than other kinds of writing.

Define your Variables: If you use variables in the solution of your problem, always say what a variable stands for before you use it. If you use an equation, say where it comes from and why it applies to this situation. Do not make your reader guess!

Use Pictures: If pictures helped you solve the problem, include nice versions of those pictures in your final solution. Even if you did not draw a picture to solve the problem, it still might help your reader understand the solution. And that is your goal!

Use Correct Spelling and Grammar: Proofread your work. A good test is to read your work aloud (this includes reading the equations and calculations aloud). There should be complete, natural-sounding sentences. Be especially careful with pronouns. Avoid using "it" and "they" for mathematical objects; use the names of the objects (or variables) instead.

Format Clearly: Do not write one long paragraph. Separate your thoughts. Put complicated equations on a single displayed line rather than in the middle of a paragraph. Do not write too small. Do not make your reader struggle to read and understand your work.

Acknowledge Collaborators: If you worked with someone else on solving the problem, give them credit!

Here is a problem you've already seen:

Problem 16

Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1's are separated by one digit, the 2's are separated by two digits, the 3's by three digits, and the 4's by four digits.

Think / Pair / Share

Below you will find several solutions that were turned in by students. Using the criteria above, how would you score these solutions on a scale of 1 to 5? Give reasons for your answers.

Solution (Solution 1).

41312432

This is the largest eight-digit b/c the #s 1, 2, 3, 4 & all separated by the given amount of spaces.

Solution (Solution 2).

41312432

You have to have the 4 in the highest place and work down from there. However unable to follow the rules the 2 and the 1 in the 10k and 100k place must switch.

Solution (Solution 3).

41312432

First, I had to start with the #4 because that is the largest digit I could start with to get the largest #. Then I had to place the next 4 five spaces away because I knew there had to be four digits separating the two 4's. Next, I place 1 in the second digit spot because 2 or 3 would interfere with the rule of how many digits could separate them, which allowed me to also place where the next 1 should be. I then placed the 3 because opening spaces showed me that I could fit three digits in between the two 3's. Lastly, I had to input the final 2's, which worked out because there were two digits separating them.

Solution (Solution 4).

1×1

2xx2

3xxx3

4xxxx4
Answer: 41312432

Solution (Solution 5).

4 3 2 4 3 2

4 2 2 4

4 1 3 1 4 3

*4 1 3 1 2 4 3 2

4 needs to be the first # to make it the biggest. Then check going down from next largest to smallest. Ex:

4 3 ___________ ×

4 2 ___________×

4 1 ___________✓

Solution (Solution 6).

41312432

I put 4 at the 10,000,000 place because the largest # should be placed at the highest value. Numbers 2 & 3 could not be placed in the 1,000,000 place because I wasn't able to separate the digits properly. So I ended up placing the #1 there. In the 100,000 place I put the #3 because it was the second highest number.

Solution (Solution 7).

41312432

Since the problem asks you for the largest 8 digit #, I knew 4 had to be the first # since it's the greatest # of the set. To solve the rest of the problem, I used the guess and test method. I tried many different combinations. First using the #3 as the second digit in the sequence, but came to no answer. Then the #2, but no combination I found correctly finished the sequence.I then finished with the #1 in the second digit in the sequence and was able to successfully fill out the entire #.

Solution (Solution 8).

4 _ _ _ _ 4 _ _

4 has to be the first digit, for the number to be the largest possible. That means the other 4 has to be the 6th digit in the number, because 4's have to be separated by four digits.

4 _ 3 _ _ 4 3 _

3 must be the third digit, in order for the number to be largest possible. 3 cannot be the second digit because the other 3 would have to be the 6th digit in the number, but 4 is already there.

4 1 3 1 _ 4 3 _

1's must be separated by one digit, so the 1's can only be the 2nd and 4th digit in the number.

4 1 3 1 2 4 3 2

This leaves the 2s to be the 5th and 8th digits.

Solution (Solution 9).

With the active rules, I tried putting the highest numbers as far left as possible. Through trying different combinations, I figured out that no two consecutive numbers can be touching in the first two digits. So I instead tried starting with the 4 then 1 then 3, since I'm going for the highest # possible.

My answer: 41312432

The Last Step

A lot of people — from Polya to the writers of the Common Core State Standards and a lot of people in between — talk about problem solving in mathematics. One fact is rarely acknowledged, except by many professional mathematicians: Asking good questions is as valuable (and as difficult) as solving mathematical problems.

After solving a mathematical problem and explaining your solution to someone else, it is a very good mathematical habit to ask yourself: What other questions can I ask?

Example: Squares on a Chess Board

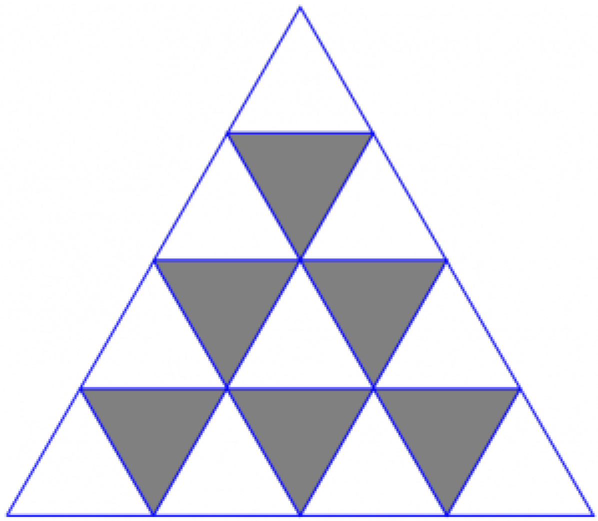
Recall Problem 3, “Squares on a Chess Board”:

> How many squares of any possible size are on a standard 8 × 8 chess board? (The answer is not 64! It’s a lot bigger!)

We have already talked about some obvious follow-up questions like “What about a 10 × 10 chess board? Or 100 × 100? Or $n \times n$?”

But there are a lot of interesting (and less obvious . . . and harder) questions you might ask:

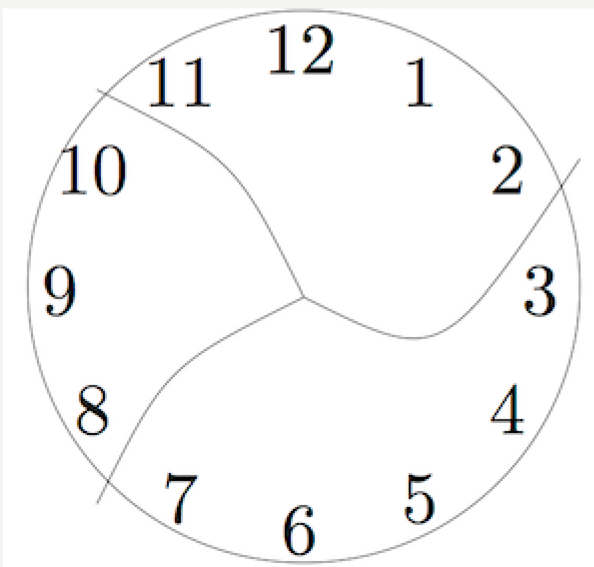
- How many rectangles of any size and shape can you find on a standard 8 × 8 chess board? (This is a lot harder, because the rectangles come in all different sizes, like 1 × 2 and 5 × 3. How could you possibly count them all?)
- How many triangles of any size and shape can you find in this picture?



Example: Broken Clock

Recall Problem 4, "Broken Clock":

> This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers. Can you break another clock into a different number of pieces so that the sums are consecutive numbers?

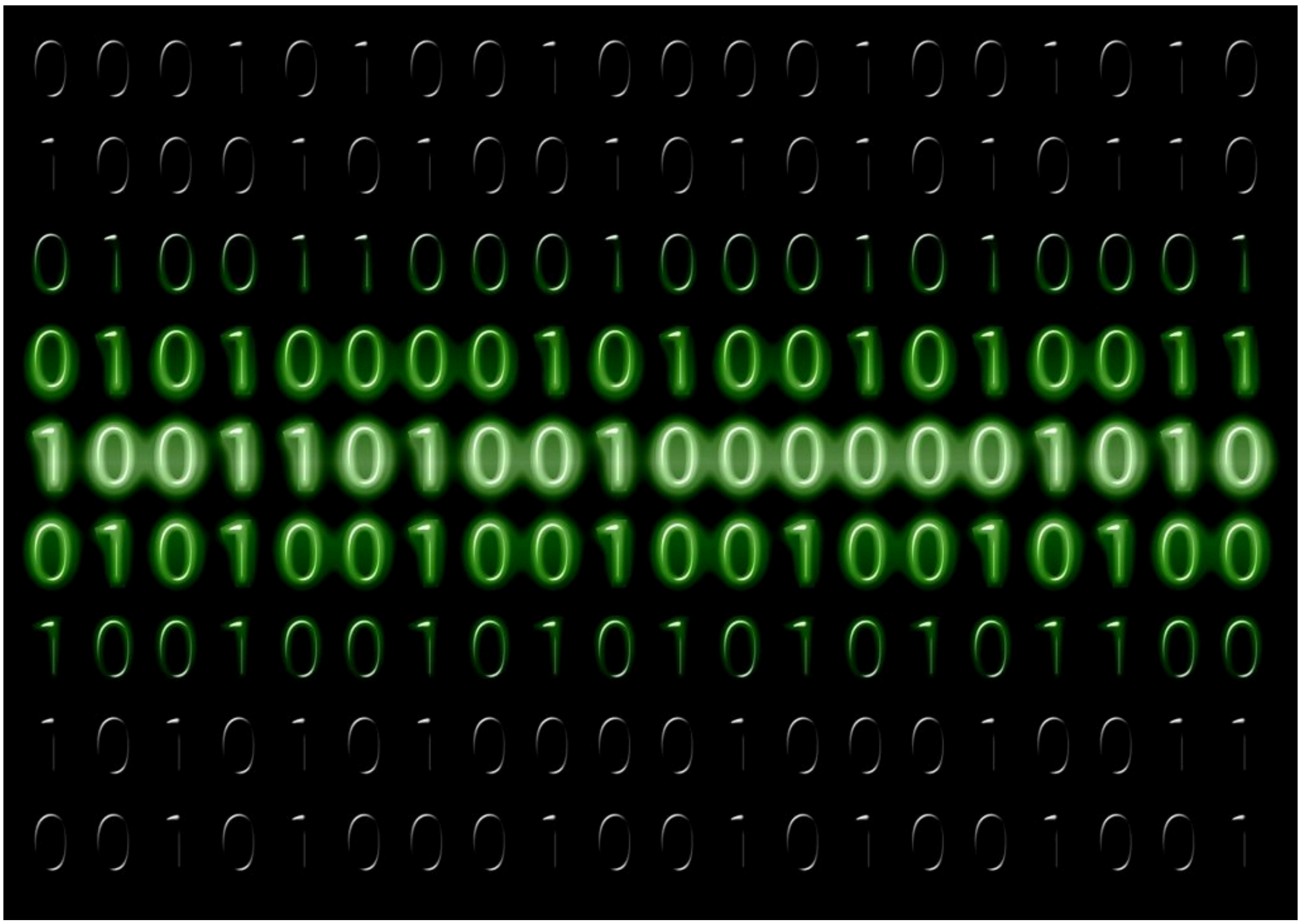


The original problem only asks if you can find one other way. The obvious follow-up question: "Find every possibly way to break the clock into some number of pieces so that the sums of the numbers on each piece are consecutive numbers. Justify that you have found every possibility."

Think / Pair / Share

Choose a problem from the Problem Bank (preferably a problem you have worked on, but that is not strictly necessary). What follow-up or similar questions could you ask?

Place Value



Binary numbers, using just 0's and 1's, are the language of computers.

1

The idea of expressing all quantities by nine digits whereby is imparted to them both an absolute value and

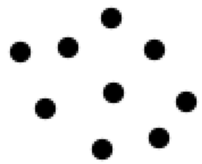
1. Images and Videos on Pixabay are released under Creative Commons CC0.

> a value of position is so simple that this very simplicity is the very reason for our not being sufficiently aware how much admiration it deserves.
>
> -Laplace

The "Dots and Boxes" approach to place value used in this part (and throughout this book) comes from James Tanton, and is used with his permission. See his development of these and other ideas at http://gdaymath.com/.

Dots and Boxes

Here are some dots; in fact there are nine of them:



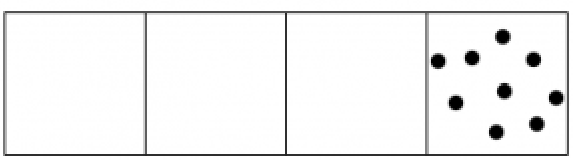
We’re going to a play an “exploding dots” game. Here’s the only rule for the game:

The 2←1 Rule

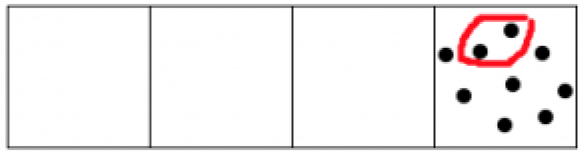
Whenever there are two dots in single box, they “explode,” disappear, and become one dot in the box to the left.

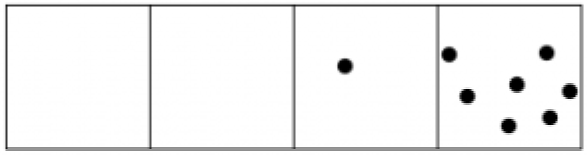
Example: Nine dots 2←1 in the system

We start by placing nine dots in the rightmost box.

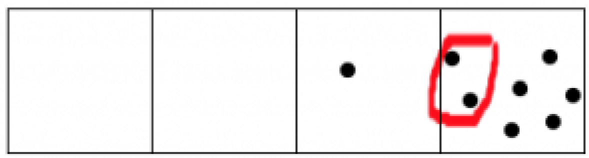


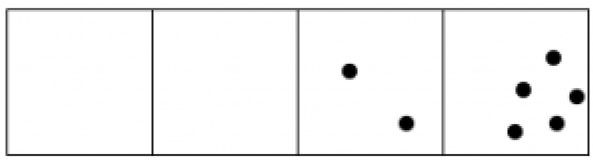
Two dots in that box explode and become one dot in the box to the left.



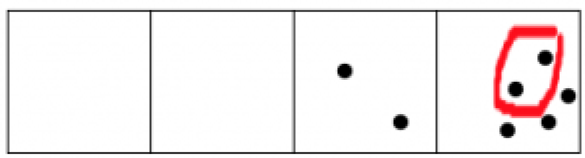


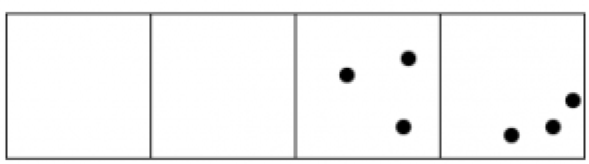
Once again, two dots in that box explode and become one dot in the box to the left.



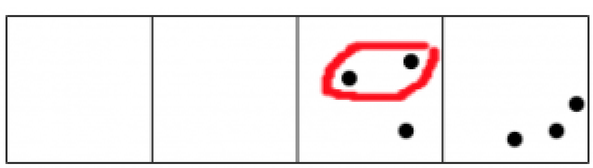


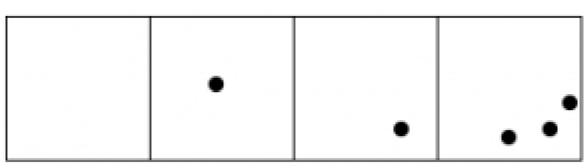
We do it again!



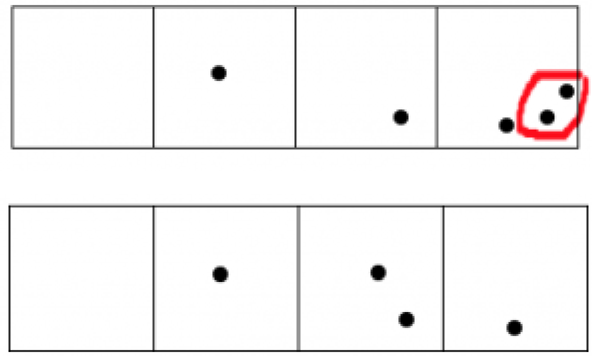


Hey, now we have more than two dots in the second box, so those can explode and move!

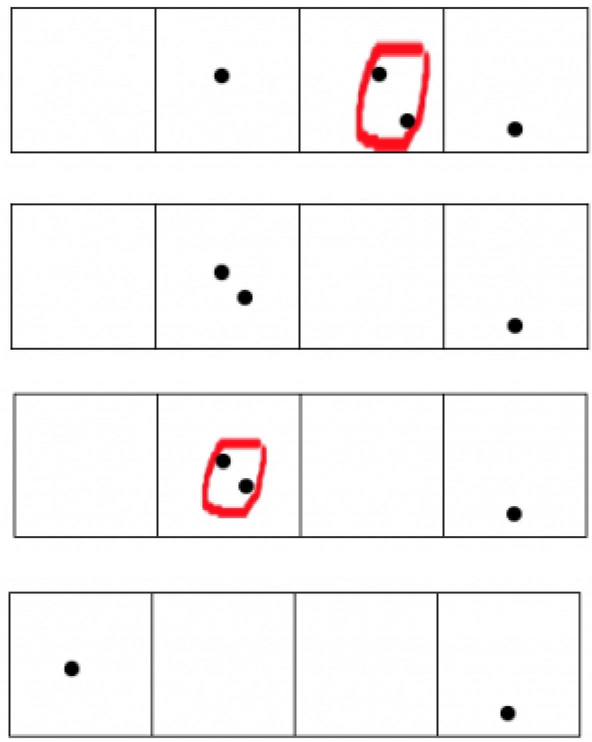




And the rightmost box still has more than two dots.



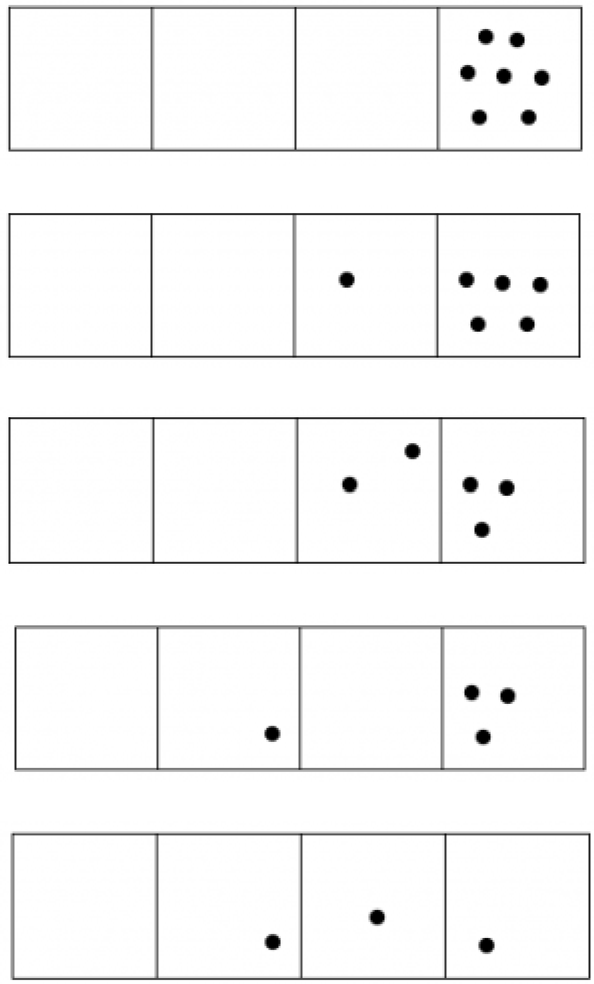
Keep going, until no box has two dots.



After all this, reading from left to right we are left with one dot, followed by zero dots, zero dots, and one final dot.

Solution: The 2 ← 1 code for nine dots is: 1001.

On Your Own. Here's a diagram showing what happens for seven dots in a 2 ← 1 box. Trace through the diagram, and circle the pairs of dots that "exploded" at each step.



Solution: The $2 \leftarrow 1$ code for seven dots is: 111.

Problem 1

Note: In solving this problem, you don't need to draw on paper; that can get tedious! Maybe you could use buttons or pennies for dots and do this by hand.

1. Draw 10 dots in the right-most box and perform the explosions. What is the $2 \leftarrow 1$ code for ten dots?
2. Find the $2 \leftarrow 1$ code for eighteen dots.
3. What number of dots has $2 \leftarrow 1$ code 101?

Think / Pair / Share

After you worked on the problem, compare your answer with a partner. Did you both get the same code? Did you have the same process?

Other Rules

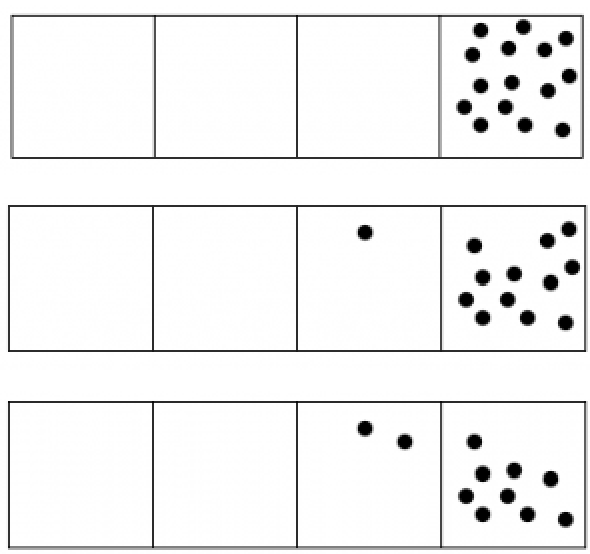
Let's play the dots and boxes game, but change the rule.

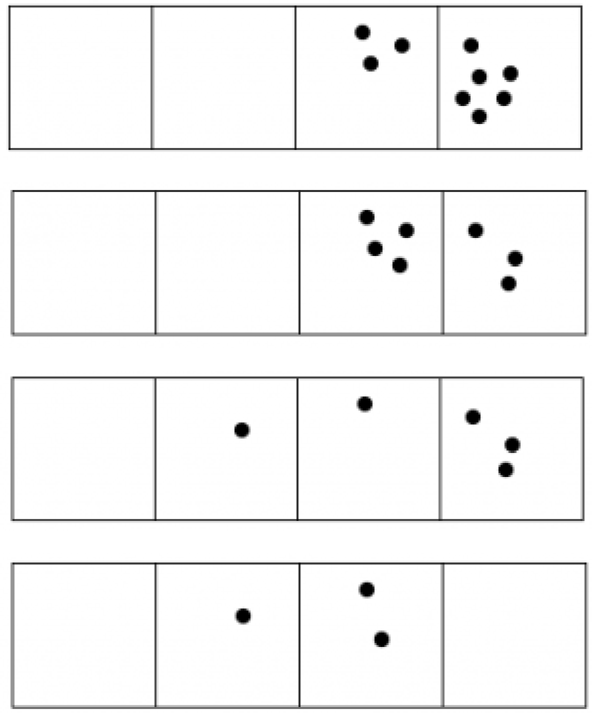
The 1←3 Rule

Whenever there are three dots in single box, they "explode," disappear, and become one dot in the box to the left.

Example: Fifteen dots in the 1←3 system

Here's what happens with fifteen dots:





Solution: The $1 \leftarrow 3$ code for fifteen dots is: 120.

Problem 2

1. Show that the $1 \leftarrow 3$ code for twenty dots is 202.
2. What is the $1 \leftarrow 3$ code for thirteen dots?
3. What is the $1 \leftarrow 3$ code for twenty-five dots?
4. What number of dots has $1 \leftarrow 3$ code 1022?
5. Is it possible for a collection of dots to have $1 \leftarrow 3$ code 2031? Explain your answer.

Problem 3

1. Describe how the $1 \leftarrow 4$ rule would work.

2. What is the $1 \leftarrow 4$ code for thirteen dots?

Problem 4

1. What is the $1 \leftarrow 5$ code for the thirteen dots?
2. What is the $1 \leftarrow 5$ code for five dots?

Problem 5

1. What is the $1 \leftarrow 9$ code for thirteen dots?
2. What is the $1 \leftarrow 9$ code for thirty dots?

Problem 6

1. What is the $1 \leftarrow 10$ code for thirteen dots?
2. What is the $1 \leftarrow 10$ code for thirty-seven dots?
3. What is the $1 \leftarrow 10$ code for two hundred thirty-eight dots?
4. What is the $1 \leftarrow 10$ code for five thousand eight hundred and thirty-three dots?

Think / Pair / Share

After you have worked on the problems on your own, compare your ideas with a partner. Can you describe what's going on in Problem 6 and why?

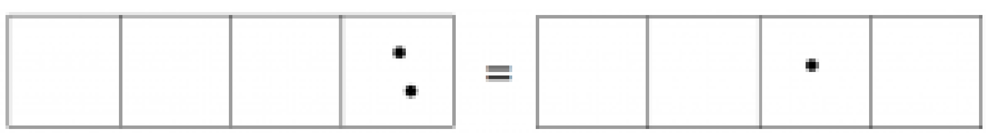
Binary Numbers

Let's go back to the 1 ← 2 rule and examine what's really going on.

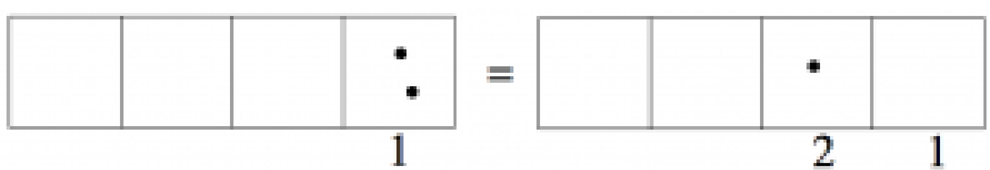
The 1←2 Rule

Whenever there are two dots in single box, they "explode," disappear, and become one dot in the box to the left.

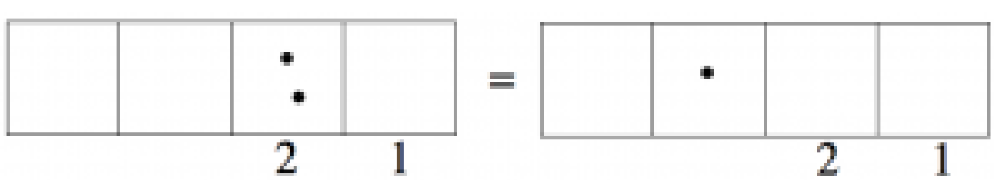
Two dots in the right-most box is worth one dot in the next box to the left.



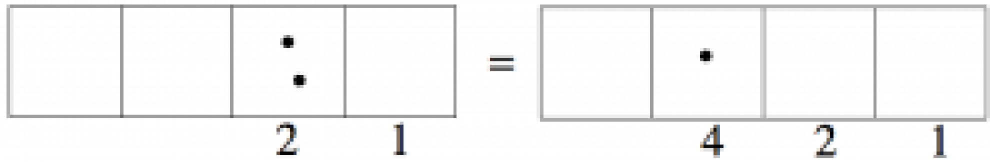
If each of the original dots is worth "one," then the single dot on the left must be worth two.



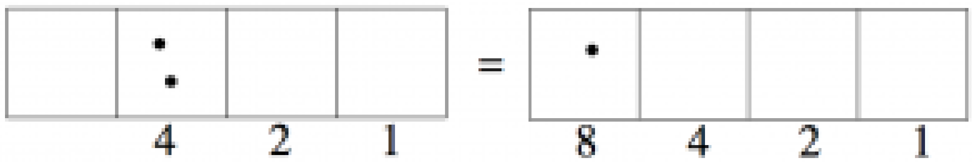
But we also have two dots in the box of value 2 is worth one dot in the box just to the left…



So that next box must be worth two 2's, which is four!



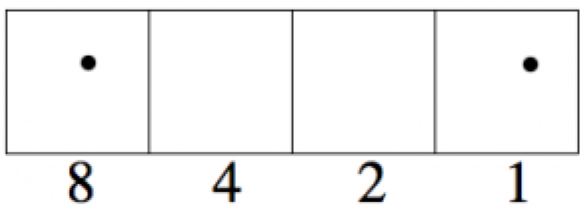
And two of these fours make eight.



Examples

Example 1: Nine dots in the 1←2 system revisited.

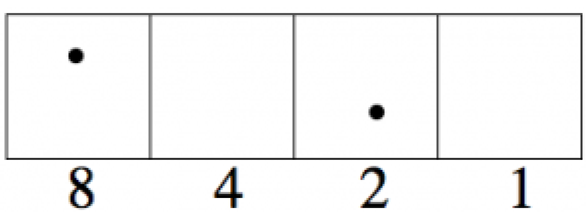
We said earlier that the 1←2 code for nine dots was 1001. Let's check:



$8 + 1 = 9$ so this works!

Example 2: Ten dots in the 1←2 system revisited.

You should have found that ten dots has 1←2 code 1010.



Yep! $8 + 2 = 10$.

Problem 7

1. If there were a box to the left of the 8 box, what would the value of that box be?
2. What would be the value of a box *two* spots to the left of the 8 box? Three spots to the left?
3. What number has $1 \leftarrow 2$ code 100101?
4. What is the $1 \leftarrow 2$ code for two hundred dots?

Definition and Notation

Numbers written in the $1 \leftarrow 2$ code are called **binary numbers** or **base two numbers**. (The prefix "bi" means "two.")

From now on, when we want to indicate that a number is written in base two, we will write a subscript "two" on the number.

So 1001_{two} means "the number of dots that has $1 \leftarrow 2$ code 1001," which we already saw was nine.

Important! When we read we say "one zero zero one base two." We don't say "one thousand and one," because "thousand" is not a binary number.

