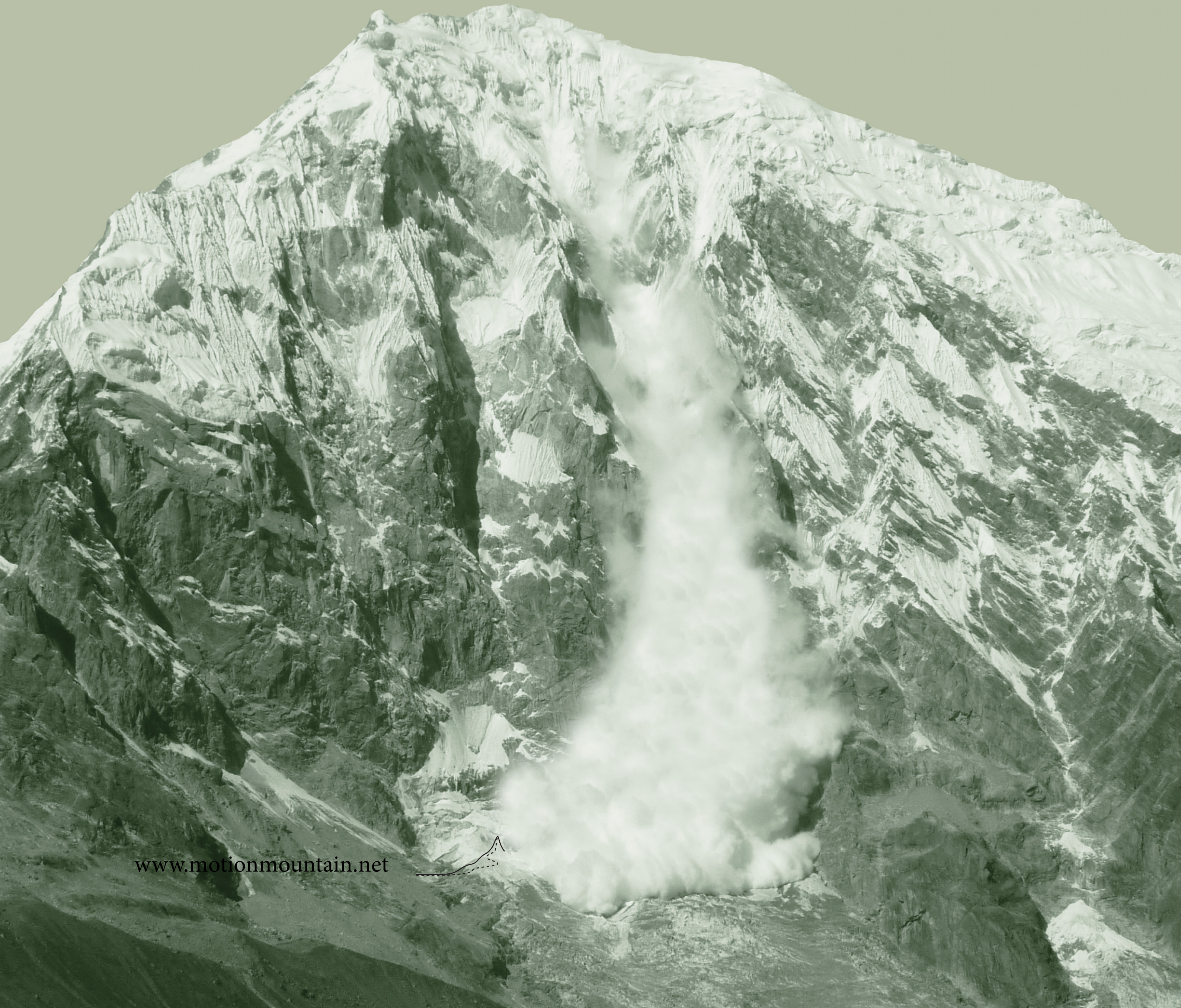
Christoph Schiller

MOTION MOUNTAIN

THE ADVENTURE OF PHYSICS – VOL.IV

THE QUANTUM OF CHANGE



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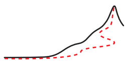

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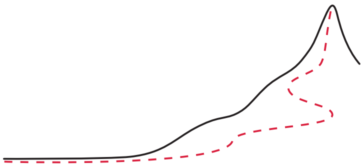

The Adventure of Physics
Volume IV

The Quantum of Change

Edition 31, available as free pdf
with films at www.motionmountain.net

To Britta, Esther and Justus Aaron

τῷ ἐμοὶ δαὶμονι

Die Menschen stärken, die Sachen klären.

PREFACE

> “Primum movere, deinde docere.*
> Antiquity”

This book series is for anybody who is curious about motion in nature. How do things, people, animals, images and empty space move? The answer leads to many adventures, and this volume presents those due to the discovery that there is a smallest possible change value in nature. This smallest change value, the quantum of action, leads to what is called *quantum physics*. In the structure of modern physics, shown in Figure 1, quantum physics covers four of eight points. The present volume introduces the foundations of quantum theory, deduces the structure of atoms and explains the appearance of probabilities, wave functions and colours.

The present introduction to quantum physics arose from a threefold aim I have pursued since 1990: to present the basics of quantum motion in a way that is simple, up to date and captivating.

In order to be *simple*, the text focuses on concepts, while keeping mathematics to the necessary minimum. Understanding the concepts of physics is given precedence over using formulae in calculations. The whole text is within the reach of an undergraduate.

In order to be *up to date*, the text is enriched by the many gems – both theoretical and empirical – that are scattered throughout the scientific literature.

In order to be *captivating*, the text tries to startle the reader as much as possible. Reading a book on general physics should be like going to a magic show. We watch, we are astonished, we do not believe our eyes, we think, and finally we understand the trick. When we look at nature, we often have the same experience. Indeed, every page presents at least one surprise or provocation for the reader to think about. Numerous interesting challenges are proposed.

The motto of the text, *die Menschen stärken, die Sachen klären*, a famous statement on pedagogy, translates as: ‘To fortify people, to clarify things.’ Clarifying things – and adhering only to the truth – requires courage, as changing the habits of thought produces fear, often hidden by anger. But by overcoming our fears we grow in strength. And we experience intense and beautiful emotions. All great adventures in life allow this, and exploring motion is one of them. Enjoy it.

Christoph Schiller

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* ‘First move, then teach.’ In modern languages, the mentioned type of *moving* (the heart) is called *motivating*; both terms go back to the same Latin root.

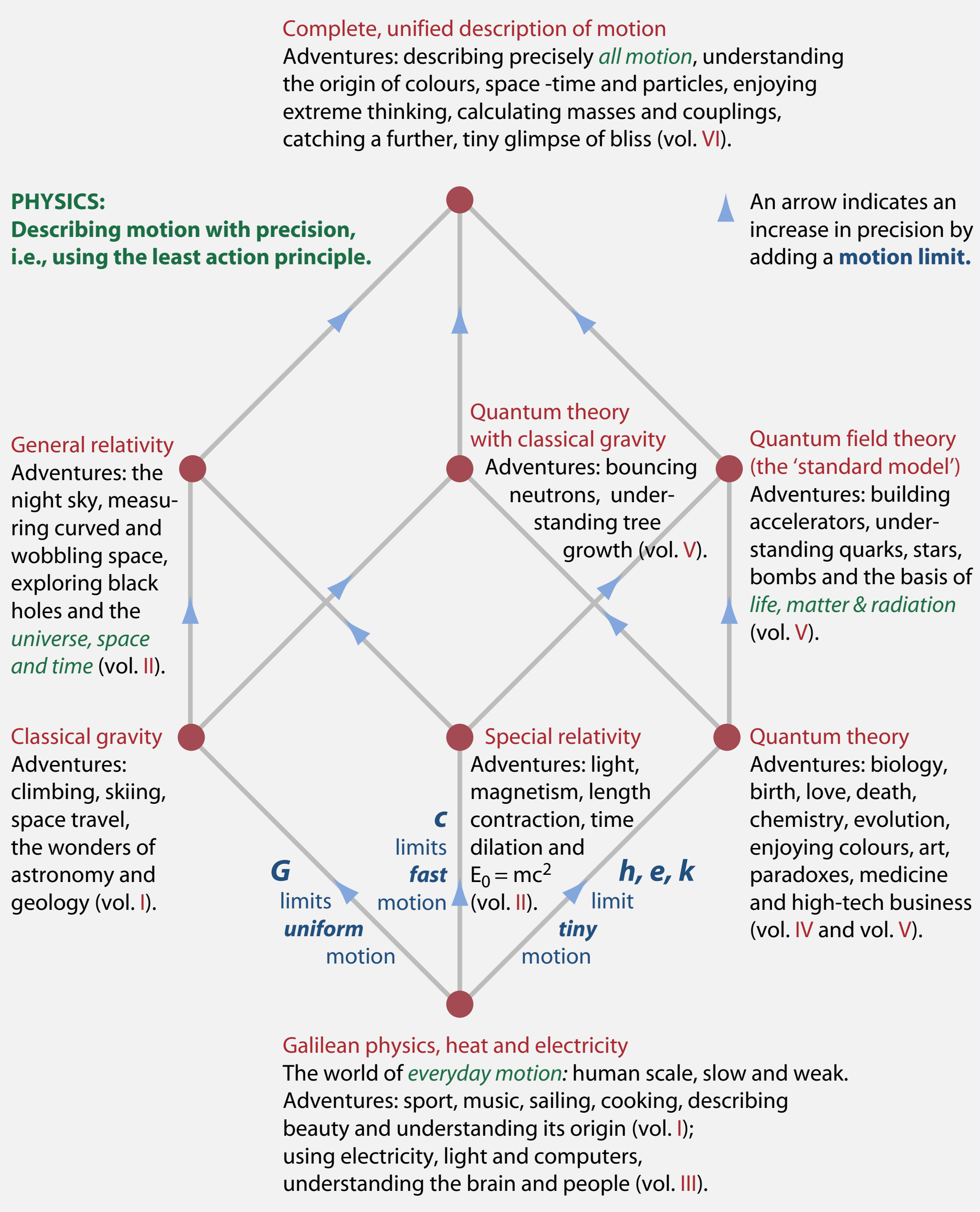


FIGURE 1 A complete map of physics, the science of motion, as first proposed by Matvei Bronshtein (b. 1907 Vinnytsia, d. 1938 Leningrad). The Bronshtein cube starts at the bottom with everyday motion, and shows the connections to the fields of modern physics. Each connection increases the precision of the description and is due to a limit to motion that is taken into account. The limits are given for uniform motion by the gravitational constant G, for fast motion by the speed of light c, and for tiny motion by the Planck constant h, the elementary charge e and the Boltzmann constant k.

USING THIS BOOK

Marginal notes refer to bibliographic references, to other pages or to challenge solutions. In the colour edition, marginal notes, pointers to footnotes and links to websites are typeset in green. Over time, links on the internet tend to disappear. Most links can be recovered via www.archive.org, which keeps a copy of old internet pages. In the free pdf edition of this book, available at www.motionmountain.net, all green pointers and links are clickable. The pdf edition also contains all films; they can be watched directly in Adobe Reader.

Solutions and hints for *challenges* are given in the appendix. Challenges are classified as easy (e), standard student level (s), difficult (d) and research level (r). Challenges for which no solution has yet been included in the book are marked (ny).

ADVICE FOR LEARNERS

Learning allows us to discover what kind of person we can be. Learning widens knowledge, improves intelligence and provides a sense of achievement. Therefore, learning from a book, especially one about nature, should be efficient and enjoyable. Avoid bad learning methods like the plague! Do not use a marker, a pen or a pencil to highlight or underline text on paper. It is a waste of time, provides false comfort and makes the text unreadable. And do not learn from a screen. In particular, never, ever, learn from the internet, from videos, from games or from a smartphone. Most of the internet, almost all videos and all games are poisons and drugs for the brain. Smartphones are dispensers of drugs that make people addicted and prevent learning. Nobody putting marks on paper or looking at a screen is learning efficiently or is enjoying doing so.

In my experience as a pupil and teacher, one learning method never failed to transform unsuccessful pupils into successful ones: if you read a text for study, summarize every section you read, *in your own words and images, aloud*. If you are unable to do so, read the section again. Repeat this until you can clearly summarize what you read in your own words and images, aloud. And *enjoy* the telling aloud! You can do this alone or with friends, in a room or while walking. If you do this with everything you read, you will reduce your learning and reading time significantly; you will enjoy learning from good texts much more and hate bad texts much less. Masters of the method can use it even while listening to a lecture, in a low voice, thus avoiding to ever take notes.

ADVICE FOR TEACHERS

A teacher likes pupils and likes to lead them into exploring the field he or she chose. His or her enthusiasm is the key to job satisfaction. If you are a teacher, before the start of a lesson, picture, feel and tell yourself how you enjoy the topic of the lesson; then picture, feel and tell yourself how you will lead each of your pupils into enjoying that topic as much as you do. Do this exercise consciously, every day. You will minimize trouble in your class and maximize your teaching success.

This book is not written with exams in mind; it is written to make teachers and students *understand* and *enjoy* physics, the science of motion.

FEEDBACK

The latest pdf edition of this text is and will remain free to download from the internet. I would be delighted to receive an email from you at fb@motionmountain.net, especially on the following issues:

Challenge 1 s

- What was unclear and should be improved?
- What story, topic, riddle, picture or film did you miss?

Also help on the specific points listed on the www.motionmountain.net/help.html web page is welcome. All feedback will be used to improve the next edition. You are welcome to send feedback by mail or by sending in a pdf with added yellow notes, to provide illustrations or photographs, or to contribute to the errata wiki on the website. If you would like to translate a chapter of the book in your language, please let me know.

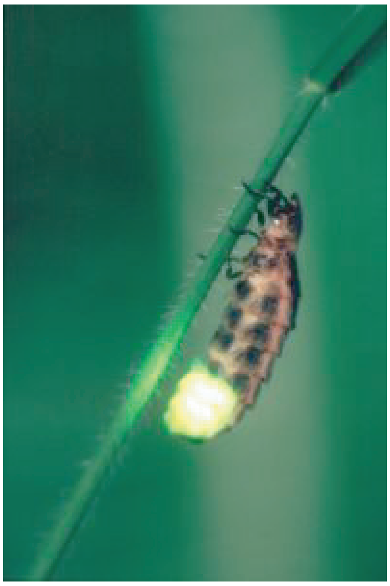
On behalf of all readers, thank you in advance for your input. For a particularly useful contribution you will be mentioned – if you want – in the acknowledgements, receive a reward, or both.

SUPPORT

Your donation to the charitable, tax-exempt non-profit organisation that produces, translates and publishes this book series is welcome. For details, see the web page www.motionmountain.net/donation.html. The German tax office checks the proper use of your donation. If you want, your name will be included in the sponsor list. Thank you in advance for your help, on behalf of all readers across the world.

The paper edition of this book is available, either in colour or in black and white, from www.amazon.com, in English and in certain other languages. And now, enjoy the reading.

CONTENTS



THE QUANTUM OF CHANGE

In our quest to understand how things move,
we discover that there is a smallest change value in nature,
implying that motion is fuzzy,
that boxes are never tight,
that matter is composed of elementary units,
and that light and interactions are streams of particles.
The smallest change value explains why antimatter exists,
why particles are unlike gloves,
why copying machines do not exist,
why probabilities are reasonable,
and how all colours in nature are formed.

CHAPTER 1

MINIMUM ACTION – QUANTUM THEORY FOR POETS

> “Natura [in operationibus suis] non facit saltus.**”
>
> 15th century

Climbing Motion Mountain up to this point, we completed three legs. We came across Galileo’s mechanics (the description of motion for kids), then continued with Einstein’s relativity (the description of motion for science-fiction enthusiasts), and finally explored Maxwell’s electrodynamics (the description of motion for business people). These three classical descriptions of motion are impressive, beautiful and useful. However, they have a small problem: they are wrong. The reason is simple: none of them describes *life*.

Whenever we observe a flower or a butterfly, such as those of Figure 2, we enjoy the bright colours, the motion, the wild smell, the soft and delicate shape or the fine details of their symmetries. However, we know:

▷ Classical physics cannot explain any characteristic length or time scale observed in nature.

Now, flowers and animals – but also many non-living systems – have characteristic sizes, size ranges and proportions; and they have characteristic rhythms. And indeed, classical physics cannot explain their origin, because

▷ The classical constants of nature – the gravitational constant G, the ideal gas constant R, the speed of light c, the vacuum permittivity ε_0 and the vacuum permeability μ_0 – do not allow defining length or time units: They cannot be combined to yield a length or time value. And they cannot be be used to build a meter bar.

In fact, the classical constants do not even allow us to measure speed or force values, even though these measurements are fractions of c and c^4/G; because in order to measure fractions, we need to define fractions first; however, defining fractions also requires length or time scales and units, which classical physics does not allow.

Without measurements, there are also no emotions! Indeed, our emotions are triggered by our senses. And all the impressions and all the information that our senses

---

Ref. 1 ** ‘Nature [in its workings] makes no jumps.’

FIGURE 2 Examples of quantum machines (© Linda de Volder).

provide us are – among others – measurements. Since classical physics does not provide measurement scales, we know:

▹ Classical physics does not allow understanding senses or emotions.

The reason for all these limitations is the following connection:

▹ Classical physics alone cannot be used to build any measurement device.

Every sense contains measurement devices. And every measurement device, like any pattern or rhythm, needs an internal scale, or, more generally, an internal measurement unit. Because classical physics does not provide any scale, classical physics does not explain how measurement devices work, not how senses work, and not how emotions appear.

To understand emotions and life, we need to go *beyond* classical physics. Take any example of a pleasant situation,* such as a beautiful evening sky, a waterfall, a happy child or a caress. Classical physics is not able to explain any aspect of the situation: First, the colours and their origin remain mysterious. Secondly, all shapes, sizes and proportions remain mysterious. Thirdly, the timing and the duration of the involved processes can-

Challenge 2 s

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* The photograph on page 14 shows a female glow worm, *Lampyris noctiluca*, as commonly found in the United Kingdom (© John Tyler, www.johntyler.co.uk/gwfacts.htm).

not be understood. Fourthly, all the sensations and emotions produced by the situation remain mysterious. To understand and explain these aspects, we need *quantum theory*. In fact, we will find out that both *life* and *every type of pleasure* are examples of quantum motion. Emotions are quantum processes.

In the early days of physics, the impossibility to describe life and pleasure was not seen as a shortcoming, because neither senses nor material properties nor scales were thought to be related to motion. And pleasure was not considered a serious subject of investigation for a respectable researcher anyway. Today, the situation is different. In our adventure we have learned that our senses of time, hearing, touch, smell and sight are primarily *detectors of motion*. Without motion, there would be no senses. Furthermore, all detectors are made of matter. During the exploration on electromagnetism we began to understand that all properties of matter are due to motions of charged constituents. Density, stiffness, colour and all other material properties result from the electromagnetic behaviour of the Lego bricks of matter: namely, the molecules, the atoms and the electrons. Thus, the properties of matter are also consequences of motion. Moreover, we saw that these tiny constituents are *not* correctly described by classical electrodynamics. We even found that light itself does not behave classically. Therefore the inability of classical physics to describe matter, light and the senses is indeed due to its intrinsic limitations.

Vol. I, page 411

Vol. III, page 231

Vol. III, page 247

Vol. III, page 149

In fact, every failure of classical physics can be traced back to a single, fundamental discovery made in 1899 by Max Planck:*

Ref. 2

▷ In nature, action values smaller than $\hbar = 1.06 \cdot 10^{-34}$ Js are not observed.

All attempts to observe physical action values smaller than this fail.** In other words, in nature – as in a good cinema film – there is always some action. The existence of a smallest action value – the so-called *quantum principle* – is in complete contrast with classical physics. (Why?) Despite this contrast, the quantum principle has passed an enorm-

Challenge 3 s

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* Max Planck (1858–1947), professor of physics in Berlin, was a central figure in thermostatics and modern physics. He discovered and named the *Boltzmann constant* k and the *quantum of action* h, often called Planck's constant. His introduction of the quantum hypothesis gave birth to quantum theory. He also made the works of Einstein known in the physical community, and later organized a job for him in Berlin. He received the Nobel Prize for physics in 1918. He was an important figure in the German scientific establishment; he also was one of the very few who had the courage to tell Adolf Hitler *face to face* that it was a bad idea to fire Jewish professors. (He got an outburst of anger as answer.) Famously modest, with many tragedies in his personal life, he was esteemed by everybody who knew him.

** In fact, this story is a slight simplification: the constant originally introduced by Planck was the (unreduced) constant $h = 2\pi\hbar$. The factor 2π leading to the final quantum principle was added somewhat later, by other researchers.

This somewhat unconventional, but didactically useful, approach to quantum theory is due to Niels Bohr. Nowadays, it is hardly ever encountered in the literature, despite its simplicity.

Ref. 3, Ref. 4

Niels Bohr (b. 1885 Copenhagen, d. 1962 Copenhagen) was one of the great figures of modern physics. A daring thinker and a polite man, he made Copenhagen University into the new centre of development of quantum theory, overshadowing Göttingen. He developed the description of the atom in terms of quantum theory, for which he received the 1922 Nobel Prize in Physics. He had to flee Denmark in 1943 after the German invasion, because of his Jewish background, but returned there after the war, continuing to attract the best physicists across the world.

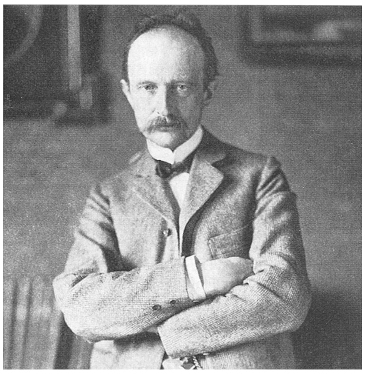

FIGURE 3 Max Planck (1858–1947)

FIGURE 4 Niels Bohr (1885–1962)

ous number of experimental tests, many of which we will encounter in this part of our mountain ascent. Above all, the quantum principle has never failed even a single test. The fundamental constant $\hbar$, which is pronounced 'aitch-bar', is called the *quantum of action*, or alternatively *Planck's constant*. Planck discovered the quantum principle when studying the properties of incandescent light, i.e., of light emanating from hot bodies. But the quantum principle also applies to motion of matter, and even, as we will see later, to motion of empty space, such as gravitational waves. Vol. III, page 149

The quantum principle states that no experiment can measure an action smaller than $\hbar$. For a long time, Einstein tried to devise experiments to overcome this limit. But he failed in all his attempts: nature does not allow it, as Bohr showed again and again. The same occurred to many other researchers.

We recall that in physics – as in the theatre – action is a measure for the *change* occurring in a system. The quantum principle can thus rephrased as Vol. I, page 248

▷ In nature, a change smaller than $\hbar = 1.06 \cdot 10^{-34}$ Js cannot be observed.

Therefore, a smallest action implies that *there is a smallest change value in nature*. If we compare two observations, there will always be change between them. Thus the quantum of action would perhaps be better named the *quantum of change*.

Can a minimum change really exist in nature? To accept the idea, we need to explore three points, detailed in Table 1. We need to show that a smaller change is *never observed* in nature, show that smaller change values *can never* be observed, and finally, show that *all consequences* of this smallest change, however weird they may be, apply to nature. In fact, this exploration constitutes all of quantum physics. Therefore, these checks are all we do in the remaining of this part of our adventure. But before we explore some of the experiments that confirm the existence of a smallest change, we directly present some of its more surprising consequences.

TABLE 1 How to convince yourself and others that there is a smallest action, or smallest change $\hbar$ in nature. Compare this table with the two tables in volume II, that about maximum speed on page 26, and that about maximum force on page 109.

STATEMENT	TEST
The smallest action value $\hbar$ is observer-invariant.	Check all observations.
Local change or action values $< \hbar$ are not observed.	Check all observations.
Local change or action values $< \hbar$ cannot be produced.	Check all attempts.
Local change or action values $< \hbar$ cannot even be imagined.	Solve all paradoxes.
The smallest local change or action value $\hbar$ is a principle of nature.	Deduce quantum theory from it. Show that all consequences, however weird, are confirmed by observation.

THE EFFECTS OF THE QUANTUM OF ACTION ON REST

Since action is a measure of change, a minimum observable action means that two successive observations of the same system always differ by at least $\hbar$. In every system, there is always *something* happening. As a consequence we find:

▷ In nature *there is no rest.*

Page 15 Everything moves, all the time, at least a little bit. *Natura facit saltus.** True, these jumps are tiny, as $\hbar$ is too small to be observable by any of our senses. Nevertheless, rest can be observed only macroscopically, and only as a long-time or many-particle average. For example, the quantum of action implies that in a mountain – an archetypal 'system at rest' – all the atoms and electrons are continually buzzing around. In short,

▷ There is motion *inside matter.*

Since there is a minimum action for all observers, and since there is no rest, we deduce:

▷ In nature *there is no perfectly straight or perfectly uniform motion.*

Forget all you have learnt so far: Inertial motion is an approximation! An object can move in straight, uniform motion only approximately, and only when observed over long distances or long times. We will see later that the more massive the object is, the better

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* 'Nature makes jumps.'

Challenge 4 s the approximation is. (Can you confirm this?) So *macroscopic* observers can still talk about space-time symmetries; and *special* relativity can thus be reconciled with quantum theory.

Also free fall, or motion along a geodesic, exists only as a long-time average. So *general* relativity, which is based on the existence of freely-falling observers, cannot be correct when actions of the order of $\hbar$ are involved. Indeed, the reconciliation of the quantum principle with general relativity – and thus with curved space – is a big challenge. (The solution is simple only for weak, everyday fields.) The issues involved are so mind-shattering that they form a separate, final, part of this adventure. We thus explore situations without gravity first.

The consequences of the quantum of action for objects

Have you ever wondered why leaves are green? You probably know that they are green because they absorb blue (short-wavelength) and red (long-wavelength) light, while allowing green (medium-wavelength) light to be reflected. How can a system filter out the small and the large, and let the middle pass through? To do so, leaves must somehow *measure* the frequency. But we have seen that classical physics does not allow measurement of time (or length) intervals, as any measurement requires a measurement unit, Vol. I, page 439 and classical physics does not allow such units to be defined. On the other hand, it takes only a few lines to confirm that with the help of the quantum of action $\hbar$ (and the Boltzmann constant k, both of which Planck discovered), fundamental units for *all* measurable quantities can be defined, including time and therefore frequency. (Can you find a combination of the speed of light c, the gravitational constant G and the quantum Challenge 5 s of action $\hbar$ that gives a time? It will only take a few minutes.)

In short, measurements are only possible at all because of the existence of the quantum of action.

▷ All measurements *are quantum effects.*

When Planck saw that the quantum of action allowed defining all units in nature, he was as happy as a child; he knew straight away that he had made a fundamental discovery, even though (in 1899) quantum theory did not yet exist. He even told his seven-year-old Ref. 5 son Erwin about it, while walking with him through the woods around Berlin. Planck explained to his son that he had made a discovery as important as universal gravity. Indeed, Planck knew that he had found the key to understanding many of the effects that were then unexplained.

▷ In nature, *all times and all frequencies are due to the quantum of action.*

All processes that take time are quantum processes. If you prefer, *waiting* is a quantum effect! In particular, without the quantum of action, oscillations and waves could not exist:

▷ Every colour is a quantum effect.

But this is not all. Planck also realized that the quantum of action allows us to understand the *size* of all things.

- ▷ Every size is a quantum effect.

Challenge 6 e Can you find the combination of c, G and $\hbar$ that yields a length? With the quantum of action, it was finally possible to determine the maximum size of mountains, of trees and Vol. I, page 338 of humans. Planck knew that the quantum of action confirmed what Galileo had already deduced long before him: that sizes are due to fundamental, smallest scales in nature.

Max Planck also understood that the quantum of action $\hbar$ was the *last missing* constant of nature. With $\hbar$, it becomes possible to define a *natural* unit for *every* observable property in nature. Together, c, G and $\hbar$ allow to define units that are independent of culture or civilization – even extraterrestrials would understand them.* In short, $\hbar$ allows understanding *all* observables. Therefore, with $\hbar$ it is possible to draw the diagram shown in Figure 1 that encompasses *all* motion in nature, and thus *all* of physics.

In our environment, the size of all objects is related and due to the size of atoms. In turn, the size of atoms is a direct consequence of the quantum of action. Can you derive an approximation for the size of atoms, knowing that it is given by the motion of electrons Challenge 8 s of mass m_e and charge e, constrained by the quantum of action? This connection, a simple formula, was discovered in 1910 by Arthur Erich Haas, 15 years before quantum theory was formulated.

- ▷ Atom sizes are quantum effects.

At the time, Haas was widely ridiculed.** Nowadays, his formula for the size of atoms is Page 186 found in all textbooks, including this one. In determining the size of atoms, the quantum of action has another important consequence:

- ▷ *Gulliver's travels* are impossible.

There are no tiny people and no giant ones. Classically, nothing speaks against the idea; Challenge 9 s but the quantum of action prevents it. Can you supply the detailed argument?

But if rest does not exist, how can *shapes* exist? Any shape of everyday life, including that of a flower, is the result of body parts remaining *at rest* with respect to each other. Now, all shapes result from interactions between the constituents of matter, as shown most clearly in the shapes of molecules. But how can a molecule, such as the water molecule H_2O, shown in Figure 5, have a shape? In fact, a molecule does not have a *fixed* shape, but its shape fluctuates, as would be expected from the quantum of action. Despite the fluctuations, every molecule does have an *average* shape, because different angles and distances correspond to different energies. Again, these average length and angle val-

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* In fact, it is also possible to define all measurement units in terms of the speed of light c, the gravitational Challenge 7 s constant G and the electron charge e. Why is this not fully satisfactory?

** Before the discovery of $\hbar$, the only simple length scale for the electron was the combination $e^2/(4\pi\varepsilon_0 m_e c^2) \approx 3\,\mathrm{fm}$; this is ten thousand times smaller than an atom. We stress that any length scale containing e is a quantum effect, and not a classical length scale, because e is the quantum of electric charge.

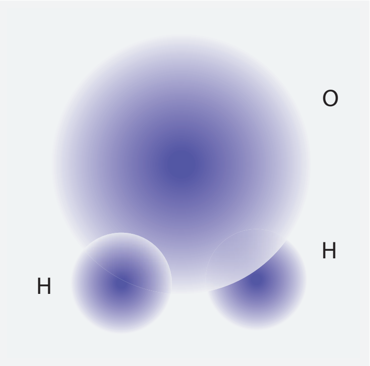

FIGURE 5 An artist's impression of a water molecule made of two hydrogen (H) and one oxygen (O) atom.

FIGURE 6 Max Born (1882–1970)

ues only exist because the quantum of action yields fundamental length scales in nature. Without the quantum of action, there would be *no* shapes in nature.

▷ All shapes are quantum effects.

All shapes in everyday life are due to molecular shapes, or to their generalizations.

The *mass* of an object is also a consequence of the quantum of action, as we will see later on. Since all material properties – such as density, colour, stiffness or polarizability – are defined as combinations of length, time and mass units, we find:

▷ All material properties arise from the quantum of action.

In short, the quantum of action determines the size, shape, colour, mass, and all other properties of objects, from stones to whipped cream.

Why 'quantum'?

Quantum effects surround us on all sides. However, since the quantum of action is so small, its effects on *motion* appear mostly, but not exclusively, in *microscopic* systems. The study of such systems was called *quantum mechanics* by Max Born, one of the major

TABLE 2 Some small systems in motion and the observed action values for their changes.

System and change	Action	Motion
Light		
Smallest amount of light absorbed by a coloured surface	$1\hbar$	quantum
Smallest impact when light reflects from mirror	$2\hbar$	quantum
Smallest consciously visible amount of light	*c.* $5\hbar$	quantum
Smallest amount of light absorbed in flower petal	$1\hbar$	quantum
Blackening of photographic film	*c.* $3\hbar$	quantum
Photographic flash	*c.* $10^{17}\hbar$	classical
Electricity		
Electron ejected from atom or molecule	*c.* $1–2\hbar$	quantum
Electron extracted from metal	*c.* $1–2\hbar$	quantum
Electron motion inside microprocessor	*c.* $2–6\hbar$	quantum
Signal transport in nerves, from one molecule to the next	*c.* $5\hbar$	quantum
Current flow in lightning bolt	*c.* $10^{38}\hbar$	classical
Materials		
Tearing apart two neighbouring iron atoms	*c.* $1–2\hbar$	quantum
Breaking a steel bar	*c.* $10^{35}\hbar$	classical
Basic process in superconductivity	$1\hbar$	quantum
Basic process in transistors	$1\hbar$	quantum
Basic magnetization process	$1\hbar$	quantum
Chemistry		
Atom collision in liquid at room temperature	$1\hbar$	quantum
Shape oscillation of water molecule	*c.* $1-5\hbar$	quantum
Shape change of molecule, e.g. in chemical reaction	*c.* $1-5\hbar$	quantum
Single chemical reaction curling a hair	*c.* $2-6\hbar$	quantum
Tearing apart two mozzarella molecules	*c.* $300\hbar$	quantum
Smelling one molecule	*c.* $10\hbar$	quantum
Burning fuel in a cylinder in an average car engine explosion	*c.* $10^{37}\hbar$	classical
Life		
Air molecule hitting eardrum	*c.* $2\hbar$	quantum
Smallest sound signal detectable by the ear	Challenge 10 ny	
Single DNA duplication step during cell division	*c.* $100\hbar$	quantum
Ovule fertilization	*c.* $10^{14}\hbar$	classical
Smallest step in molecular motor	*c.* $5\hbar$	quantum
Sperm motion by one cell length	*c.* $10^{15}\hbar$	classical
Cell division	*c.* $10^{19}\hbar$	classical
Fruit fly's wing beat	*c.* $10^{24}\hbar$	classical
Person walking one body length	*c.* $2\cdot10^{36}\hbar$	classical
Nuclei and stars		
Nuclear fusion reaction in star	*c.* $1-5\hbar$	quantum
Explosion of gamma-ray burster	*c.* $10^{80}\hbar$	classical

contributors to the field.* Later, the term *quantum theory* became more popular.

Quantum theory arises from the existence of *smallest* measurable values in nature, generalizing the idea that Galileo had in the seventeenth century. As discussed in detail earlier on, it was Galileo's insistence on 'piccolissimi quanti' – smallest quanta – of matter that got him into trouble. We will soon discover that the idea of a smallest change is necessary for a precise and accurate description of matter and of nature as a whole. Therefore Born adopted Galileo's term for the new branch of physics and called it 'Quantentheorie' or 'theory of quanta'. The English language adopted the Latin singular 'quantum' instead of the plural used in most other languages. Vol. I, page 335

Note that the term 'quantum' does *not* imply that all measurement values are *multiples* of a smallest one: this is so only in a few cases.

Quantum theory is the description of microscopic motion. Quantum theory is *necessary* whenever a process produces an action value of the order of the quantum of action. Table 2 shows that all processes on atomic and molecular scales, including biological and chemical processes, are quantum processes. So are processes of light emission and absorption. These phenomena can *only* be described with quantum theory.

Table 2 also shows that the term 'microscopic' has a different meaning for a physicist and for a biologist. For a biologist, a system is 'microscopic' if it requires a microscope for its observation. For a physicist, a system is *microscopic* if its characteristic action is of the order of the quantum of action. In other words, for a physicist a system is usually microscopic if it is *not even* visible in a (light) microscope. To increase the confusion, some quantum physicists nowadays call their own class of microscopic systems 'mesoscopic', while others call their systems 'nanoscopic'. Both terms were introduced only to attract attention and funding: they are conceptually useless.

THE EFFECT OF THE QUANTUM OF ACTION ON MOTION

There is another way to characterize the difference between a microscopic, or quantum, system and a macroscopic, or classical, one. A smallest action implies that the difference between the action values S of two successive observations of the same system, a time Δt apart, cannot vanish. We have

$$|S(t+\Delta t) - S(t)| = |(E \pm \Delta E)(t+\Delta t) - Et| = |E\Delta t \pm t\Delta E \pm \Delta E\Delta t| \geqslant \frac{\hbar}{2} \,. \qquad (1)$$

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* Max Born (b. 1882 Breslau, d. 1970 Göttingen) first studied mathematics, then turned to physics. A professor at Göttingen University, he made the city one of the world centres of physics. He developed quantum mechanics with his assistants Werner Heisenberg and Pascual Jordan, and then applied it to scattering, solid-state physics, optics and liquids. He was the first to understand that the wave function, or state function, describes a probability amplitude. Later, Born and Emil Wolf wrote what is still the main textbook on optics. Many of Born's books were classics and read all over the world. Ref. 6

Born attracted to Göttingen the most brilliant talents of the time, receiving as visitors Hund, Pauli, Nordheim, Oppenheimer, Goeppert-Mayer, Condon, Pauling, Fock, Frenkel, Tamm, Dirac, Mott, Klein, Heitler, London, von Neumann, Teller, Wigner, and dozens of others. Being Jewish, Born lost his job in 1933, when criminals took over the German government. He emigrated, and became professor in Edinburgh, where he stayed for 20 years. Physics at Göttingen never recovered from this loss. For his elucidation of the meaning of the wave function he received the 1954 Nobel Prize in Physics.


FIGURE 7 Werner Heisenberg (1901–1976)

The factor 1/2 arises because a smallest action $\hbar$ automatically implies an action indeterminacy of half its value. Now the values of the energy E and time t – but not of (the positive) ΔE or Δt – can be set to zero if we choose a suitable observer. Thus, the existence of a quantum of action implies that in any system the evolution is constrained by

$$\Delta E \Delta t \geqslant \frac{\hbar}{2} , \qquad (2)$$

where E is the energy of the system and t is its age, so that ΔE is the change of energy and Δt is the time between two successive observations.

Challenge 11 e By a similar reasoning, we find that for any physical system the position and momentum are constrained by

$$\Delta x \Delta p \geqslant \frac{\hbar}{2} , \qquad (3)$$

where Δx is the indeterminacy in position and Δp is the indeterminacy in momentum. These two famous relations were called *indeterminacy relations* by their discoverer, Werner Heisenberg.* In English they are often called 'uncertainty relations'; however, this term is incorrect. The quantities are not uncertain, but *undetermined*. Because of the quantum of action, system observables have *no* definite value. There is no way to ascribe

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* It is often said that the indeterminacy relation for energy and time has a different weight from that for momentum and position. This is a wrong idea, propagated by the older generation of physicists, which has survived through many textbooks for over 70 years. Just forget it. It is essential to remember that all four quantities appearing in the inequalities describe the *internal* properties of the system. In particular, t is a time variable deduced from changes observed *inside* the system, and not the time coordinate measured by an outside clock; similarly, the position x is not the external space coordinate, but the position characterizing the system. Ref. 7

Werner Heisenberg (1901–1976) was an important theoretical physicist and an excellent table-tennis and tennis player. In 1925, as a young man, he developed, with some help from Max Born and Pascual Jordan, the first version of quantum theory; from it he deduced the indeterminacy relations. For these achievements he received the Nobel Prize in Physics in 1932. He also worked on nuclear physics and on turbulence. During the Second World War, he worked on the nuclear-fission programme. After the war, he published several successful books on philosophical questions in physics, slowly turned into a crank, and tried unsuccessfully – with some half-hearted help from Wolfgang Pauli – to find a unified description of nature based on quantum theory, the 'world formula'.

a precise value to momentum, position, or any other observable of a quantum system. We will use the term 'indeterminacy relation' throughout. The habit to call the relation a 'principle' is even more mistaken.

Any system whose indeterminacy is of the order of $\hbar$ is a quantum system; if the indeterminacy product is much larger, the system is classical, and then classical physics is sufficient for its description. So even though classical physics assumes that there are *no* measurement indeterminacies in nature, a system is *classical* only if its indeterminacies are *large* compared to the minimum possible ones!

In other terms, quantum theory is necessary whenever we try to measure some quantity as precisely as possible. In fact, every measurement is itself a quantum process. And the indeterminacy relation implies that measurement precision is limited. The quantum of action shows that

▷ *Motion cannot be observed to infinite precision.*

In other words, the microscopic world is *fuzzy*. This fact has many important consequences and many strange ones. For example, if motion cannot be observed with infinite precision, the very concept of motion needs to be handled with great care, as it cannot be applied in certain situations. In a sense, the rest of our quest is just an exploration of the implications of this result.

In fact, as long as space-time is *flat*, it turns out that we *can* retain the concept of motion to describe observations, provided we remain aware of the limitations implied by the quantum principle.

THE SURPRISES OF THE QUANTUM OF ACTION

The quantum of action $\hbar$ implies a fuzziness of all motion. This fuzziness also implies the existence of short-time deviations from energy, momentum and angular-momentum conservation in microscopic systems. For general assurance it must be stressed that for long observation times – surely for all times longer than a microsecond – conservation holds. But in the first part of our adventure, we realized that any type of non-conservation implies the existence of *surprises* in nature. Well, here are some of them.

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Since precisely uniform motion does not exist, a system moving in one dimension only – such as the hand of a clock – always has the possibility of moving a bit in the opposite direction, thus leading to incorrect readings. Indeed, quantum theory predicts that clocks have essential limitations:

▷ Perfect clocks do not exist.

The deep implications of this statement will become clear step by step.

It is also impossible to avoid that an object makes small displacement sideways. In fact, quantum theory implies that, strictly speaking,

▷ Neither uniform nor one-dimensional motion exists.

Also this statement harbours many additional surprises.

Quantum limitations apply also to metre rules. It is impossible to ensure that the rule is completely at rest with respect to the object being measured. Thus the quantum of action implies again, on the one hand, that measurements are possible, and on the other hand:

▷ Measurement accuracy is limited.

It also follows from the quantum of action that any inertial or freely-falling observer must be *large*, as only large systems approximate inertial motion.

▷ An observer cannot be microscopic.

If humans were not macroscopic, they could neither observe nor study motion.

Because of the finite accuracy with which microscopic motion can be observed, we discover that

▷ *Faster-than-light motion* is possible in the microscopic domain.

Quantum theory thus predicts *tachyons*, at least over short time intervals. For the same reason,

▷ *Motion backwards in time* is possible over microscopic times and distances.

In short, a quantum of action implies the existence of microscopic time travel. However, this remains impossible in the macroscopic domain, such as everyday life.

But there is more. Imagine a moving car suddenly disappearing for good. In such a situation, neither momentum nor energy would be conserved. The action change for such a disappearance is large compared to $\hbar$, so that its observation would contradict even classical physics – as you may wish to check. However, the quantum of action allows a *microscopic* particle, such as an electron, to disappear for a *short* time, provided it reappears afterwards.

Challenge 12 s

▷ The quantum of action implies that there is no permanence in nature.

The quantum of action also implies:

▷ The vacuum is not empty.

If we look at empty space twice, the two observations being separated by a tiny time interval, some energy will be observed the second time. If the time interval is short enough, the quantum of action will lead to the observation of radiation or matter particles. Indeed, particles can appear anywhere from nowhere, and disappear just afterwards: the action limit requires it. In summary, nature exhibits short-term appearance and disappearance of matter and radiation. In other words, the classical idea of an *empty* vacuum is correct only when the vacuum is observed over a *long* time.

The quantum of action implies that compass needles cannot work. If we look twice in

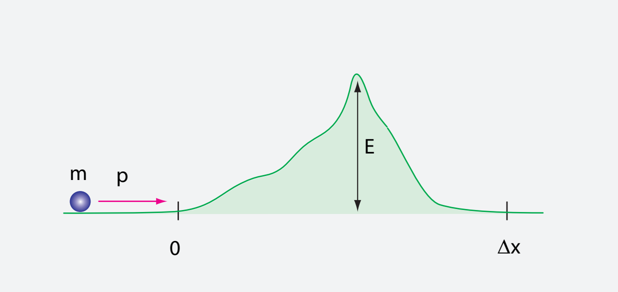

FIGURE 8 Hills are never high enough.

quick succession at a compass needle, or even at a house, we usually observe that it stays oriented in the same direction. But since physical action has the same dimensions as angular momentum, a minimum value for action implies a minimum value for angular momentum. Even a macroscopic object has a minimum value for its rotation. In other words, quantum theory predicts

Challenge 13 e

▷ Everything rotates.

An object can be non-rotating only approximately, when observations are separated by long time intervals.

For microscopic systems, the quantum limits on rotation have specific effects. If the rotation angle can be observed – as for molecules – the system behaves like a macroscopic object: its position and orientation are fuzzy. But for a system whose rotation angle cannot be observed, the quantum of action limits the angular momentum to multiples of $\hbar/2$. In particular, all microscopic bound systems – such as molecules, atoms, or nuclei – contain rotational motion and rotating components.

Transformation, life and Democritus

At the beginning of our adventure, we mentioned that the Greeks distinguished three types of changes: transport, growth, and transformation. We also mentioned that Democritus had deduced that all these types of changes – including life and death – were in fact the same, and due to the motion of atoms. The quantum of action makes exactly this point.

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First of all, a minimum action implies that cages in zoos are dangerous and banks are not safe. A cage is a feature that needs a lot of energy to overcome. Physically speaking, the wall of a cage is an energy hill, resembling the real hill shown in Figure 8. Imagine that a particle with momentum p approaches one side of the hill, which is assumed to have width Δx.

In everyday life – and thus in classical physics – the particle will never be observed on the other side of the hill if its kinetic energy $p^2/2m$ is *less* than the height E of the hill. But imagine that the missing momentum to overcome the hill, $\Delta p = \sqrt{2mE} - p$, satisfies $\Delta x \Delta p \leqslant \hbar/2$. The particle will have the possibility to overcome the hill, despite

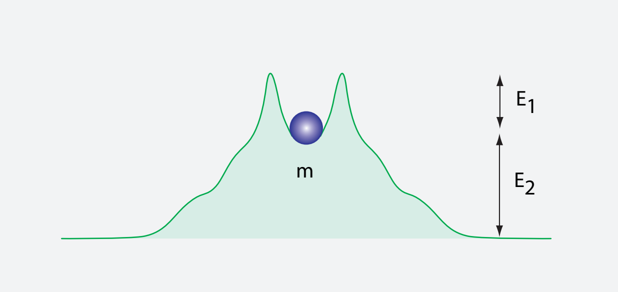

FIGURE 9 Leaving enclosures.

its insufficient energy. The quantum of action thus implies that a hill of width

$$\Delta x \leqslant \frac{\hbar/2}{\sqrt{2mE} - p} \tag{4}$$

is *not* an obstacle to a particle of mass m. But this is not all. Since the value of the particle momentum p is itself undetermined, a particle can overcome the hill even if the hill is *wider* than the value (4) – although the broader it is, the lower the probability will be. So any particle can overcome *any* obstacle. This is called the *tunnelling effect*, for obvious reasons. Classically, tunnelling is impossible. In quantum theory, the feat is possible, because the wave function does not vanish at the location of the hill; sloppily speaking, the wave function is non-zero inside the hill. It thus will be also non-zero behind the hill. As a result, quantum systems can penetrate or 'tunnel' through hills.

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In short, the minimum-action principle implies that there are no tight boxes in nature. Thanks to the tunnelling effect,

▷ Matter is not impenetrable.

The penetrability of all matter is in contrast to everyday, classical observation. Can you explain why lion cages work *despite* the quantum of action?

Challenge 14 s

By the way, the quantum of action also implies that a particle with a kinetic energy greater than the energy height of a hill can be *reflected* by the hill. Also this effect is impossible in classical physics.

The minimum-action principle also implies that bookshelves are dangerous. Why? Shelves are obstacles to motion. A book on a shelf is in the same situation as the mass in Figure 9: the mass is surrounded by energy hills hindering its escape to the outer, lower-energy world. But thanks to the tunnelling effect, escape is always possible. The same picture applies to a branch of a tree, a nail in a wall, or anything attached to anything else. Things can never be permanently fixed together. In particular, we will discover that every example of light emission – even radioactivity – results from this effect.

In summary, the quantum of action thus implies that

▷ Decay is part of nature.

Note that decay often appears in everyday life, under a different name: *breaking*. In fact,

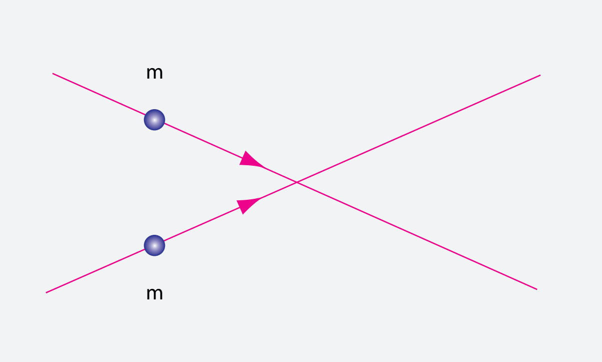

FIGURE 10 Identical objects with crossing paths.

Ref. 8 all breakages require the quantum of action for their description. Obviously, the cause of breaking is often classical, but the *mechanism* of breaking is always quantum. Only objects that obey quantum theory can break. In short, there are no stable excited systems in nature. For the same reason, by the way, no memory can be perfect. (Can you confirm this?) Challenge 15 s

Taking a more general view, *ageing* and *death* also result from the quantum of action. Death, like ageing, is a composition of breaking processes. When dying, the mechanisms in a living being break. Breaking is a form of decay, and is due to tunnelling. Death is thus a quantum process. Classically, death does not exist. Might this be the reason why so many people believe in immortality or eternal youth? Challenge 16 s

We will also discover that the quantum of action is the reason for the importance of the action observable in classical physics. In fact, the existence of a *smallest* action is the reason for the *least*-action principle of classical physics.

A minimum action also implies that matter cannot be continuous, but must be composed of smallest entities. Indeed, any flow of a truly continuous material would contradict the quantum principle. Can you give the precise argument? Challenge 17 s Of course, at this point in our adventure, the non-continuity of matter is no longer a surprise. But the quantum of action implies that even *radiation* cannot be continuous. As Albert Einstein was the first to state clearly, light is made of quantum particles.

Even more generally, the quantum of action implies that in nature

▷ All flows and all waves are made of microscopic particles.

The term 'microscopic' (or 'quantum') is essential, as such particles do *not* behave like little stones. We have already encountered several differences, and we will encounter others shortly. For these reasons, there should be a special name for microscopic particles; but so far all proposals, of which *quanton* is the most popular, have failed to catch on.

The quantum of action has several strange consequences for microscopic particles. Take two such particles with the same mass and composition. Imagine that their paths cross, and that at the crossing they approach each other very closely, as shown in Figure 10. A minimum action implies that in such a situation, if the distance becomes small enough, the two particles can switch roles, without anybody being able to avoid, or notice, it. Thus, in a volume of gas it is *impossible* – thanks to the quantum of action – to

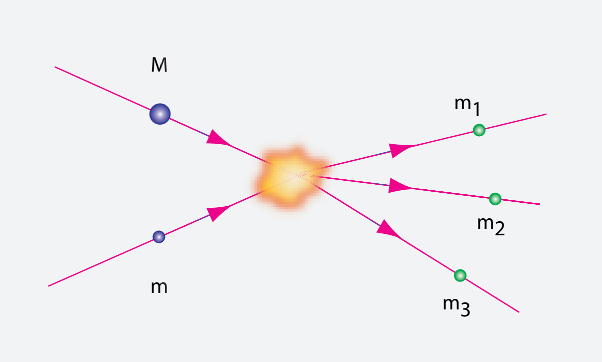

FIGURE 11 Transformation through reaction.

follow particles moving around and to say which particle is which. Can you confirm this deduction, and specify the conditions, using the indeterminacy relations? In summary

Challenge 18 s

▷ In nature it is impossible to distinguish between identical particles.

Challenge 19 s

Can you guess what happens in the case of light?