Think / Pair / Share

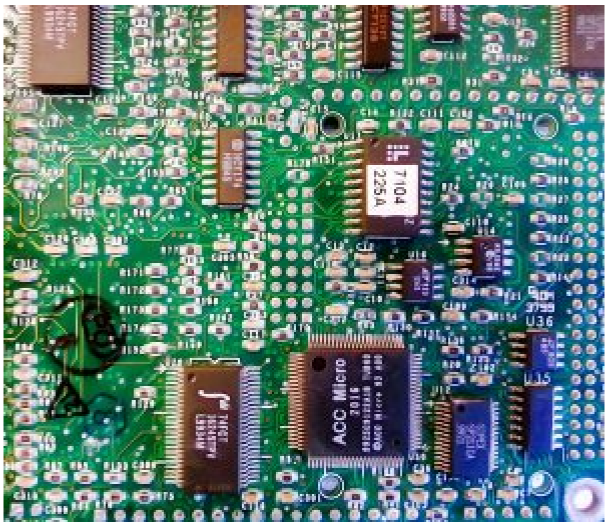
1. Your first goal: come up with a *general method* to find the number of dots represented by any binary number. Clearly describe your method. Test your method out on these numbers, and check your work by actually "unexploding" the dots.
   1_{two} 101_{two} 1011_{two} 11111_{two}
2. Explain why binary numbers only contain the digits 0 and 1.
3. Here is a new (harder) goal: come up with a *general method* to find the binary number related to any number of dots *without actually going through the "exploding dot" process*. Clearly describe your method. Test your method out on these numbers, and find a way to check your work.

two dots = $\_\_\__{\text{two}}$ seventeen dots = $\_\_\__{\text{two}}$

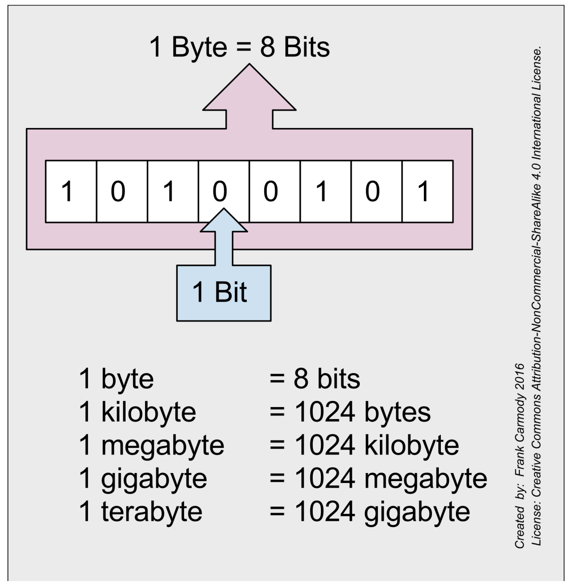
sixty-three dots = $\_\_\__{\text{two}}$ one hundred dots = $\_\_\__{\text{two}}$

History

You probably realize by now that a number is an abstract concept with many representations. The standard decimal representation of a number is only one of these. For computers, numbers are always represented in binary. The basic units are transistors which are either on (1) or off (0).



Electronic circuit board from Samsung Galaxy S III.



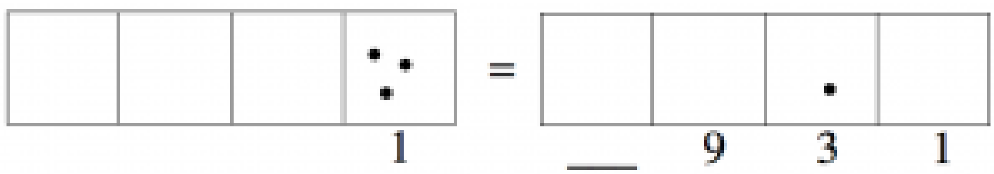
A transistor[1] is said to store ***one bit*** of information. Eight bits make a byte and a typical home computer's central processing unit performs computations on registries that are each 8 bytes (64-bits).

Using the $1 \leftarrow 2$ rule we can represent the numbers 0 through 18,446,744,073,709,551,615 with 64 bits.

1. Image of circuit board from http://www.publicdomainpictures.net/, licensed under CC0 Public Domain.

Other Bases

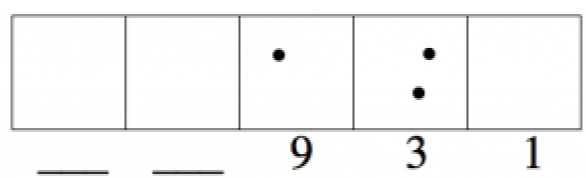
In the $1 \leftarrow 3$ system, three dots in one box is worth one dot in the box one spot to the left. This gives a new picture:



Each dot in the second box from the left is worth three ones. Each dot in the third box is worth three 3's, which is nine, and so on.

Example

We said that the $1 \leftarrow 3$ code for fifteen is 120. We see that this is correct because:



$$1 \cdot 9 + 2 \cdot 3 + 0 \cdot 1 = 15.$$

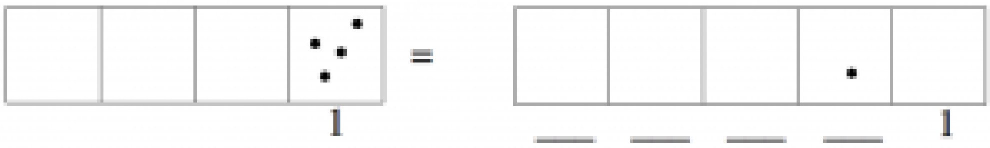
Problem 8

Answer these questions about the $1 \leftarrow 3$ system.

1. What label should go on the box to the left of the 9 box?
2. What would be the value of a box two spots to the left of the 9 box?

3. What number has 1 ← 3 code 21002?
4. What is the 1 ← 3 code for two hundred dots?

In the 1 ← 4 system, four dots in one box are worth one dot in the box one place to the left.



Problem 9

Answer these questions about the 1 ← 4 system.

1. What is the value of each box in the picture above?
2. What is the 1 ← 4 code for twenty-nine dots?
3. What number has 1 ← 4 code 132?

Problem 10

In the 1 ← 10 system, ten dots in one box are worth one dot in the box one place to the left.

1. Draw a picture of the 1 ← 10 and label the first four boxes with their values.
2. What is the 1 ← 10 code for eight thousand four hundred and twenty-two?
3. What number has 1 ← 10 code 95,753?
4. When we write the number 7,842, what does the "7" represent?
   The "4" is four groups of what value?
   The "8" is eight groups of what value?
   The "2" is two groups of what value?
5. Why do you think we use the 1 ← 10 system for writing numbers?

Definition

Recall that numbers written in the $1 \leftarrow 2$ system are called **binary** or **base two** numbers.

Numbers written in the $1 \leftarrow 3$ system are called **base three** numbers.

Numbers written in the $1 \leftarrow 4$ system are called **base four** numbers.

Numbers written in the $1 \leftarrow 10$ system are called **base ten** numbers.

In general, numbers written in the $1 \leftarrow b$ system are called **base *b*** numbers.

In a base b number system, each place represents a power of b, which means b^n for some whole number n. Remember this means b multiplied by itself n times:

- The right-most place is the units or ones place. (Why is this a power of b?)
- The second spot is the "b" place. (In base ten, it's the tens place.)
- The third spot is the "b^2" place. (In base ten, that's the hundreds place. Note that $10^2 = 100$.)
- The fourth spot is the "b^3" place. (In base ten, that's the thousands place, since $10^3 = 1000$.)
- And so on.

Notation

Whenever we're dealing with numbers written in different bases, we use a subscript to indicate the base so that there can be no confusion. So:

- 102_{three} is a base three number (read it as "one-zero-two base three"). This is the base three code for the number eleven.
- 222_{four} is a base four number (read it as "two-two-two base four"). This is the base four code for the number forty-two.
- 54321_{ten} is a base ten number. (It's ok to say "fifty-four thousand three hundred and twenty-one." Why?)

If the base is not written, we assume it's base ten.

Remember: when you see the subscript, you are seeing the **code** for some number of dots.

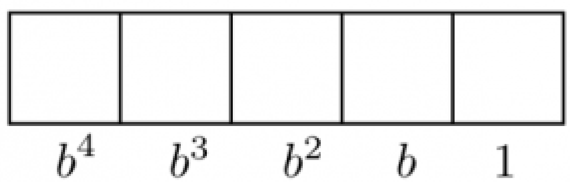
Think / Pair / Share

1. Find the number of dots represented by each of these:
   222_{three} $\quad 310_{\text{four}}$ $\quad 54321_{\text{ten}}$.
2. Represent nine dots in each base:
   three, five, eight, nine, and eleven.
3. Which digits are used in the base two system? The base three system? The base four system? The base five system? The base six system? The base ten system?
4. What does the *base* tell you about the number system? (Think of as many answers as you can!)

Base *b* to Base Ten

We're now going to describe some general methods for converting from base *b* to base ten, where *b* can represent any whole number bigger than one.

If the base is *b*, that means we're in a $1 \leftarrow b$ system. A dot in the right-most box is worth 1. A dot in the second box is worth *b*. A dot in the third box is worth b^2, and so on.



So, for example, the number 10123_b represents

$$1 \cdot b^4 + 0 \cdot b^3 + 1 \cdot b^2 + 2 \cdot b + 3 \cdot 1,$$

because we imagine three dots in the right-most box (each worth one), two dots in the second box (each representing *b* dots), one dot in the third box (representing b^2 dots), and so on. That means we can just do a short calculation to find the total number of dots, without going through all the trouble of drawing the picture and "unexploding" the dots.

Examples

Example 1: Consider the number 123_{five}.

This represents the number

$$1 \cdot 5^2 + 2 \cdot 5 + 3 = 25 + 10 + 3 = 38.$$

Example 2: Consider the number 123_{seven}.

$$1 \cdot 7^2 + 2 \cdot 7 + 3 = 49 + 14 + 3 = 66.$$

Base Ten to Base *b*

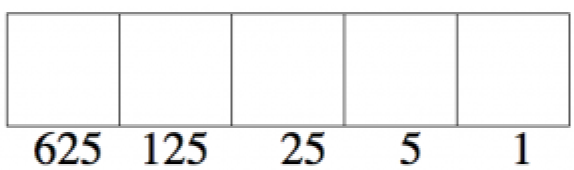
We're now going to describe some general methods for converting from base ten to base *b*, where *b* can represent any whole number bigger than one.

There are two general methods for doing these conversions. For each method, we'll work out an example, and then describe the general method. The first method we describe fills in the boxes from left to right.

Example: Method 1 (left-to-right)

To convert 321_{ten} to a base five number (without actually going through the tedious process of exploding dots in groups of five).

Find the largest power of five that is smaller than 321. We'll just list powers of five:

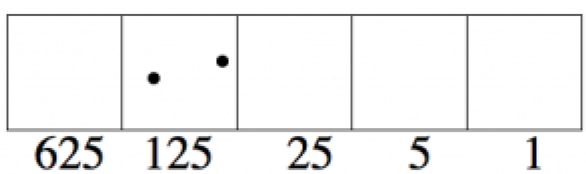


So we know that the left-most box we'll use is the 125 box, because 625 is too big.

How many dots will be in the 125 box? That's the same as asking how many 125's are in 321. Since

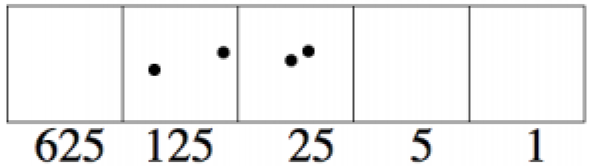
$$2 \cdot 125 = 250 \quad \text{and} \quad 3 \cdot 125 = 375,$$

we should put two dots in the 125 box. Three dots would be too much.



How many dots are left unaccounted for? $321 - 250 = 71$ dots are left.

Now repeat the process: The largest power of five that's less than 71 is $5^2 = 25$. If we put two in the 25 box, that takes care of 50 dots. (Three dots would be 75, which is too much.)



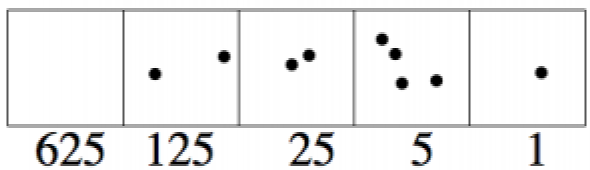
So far we have two dots in the 5^3 box and two dots in the 5^2 box, so that's a total of

$$2 \cdot 125 + 2 \cdot 25 = 300 \text{ dots.}$$

We have $321 - 300 = 21$ dots left to account for.

Repeat the process again: The biggest power of 5 that's less than 21 is 5. How many dots can go in the 5 box? $5 \cdot 4 = 20$, so we can put four dots in the 5 box.

We have one dot left to account for. If we put one dot in the 1 box, we're done.



$$2 \cdot 125 + 2 \cdot 25 + 4 \cdot 5 + 1 = 250 + 50 + 20 + 1 = 321.$$

So $321_{\text{ten}} = 2241_{\text{five}}$.

The general algorithm to convert from base ten to base :

1. Start with your base ten number *n*. Find the largest *power of b* that's less than *n*, say that power is b^k.
2. Figure out how many dots can go in the b^k box without going over *n*. Say that number is *a*. Put the digit *a* in the b^k box, and then subtract $n - a \cdot b^k$ to figure out how many dots are left.
3. If your number is now zero, you've accounted for all the dots. Put zeros in any boxes that remain, and you have the number. Otherwise, start over at step (1) with the number of dots you have left.

The method is a little tricky to describe in complete generality. It's probably better to try a few examples on your own to get the hang of it.

Think / Pair / Share

Use the method above to convert 99_{ten} to base three, to base four, and to base five.

Here's another method to convert base ten numbers to another base, and this method fills in the digits from right to left. Again, we'll start with an example and then describe the general method.

Example: Method 2 (right-to-left)

To convert 712_{ten} to a base seven number, imagine there are 712 dots in the ones box. We'll write the number, but imagine it as dots.

				712
	343	49	7	1

Find out how many groups of 7 you can make, and how many dots would be left over.

$$712 \div 7 = 101 \text{ R}5; \quad \text{that is, } 712 = 101 \cdot 7 + 5.$$

That means we have 101 groups of 7 dots, with 5 dots left over.

"Explode" the groups of 7 one box to the left, and leave the 5 dots behind.

			101	5
	343	49	7	1

Now repeat the process: How many groups of 7 can you make from the 101 dots?

$$101 \div 7 = 14 \text{ R}3, \quad \text{meaning } 101 = 14 \cdot 7 + 3.$$

"Explode" the groups of 7 one box to the left, and leave the 3 dots behind.

		14	3	5
	343	49	7	1

Repeat:

$$14 \div 7 = 2 \text{ R}0, \quad \text{so } 14 = 2 \cdot 7 + 0.$$

"Explode" the groups of 7 one box to the left, and leave 0 dots behind.

	2	0	3	5
	343	49	7	1

Since there are fewer than 7 dots in each box, we're done.

$$712_{\text{ten}} = 2035_{\text{seven}}.$$

Of course, we can (and should!) check our calculation by converting the answer back to base ten:

$$2035_{\text{seven}} = 2 \cdot 7^3 + 0 \cdot 7^2 + 3 \cdot 7 + 5 = 686 + 0 + 21 + 5 = 712_{\text{ten}}.$$

So here's a second general method for converting base ten numbers to an arbitrary base *b*:

1. Divide the base ten number by *b* to get a quotient and a remainder.
2. Put the remainder in the right-most space in the base *b* number.
3. If the quotient is less than *b*, it goes in the space one spot to the left. Otherwise, go back to step (1) and repeat it with the quotient, filling in the remainders from right to left in the base number.

Again, the method probably makes more sense if you try it out a few times.

Think / Pair / Share

Use the method described above to convert 250_{ten} to base three, four, five, and six.

Number Systems

Our number system is a western adaptation of the Hindu-Arabic numeral system developed somewhere between the first and fourth centuries AD. However, numbers have been recorded with tally marks throughout history. The Ishango Bone[1] from Africa is about 25,000 years old. It's the lower leg bone from a baboon, and contains tally marks. We know the marks were used for counting because they appear in distinct groups.

This reindeer antler[2] from France is about 15,000 years old, and also shows clearly grouped tally marks.

1. Image of Ishango bone by Ben2 (Own work), CC-BY-SA-3.0 or CC BY-SA 2.5-2.0-1.0], via Wikimedia Commons.
2. Image of antler By Ryan Somma from Occoquan, USA [CC BY-SA 2.0], via Wikimedia Commons

Of course, we still use tally marks today![3]

3. Image of Hanakapiai beach warning sign by God of War at the English language Wikipedia [GFDL or CC-BY-SA-3.0], via Wikimedia Commons.

Base ten numbers (the ones you have probably been using your whole life), and base *b* numbers (the ones you've been learning about in this chapter) are both positional number systems.

Definition

A **positional number system** is one way of writing numbers. It has unique symbols for 1 through $b - 1$, where b is the base of the system. Modern positional number systems also include a symbol for 0.

The **positional value** of each symbol depends on its position in the number:

- The positional value of a symbol in the first position is just its face value.
- The positional value of a symbol in the second position is b times its value.
- The positional value of a symbol in the third position is b^2 times its value.
- And so on.

The value of a number is the sum of the positional values of its digits.

Definition

In an **additive number system,** the value of a written number is the sum of the face values of the symbols that make up the number. The only symbol necessary for an additive number system is a symbol for 1, however many additive number systems contain other symbols.

History: Roman numerals

The ancient Romans used a version of an additive number systems. The Romans represented numbers this way:

number	Roman Numeral
1	I
5	V
10	X
50	L
100	C
500	D
1,000	M

So the number 2013 would be represented as MMXIII. This is read as 2,000 (two M's), one ten (one X), and three ones (three I's).

For any additive number system very large numbers become impractical to write. To represent the number one million in Roman numerals it would take one thousand M's!

However, the Roman numerals did have one efficiency advantage: The order of the symbols mattered. If a symbol to the left was smaller than the symbol to the right, it would be subtracted instead of added. So for example nine is represented as IX rather than VIIII.

Think / Pair / Share

If you don't already know how to use Roman numerals, research it a little bit. Then answer these questions.

- Write the numbers 1–20 in Roman numerals.
- What is the maximum number of symbols needed to write any number between 1 and 1,000 in Roman Numerals? Justify your answer.

The earliest positional number systems are attributed to the Babylonians (base 60) and the Mayans (base 20). These positional systems were both developed before they had a symbol or a clear concept for zero. Instead of using 0, a blank space was used to indicate skipping a particular place value. This could lead to ambiguity.

Suppose we didn't have a symbol for 0, and someone wrote the number

$$2\quad 3.$$

It would be impossible to tell if they mean 23, 203, 2003, or maybe two separate numbers (two and three).

Leonardo Pisano Bigollo, more commonly known as **Fibonacci**[4], played a pivotal role in guiding Europe out of a long period in which the importance and development of math was in marked decline. He was born in Italy around 1170 CE to Guglielmo Bonacci, a successful merchant. Guglielmo brought his son with him to what is now Algeria, and Leonardo was educated in mathematics mathematics there.

Fibonacci

At the time, Roman Numerals dominated Europe, and the official means of calculations was the abacus. Muḥammad ibn Mūsā al-Khwārizmī[5] described the use of Hindu-Arabic system in his book *On the Calculation with Hindu Numerals* in 825 CE, but it was not well-known in Europe.

4. Image of Fibonacci via Wikimedia Commons, in the public domain.
5. Image of al-Khwarizmi statue by M. Tomczak [CC BY-SA 3.0], via Wikimedia Commons.

Statue of al-Khwarizmi at Amirkabir University of Technology

Fibonacci's book *Liber Abaci* described the Hindu-Arabic system and its business applications for a European readership. His book was well-received throughout Europe, and it marked the beginning of a reawakening of European mathematics.

History: Hawaiian numbers

The Hindu-Arabic number system is now used nearly exclusively throughout the globe. But many cultures had their own number systems before contact and trade with other countries spread the work of al-Khwārizmī throughout the world.

There is evidence that pre-contact Hawaiians actually used two different number systems. Depending on what they were counting, they might use base 4 instead (or a mixed base-10 and base-4 system). One theory is that certain objects (fish, taro, etc.) were often put in bundles of 4, so were more natural to count by 4's than by 10's. The number four also had spiritual significance in Hawaiian culture.

Humans have 5 fingers on each hand[6], making base ten a natural choice for counting. Bu there are 4 gaps between the fingers, meaning that a hand can carry four fish or taro plants similar objects, making base four a natural choice for some cultures.

In the mixed base system, instead of powers of 10, numbers are broken down into sums of numbers that look like 4 times a power of 10 (40, 400, 4000, etc.).

1	‘ekahi
2	‘elua
3	‘ekolu
4	‘ehā (or kauna)
5	‘elima
6	‘eono
7	‘ehiku
8	‘ewalu
9	‘eiwa
10	‘umi
11–19	‘umi kumamā {kahi, lua, kolu, hā, etc.}
20	iwakālua
21–29	Iwakālua kumamā {kahi, lua, kolu, hā, etc.}
30	kanakolu
31–39	kanakolu kumamā {kahi, lua, etc.}
40	kanahā
400	lau
4,000	mano
40,000	kini
400,000	lehu

6. Image of hand from https://pixabay.com, licensed under CC0 Creative Commons.

Here are a few examples (refer to the table above for the Hawaiian names of the numbers):

Example

ʻekolu kini, ʻewalu lau me ʻekahi

translates to three 40,000's, eight 400's, and one;

$3 \cdot 40000 + 8 \cdot 400 + 1 = 123201$

Example

$5207 = 1 \cdot 4000 + 3 \cdot 400 + 7$

would be ʻekahi mano, ʻekolu lau me ʻehiku

On Your Own

Work on the following exercises on your own or with a partner.

1. Translate this Hawaiian number to English and then write it in base ten.

ʻekahi kanahā me kanakolu kumamāiwa

2. Translate this base-ten number to Hawaiian.

1,573

Think / Pair / Share

How is learning about different number systems (including representing numbers in different bases) valuable to you as a future teacher?

Even Numbers

How do we know if a number is even? What does it mean?

Definition

Some number of dots is **even** if I can divide the dots into pairs, and every dot has a partner.

Some number of dots is **odd** if, when I try to pair up the dots, I always have a single dot left over with no partner.

The number of dots is either even or odd. It's a property of the quantity and is doesn't change when you represent that quantity in different bases.

Problem 13

Which of these numbers represent an even number of dots? Explain how you decide.

22_{ten} 319_{ten} 133_{five} 222_{five} 11_{seven} 11_{four}

Think / Pair / Share

Compare your answers to problem 13 with a partner. Then try these together:

1. Count by twos to 20_{ten}.
2. Count by twos to 30_{four}.
3. Count by twos to 51_{seven}.

You know that you can tell if a base ten number is even just by looking at the ones place. But why is that true? That's not the definition of an even number. There are a few key ideas behind this handy trick:

- In base ten, every number looks like

  (some multiple of ten) + (ones digit)

$$53 = 50 + 3$$
$$492 = 490 + 2$$
$$45637289108 = 45637289100 + 8$$

- Every multiple of ten is an even number, since

  $$10n = 2(5n),$$

  and two times a whole number is always even.

- Your whole number looks like this:

  (some multiple of ten) + (ones digit)

  (even number) + (ones digit),

- Even plus even is even, and even plus odd is odd, so your whole number is even when the ones digit is even, and it's odd when the ones digit is odd.

Think / Pair / Share

- Make sure you understand the explanation above. Does each piece make sense to you?
- In particular: Use the definition of even and odd above to explain the last step. Why is it true that even + even = even and even + odd = odd?
- What about odd + odd? Is that odd or even? Justify what you say.

Problem 14

1. Write the numbers zero through fifteen in base seven:

base ten	base seven
0	0_{seven}
1	1_{seven}
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	

2. Circle all of the even numbers in your list. How do you know they are even?

3. Find a rule: how can you tell if a number is even when it's written in base seven?

Problem 15

1. Write the numbers zero through fifteen in base four:

base ten	base four
0	0_{four}
1	1_{four}
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	

2. Circle all of the even numbers in your list. How do you know they are even?

3. Find a rule: how can you tell if a number is even when it's written in base four?

Think / Pair / Share

- Why are the rules for recognizing even numbers different in different bases?
- For either your base four rule or your base seven rule, can you explain *why* it works that way?

Problem Bank

Problem 28

1. If you were counting in base four, what number would you say just before you said 100_{four}?
2. What number is one more than 133_{four}?
3. What is the greatest three-digit number that can be written in base four? What numbers come just before and just after that number?

Problem 29

Explain what is wrong with writing 313_{two} or 28_{eight}.

Problem 30

1. Write out the base three numbers from 1_{three} to 200_{three}.
2. Write out the base five numbers from 1_{five} to 100_{five}.
3. Write the four base six numbers that come after 154_{six}.

Problem 31

Convert each base ten number to a base four number. Explain how you did it.

$13, \qquad 8, \qquad 24, \qquad 49$

Challenges:

$0.125, \qquad 0.11111\cdots = 0.\overline{1}$

Problem 32

In order to use base sixteen, we need sixteen digits — they will represent the numbers zero through fifteen. We can use our usual digits 0–9, but we need *new symbols* to represent the *digits* ten, eleven, twelve, thirteen, fourteen, and fifteen. Here's one standard convention:

base ten	base sixteen
7	7_{sixteen}
8	8_{sixteen}
9	9_{sixteen}
10	$\text{A}_{\text{sixteen}}$
11	$\text{B}_{\text{sixteen}}$
12	$\text{C}_{\text{sixteen}}$
13	$\text{D}_{\text{sixteen}}$
14	$\text{E}_{\text{sixteen}}$
15	$\text{F}_{\text{sixteen}}$
16	10_{sixteen}

1. Convert these numbers from base sixteen to base ten, and show your work:
   $6\text{D}_{\text{sixteen}} \qquad \text{AE}_{\text{sixteen}} \qquad 9\text{C}_{\text{sixteen}} \qquad 2\text{B}_{\text{sixteen}}$
2. Convert these numbers from base ten to base sixteen, and show your work:
   $97 \qquad 144 \qquad 203 \qquad 890$

Problem 33

How many different symbols would you need for a base twenty-five system? Justify your answer.

Problem 34

All of the following numbers are multiples of three.

$$3, \quad 6, \quad 9, \quad 12, \quad 21, \quad 27, \quad 33, \quad 60, \quad 81, \quad 99.$$

1. Identify the *powers of* 3 in the list. Justify your answer.
2. Write each of the numbers above in base three.
3. In base three: how can you recognize a *multiple of* 3? Explain your answer.
4. In base three: how can you recognize a *power of* 3? Explain your answer.

Problem 35

All of the following numbers are multiples of five.

$$5, \quad 10, \quad 15, \quad 25, \quad 55, \quad 75, \quad 100, \quad 125, \quad 625, \quad 1000.$$

1. Identify the *powers of* 5 in the list. Justify your answer.
2. Write each of the numbers above in base five.
3. In base five: how can you recognize a *multiple of* 5? Explain your answer.
4. In base five: how can you recognize a *power of* 5? Explain your answer.

Problem 36

Convert each number to the given base.

1. 395_{ten} into base eight.
2. 52_{ten} into base two.
3. 743_{ten} into base five.

Problem 37

What bases makes theses equations true? Justify your answers.

1. $35 = 120\_\_$
2. $41_{\text{six}} = 27\_\_$
3. $52_{\text{seven}} = 34\_\_$

Problem 38

What bases makes theses equations true? Justify your answers.

1. $32 = 44\_\_$
2. $57_{\text{eight}} = 10\_\_$
3. $31_{\text{four}} = 11\_\_$
4. $15_x = 30_y$

Problem 39

1. Find a base ten number that is twice the product of its two digits. Is there more than one answer? Justify what you say.
2. Can you solve this problem in any base other than ten?

Problem 40

1. I have a four-digit number written in base ten. When I multiply my number by four, the digits get reversed. Find the number.
2. Can you solve this problem in any base other than ten?

Problem 41

Convert each base four number to a base ten number. Explain how you did it.

13_{four} 322_{four} 101_{four} 1300_{four}

Challenges:

0.2_{four} $0.111\ldots_{\text{four}} = 0.\overline{1}_{\text{four}}$

Problem 42

Consider this base ten number (I got this by writing the numbers from 1 to 60 in order next to one another):

$12345678910111213\ldots57585960.$

1. What is the largest number that can be produced by erasing one hundred digits of the number? (When you erase a digit it goes away. For example, if you start with the number 12345 and erase the middle digit, you produce the number 1245.) How do you *know* you got the largest possible number?
2. What is the smallest number that can be produced by erasing one hundred digits of the number? How do you *know* you got the smallest possible number?

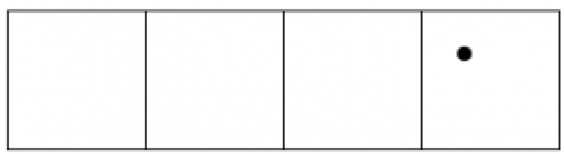
Problem 43

Can you find two different numbers (not necessarily single digits!) a and b so that $a_b = b_a$? Can you find more than one solution? Justify your answers.

Exploration

Problem 44

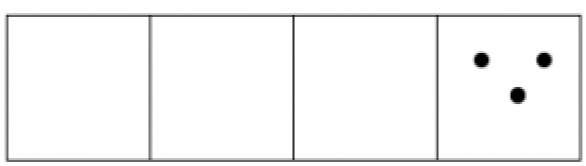
Jay decides to play with a system that follows a $1 \leftarrow 1$ rule. He puts one dot into the right-most box. What happens?



Problem 45

Poindexter decides to play with a system that follows the rule $2 \leftarrow 3$.

1. Describe what this rule does when there are three dots in the right-most box.



2. Draw diagrams or use buttons or pennies to find the codes for the following numbers:

$$1 \text{ through } 20, 24, 27, 30, 33, 36, \text{ and } 39$$

3. Can you find (and *explain*) any patterns?

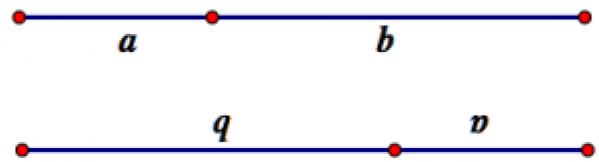
Problem 46

Repeat problem 45 for your own rule. Choose two numbers $a \neq 1$ and b. For each of the numbers

$$1 \text{ through } 20, 24, 27, 30, 33, 36, \text{ and } 39,$$

figure out the $a \leftarrow b$ code. Look for patterns, and explain them if you can!

Number and Operations



$a + b = b + a$

> The essence of mathematics is not to make simple things complicated, but to make complicated things simple.
>
> -S. Gudder

The "Dots and Boxes" approach to understanding operations used in this part (and throughout this book) comes from James Tanton, and is used with his permission. See his development of these and other ideas at http://gdaymath.com/.

Introduction

When learning and teaching about arithmetic, it helps to have mental and physical *models* for what the operations mean. That way, when you are presented with an unfamiliar problem or a question about why something is true, you can often work it out using the model — this might mean drawing pictures, using physical materials (manipulatives), or just thinking about the model to help you reason out the answer.

Think / Pair / Share

Write down your mental models for each of the four basic operations. What do they actually *mean*? How would you explain them to a second grader? What pictures could you draw for each operation? Think about each one separately, as well as how they relate to each other:

- addition
- subtraction
- multiplication, and
- division.