But matter deserves still more attention. Imagine again two particles – even two different ones – approaching each other very closely, as shown in Figure 11. We know that if the approach distance gets small, things get fuzzy. Now, the minimum-action principle makes it possible for something to happen in that small domain as long as resulting outgoing products have the same total linear momentum, angular momentum and energy as the incoming ones. Indeed, ruling out such processes would imply that arbitrarily small actions could be observed, thus eliminating nature's fuzziness, as you may wish to check for yourself. In short,

Challenge 20 e

▷ The quantum of action allows transformation of matter.

One also says that the quantum of action allows particle *reactions*. In fact, we will discover that *all* kinds of reactions in nature, including breathing, digestion, and all other chemical and nuclear reactions, are due just to the existence of the quantum of action.

One type of process that is especially dear to us is *growth*. The quantum of action implies that all growth happens in small steps. Indeed,

▷ All growth processes in nature are quantum processes.

Above all, as mentioned already, the quantum of action explains life. Only the quantum of action makes reproduction and heredity possible. Birth, sexuality and death are consequences of the quantum of action.

So Democritus was both right and wrong. He was right in deducing fundamental constituents for matter and radiation. He was right in unifying all change in nature – from transport to transformation and growth – as motion of particles. But he was wrong in assuming that the small particles behave like stones. As we will show in the following, the smallest particles behave like *quantons*: they behave randomly, and they behave partly as waves and partly as particles.

RANDOMNESS – A CONSEQUENCE OF THE QUANTUM OF ACTION

What happens if we try to measure a change smaller than the quantum of action? Nature has a simple answer: we get *random results*. If we build an experiment that tries to produce a change or action of the size of a quarter of the quantum of action, the experiment will produce, for example, a change of *one* quantum of action in a quarter of the cases, and *no* change in three quarters of the cases,* thus giving an *average* of one quarter of $\hbar$.

▷ Attempts to measure actions below $\hbar$ lead to random results.

If you want to condense quantum physics in one key statement, this is it.

The quantum of action leads to randomness at microscopic level. This connection can be seen also in the following way. Because of the indeterminacy relations, it is impossible to obtain definite values for both the momentum and the position of a particle. Obviously, definite values are also impossible for the individual components of an experimental set-up or an observer. Therefore, initial conditions – both for a system and for an experimental set-up – cannot be exactly duplicated. The quantum of action thus implies that whenever an experiment on a microscopic system is performed twice, the outcomes will (usually) be different. The outcomes could only be the same if both the system and the observer were in exactly the same configuration each time. However, because of the second principle of thermodynamics, and because of the quantum of action, reproducing a configuration is impossible. Therefore,

▷ Microscopic systems behave randomly.

Obviously, there will be some *average* outcome; but in all cases, microscopic observations are *probabilistic*. Many find this conclusion of quantum theory the most difficult to swallow. But fact is: the quantum of action implies that the behaviour of quantum systems is strikingly different from that of classical systems. The conclusion is unavoidable:

▷ Nature behaves randomly.

Can we observe randomness in everyday life? Yes. Every *window* proves that nature behaves randomly on a microscopic scale. Everybody knows that we can use a train window either to look at the *outside* landscape or, by concentrating on the reflected image, to observe some interesting person *inside* the carriage. In other words, observations like that of Figure 12 show that glass reflects some of the light particles and lets some others pass through. More precisely, glass reflects a random selection of light particles; yet the average proportion is constant. In these properties, partial reflection is similar to the tunnelling effect. Indeed, the partial reflection of photons in glass is a result of the quantum of action. Again, the situation can be *described* by classical physics, but the precise amount of reflection cannot be *explained* without quantum theory. We retain:

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* In this context, 'no change' means 'no change' in the physical variable to be measured; generally speaking, there is always some change, but not necessarily in the variable being measured.

FIGURE 12 A famous quantum effect: how do train windows manage to show two superimposed images? (Photo © Greta Mansour)

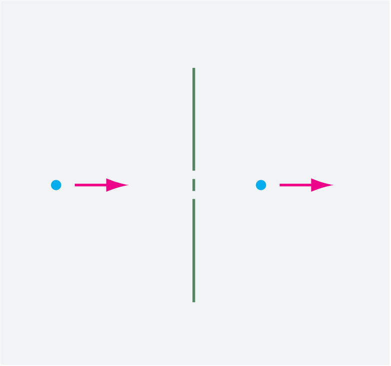

FIGURE 13 A particle and a screen with two nearby slits.

▷ Quantons move randomly.

Without the quantum of action, train journeys would be much more boring.

WAVES – A CONSEQUENCE OF THE QUANTUM OF ACTION

The quantum of action implies an important result about the *paths* of particles. If a particle travels from one point to another, there is no way to say which path it has taken in between. Indeed, in order to distinguish between two possible, but slightly different, paths, actions smaller than $\hbar$ would have to be measured reliably. In particular, if a particle is sent through a screen with two sufficiently close slits, as illustrated in Figure 13, it is impossible to say which slit the particle passed through. This impossibility is fundamental.

We already know phenomena of motion for which it is not possible to say with precision how something moves or which path is taken behind two slits: *waves* behave in this way. All waves are subject to the indeterminacy relations

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$$\Delta\omega\Delta t \geqslant \frac{1}{2} \quad \text{and} \quad \Delta k\Delta x \geqslant \frac{1}{2} . \tag{5}$$

A wave is a type of motion described by a *phase* that changes over space and time. This turns out to hold for all motion. In particular, this holds for matter.

We saw above that quantum systems are subject to

$$\Delta E \Delta t \geqslant \frac{\hbar}{2} \quad \text{and} \quad \Delta p \Delta x \geqslant \frac{\hbar}{2} . \tag{6}$$

We are thus led to ascribe a frequency and a wavelength to a quantum system:

$$E = \hbar\omega \quad \text{and} \quad p = \hbar k = \hbar\frac{2\pi}{\lambda} . \tag{7}$$

The energy–frequency relation for *light* and the equivalent momentum–wavelength relation were deduced by Max Planck in 1899. In the years from 1905 onwards, Albert Einstein confirmed that the relations are valid for all examples of emission and absorption of light. In 1923 and 1924, Louis de Broglie* predicted that the relation should hold also for all quantum *matter* particles. The experimental confirmation came a few years later. (This is thus another example of a discovery that was made about 20 years too late.) In short, the quantum of action implies:

Page 76

▷ Matter particles behave like waves.

In particular, the quantum of action implies the existence of *interference* for streams of matter.

PARTICLES – A CONSEQUENCE OF THE QUANTUM OF ACTION

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The quantum of action, the smallest change, implies that flows cannot be arbitrarily weak. This applies to *all* flows: in particular, it applies to rivers, solid matter flows, gas flows, light beams, energy flows, entropy flows, momentum flows, angular momentum flows, probability flows, signals of all kind, electrical charge flows, colour charge flows and weak charge flows.

Water flows in rivers, like any other matter flow, cannot be arbitrary small: the quantum of action implies that there is a smallest matter flow in nature. Depending on the situation, the smallest matter flow is a molecule, an atom or a smaller particle. Indeed, the quantum of action is also at the origin of the observation of a smallest charge in electric current. Since all matter can flow, the quantum of action implies:

▷ All matter has particle aspects.

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* Louis de Broglie (b. 1892 Dieppe, d. 1987 Paris), physicist and professor at the Sorbonne. The energy–frequency relation for light had earned Max Planck and Albert Einstein the Nobel Prize in Physics, in 1918 and 1921. De Broglie expanded the relation to predict the wave nature of the electron (and of all other quantum matter particles): this was the essence of his doctoral thesis. The prediction was first confirmed experimentally a few years later, in 1927. For the prediction of the wave nature of matter, de Broglie received the Nobel Prize in Physics in 1929. Being an aristocrat, he did no more research after that. For example, it was Schrödinger who then wrote down the wave equation, even though de Broglie could equally have done so.

In the same way, the quantum of action, the smallest change, implies that light cannot be arbitrarily faint. There is a smallest illumination in nature; it is called a *photon* or a *light quantum*. Now, light is a wave, and the argument can be made for any other wave as well. In short, the quantum of action thus implies:

▷ All waves have particle aspects.

This has been proved for light waves, water waves, X-rays, sound waves, plasma waves, fluid whirls and any other wave type that has ever been observed. There is one exception: gravitational waves have finally been observed in 2016, many decades after their prediction; it is expected that their particle-like aspects, the *gravitons*, also exist, though this might take a long time to prove by experiment.

In summary, the quantum of action states:

▷ If something moves, it is made of quantum particles, or quantons.

Later on we will explore and specify the exact differences between a quantum particle and a small stone or a grain of sand. We will discover that matter quantons move differently, behave differently under rotation, and behave differently under exchange.

QUANTUM INFORMATION

In computer science, the smallest unit of change is called a 'bit change'. The existence of a smallest change in nature implies that computer science – or information science – can be used to describe nature, and in particular quantum theory. This analogy has attracted much research in the past decades, and explored many interesting questions: Is unlimited information storage possible? Can information be read out and copied completely? Can information be transmitted while keeping it secret? Can information transmission and storage be performed independently of noise? Can quantum physics be used to make new types of computers? So far, the answer to all these questions is negative; but the hope to change the situation is not dead yet.

The analogy between quantum theory and information science is limited: information science can describe only the 'software' side of devices. For a physicist, the 'hardware' side of nature is central. The hardware of nature enters the description whenever the actual value $\hbar$ of the quantum of action must be introduced.

As we explore the similarities and differences between nature and information science, we will discover that the quantum of action implies that macroscopic physical systems cannot be copied – or 'cloned', as quantum theorists like to say. Nature does not allow copies of macroscopic objects. In other words:

▷ Perfect copying machines do not exist.

The quantum of action makes it impossible to gather and use all information in a way that allows production of a perfect copy.

The exploration of copying machines will remind us again that the precise order in which measurements are performed in an experiment matters. When the order of measurements can be reversed without affecting the net result, physicists speak of 'commutation'. The quantum of action implies:

▷ Physical observables do not commute.

Page 152 We will also find that the quantum of action implies that systems are not always independent, but can be *entangled*. This term, introduced by Erwin Schrödinger, describes one of the most absurd consequences of quantum theory. Entanglement makes everything in nature connected to everything else. Entanglement produces effects that *seem* (but are not) faster than light.

▷ Entanglement produces a (fake) form of non-locality.

Ref. 9 Entanglement implies that trustworthy communication cannot exist.

We will also discover that *decoherence* is an ubiquitous process in nature that influences all quantum systems. For example, it allows measurements on the one hand and makes quantum computers impossible on the other. Page 157

CURIOSITIES AND FUN CHALLENGES ABOUT THE QUANTUM OF ACTION

Even if we accept that no experiment performed so far contradicts the minimum action, we still have to check that the minimum action does not contradict reason. In particular, the minimum action must also be consistent with all *imagined* experiments. This is not self-evident.

* *

Challenge 21 s Where is the quantum scale in a pendulum clock?

* *

When electromagnetic fields come into play, the value of the action (usually) depends on the choice of the vector potential, and thus on the choice of gauge. We saw in the part on electrodynamics that a suitable choice of gauge can change the value of the action by adding or subtracting any desired amount. Nevertheless, there is a smallest action in nature. This is possible, because in quantum theory, physical gauge changes cannot add or subtract any amount, but only multiples of *twice* the minimum value. Thus they do not allow us to go below the minimum action. Vol. III, page 85

* *

Adult plants stop growing in the dark. Without light, the reactions necessary for growth cease. Can you show that this is a quantum effect, not explainable by classical physics? Challenge 22 s

* *

Most quantum processes in everyday life are electromagnetic. Can you show that the quantum of action must also hold for nuclear processes, i.e., for processes that are not electromagnetic? Challenge 23 s

* *

Challenge 24 s Is the quantum of action independent of the observer, even near the speed of light? This question was the reason why Planck contacted the young Einstein, inviting him to Berlin, thus introducing him to the international physics community.

* *

The quantum of action implies that tiny people, such as *Tom Thumb*, cannot exist. The quantum of action implies that fractals cannot exist in nature. The quantum of action implies that ‘*Moore’s law*’ of semiconductor electronics, which states that the number of transistors on a chip doubles every two years, cannot be valid for ever. Why not? Challenge 25 s

* *

Take a horseshoe. The distance between the two ends is not fixed, since otherwise their position and velocity would be known at the same time, contradicting the indeterminacy relation. Of course, this reasoning is also valid for any other solid object. In short, both quantum mechanics and special relativity show that rigid bodies do not exist, albeit for different reasons.

* *

Angular momentum has the same dimensions as action. A smallest action implies that there is a smallest angular momentum in nature. How can this be, given that some particles have spin zero, i.e., have no angular momentum? Challenge 26 s

* *

Could we have started the whole discussion of quantum theory by stating that there is a minimum angular momentum instead of a minimum action? Challenge 27 s

* *

Niels Bohr, besides propagating the idea of a minimum action, was also an enthusiast of the so-called *complementarity principle*. This is the idea that certain pairs of observables of a system – such as position and momentum – have linked *precision*: if one observable of the pair is known to high precision, the other observable is necessarily known with low precision. Can you deduce this principle from the minimum action? Challenge 28 s

The dangers of buying a can of beans

Another way to show the absurd consequences of quantum theory is given by the ultimate product warning, which according to certain well-informed lawyers should be printed on every can of beans and on every product package. Ref. 10 It shows in detail how deeply our human condition fools us.

Warning: care should be taken when **looking** at this product:

- It emits heat radiation.
- Bright light has the effect to compress this product.

Warning: care should be taken when **touching** this product:

- Part of it could heat up while another part cools down, causing severe burns.

Warning: care should be taken when **handling** this product:

- This product consists of at least 99.999 999 999 999 % empty space.
- This product contains particles moving with speeds higher than one million kilometres per hour.
- Every kilogram of this product contains the same amount of energy as liberated by about one hundred nuclear bombs.*
- In case this product is brought in contact with antimatter, a catastrophic explosion will occur.
- In case this product is rotated, it will emit gravitational radiation.

Warning: care should be taken when **transporting** this product:

- The force needed depends on its velocity, as does its weight.
- This product will emit additional radiation when accelerated.
- This product attracts, with a force that increases with decreasing distance, every other object around, including its purchaser's kids.

Warning: care should be taken when **storing** this product:

- It is impossible to keep this product in a specific place and at rest at the same time.
- Except when stored underground at a depth of several kilometres, over time cosmic radiation will render this product radioactive.
- This product may disintegrate in the next 10^{35} years.
- It could cool down and lift itself into the air.
- This product warps space and time in its vicinity, including the storage container.
- Even if stored in a closed container, this product is influenced and influences all other objects in the universe, including your parents in law.
- This product can disappear from its present location and reappear at any random place in the universe, including your neighbour's garage.

Warning: care should be taken when **travelling away from** this product:

- It will arrive at the expiration date before the purchaser does so.

Warning: care should be taken when **using** this product:

- Any use whatsoever will increase the entropy of the universe.
- The constituents of this product are exactly the same as those of any other object in the universe, including those of rotten fish.

All these statements are correct. The impression of a certain paranoid side to quantum physics is purely coincidental.

---

* A standard nuclear warhead has an explosive yield of about 0.2 megatons (implied is the standard explosive trinitrotoluene or TNT), about thirteen times the yield of the Hiroshima bomb, which was 15 kilotonne. A megatonne is defined as 1 Pcal=4.2 PJ, even though TNT delivers about 5 % slightly less energy than this value. In other words, a megaton is the energy content of about 47 g of matter. That is less than a handful for most solids or liquids. Ref. 11

A summary: quantum physics, the law and indoctrination

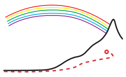
The mere existence of a quantum of action, a quantum of change, has many deep consequences: randomness, wave-particle duality, matter transformation, death, and, above all, new thinking habits.

Don't all the deductions from the quantum of action presented so far look wrong, or at least crazy? In fact, if you or your lawyer made some of the statements on quantum physics in court, maybe even under oath, you might end up in prison! However, all the above statements are correct: they are all confirmed by experiment. And there are many more surprises to come. You may have noticed that, in the preceding examples, we have made no explicit reference to electricity, to the nuclear interactions or to gravity. In these domains the surprises are even more astonishing. Observation of antimatter, electric current without resistance, the motion inside muscles, vacuum energy, nuclear reactions in stars, and – maybe one day – the boiling of empty space, will fascinate you as much as they have fascinated, and still fascinate, thousands of researchers.

Challenge 29 d In particular, the consequences of the quantum of action for the early universe are mind-boggling. Just try to explore for yourself its consequences for the big bang. Together, all these topics will lead us a long way towards the aim of our adventure. The consequences of the quantum of action are so strange, so incredible, and so numerous, that quantum physics can rightly be called the description of motion for *crazy* scientists. In a sense, this generalizes our previous definition of quantum physics as the description of motion related to pleasure.

Page 167 Unfortunately, it is sometimes claimed that 'nobody understands quantum theory'. This is wrong. In fact, it is worse than wrong: it is indoctrination and disinformation. Indoctrination and disinformation are methods that prevent people from making up their own mind and from enjoying life. In reality, the consequences of the quantum of action can be understood and enjoyed by everybody. In order to do so, our first task on our way towards completing our adventure will be to use the quantum of action to study our classical standard of motion: the motion of light.

> “Nie und nirgends hat es Materie ohne
> Bewegung gegeben, oder kann es sie geben.”
> Friedrich Engels, *Anti-Dühring*.*



Ref. 12 * 'Never and nowhere has matter existed, nor can it exist, without motion.' Friedrich Engels (1820–1895) was one of the theoreticians of Marxism.

Chapter 2

LIGHT – THE STRANGE CONSEQUENCES OF THE QUANTUM OF ACTION

> “Alle Wesen leben vom Lichte,
> jedes glückliche Geschöpfe.”
> Friedrich Schiller, *Wilhelm Tell*.**

Since all the colours of materials are quantum effects, it becomes mandatory to study the properties of light itself. If a smallest change really exists, then there should also be a smallest illumination in nature. This conclusion was already drawn Ref. 13 in ancient Greece, for example by Epicurus (341–271 BCE), who stated that light is a stream of little particles. The smallest possible illumination would then be that due to a single light particle. Today, the particles are called *light quanta* or *photons*. Incredibly, Epicurus himself could have checked his prediction with an experiment.

How do faint lamps behave?

Ref. 14 Around 1930, Brumberg and Vavilov found a beautiful way to check the existence of photons using the naked eye and a lamp. Our eyes do not allow us to *consciously* detect single photons, but Brumberg and Vavilov found a way to circumvent this limitation. In fact, the experiment is so simple that it could have been performed many centuries earlier; but nobody had had a sufficiently daring imagination to try it.

Brumberg and Vavilov constructed a mechanical shutter that could be opened for time intervals of 0.1 s. From the other side, in a completely dark room, they illuminated the opening with extremely weak green light: about 200 aW at 505 nm, as shown in Figure 14. At that intensity, whenever the shutter opens, on average about 50 photons can pass. This is just the sensitivity threshold of the eye. To perform the experiment, they repeatedly looked into the open shutter. The result was simple but surprising. Sometimes they observed light, and sometimes they did not. Whether they did or did not was completely random. Brumberg and Vavilov gave the simple explanation that at low lamp powers, because of fluctuations, the number of photons is above the eye threshold half the time, and below it the other half. The fluctuations are random, and so the conscious detection of light is as well. This would *not* happen if light were a *continuous* stream: in that case, the eye would detect light at each and every opening of the shutter. (At higher light intensities, the percentage of non-observations quickly decreases, in accordance with the explanation given.)

In short, a simple experiment proves:

** ‘From light all beings live, each fair-created thing.’ Friedrich Schiller (b. 1759 Marbach, d. 1805 Weimar), poet, playwright and historian.

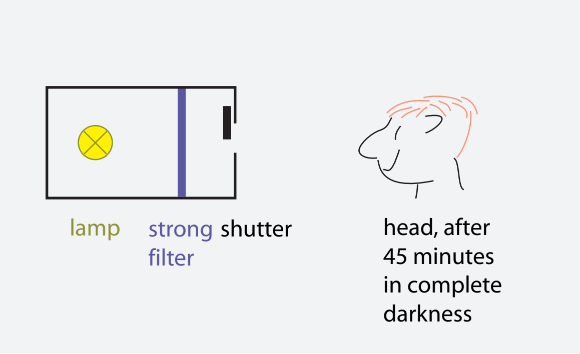


FIGURE 14 How to experience single photon effects (see text).

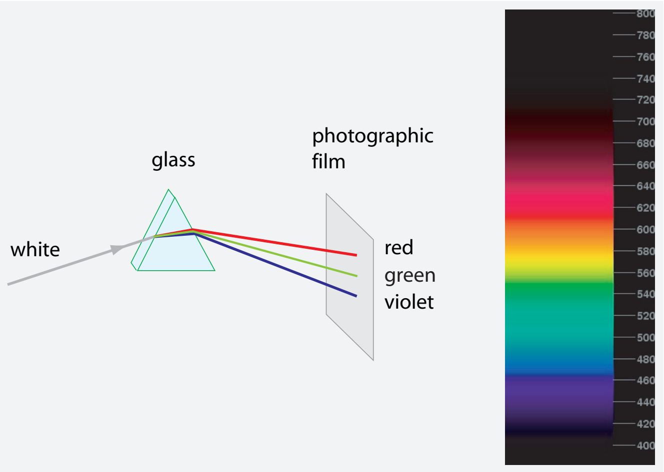


FIGURE 15 How does a white-light spectrum appear at extremely long screen distances? (The short-screen-distance spectrum shown, © Andrew Young, is optimized for CRT display, not for colour printing, as explained on mintaka.sdsu.edu/GF/explain/optics/rendering.html.)

▷ Light is made of photons.

Nobody knows how the theory of light would have developed if this simple experiment had been performed 100 or even 2500 years earlier.

The reality of photons becomes more convincing if we use devices to help us. A simple way is to start with a screen behind a prism illuminated with white light, as shown in Figure 15. The light is split into colours. As the screen is placed further and further away, the illumination intensity cannot become arbitrarily small, as that would contradict the quantum of action. To check this prediction, we only need some black-and-white photographic film. Film is blackened by daylight of any colour; it becomes dark grey at medium intensities and light grey at lower intensities. Looking at an extremely light grey film under the microscope, we discover that, even under uniform illumination, the grey shade is actually composed of black spots, arranged more or less densely. All these spots have the same size, as shown in Figure 16. This regular size suggests that a photographic film reacts to single photons. Detailed research confirms this conjecture; in the twentieth century, the producers of photographic films have elucidated the underlying atomic mechanism in all its details.

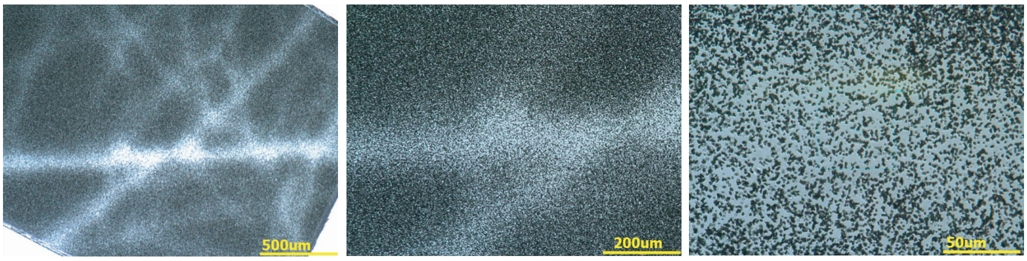


FIGURE 16 Exposed photographic film at increasing magnification (© Rich Evans).

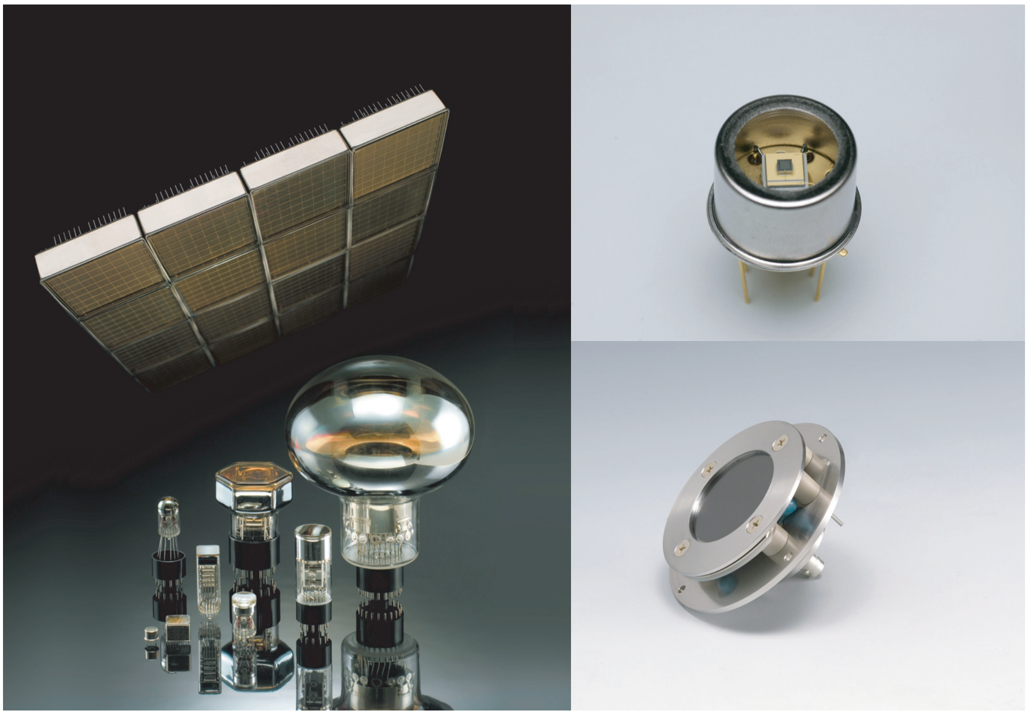


FIGURE 17 Detectors that allow photon counting: photomultiplier tubes (left), an avalanche photodiode (top right, *c.* 1 cm) and a microchannel plate (bottom right, *c.* 10 cm) (© Hamamatsu Photonics).

Ref. 15

Single photons can be detected most elegantly with electronic devices. Such devices can be photomultipliers, photodiodes, microchannel plates or rod cells in the eye; a selection is shown in Figure 17. Also these detectors show that low-intensity light does *not* produce a homogeneous colour: on the contrary, low-intensity produces a random pattern of equal spots, even when observing typical wave phenomena such as interference patterns, as shown in Figure 18. Today, recording and counting individual photons is a standard experimental procedure. Photon counters are part of many spectroscopy set-ups, such as those used to measure tiny concentrations of materials. For example, they are used to detect drugs in human hair.

All experiments thus show the same result: whenever sensitive light detectors are constructed with the aim of 'seeing' as accurately as possible – and thus in environments as

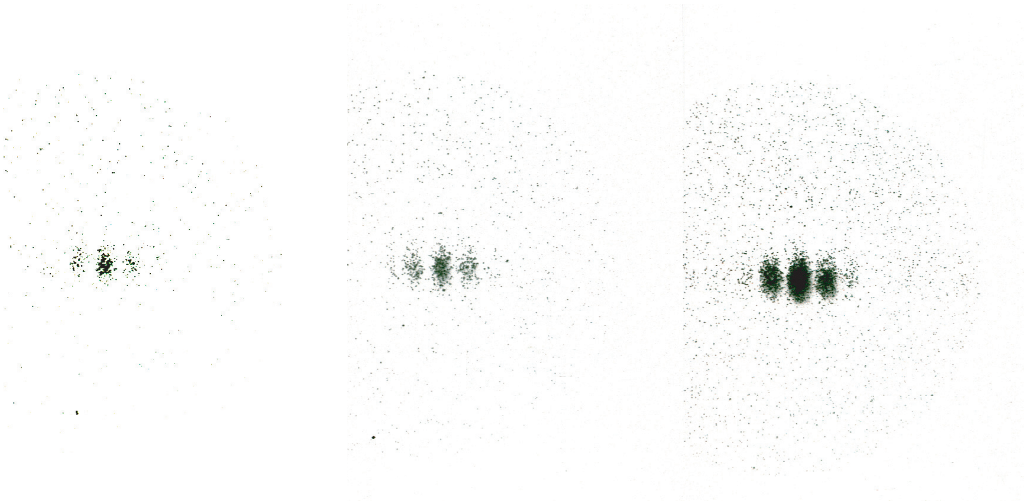


FIGURE 18 Light waves are made of particles: observation of photons – black spots in these negatives – in a low intensity double slit experiment, with exposure times of 1, 2 and 5 s, using an image intensifier (© Delft University of Technology).

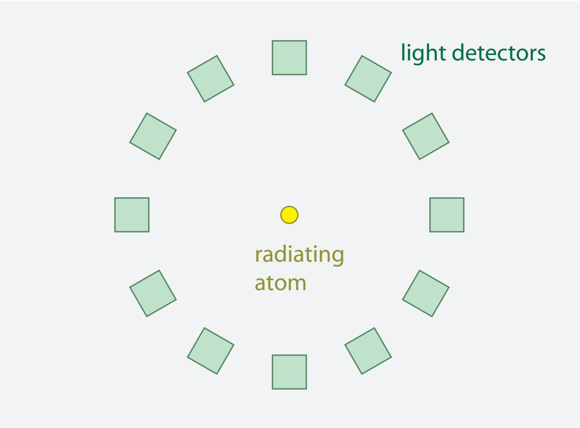


FIGURE 19 An atom radiating *one* photon triggers only *one* detector and recoils in only *one* direction.

dark as possible – one finds that light manifests as a stream of *light quanta*. Nowadays they are usually called *photons*, a term that appeared in 1926. Light of low or high intensity corresponds to a stream with a small or large number of photons.

A particularly interesting example of a low-intensity source of light is a single atom. Atoms are tiny spheres. When atoms radiate light or X-rays, the radiation should be emitted as a spherical wave. But in all experiments – see Figure 19 for a typical set-up – the light emitted by an atom is *never* found to form a spherical wave, in contrast to what we might expect from everyday physics. Whenever a radiating atom is surrounded by many detectors, only a *single* detector is triggered. Only the average over many emissions and detections yields a spherical shape. The experiments shows clearly that partial photons cannot be detected.

All experiments in dim light thus show that the continuum description of light is incorrect. All such experiments thus prove directly that light is a stream of particles, as Epicurus had proposed in ancient Greece. More precise measurements confirm the role of the quantum of action: every photon leads to the *same* amount of change. All photons of the same frequency blacken a film or trigger a scintillation screen in the same way. In short, the amount of change induced by a single photon is indeed the *smallest* amount of change that light can produce.

If there were no smallest action value, light could be packaged into arbitrarily small amounts. But nature is different. In simple terms: the classical description of light by a *continuous* vector potential $A(t,x)$, or electromagnetic field $F(t,x)$, whose evolution is described by a principle of least action, is *wrong*. Continuous functions do not describe the observed particle effects. A modified description is required. The modification has to be significant only at low light intensities, since at high, everyday intensities the classical Lagrangian describes all experimental observations with sufficient accuracy.*

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At which intensities does light cease to behave as a continuous wave? Human eyesight does not allow us to consciously distinguish single photons, although experiments show that the hardware of the eye is in principle able to do so. The faintest stars that can be seen at night produce a light intensity of about 0.6 nW/m^2. Since the pupil of the eye is small, and we are not able to see individual photons, photons must have energies smaller than 100 aJ. Brumberg and Vavilov's experiment yields an upper limit of around 20 aJ.

Ref. 16

An exact value for the quantum of action found in light must be deduced from laboratory experiment. Some examples are given in the following.

PHOTONS

In general, all experiments show that a beam of light of frequency f or angular frequency ω, which determines its colour, is accurately described as a stream of photons, each with the same energy E given by

$$E = \hbar\, 2\pi f = \hbar\,\omega\,. \tag{8}$$

This relation was first deduced by Max Planck in 1899. He found that for light, the smallest measurable action is given by the quantum of action $\hbar$. In short, *colour* is a property of photons. A coloured light beam is a hailstorm of corresponding photons.

The value of *Planck's constant* can be determined from measurements of black bodies or other light sources. All such measurements coincide and yield

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$$\hbar = 1.054\,571\,726(47) \cdot 10^{-34}\,\mathrm{Js}\,, \tag{9}$$

a value so small that we can understand why photons go unnoticed by humans. For example, a green photon with a wavelength of 555 nm has an energy of 0.37 aJ. Indeed, in normal light conditions the photons are so numerous that the continuum approximation for the electromagnetic field is highly accurate. In the dark, the insensitivity of the signal processing of the human eye – in particular the slowness of the light receptors – makes photon counting impossible. However, the eye is not far from the maximum possible sensitivity. From the numbers given above about dim stars, we can estimate that humans

Challenge 30 e
Ref. 16
Challenge 31 ny

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* The transition from the classical case to the quantum case used to be called *quantization*. This concept, and the ideas behind it, are only of historical interest today.

are able to see *consciously*, under ideal conditions, flashes of about half a dozen photons; in normal conditions, the numbers are about ten times higher.

Let us explore the other properties of photons. Above all, photons have no measurable (rest) mass and no measurable electric charge. Can you confirm this? In fact, experiments can only provide an upper limit for both quantities. The present experimental upper limit for the (rest) mass of a photon is 10^{-52} kg, and for the charge is $5 \cdot 10^{-30}$ times the electron charge. These limits are so small that we can safely say that both the mass and the charge of the photon vanish.

Challenge 32 s

Ref. 17

We know that intense light can *push* objects. Since the energy, the lack of mass and the speed of photons are known, we deduce that the photon momentum is given by

Challenge 33 e

$$p = \frac{E}{c} = \hbar \frac{2\pi}{\lambda} \quad \text{or} \quad \boldsymbol{p} = \hbar \boldsymbol{k} . \tag{10}$$

In other words, if light is made of particles, we should be able to play billiard with them. This is indeed possible, as Arthur Compton showed in a famous experiment in 1923. He directed X-rays, which are high-energy photons, onto graphite, a material in which electrons move almost freely. He found that whenever the electrons in the material are hit by the X-ray photons, the deflected X-rays change colour. His experiment is shown in Figure 20. As expected, the strength of the hit is related to the deflection angle of the photon. From the colour change and the deflection angle, Compton confirmed that the photon momentum indeed satisfies the expression $\boldsymbol{p} = \hbar \boldsymbol{k}$.

Ref. 18

All other experiments agree that photons have momentum. For example, when an atom *emits* light, the atom feels a *recoil*. The momentum again turns out to be given by the expression $\boldsymbol{p} = \hbar \boldsymbol{k}$. In short, the quantum of action determines the momentum of the photon.

The value of a photon's momentum respects the indeterminacy relation. Just as it is impossible to measure exactly both the wavelength of a wave and the position of its crest, so it is impossible to measure both the momentum and the position of a photon. Can you confirm this? In other words, the value of the photon momentum is a direct consequence of the quantum of action.

Challenge 34 s

From our study of classical physics, we know that light has a property beyond its colour: light can be *polarized*. That is only a complicated way to say that light can *turn* the objects that it shines on. In other words, light has an angular momentum oriented (mainly) along the axis of propagation. What about photons? Measurements consistently find that each light quantum carries an *angular momentum* given by $L = \hbar$. It is called its *helicity*. The quantity is similar to one found for massive particles: one therefore also speaks of the *spin* of a photon. In short, photons somehow 'turn' – in a direction either parallel or antiparallel to their direction of motion. Again, the magnitude of the photon helicity, or spin, is no surprise; it confirms the classical relation $L = E/\omega$ between energy and angular momentum that we found in the section on classical electrodynamics. Note that, counterintuitively, the angular momentum of a single photon is fixed, and thus independent of its energy. Even the most energetic photons have $L = \hbar$. Of course, the value of the helicity also respects the limit given by the quantum of action. The many consequences of the helicity (spin) value $\hbar$ will become clear in the following.

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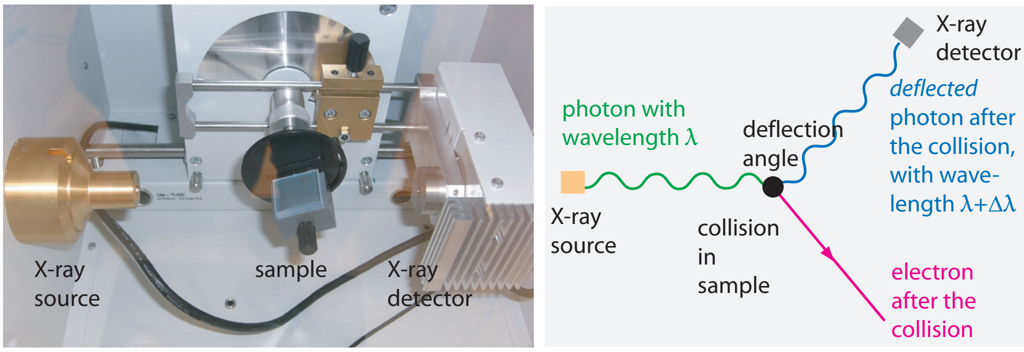

FIGURE 20 A modern version of Compton's experiment fits on a table. The experiment shows that photons have momentum: X-rays – and thus the photons they consist of – change frequency when they hit the electrons in matter in exactly the same way as predicted from colliding particles (© Helene Hoffmann).

WHAT IS LIGHT?

> “La lumière est un mouvement luminaire de corps lumineux.”
>
> Blaise Pascal*

In the seventeenth century, Blaise Pascal used the above statement about light to make fun of certain physicists, ridiculing the blatant use of a circular definition. Of course, he was right: in his time, the definition was indeed circular, as no meaning could be given to any of the terms. But whenever physicists study an observation with care, philosophers lose out. All those originally undefined terms now have a definite meaning and the circular definition is resolved. Light is indeed a type of motion; this motion can rightly be called 'luminary' because, in contrast to the motion of material bodies, it has the unique property $v = c$; the luminous bodies, called *light quanta* or *photons*, are characterized, and differentiated from all other particles, by their dispersion relation $E = cp$, their energy $E = \hbar\omega$, their spin $L = \hbar$, the vanishing of all other quantum numbers, and the property of being the quanta of the electromagnetic field.

In short, *light is a stream of photons*. It is indeed a 'luminary movement of luminous bodies'. Photons provide our first example of a general property of the world on small scales: *all waves and all flows in nature are made of quantum particles*. Large numbers of (coherent) quantum particles – or *quantons* – behave as and form waves. We will see shortly that this is the case even for matter. Quantons are the fundamental constituents of *all* waves and *all* flows, without exception. Thus, the everyday continuum description of light is similar in many respects to the description of water as a continuous fluid: photons are the atoms of light, and continuity is an approximation valid for large numbers of particles. Single quantons often behave like classical particles.

Physics books used to discuss at length a so-called *wave–particle duality*. Let us be clear from the start: quantons, or quantum particles, are *neither* classical waves *nor* clas-

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* 'Light is the luminary movement of luminous bodies.' Blaise Pascal (b. 1623 Clermont, d. 1662 Paris), important mathematician and physicist up to the age of 26, after which he became a theologian and philosopher.

sical particles. In the microscopic world, quantons are the fundamental objects.

However, there is much that is still unclear. Where, inside matter, do these monochromatic photons come from? Even more interestingly, if light is made of quantons, all electromagnetic fields, even static ones, must be made of photons as well. However, in static fields nothing is flowing. How is this apparent contradiction resolved? And what implications does the particle aspect have for these static fields? What is the difference between quantons and classical particles? The properties of photons require more careful study.

The size of photons

First of all, we might ask: what are these photons made of? All experiments so far, performed down to the present limit of about 10^{-20} m, give the same answer: 'we can't find anything'. This is consistent with both a vanishing mass and a vanishing size of photons. Indeed, we would intuitively expect a body with a finite size to have a finite mass. Thus, although experiments can give only an upper limit, it is consistent to claim that a photon has *zero size*.

A particle with zero size cannot have any constituents. Thus a photon cannot be divided into smaller entities: photons are not composite. For this reason, they are called *elementary* particles. We will soon give some further strong arguments for this result. (Can you find one?) Nevertheless, the conclusion is strange. How can a photon have vanishing size, have no constituents, and still be *something*? This is a hard question; the answer will appear only in the last volume of our adventure. At the moment we simply have to accept the situation as it is. We therefore turn to an easier question.

Challenge 35 s

Are photons countable? – Squeezed light

> Also gibt es sie doch.
>
> Max Planck*

We saw above that the simplest way to count photons is to distribute them across a large screen and then to absorb them. But this method is not entirely satisfactory, as it destroys the photons. How can we count photons without destroying them?

One way is to reflect photons in a mirror and measure the recoil of the mirror. It seems almost unbelievable, but nowadays this effect is becoming measurable even for small numbers of photons. For example, it has to be taken into account in relation to the laser mirrors used in gravitational wave detectors, whose position has to be measured with high precision.

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Another way of counting photons without destroying them involves the use of special high-quality laser cavities. It is possible to count photons by the effect they have on atoms cleverly placed inside such a cavity.

In other words, light intensity can indeed be measured without absorption. These measurement show an important issue: even the best light beams, from the most sophist-

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* 'Thus they do exist after all.' Max Planck, in his later years, said this after standing silently, for a long time, in front of an apparatus that counted single photons by producing a click for each photon it detected. For a large part of his life, Planck was sceptical of the photon concept, even though his own experiments and conclusions were the starting point for its introduction.

icated lasers, *fluctuate* in intensity. There are *no* steady beams. This comes as no surprise: if a light beam did *not* fluctuate, observing it twice would yield a vanishing value for the action. However, there is a minimum action in nature, namely $\hbar$. Thus any beam and any flow in nature *must* fluctuate. But there is more.

A light beam is described, in a cross section, by its intensity and phase. The change – or action – that occurs while a beam propagates is given by the product of intensity and phase. Experiments confirm the obvious deduction: the intensity and phase of a beam behave like the momentum and position of a particle in that they obey an indeterminacy relation. You can deduce it yourself, in the same way as we deduced Heisenberg's relations. Using as characteristic intensity $I = E/\omega$, the beam energy divided by the angular frequency, and calling the phase φ, we get*

$$\Delta I \, \Delta\varphi \geqslant \frac{\hbar}{2} \,. \tag{12}$$

Equivalently, the indeterminacy product for the average photon number $n = I/\hbar = E/\hbar\omega$ and the phase φ obeys:

$$\Delta n \, \Delta\varphi \geqslant \frac{1}{2} \,. \tag{13}$$

For light emitted from an ordinary lamp, so-called *thermal light*, the indeterminacy product on the left-hand side of the above inequality is a large number. Equivalently, the indeterminacy product for the action (12) is a large multiple of the quantum of action.

For laser beams, i.e., beams of *coherent light*,** the indeterminacy product is close to 1/2. An illustration of coherent light is given in Figure 22.

Today it is possible to produce light for which the product of the two indeterminacies in equation (13) is near 1/2, but whose two values *differ* (in the units of the so-called *phasor space* illustrated in Figure 21). Such light is called *non-classical* or *squeezed*. The photon statistics is either hyper- or sub-Poissonian. Such light beams require involved laboratory set-ups for their production and are used in many modern research applications. Non-classical light has to be treated extremely carefully, as the smallest disturbances transforms it back into ordinary coherent (or even thermal light), in which Poisson (or even Bose-Einstein) statistics hold again. A general overview of the main types of light beams is given in Figure 21, together with their intensity and phase behaviour. (Several properties shown in the figure are defined for a single phase space cell only.) Ref. 19

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* A large photon number is assumed in the expression. This is obvious, as $\Delta\varphi$ cannot grow beyond all bounds, more precisely, not beyond 2π. The exact relations are

$$\begin{aligned} \Delta I \, \Delta\cos\varphi &\geqslant \frac{\hbar}{2}|\langle\sin\varphi\rangle| \\ \Delta I \, \Delta\sin\varphi &\geqslant \frac{\hbar}{2}|\langle\cos\varphi\rangle| \end{aligned} \tag{11}$$

where $\langle x\rangle$ denotes the expectation value of the observable x.

** Coherent light is light for which the photon number probability distribution is Poissonian; in particular, the variance is equal to the mean photon number. Coherent light is best described as composed of photons in coherent quantum states. Such a (canonical) *coherent state*, or *Glauber state*, is formally a state with $\Delta\varphi \to 1/n$ and $\Delta n \to n$.

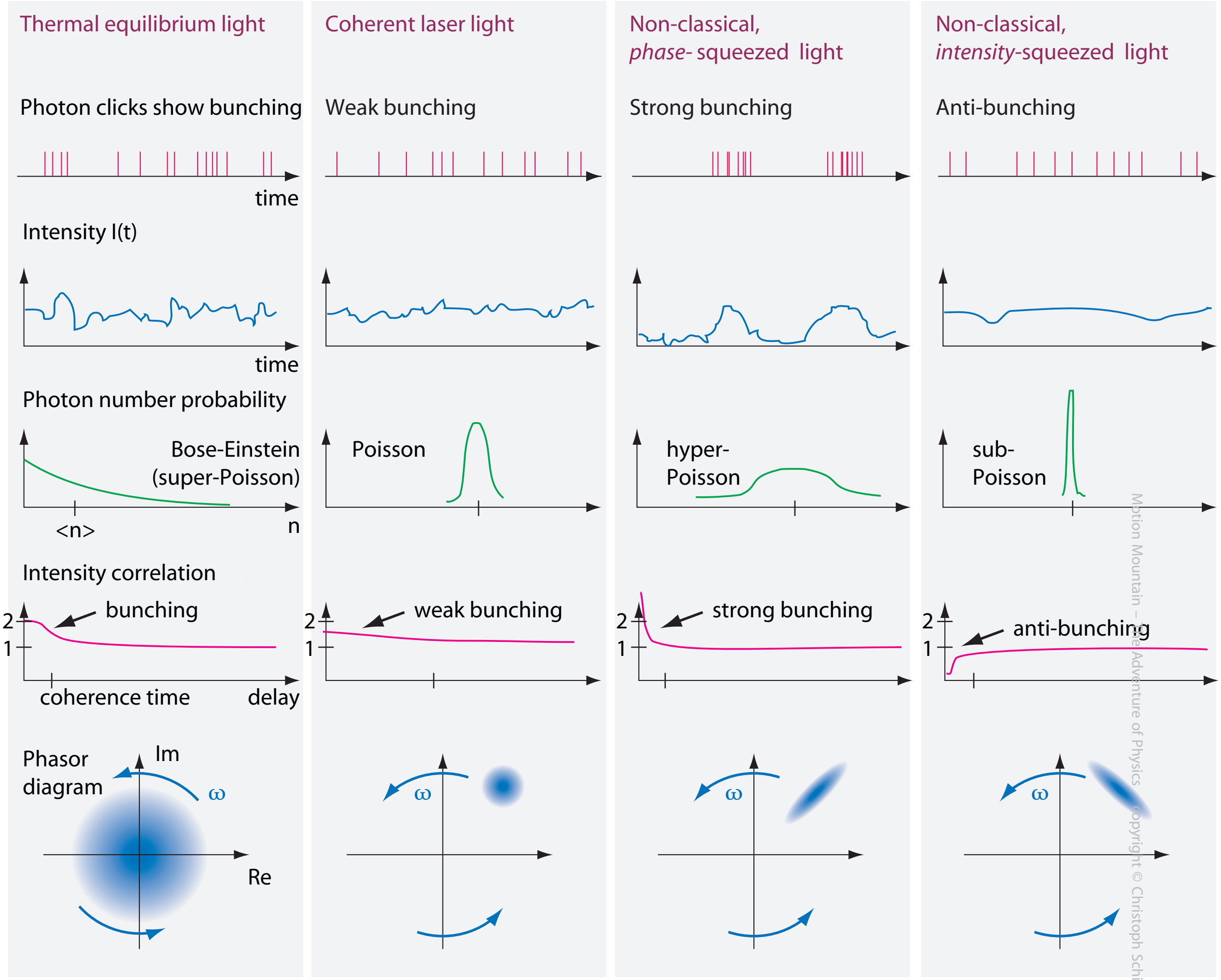

FIGURE 21 Four types of light and their photon properties: thermal light, laser light, and two extreme types of non-classical, squeezed light.

One extreme of non-classical light is *phase-squeezed light*. Since a phase-squeezed light beam has an (almost) determined phase, the photon number in such a beam fluctuates from zero to (almost) infinity. In other words, in order to produce coherent laser light that approximates a pure sine wave as perfectly as possible, we must accept that the photon number is as undetermined as possible. Such a beam has extremely small phase fluctuations that provide high precision in interferometry; the phase noise is as low as possible.

The other extreme of non-classical light is a beam with a given, fixed number of photons, and thus with an extremely high phase indeterminacy. In such an *amplitude-squeezed light beam*, the phase fluctuates erratically.* This sort of squeezed, non-classical

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* The most appropriate quantum states to describe such light are called *number states*, sometimes *Fock states*. These states are stationary, thus eigenstates of the Hamiltonian, and contain a fixed number of photons.

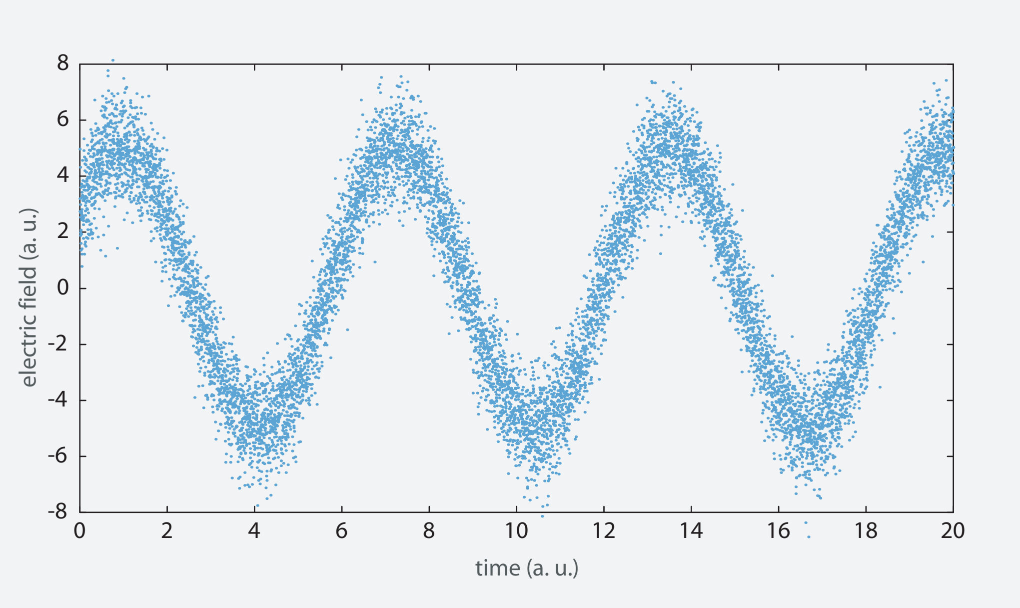

FIGURE 22 A simple way to illustrate the indeterminacy of a light beam's intensity and phase: the measured electric field of a coherent electromagnetic wave with low intensity, consisting of about a dozen photons. The cloudy sine wave corresponds to the phasor diagram at the bottom of the second column in the previous overview. For large number of photons, the relative noise amplitude is negligible. (© Rüdiger Paschotta)

light is ideal for precision intensity measurements as it provides the lowest intensity noise available. This kind of light shows *anti-bunching* of photons. To gain more insight, sketch the graphs corresponding to Figure 22 for phase-squeezed and for amplitude-squeezed light. Challenge 36 s

In contrast, the *coherent light* that is emitted by laser pointers and other lasers lies between the two extreme types of squeezed light: the phase and photon number indeterminacies are of similar magnitude.