After writing down you own ideas, share them with a partner. Do you and your partner have the same models for each of the operations or do you think about them differently?

Teachers should have lots of mental models — lots of ways to explain the same concept. In this chapter, we'll look at some different ways to understand the four basic arithmetic operations. First, let's define some terms:

Definition

Counting numbers are literally the numbers we use for counting: 1, 2, 3, 4, 5… These are sometimes called the *natural numbers* by mathematicians, and they are represented by the symbol $\mathbb{N}$.

Whole numbers are the counting numbers together with zero.

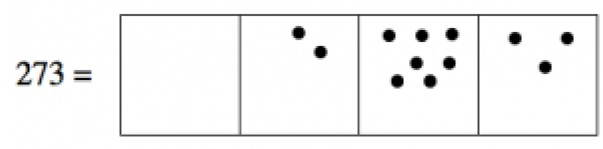
Integers include the positive and negative whole numbers, and mathematicians represent these with the symbol $\mathbb{Z}$. (This comes from German, where the word for "number" is "zählen.")

We already have a natural model for thinking about counting numbers: a number is a quantity of dots. Depending on which number system you use — Roman numerals, base ten, binary, etc. — you might write down the number in different ways. But the quantity of dots is a counting number, however you write it down.

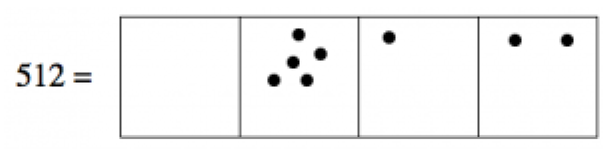
Addition: Dots and Boxes

Addition as combining

For now, we'll focus on the base-10 system. Here's how we think about the number 273 in that system:

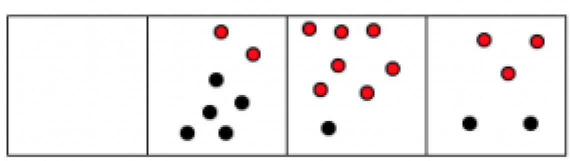


And here is the number 512:



Example: 273+512

We can add these in the natural way: just combine the piles of dots. Since they're already in place-value columns, we can combine dots from the two numbers that are in the same place-value box.



We can count up the answer: there are 7 dots in the hundreds box, 8 dots in the tens box, and 5 dots in the ones box.

$$\begin{array}{r} 273 \\ +512 \\ \hline 785 \end{array}$$

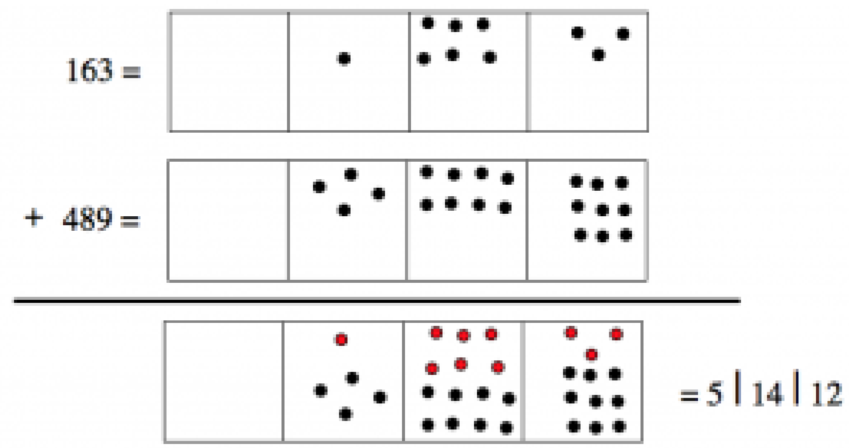
And saying out the long way we have:

- Two hundreds plus five hundreds gives 7 hundreds.
- Seven tens plus one ten gives 8 tens.
- Three ones plus two ones gives 5 ones.

This gives the answer: 785.

Example: 163+489

Let’s do another one. Consider 163+489.



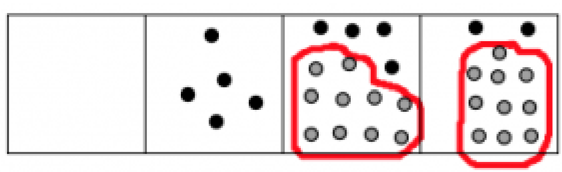
$$\begin{array}{r} 1 \quad 6 \quad 3 \\ +\,4 \quad 8 \quad 9 \\ \hline 5\,|\,14\,|\,12 \end{array}$$

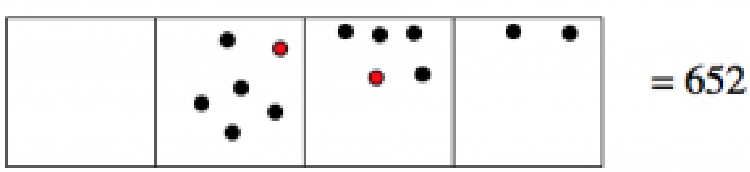
And this is absolutely correct:

- One hundred plus four hundreds is 5 hundreds.
- Six tens plus eight tens is 14 tens.
- Three ones plus nine ones is 12 ones.

The answer is 5 | 14 | 12, which we might try to pronounce as "five hundred and fourteeny-tenty twelvety." The trouble with this answer is that most of the rest of the world wouldn't understand what we are talking about.

Since this is a base 10 system, we can do some explosions.





The answer is "six hundred fifty two." Okay, the world can understand this one!

$$\begin{array}{cccc} & 1 & 6 & 3 \\ + & 4 & 8 & 9 \\ \hline \end{array}$$

$$5 \,|\, 14 \,|\, 12 = \; 652$$

Think / Pair / Share

Solve the following exercises by thinking about the dots and boxes. (You can draw the pictures, or just imagine them.) Then translate the answer into something the rest of the world can understand.

$$\begin{array}{r} 148 \\ +323 \\ \hline \end{array} \qquad \begin{array}{r} 567 \\ +271 \\ \hline \end{array} \qquad \begin{array}{r} 310462872 \\ +389107123 \\ \hline \end{array}$$

Problem 1

Use the dots and boxes technique to solve these problems. *Do not covert to base 10! Try to work directly in the base given.* It might help to actually draw the pictures.

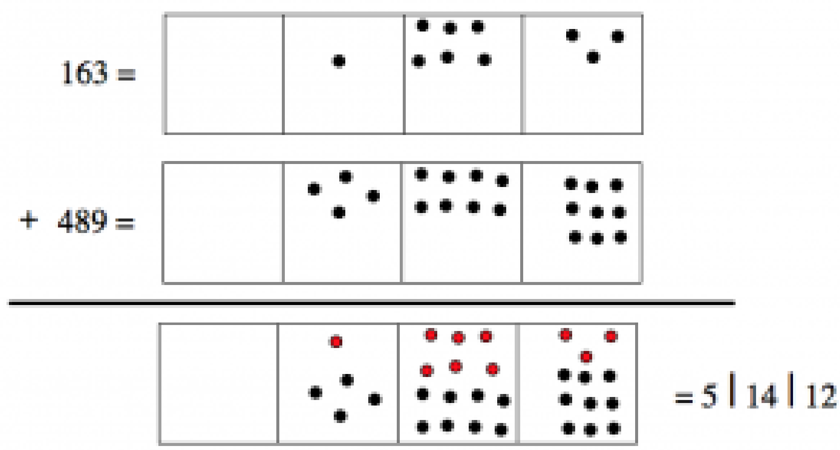
$$\begin{array}{r} 20413_{\text{five}} \\ \underline{+13244_{\text{five}}} \end{array} \qquad \begin{array}{r} 4052_{\text{nine}} \\ \underline{+6288_{\text{nine}}} \end{array} \qquad \begin{array}{r} 3323_{\text{seven}} \\ \underline{+3555_{\text{seven}}} \end{array}$$

The Standard Algorithm for Addition

Let's go back to the example 163+489. Some teachers don't like writing:

$$\begin{array}{rrr} 1 & 6 & 3 \\ \underline{+4} & \underline{8} & \underline{9} \\ \end{array}$$
$$5 \mid 14 \mid 12 = 652$$

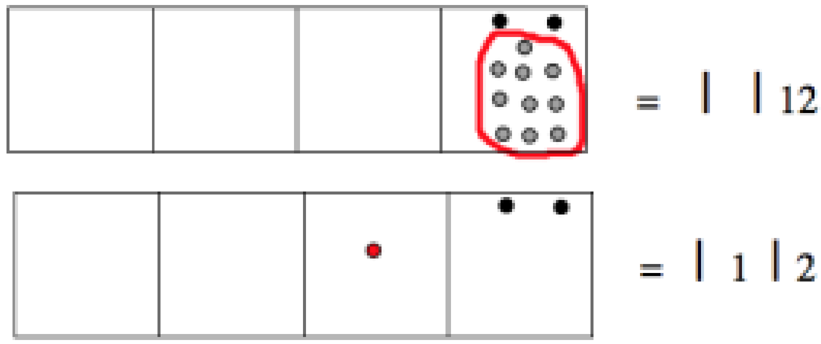
They prefer to teach their students to start with the 3 and 9 at the end and sum those to get 12. This is of course correct — we got 12 as well.



But they don't want students to write or think "twelvety," so they have their students write something like this:

$$\begin{array}{r} 1 \\ 163 \\ \underline{+489} \\ 2 \end{array}$$

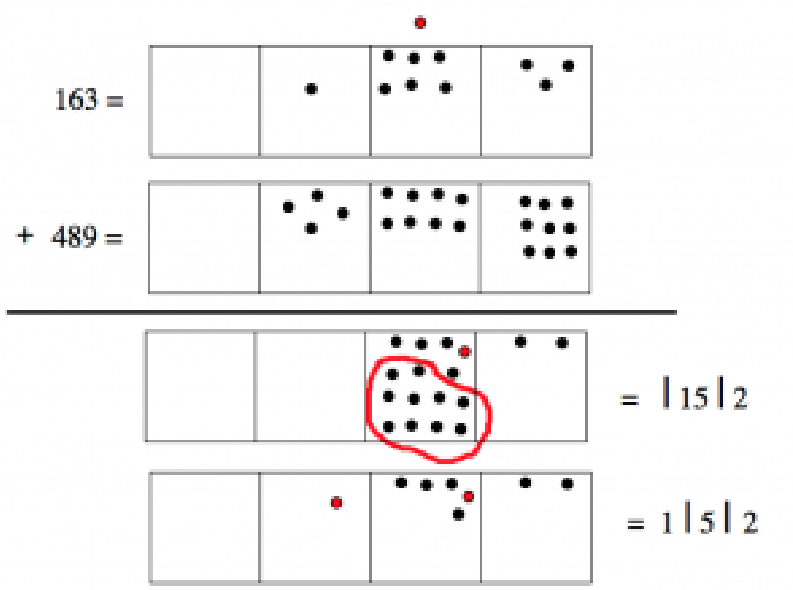
This can seem completely mysterious. What's really going on? They are exploding ten dots, of course!



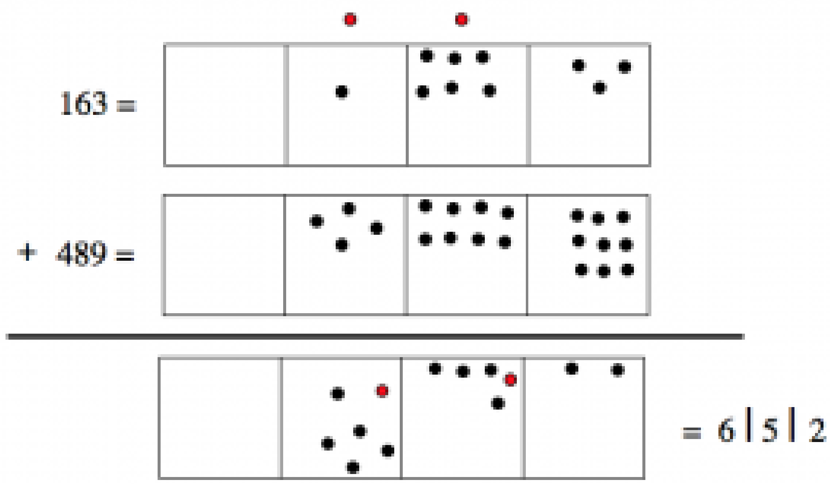
Now we carry on with the problem and add the tens. Students are taught to write:

$$\begin{array}{r} 1 \\ 163 \\ \underline{+489} \\ 52 \end{array}$$

But what this means is better shown in this next picture. Notice the "exploded" (or regrouped) dot at the very top, which is added to the tens box in the answer.



And now we finish the problem by combining the dots in the hundreds boxes:



$$\begin{array}{r} 1 \\ 163 \\ \underline{+489} \\ 652 \end{array}$$

In the standard algorithm, we work from right to left, doing the "explosions" as we go along. This means that we start adding at the ones place and work towards the left-most place value, "carrying" digits that come from the explosions. (This is really not carrying; a better term for it is *regrouping*. Ten ones become one ten. Ten tens become one hundred. And so on.)

In the dots and boxes method, we add in any direction or order we like and then we do the explosions at the end.

- **Why do we like the standard algorithm?** Because it is efficient.
- **Why do we like the dots and boxes method?** Because it is easy to understand.

Subtraction: Dots and Boxes

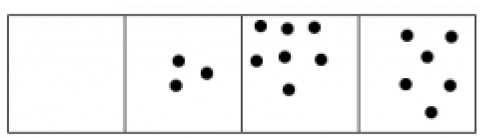
Subtraction as Take-Away

To model addition, we started with two collections of dots (two numbers), and we *combined* them to form one bigger collection. That's pretty much the definition of addition: combining two collections of objects. In subtraction, we start with one collection of dots (one number), and we take some dots away.

Example: 376 – 125

Suppose we want to find 376-125 in the dots and boxes model. We start with the representation of 376:

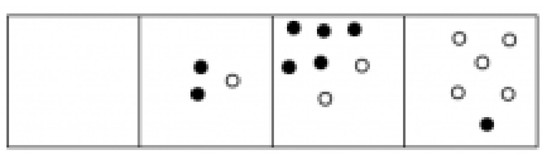
376 =



Since we want to "take away" 125, that means:

- We take away one dot from the hundreds box, leaving two dots.
- We take away two dots from the tens box, leaving five dots.
- And we take away five dots from the ones box, leaving one dot.

376 – 125 =



So the answer is:

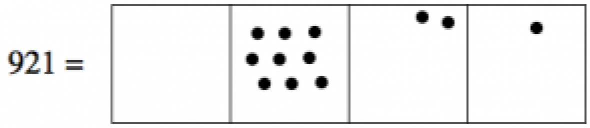
$$\begin{array}{r} 376 \\ -125 \\ \hline 251 \end{array}$$

And saying it out the long way we have:

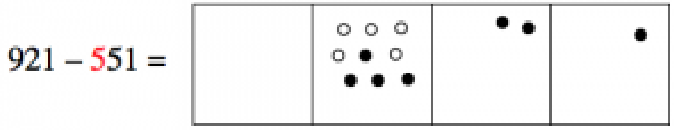
- Three hundreds take away one hundred leaves 2 hundreds.
- Seven tens take away two tens gives 5 tens.
- Six ones take away five ones gives 1 one.

Example: 921 – 551

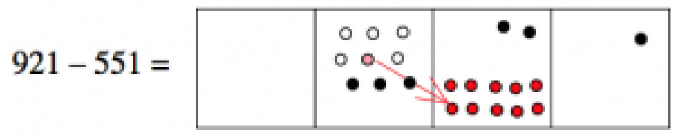
Let's try a somewhat harder example: 921-551. We start with the representation of 921:

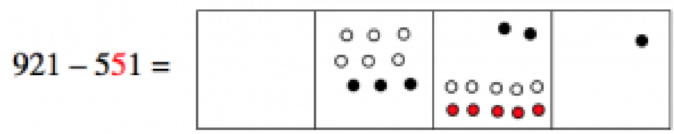


Since we want to "take away" 551, that means we take away five dots from the hundreds box, leaving four dots.



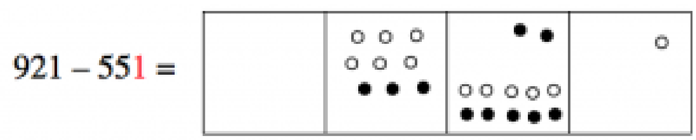
Now we want to take away five dots from the tens box, but we can't do it! There are only two dots there. What can we do? Well, we still have some hundreds, so we can "unexplode" a hundreds dot, and put ten dots in the tens box instead. Then we'll be able to take five of them away, leaving seven.





(Notice that we also have one less dot in the hundreds box; there’s only three dots there now.)

Now we want to take one dot from the ones box, and that leaves no dots there.



So the answer is:

$$\begin{array}{r} 921 \\ -551 \\ \hline 370 \end{array}$$

Think / Pair / Share

Solve the following exercises by thinking about dots and boxes. (You can draw pictures, or just imagine them.)

$$\begin{array}{r} 323 \\ \underline{-148} \end{array} \qquad \begin{array}{r} 567 \\ \underline{-271} \end{array} \qquad \begin{array}{r} 389107123 \\ \underline{-310462872} \end{array}$$

Problem 2

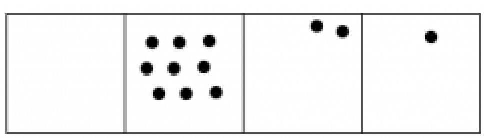
Use the dots and boxes technique to solve these problems. *Do not covert to base 10! Try to work directly in the base given.* It might help to actually draw the pictures.

$$\begin{array}{r} 20413_{\text{five}} \\ \underline{-13244_{\text{five}}} \end{array} \qquad \begin{array}{r} 6252_{\text{nine}} \\ \underline{-4088_{\text{nine}}} \end{array} \qquad \begin{array}{r} 4323_{\text{seven}} \\ \underline{-3524_{\text{seven}}} \end{array}$$

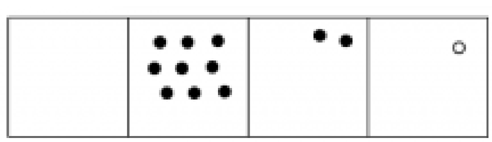
The Standard Algorithm for Subtraction

Just like in addition, the standard algorithm for subtraction requires you to work from right to left, and "borrow" (this is really *regrouping*!) whenever necessary. Notice that in the dots and boxes approach, you don't need to go in any particular order when you do the subtraction. You just "unexplode" the dots as necessary when computing.

Here's how the standard algorithm looks with the dots and boxes model for 921 – 551: Start with 921 dots.

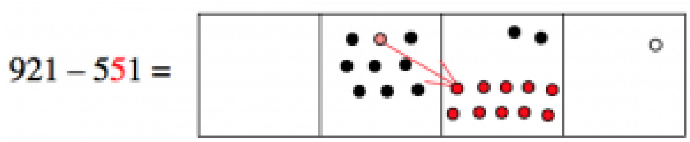
921 =


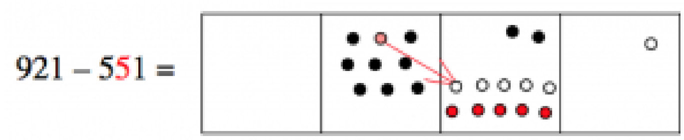
Then take away one dot from the ones box.

921 – 551 =


$$\begin{array}{r} 921 \\ -551 \\ \hline 0 \end{array}$$

Now we want to take away five dots from the tens box. But there aren't five dots there. So we "unexplode" one of the hundreds dots to get more tens:



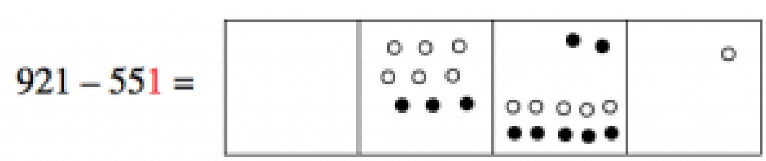


In the standard algorithm, we show the unxplosion as a regrouping, subtracting one from the hundreds place of 921 and adding ten to the tens place. So we are rewriting

$$9 \mid 2 \mid 1 = 8 \mid 12 \mid 1.$$

$$\begin{array}{rrrr} & 8 & 12 & \\ & \cancel{9} & \cancel{2} & 1 \\ - & 5 & 5 & 1 \\ \hline & & 7 & 0 \end{array}$$

Finally, we want to take away five from the eight dots left in the hundreds column.



$$\begin{array}{rrrr} & 8 & 12 & \\ & \cancel{9} & \cancel{2} & 1 \\ - & 5 & 5 & 1 \\ \hline & 5 & 7 & 0 \end{array}$$

Multiplication: Dots and Boxes

Multiplication as Repeated Addition

Problem 3

Jenny was asked to compute 243192×4. She wrote:

$$243192 \times 4 = 8 \,|\, 16 \,|\, 12 \,|\, 4 \,|\, 36 \,|\, 8.$$

1. What was Jenny thinking about? Is her answer correct?
2. Translate Jenny's answer into a number that the rest of the world can understand.
3. Use Jenny's method to find the answers to these multiplication exercises. Be sure to translate your answers into familiar base 10 numbers.

$156 \times 3 =$ $\quad 2873 \times 2 =$ $\quad 71181 \times 5 =$ $\quad 3726510392 \times 2 =$

Problem 4

Can you adapt Jenny's method to solve these problems? Write your answers in base eight. Try to work directly in base eight rather than converting to base 10 and back again!

$156_{\text{eight}} \times 3_{\text{eight}} =$

$2673_{\text{eight}} \times 4_{\text{eight}} =$

$36255772_{\text{eight}} \times 2_{\text{eight}} =$

Jenny might have been thinking about multiplication as repeated addition. If we have some number N and we multiply that number by 4, what we mean is:

$$4 \cdot N = N + N + N + N.$$

If we take the number 243192 and add it to itself four times using the “combining method,” we get

- 2 + 2 + 2 + 2 = 8 ones,
- 9 + 9 + 9 + 9 = 36 tens,
- 1 + 1 + 1 + 1 = 4 hundreds,
- and so on.

Notation

Notice that we have used both × and · to represent multiplication. It’s a bit awkward to use × when you’re also using variables. Is it the letter x? Or the multiplication symbol ×? It can be hard to tell! In this case, the symbol · is more clear.

We can even simplify the notation further, writing 4N instead of 4 · N. But of course we only do that when we are multiplying *variables* by some quantity. (We wouldn’t want 34 to mean 3 · 4, would we?)

Problem 5

Here is a strange addition table. Use it to solve the following problems. Important: Don’t try to assign numbers to A, B, and C. Solve the problems just using what you know about the operations!

+	A	B	C
A	C	A	B
B	A	B	C
C	B	C	A

A + B B + C 2A 5C 3A + 4B

Think / Pair / Share

How does an addition table help you solve multiplication problems like 5C?

Division: Dots and Boxes

Quotative Model of Division

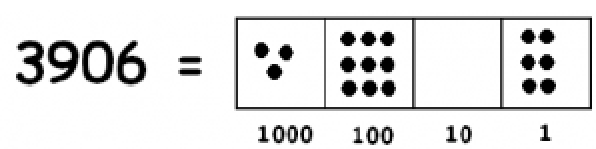
Suppose you are asked to compute 3906 ÷ 3. One way to interpret this question (there are others) is:

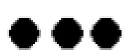
"How many groups of 3 fit into 3906?"

Definition

In the **quotative model of division**, you are given a **dividend** (here it is 3906), and you are asked to split it into equal-sized groups, where the size of the group is given by the **divisor** (here it is 3).

In our dots and boxes model, the dividend 3906 looks like this:



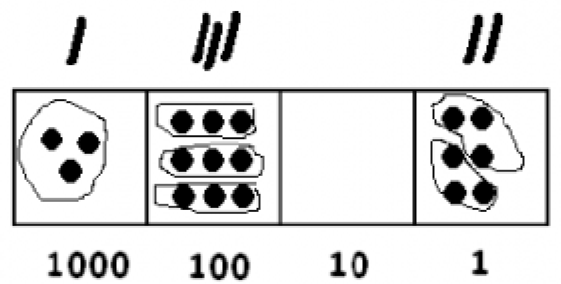
and three dots looks like this: 

So we are really asking:

"How many groups of  fit into the picture of 3906?"

Example: 3906 ÷ 3

There is one group of 3 at the thousands level, and three at the hundreds level, none at the tens level, and two at the ones level.



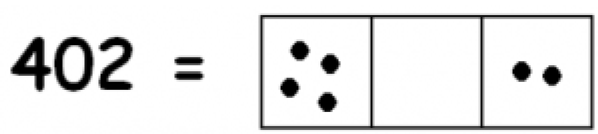
Notice what we have in the picture:

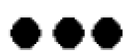
- One group of 3 in the thousands box.
- Three groups of 3 in the hundreds box.
- Zero groups of 3 in the tens box.
- Two groups of 3 in the ones box.

This shows that 3 goes into 3906 one thousand, three hundreds and two ones times. That is,

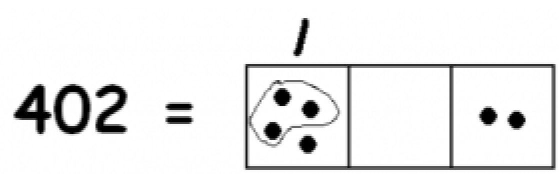
$$3906 \div 3 = 1302.$$

Let’s try a harder one! Consider 402 ÷ 3. Here’s the picture:



We are still looking for groups of three dots: 

There is certainly one group at the 100’s level.



and now it seems we are stuck there are no more groups of three!

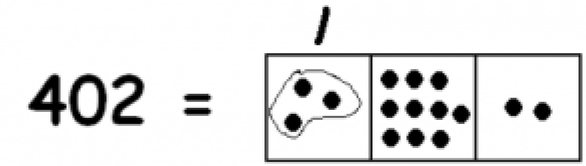
Think / Pair / Share

What can we do now? Are we really stuck? Can you finish the division problem?

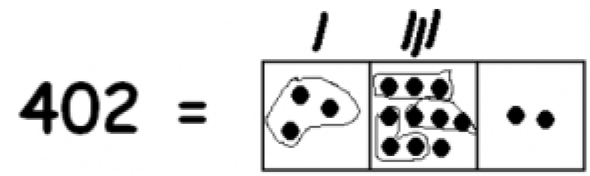
Example: 402 ÷ 3

Here are the details worked out for 402 ÷ 3. But don't read this until you've thought about it yourself!

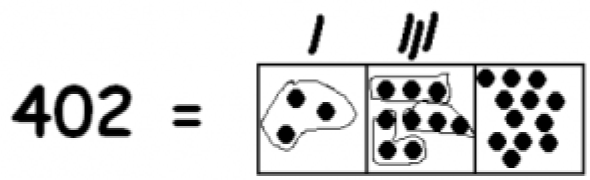
Since each dot is worth ten dots in the box to the right we can write:



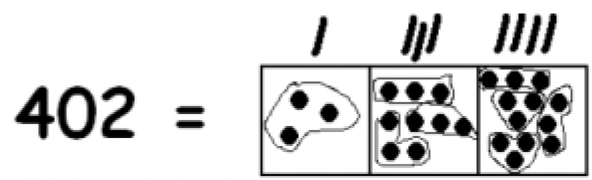
Now we can find more groups of three:



There is still a troublesome extra dot. Let's unexplode it too



This gives us more groups of three:



In the picture we have:

- One group of 3 in the hundreds box.
- Three groups of 3 in the tens box.
- Four groups of 3 in the ones box.

Finally we have the answer!

$$402 \div 3 = 134.$$

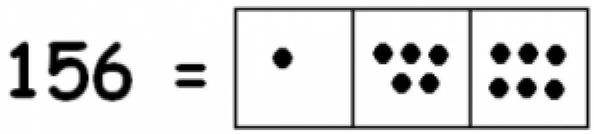
Think / Pair / Share

Solve each of these exercises using the dots and boxes method:

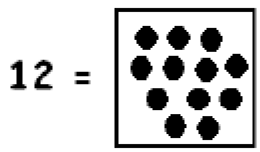
62124 ÷ 3   61230 ÷ 5

Example: 156 ÷ 12

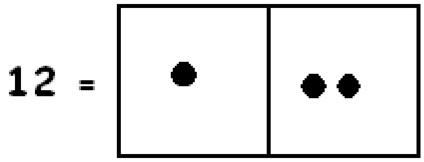
Let's turn up the difficulty a notch. Consider 156 ÷ 12. Here we are looking for groups of 12 in this picture:



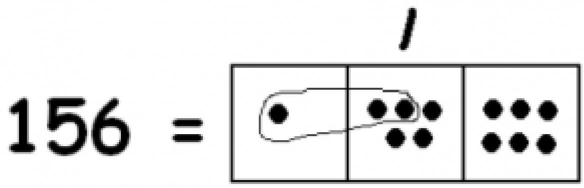
What does 12 look like? It can be twelve dots in a single box:



But most often we would write 12 this way, as a ten and 2 ones:

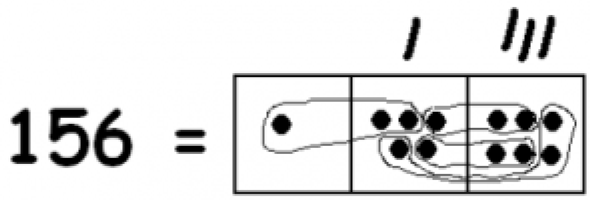


We certainly see some of these in the picture. There is certainly one at the tens level:



Note: With an unexplosion this would be twelve dots in the tens box, so we mark one group of 12 above the tens box.

We also see three groups of twelve ones:



So in the picture we have:

- One group of 12 dots in the tens box.
- Three groups of 12 dots in the ones box.

That means

$$156 \div 12 = 13.$$

Problem 6

Use the dots and boxes model to compute each of the following:

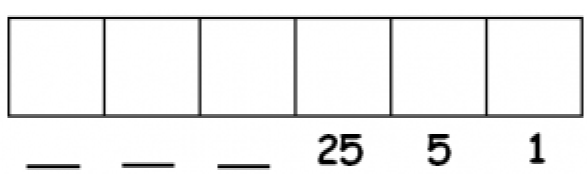
$$13453 \div 11$$

$$4853 \div 23$$

$$214506 \div 102$$

Problem 7

Remember that base five numbers are in a $1 \leftarrow 5$ dots-and-boxes system. What are the place values in the $1 \leftarrow 5$ system? Fill in the blanks:



1. Draw a dots-and-boxes picture of the number 424_{five}.
2. Draw a dots-and-boxes picture of the number 11_{five}.
3. Use the dots and boxes method to find $424_{\text{five}} \div 11_{\text{five}}$. < /li >< li > $Rewritethedivisionsentence$424_\text{five} \div

11_\text{five} = 34_\text{five}
inbaseten, andcheckthatit′scorrect. < /li >< li > Usedots − and − boxestofind
2021_\text{five} \div 12_\text{five}$. *Don't convert to base 10!*

Think / Pair / Share

- Use dots and boxes to compute these.

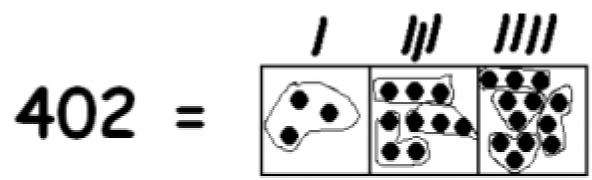
$$2130 \div 10$$

$$41300 \div 100$$

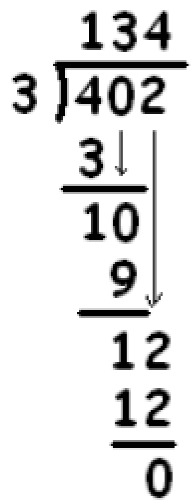
- What pictures did you use for 10 and for 100? Can you describe in words what happens when dividing by 10 and by 100 and why?

The Standard Algorithm for Division

We used dots and boxes to show that 402 ÷ 3 = 134.

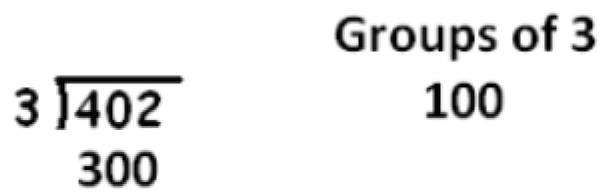


In elementary school, you might have learned to solve this division problem by using a diagram like the following:

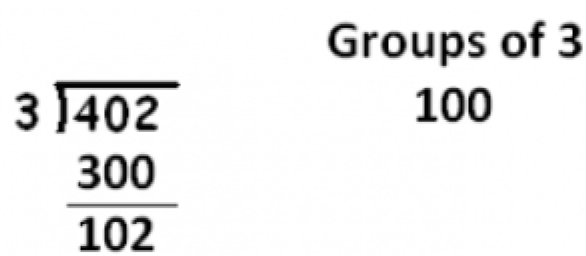


At first glance this seems very mysterious, but it is really no different from the dots and boxes method. Here is what the table means.

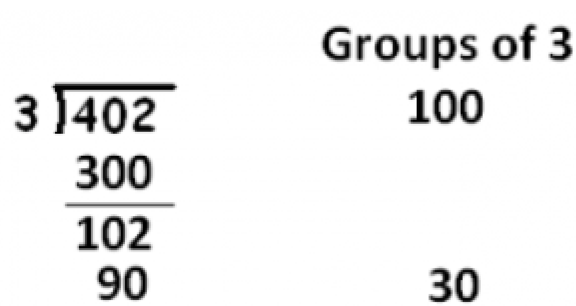
To compute 402 ÷ 3, we first make a big estimation as to how many groups of 3 there are in 402. Let's guess that there are 100 groups of three.



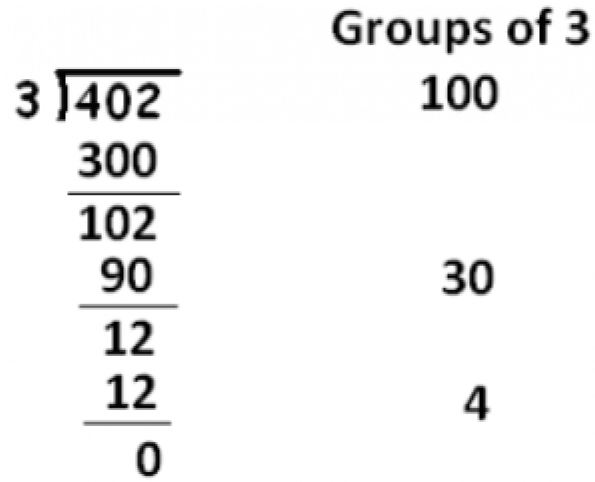
How much is left over after taking away 100 groups of 3? We subtract to find that there is 102 left.



How many groups of 3 are in 102? Let's try 30:



How many are left? There are 12 left and there are four groups of 3 in 12.

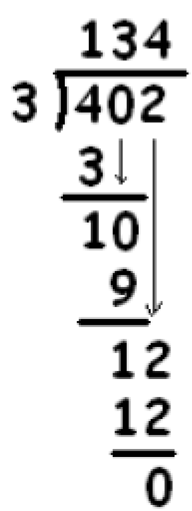


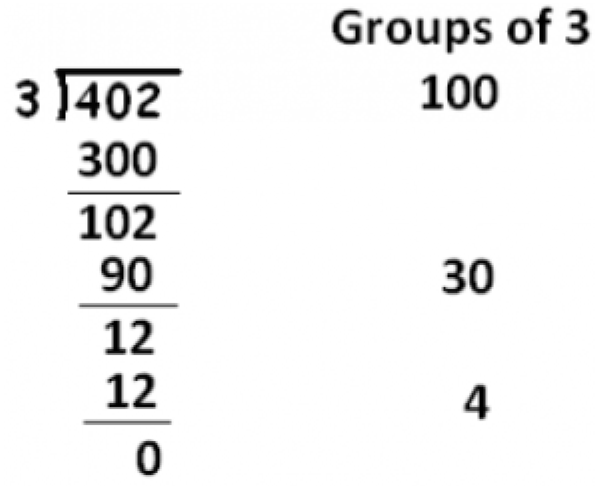
That accounts for entire number 402. And where do we find the final answer? Just add the total count of groups of three that we tallied:

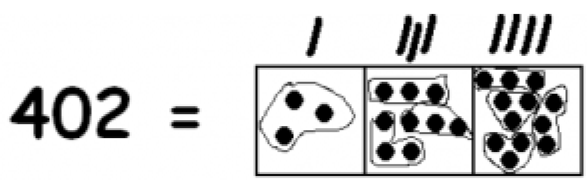
$$402 \div 3 = 100 + 30 + 4 = 134.$$

Think / Pair / Share

- Compare the two division diagrams below. In what way are they the same? In what way are they different?
- Also look at the dots and boxes method. In what way is it the same or different from the two diagrams?







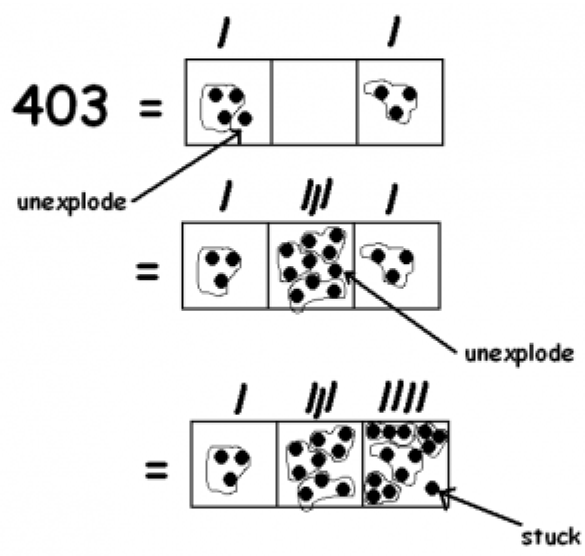
- **Why do we like the standard algorithm?** Because it is quick, not too much to write down, and it works every time.
- **Why do we like the dots and boxes method?** Because it easy to understand. (And drawing dots and boxes is kind of fun!)

Division with Remainders

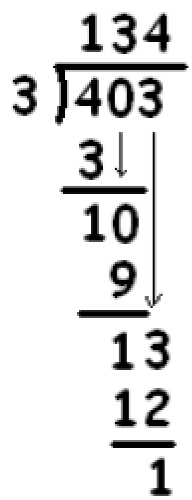
We saw that 402 is evenly divisible by 3: 402 ÷ 3 = 134. This means that 403, one more, shouldn't be divisible by three. It should be one dot too big.

Example: 403 ÷ 3

Do we see the extra dot if we compute 402 ÷ 3 with dots and boxes?



Yes we do! We have one dot left at the end that can't be divided. This is how it looks in the standard algorithm.



In school, we say that we have a *remainder* of one and sometimes write:

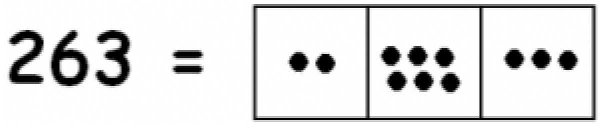
$$403 \div 3 = 134 \text{ R}1.$$

But what does that really mean? It means that we have 134 groups of three with one dot left over. So

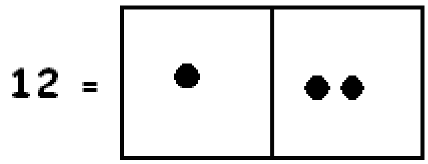
$$402 = 134 \cdot 3 + 1.$$

Example: 263 ÷ 12

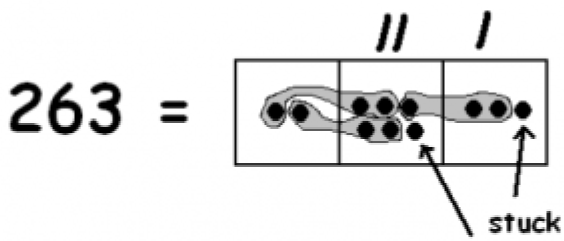
Let's try another one: 263 ÷ 12. Here's what we have:



And we are looking for groups like this:



Here goes!



Unexploding won't help any further and we are indeed left with one remaining dot in the tens position and a dot in the ones position. We have 21 groups of twelve, and a remainder of eleven.

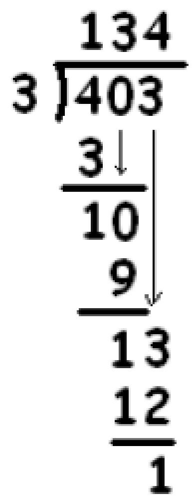
$$263 = 21 \cdot 12 + 11.$$

Think / Pair / Share

- Use the dots and boxes method to compute each quotient and remainder:

$$5210 \div 4$$
$$4857 \div 23$$
$$31533 \div 101$$

- Now use the standard algorithm (an example is shown below) to compute each of the quotients and remainders above.



$402 = 134 \cdot 3 + 1.$

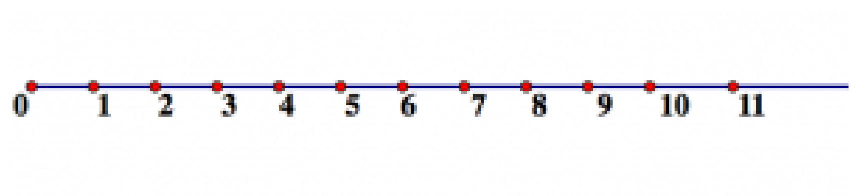
- Which method do you like better: dots and boxes or the standard algorithm method? Or does it depend on the problem you are doing?

Number Line Model

Another way we often think about numbers is as abstract quantities that can be measured: length, area, and volume are all examples.

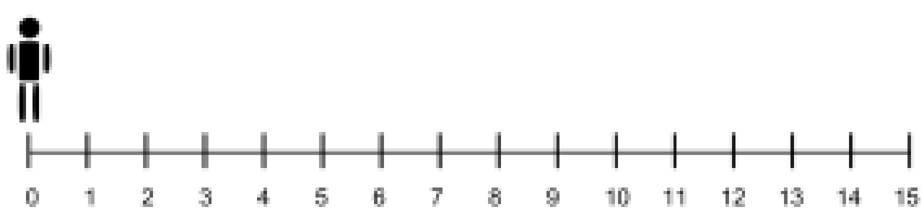
In a measurement model, you have to pick a *basic unit*. The basic unit is a quantity — length, area, or volume — that you assign to the number one. You can then assign numbers to other quantities based on how many of your basic unit fit inside.

For now, we'll focus on the quantity length, and we'll work with a number line where the basic unit is already marked off.



Addition and Subtraction on the Number Line

Imagine a person — we'll call him Zed — who can stand on the number line. We'll say that the distance Zed walks when he takes a step is exactly one unit.



When Zed wants to add or subtract with whole numbers on the number line, he always starts at 0 and faces the positive direction (towards 1). Then what he does depends on the calculation.

If Zed wants to *add* two numbers, he walks forward (to the right of the number line) however many steps are indicated by the first number (the first *addend*). Then he walks forward (to your right on the number line) the number of steps indicated by the second number (the second *addend*). Where he lands is the *sum* of the two numbers.

Example: 3 + 4

If Zed wants to add 3 + 4, he starts at 0 and faces towards the positive numbers. He walks forward 3 steps, then he walks forward 4 more steps.

Zed ends at the number 7, so the sum of 3 and 4 is 7. 3 + 4 = 7. (But you knew that of course! The point right now is to make sense of the *number line model*.)

When Zed wants to *subtract* two numbers, he he walks forward (to the right on the number line) however many steps are indicated by the first number (the *minuend*). Then he walks *backwards* (to the left on the number line) the number of steps indicated by the second number (the *subtrahend*). Where he lands is the *difference* of the two numbers.

Example: 11 – 3

If Zed wants to subtract 11 – 3, he starts at 0 and faces the positive numbers (the right side of the number line). He walks forward 11 steps on the number line, then he walks backwards 3 steps.

Zed ends at the number 8, so the difference of 11 and 3 is 8. 11 – 3 = 8. (But you knew that!)

Think / Pair / Share

- Work out each of these exercises on a number line. You can actually pace it out on a life-sized number line or draw a picture:

$$4+5 \qquad 6+9 \qquad 10-7 \qquad 8-1$$

- Why does it make sense to walk forward for addition and walk backwards for subtraction? In what way is this the same as “combining” for addition and “take away” for subtraction”?
- What happens if you do these subtraction problems on a number line? Explain your answers.

$$6-9 \qquad 1-7 \qquad 4-11 \qquad 0-1$$

- Could you do the subtraction problems above with the dots and boxes model?

Multiplication and Division on the Number Line

Since multiplication is really repeated addition, we can adapt our addition model to become a multiplication model as well. Let’s think about 3×4. This means to add four to itself three times (that’s simply the definition of multiplication!):

$$3 \times 4 = 4 + 4 + 4.$$

So to multiply on the number line, we do the process for addition several times.

To multiply two numbers, Zed starts at 0 as always, and he faces the positive direction. He walks forward the number of steps given by the second number (the second *factor*). He repeats that process the number of times given by the first number (the first *factor*). Where he lands is the *product* of the two numbers.

Example: 3 × 4

If Zed wants to multiply 3 × 4, he can think of it this way:

$$3 \quad \times \quad 4$$

$$\swarrow \qquad \searrow$$

how many times to repeat it — how many steps to take forward

Zed starts at 0, facing the positive direction. The he repeats this three times: take four steps forward.

He ends at the number 12, so the product of 3 and 4 is 12. That is, 3 × 4 = 12.

Remember our quotative model of division: One way to interpret 15 ÷ 5 is:

> How many groups of 5 fit into 15?

Thinking on the number line, we can ask it this way:

> Zed takes 5 steps at a time. If Zed lands at the number 15, how many times did he take 5 steps?

To calculate a division problem on the number line, Zed starts at 0, facing the positive direction. He walks forward the number of steps given by the second number (the *divisor*). He repeats that process until he lands at the first number (the *dividend*). The number of times he repeated the process gives the *quotient* of the two numbers.

Example: 15 ÷ 5

If Zed wants to compute 15 ÷ 5, he can think of it this way:

He starts at 0, facing the positive direction.

- Zed takes 5 steps forward. He is now at 5, not 15. So he needs to repeat the process.
- Zed takes 5 steps forward again. He is now at 10, not 15. So he needs to repeat the process.
- Zed takes 5 more steps forward. He is at 15, so he stops.

Since he repeated the process three times, we see there are 3 groups of 5 in 15. So the quotient of 15 and 5 is 3. That is, 15 ÷ 5 = 3.

Think / Pair / Share

- Work out each of these exercises on a number line. You can actually pace it out on a life-sized number line or draw a picture:

$2 \times 5 \qquad 7 \times 1 \qquad 10 \div 2 \qquad 6 \div 1$

- Can you think of a way to interpret these multiplication problems on a number line? Explain your ideas.

4×0 0×5 $3 \times (-2)$ $2 \times (-1)$

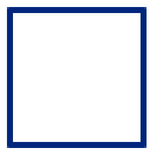
- What happens if you try to solve these division problems on a number line? Can you do it? Explain your ideas.

$0 \div 2$ $0 \div 10$ $3 \div 0$ $5 \div 0$

Area Model for Multiplication

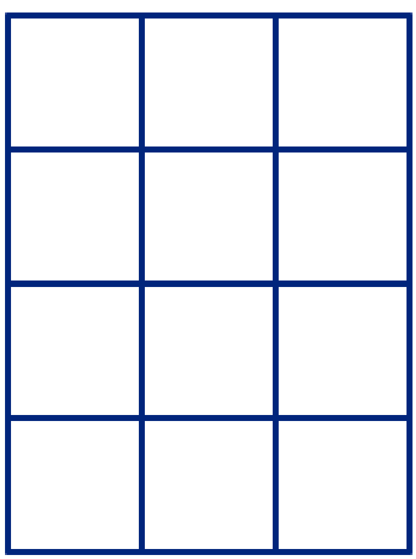
So far we have focused on a linear measurement model, using the number line. But there's another common way to think about multiplication: using area.

For example, suppose our basic unit is one square:



We can picture 4 × 3 as 4 groups, with 3 squares in each group, all lined up:

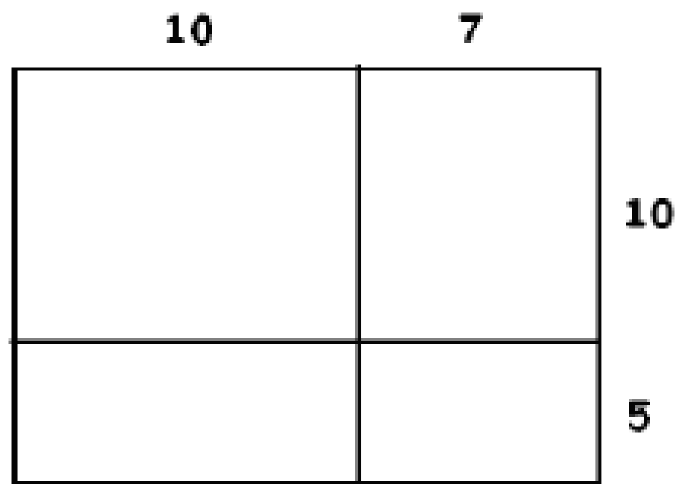
But we can also picture them stacked up instead of lined up. We would have 4 rows, with 3 squares in each row, like this:



So we can think about 4 × 3 as a rectangle that has length 3 and width 4. The product, 12, is the total number of squares in that rectangle. (That is also the area of the rectangle, since each square was one unit!)

Think / Pair / Share

Vera drew this picture as a model for 15 × 17. Use her picture to help you compute 15 × 17. Explain your work.



Problem 8

Draw pictures like Vera's for each of these multiplication exercises. Use your pictures to find the products without using a calculator or the standard algorithm.

23 × 37 8 × 43 371 × 42

The Standard Algorithm for Multiplication

How were you taught to compute 83 × 27 in school? Were you taught to write something like the following?

$$
\begin{array}{lr}
 & 83 \\
\times & 27 \\
\hline
 & 21 \\
 & 56 \\
 & 6 \\
 & 16 \\
\hline
 & 2241
\end{array}
$$

Or maybe you were taught to put in the extra zeros rather than leaving them out?

$$
\begin{array}{lr}
 & 83 \\
\times & 27 \\
\hline
 & 21 \\
 & 560 \\
 & 60 \\
 & 1600 \\
\hline
 & 2241
\end{array}
$$

This is really no different than drawing the rectangle and using Vera's picture for calculating!

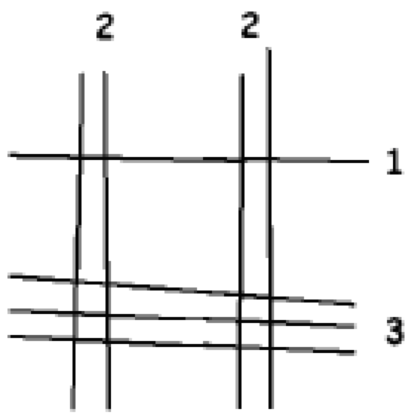
Think / Pair / Share

- Use the example above to explain why Vera's rectangle method and the standard algorithm are really the same.
- Calculate the products below using both methods. Explain where you're computing the same pieces in each algorithm.

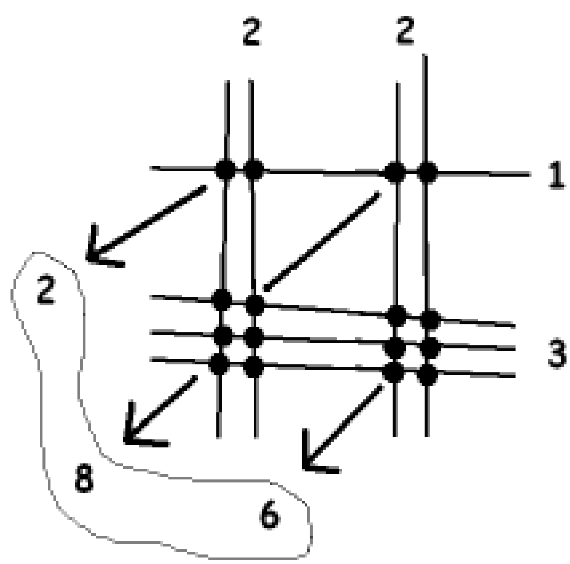
23 × 14   106 × 21   213 × 31

Lines and Intersections

Here's an unusual way to perform multiplication. To compute 22 × 13, for example, draw two sets of vertical lines, the left set containing two lines and the right set two lines (for the digits in 22) and two sets of horizontal lines, the upper set containing one line and the lower set three (for the digits in 13).

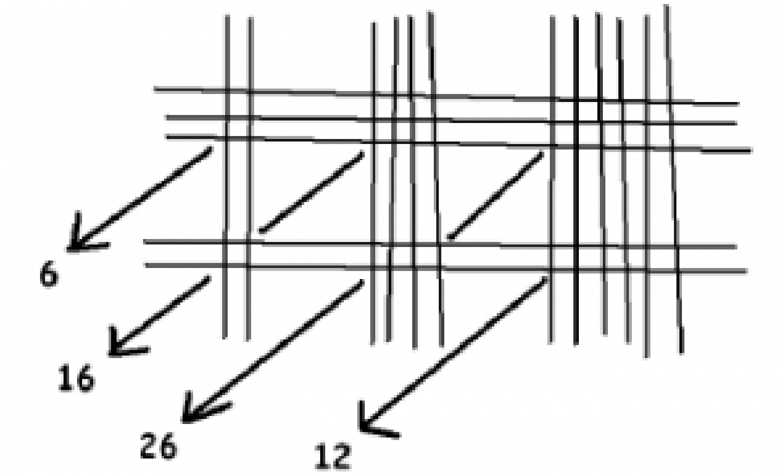


There are four sets of intersection points. Count the number of intersections in each and add the results diagonally as shown:



The answer 286 appears!

There is one possible glitch as illustrated by the computation 246 × 32:



Although the answer 6 thousands, 16 hundreds, 26 tens, and 12 ones is absolutely correct, one needs to carry digits and translate this as 7,872.

Problem 9

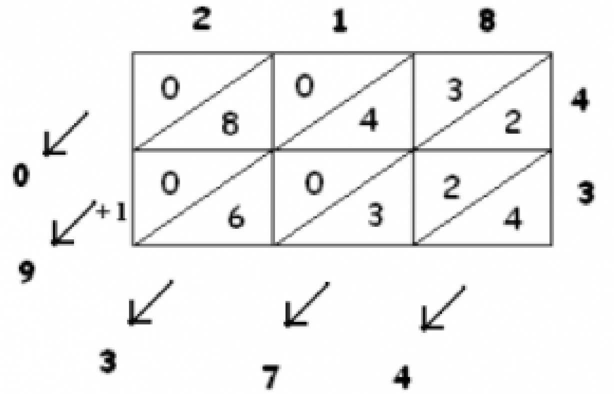
1. Compute 131 × 122 via this method. Check your answer using another method.
2. Compute 15 × 1332 via this method. Check your answer using another method.
3. Can you adapt the method to compute 102 × 3054? (Why is some adaptation necessary?)
4. Why does the method work in general?

Lattice Multiplication

In the 1500s in England, students were taught to compute multiplication using following galley method, now more commonly known as the *lattice method*.

To multiply 43 and 218, for example, draw a 2 × 3 grid of squares. Write the digits of the first number along the right side of the grid and the digits of the second number along the top.

Divide each cell of the grid diagonally and write in the product of the column digit and row digit of that cell, separating the tens from the units across the diagonal of that cell. (If the product is a one digit answer, place a 0 in the tens place.)



To get the answer, add the entries in each diagonal, carrying tens digits over to the next diagonal if necessary. In our example, we have

$$218 \times 43 = 9374.$$

Problem 10

1. Compute 5763 × 345 via the lattice method.
2. Explain why the lattice method is really the standard algorithm in disguise.
3. What is the specific function of the diagonal lines in the grid?

Properties of Operations

So far, you have seen a couple of different *models* for the operations: addition, subtraction, multiplication, and division. But we haven't talked much about the operations themselves — how they relate to each other, what properties they have that make computing easier, and how some special numbers behave. There's lots to think about!

The goal in this section is to use the models to understand why the operations behave according to the rules you learned back in elementary school. We're going to keep asking ourselves "Why does it work this way?"

Think / Pair / Share

Each of these models lends itself to thinking about the operation in a slightly different way. Before we really dig in to thinking about the operations, discuss with a partner:

- Of the models we discussed so far, do you prefer one of them?
- How well do the models we discussed match up with how you usually think about whole numbers and their operations?
- Which models are useful for computing? Why?
- Which models do you think will be useful for explaining how the operations work? Why?

Connections Between the Operations

We defined addition as combining two quantities and subtraction as "taking away." But in fact, these two operations are intimately tied together. These two questions are exactly the same:

$$27 - 13 = \_\_\_\_ \qquad\qquad 27 = 13 + \_\_\_\_\_.$$

More generally, for any three whole numbers a, b, and c, these two equations express the same fact. (So either both equations are true or both are false. Which is the case depends on the values you choose for a, b, and c!)

$$c - b = a \qquad\qquad c = a + b.$$

In other words, we can think of every subtraction problem as a "missing addend" addition problem. Try it out!

Problem 11

Here is a strange addition table. Use it to solve the following problems. Justify your answers. Important: Don't try to assign numbers to A, B, and C. Solve the problems just using what you know about the operations!

+	A	B	C
A	C	A	B
B	A	B	C
C	B	C	A

$A + C$ $\quad B + C$ $\quad A - C$ $\quad C - A$ $\quad A - A$ $\quad B - C$

Think / Pair / Share

How does an addition table help you solve subtraction problems?

We defined multiplication as repeated addition and division as forming groups of equal size. But in fact, these two operations are also tied together. These two questions are exactly the same:

$$27 \div 3 = \_\_\_\_\_ \qquad\qquad 27 = \_\_\_\_\_ \times 3.$$

More generally, for any three whole numbers a, b, and c, these two equations express the same fact. (So either both equations are true or both are false. Which is the case depends on the values you choose for a, b, and c!)

$$c \div b = a \qquad\qquad c = a \cdot b.$$

In other words, we can think of every division problem as a "missing factor" multiplication problem. Try it out!

Problem 12

Rewrite each of these division questions as a "missing factor" multiplication question. Which ones can you solve and which can you not solve? Explain your answers.

$9 \div 3$ $100 \div 25$ $0 \div 3$ $9 \div 0$ $0 \div 0$

Problem 13

Here's a multiplication table.

×	**A**	**B**	**C**	**D**	**E**
A	A	A	A	A	A
B	A	B	C	D	E
C	A	C	E	B	D
D	A	D	B	E	C
E	A	E	D	C	B

- Use the table to solve the problems below. Justify your answers. Important: Don't try to assign numbers to the letters. Solve the problems just using what you know about the operations!

$$C \times D \qquad C \times A \qquad A \times A \qquad C \div D \qquad D \div C \qquad D \div E$$

- Can you use the table to solve these problems? Explain your answers. Recall that x^n means n copies of x multiplied together, $x \cdot x \cdot x \cdot \ldots \cdot x$

$$D^2 \qquad C^3 \qquad A \div C \qquad A \div D \qquad D \div A \qquad A \div A$$

Think / Pair / Share

How does a multiplication table help you solve division (and exponentiation) problems?

Throughout this course, our focus is on explanation and justification. As teachers, you need to know what is true in mathematics, but you also need to know *why* it is true. And you will need lots of ways to explain *why*, since different explanations will make sense to different students.

Think / Pair / Share

Arithmetic Fact: $a + b = c$ and $c – b = a$ are the same mathematical fact.

Why is this *not* a good explanation?

> "I can check that this is true! For example, 2+3 = 5 and 5 – 3 = 2. And 3 + 7 = 10 and 10 – 7 = 3. It works for whatever numbers you try."

Addition and Subtraction: Explanation 1

Arithmetic Fact:

$a + b = c$ and $c – b = a$ are the same mathematical fact.

Why It's True, Explanation 1:

First we'll use the definition of the operations.

Suppose we know $c – b = a$ is true. Subtraction means "take away." So

$$c – b = a$$

means we start with quantity c and take away quantity b, and we end up with quantity a. Start with this equation, and imagine adding quantity b to both sides.

On the left, that mans we started with quantity c, took away b things, and then put those b things right back! Since we took away some quantity and then added back the exact same quantity, there's no overall change. We're left with quantity c.

On the right, we would be combining (adding) quantity a with quantity b. So we end up with: c = a + b.

On the other hand, suppose we know the equation $a + b = c$ is true. Imagine taking away (subtracting) quantity b from both sides of this equation: a + b = c.

On the left, we started with a things and combined that with b things, but then we immediately take away those b things. So we're left with just our original quantity of a.

On the right, we start with quantity c and take away b things. That's the very definition of $c – b$. So we have the equation:

$$a = c – b.$$

Why It's True, Explanation 2:

Let's use the measurement model to come up with another explanation.