The observations about thermal light, coherent laser light and non-classical light highlight an important property of nature: the number of photons in a light beam is not a well-defined quantity. In general, it is undetermined, and it *fluctuates*. Photons, unlike stones, cannot be counted precisely – as long as they are propagating and not absorbed. In flight, it is only possible to determine an approximate, average photon number, within the limits set by indeterminacy. Is it correct to claim that the number of photons at the beginning of a beam is not necessarily the same as the number at the end of the beam? Challenge 37 ny

The fluctuations in the number of photons are of most importance at optical frequencies. At radio frequencies, the photon number fluctuations are usually negligible, due to the low photon energies and the usually high photon numbers involved. Conversely, at gamma-ray energies, wave effects play little role. For example, we saw that in deep, dark intergalactic space, far from any star, there are about 400 photons per cubic centimetre; they form the cosmic background radiation. This photon density number, like the number of photons in a light beam, also has a measurement indeterminacy. Can you estimate it? Challenge 38 s

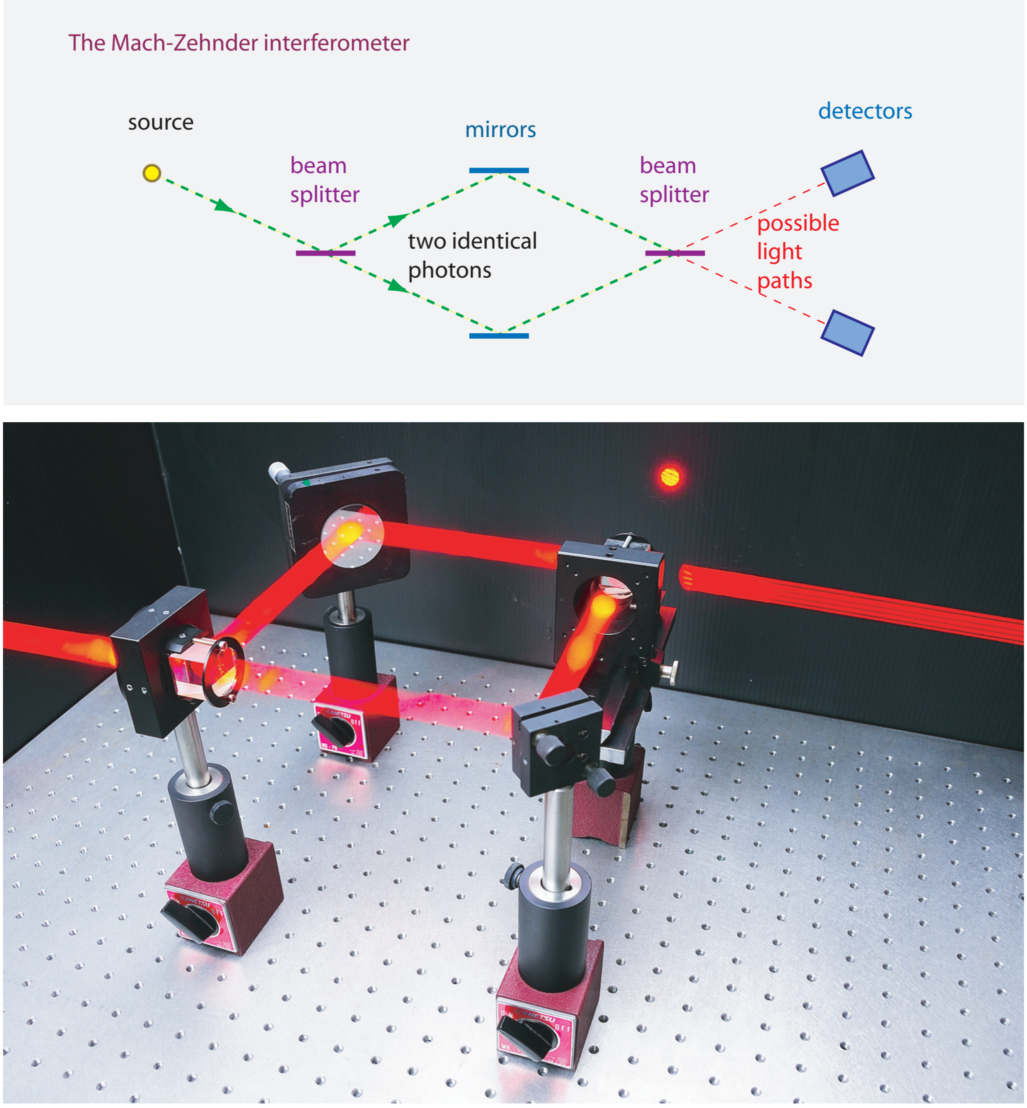

FIGURE 23 The Mach–Zehnder interferometer and a practical realization, about 0.5 m in size (© Félix Dieu and Gaël Osowiecki).

In short, unlike pebbles, *photons are countable, but their number is not fixed.* And this is not the only difference between photons and pebbles.

THE POSITIONS OF PHOTONS

Where is a photon when it moves in a beam of light? Quantum theory gives a simple answer: nowhere in particular. This is proved most spectacularly by experiments with interferometers, such as the basic interferometer shown in Figure 23. Interferometers show that even a beam made of a *single* photon can be split, led along two different paths, and then recombined. The resulting interference shows that the single photon cannot be said to have taken either of the two paths. If one of the two paths is blocked, the pattern on the screen changes. In other words, somehow the photon must have taken both paths at the same time. Photons cannot be localized: they have no position.*

We come to the conclusion that macroscopic light pulses have paths, but the individual photons in it do not. Photons have neither position nor paths. Only large numbers of photons can have positions and paths, and then only approximately.

The impossibility of localizing photons can be quantified. Interference shows that it is impossible to localize photons in the direction *transverse* to the motion. It might seem less difficult to localize photons *along* the direction of motion, when it is part of a light pulse, but this is a mistake. The quantum of action implies that the indeterminacy in the longitudinal position is given at least by the wavelength of the light. Can you confirm this? Challenge 39 e It turns out that photons can only be localized within a *coherence length*. In fact, the transversal and the longitudinal coherence length differ in the general case. The longitudinal coherence length (divided by c) is also called temporal coherence, or simply, the *coherence time*. Page 49 It is also indicated in Figure 21. The impossibility of localizing photons is a consequence of the quantum of action. For example, the transverse coherence length is due to the indeterminacy of the transverse momentum; the action values for paths leading to points separated by less than a coherence length differ by less than the quantum of action $\hbar$. Whenever a photon is detected somewhere, e.g., by absorption, a *precise* statement on its direction or its origin *cannot* be made. Sometimes, in special cases, there can be a high probability for a certain direction or source, though.

Lack of localisation means that photons *cannot* be simply visualized as short wave trains. For example, we can increase the coherence length by sending light through a narrow filter. Photons are truly *unlocalizable* entities, specific to the quantum world. Photons are neither little stones nor little wave packets. Conversely, 'light path', 'light pulse position' and 'coherence' are properties of a photon ensemble, and do *not* apply to a single photon.

Whenever photons can *almost* be localized along their direction of motion, as in coherent light, we can ask how photons are lined up, one after the other, in a light beam. Of course, we have just seen that it does not make sense to speak of their *precise* position. But do photons in a perfect beam arrive at almost-regular intervals?

To the shame of physicists, the study of photon correlations was initiated by two astronomers, Robert Hanbury Brown and Richard Twiss, in 1956, and met with several years Ref. 20 of disbelief. They varied the transversal distance of the two detectors shown in Figure 24 – from a few to 188 m – and measured the intensity correlations between them. Hanbury Brown and Twiss found that the intensity fluctuations within the volume of coherence are correlated. Thus the photons themselves are correlated. With this experiment, they were able to measure the diameter of numerous distant stars.

Inspired by the success of Hanbury Brown and Twiss, researchers developed a simple method to measure the probability that a second photon in a light beam arrives at a given time after the first one. They simply split the beam, put one detector in the first branch, and varied the position of a second detector in the other branch. The set-up is sketched in Figure 25. Such an experiment is nowadays called a *Hanbury Brown Twiss experiment*.

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* We cannot avoid this conclusion by saying that photons are *split* at the beam splitter: if we place a detector in each arm, we find that they never detect a photon at the same time. Photons cannot be divided.

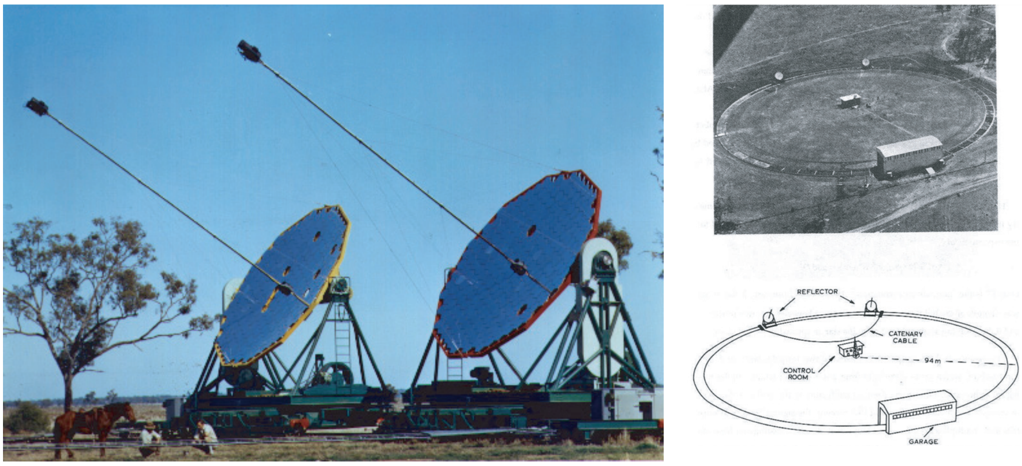

FIGURE 24 The original experimental set-up with which Hanbury Brown and Twiss measured stellar diameters at Narrabri in Australia. The distance between the two light collectors could be changed by moving them on rails. The light detectors are at the end of the poles and each of them, as they wrote, 'collected light as rain in a bucket.' (© John Davis).

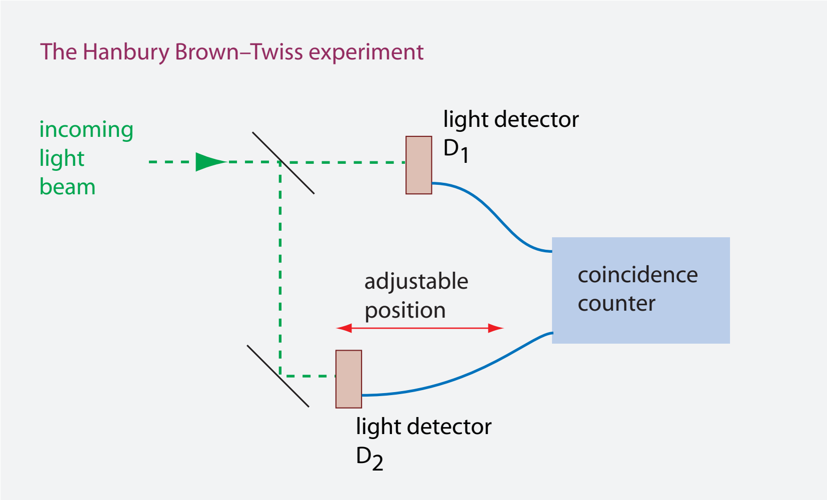

FIGURE 25 How to measure photon statistics with an electronic intensity correlator or coincidence counter, the variation being measured by varying the position of a detector.

One finds that, for coherent light within the volume of coherence, the clicks in the two counters – and thus the photons themselves – are *correlated*. To be more precise, such experiments show that whenever the first photon hits, the second photon is most likely to hit just afterwards. Thus, photons in light beams are *bunched*. Bunching is one of the many results showing that photons are quantons, that they are indeed necessary to describe light, and that they are unlocalizable entities. As we will see below, the result also implies that photons are bosons.

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Every light beam has an upper time limit for bunching: the *coherence time*. For times longer than the coherence time, the probability for bunching is low, and independent of the time interval, as shown in Figure 25. The coherence time characterizes every light

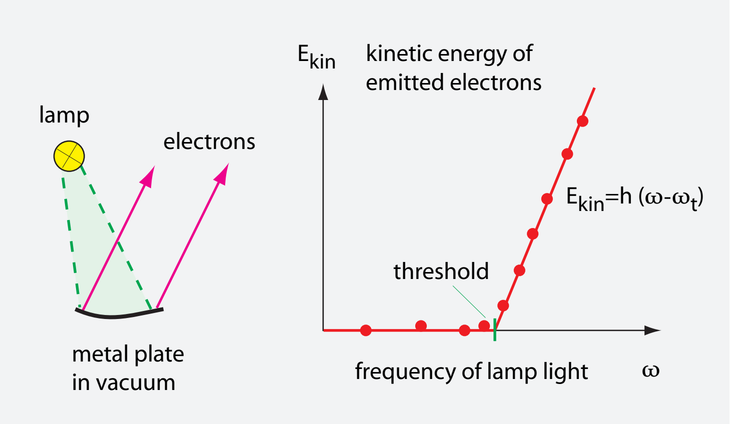

FIGURE 26 The kinetic energy of electrons emitted in the photoelectric effect.

beam. In fact, it is often easier to think in terms of the *coherence length* of a light beam. For thermal light, the coherence length is only a few micrometres: a small multiple of the wavelength. The largest coherence lengths, of over 300 000 km, are obtained with research lasers that have an extremely narrow laser bandwith of just 1 Hz. Interestingly, coherent light is even found in nature: several special stars have been found to emit it. Ref. 21

Although the intensity of a good laser beam is almost constant, the photons do not arrive at regular intervals. Even the best laser light shows bunching, though with different statistics and to a lesser degree than lamp light, as illustrated in Figure 21. Light whose photons arrive regularly, thus exhibiting so-called *(photon) anti-bunching*, is obviously non-classical in the sense defined above; such light can be produced only by special experimental arrangements. Extreme examples of this phenomenon are being investigated at present by several research groups aiming to construct light sources that emit one photon at a time, at regular time intervals, as reliably as possible. In short, we can state that the precise photon statistics in a light beam depends on the mechanism of the light source.

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In summary, experiments force us to conclude that light is made of photons, but also that photons cannot be localized in light beams. It makes no sense to talk about the position of a photon in general; the idea makes sense only in some special situations, and then only approximately and as a statistical average.

Are photons necessary?

In light of the results uncovered so far, the answer to the above question is obvious. But the issue is tricky. In textbooks, the *photoelectric effect* is usually cited as the first and most obvious experimental proof of the existence of photons. In 1887, Heinrich Hertz observed that for certain metals, such as lithium or caesium, incident ultraviolet light leads to charging of the metal. Later studies of the effect showed that the light causes emission of electrons, and that the energy of the ejected electrons does not depend on the intensity of the light, but only on the difference between $\hbar$ times its frequency and a material-dependent threshold energy. Figure 26 summarizes the experiment and the measurements.

In classical physics, the photoelectric effect is difficult to explain. But in 1905, Albert Einstein deduced the measurements from the assumption that light is made of photons of energy $E = \hbar\omega$. He imagined that this energy is used partly to take the electron over the threshold, and partly to give it kinetic energy. More photons only lead to more electrons, not to faster ones. In 1921, Einstein received the Nobel Prize for the explanation of the photoelectric effect. But Einstein was a *genius*: he deduced the correct result by a somewhat incorrect reasoning. The (small) mistake was the assumption that a classical, continuous light beam would produce a different effect. In fact, it is easy to see that a classical, continuous electromagnetic field interacting with discrete matter, made of discrete atoms containing discrete electrons, would lead to exactly the same result, as long as the motion of electrons is described by quantum theory. Several researchers confirmed this early in the twentieth century. The photoelectric effect by itself does *not* imply the existence of photons.

Ref. 22

Ref. 23

Indeed, many researchers in the past were unconvinced that the photoelectric effect shows the existence of photons. Historically, the most important argument for the *necessity* of light quanta was given by Henri Poincaré. In 1911 and 1912, aged 57 and only a few months before his death, he published two influential papers proving that the radiation law of black bodies – in which the quantum of action had been discovered by Max Planck – *requires* the existence of photons. He also showed that the amount of radiation emitted by a hot body is *finite* only because of the *quantum* nature of the processes leading to light emission. A description of these processes in terms of classical electrodynamics would lead to (almost) *infinite* amounts of radiated energy. Poincaré's two influential papers convinced most physicists that it was worthwhile to study quantum phenomena in more detail. Poincaré did not know about the action limit $S \geqslant \hbar$; yet his argument is based on the observation that light of a given frequency has a minimum intensity, namely a single photon. Such a one-photon beam may be split into two beams, for example by using a half-silvered mirror. However, taken together, those two beams never contain more than a single photon.

Ref. 24

Another interesting experiment that requires photons is the observation of 'molecules of photons'. In 1995, Jacobson et al. predicted that the de Broglie wavelength of a *packet* of photons could be observed. According to quantum theory, the packet wavelength is given by the wavelength of a single photon divided by the number of photons in the packet. The team argued that the packet wavelength could be observable if such a packet could be split and recombined without destroying the cohesion within it. In 1999, this effect was indeed observed by de Pádua and his research group in Brazil. They used a careful set-up with a nonlinear crystal to create what they call a *biphoton*, and observed its interference properties, finding a reduction in the effective wavelength by the predicted factor of two. Since then, packages with three and even four entangled photons have been created and observed.

Ref. 25

Ref. 26

Yet another argument for the necessity of photons is the above-mentioned recoil felt by atoms emitting light. The recoil measured in these cases is best explained by the emission of a photon in a particular direction. In contrast, classical electrodynamics predicts the emission of a spherical wave, with no preferred direction.

Page 43

Obviously, the observation of *non-classical light*, also called *squeezed light*, also argues for the existence of photons, as squeezed light proves that photons are indeed an *intrinsic* aspect of light, necessary even when interactions with matter play no role. The same is

Page 49

Ref. 27

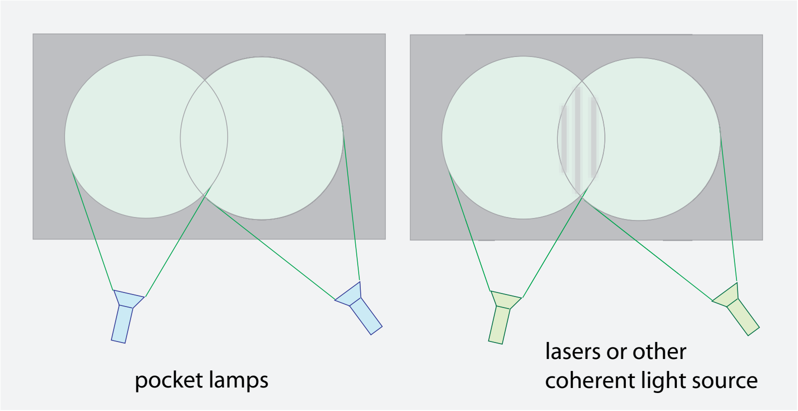

FIGURE 27 Two situations in which light crosses light: different light sources lead to different results.

true for the Hanbury Brown–Twiss effect.

Finally, the spontaneous decay of excited atomic states also requires the existence of photons. This cannot be explained by a continuum description of light.

In summary, the concept of a photon is indeed necessary for a precise description of light; but the details are often subtle, as the properties of photons are unusual and require a change in our habits of thought. To avoid these issues, most textbooks stop discussing photons after coming to the photoelectric effect. This is a pity, as it is only then that things get interesting. Ponder the following. Obviously, all electromagnetic fields are made of photons. At present, photons can be counted for gamma rays, X-rays, ultraviolet light, visible light and infrared light. However, for lower frequencies, such as radio waves, photons have not yet been detected. Can you imagine what would be necessary to count the photons emitted from a radio station? This issue leads directly to the most important question of all: Challenge 40 s

INTERFERENCE: HOW CAN A WAVE BE MADE UP OF PARTICLES?

> Die ganzen fünfzig Jahre bewusster Grübelei haben mich der Antwort auf die Frage ‘Was sind Lichtquanten?’ nicht näher gebracht. Heute glaubt zwar jeder Lump er wisse es, aber er täuscht sich.
>
> Albert Einstein, 1951 *

If a light wave is made of particles, we must be able to explain each and every wave property in terms of photons. The experiments mentioned above already hint that this is possible only because photons are *quantum* particles. Let us take a more detailed look at this connection.

Light can *cross* other light undisturbed, for example when the light beams from two pocket lamps shine through each other. This observation is not hard to explain with

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* ‘Fifty years of conscious brooding have not brought me nearer to the answer to the question ‘What are light quanta?’ Nowadays every bounder thinks he knows it, but he is wrong.’ Einstein wrote this a few years before his death in a letter to Michele Besso. Ref. 28

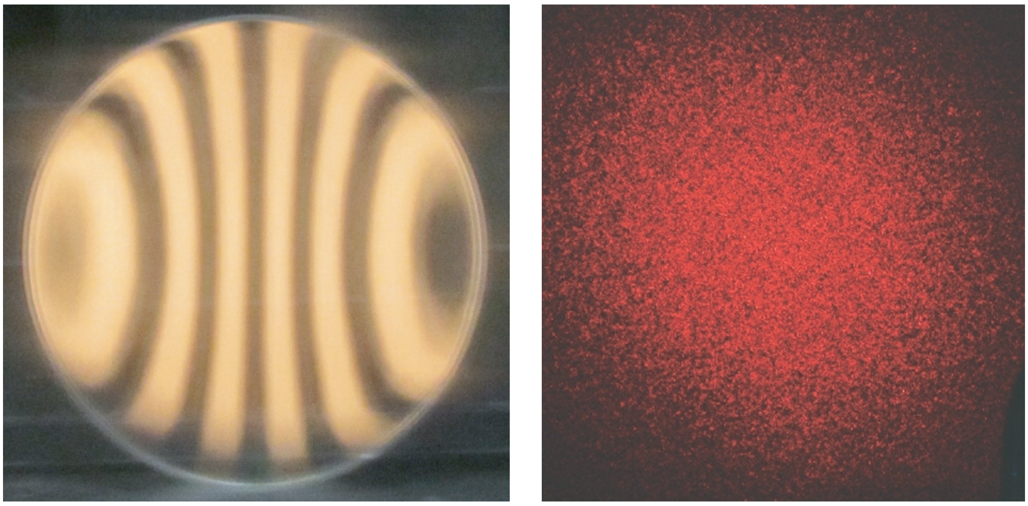

FIGURE 28 Examples of interference patterns that appear when coherent light beams cross: the interference produced by a self-made parabolic telescope mirror of 27 cm diameter, and a speckle laser pattern on a rough surface (© Mel Bartels, Epzcaw).

photons; since photons do not interact with each other, and are point-like, they 'never' hit each other. In fact, there *is* an extremely small positive probability for their interaction, as we will find out later, but this effect is not observable in everyday life. Vol. V, page 130

But if two *coherent* light beams, i.e., two light beams of identical frequency and fixed phase relation cross, we observe alternating bright and dark regions: so-called *interference fringes*. The schematic set-up is shown in Figure 27. Examples of actual interference effects are given in Figure 28 and Figure 29. How do these interference fringes appear?* How can it be that photons are not detected in the dark regions? We already know the only possible answer: the brightness at a given place corresponds to the *probability* that a photon will arrive there. The fringes imply:

▷ Photons behave like moving little *arrows*.

Some further thought leads to the following description:

- The arrow is always perpendicular to the direction of motion.
- The arrow's direction stays fixed in space when the photons move.
- The length of an arrow shrinks with the square of the distance travelled.
- *The probability of a photon arriving somewhere is given by the square of an arrow.*
- *The final arrow is the* sum *of all the arrows arriving there by all possible paths.*
- Photons emitted by single-coloured sources are emitted with arrows of constant length pointing in the direction ωt; in other words, such sources spit out photons with a *rotating* mouth.
- Photons emitted by incoherent sources – e.g., thermal sources, such as pocket lamps – are emitted with arrows of constant length pointing in *random* directions.

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* If lasers are used, fringes can only be observed if the two beams are derived from a single beam by splitting, or if two expensive high-precision lasers are used. (Why?) Challenge 41 s

FIGURE 29 Top: calculated interference patterns – and indistinguishable from observed ones under ideal, "textbook" conditions – produced by two parallel narrow slits illuminated with green light and with white light. Bottom: two Gaussian beams interfering at an angle (© Dietrich Zawischa, Rüdiger Paschotta).

With this simple model[*] we can explain the wave behaviour of light. In particular, we can describe the interference stripes seen in laser experiments, as shown schematically in Figure 30. You can check that in some regions the two arrows travelling through the two slits add up to zero *for all times*. No photons are detected there: those regions are black. In other regions, the arrows always add up to the maximal value. These regions are always bright. Regions in between have intermediate shades. Obviously, in the case of usual pocket lamps, shown in the left-hand diagram of Figure 27, the brightness in the common region also behaves as expected: the averages simply add up.

Obviously, the photon model implies that an interference pattern is built up as the sum of a large number of single-photon hits. Using low-intensity beams, we should therefore be able to see how these little spots slowly build up an interference pattern by accumulating in the bright regions and never hitting the dark regions. This is indeed the case, as

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[*] The model gives a correct description of light except that it neglects polarization. To add polarization, it is necessary to combine arrows that rotate in both senses around the direction of motion.

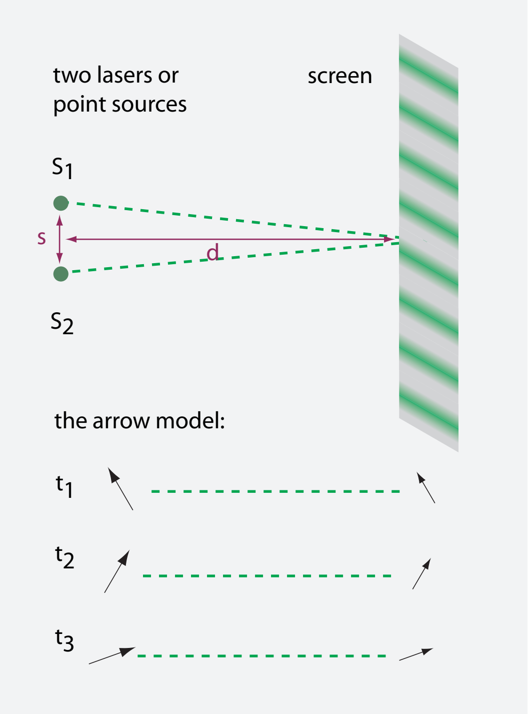

FIGURE 30 Interference and the description of light with arrows (at three instants of time).