The equation $a + b = c$ means Zed starts at 0, walks forward a steps, and then walks forward b steps, and he ends at c.

If Zed wants to compute $c - b$, he starts at 0, walks forward c steps, and then walks backwards b steps. But we know that to walk forward c steps, he can first walk forward a steps and then walk forward b steps. So Zed can compute $c - b$ this way:

- Start at 0.
- Walk forward a steps.
- Walk forward b steps. (Now at c, since $a + b = c$.)
- Walk backwards b steps.

The last two sets of steps cancel each other out, so Zed lands back at a. That means $c - b = a$.

On the other hand, the equation $c - b = a$ means that Zed starts at 0, walks forward c steps, then walks backwards b steps, and he ends up at a.

If Zed wants to compute $a + b$, he starts at 0, walks forward a steps, and then walks forwards b additional steps. But we know that to walk forward a steps, he can first walk forward c steps and then walk backwards b steps. So Zed can compute $a + b$ this way:

- Start at 0.
- Walk forward c steps.
- Walk backwards b steps. (Now at a, since $c - b = a$.)
- Walk forward b steps.

The last two sets of steps cancel each other out, so Zed lands back at c. That means $a + b = c$.

Think / Pair / Share

- Read over the two explanations in the example above. Do you think either one is more clear than the other?
- Come up with your own **explanation** (not examples!) to explain:

$$c \div b = a \quad \text{is the same fact as} \quad c = a \times b.$$

Properties of Addition and Subtraction

You probably know several properties of addition, but you may never have stopped to wonder: *Why is that true?!* Now's your chance! In this section, you'll use the definition of the operations of addition and subtraction and the models you've learned to explain why these properties are always true.

Here are the three properties you'll think about:

- Addition of whole numbers is *commutative*.
- Addition of whole numbers is *associative*.
- The number 0 is an *identity* for addition of whole numbers.

For each of the properties, we don't want to confuse these three ideas:

- what the property is called and what it means (the definition),
- some examples that *demonstrate* the property, and
- an explanation for *why* the property holds.

Notice that *examples* and *explanations* are not the same! It's also very important not to confuse the *definition* of a property with the *reason* it is true!

These properties are all universal statements — statements of the form "for all," "every time," "always," etc. That means that to show they are true, you either have to check every case or find a *reason why* it must be so.

Since there are infinitely many whole numbers, it's impossible to check every case. You'd never finish! Our only hope is to look for *general explanations*. We'll work out the explanation for the first of these facts, and you will work on the others.

Addition is Commutative

Example: Commutative Law

Property:

Addition of whole numbers is commutative.

What it Means (words):

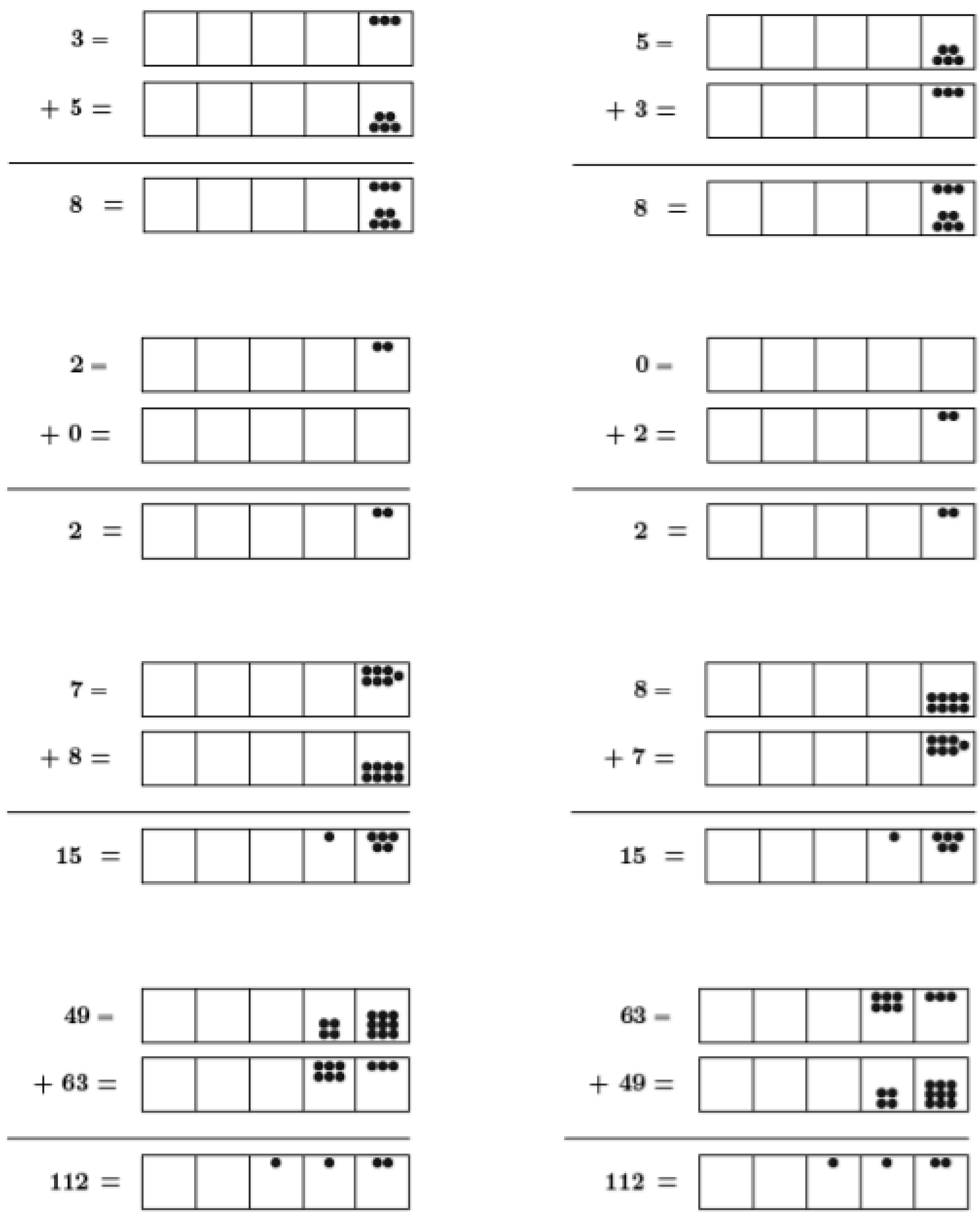
When I add two whole numbers, the order I add them doesn't affect the sum.

What it Means (symbols):

For any two whole numbers a and b,

$$a + b = b + a.$$

Examples:



Now we need a *justification*. *Why* is addition of whole numbers commutative?

Why It's True, Explanation 1:

Let's think about addition as combining two quantities of dots.

- To add $a + b$, we take a dots and b dots, and we combine them in a box. To keep things straight, lets imagine the a dots are colored red and the b dots are colored blue. So in the box we have a red dots,

b blue dots and $a + b$ total dots.

- To add $b + a$, let's take b blue dots and a red dots, and put them all together in a box. We have b blue dots, a red dots and $b + a$ total dots.
- But the total number of dots are the same in the two boxes! How do we know that? Well, there are a red dots in each box, so we can match them up. There are b blue dots in each box, so we can match them up. That's it! If we can match up the dots one-for-one, there must be the same number of them!

- That means $a + b = b + a$.

Why It's True, Explanation 2:

We can also use the measurement model to explain why $a + b = b + a$ no matter what numbers we choose for a and b. Imagine taking a segment of length a and combining it linearly with a segment of length b. That's how we get a length of $a + b$.

But if we just rotate that segment so it's upside down, we see that we have a segment of length b combined with a segment of length a, which makes a length of $b + a$.

But of course it's the same segment! We just turned it upside down! So the lengths must be the same. That is, $a + b = b + a$.

Addition is Associative

Your turn! You'll answer the question, "Why is addition of whole numbers associative?"

Property: Addition of whole numbers is associative.

What it Means (words): When I add three whole numbers in a given order, the way I group them (to add two at a time) doesn't affect the sum.

What it Means (symbols): For any three whole numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

Problem 14

1. Come up with at least three *examples* to demonstrate associativity of addition.
2. Use our models of addition to come up with an *explanation*. Why does associativity hold in *every*

case? **Note:** your explanation should not use specific numbers. It is not an example!

0 is an Identity for Addition

Property: The number 0 is an *identity* for addition of whole numbers.

What it Means (words): When I add any whole number to 0 (in either order), the sum is the very same whole number I added to 0.

What it Means (symbols): For any whole numbers n,

$$n + 0 = n \quad \text{and} \quad 0 + n = n.$$

Problem 15

1. Come up with at least three *examples* to demonstrate that 0 is an identity for addition.
2. Use our models of addition to come up with an *explanation*. Why does this property of 0 hold in *every possible case*?

Properties of Subtraction

Since addition and subtraction are so closely linked, it's natural to wonder if subtraction has some of the same properties as addition, like commutativity and associativity.

Example: Is subtraction commutative?

Justin asked if the operation of subtraction is commutative. That would mean that the difference of two whole numbers doesn't depend on the order in which you subtract them.

In symbols: *for every choice* of whole numbers a and b we would have $a - b = b - a$.

Jared says that subtraction is *not* commutative since 4 – 3 = 1, but 3 – 4 ≠ 1. (In fact, 3 – 4 = -1.)

Since the statement "subtraction is commutative" is a *universal statement*, one counterexample is enough to show it's not true. So Jared's counterexample lets us say with confidence:

Subtraction is **not** commutative.

Think / Pair / Share

Can you find any examples of whole numbers a and b where $a - b = b - a$ is true? Explain your answer.

Problem 16

Lyle asked if the operation of subtraction is associative.

1. State what it would mean for subtraction to be associative. You should use words and symbols.
2. What would you say to Lyle? Decide if subtraction is associative or not. Carefully explain how you made your decision and *how you know you're right*.

Problem 17

Jess asked if the number 0 is an identity for subtraction.

1. State what it would mean for 0 to be an identity for subtraction. You should use words and symbols.
2. What would you say to Jess? Decide if 0 is an identity for subtraction or not. Carefully explain how you made your decision and *how you know you're right*

Properties of Multiplication and Division

Now we're going to turn our attention to familiar properties of multiplication and division, with the focus still on explaining why these properties are always true.

Here are the four properties you'll think about:

- Multiplication of whole numbers is *commutative*.
- Multiplication of whole numbers is *associative*.
- Multiplication of whole numbers *distributes over addition*
- The number 1 is an *identity* for multiplication of whole numbers

For each of the properties, remember to keep straight:

- what the property is called and what it means (the definition),
- some examples that *demonstrate* the property, and
- an explanation for *why* the property holds.

Once again, it's important to distinguish between *examples* and *explanations*. They are not the same! Since there are infinitely many whole numbers, it's impossible to check every case, so examples will never be enough to explain why these properties hold. You have to figure out *reasons* for these properties to hold, based on what you know about the operations.

1 is an Identity for Multiplication

We'll work out the explanation for the last of these facts, and you will work on the others.

Example: 1 is an Identity for multiplication

Property:

The number 1 is an identity for multiplication of whole numbers.

What it Means (words):

When I multiply a number by 1 (in either order), the product is that number.

What it Means (symbols):

For any whole number m,

$$m \times 1 = m \quad \text{and} \quad 1 \times m = m.$$

Examples:

$$1 \times 5 = 5, \qquad 19 \times 1 = 19, \qquad \text{and } 1 \times 1 = 1.$$

Why does the number 1 act this way with multiplication?

Why It's True, Explanation 1:

Let's think first about the definition of multiplication as repeated addition:

- $m \times 1$ means to add the number one to itself m times:

$$\underbrace{1 + 1 + \cdots + 1}_{m \text{ times}}$$

So we see that $m \times 1 = m$ for any whole number m.

- On the other hand, $1 \times m$ means to add the number m to itself just one time. So $1 \times m = m$ also.

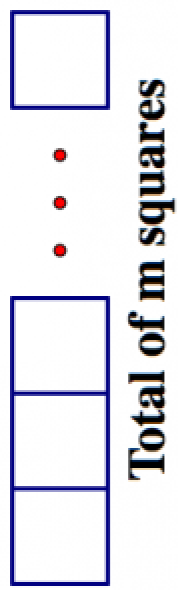
Why It's True, Explanation 2:

We can also use the number line model to create a justification. If Zed calculates $1 \times m$, he will start at 0 and face the positive direction. He will then take m steps forward, and he will do it just one time. So he lands at m, which means $1 \times m = m$.

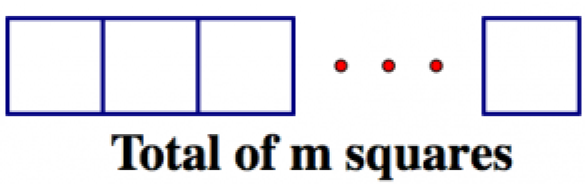
If Zed calculates $m \times 1$, he starts at 0 and faces the positive direction. Then he takes one step forward, and he repeats that m times. So he lands at m. We see that $m \times 1 = m$.

Why It's True, Explanation 3:

In the area model, $m \times 1$ represents m rows with one square in each row. That makes a total of m squares. So $m \times 1 = m$.



Similarly, $1 \times m$ represents one row of m squares. That's also a total of m squares. So $1 \times m = m$.



Think / Pair / Share

The example presented several different explanations. Do you think one is more convincing than the others? Or more clear and easier to understand?

Multiplication is Commutative

Property: Multiplication whole numbers is commutative.

What it Means (words): When I multiply two whole numbers, switching the order in which I multiply them does not affect the product.

What it Means (symbols): For any two whole numbers a and b,

$$a \cdot b = b \cdot a.$$

Problem 18

1. Come up with at least three *examples* to demonstrate the commutativity of multiplication.
2. Use our models of multiplication to come up with an *explanation*. Why does commutativity hold in *every case*? **Note:** Your explanation should not use particular numbers. It is not an example!

Multiplication is Associative

Property: Multiplication of whole numbers is associative.

What it Means (words): When I multiply three whole numbers in a given order, the way I group them (to multiply two at a time) doesn't affect the product.

What it Means (symbols): For any three whole numbers a, b, and c,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

Problem 19

1. Come up with at least three *examples* to demonstrate the associativity of multiplication.
2. Use our models of multiplication to come up with an *explanation*. Why does associativity hold in

every case?

Multiplication Distributes over Addition

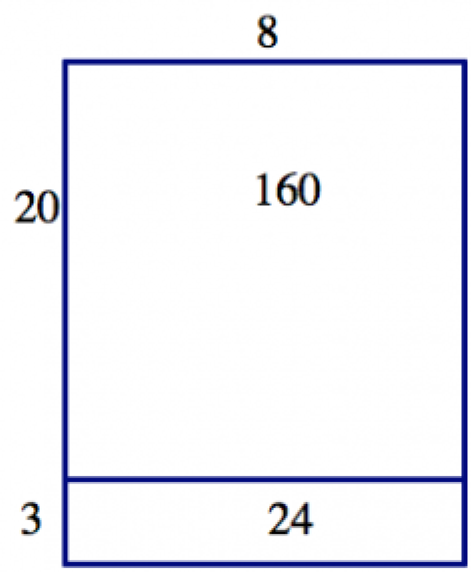
Property: Multiplication distributes over addition.

What it means: The distributive law for multiplication over addition is a little hard to state in words, so we'll jump straight to the symbols. For any three whole numbers *x*, *y*, and *z*:

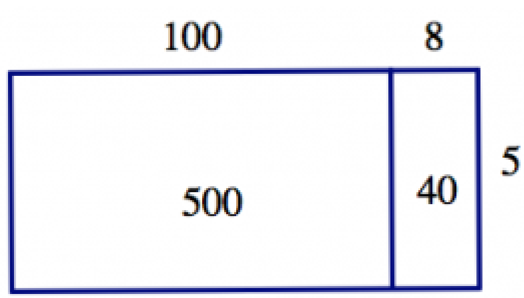
$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

Examples: We actually did calculations very much like the examples above, when we looked at the area model for multiplication.

$8 \cdot (23) = 8 \cdot (20 + 3) = 8 \cdot 20 + 8 \cdot 3 = 160 + 24 = 184$



$5 \cdot (108) = 5 \cdot (100 + 8) = 5 \cdot 100 + 5 \cdot 8 = 500 + 40 = 540$



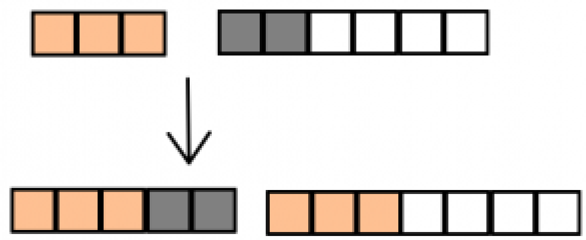
Problem 20

Which of the following pictures best represents the distributive law in the equation

$$3 \cdot (2 + 4) = 3 \cdot 2 + 3 \cdot 4?$$

Explain your choice.

(a)



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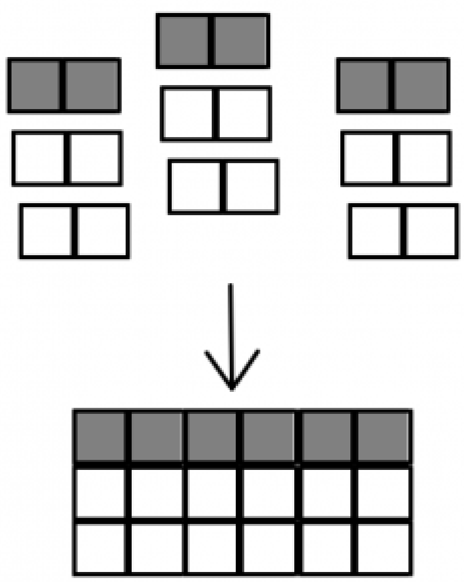

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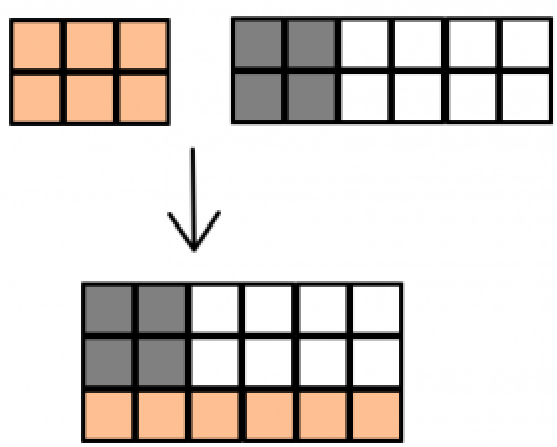

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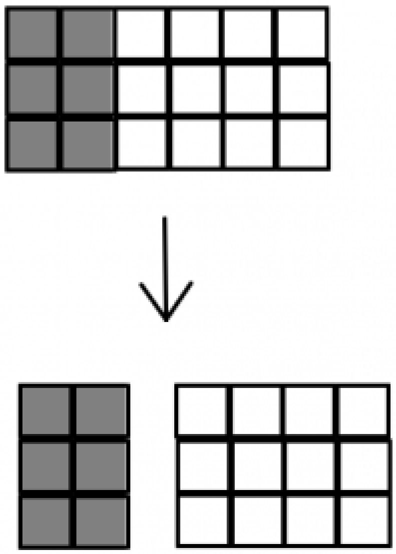
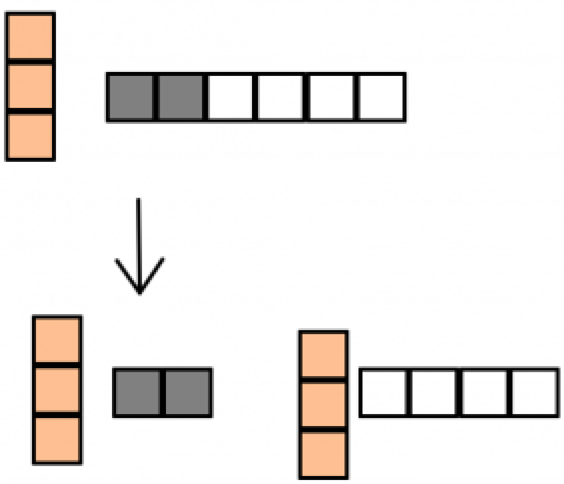


(e)



Problem 21

Use the distributive law to easily compute each of these in your head (no calculators!). Explain your solutions.

45×11 $\quad$ 63×101 $\quad$ $172 \times 1001.$

Think / Pair / Share

Use one of our models for multiplication and addition to explain why the distributive rule works every time.

Properties of Division

It's natural to wonder which, if any, of these properties also hold for division (since you know that the operations of multiplication and division are connected).

Example: Is Division Associative?

If division were associative, then for any choice of three whole numbers a, b, and c, we would have

$$a \div (b \div c) = (a \div b) \div c.$$

Remember, the parentheses tell you which two numbers to divide first.

Let's try the example $a = 9$, $b = 3$, and $c = 1$. Then we have:

$$9 \div (3 \div 1) = 9 \div 3 = 3$$

and

$$(9 \div 3) \div 1 = 3 \div 1 = 3.$$

So is it true? Is division associative? Well, we can't be sure. This is just one example. But "division is associative" is a *universal statement*. If it's true, it has to work for *every possible example*. Maybe we just stumbled on a good choice of numbers, but it won't always work.

Let's keep looking. Try $a = 16$, $b = 4$, and $c = 2$.

$$16 \div (4 \div 2) = 16 \div 2 = 8$$

and

$$(16 \div 4) \div 2 = 4 \div 2 = 2.$$

That's all we need! A single counterexample lets us conclude:

Division is **not** associative.

What about the other properties? It's your turn to decide!

Problem 22

1. State what it would mean for division to be commutative. You should use words and symbols.
2. Decide if division is commutative or not. Carefully explain how you made your decision and *how you know you're right*.

Problem 23

1. State what it would mean for division to distribute over addition. You definitely want to use symbols!
2. Decide if division distributes over addition or not. Carefully explain how you made your decision and *how you know you're right*.

Problem 24

1. State what it would mean for the number 1 to be an identity for division. You should use words and symbols.
2. Decide if 1 is an identity for division or not. Carefully explain how you made your decision and *how you know you're right.*

Zero Property for Multiplication and Division

Problem 25

You probably know another property of multiplication that hasn't been mentioned yet:

If I multiply any number times 0 (in either order), the product is 0. This is sometimes called the zero property of multiplication. Notice that the *zero property* is very different from the property of being an identity!

1. Write what the zero property means using both words and symbols:

> For every whole number *n* . . .

2. Give at least three examples of the zero property for multiplication.
3. Use one of our models of multiplication to explain why the zero property holds.

Think / Pair / Share

- For each division problem below, turn it into a multiplication problem. Solve those problems if you can. If you can't, explain what is wrong.

$$5 \div 0 \qquad 0 \div 5 \qquad 7 \div 0 \qquad 0 \div 7 \qquad 0 \div 0$$

- Use your work to explain why we say that *division by* 0 *is undefined*.
- Use one of our models of division to explain why *division by* 0 *is undefined*.

Four Fact Families

In elementary school, students are often encouraged to memorize "four fact families," for example:

$$2 + 3 = 5 \qquad 5 - 3 = 2$$

$$3 + 2 = 5 \qquad 5 - 2 = 3$$

Here's a different "four fact family":

$$2 \cdot 3 = 6 \qquad 6 \div 3 = 2$$

$$3 \cdot 2 = 6 \qquad 6 \div 2 = 3$$

Think / Pair / Share

- In what sense are these groups of equations "families"?
- Write down at least two more addition / subtraction four fact families.
- Use properties of addition and subtraction to explain *why* these four fact families are each really one fact.

- Write down at least two more multiplication / division four fact families.
- Use properties of multiplication and division to explain *why* these four fact families are each really one fact.

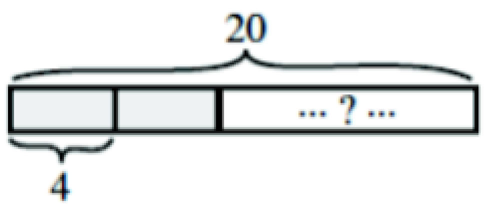
Problem 26

1. Here's a true fact in base six: $2_{\text{six}} + 3_{\text{six}} = 5_{\text{six}}$. Write the rest of this four fact family.
2. Here's a true fact in base six: $11_{\text{six}} - 5_{\text{six}} = 2_{\text{six}}$. Write the rest of this four fact family.

Going Deeper with Division

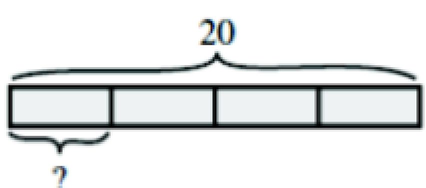
So far we've been thinking about division in what's called the *quotative model*. In the quotative model, we want to make groups of equal size. We know the *size of the group*, and we ask *how many groups*. For example, we think of 20 ÷ 4 as:

How many groups of 4 are there in a group of 20?



Thinking about four fact families, however, we realize we can turn the question around a bit. We could think about the *partitive model* of division. In the partitive model, we want to make an equal number of groups. We know *how many groups*, and we ask the *size of the group*. In the partitive model, we think of 20 ÷ 4 as:

20 is 4 groups of what size?



When we know the original amount and the number of parts, we use partitive division to find the size of each part.

When we know the original amount and the size of each part, we use quotative division to find the number of parts.

Here are some examples in word problems:

Partitive	**Quotative**
number of groups known find the number in each group	number in each group known find the number of groups
A movie theater made \$6450 in one night of ticket sales. 430 people purchased a ticket. How much does one ticket cost?	A movie theater made \$6450 in one night of ticket sales. Each ticket cost \$12.50. How many people purchased a ticket?

Think / Pair / Share

For each word problem below:

- Draw a picture to show what the problem is asking.
- Use your picture to help you decide if it is a quotative or a partitive division problem.
- Solve the problem using any method you like.

1. David made 36 cookies for the bake sale. He packaged the cookies in boxes of 9. How many boxes did he use?
2. David made 36 cookies to share with his friends at lunch. There were 12 people at his lunch table (including David). How many cookies did each person get?
3. Liz spent one summer hiking the Appalachin trail. She completed 1,380 miles of the trail and averaged 15 miles per day. How many days was she out hiking that summer?
4. On April 1, 2012, **Chase Norton** became the first person to hike the entire Koʻolau summit in a single trip. (True story!) It took him eight days to hike all 48 miles from start to finish. If he kept a steady pace, how many miles did he hike each day?

Think / Pair / Share

Write your own word problems: Write one partitive division problem and one quotative division problem. Choose your numbers carefully so that the answer works out nicely. Be sure to solve your problems!

Why think about these two models for division? You won't be teaching the words *partitive* and *quotative* to your students. But recognizing the two kinds of division problems (and being able to come up with examples of each) will make you a better teacher.

It's important that your students are exposed to both ways of thinking about division, and to problems of both types. Otherwise, they may think about division too narrowly and not really understand what's going on. If you understand the two kinds of problems, you can more easily diagnose and remedy students' difficulties.

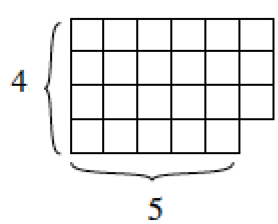
Most of the division problems we've looked at so far have come out evenly, with no remainder. But of course, that doesn't always happen! Sometimes, a whole number answer makes sense, and the context of the problem should tell you which whole number is the right one to choose.

Problem 27

What is 43 ÷ 4?

1. Write a problem that uses the computation 43 ÷ 4 and gives 10 as the correct answer.
2. Write a problem that uses the computation 43 ÷ 4 and gives 11 as the correct answer.
3. Write a problem that uses the computation 43÷4 and gives 10.75 as the correct answer.

We can think about division with remainder in terms of some of our models for operations. For example, we can calculate that 23 ÷ 4 = 5 R3. We can picture it this way:



$$23 = 5 \cdot 4 + 3.$$

Think / Pair / Share

- Explain how the picture above illustrates $23 = 5 \cdot 4 + 3$. Where do you see the remainder of 3 in the picture?
- Explain the connection between these two equations.

$$23 \div 4 = 5 \text{ R3} \quad \text{and} \quad 23 = 5 \cdot 4 + 3.$$

- How could you use the number line model to show the calculation $23 = 5 \cdot 4 + 3$? What does a “remainder” look like in this model?
- Draw area models for each of these division problems. Find the quotient and remainder.

$$40 \div 12 \qquad 59 \div 10 \qquad 91 \div 16$$

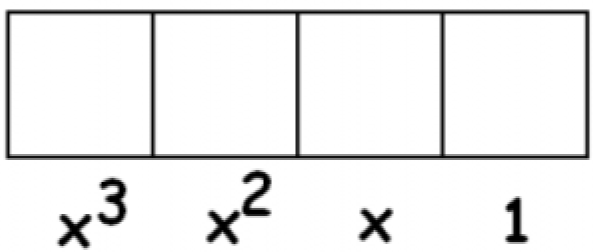
Division Explorations

Anu refuses to tell anyone if she is working in a 1 ← 10 system, or a 1 ← 5 system, or any other system. She makes everyone call it a 1 ← *x* system but won't tell anyone what *x* stands for.

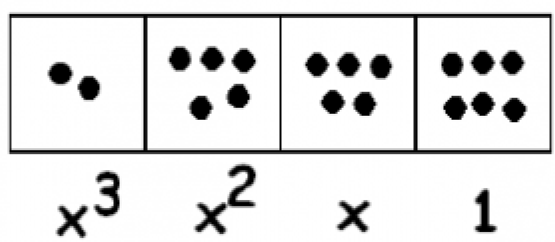
We know that boxes in a 1 ← 10 have values that are powers of ten: 1, 10, 100, 1000, 10000…

And boxes in a 1 ← 5 system are powers of five: 1, 5, 25, 125, 625…

So Anu's system, whatever it is, must be powers of *x*: $1, x, x^2, x^3, x^4 \ldots$

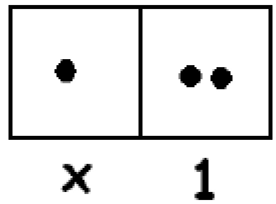


When Anu writes 2556_x she must mean:



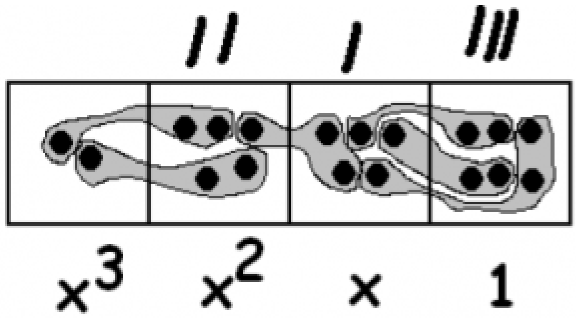
$$2x^3 + 5x^2 + 5x + 6.$$

And when she writes 12_x she means:



$x + 2.$

Anu decides to compute $2556_x \div 12_x$.