Page 43 we have seen earlier on. All experiments confirm this description.

In other words, interference is the superposition of coherent light fields or, more generally, of coherent electromagnetic fields. Coherent light fields have specific, more regular photon behaviour, than incoherent light fields. We will explore the details of photon statistics in more detail shortly.

In summary, photons are quantum particles. Quantum particles can produce interference patterns – and all other wave effects – when they appear in large numbers, because they are described by an arrow whose length squared gives the probability for its detection.

INTERFERENCE OF A SINGLE PHOTON

It is important to point out that interference between two light beams is *not* the result of two different photons cancelling each other out or being added together. Such cancellation would contradict conservation of energy and momentum. Interference is an effect applicable to each photon separately – as shown in the previous section – because each photon is spread out over the whole set-up: each photon takes all possible paths. As Paul Dirac stressed: Ref. 29

▷ Each photon interferes only with itself.

Interference of a photon with itself only occurs because photons are quantons, and not classical particles.

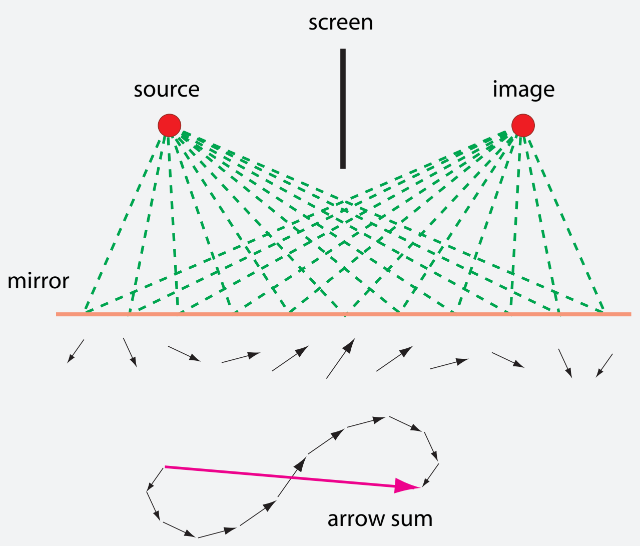

FIGURE 31 Light reflected by a mirror, and the corresponding arrows (at an instant of time).

Dirac's oft-quoted statement leads to a famous paradox: if a photon can interfere only with itself, how can two laser beams from two different lasers interfere with each other? The answer given by quantum physics is simple but strange: in the region where the beams interfere – as mentioned above – it is impossible to say from which source a photon has come. The photons in the crossing region *cannot* be said to come from a specific source. Photons, also in the interference region, are quantons, and they indeed interfere only with themselves.

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Another description of the situation is the following:

▷ A photon interferes only within its volume of coherence. And in that volume, it is impossible to distinguish photons.

In the coherence volume formed by the longitudinal and transversal coherence length – sometimes also called a *phase space cell* – we cannot completely say that light is a flow of photons, because a flow cannot be defined in it. Despite regular claims to the contrary, Dirac's statement is correct, as we will see below. It is a strange consequence of the quantum of action.

Page 66

Reflection and diffraction deduced from photon arrows

Waves also show *diffraction*. Diffraction is the change of propagation direction of light or any other wave near edges. To understand this phenomenon with photons, let us start with a simple mirror, and study *reflection* first. Photons (like all quantum particles) move from source to detector by *all* possible paths. As Richard Feynman,* who discovered this explanation, liked to stress, the term 'all' has to be taken literally. This is not a big deal in

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* Richard ('Dick') Phillips Feynman (b. 1918 New York City, d. 1988 Los Angeles), physicist, was one of the founders of quantum electrodynamics. He also discovered the 'sum-over-histories' reformulation of quantum theory, made important contributions to the theory of the weak interaction and to

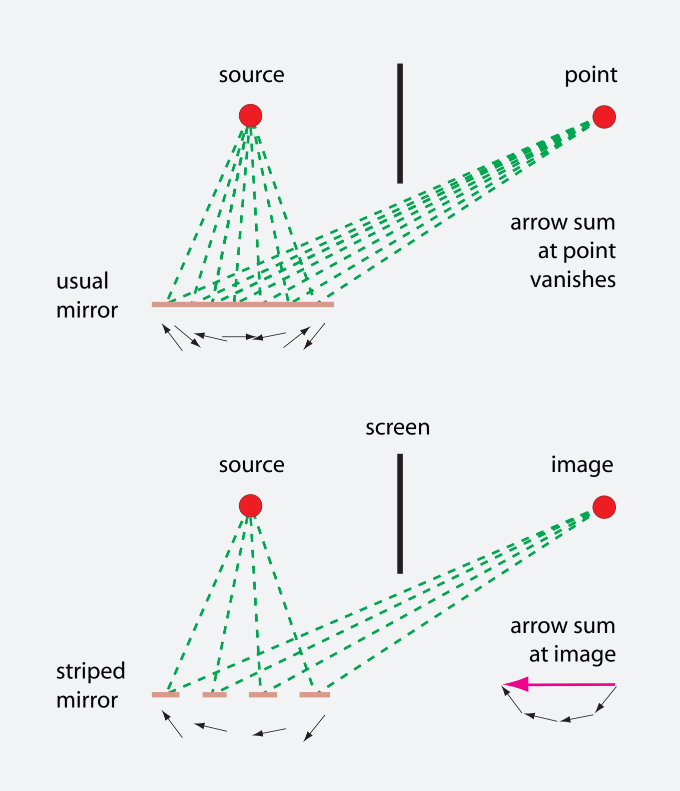

FIGURE 32 The light reflected by a badly-placed mirror and by a grating.

the explanation of interference. But in order to understand a mirror, we have to include all possibilities, however crazy they seem, as shown in Figure 31.

As stated above, a light source emits rotating arrows. To determine the probability that light arrives at a certain location within the image, we have to add up all the arrows arriving at the same time at that location. For each path, the arrow orientation at the image is shown – for convenience only – below the corresponding segment of the mirror. The angle and length of the arriving arrow depends on the path. Note that the sum of all the arrows does not vanish: light does indeed arrive at the image. Moreover, the largest contribution comes from the paths near to the middle. If we were to perform the same calculation for another image location, (almost) no light would get there.

In short, the rule that reflection occurs with the incoming angle equal to the outgoing angle is an approximation, following from the arrow model of light. In fact, a detailed

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quantum gravity, and co-authored a famous textbook, the *Feynman Lectures on Physics*, now online at www.feynmanlectures.info. He is one of those theoretical physicists who made his career mainly by performing complex calculations – but he backtracked with age, most successfully in his teachings and physics books, which are all worth reading. He was deeply dedicated to physics and to enlarging knowledge, and was a collector of surprising physical explanations. He helped building the nuclear bomb, wrote papers in topless bars, avoided to take any professional responsibility, and was famously arrogant and disrespectful of authority. He wrote several popular books on the events of his life. Though he tried to surpass the genius of Wolfgang Pauli throughout his life, he failed in this endeavour. He shared the 1965 Nobel Prize in Physics for his work on quantum electrodynamics.

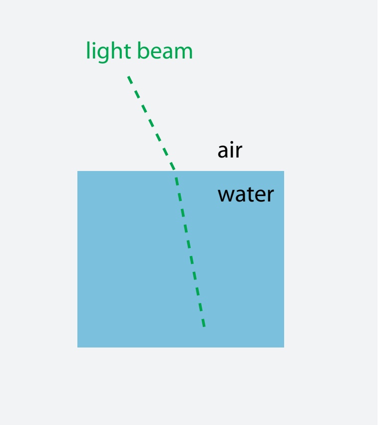

FIGURE 33 If light were made of little stones, they would move faster in water.

calculation, with more arrows, shows that the approximation is quite precise: the errors are much smaller than the wavelength of the light.

The proof that light does indeed take all these strange paths is given by a more specialized mirror. As show in Figure 32, we can repeat the experiment with a mirror that reflects only along certain stripes. In this case, the stripes have been carefully chosen so that the corresponding path lengths lead to arrows with a bias in one direction, namely to the left. The arrow addition now shows that such a specialized mirror – usually called a *grating* – allows light to be reflected in unusual directions. Indeed, this behaviour is standard for waves: it is called *diffraction*. In short, the arrow model for photons allows us to describe this wave property of light, provided that photons follow the 'crazy' probability scheme. Do not get upset! As was said above, quantum theory is the theory for crazy people.

You may wish to check that the arrow model, with the approximations it generates by summing over all possible paths, automatically ensures that the quantum of action is indeed the smallest action that can be observed. Challenge 42 s

Refraction and partial reflection from photon arrows

All waves have a *signal velocity*. The signal velocity also depends on the medium in which they propagate. As a consequence, waves show *refraction* when they move from one medium into another with different signal velocity. Interestingly, the naive particle picture of photons as little stones would imply that light is faster in materials with high refractive indices: the so-called *dense* materials. (See Figure 33.) Can you confirm this? Challenge 43 e However, experiments show that light in dense materials moves *slowly*. The wave picture has no difficulty explaining this observation. (Can you confirm this?) Challenge 44 e Historically, this was one of the arguments *against* the particle theory of light. In contrast, the arrow model of light presented above is able to explain refraction properly. It is not difficult: try it. Challenge 45 e

Waves also *reflect partially* from materials such as glass. This is one of the most difficult wave properties to explain with photons. But it is one of the few effects that is *not* explained by a classical wave theory of light. However, it *is* explained by the arrow model, as we will find out. Partial reflection confirms the first two rules of the arrow model. Par-

Page 57 tial reflection shows that photons indeed behave *randomly*: some are reflected and other are not, without any selection criterion. The distinction is purely statistical. More about this issue shortly.

FROM PHOTONS TO WAVES

In waves, the fields *oscillate in time and space*. One way to show how waves can be made of particles is to show how to build up a sine wave using a large number of photons. A sine wave is a coherent state of light. The way to build them up was explained in detail by Roy Glauber. Ref. 30 In fact, to build a pure sine wave, we need a superposition of a beam with one photon, a beam with two photons, a beam with three photons, and so on. Together, they give a perfect sine wave. As expected, its photon number fluctuates to the highest possible degree.

If we repeat the calculation for non-ideal beams, we find that the indeterminacy relation for energy and time is respected: every emitted beam will possess a certain spectral width. Purely monochromatic light does not exist. Similarly, no system that emits a wave *at random* can produce a monochromatic wave. All experiments confirm these results.

In addition, waves can be *polarized*. So far, we have disregarded this property. In the photon picture, polarization is the result of carefully superposing beams of photons spinning clockwise and anticlockwise. Indeed, we know that linear polarization can be seen as a result of superposing circularly-polarized light of both signs, using the proper phase. What seemed a curiosity in classical optics turns out to be a fundamental justification for quantum theory.

Finally, photons are *indistinguishable*. When two photons of the same colour cross, there is no way to say afterwards which of the two is which. The quantum of action makes this impossible. The indistinguishability of photons has an interesting consequence. It is impossible to say which emitted photon corresponds to which arriving photon. In other words, there is no way to follow the path of a photon, as we are used to following the path of a billiard ball. Page 52 Photons are indeed indistinguishable. In addition, the experiment by Hanbury Brown and Twiss implies that photons are bosons. Ref. 31 We will discover more details about the specific indistinguishability of bosons later on. Page 112

In summary, we find that light waves *can* indeed be described as being built of particles. However, this is only correct with the proviso that *photons*

- are not precisely countable – never with a precision better than $\sqrt{N}$,
- are not localizable – never with a precision better than the coherence length,
- have no size, no charge and no (rest) mass,
- show a phase that increases as ωt, i.e., as the product of frequency and time,
- carry spin 1,
- of the same frequency are indistinguishable bosons – within a coherence volume,
- can take any path whatsoever – as long as allowed by the boundary conditions,
- have no discernable origin, and
- have a detection probability given by the square of the sum of amplitudes* for all allowed paths leading to the point of detection.

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* The amplitude of a photon field, however, cannot and should not be identified with the wave function of any massive spin 1 particle.

In other words, light can be described as made of particles *only* if these particles have special, *quantum* properties. These quantum properties differ from everyday particles and allow photons to behave like waves whenever they are present in large numbers.

Can light move faster than light? – Real and virtual photons

In a vacuum, light can move faster than c, as well as slower than c. The quantum principle provides the details. As long as this principle is obeyed, the speed of a short light flash can differ – though only by a tiny amount – from the 'official' value. Can you estimate the allowable difference in arrival time for a light flash coming from the dawn of time? Challenge 46 ny

The arrow description for photons gives the same result. If we take into account the crazy possibility that photons can move with any speed, we find that all speeds very different from c cancel out. The only variation that remains, translated into distances, is the indeterminacy of about one wavelength in the longitudinal direction, which we mentioned above. Challenge 47 ny

In short, light, or real photons, can indeed move faster than light, though only by an amount allowed by the quantum of action. For everyday situations, i.e., for high values of the action, all quantum effects average out, including light and photon velocities different from c.

Ref. 32 Not only the position, but also the energy of a single photon can be undefined. For example, certain materials split one photon of energy $\hbar\omega$ into two photons, whose two energies add up to the original one. Quantum mechanics implies that the energy partitioning is known only when the energy of one of the two photons is measured. Only at that very instant is the energy of the second photon known. Before the measurement, both photons have undefined energies. The process of energy fixing takes place *instantaneously*, even if the second photon is far away. We will explain below the background to this and similar strange effects, which *seem* to be faster than light. In fact, despite the appearance, these observations do *not* involve faster-than-light transmission of energy or information. Page 153 Challenge 48 s

More bizarre consequences of the quantum of action appear when we study *static* electric fields, such as the field around a charged metal sphere. Obviously, such a field must also be made of photons. How do they move? It turns out that static electric fields are made of *virtual* photons. Virtual photons are photons that do not appear as free particles: they only appear for an extremely short time before they disappear again. In the case of a static electric field, they are *longitudinally* polarized, and do not carry energy away. Virtual photons, like other virtual particles, are 'shadows' of particles that obey

$$\Delta x \Delta p \leqslant \hbar/2 \,. \tag{14}$$

Rather than obeying the usual indeterminacy relation, they obey the opposite relation, which expresses their very brief appearance. Despite their intrinsically short life, and despite the impossibility of detecting them directly, virtual particles have important effects. We will explore them in detail shortly. Page 193

In fact, the vector potential A allows *four* polarizations, corresponding to the four coordinates (t, x, y, z). It turns out that for the photons one usually talks about – the free or *real* photons – the polarizations in the t and z directions cancel out, so that one observes only the x and y polarizations in actual experiments.

For bound or *virtual* photons, the situation is different. All four polarizations are possible. Indeed, the z and t polarizations of virtual photons – which do not appear for real photons, i.e., for free photons – are the ones that can be said to be the building blocks of *static* electric and magnetic fields.

In other words, static electric and magnetic fields are continuous flows of virtual photons. In contrast to real photons, virtual photons can have mass, can have spin directions not pointing along the path of motion, and can have momentum opposite to their direction of motion. Exchange of virtual photons leads to the *attraction* of bodies of different charge. In fact, virtual photons *necessarily* appear in any description of electromagnetic interactions. Later on we will discuss their effects further – including the famous attraction of neutral bodies. Vol. V, page 122

Vol. II, page 72 We have seen already early on that virtual photons, for example those that are needed to describe collisions of charges, must be able to move with speeds higher than that of light. This description is required in order to ensure that the speed of light remains a limit in all experiments.

In summary, it might be intriguing to note that virtual photons, in contrast to real photons, are not bound by the speed of light; but it is also fair to say that virtual photons move faster-than-light only in a *formal* sense.

INDETERMINACY OF ELECTRIC FIELDS

We have seen that the quantum of action implies an indeterminacy for light intensity. Since light is an electromagnetic wave, this indeterminacy implies similar, separate limits for electric and magnetic fields at a given point in space. This conclusion was first drawn in 1933 by Bohr and Rosenfeld. They started from the effects of the fields on a test particle of mass m and charge q, which are described by: Ref. 33

$$m\boldsymbol{a} = q\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{b}) \;. \tag{15}$$

Since it is impossible to measure both the momentum and the position of a particle, they deduced an indeterminacy for the electrical field, given by Challenge 49 ny

$$\Delta E = \frac{\hbar}{q\,\Delta x\, t} \;, \tag{16}$$

where t is the measurement time and Δx is the position indeterminacy. Thus every value of an electric field, and similarly of a magnetic field, possesses an indeterminacy. The state of the electromagnetic field behaves like the state of matter in this respect: both follow an indeterminacy relation.

HOW CAN VIRTUAL PHOTON EXCHANGE LEAD TO ATTRACTION?

Exchange of real photons always leads to recoil. But exchange of *virtual* photons can lead either to attraction or repulsion, depending on the signs of the two charges involved. This is worth looking at.

We start with two localized charges of same sign, located both on the x-axis, and want to determine the momentum transferred from the charge on the right side via a virtual photon to the charge on the left side.

For the virtual photon, the important part of its state *in momentum space* is its imaginary part, which, *if emitted by a negative charge,* has a positive peak (delta function shape) at the negative of its momentum value and a negative peak at its positive momentum value.

When the virtual photon hits the other charged particle, on the left, it can push it either to the left or to the right. The probability amplitude for each process is given by the particle charge times the photon momentum value times i times time. Both amplitudes need to be added.

In the case that the second particle has the *same* charge as the first, the effect of the virtual photon absorption in momentum space is to add a wave function that originally was antisymmetric and positively valued on the positive axis, and that is then shifted to the left, to a second wave function which originally was the negative of the first, but is then shifted to the right. The result for this one-photon absorption process is a real-valued, antisymmetric function in momentum space, with positive values for negative momenta, and negative values for positive momenta.

To understand repulsion, we need to add the wave function for this one-photon process to the zero-photon (thus unmodified) function of the second particle, and then square the sum. This unmodified function was positive in the case of same charges. The squaring process of the sum yields a probability distribution in momentum space whose maximum is at a negative momentum value; thus the second particle has been *repelled* from the first.

If the charges had different signs, the maximum of the sum would be at a positive momentum value, and the second particle would be *attracted* to the first. In short, attraction or repulsion is determined by the interference between the wave function for one-photon absorption (more precisely, for odd-photon-number absorption) and the wave function for zero-photon absorption (more precisely, for even-photon-number absorption).

Can two photons interfere?

Page 59 In 1930, Paul Dirac made the famous statement already mentioned earlier on:

> ▷ Each photon interferes only with itself. Interference between two different photons never occurs. Ref. 29

Often this statement is misinterpreted as implying that light from two separate photon *sources* cannot interfere. Unfortunately, this false interpretation has spread through a part of the literature. Everybody can check that this statement is incorrect with a radio: signals from two distant radio stations transmitting on the same frequency lead to beats in amplitude, i.e., to *wave interference*. (This should not to be confused with the more common *radio interference*, which usually is simply a superposition of intensities.) Radio transmitters are coherent sources of photons, and any radio receiver shows that signals form two different sources can indeed interfere. Ref. 34

In 1949, interference of fieds emitted from two different photon sources has been demonstrated also with *microwave* beams. From the nineteen fifties onwards, numerous experiments with two lasers and even with two thermal light sources have shown light interference. For example, in 1963, Magyar and Mandel used two ruby lasers emitting light pulses and a rapid shutter camera to produce spatial interference fringes. Ref. 35

However, all these experimental results with two interfering sources do not contradict the statement by Dirac. Indeed, two photons cannot interfere for several reasons.

- Interference is a result of the space-time propagation of waves; photons appear only when the energy–momentum picture is used, mainly when interaction with matter takes place. The description of space-time propagation and the particle picture are mutually exclusive – this is one aspect of the complementary principle. Why does Dirac seem to mix the two in his statement? Dirac employs the term 'photon' in a very general sense, as quantized state of the electromagnetic field. When two coherent beams are superposed, the quantized entities, the photons, cannot be ascribed to either of the sources. Interference results from superposition of two coherent states, not of two particles.
- Interference is only possible if one *cannot know* where the detected photon comes from. The quantum mechanical description of the field in a situation of interference never allows ascribing photons of the superposed field to one of the sources. In other words, if it is possible to say from which source a detected photon comes from, interference *cannot* be observed.
- Interference between two coherent beams requires a correlated or fixed phase between them, i.e., an undetermined particle number; in other words, interference is possible if and only if the photon number for each of the two beams is unknown. And a beam has an unknown photon number when the number indeterminacy is of similar size as the average number.

The statement of Dirac thus depends on the definition of the term 'photon'. A better choice of words is to say that interference is always between two (indistinguishable) state histories, but never between two quantum particles. Or, as expressed above:

▷ A photon interferes only *within* its volume of coherence, i.e., within its own cell of phase space. Outside, there is no interference. And inside that volume or cell, it is impossible to distinguish photons, states or histories.

The concept of 'photon' remains deep even today. The quantum particle model of coherence and light remains fascinating to this day. Summarizing, we can say: *Two different electromagnetic beams can interfere, but two different photons cannot.*

CURIOSITIES AND FUN CHALLENGES ABOUT PHOTONS

Can one explain refraction with photons? Newton was not able to do so, but today we can. In refraction by a horizontal surface, as shown in Figure 34, the situation is translationally invariant along the horizontal direction. Therefore, the momentum component along this direction is conserved: $p_1 \sin \alpha_1 = p_2 \sin \alpha_2$. The photon energy $E = E_1 = E_2$ is obviously conserved. The index of refraction n is defined in terms of momentum and

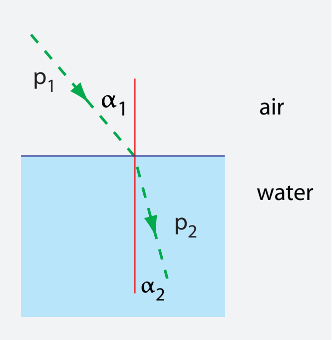


FIGURE 34 Refraction and photons.

energy as

$$n = \frac{cp}{E} \,. \qquad (17)$$

Challenge 50 e

The 'law' of refraction follows:

$$\frac{\sin \alpha_1}{\sin \alpha_2} = n \,. \qquad (18)$$

The relation is known since the middle ages.

There is an important issue here. In a material, the velocity of a photon $v = \delta E/\delta p$ in a light ray *differs* from the phase velocity $u = E/p$ that enters into the calculation. In summary, inside matter, the concept of photon must be used with extreme care.

* *

If an electromagnetic wave has amplitude A, the photon density d is

$$d = \frac{A^2}{\hbar\omega} \,. \qquad (19)$$

Challenge 51 ny

Can you show this?

* *

Challenge 52 e

Show that for a laser pulse in vacuum, the coherence volume increases during propagation, whereas the volume occupied in phase space remains constant. Its entropy is constant, as its path is reversible.

* *

Challenge 53 s

A typical effect of the quantum 'laws' is the yellow colour of the lamps used for street illumination in most cities. They emit pure yellow light of (almost) a single frequency; that is why no other colours can be distinguished in their light. According to classical electrodynamics, harmonics of that light frequency should also be emitted. Experiments show, however, that this is not the case; classical electrodynamics is thus wrong. Is this argument correct?

FIGURE 35 The blue shades of the sky and the colours of clouds are due to various degrees of Rayleigh, Mie and Tyndall scattering (© Giorgio di Iorio).

* *

How can you check whether a single-photon-triggered bomb is functional without exploding it? This famous puzzle, posed by Avshalom Elitzur and Lev Vaidman, requires interference for its solution. Can you find a way? Challenge 54 ny

* *

What happens to photons that hit an object but are not absorbed or transmitted? Generally speaking, they are scattered. *Scattering* is the name for any process that changes the motion of light (or that of any other wave). The details of the scattering process depend on the object; some scattering processes only change the direction of motion, others also change the frequency. Table 3 gives an overview of processes that scatter light.

All scattering properties depend on the material that produces the deflection of light. Among others, the study of scattering processes explains many colours of transparent materials, as we will see below. Page 171

We note that the bending of light due to gravity is not called scattering. Why not? Challenge 55 e

A SUMMARY ON LIGHT: PARTICLE AND WAVE

In summary, light is a stream of light quanta or photons. A single photon is the smallest possible light intensity of a given colour. Photons, like all quantons, are quite different from everyday particles. In fact, we can argue that the only (classical) particle aspects of photons are their quantized energy, momentum and spin. In all other respects, photons are *not* like little stones. Photons move with the speed of light. Photons cannot be localized in light beams. Photons are indistinguishable. Photons are bosons. Photons have no mass, no charge and no size. It is more accurate to say that *photons are calculating devices to precisely describe observations about light.* Ref. 36

TABLE 3 Types of light scattering.

Scattering type	Scatterer	Details	Examples
Rayleigh scattering	atoms, molecules	elastic, intensity changes as $1/\lambda^4$, scatterers smaller than $\lambda/10$	blue sky, red evening sky, blue cigarette smoke
Mie scattering	transparent objects, droplets	elastic, intensity changes as $1/\lambda^{0.5}$ to $1/\lambda^2$, scatterer size around λ	blue sky, red evenings, blue distant mountains
Geometric scattering	edges	elastic, scatterer size larger than λ	better called *diffraction*, used in interference
Tyndall scattering	non-transparent objects	elastic, angle weakly or not wavelength-dependent	smog, white clouds, fog, white cigarette smoke
Smekal–Raman scattering	excited atoms, molecules	inelastic, light gains energy	used in lidar investigations of the atmosphere
Inverse Raman scattering	atoms, molecules	inelastic, light loses energy	used in material research
Thomson scattering	electrons	elastic	used for electron density determination
Compton scattering	electrons	inelastic, X-ray lose energy	proves particle nature of light (see page 46)
Brillouin scattering	acoustic phonons, density variations in solids/fluids	inelastic, frequency shift of a few GHz	used to study phonons and to diagnose optical fibres
Von Laue or X-ray scattering	crystalline solids	elastic, due to interference at crystal planes	used to determine crystal structures; also called *Bragg diffraction*

The strange properties of photons are the reason why earlier attempts to describe light as a stream of (classical) particles, such as the attempt of Newton, failed miserably, and were rightly ridiculed by other scientists. Indeed, Newton upheld his theory against all experimental evidence – especially with regard to light's wave properties – which is something that a physicist should never do. Only after people had accepted that light is a wave, and then discovered and understood that quantum particles are fundamentally different from classical particles, was the quanton description successful.