She obtains:

$$(2x^3 + 5x^2 + 5x + 6) \div (x + 2) = 2x^2 + x + 3.$$

Problem 28

1. Check Anu's division by computing the product $(x + 2)(2x^2 + x + 3)$.

   Did it work?
2. Use Anu's method to find $(3x^2 + 7x + 2) \div (x + 2)$.
3. Use Anu's method to find $(2x^4 + 3x^3 + 5x^2 + 4x + 1) \div (2x + 1)$.
4. Use Anu's method to find $(x^4 + 3x^3 + 6x^2 + 5x + 3) \div (x^2 + x + 1)$.

Anu later tells use that she really was thinking of a $1 \leftarrow 10$ system so that *x* does equal ten. Then her number 2556 really was two thousand, five hundred and fifty six and 12 really was twelve. Her statement:

$$(2x^3 + 5x^2 + 5x + 6) \div (x + 2) = 2x^2 + x + 3$$

is actually 2556 ÷ 12 = 213.

Problem 29

1. Check that 2556 ÷ 12 = 213 is correct in base 10.
2. Keeping with the 1 ← 10 system, what division problems did you actually solve in parts (b), (c), and (d) of Problem 28? Check that your answers are correct.

Uh Oh! Anu has changed her mind. She now says she was thinking of a 1 ← 11 system.

Now 2556_x means $2 \cdot 11^3 + 5 \cdot 11^2 + 5 \cdot 11 + 6 = 3328_{\text{ten}}$.

Similarly, 12_x means $1 \cdot 11 + 2 = 13_{\text{ten}}$, and 213_x means $2 \cdot 11^2 + 1 \cdot 11 + 3 = 256_{\text{ten}}$.

So Anu's computation $2556_x \div 12_x = 213_x$ is actually the (base 10) statement:

$$3328 \div 13 = 256.$$

Problem 30

1. Check that 3328 ÷ 13 = 256 is also correct in base ten.
2. Keeping with the 1 ← 11 system, what division problems did you actually solve in parts (b), (c), and (d) of Problem 28? Check that they are correct.

Problem 31

1. Use Anu's method to show that
$$(x^4 + 4x^3 + 6x^2 + 4x + 1) \div (x + 1) = (x^3 + 3x^2 + 3x + 1).$$
2. What is this saying for $x = 10$? Check that the division is correct.
3. What is this saying for $x = 2$? Check that the division is correct.
4. What is this saying for x equal to each of 3, 4, 5, 6, 7, 8, 9, and 11? Check that each division is correct.
5. What is this saying for $x = 0$?

Problem Bank

Problem 32

Compute the following using dots and boxes:

$$64212 \div 3$$

$$44793 \div 21$$

$$6182 \div 11$$

$$99916131 \div 31$$

$$637824 \div 302$$

$$2125122 \div 1011$$

Problem 33

1. Fill in the squares using the digits 4, 5, 6, 7, 8, and 9 exactly one time each to make the largest possible sum:
$$\begin{array}{r} \square\ \square\ \square \\ +\square\ \square\ \square \\ \hline \end{array}$$

2. Fill in the squares using the digits 4, 5, 6, 7, 8, and 9 exactly one time each to make the smallest possible (positive) difference:
$$\begin{array}{r} \square\ \square\ \square \\ -\square\ \square\ \square \\ \hline \end{array}$$

Problem 34

1. Make a base six addition table.

$+$	0_{six}	1_{six}	2_{six}	3_{six}	4_{six}	5_{six}
0_{six}	0_{six}	1_{six}				
1_{six}						10_{six}
2_{six}						
3_{six}						
4_{six}						
5_{six}				12_{six}		

2. Use the table to solve these subtraction problems.

$13_{\text{six}} - 5_{\text{six}}$ $\quad$ $12_{\text{six}} - 3_{\text{six}}$ $\quad$ $10_{\text{six}} - 4_{\text{six}}$.

Problem 35

Do these calculations in base four. Don't translate to base 10 and then calculate there — try to work in base four.

1. $33_{\text{four}} + 11_{\text{four}}$
2. $123_{\text{four}} + 22_{\text{four}}$
3. $223_{\text{four}} - 131_{\text{four}}$
4. $112_{\text{four}} - 33_{\text{four}}$

Problem 36

1. Make a base five multiplication table.

$\times$	0_{five}	1_{five}	2_{five}	3_{five}	4_{five}
0_{five}	0_{five}	0_{five}			
1_{five}					
2_{five}					
3_{five}			11_{five}		
4_{five}				22_{five}	

2. Use the table to solve these division problems.
$11_{\text{five}} \div 2_{\text{five}}$ $22_{\text{five}} \div 3_{\text{five}}$ $13_{\text{five}} \div 4_{\text{five}}$.

Problem 37

1. Here is a true fact in base five:
$2_{\text{five}} \cdot 3_{\text{five}} = 11_{\text{five}}$

   Write the rest of this four fact family.

2. Here is a true fact in base five:
$13_{\text{five}} \div 2_{\text{five}} = 4_{\text{five}}$

   Write the rest of this four fact family.

Directions for AlphaMath Problems (Problems 38 – 41):

- Letters stand for digits 0–9.
- In a given problem, the same letter always represents the same digit, and different letters always represent different digits.
- There is no relation between problems (so "A" in part 1 and "A" in part 3 might be different).
- Two, three, and four digit numbers never start with a zero.
- Your job: Figure out what digit each letter stands for, so that the calculation shown is correct.

Problem 38

Notes: In part 2, "O" represents the letter "oh," not the digit zero.

1.
$$\begin{array}{r} A \\ A \\ +A \\ \hline H\ A \end{array}$$

2.
$$\begin{array}{r} O\ N\ E \\ +O\ N\ E \\ \hline T\ W\ O \end{array}$$

3.
$$\begin{array}{r} A\ B\ C \\ +A\ C\ B \\ \hline C\ B\ A \end{array}$$

Problem 39

Here's another AlphaMath problem.

$$\begin{array}{r} T\ E\ N \\ +N\ O\ T \\ \hline N\ I\ N\ E \end{array}$$

1. Solve this AlphaMath problem in base 10.
2. Now solve it in base 6.

Problem 40

Find all solutions to this AlphaMath problem **in base 9**.

Notes: Even though this is two calculations, it is a *single problem*. All T's in both calculations represent the same digit, all B's represent the same digit, and so on.

Remember that "O" represents the letter "oh" and not the digit zero, and that two and three digit numbers never start with the digit zero

$$\begin{array}{r} T\ O \\ -B\ E \\ \hline O\ R \end{array} \qquad \begin{array}{r} N\ O\ T \\ -\ \ T\ O \\ \hline B\ E \end{array}$$

Problem 41

This is a single AlphaMath problem. (So all G's represent the same digit. All A's represent the same digit. And so on.)

Solve the problem in **base 6**.

$$GALON = (GOO)^2 \qquad ALONG = (OOG)^2$$

Problem 42

A *perfect square* is a number that can be written as $a \cdot a$ or a^2 (some number times itself).

1. Which of the following *base seven numbers* are perfect squares? For each number, answer **yes** (it is a perfect square) or **no** (it is not a perfect square) and give a justification of your answer.
   4_{seven} $\quad 25_{\text{seven}}$ $\quad 51_{\text{seven}}$
2. For which choices of base b is the number $b^2 a perfect square? Justify your answer

Problem 43

Geoff spilled coffee on his homework. The answers were correct. Can you determine the missing digits and the bases?

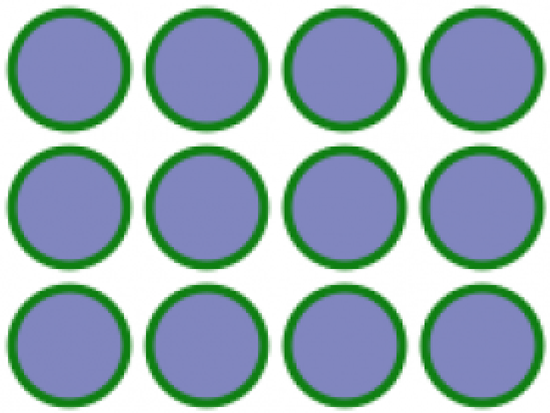
Problem 44

1. Rewrite each subtraction problem as an addition problem:
   $x - 156 = 279$ $\quad$ $279 - 156 = x$ $\quad$ $a - x = b$.
2. Rewrite each division problem as a multiplication problem:
   $24 \div x = 12$ $\quad$ $x \div 3 = 27$ $\quad$ $a \div b = x$.

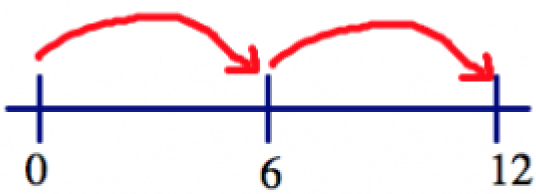
Problem 45

Which of the following models represent the same multiplication problem? Explain your answer.

(a)



(b)



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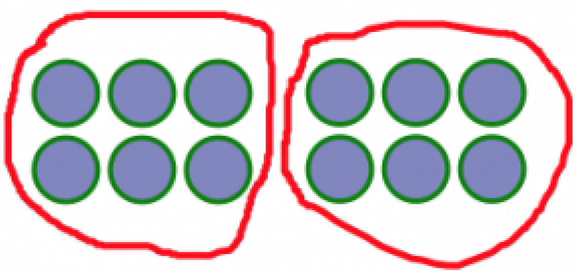
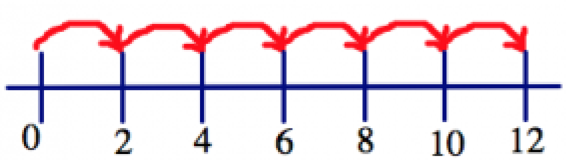


(d)



Problem 46

Show an area model for each of these multiplication problems. Write down the standard computation next to the area model and see how it compares.

20×33 $\quad$ 24×13 $\quad$ 17×11

Problem 47

Suppose the 2 key on your calculator is broken. How could you still use the calculator compute these products? Think about what properties of multiplication might be helpful. (Write out the calculation you would do on the calculator, not just the answer.)

1592×3344 $\quad$ 2008×999 $\quad$ 655×525

Problem 48

Today is Jennifer's birthday, and she's twice as old as her brother. When will she be twice as old as him again? Choose the best answer and justify your choice.

1. Jennifer will always be twice as old as her brother.
2. It will happen every two years.
3. It depends on Jennifer's age.
4. It will happen when Jennifer is twice as old as she is now.
5. It will never happen again.

Problem 49

1. Find the quotient and remainder for each problem.

$7 \div 3$ $\quad$ $3 \div 7$ $\quad$ $7 \div 1$ $\quad$ $1 \div 7$ $\quad$ $15 \div 5$

2. How many possible remainders are there when dividing by these numbers? Justify what you say.
   2 $\quad$ 12 $\quad$ 62 $\quad$ 23

Problem 50

Identify each problem as either partitive or quotative division and say why you made that choice. Then solve the problem.

1. Adriana bought 12 gallons of paint. If each room requires three gallons of paint, how many rooms can she paint?
2. Chris baked 15 muffins for his family of five. How many muffins does each person get?
3. Prof. Davidson gave three straws to each student for an activity. She used 51 straws. How many students are in her class?

Problem 51

Use the digits 1 through 9. Use each digit exactly once. Fill in the squares to make all of the equations true.

$$\square - \square = \square$$
$$\times$$
$$\square \div \square = \square$$
$$=$$
$$\square + \square = \square$$

Fractions

> A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator, the smaller the fraction.
>
> —Leo Tolstoy

The “Pies Per Child[1]” approach to fractions used in this part comes from James Tanton, and is used with his permission. See his development of these and other ideas at http://gdaymath.com/.

1. Pie image by Claus Ableiter (Own work) [GFDL, CC-BY-SA-3.0 or CC BY-SA 2.5-2.0-1.0], via Wikimedia Commons

Introduction

Fractions are one of the hardest topics to teach (and learn!) in elementary school. What is the reason for this? In this part of the book, will try to provide you with some insight about this (as well as some better ways for understanding, teaching, and learning about fractions). But for now, think about what makes this topic so hard.

Think / Pair / Share

You may have struggled learning about fractions in elementary school. Maybe you still find them confusing. Even if you were one of the lucky ones who did not struggle when learning about fractions, you probably had friends who did struggle.

With a partner, talk about why this is. What is so difficult about understanding fractions? Why is the topic harder than other ones we tackle in elementary schools?

Remember that teachers should have lots of mental models — lots of ways to explain the same concept. In this chapter, we will look at some different ways to understand the idea of fractions as well as basic operations on them.

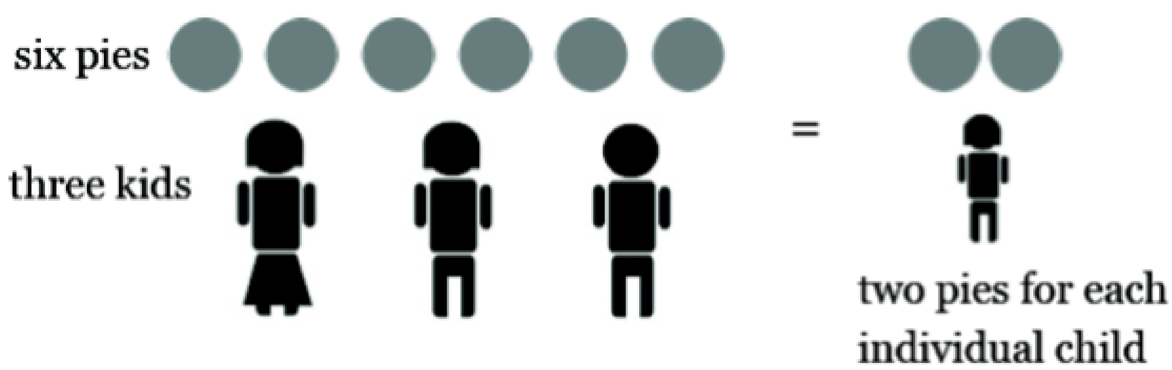
What is a Fraction?

One of the things that makes fractions such a difficult concept to teach and to learn is that you have to think about them in a lot of different ways, depending on the problem at hand. For now, we are going to think of a fraction as the answer to a division problem.

Example: Pies per child

Suppose 6 pies are to be shared equally among 3 children. This yields 2 pies per kid. We write

$$\frac{6}{3} = 2.$$



The fraction $\frac{6}{3}$ is equivalent to the division problem $6 \div 3 = 2$. It represents the number of pies one whole child receives when three kids share six pies equally.

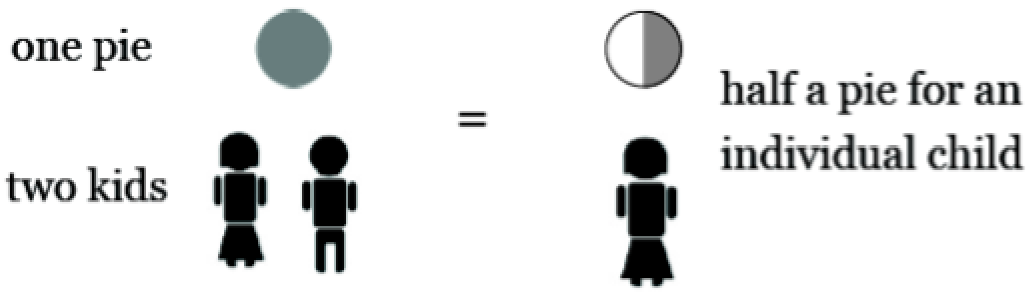
In the same way …

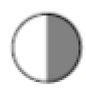
- Sharing 10 pies among 2 kids yields $\frac{10}{2} = 5$ pies per kid.
- Sharing 8 pies among 2 children yields $\frac{8}{2} = 4$ pies per child.
- Sharing 5 pies among 5 kids yields $\frac{5}{5} = 1$ pie per kid.

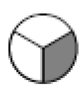
- Sharing 1 pie among 2 children yields $\frac{1}{2}$, which we call "one-half."

This final example is actually saying something! It also represents how fractions are usually taught to students:

If one pie is shared *equally* between two kids, then each child receives a portion of a pie which we choose to call "half."



Thus students are taught to associate the number "$\frac{1}{2}$" to the picture .

In the same way, the picture  is said to represent "one-third," that is, $\frac{1}{3}$. (And this is indeed the amount of pie an individual child would receive if one pie is shared among three.)

The picture  is called "one-fifth" and is indeed $\frac{1}{5}$, the amount of pie an individual receives if three pies are shared among five children.

And the picture  is called "three-fifths" to represent $\frac{3}{5}$, the amount of pie an individual receives if three pies are shared among five children.

Think / Pair / Share

Carefully explain why this is true: If five kids share three pies equally, each child receives an amount that looks like this: .

Your explanation will probably require both words and pictures.

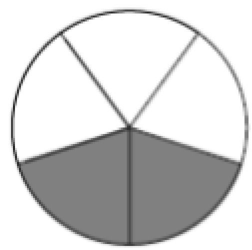
On Your Own

Work on the following exercises on your own or with a partner.

1. Draw a picture associated with the fraction $\frac{1}{6}$.

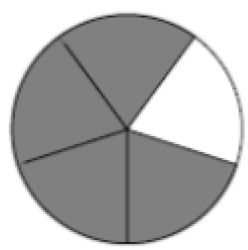
2. Draw a picture associated with the fraction $\frac{3}{7}$. Is your picture really the amount of pie an individual would receive if three pies are shared among seven kids? Be very clear on this!

3. Let's work backwards! Here's the answer to a division problem:



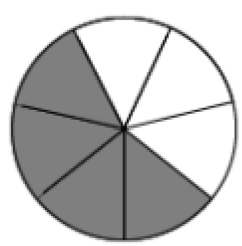
This represents the amount of pie an individual kid receives if some number of pies is shared among some number of children. How many pies? How many children? How can you justify your answers?

4. Here's another answer to a division problem:



How many pies? How many children? How can you justify your answers?

5. Here is another answer to a division problem:



How many pies? How many children? How can you justify your answers?

6. Leigh says that "$\frac{3}{5}$ is three times as big as $\frac{1}{5}$." Is this right? Explain your answer.

7. Draw a picture for the answer to the division problem $\frac{4}{8}$. Describe what you notice about the answer.

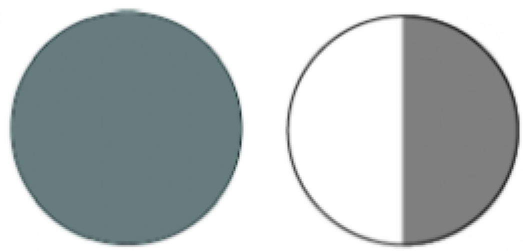
8. Draw a picture for the answer to the division problem $\frac{2}{10}$. Describe what you notice about the answer.

9. What does the division problem $\frac{1}{1}$ represent? How much pie does an individual child receive?

10. What does the division problem $\frac{5}{1}$ represent? How much pie does an individual child receive?

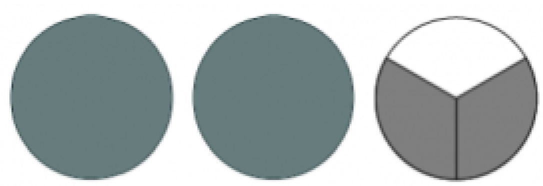
11. What does the division problem $\frac{5}{5}$ represent? How much pie does an individual child receive?

12. Here is the answer to another division problem. This is the amount of pie an individual child receives:



How many pies were in the division problem? How many kids were in the division problem? Justify your answers.

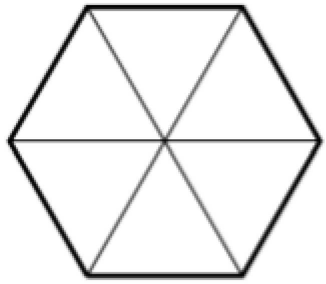
13. Here is the answer to another division problem. This is the amount of pie an individual child receives:



How many pies were in the division problem? How many kids were in the division problem? Justify your answers

14. Many teachers have young students divide differently shaped pies into fractions. For example, a hexagonal pie is good for illustrating the fractions:

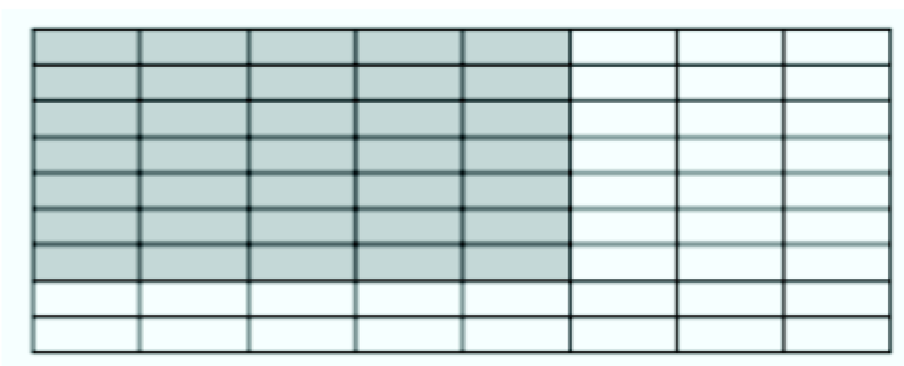
$$\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \text{ and } \frac{6}{6}.$$



- Why is this shape used? What does $\frac{1}{6}$ of a pie look like?
- What does $\frac{6}{6}$ of a pie look like?
- What shape pie would be good for illustrating the fractions $\frac{1}{8}$ up to $\frac{8}{8}$?

Problem 1

Some rectangular pies are distributed to some number of kids. This picture represents the amount of pie an individual child receives. The large rectangle represents one whole pie.



How many pies? How many kids? Carefully justify your answers!

Pies Per Child Model

In our model, a fraction $\frac{a}{b}$ represents the amount of pie an individual child receives when a pies are shared equally by b kids.

$$\begin{array}{l} \#\textbf{pies} \longrightarrow \\ \#\textbf{kids} \longrightarrow \end{array} \frac{a}{b} = \textbf{amount per individual child}$$

Think / Pair / Share

- What is $\frac{2}{2}$? What is $\frac{7}{7}$? What is $\frac{100}{100}$? How can you use the "Pies Per Child Model" to make sense of $\frac{a}{a}$ for any positive whole number a?
- What is $\frac{2}{1}$? What is $\frac{7}{1}$? What is $\frac{1876}{1}$? How can you use the "Pies Per Child Model" to make sense of $\frac{b}{1}$ for any positive whole number b?
- Write the answer to this division problem: "I have no pies to share among thirteen kids." How can you generalize this division problem to make a general statement about fractions?

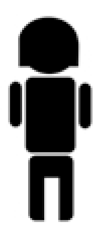
Definition

For a fraction $\frac{a}{b}$, the top number a (which, for us, is the number of pies) is called the **numerator** of the fraction, and the bottom number b (the number of kids), is called the **denominator** of the fraction.

Most people insist that the numerator and denominator each be whole numbers, but they do not have to be.

Think / Pair / Share

To understand why the numerator and denominator need not be whole numbers, we must first be a little gruesome. Instead of dividing pies, let's divide kids! Here is one child:



- What would half a kid look like?

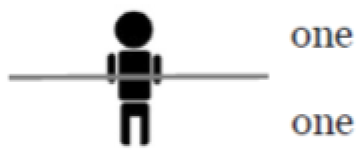
- What would one-third of a kid look like?
- What would three-fifths of a child look like?

So, what would

$$\frac{1}{\left(\frac{1}{2}\right)}$$

represent?

This means assigning one pie to each "group" of half a child. So how much would a whole child receive? Well, we would have a picture like this:



The whole child gets two pies, so we have:

$$\frac{1}{\left(\frac{1}{2}\right)} = 2.$$

Think / Pair / Share

Draw pictures for these problems if it helps!

1. What does
$$\frac{1}{\left(\frac{1}{3}\right)}$$
represent? Justify your answer using the "Pies Per Child Model."
2. What is

$$\frac{1}{\left(\frac{1}{6}\right)}?$$

Justify your answer.

3. Explain why the fraction

$$\frac{5}{\left(\frac{1}{2}\right)}$$

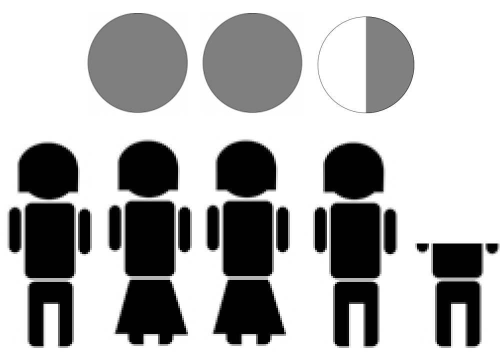
represents the number 10. (How much pie is given to half a kid? To a whole kid?)

4. What is

$$\frac{4}{\left(\frac{1}{3}\right)}?$$

Justify your answer.

5. **Challenge:** Two-and-a-half pies are to be shared equally among four-and-a-half children. How much pie does an individual (whole) child receive? Justify your answer.



Jargon: Improper fractions

A fraction with a numerator smaller than its denominator is called (in school math jargon) a *proper fraction*. For example, $\frac{45}{58}$ is "proper."

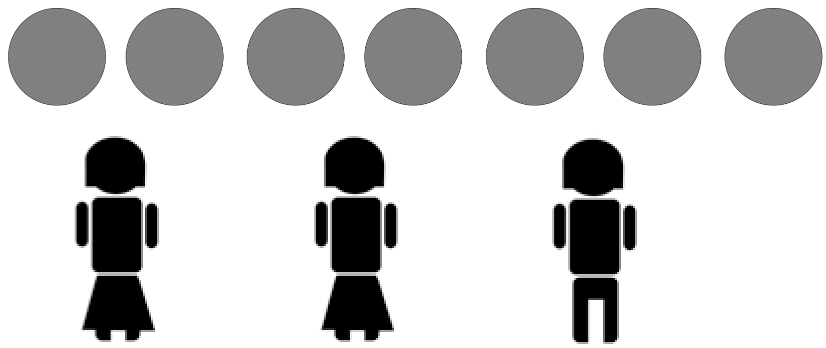
A fraction with numerator larger than its denominator is called (in school math jargon) an *improper fraction*. For example, $\frac{7}{3}$ is "improper." (In the 1800's, these fractions were called *vulgar fractions*.)

For some reason, improper fractions are considered, well, *improper* by some teachers. So students are often asked to write improper fractions as a combination of a whole number and a proper fraction (often called "mixed numbers"). Despite their name and these prejudices, improper fractions are useful nonetheless!

With a mixed number, you have a good sense of the overall size of the number: "a little more than five," or "a bit less than 17." But it is often easier to do calculations with improper fractions (why do you think that is?).

Example: 7/3

If seven pies are shared among three kids, then each kid will certainly receive two whole pies, leaving one pie to share among the three children.



Thus, $\frac{7}{3}$ equals 2 plus $\frac{1}{3}$. People write:

$$\frac{7}{3} = 2\frac{1}{3}$$

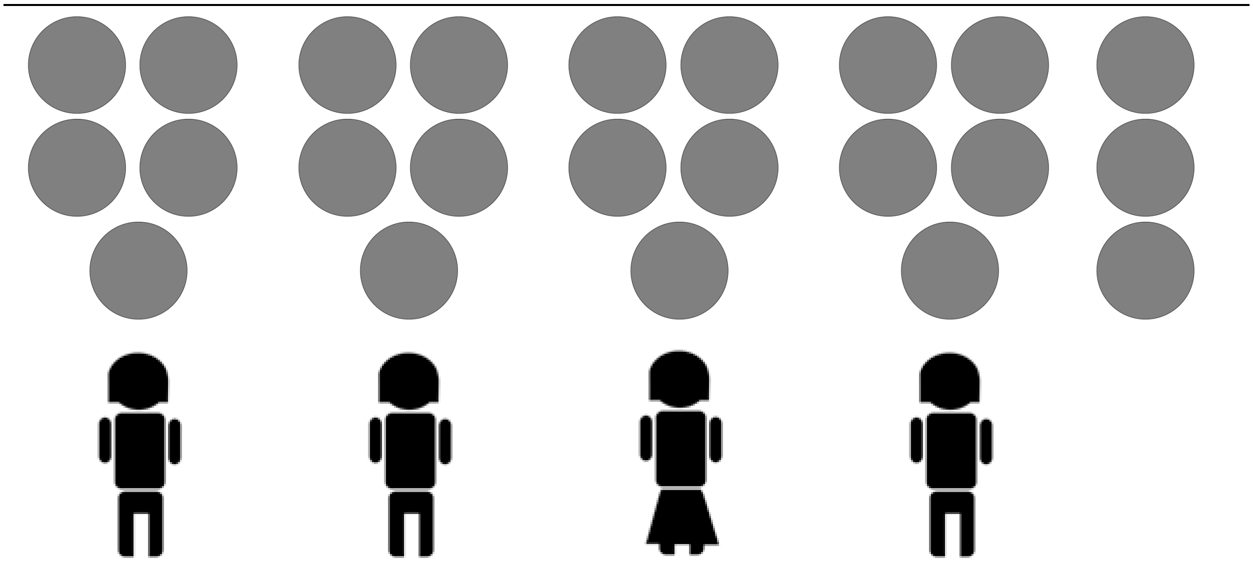
and call the result a *mixed number*. One can also write:

$$2 + \frac{1}{3},$$

which is what $2\frac{1}{3}$ really means. But most people choose to omit the plus sign.

Example: 23/4

If 4 children share 23 pies, we can give them each 5 whole pies. That uses 20 pies, and there are 3 pies left over.



Those three pies are still to be shared equally by the four kids. We have:

$$\frac{23}{4} = 5\tfrac{3}{4}.$$

Example: $2\frac{1}{5}$

For fun, let us write the number 2 as a fraction with denominator 5:

$$2 = \frac{10}{5}.$$

So:

$$2\tfrac{1}{5} \quad = \quad 2 + \frac{1}{5} \quad = \quad \frac{10}{5} + \frac{1}{5} \quad = \quad \frac{11}{5}.$$

We have written the mixed number $2\frac{1}{5}$ as the improper fraction $\frac{11}{5}$.