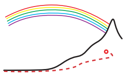
The quantum of action implies that *all waves are streams of quantons*. In fact, all waves are *correlated* streams of quantons. This is true for light, for any other form of radiation, and for all forms of matter waves.

The indeterminacy relations show that even a single quanton can be regarded as a wave; however, whenever it interacts with the rest of the world, it behaves as a particle. In fact, it is *essential* that all waves be made of quantons: if they were not, then interactions would be non-local, and objects could not be localized at all, contrary to experience.

To decide whether the wave or the particle description is more appropriate, we can use the following criterion. Whenever matter and light interact, it is more appropriate to describe electromagnetic radiation as a wave if the wavelength λ satisfies

$$\lambda \gg \frac{\hbar c}{kT} , \qquad (20)$$

where $k = 1.4 \cdot 10^{-23}\,\mathrm{J/K}$ is Boltzmann's constant and T is the temperature of the particle. If the wavelength is much *smaller* than the quantity on the right-hand side, the particle description is most appropriate. If the two sides are of the same order of magnitude, both descriptions play a role. Can you explain the criterion? Challenge 56 e



CHAPTER 3

MOTION OF MATTER – BEYOND CLASSICAL PHYSICS

> All great things begin as blasphemies.
>
> George Bernard Shaw

The existence of a smallest action has numerous important consequences for the motion of matter. We start with a few experimental results that show that the quantum of action is indeed the smallest measurable action value, also in the case of matter. Then we show that the quantum of action implies the existence of a phase and thus of the wave properties of matter. Finally, from the quantum of action, we deduce for the motion of matter the same description that we already found for light: matter particles behave like rotating arrows.

WINE GLASSES, PENCILS AND ATOMS – NO REST

> Otium cum dignitate.[**]
>
> Cicero, *De oratore*.

If the quantum of action is the smallest observable change in a physical system, then two observations of the same system must always differ. Thus there cannot be perfect rest in nature. Is that true? Experiments show that this is indeed the case.

A simple consequence of the lack of perfect rest is the impossibility of completely filling a glass of wine. If we call a glass at maximum capacity (including surface tension effects, to make the argument precise) 'full', we immediately see that the situation requires the liquid's surface to be completely at rest. This is never observed. Indeed, a completely quiet surface would admit two successive observations that differ by less than $\hbar$. We could try to reduce all motions by reducing the temperature of the system. To achieve absolute rest we would need to reach absolute zero temperature. Experiments show that this is impossible. (Indeed, this impossibility, the so-called *third 'law' of thermodynamics*, is equivalent to the existence of a minimum action.) All experiments confirm: *There is no rest in nature.* In other words, the quantum of action proves the old truth that a glass of wine is always partially empty *and* partially full.

The absence of microscopic rest, predicted by the quantum of action, is confirmed in many experiments. For example, a pencil standing on its tip *cannot* remain vertical, as shown in Figure 36, even if it is isolated from all disturbances, such as vibrations, air molecules and thermal motion. This – admittedly very academic – conclusion follows

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[**] 'Rest with dignity.'

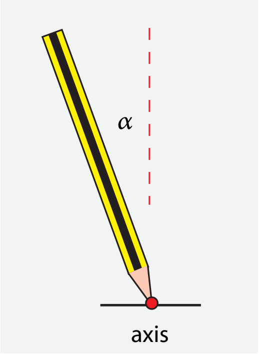

FIGURE 36 A falling pencil.

from the indeterminacy relation. In fact, it is even possible to calculate the time after which a pencil *must* have fallen over. In practice however, pencils fall over much earlier, because in usual conditions, external disturbances are much larger than the effects of the quantum of action. Challenge 57 d

But the most important consequence of the absence of rest is another. The absence of rest for the electrons inside atoms prevents them from falling into the nuclei, despite their mutual attraction. In other words, the existence and the size of atoms, and thus of all matter, is a direct consequence of the absence of microscopic rest! We will explore this consequence in more detail below. Since we are made of atoms, we can say: we only exist and live because of the quantum of action. Page 79

No infinite measurement precision

Not only does the quantum of action prevent the existence of rest; the quantum of action also prevents the observation or measurement of rest. In order to check whether an object is at rest, we need to observe its position with high precision. Because of the wave properties of light, we need a high-energy photon: only a high-energy photon has a small wavelength and thus allows a precise position measurement. As a result of this high energy, however, the object is disturbed. Worse, the disturbance itself is not precisely measurable; so there is no way to determine the original position even by taking the disturbance into account. In short, perfect rest cannot be observed – even if it existed.

Indeed, all experiments in which systems have been observed with highest precision confirm that perfect rest does not exist. The absence of rest has been confirmed for electrons, neutrons, protons, ions, atoms, molecules, atomic condensates and crystals. The absence of rest has been even confirmed for objects with a mass of about a tonne, as used in certain gravitational wave detectors. No object is ever at rest.

The same argument on measurement limitations also shows that no measurement, of any observable, can ever be performed to infinite precision. This is another of the far-reaching consequences of the quantum of action.

Cool gas

The quantum of action implies that rest is impossible in nature. In fact, even at extremely low temperatures, all particles inside matter are in motion. This fundamental lack of rest is said to be due to the so-called *zero-point fluctuations*. A good example is provided by the recent measurements of Bose–Einstein condensates. They are trapped gases, with a small number of atoms (between ten and a few million), cooled to extremely low temperatures (around 1 nK). The traps allow to keep the atoms suspended in mid-vacuum. These cool and trapped gases can be observed with high precision. Using elaborate experimental techniques, Bose–Einstein condensates can be put into states for which $\Delta p \Delta x$ is almost exactly equal to $\hbar/2$ – though never lower than this value. These experiments confirm directly that there is no observable rest, but a fundamental *fuzziness* in nature. And the fuzziness is described by the quantum of action.

This leads to an interesting puzzle. In a normal object, the distance between the atoms is much larger than their de Broglie wavelength. (Can you confirm this?) But today it is possible to cool objects to extremely low temperatures. At sufficiently low temperatures, less than 1 nK, the wavelength of the atoms may be larger than their separation. Can you imagine what happens in such cases?

Challenge 58 s

Ref. 37

Challenge 59 s

FLOWS AND THE QUANTIZATION OF MATTER

> Die Bewegung ist die Daseinsform der Materie.
> Friedrich Engels, *Anti-Dühring*.*

Not only does the quantum of action make rest impossible, it also makes impossible any situation that does not change in time. The most important examples of (apparently) stationary situations are *flows*. The quantum of action implies that *no* flow can be stationary. More precisely, a smallest action implies that no flow can be continuous. *All flows fluctuate*. In nature, all flows are made of smallest entities: all flows are made of quantum particles. We saw above that this is valid for light; it also applies to matter flows. Two simple kinds of flow from our everyday experience directly confirm this consequence from the quantum of action: flows of fluids and flows of electricity.

FLUID FLOWS AND QUANTONS

The flow of matter also exhibits smallest units. We mentioned early on in our adventure that a consequence of the particulate structure of liquids is that oil or any other smooth liquid produces *noise* when it flows through even the smoothest of pipes. We mentioned that the noise we hear in our ears in situations of absolute silence – for example, in a snowy and windless landscape in the mountains or in an anechoic chamber – is partly due to the granularity of blood flow in the veins. All experiments confirm that all flows of matter produce vibrations. This is a consequence of the quantum of action, and of the resulting granularity of matter. In fact, the quantum of action can be determined from noise measurements in fluids.

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KNOCKING TABLES AND QUANTIZED CONDUCTIVITY

If electrical current were a continuous flow, it would be possible to observe action values as small as desired. The simplest counter-example was discovered in 1996, by José Costa-Krämer and his colleagues. They put two metal wires on top of each other on a kitchen

Ref. 38, Ref. 39

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* 'Motion is matter's way of being.'

Ref. 12

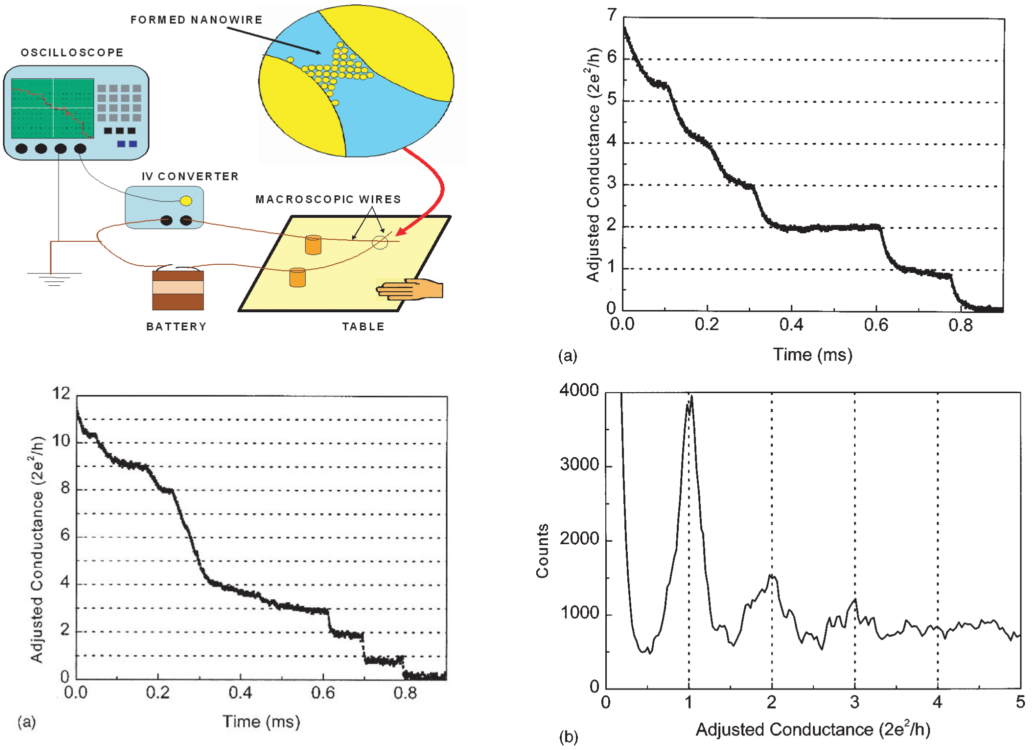

FIGURE 37 Steps in the flow of electricity in metal wire crossings: the set-up, the nanowires at the basis of the effect, and three measurement results (© José Costa-Krämer, AAPT from Ref. 39).

table and attached a battery, a current-voltage converter – or simply a resistor – and a storage oscilloscope to them. Then they measured the electrical current while *knocking* on the table. That is all.

Knocking the table breaks the contact between the two wires. In the last millisecond before the wires detach, the conductivity and thus the electrical current diminishes in regular steps of about 7 μA, as can easily be seen on the oscilloscope. Figure 37 shows such a measurement. This simple experiment could have beaten, if it had been performed a few years earlier, a number of other, enormously expensive experiments which discovered this same quantization at costs of several million euro each, using complex set-ups at extremely low temperatures.

In fact, the quantization of conductivity appears in any electrical contact with a small cross-section. In such situations the quantum of action implies that the conductivity can only be a multiple of $2e^2/\hbar \approx (12\,906\,\Omega)^{-1}$. Can you confirm this result? Note that electrical conductivity can be as small as required; only the *quantized* electrical conductivity has the minimum value of $2e^2/\hbar$. Challenge 60 e

Many more elaborate experiments confirm the observation of conductance steps. They force us to conclude that there is a *smallest electric charge* in nature. This smallest charge has the same value as the charge of an electron. Indeed, electrons turn out to be part of every atom, in a construction to be explained shortly. In metals, a large number of electrons can move freely: that is why metals conduct electricity so well and work as

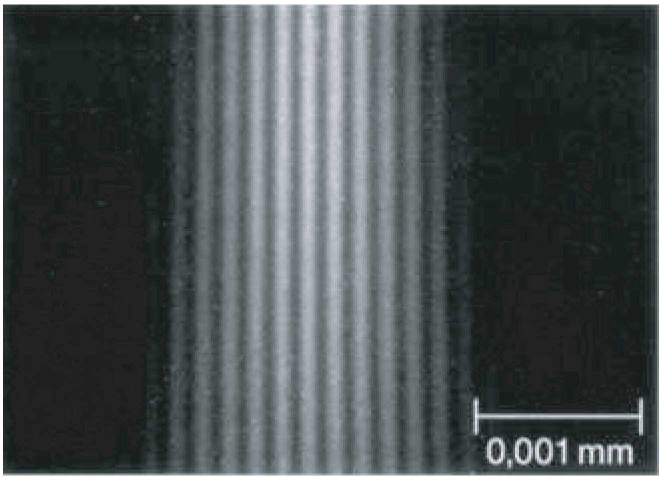

FIGURE 38 Electrons beams diffract and interfere at multiple slits (© Claus Jönsson).

mirrors.

In short, matter and electricity flow in smallest units. Depending on the flowing material, the smallest flowing units of matter may be 'molecules', 'atoms', 'ions', or 'electrons'. All of them are *quantum* particles, or *quantons*. In short, the quantum of action implies that matter is made of quantons. Matter quantons share some properties with ordinary stones, but also differ from them in many ways. A stone has position and momentum, mass and acceleration, size, shape, structure, orientation and angular momentum, and colour. We now explore each of these properties for quantons, and see how they are related to the quantum of action.

Matter quantons and their motion – matter waves

Ref. 40 In 1923 and 1924, the influential physicist Louis de Broglie pondered the consequences of the quantum of action for matter particles. He knew that in the case of light, the quantum of action connects wave behaviour to particle behaviour. He reasoned that the same should apply to matter. It dawned to him that streams of matter particles with the same momentum should behave as waves, just as streams of light quanta do. He thus predicted that like for light, coherent matter flows should have a wavelength λ and angular frequency ω given by

$$\lambda = \frac{2\pi\hbar}{p} \quad \text{and} \quad \omega = \frac{E}{\hbar} , \tag{21}$$

where p and E are the momentum and the energy, respectively, of the single particles. Equivalently, we can write the relations as

$$\boldsymbol{p} = \hbar \boldsymbol{k} \quad \text{and} \quad E = \hbar\omega . \tag{22}$$

All these relations state that matter quantons also behave as waves. For everyday objects, the predicted wavelength is unmeasurably small – though not for microscopic particles.

Soon after de Broglie's prediction, experiments began to confirm it. Matter streams

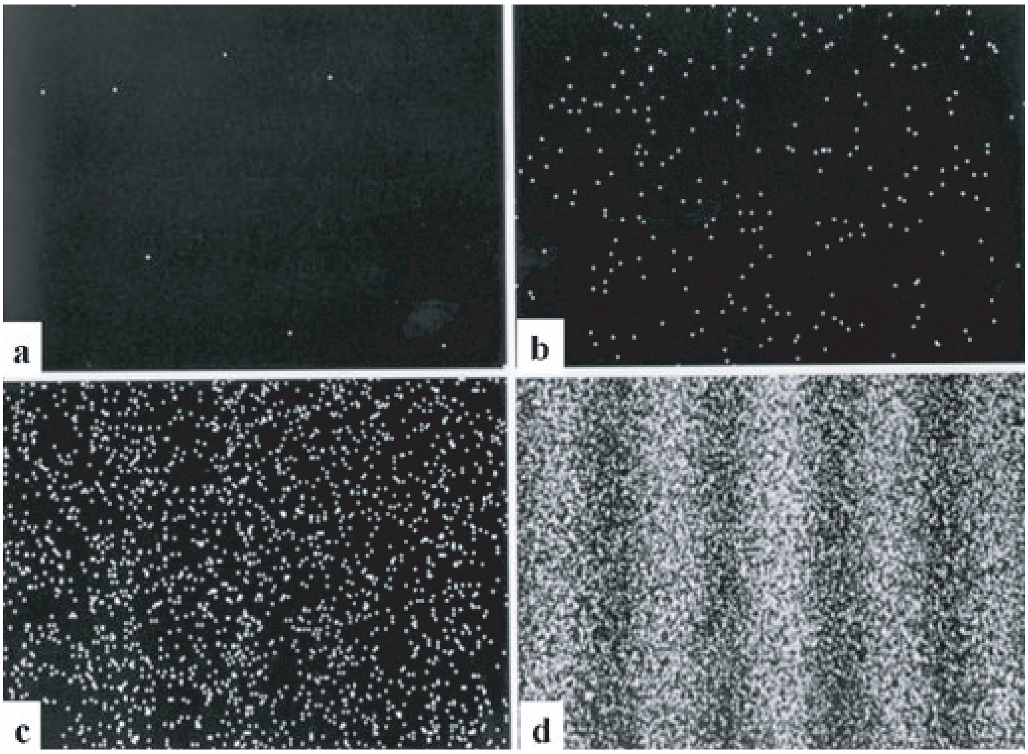

FIGURE 39 Formation over time of the interference pattern of electrons, here in a low-intensity double-slit experiment: (a) 8 electrons, (b) 270 electrons, (c) 2000 electrons, (d) 6000 electrons, after 20 minutes of exposure. The last image corresponds to the situation shown in the previous figure. (© Tonomura Akira/Hitachi).

were observed to diffract, refract and interfere; and all observations matched the values predicted by de Broglie. Because of the smallness of the wavelength of quantons, careful experiments are needed to detect these effects. But one by one, all experimental confirmations of the wave properties of light were repeated for matter beams. For example, just as light is diffracted when it passes around an edge or through a slit, matter is also diffracted in these situations. This is true even for electrons, the simplest particles of everyday matter, as shown in Figure 38. In fact, the experiment with electrons is quite difficult. It was first performed by Claus Jönsson in Tübingen in 1961; in the year 2002 it was voted the most beautiful experiment in all of physics. Many years after Jönsson, the experiment was repeated with a modified electron microscope, as shown in Figure 39. Ref. 41

Inspired by light interferometers, researchers began to build *matter* interferometers. Matter interferometers have been used in many beautiful experiments, as we will find out. Today, matter interferometers work with beams of electrons, nucleons, nuclei, atoms, or even large molecules. Just as observations of light interference prove the wave character of light, so the interference patterns observed with matter beams prove the wave character of matter. They also confirm the value of $\hbar$. Vol. V, page 142 Ref. 42 Vol. III, page 101

Like light, matter is made of particles; like light, matter behaves as a wave when large numbers of particles with the same momentum are involved. But although beams of large molecules behave as waves, everyday objects – such as cars on a motorway – do not. There are several reasons for this. First, for cars on a motorway the relevant wavelength is extremely small. Secondly, the speeds of the cars vary too much. Thirdly, cars can be counted. In summary, streams of cars with the same speed cannot be made coherent.

If matter behaves like a wave, we can draw a strange conclusion. For any wave, the position and the wavelength cannot both be sharply defined simultaneously: the indeterminacies of the wave number $k = 2\pi/\lambda$ and of the position X obey the relation

$$\Delta k \Delta X \geq \frac{1}{2} . \qquad (23)$$

Similarly, for every wave the angular frequency $\omega = 2\pi f$ and the instant T of its peak amplitude cannot both be sharply defined. Their indeterminacies are related by

$$\Delta \omega \Delta T \geq \frac{1}{2} . \qquad (24)$$

Using de Broglie's wave properties of matter (22), we get

$$\Delta p \Delta X \geqslant \frac{\hbar}{2} \qquad \text{and} \qquad \Delta E \Delta T \geqslant \frac{\hbar}{2} . \qquad (25)$$

These famous relations are called *Heisenberg's indeterminacy relations*. They were discovered by Werner Heisenberg in 1925. They are valid for all quantum particles, be they matter or radiation. The indeterminacy relations state that there is no way to simultaneously ascribe a precise momentum and position to a quantum system, nor to simultaneously ascribe a precise energy and age. The more accurately one quantity is known, the less accurately the other is.* As a result, matter quantons – rather like stones – can *always* be localized, but always only approximately. On the other hand, we saw that photons often cannot be localized.

Both indeterminacy relations have been checked experimentally in great detail. All experiments confirm them. In fact, every experiment proving that matter behaves like a wave is a confirmation of the indeterminacy relation – and vice versa.

When two variables are linked by indeterminacy relations, one says that they are *complementary* to each other. Niels Bohr systematically explored all possible such pairs. You can also do that for yourself. Bohr was deeply fascinated by the existence of a complementarity principle, and he later extended it in philosophical directions. In a well-known scene, somebody asked him what was the quantity complementary to precision. He answered: 'clarity'.

Challenge 61 s

We remark that the usual, *real*, matter quantons always move more slowly than light. Due to the inherent fuzziness of quantum motion, it should not come to a surprise that exceptions exist. Indeed, in some extremely special cases, the quantum of action allows the existence of particles that move faster than light – so-called *virtual* particles – which we will meet later on.

Page 193

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* A policeman stopped the car being driven by Werner Heisenberg. 'Do you know how fast you were driving?' 'No, but I know exactly where I was!'

In summary, the quantum of action means that matter quantons do not behave like point-like stones, but as waves. In particular, like for waves, the values of position and momentum cannot both be exactly defined for quantons. The values are fuzzy – position and momentum are undetermined. The more precisely one of the two is known, the less precisely the other is known.

MASS AND ACCELERATION OF QUANTONS

Matter quantons, like stones, have mass. Indeed, hits by single electrons, atoms or molecules can be detected, if sensitive measurement set-ups are used. Quantons can also be slowed down or accelerated. We have already explored some of these experiments in Vol. III, page 30 the section on electrodynamics. However, quantons differ from pebbles. Using the time–energy indeterminacy relation, you can deduce that Challenge 62 s

$$a \leqslant \frac{2mc^3}{\hbar} . \tag{26}$$

Thus there is a *maximum acceleration* for quantons.* Indeed, no particle has ever been observed with a higher acceleration than this value. Ref. 43 In fact, no particle has ever been observed with an acceleration anywhere *near* this value. The quantum of action thus prevents rest but also limits acceleration.

WHY ARE ATOMS NOT FLAT? WHY DO SHAPES EXIST?

The quantum of action determines all sizes in nature. In particular, it determines all *shapes*. Let us start to explore this topic.

Experiments show that all *composed* quantons, such as atoms or molecules, have structures of finite size and often with complex shape. The size and the shape of every composed quanton are due to the motion of their constituents. The motion of the constituents is due to the quantum of action; but how do they move?

In 1901, Jean Perrin and independently, in 1904, Nagaoka Hantaro proposed that atoms are small 'solar systems'. Ref. 44 In 1913, Niels Bohr used this idea, combining it with the quantum of action, and found that he could predict the size and the colour of hydrogen atoms, two properties that had not until then been understood. Ref. 45 We will perform the calculations below. Page 181 Even Bohr knew that the calculations were not completely understood, because they seemed to assume that hydrogen atoms were flat, like the solar system is. But first of all, atoms are observed to be spherical. Secondly, a flat shape would contradict the quantum of action. Challenge 64 e Indeed, the quantum of action implies that the motion of quantum constituents is fuzzy. Therefore, all composed quantons, such as atoms or molecules, must be made of *clouds* of constituents.

---

* We note that this acceleration limit is different from the acceleration limit due to general relativity:

$$a \leqslant \frac{c^4}{4Gm} . \tag{27}$$

In particular, the quantum limit (26) applies to microscopic particles, whereas the general-relativistic limit applies to macroscopic systems. Can you confirm that in each domain the relevant limit is the smaller of the two? Challenge 63 e

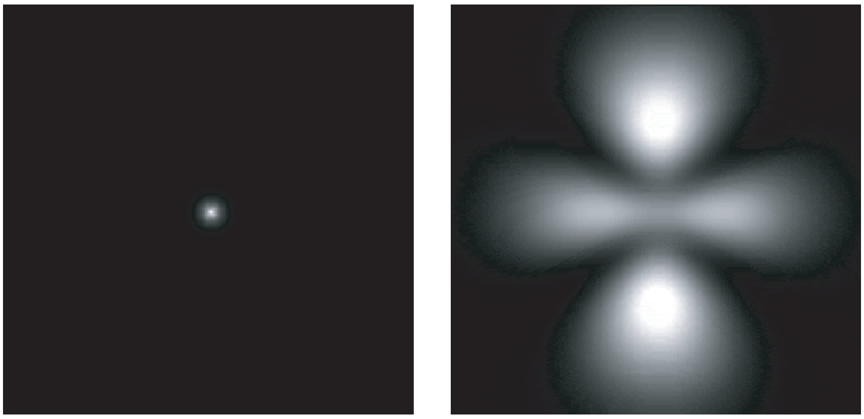

FIGURE 40 Probability clouds: a hydrogen atom in its spherical ground state (left) and in a non-spherical excited state (right) as seen by an observer travelling around it (QuickTime film produced with Dean Dauger's software package 'Atom in a Box', available at daugerresearch.com).

In short, the quantum of action predicts:

▷ Atoms are spherical clouds.

Experiment and theory confirm that the shape of any atom is due to the cloud, or probability distribution, of its lightest components, the electrons. The quantum of action thus states that atoms or molecules are not hard balls, as Democritus or Dalton believed, but that they are clouds. *Matter is made of clouds.*

Atomic electron clouds are not infinitely hard, but can to a certain degree interpenetrate and be deformed. The region where this deformation occurs is called a chemical *bond*. Bonds lead to molecules. Molecules, being composed of atoms, are composed of (deformed) spherical clouds. Bonds also lead to liquids, solids, flowers and people. A detailed exploration confirms that all shapes, from the simplest molecules to the shape of people, are due to the interactions between electrons and nuclei of the constituent atoms. Nowadays, molecular shapes can be calculated to high precision. Small molecules, like water, have shapes that are fairly rigid, though endowed with a certain degree of elasticity. Large molecules, such as polymers or peptides, have flexible shapes. These shape changes are essential for their effects inside cells and thus for our survival. A large body of biophysical and biochemical research is exploring molecular shape effects.