Think / Pair / Share

- Write each of the following as a mixed number. Explain how you got your answer.
  $\frac{17}{3}, \quad \frac{8}{5}, \quad \frac{100}{3}, \quad \frac{200}{199}.$
- Convert each of these mixed numbers into "improper" fractions. Explain how you got your answer.
  $3\frac{1}{4}, \quad 5\frac{1}{6}, \quad 1\frac{3}{11}, \quad 200\frac{1}{200}.$

Students are often asked to memorize the names "proper fractions," "improper fractions," and "mixed number" so that they can follow directions on tests and problem sets.

But, to a mathematician, these names are not at all important! There is no "correct" way to express an answer (assuming, that the answer is mathematically the right number). We often wish to express our answer in a simpler form, but sometimes the context will tell you what form is "simple" and what form is more complicated.

As you work on problems in this chapter, decide for yourself which type of fraction would be best to work with as you do your task.

The Key Fraction Rule

We know that $\frac{a}{b}$ is the answer to a division problem:

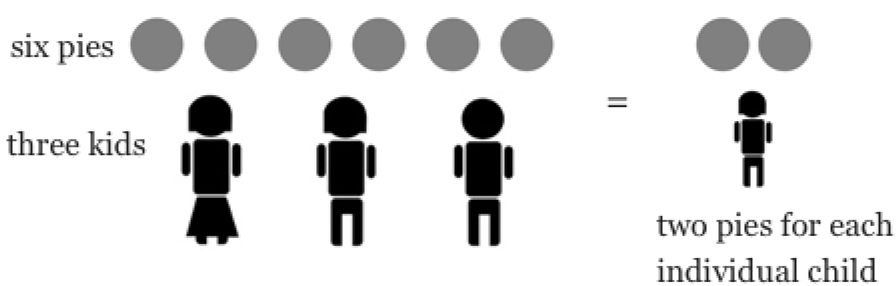
$$\frac{a}{b}$$

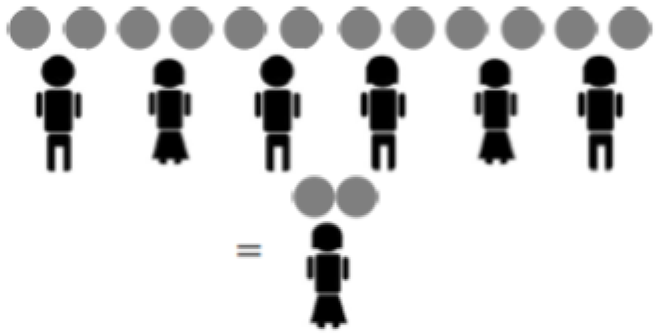
represents the amount of pie an individual child receives when a pies are shared equally by b children.

What happens if we double the number of pie and double the number of kids? Nothing! The amount of pie per child is still the same:

$$\frac{2a}{2b} = \frac{a}{b}.$$

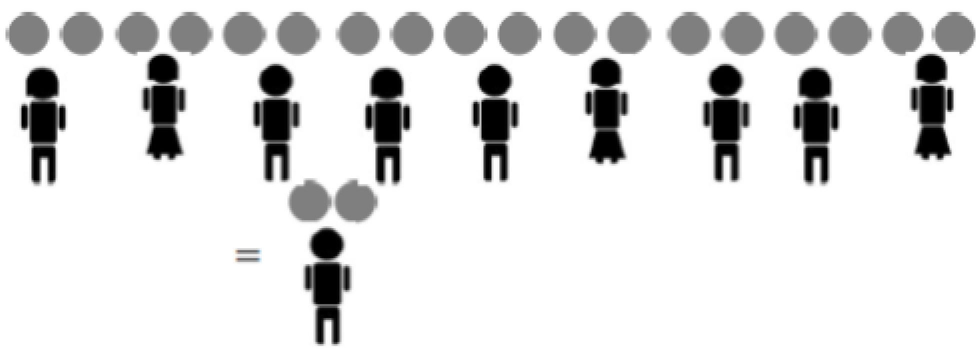
For example, as the picture shows, $\frac{6}{3}$ and 126 both give two pies for each child.





And tripling the number of pies and the number of children also does not change the final amount of pies per child, nor does quadrupling each number, or one trillion-billion-tupling the numbers!

$$\frac{6}{3} \quad = \quad \frac{12}{6} \quad = \quad \frac{18}{9} \quad = \quad \ldots \quad = \quad \text{two pies per child.}$$



This leads us to want to believe:

Key Fraction Rule

$$\frac{xa}{xb} = \frac{a}{b},$$

(at least for positive whole numbers x).

We say that the fractions $\frac{xa}{xb}$ and $\frac{a}{b}$ are **equivalent**.

Example: Fractions equivalent to 3/5

For example,

$$\frac{3}{5} \text{ (sharing three pies among five kids)}$$

yields the same result as

$$\frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10} \text{ (sharing six pies among ten kids)}$$

and as

$$\frac{3 \cdot 100}{5 \cdot 100} = \frac{300}{500} \text{ (sharing 300 pies among 500 kids).}$$

Think / Pair / Share

Write down a lot of equivalent fractions for $\frac{1}{2}$, for $\frac{10}{3}$, and for 1.

Example: Going Backwards

$$\frac{20}{32} \text{ (sharing 20 pies among 32 kids)}$$

is the same problem as:

$$\frac{5 \cdot 4}{8 \cdot 4} = \frac{5}{8} \text{ (sharing five pies among eight kids).}$$

Most people say we have *cancelled* or taken a common factor 4 from the numerator and denominator.

Mathematicians call this process *reducing* the fraction to lowest terms. (We have made the numerator and denominator smaller, in fact as small as we can make them!)

Teachers tend to say that we are *simplifying* the fraction. (You have to admit that $\frac{5}{8}$ does look simpler than $\frac{20}{32}$.)

Example: How Low Can You Go?

As another example, $\frac{280}{350}$ can certainly be simplified by noticing that there is a common factor of 10 in both the numerator and the denominator:

$$\frac{280}{350} = \frac{28 \cdot 10}{35 \cdot 10} = \frac{28}{35}.$$

We can go further as 28 and 35 are both multiples of 7:

$$\frac{28}{35} = \frac{4 \cdot 7}{5 \cdot 7} = \frac{4}{5}.$$

Thus, sharing 280 pies among 350 children gives the same result as sharing 4 pies among 5 children!

$$\frac{280}{350} = \frac{4}{5}.$$

Since 4 and 5 share no common factors, this is as far as we can go with this example (while staying with whole numbers).

On Your Own

Mix and Match: On the top are some fractions that have not been simplified. On the bottom are the simplified answers, but in random order. Which simplified answer goes with which fraction? (Notice that there are fewer answers than questions!)

$1.\ \frac{10}{20} \quad 2.\ \frac{50}{75} \quad 3.\ \frac{24000}{36000} \quad 4.\ \frac{24}{14} \quad 5.\ \frac{18}{32} \quad 6.\ \frac{1}{40}$

$a.\ \frac{2}{3} \quad b.\ \frac{9}{16} \quad c.\ \frac{12}{7} \quad d.\ \frac{1}{40} \quad e.\ \frac{1}{2}$

Think / Pair / Share

Use the "Pies Per Child Model" to explain **why** the key fraction rule holds. That is, explain why each individual child gets the same amount of pie in these two situations:

- if you have a pies and b kids, or
- if you have xa pies and xb kids.

Adding and Subtracting Fractions

Here are two very similar fractions: $\frac{2}{7}$ and $\frac{3}{7}$. What might it mean to add them? It might seem reasonable to say:

$$\frac{2}{7} \text{ represents 2 pies shared by 7 kids.}$$

$$\frac{3}{7} \text{ represents 3 pies shared by 7 kids.}$$

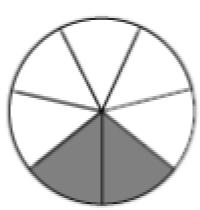
So maybe $\frac{2}{7} + \frac{3}{7}$ represents 5 pies among 14 kids, giving the answer $\frac{5}{14}$. It is very tempting to say that "adding fractions" means "adding pies and adding kids."

The trouble is that a fraction is not a pie, and a fraction is not a child. So adding pies and adding children is not actually adding fractions. A fraction is something different. It is related to pies and kids, but something more subtle. A fraction is an *amount of pie per child*.

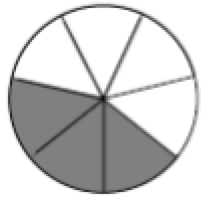
One cannot add pies, one cannot add children. One must add instead the amounts individual kids receive.

Example: 2/7 + 3/7

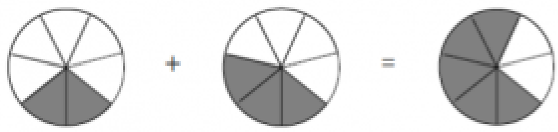
Let us take it slowly. Consider the fraction $\frac{2}{7}$. Here is a picture of the amount an individual child receives when two pies are given to seven kids:



Consider the fraction $\frac{3}{7}$. Here is the picture of the amount an individual child receives when three pies are given to seven children:



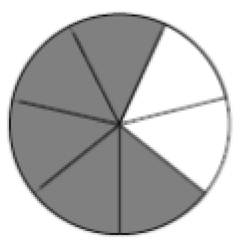
The sum $\frac{2}{7} + \frac{3}{7}$ corresponds to the sum:



The answer, from the picture, is $\frac{5}{7}$.

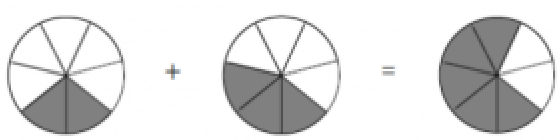
Think / Pair / Share

Remember that $\frac{5}{7}$ means "the amount of pie that one child gets when five pies are shared by seven children." Carefully explain *why* that is the same as the picture given by the sum above:



Your explanation should use both words and pictures!

Most people read this as "two sevenths plus three sevenths gives five sevenths" and think that the problem is just as easy as saying "two apples plus three apples gives five apples." And, in the end, they are right!



This is how the addition of fractions is first taught to students: Adding fractions with the same denominator seems just as easy as adding apples:

4 tenths + 3 tenths + 8 tenths = 15 tenths.

$$\frac{4}{10}+\frac{3}{10}+\frac{8}{10}=\frac{15}{10}.$$

(And, if you like, $\frac{15}{10}=\frac{5\cdot 3}{5\cdot 2}=\frac{3}{2}$.)

82 sixty-fifths + 91 sixty-fifths = 173 sixty-fifths:

$$\frac{82}{65}+\frac{91}{65}\quad=\quad\frac{173}{65}.$$

We are really adding **amounts per child** not amounts, but the answers match the same way.

We can use the "Pies Per Child Model" to explain *why* adding fractions with like denominators works in this way.

Example: 2/7 + 3/7

Think about the addition problem $\frac{2}{7}+\frac{3}{7}$:

$$\begin{array}{r} \text{amount of pie each kid gets when 7 kids share 2 pies} \\ +\ \text{amount of pie each kid gets when 7 kids share 3 pies} \\ \hline ????? \end{array}$$

Since in both cases we have 7 kids sharing the pies, we can imagine that it is the same 7 kids in both cases. First, they share 2 pies. Then they share 3 more pies. The total each child gets by the time all the pie-sharing is done is the same as if the 7 kids had just shared 5 pies to begin with. That is:

$$\begin{array}{r} \text{amount of pie each kid gets when 7 kids share 2 pies} \\ +\ \text{amount of pie each kid gets when 7 kids share 3 pies} \\ \hline \text{amount of pie each kid gets when 7 kids share 5 pies.} \end{array}$$

$$\frac{2}{7}+\frac{3}{7}=\frac{5}{7}.$$

Now let us think about the general case. Our claim is that

$$\frac{a}{d}+\frac{b}{d}=\frac{a+b}{d}.$$

Translating into our model, we have d kids. First, they share a pies between them, and $\frac{a}{d}$ represents the amount each child gets. Then they share b more pies, so the additional amount of pie each child gets is $\frac{b}{d}$. The total each kid gets is $\frac{a}{d}+\frac{b}{d}$.

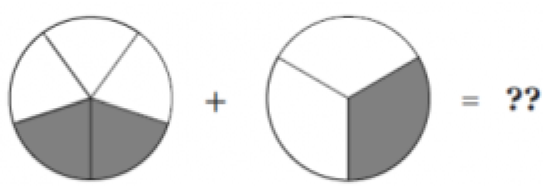
But it does not really matter that the kids first share a pies and then share b pies. The amount each child gets is the same as if they had started with all of the pies — all $a+b$ of them — and shared them equally. That amount of pie is represented by $\frac{a+b}{d}$.

Think / Pair / Share

- How can you *subtract* fractions with the same denominator? For example, what is $$\frac{400}{903}-\frac{170}{903}?$$
- Use the "Pies Per Child" model to *carefully explain why* $$\frac{a}{d}-\frac{b}{d} = \frac{a-b}{d}.$$
- Explain why the fact that the denominators are the same is *essential* to this addition and subtraction method. Where is that fact used in the explanations?

Fractions with Different Denominators

This approach to adding fractions suddenly becomes tricky if the denominators involved are not the same common value. For example, what is $\frac{2}{5}+\frac{1}{3}$?



Let us phrase this question in terms of pies and kids:

Suppose Poindexter is part of a team of five kids that shares two pies. Then later he is part of a team of three kids that shares one pie. How much pie does Poindexter receive in total?

Think / Pair / Share

Talk about these questions with a partner before reading on. It is actually a very difficult problem! What might a student say, if they do not already know about adding fractions? Write down any of your thoughts.

1. Do you see that this is the same problem as computing $\frac{2}{5} + \frac{1}{3}$?
2. What might be the best approach to answering the problem?

One way to think about answering this addition question is to write $\frac{2}{5}$ in a series of alternative forms using our key fraction rule (that is, multiply the numerator and denominator each by 2, and then each by 3, and then each by 4, and so on) and to do the same for $\frac{1}{3}$:

$$\begin{array}{cc} \frac{2}{5} & + \quad \frac{1}{3} \\ \\ \frac{4}{10} & \frac{2}{6} \\ \\ \frac{6}{15} & \frac{3}{9} \\ \\ \frac{8}{20} & \frac{4}{12} \\ \\ \frac{10}{25} & \frac{5}{15} \\ \\ \vdots & \vdots \end{array}$$

We see that the problem $\frac{2}{5} + \frac{1}{3}$ is actually the same as $\frac{6}{15} + \frac{5}{15}$. So we can find the answer using the same-denominator method:

$$\frac{2}{5} + \frac{1}{3} \quad = \quad \frac{6}{15} + \frac{5}{15} \quad = \quad \frac{11}{15}.$$

Example: 3/8 + 3/10

Here is another example of adding fractions with unlike denominators: $\frac{3}{8} + \frac{3}{10}$. In this case, Valerie is part of a group of 8 kids who share 3 pies. Later she is part of a group of 10 kids who share 3 different pies. How much total pie did Valerie get?

$$\frac{3}{8} \quad + \quad \frac{3}{10}$$

$$\frac{6}{16} \qquad \frac{6}{20}$$

$$\frac{9}{24} \qquad \frac{9}{30}$$

$$\frac{12}{32} \qquad \frac{12}{40}$$

$$\frac{15}{40} \qquad \frac{15}{50}$$

$$\vdots \qquad \vdots$$

$$\frac{3}{8} + \frac{3}{10} \quad = \quad \frac{15}{40} + \frac{12}{40} \quad = \quad \frac{17}{40}.$$

Of course, you do not need to list all of the equivalent forms of each fraction in order to find a common denominator. If you can see a denominator right away (or think of a faster method that always works), go for it!

Think / Pair / Share

Cassie suggests the following method for the example above:

> *When the denominators are the same, we just add the numerators. So when the numerators are the same, shouldn't we just add the denominators? Like this:*
>
> $$\frac{3}{8} + \frac{3}{10} = \frac{3}{18}.$$

What do you think of Cassie's suggestion? Does it make sense? What would you say if you were Cassie's teacher?

On Your Own

Try these exercises on your own. For each addition exercise, also write down a "Pies Per Child" interpretation of the problem. You might also want to draw a picture.

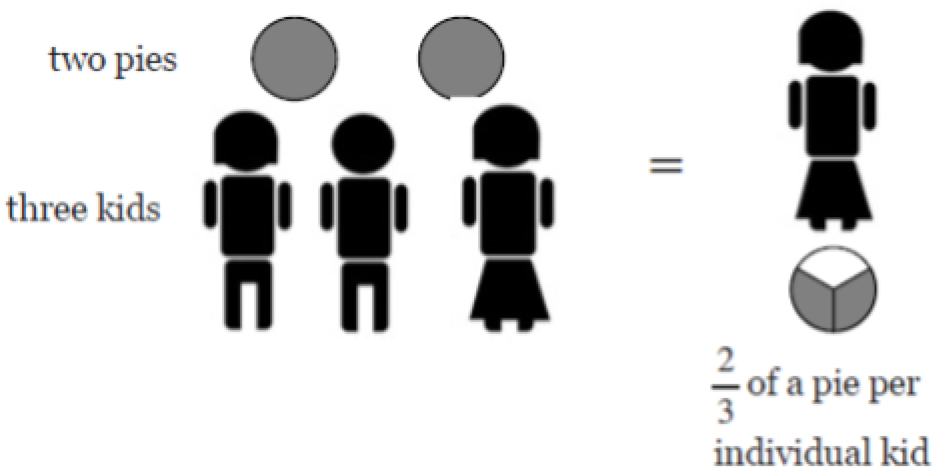
1. What is $\frac{1}{2} + \frac{1}{3}$?
2. What is $\frac{2}{5} + \frac{37}{10}$?
3. What is $\frac{1}{2} + \frac{3}{10}$?
4. What is $\frac{2}{3} + \frac{5}{7}$?
5. What is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$?
6. What is $\frac{3}{10} + \frac{4}{25} + \frac{7}{20} + \frac{3}{5} + \frac{49}{50}$?

Now try these subtraction exercises.

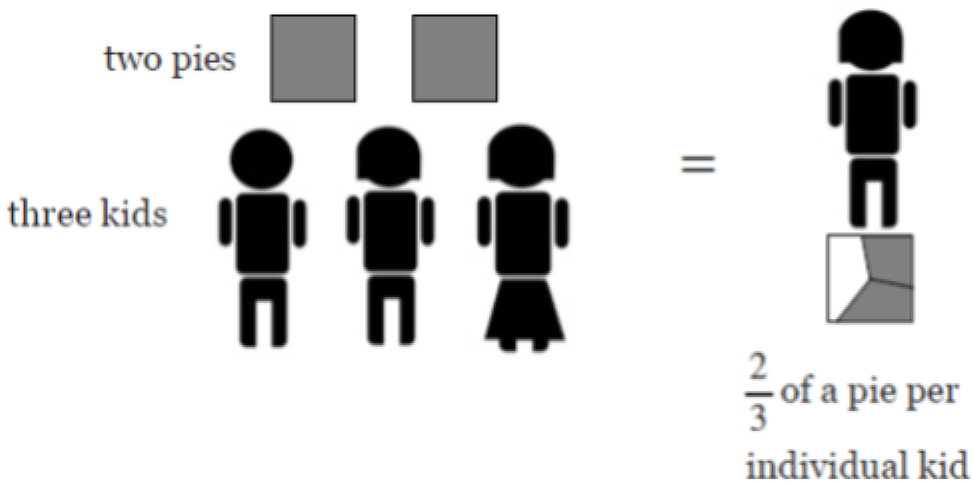
1. What is $\frac{7}{10} - \frac{3}{10}$?
2. What is $\frac{7}{10} - \frac{3}{20}$?
3. What is $\frac{1}{3} - \frac{1}{5}$?
4. What is $\frac{2}{35} - \frac{2}{7} + \frac{2}{5}$?
5. What is $\frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16}$?

What is a Fraction? Revisited

So far, we have been thinking about a fraction as the answer to a division problem. For example, $\frac{2}{3}$ is the result of sharing two pies among three children.

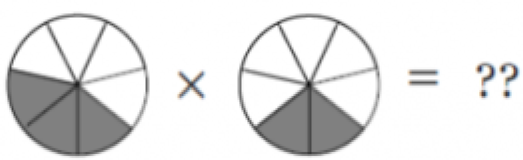


Of course, pies do not have to be round. We can have square pies, or triangular pies or squiggly pies or any shape you please.



This "Pies Per Child Model" has served us perfectly well in thinking about the meaning of fractions, equivalent fractions, and even adding and subtracting fractions.

However, there is no way to use this model to make sense of multiplying fractions! What would this mean?



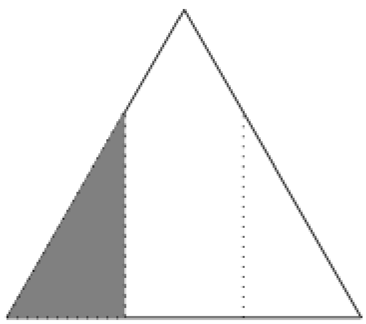
So what *are* fractions, if we are asked to multiply them? We are forced to switch models and think about fractions in a new way.

This switch is fundamentally perturbing. Think about students learning this for the first time. We keep switching concepts and models, and speak of fractions in each case as though all is naturally linked and obvious. None of this is obvious, it is all absolutely confusing. This is just one of the reasons that fractions can be such a difficult concept to teach and to learn in elementary school!

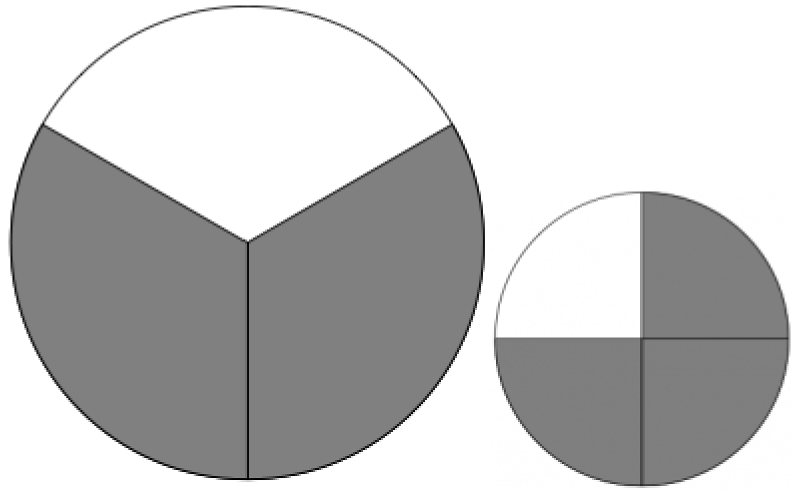
Think / Pair / Share (What's wrong here?)

For each of the following visual representations of fractions, there is a corresponding **incorrect** symbolic expression.

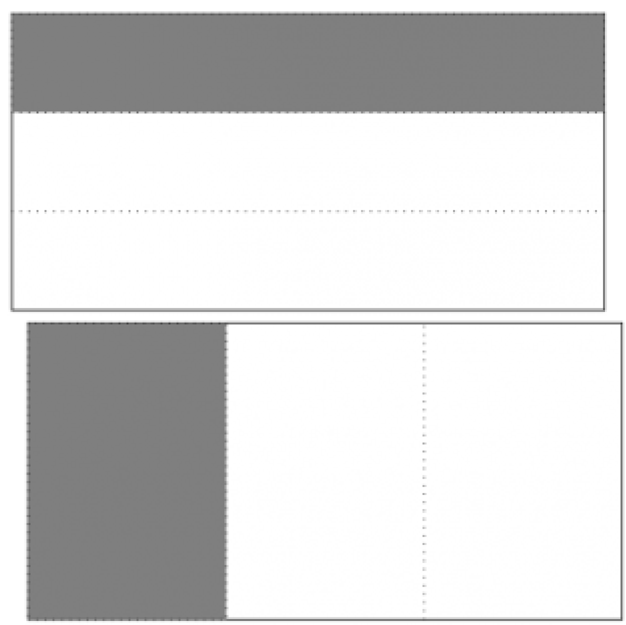
- Why is the symbolic representation incorrect?
- What might elementary students find confusing in these visual representations?



$\frac{1}{3}$



$\frac{2}{3} > \frac{3}{4}$



$\frac{1}{3} \neq 13$

Units and unitizing

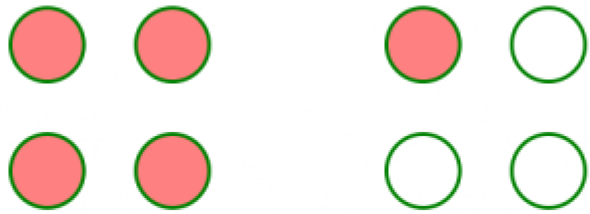
In thinking about fractions, it is important to remember that there are always *units* attached to a fraction, even if the units are hidden. If you see the number $\frac{1}{2}$ in a problem, you should ask yourself “half of what?” The answer to that question is your unit, the amount that equals 1.

So far, our units have been consistent: the “whole” (or unit) was a whole pie, and fractions were represented by pies cut into equal-sized pieces. But this is just a model, and we can take anything, cut it into equal-sized pieces, and talk about fractions *of that whole*.

One thing that can make fraction problems so difficult is that the fractions in the problem may be given in different *units* (they may be "parts" of different "wholes").

Example (Everyone is right!)

Mr. Li shows this picture to his class and asks what number is shown by the shaded region.

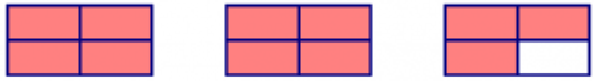


- Kendra says the shaded region represents the number 5.
- Dylan says it represents $2\frac{1}{2}$.
- Kiana says it represents $\frac{5}{8}$.
- Nate says it is $1\frac{1}{4}$.

Mr. Li exclaims, "Everyone is right!"

Think / Pair / Share

1. How can it be that everyone is right? Justify each answer by explaining what each student thought was the *unit* in Mr. Li's picture.
2. Now look at this picture:



- If the shaded region represents $3\frac{2}{3}$, what is the unit?
- Find three other numbers that could be represented by the shaded region, and explain what the unit is for each answer.

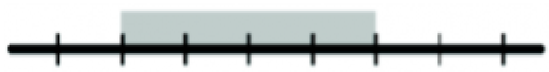
Example (Segments)

This picture

represents $\frac{2}{3}$. The whole segment (the *unit*) is split into three equal pieces by the tick marks, and two of those three equal pieces are shaded.

Think / Pair / Share

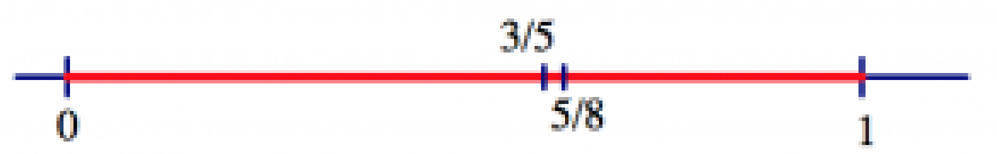
For each picture below, say what fraction it represents and how you know you are right.



Ordering Fractions

If we think about fractions as “portions of a segment,” then we can talk about their locations on a number line. We can start to treat fractions like numbers. In the back of our minds, we should remember that fractions are always relative to some unit. But on a number line, the unit is clear: it is the distance between 0 and 1.

This measurement model makes it much easier to tackle questions about the relative size of fractions based on where they appear on the number line. We can mark off different fractions as parts of the unit segment. Just as with whole numbers, fractions that appear farther to the right are larger.



3/5 and 5/8 are very close, but 5/8 is just a bit bigger.

Think / Pair / Share (Ordering fractions)

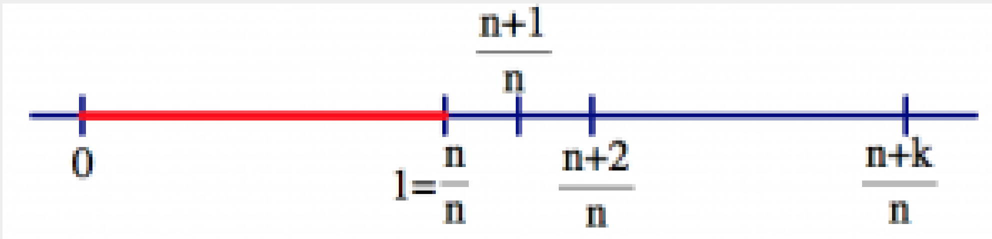
1. What quick method can you use to determine if a fraction is greater than 1?
2. What quick method can you use to determine if a fraction is greater than $\frac{1}{2}$?
3. Organize these fractions from smallest to largest using *benchmarks*: 0 to $\frac{1}{2}$, $\frac{1}{2}$ to 1, and greater than 1. Justify your choices.
   $\frac{25}{23}, \quad \frac{4}{7}, \quad \frac{17}{35}, \quad \frac{2}{9}, \quad \frac{14}{15}.$
4. Arrange each group of fractions in *ascending order*. Keep track of your thinking and your methods.

- $\frac{7}{17}, \quad \frac{4}{17}, \quad \frac{12}{17}.$
- $\frac{3}{7}, \quad \frac{3}{4}, \quad \frac{3}{8}.$
- $\frac{5}{6}, \quad \frac{7}{8}, \quad \frac{3}{4}.$
- $\frac{8}{13}, \quad \frac{12}{17}, \quad \frac{1}{6}.$
- $\frac{5}{6}, \quad \frac{10}{11}, \quad \frac{2}{3}.$

You probably came up with benchmarks and intuitive methods to think about the relative sizes of fractions. Here are some of these methods. (Did you come up with others?)

Fraction Intuition

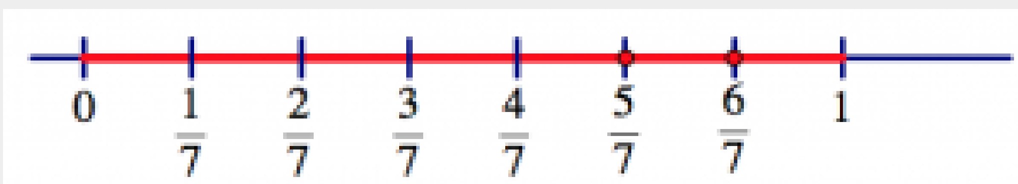
Greater than 1: A fraction is greater than 1 if its numerator is greater than its denominator. How can we see this? Well, the denominator represents how many pieces in one whole (one unit). The numerator represents how many pieces in your portion. So if the numerator is bigger, that means you have more than the number of pieces needed to make one whole.



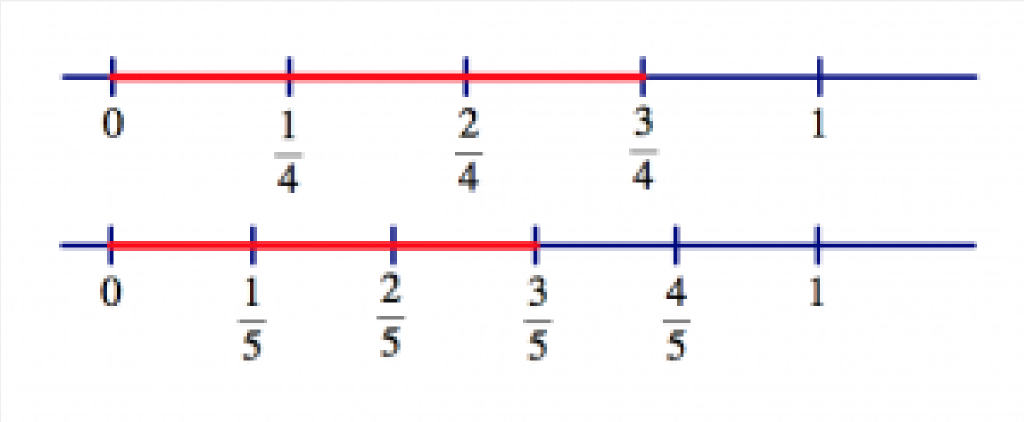
Greater than $\frac{1}{2}$: A fraction is greater than $\frac{1}{2}$ if the numerator is more than half the denominator. Another way to check (which might be an easier calculation): a fraction is greater than $\frac{1}{2}$ if twice the numerator is bigger than the denominator.

Why? Well, if we double the fraction and get something bigger than 1, then the original fraction must be bigger than $\frac{1}{2}$.

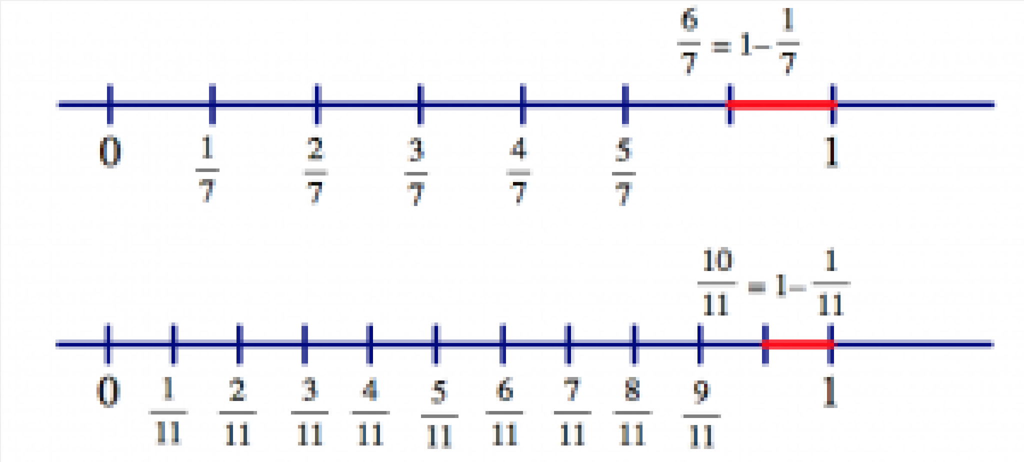
Same denominators: If two fractions have the same denominator, just compare the numerators. The fractions will be in the *same order as the numerators*. For example, $\frac{5}{7} < \frac{6}{7}$. Why? Well, the pieces are the same size since the denominators are the same. If you have more pieces of the same size, you have a bigger number.



Same numerators: If the numerators of two fractions are the same, just compare the denominators. The fractions should be in the *reverse order of the denominators*. For example, $\frac{3}{4} > \frac{3}{5}$. The justification for this one is a little trickier: The denominator tells you how many pieces make up one whole. If there are more pieces in a whole (if the denominator is bigger), then the pieces must be smaller. And if you take the same number of pieces (same numerator), then the bigger piece wins.



Numerator = denominator-1: You can easily compare two fractions whose numerators are both one less than their denominators. The fractions will be in the same order as the denominators. Think of each fraction as a pie with one piece missing. The greater the denominator, the smaller the missing piece, so the greater the amount remaining. For example, $\frac{6}{7} < \frac{10}{11}$, since $\frac{6}{7} = 1 - \frac{1}{7}$ and $\frac{10}{11} = 1 - \frac{1}{11}$.



Numerator = denominator − constant: You can extend the test above to fractions whose numerators are both the same amount less than their denominators. The fractions will again be in the same order as the denominators, for exactly the same reason. For example, $\frac{3}{7} < \frac{7}{11}$, because both are four "pieces" less than one whole, and the $\frac{1}{11}$ pieces are smaller than the $\frac{1}{7}$ pieces.

Equivalent fractions: Find equivalent fractions that lets you compare numerators or denominators, and then use one of the above rules.

Arithmetic Sequences

Consider the patterns below.

Pattern 1: $5, \quad 8, \quad 11, \quad 14, \quad 17, \quad 20, \quad 23, \quad 26, \ldots$

Pattern 2: $2, \quad 9, \quad 16, \quad 23, \quad 30, \quad 37, \quad 44, \quad 51, \ldots$

Pattern 3: $\frac{1}{5}, \quad \frac{3}{5}, \quad 1, \quad \frac{7}{5}, \quad \frac{9}{5}, \quad \frac{11}{5}, \quad \frac{13}{5}, \quad 3, \ldots$

Think / Pair / Share

Answer these questions about each of the patterns.

- Can you predict the next five numbers?
- Can you predict the 100th number?
- What do these sequences have in common? Describe the pattern in words.

The patterns above are called **arithmetic sequences**: a sequence of numbers where the difference between consecutive terms is a constant. Here are some other examples:

Pattern A: $\underbrace{1, \quad 2,}_{+1} \underbrace{\quad 3,}_{+1} \underbrace{\quad 4,}_{+1} \underbrace{\quad 5,}_{+1} \ldots$

Pattern B: $\underbrace{2, \quad 4,}_{+2} \underbrace{\quad 6,}_{+2} \underbrace{\quad 8,}_{+2} \underbrace{\quad 10,}_{+2} \ldots$

Pattern C: $\underbrace{\frac{1}{3}, \quad 1,}_{+\frac{2}{3}} \underbrace{\quad \frac{5}{3},}_{+\frac{2}{3}} \underbrace{\quad \frac{7}{3},}_{+\frac{2}{3}} \underbrace{\quad 3,}_{+\frac{2}{3}} \ldots$

Think / Pair / Share

If you have not done so already, find the common difference between terms for Patterns 1, 2, and 3. Are they really arithmetic sequences?

Then make up your own arithmetic sequence using whole numbers. Exchange sequences with a partner, and check if your partner's sequence is really an arithmetic sequence.

Here are several more number patterns:

Pattern 4: $1, \quad 2, \quad 4, \quad 8, \quad 16, \quad 32, \quad 64, \quad 128, \ldots$

Pattern 5: $1, \quad 3, \quad 6, \quad 10, \quad 15, \quad 21, \quad 28, \quad 36, \ldots$

Pattern 6: $\frac{2}{5}, \quad \frac{7}{10}, \quad 1, \quad \frac{13}{10}, \quad \frac{8}{5}, \quad \frac{19}{10}, \quad \frac{11}{5}, \quad \frac{5}{2}, \ldots$

Pattern 7: $\frac{3}{5}, \quad \frac{6}{5}, \quad \frac{12}{5}, \quad \frac{24}{5}, \quad \frac{48}{5}, \quad \frac{96}{5}, \ldots$

Think / Pair / Share

For each of the sequences above, decide if it is an arithmetic sequence or not. Justify your answers.

Problem 2

$$\frac{1}{4}, \quad \_\_, \quad \_\_, \quad \frac{1}{3}$$

1. Find two fractions between $\frac{1}{4}$ and $\frac{1}{3}$.
2. Are the resulting four fractions in an arithmetic sequence? Justify your answer.

Problem 3

Find two fractions between $\frac{1}{6}$ and $\frac{1}{5}$ so the resulting four numbers are in an arithmetic sequence.

$$\frac{1}{6}, \quad \_\_, \quad \_\_, \quad \frac{1}{5}$$

Problem 4

Find three fractions between $\frac{2}{5}$ and $\frac{5}{7}$ so the resulting four numbers are in an arithmetic sequence.

$$\frac{2}{5}, \quad \_\_, \quad \_\_, \quad \_\_, \quad \frac{5}{7}$$

Think / Pair / Share

Make up two fraction sequences of your own, one that *is* an arithmetic sequence and one that *is not* an arithmetic sequence.

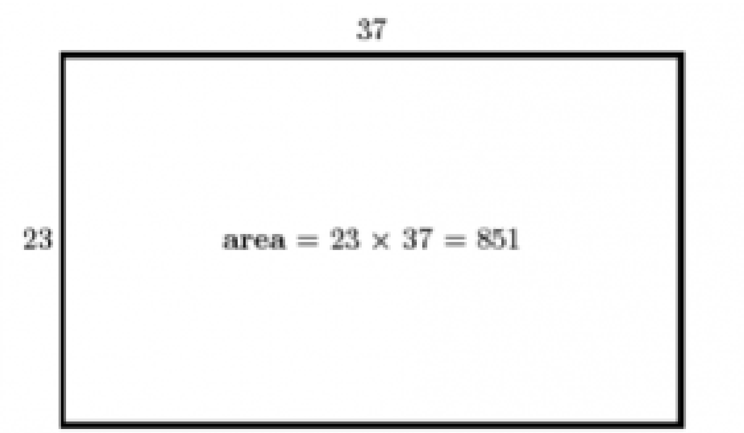
Exchange your sequences with a partner, but do not tell your partner which is which.

When you get your partner's sequences: decide which is an arithmetic sequence and which is not. Check if you and your partner agree.

Multiplying Fractions

Area Model

One of our models for multiplying whole numbers was an area model. For example, the product 23×37 is the area (number of 1 × 1 squares) of a 23-by-37 rectangle:

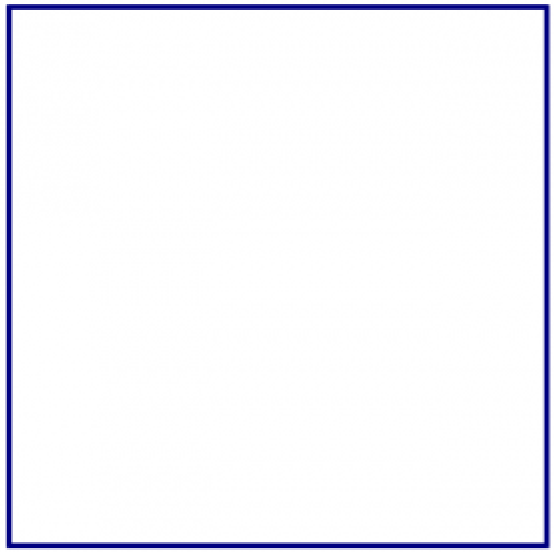


So the product of two fractions, say, $\frac{4}{7} \times \frac{2}{3}$ should also correspond to an area problem.

Example (4/7 × 2/3)

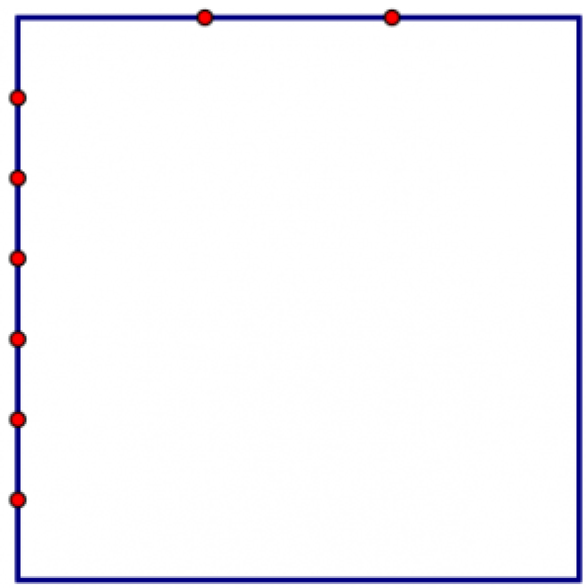
Let us start with a segment of some length that we call 1 unit:

Now, build a square that has one unit on each side:

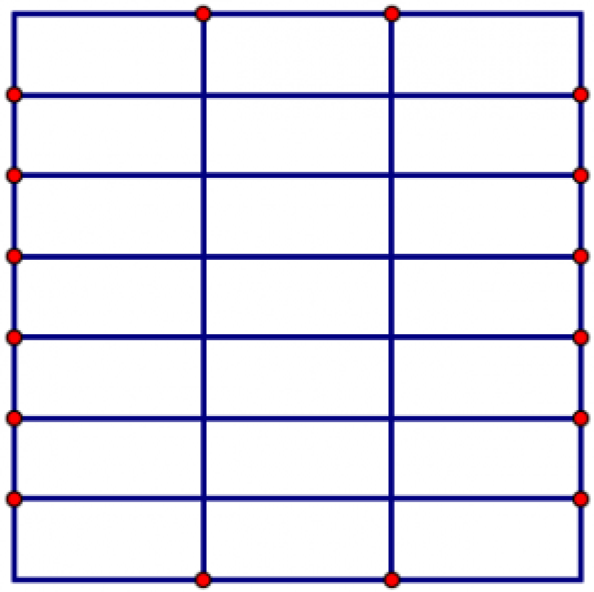


The area of the square, of course, is $1 \times 1 = 1$ square unit.

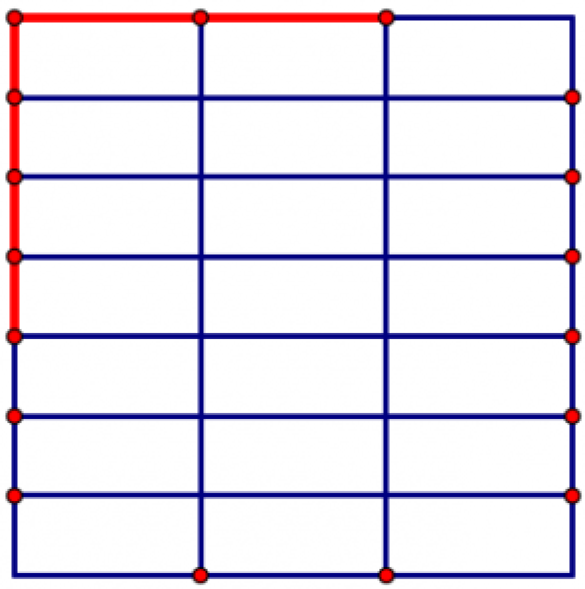
Now, let us divide the segment on top into three equal-sized pieces. (So each piece is $\frac{1}{3}$.) And we will divide the segment on the side into seven equal-sized pieces. (So each piece is $\frac{1}{7}$.)



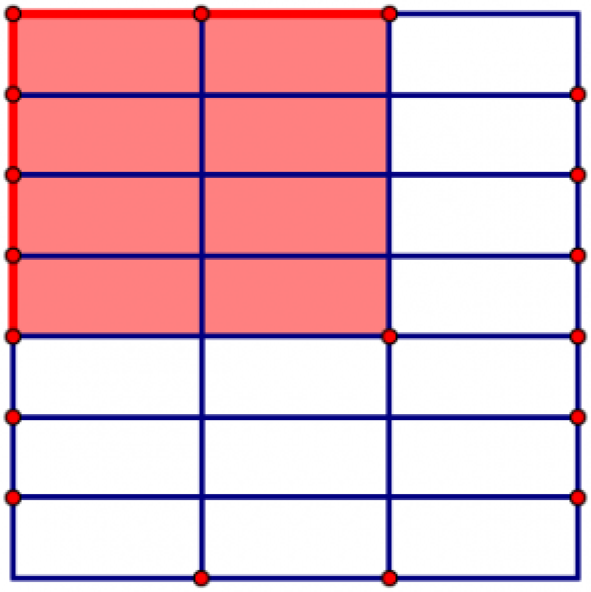
We can use those marks to divide the whole square into small, equal-sized rectangles. (Each rectangle has one side that measures $\frac{1}{3}$ and another side that measures $\frac{1}{7}$.)



We can now mark off four sevenths on one side and two thirds on the other side.



The result of the multiplication should be the area of the rectangle with $\frac{4}{7}$ on one side and $\frac{2}{3}$ on the other. What is that area?



Remember, the whole square was one unit. That one-unit square is divided into 21 equal-sized pieces, and our rectangle (the one with sides $\frac{4}{7}$ and $\frac{2}{3}$) contains eight of those rectangles. Since the shaded area is the answer to our multiplication problem we conclude that

$$\frac{4}{7} \times \frac{2}{3} \quad = \quad \frac{8}{21}.$$

Think / Pair / Share

1. Use are model to compute each of the following products. Draw the picture to see the answer clearly.
   $$\frac{3}{4} \times \frac{5}{6}, \qquad \frac{3}{8} \times \frac{4}{5}, \qquad \frac{5}{8} \times \frac{3}{7}.$$
2. The area problem $\frac{4}{7} \times \frac{2}{3}$ yielded a diagram with a *total* of 21 small rectangles. Explain why 21 appears as the total number of equal-sized rectangles.
3. The area problem $\frac{4}{7} \times \frac{2}{3}$ yielded a diagram with 8 small *shaded* rectangles. Explain why 8 appears as the number of shaded rectangles.

Problem 5

How can you extend the area model for fractions greater than 1? Try to draw a picture for each of these:

$$\frac{3}{4} \cdot \frac{3}{2}, \qquad \frac{2}{5} \cdot \frac{4}{3}, \qquad \frac{3}{10} \cdot \frac{5}{4}, \qquad \frac{5}{2} \cdot \frac{7}{4}.$$

On Your Own

Work on the following exercises on your own or with a partner.

1. Compute the following products, simplifying each of the answers as much as possible. You do not need to draw pictures, but you may certainly choose to do so if it helps!

$$\frac{5}{11} \times \frac{7}{12}, \qquad \frac{4}{7} \times \frac{4}{8}, \qquad \frac{1}{2} \times \frac{1}{3}, \qquad \frac{2}{1} \times \frac{3}{1}, \qquad \frac{1}{5} \times \frac{5}{1}.$$

2. Compute the following products. (Do n0t work too hard!)

$$\frac{3}{4} \times \frac{1}{3} \times \frac{2}{5}, \qquad \frac{5}{5} \times \frac{7}{8}, \qquad \frac{88}{88} \times \frac{541}{788}, \qquad \frac{77876}{311} \times \frac{311}{77876}.$$

3. Try this one. Can you make use of the fraction rule $\frac{xa}{xb} = \frac{a}{b}$ to help you calculate? How?

$$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{5}{6} \times \frac{6}{7} \times \frac{7}{8} \times \frac{8}{9} \times \frac{9}{10}.$$

Think / Pair / Share

How are these two problems different? Draw a picture of each.

1. Pam had $\frac{2}{3}$ of a cake in her refrigerator, and she ate $\frac{1}{2}$ of it. How much total cake did she eat?
2. On Monday, Pam ate $\frac{2}{3}$ of a cake. On Tuesday, Pam ate $\frac{1}{2}$ of a cake. Both cakes were the same size. How much total cake did she eat?

When a problem includes a phrase like "$\frac{2}{3}$ of …," students are taught to treat "of" as multiplication, and to use that to solve the problem. As the above problems show, in some cases this makes sense, and in some cases it does not. It is important to read carefully and understand what a problem is asking, not memorize rules about "translating" word problems.

Explaining the Rule

You probably simplified your work in the exercises above by using a multiplication rule like the following.

Multiplying Fractions

$$\frac{a}{b} \cdot \frac{c}{d} \quad = \quad \frac{a \cdot c}{b \cdot d}.$$

Of course, you may then choose to simplify the final answer, but the answer is always *equivalent* to this one. Why? The area model can help us explain what is going on.

First, let us clearly write out how the area model says to multiply $\frac{a}{b} \cdot \frac{c}{d}$. We want to build a rectangle where one side has length $\frac{a}{b}$ and the other side has length $\frac{c}{d}$. We start with a square, one unit on each side.

- Divide the top segment into b equal-sized pieces. Shade a of those pieces. (This will be the side of the rectangle with length $\frac{a}{b}$.)
- Divide the left segment into d equal-sized pieces. Shade c of those pieces. (This will be the side of the rectangle with length $\frac{c}{d}$.)
- Divide the whole rectangle according to the tick marks on the sides, making equal-sized rectangles.
- Shade the rectangle bounded by the shaded segments.

If the answer is $\frac{a \cdot c}{b \cdot d}$, that means there are $b \cdot d$ total equal-sized pieces in the square, and $a \cdot c$ of them are shaded. We can see from the model why this is the case:

- The top segment was divided into b equal-sized pieces. So there are b columns in the rectangle.
- The side segment was divided into d equal-sized pieces. So there are d rows in the rectangle.
- A rectangle with b columns and d rows has $b \cdot d$ pieces. (The area model for whole-number multiplication!)

Think / Pair / Share

Stick with the general multiplication rule

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

Write a clear explanation for why $a \cdot c$ of the small rectangles will be shaded.

Multiplying Fractions by Whole Numbers

Often, elementary students are taught to multiply fractions by whole numbers using the fraction rule.

Example: Multiply Fractions

For example, to multiply $2 \cdot \frac{3}{7}$, we think of "2" as $\frac{2}{1}$, and compute this way

$$2 \cdot \frac{3}{7} = \frac{2}{1} \cdot \frac{3}{7} = \frac{2 \cdot 3}{1 \cdot 7} = \frac{6}{7}.$$

We can also think in terms of our original "Pies Per Child" model to answer questions like this.

Example: Pies per Child

We know that $\frac{3}{7}$ means the amount of pie each child gets when 7 children evenly share 3 pies**.**

If we compute $2 \cdot \frac{3}{7}$ that means we double the amount of pie each kid gets. We can do this by doubling the number of pies. So the answer is the same as $\frac{6}{7}$: the amount of pie each child gets when 7 children evenly share 6 pies.

Finally, we can think in terms of units and unitizing.

Example: Units

The fraction $\frac{3}{7}$ means that I have 7 equal pieces (of *something*), and I take 3 of them.

So $2 \cdot \frac{3}{7}$ means do that twice. If I take 3 pieces and then 3 pieces again, I get a total of 6 pieces. There are still 7 equal pieces in the whole, so the answer is $\frac{6}{7}$.

Think / Pair / Share

1. Use all three methods to explain how to find each product:
   $$3 \cdot \frac{2}{5}, \qquad 4 \cdot \frac{3}{8}, \qquad 6 \cdot \frac{1}{5}.$$
2. Compare these different ways of thinking about fraction multiplication. Are any of them more natural to you? Does one make more sense than the others? Do the particular numbers in the problem affect your answer? Does your partner agree?

Explaining the Key Fraction Rule

Roy says that the fraction rule

$$\frac{xa}{xb} = \frac{a}{b}$$

is "obvious" if you think in terms of multiplying fractions. He reasons as follows:

We know multiplying anything by 1 does not change a number:

$$1 \cdot 4 = 4$$
$$1 \cdot 2014 = 2014$$
$$1 \cdot \frac{5}{7} = \frac{5}{7}$$

So, in general,

$$1 \cdot \frac{a}{b} = \frac{a}{b}.$$

Now, $\frac{2}{2} = 1$*, so that means that*

$$\frac{2}{2} \cdot \frac{a}{b} = 1 \cdot \frac{a}{b} = \frac{a}{b},$$

which means

$$\frac{2a}{2b} = \frac{a}{b}.$$

By the same reasoning, $\frac{3}{3} = 1$*, so that means that*

$$\frac{3}{3} \cdot \frac{a}{b} = 1 \cdot \frac{a}{b} = \frac{a}{b},$$

which means

$$\frac{3a}{3b} = \frac{a}{b}.$$

Think / Pair / Share

What do you think about Roy's reasoning? Does it make sense? How would Roy explain the general rule for positive whole numbers x:

$$\frac{xa}{xb} = \frac{a}{b}?$$

Dividing Fractions: Meaning

Dividing fractions is one of the hardest ideas in elementary school mathematics. By now, you are used to the rule: to divide by a fraction, multiply by its reciprocal. ("invert and multiply"). But ask yourself: Why does this rule work? Does it really make sense to you? Can you explain why it makes sense to a third grader?

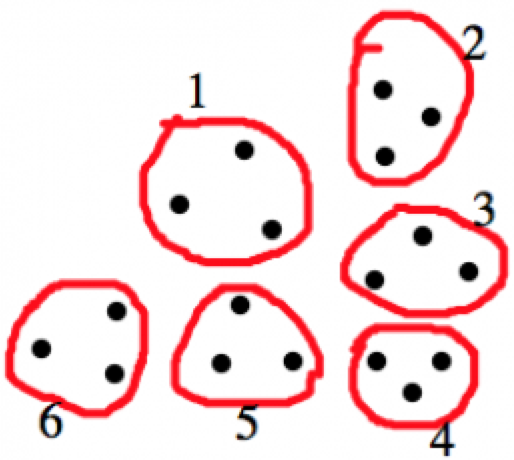
We are going to build up to the "invert and multiply" rule, but along the way, we'll find some more meaningful ways to understand division of fractions. So please play along: **pretend that you don't already know the "invert and multiply" rule,** and solve the problems in this chapter with other methods.

Groups of Equal Size

Remember the quotative model for division: $18 \div 3$ means:

> How many groups of 3 can I find in 18?

We start with 18 dots (or candy bars or molecules), and we make groups of 3 dots (or 3 whatevers). We ask: how many groups can we make?



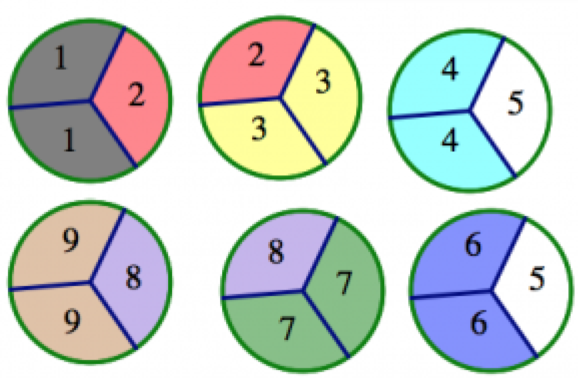
18 dots, split into groups of 3 dots. Since there are 6 groups, we have 18 ÷ 3 = 6.

This same idea applies when we divide fractions. For example, $6 \div 23$ means:

> How many groups of $\frac{2}{3}$ can I find in 6?

Example: 6 ÷ 2/3

Let's draw a picture of 6 pies, and see how many groups of $\frac{2}{3}$ we can find:



We found nine equal groups of size $\frac{2}{3}$, so we conclude that

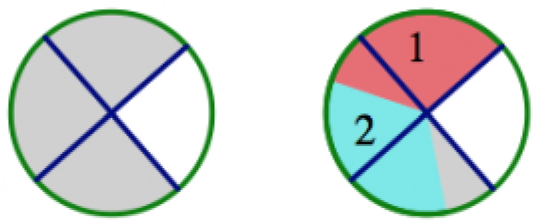
$$6 \div \frac{2}{3} = 9.$$

Unfortunately, it's not always quite so straightforward to find the equal groups. For example, $\frac{3}{4} \div \frac{1}{3}$ asks the question:

> How many groups of $\frac{1}{3}$ can I find in $\frac{3}{4}$?

Example: 3/4 ÷ 1/3

Let's draw a picture of $\frac{3}{4}$ of a pie, and see how many groups of $\frac{1}{3}$ we can find:



The first pictures shows $\frac{3}{4}$ of a pie. The second picture shows two equal groups of $\frac{1}{3}$ inside of $\frac{3}{4}$, but there's a little bit left over. We conclude

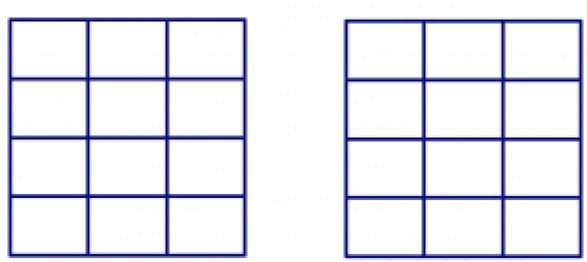
$$\frac{3}{4} \div \frac{1}{3} = 2 + \text{ a tiny bit more.}$$

But how much more? Can we figure it out exactly?

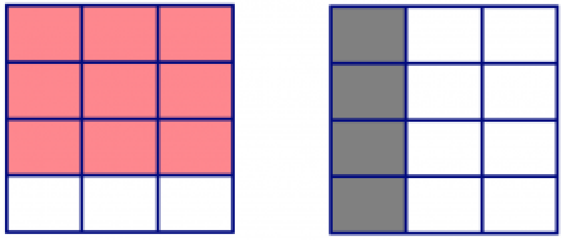
Here's a method that will let you do the computation exactly. We'll use rectangular pies, and divide them up into rows and columns based on the denominators of the numbers we're dividing.

Example: 3/4 ÷ 1/3

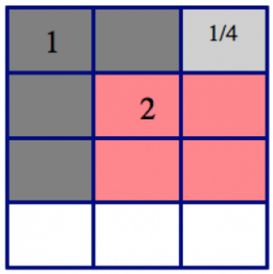
Start by drawing two identical rectangles, each with 4 rows (from the denominator of $\frac{3}{4}$ and 3 columns (from the denominator of $\frac{1}{3}$).



Shade $\frac{3}{4}$ of the first rectangle (this is exactly three rows), and shade $\frac{1}{3}$ of the second rectangle (so that's one column).



Now ask: how many copies of $\frac{1}{3}$ can I find in $\frac{3}{4}$? Well, $\frac{1}{3}$ is equal to four of the smaller squares. So we find groups equal to that:



In the picture of $\frac{3}{4}$, we can find:

- two groups of four squares (two groups of $\frac{1}{3}$), and
- one square left over, which is $\frac{1}{4}$ of the group we're looking for.

We conclude:

$$\frac{3}{4} \div \frac{1}{3} = 2\frac{1}{4}.$$

Think / Pair / Share

Use either method above to find the following quotients. Remember, pretend that you don't know any method to divide fractions except finding equal-sized groups.

$$\frac{3}{4} \div \frac{1}{2} \qquad \frac{1}{3} \div \frac{1}{2} \qquad \frac{4}{9} \div \frac{1}{3} \qquad \frac{4}{5} \div \frac{1}{3}$$

$$\frac{3}{5} \div \frac{3}{4} \qquad \frac{3}{2} \div \frac{1}{2} \qquad \frac{2}{3} \div \frac{1}{2}$$