In summary, the quantum of action implies that shapes exist – and that they fluctuate. For example, if a long molecule is held fixed at its two ends, it cannot remain at rest in between. Such experiments are easy to perform nowadays, for example with DNA. In fact, all experiments confirm that the quantum of action prevents rest, produces sizes and shapes, and enables chemistry and life.

In nature, all sizes and shapes are due to the quantum of action. Now, every macroscopic object and every quantum object with a non-spherical shape is able to rotate. We therefore explore what the quantum of action can say about rotation.

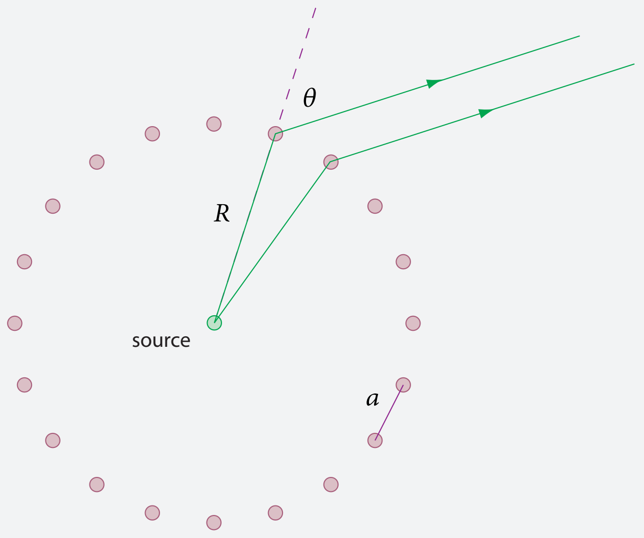

FIGURE 41 The quantization of angular momentum.

ROTATION, QUANTIZATION OF ANGULAR MOMENTUM, AND THE LACK OF NORTH POLES

> “Tristo è quel discepolo che non avanza il suo maestro.”
>
> Leonardo da Vinci*

In everyday life, rotation is a frequent type of motion. Wheels are all around us. It turns out that the quantum of action has important consequences for rotational motion. First of all, we note that action and angular momentum have the same physical dimension: both are measured in Js or Nms. It only takes a little thought to show that if matter or radiation has a momentum and wavelength related by the quantum of action, then angular momentum is fixed in multiples of the quantum of action. This beautiful argument is due to Dicke and Wittke. Ref. 46

Imagine a circular fence, made of N vertical steel bars spaced apart at a distance $a = 2\pi R/N$, as shown in Figure 41. At the centre of the fence, imagine a source of matter or radiation that can emit particles towards the fence in any chosen direction. The linear momentum of such a particle is $p = \hbar k = 2\pi\hbar/\lambda$. At the fence slits, the wave will interfere. Outside the fence, the direction of the motion of the particle is determined by the condition of constructive interference. In other words, the angle θ, describing the direction of motion outside the fence, is given by $a \sin\theta = M\lambda$, where M is an integer. Through the deflection due to the interference process, the fence receives a linear momentum $p \sin\theta$, or an angular momentum $L = pR \sin\theta$. Combining all these expressions, we find that the angular momentum transferred to the fence is Challenge 65 e

$$L = NM\hbar\,. \tag{28}$$

---

* ‘Sad is that disciple who does not surpass his master.’ This statement from one of his notebooks, the Codice Forster III, is sculpted in large letters in the chemistry aula of the University of Rome *La Sapienza*.

In other words, the angular momentum of the fence is an integer multiple of $\hbar$. Fences can only have integer intrinsic angular momenta (in units of $\hbar$). The generalization of the argument to all bodies is also correct. (Of course, this latter statement is only a hint, not a proof.)

▷ The measured intrinsic angular momentum of bodies is always a multiple of $\hbar$.

Quantum theory thus states that every object's angular momentum increases in steps. Angular momentum is quantized. This result is confirmed by all experiments.

But rotation has more interesting aspects. Thanks to the quantum of action, just as linear momentum is usually fuzzy, so is angular momentum. There is an indeterminacy relation for angular momentum L. The complementary variable is the phase angle φ of the rotation. The indeterminacy relation can be expressed in several ways. The simplest approximation – and thus *not* the exact expression – is Ref. 47 Ref. 48 Page 48

$$\Delta L\,\Delta\varphi \geqslant \frac{\hbar}{2}\ . \qquad (29)$$

This is obviously an approximation: the relation is only valid for large angular momenta. In any case, the expression tells us that rotation behaves similarly to translation. The expression cannot be valid for small angular momentum values, as $\Delta\varphi$ by definition cannot grow beyond 2π. In particular, angular-momentum eigenstates have $\Delta L = 0$.*

The indeterminacy of angular momentum appears for all macroscopic bodies. We can say that the indeterminacy appears for all cases when the angular phase of the system can be measured.

The quantization and indeterminacy of angular momentum have important consequences. Classically speaking, the poles of the Earth are the places that do not move when observed by a non-rotating observer. Therefore, at those places matter would have a defined position and a defined momentum. However, the quantum of action forbids this. There cannot be a North Pole on Earth. More precisely, the idea of a *fixed* rotational axis is an approximation, not valid in general. This applies in particular to rotating quantum particles.

Rotation of quantons

The effects of the quantum of action on the rotation of microscopic particles, such as atoms, molecules or nuclei, are especially interesting. We note again that action and angular momentum have the same units. The precision with which angular momentum

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* An *exact* formulation of the indeterminacy relation for angular momentum is

$$\Delta L\,\Delta\varphi \geqslant \frac{\hbar}{2}|1-2\pi P(\pi)|\ , \qquad (30)$$

where $P(\pi)$ is the normalized probability that the angular position has the value π. For an angular-momentum eigenstate, one has $\Delta\varphi = \pi/\sqrt{3}$ and $P(\pi) = 1/2\pi$. This exact expression has been tested and confirmed by experiments. Ref. 49

can be measured depends on the precision of the rotation angle. But if a *microscopic* particle rotates, this rotation might be unobservable: a situation in fundamental contrast with the case of *macroscopic* objects. Experiments indeed confirm that many microscopic particles have unobservable rotation angles. For example, in many (but not all) cases, an atomic nucleus rotated by half a turn cannot be distinguished from the unrotated nucleus.

If a microscopic particle has a *smallest* unobservable rotation angle, the quantum of action implies that the angular momentum of that particle *cannot* be zero. It must always be rotating. Therefore we need to check, for each particle, what its smallest unobservable angle of rotation is. Physicists have checked all particles in nature in experiments, and found smallest unobservable angles (depending on the particle type) of 0, 4π, 2π, $4\pi/3$, π, $4\pi/5$, $2\pi/3$ etc.

Let us take an example. Certain nuclei have a smallest unobservable rotation angle of *half* a turn. This is the case for a prolate nucleus (one that looks like a rugby ball) turning around its short axis, such as a ^{23}Na nucleus. In this case, both the largest observable rotation angle and the indeterminacy are thus a *quarter* turn. Since the change, or action, produced by a rotation is the number of turns multiplied by the angular momentum, we find that the angular momentum of this nucleus is $2 \cdot \hbar$.

As a general result, we deduce from the minimum angle values that the angular momentum of a microscopic particle can be 0, $\hbar/2$, $\hbar$, $3\hbar/2$, $2\hbar$, $5\hbar/2$, $3\hbar$ etc. In other words, the intrinsic angular momentum of a particle, usually called its *spin*, is an integer multiple of $\hbar/2$. Spin describes how a particle behaves under rotations.

How can a particle rotate? At this point, we do not yet know how to *picture* the rotation. But we can *feel* it – just as we showed that light is made of rotating entities: all matter, including electrons, can be *polarized*. This is shown clearly by the famous Stern–Gerlach experiment.

Silver, Stern and Gerlach – polarization of quantons

After a year of hard work, in 1922, Otto Stern and Walther Gerlach* completed a beautiful experiment to investigate the polarization of matter quantons. They knew that inhomogeneous magnetic fields act as polarizers for rotating charges. Rotating charges are present in every atom. Therefore they let a beam of silver atoms, extracted from an oven by evaporation, pass an inhomogeneous magnetic field. They found that the beam splits into *two separate* beams, as shown in Figure 42. No atoms leave the magnetic field region in intermediate directions. This is in full contrast to what would be expected from classical physics. Ref. 50

The splitting into *two* beams is an intrinsic property of silver atoms; today we know that it is due to their spin. Silver atoms have spin $\hbar/2$, and depending on their orientation in space, they are deflected either in the direction of the field inhomogeneity or against it. The splitting of the beam is a pure quantum effect: there are no intermediate options. Indeed, the Stern–Gerlach experiment provides one of the clearest demonstrations that classical physics does not work well in the microscopic domain. In 1922, the

---

* Otto Stern (1888–1969) and Walther Gerlach (1889–1979) worked together at the University of Frankfurt. For his subsequent measurement of the anomalous magnetic moment of the proton, Stern received the Nobel Prize in Physics in 1943, after he had to flee National Socialism.

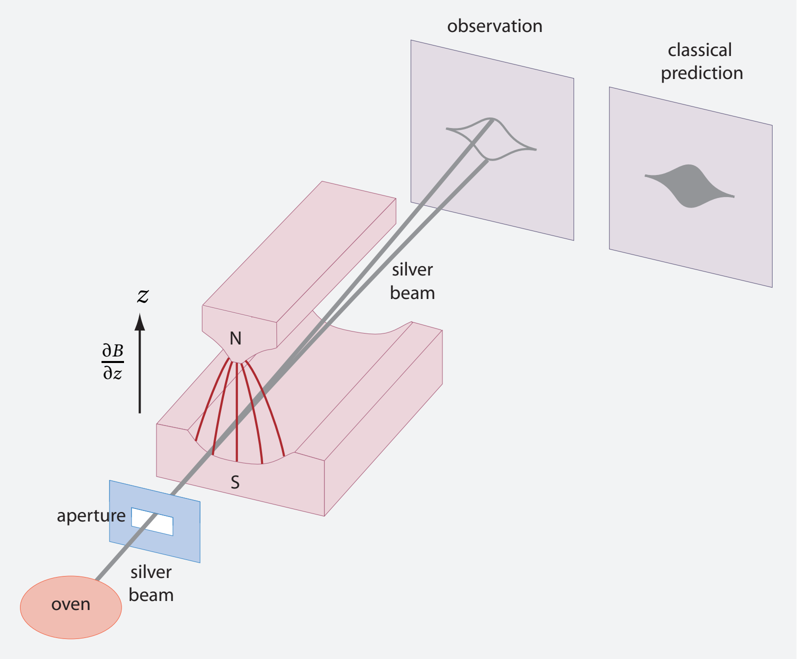


FIGURE 42 The Stern–Gerlach experiment.

result seemed so strange that it was studied in great detail all over the world.

When one of the two beams – say the ‘up’ beam – is passed through a second set-up, all the atoms end up in the ‘up’ beam. The other possible exit, the ‘down’ beam, remains unused in this case. In other words, the up and down beams, in contrast to the original beam, cannot be split further. This is not surprising.

But if the second set-up is rotated by $\pi/2$ with respect to the first, again two beams – ‘right’ and ‘left’ – are formed, and it does not matter whether the incoming beam is directly from the oven or from the ‘up’ part of the beam. A partially-rotated set-up yields a partial, uneven split. The proportions of the two final beams depend on the angle of rotation of the second set-up.

We note directly that if we split the beam from the oven first vertically and then horizontally, we get a different result from splitting the beam in the opposite order. (You can Challenge 66 e check this yourself.) Splitting processes do not commute. When the order of two operations makes a difference to the net result, physicists call them *non-commutative*. Since all measurements are also physical processes, we deduce that, in general, measurements and processes in quantum systems are non-commutative.

Beam splitting is direction-dependent. Matter beams behave almost in the same way as polarized light beams. Indeed, the inhomogeneous magnetic field acts on matter somewhat like a polarizer acts on light. The up and down beams, taken together, define a polarization direction. Indeed, the polarization direction can be rotated, with the help of a *homogeneous* magnetic field. And a rotated beam in a unrotated magnet behaves like an unrotated beam in a rotated magnet.

In summary, matter quantons can be polarized. We can picture polarization as the orientation of an internal rotation axis of the massive quanton. To be consistent, the

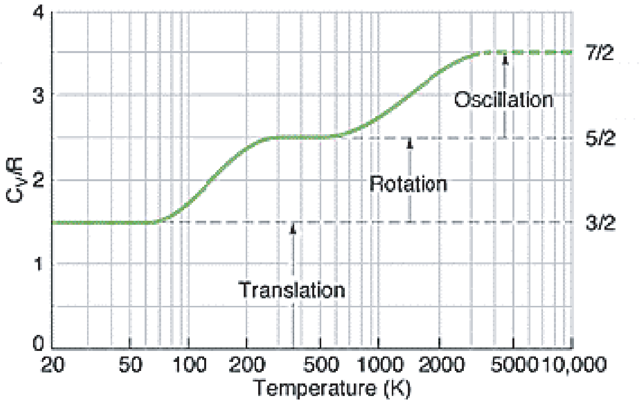


FIGURE 43 An idealized graph of the heat capacity of hydrogen over temperature (© Peter Eyland).

rotation axis must be imagined to precess around the direction of polarization. Thus, massive quantum particles resemble photons also in their polarizability.

Curiosities and fun challenges about quantum matter

> It is possible to walk while reading, but not to read while walking.
>
> Serge Pahaut

The quantum of action implies that there are *no fractals* in nature. Everything is made of particles. And particles are clouds. Quantum theory requires that all shapes in nature be 'fuzzy' *clouds*.

∗ ∗

Can atoms rotate? Can an atom that falls on the floor roll under the table? Can atoms be put into high-speed rotation? The answer is 'no' to all these questions, because angular Ref. 51 momentum is quantized; moreover, atoms are not solid objects, but clouds. The macroscopic case of an object turning more and more slowly until it stops does not exist in the microscopic world. The quantum of action does not allow it.

∗ ∗

Light is refracted when it enters dense matter. Do matter waves behave similarly? Yes, they do. In 1995, David Pritchard showed this for sodium waves entering a gas of helium Ref. 52 and xenon.

∗ ∗

Many quantum effects yield curves that show steps. An important example is the molar heat of hydrogen H_2 gas, shown in Figure 43. In creasing the temperature from 20 to 8 000 K, the molar heat is shows two steps, first from $3R/2$ to $5R/2$, and then to $7R/2$. Can you explain the reason?

∗ ∗

Most examples of quantum motion given so far are due to electromagnetic effects. Can you argue that the quantum of action must also apply to nuclear motion, and in particular, to the nuclear interactions? Challenge 67 s

* *

There are many other formulations of the indeterminacy principle. An interesting one is due to de Sabbata and Sivaram, who explained in 1992 that the following intriguing relation between temperature and time also holds:

$$\Delta T \Delta t \geqslant \hbar / k . \qquad (31)$$

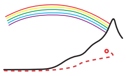
Ref. 53 Here, k is the Boltzmann constant. All experimental tests so far have confirmed the result.

* *

Challenge 68 e Here is a trick question: what is the moment of inertia of an electron? Why?

FIRST SUMMARY ON THE MOTION OF QUANTUM PARTICLES

In summary, the 'digital' beam splitting seen in the Stern–Gerlach experiment and the wave properties of matter force us to rethink our description of motion. They show that microscopic matter motion follows from the quantum of action, the smallest observable action value. In special relativity, the existence of a maximum speed forced us to introduce the concept of space-time, and then to refine our description of motion. In general relativity, the maximum force obliged us to introduce the concepts of horizon and curvature, and then again to refine our description of motion. At the present point, the existence of the quantum of action and the wave behaviour of matter force us to take two similar steps: we first introduce the concept of a wave function, and then we refine our description of matter motion.



CHAPTER 4

THE QUANTUM DESCRIPTION OF MATTER AND ITS MOTION

Ref. 54

> “Die Quanten sind doch eine hoffnungslose Schweinerei![**] Max Born

In everyday life and in classical physics, we say that a system *has* a position, that it *is* oriented in a certain direction, that it *has* an axis of rotation, and that it *is* in a state with specific momentum. In classical physics, we can talk in this way because the *state* – the situation a system ‘is’ in and the properties a system ‘has’ – coincide with the *results of measurement*. They coincide because measurements can always be imagined to have a negligible effect on the system.

However, because of the existence of a smallest action, the interaction necessary to perform a measurement on a system *cannot* be made arbitrarily small. Therefore, the quantum of action makes it *impossible* for us to continue saying that a system *has* momentum, *has* position or *has* an axis of rotation. The quantum of action forces us to use the idea of the rotating arrow and to introduce the concept of *wave function* or *state function*. Let us see why and how.

STATES AND MEASUREMENTS – THE WAVE FUNCTION

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The Stern–Gerlach experiment shows that the measured values of spin *orientation* are *not* intrinsic, but result from the measurement process (in this case, from the interaction with the applied inhomogeneous field). This is in contrast to the spin *magnitude*, which is intrinsic and independent of state and measurement. In short, the quantum of action forces us to distinguish carefully three concepts:

- the state of the system;
- the operation of measurement;
- the result or outcome of the measurement.

In contrast to the classical, everyday case, the *state* of a quantum system – the properties a system ‘has’ – is *not* described by the outcomes of measurements. The simplest illustration of this difference is the system made of a single particle in the Stern–Gerlach experiment. The experiment shows that a spin measurement on a general (oven) particle state sometimes gives ‘up’ (say $+1$), and sometimes gives ‘down’ (say -1). So a general atom, in an oven state, has no intrinsic orientation. Only *after* the measurement, an atom is either in an ‘up’ state or in a ‘down’ state.

[**] ‘Those quanta are a hopeless dirty mess!’

It is also found that feeding ‘up’ states into a second measurement apparatus gives only ‘up’ states: thus certain special states, called *eigenstates*, do remain unaffected by measurement.

Finally, the Stern–Gerlach experiment and its variations show that states can be rotated by applied fields: atom states have a direction or orientation in space. The experiments also show that the states rotate as the atoms move through space.

The experimental observations can be described in a straightforward way. Since measurements are operations that take a state as input and produce an output state and a measurement result, we can say:

- *States* are described by rotating arrows, or rotating vectors.
- *Measurements* of observables are operations on the state vectors.
- *Measurement results* are *real numbers*; and like in classical physics, they usually depend on the observer.

In particular, we have distinguished two quantities that are not distinguished in classical physics: states and measurement results. Given this distinction, quantum theory follows quite simply, as we shall see.

Given that the quantum of action is not vanishingly small, any measurement of an observable quantity is an interaction with a system and thus a transformation of its state. Therefore, quantum physics describes physical observables as *operators*, or equivalently, as transformations. The Stern–Gerlach experiment shows this clearly: the interaction with the field influences the atoms: some in one way, and some in another way. In fact, all experiments show:

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- Mathematically, states are *complex vectors*, or rotating arrows, in an abstract space. This space of all possible states or arrows is a *Hilbert space*.
- Mathematically, measurements are linear transformations, more precisely, they are described by self-adjoint, or *Hermitean*, *operators* (or matrices).
- Mathematically, *changes of viewpoint* are described by *unitary operators* (or matrices) that act on states, or arrows, and on measurement operators.

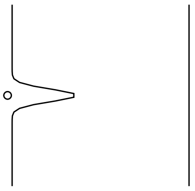
Quantum-mechanical experiments also show that a measurement of an observable can only give a result that is an *eigenvalue* of the corresponding transformation. The resulting states after the measurement, those exceptional states that are *not* influenced when the corresponding variable is measured, are the *eigenvectors*. In short, every expert on motion must know what an eigenvalue and an eigenvector is.

For any linear transformation T, those special vectors ψ that are transformed into multiples of themselves,

$$T\psi = \lambda\psi \tag{32}$$

are called *eigenvectors* (or *eigenstates*), and the multiplication factor λ is called the associated *eigenvalue*. Experiments show:

- The state of the system *after* a measurement is given by the eigenvector corresponding to the measured eigenvalue.



In the Stern–Gerlach experiment, the eigenstates are the ‘up’ and the ‘down’ states. In general, the eigenstates are those states that do not change when the corresponding variable is measured. Eigenvalues of Hermitean operators are always real, so that consistency is ensured: all measurement results are real numbers.

In summary, the quantum of action obliges us to distinguish between three concepts that are mixed together in classical physics: the *state* of a system, a *measurement* on the system, and the *measurement result*. The quantum of action forces us to change the vocabulary with which we describe nature, and obliges to use more differentiated concepts. Now follows the main step: the description of *motion* with these concepts. This is what is usually called ‘quantum theory’.

Visualizing the wave function: rotating arrows and probability clouds

We just described the state of a quanton with an arrow. In fact, this is an approximation for localized quantons. More precisely,

> ▷ The state of a quantum particle is described by a spatial distribution of arrows, a so-called *wave function*.

To develop a visual image of the wave function, we first imagine a quantum particle that is localized as much as possible. In this case, the wave function for a free quanton can be described simply by a single rotating arrow.

Experiments show that when a *localized* quanton travels through space, the attached arrow rotates. If the particle is non-relativistic and if spin can be neglected, the rotation takes place in a plane perpendicular to the direction of motion. The end of the arrow then traces a *helix* around the direction of motion. In this case, the state at a given time is described by the angle of the arrow. This angle is the *quantum phase*. The quantum phase is responsible for the wave properties of matter, as we will see. The wavelength and the frequency of the helix are determined by the momentum and the kinetic energy of the particle.

If the particle is *not localized* – but still non-relativistic and still with negligible spin effects – the state, or the wave function, defines a rotating arrow *at each point in space*. The rotation still takes place in a plane perpendicular to the direction of motion. But now we have a distribution of arrows that all trace helices parallel to the direction of motion. At each point in space and time, the state has a quantum phase and a length of the arrow. The arrow lengths decrease towards spatial infinity.

Figure 44 shows an example of evolution of a wave function for non-relativistic particles with negligible spin effects. The direction of the arrow at each point is shown by the colour at the specific point. The length of the arrow is shown by the brightness of the colour. For non-relativistic particles with negligible spin effects, the wave function $\psi(t, x)$ is thus described by a length and a phase: it is a *complex* number at each point in space. The phase is essential for interference and many other wave effects. What measurable property does the amplitude, the length of the local arrow, describe? The answer was given by the famous physicist Max Born:

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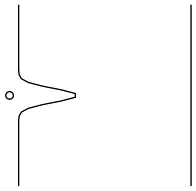

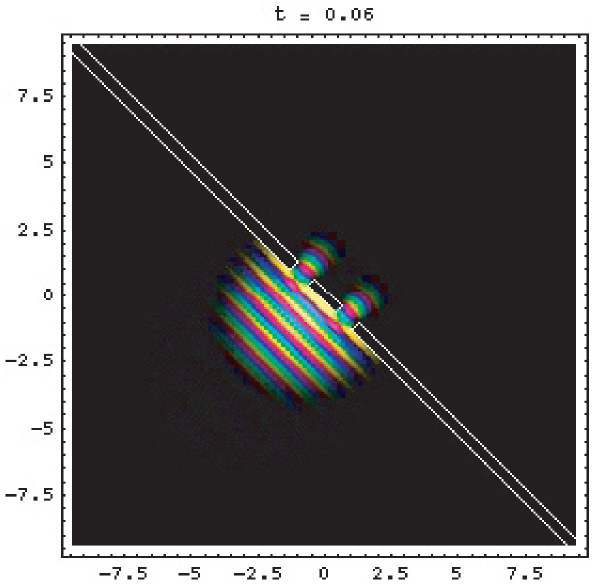

FIGURE 44 The motion of a wave function, the quantum state, through a double slit, showing both the particle and the wave properties of matter. The density of the state, related to the arrow length, is displayed by brightness, and the local phase is encoded in the colour. (QuickTime film © Bernd Thaller)

▷ The amplitude of the wave function is a *probability amplitude*. The square of the amplitude, i.e., the quantity $|\psi(t,x)|^2$, gives the probability to find the particle at the place x at time t.

In other terms, a wave function is a combination of two ideas: on the one hand, a wave function is a *cloud*; on the other hand, at each point of the cloud one has to imagine an arrow. Over time, the arrows rotate and the cloud changes shape.

▷ A wave function is a cloud of rotating arrows.

Describing the state of a matter particle with a cloud of rotating arrows is the essential step to picture the wave properties of matter.

We can clarify the situation further.

▷ In every process in which the phase of the wave function is not important, the cloud image of the wave function is sufficient and correct.

For example, the motion of atoms of molecules in gases or liquids can be imagined as the motion of cloudy objects. It needs to be stressed that the clouds in question are quite hard: it takes a lot of energy to deform atomic clouds. The hardness of a typical crystal is directly related to the hardness of the atomic clouds that are found inside. Atoms are extremely stiff, or hard clouds.

On the other hand,

▷ In every process in which the phase of the wave function does play a role, the cloud image of the wave function needs to be expanded with rotating arrows at each point.

This is the case for interference processes of quantons, but also for the precise description of chemical bonds.

Teachers often discuss the best way to explain wave functions. Some teachers prefer to use the cloud model only, others prefer not to use any visualization at all. Both approaches are possible; but the most useful and helpful approach is to imagine the state or wave function of a *non-relativistic* quantum particle as an arrow at every point in space. The rotation frequency of the set of arrows is the *kinetic energy* of the particle; the wavelength of the arrow motion – the period of the helical curve that the tip of the arrows – or of the average arrow – traces during motion – is the *momentum* of the quantum particle.

An arrow at each point in space is a (mathematical) *field*. The field is concentrated in the region where the particle is located, and the amplitude of the field is related to the probability to find the particle. Therefore the state field, the wave function or state function, is an *arrow cloud*. It is usually called with the greek letter ψ.

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Challenge 69 e

Note that even though the wave function can be seen as defining an arrow at every point in space, the wave function as a whole can also be described as one, single vector, this time in a Hilbert space. For free particles, i.e., particles that are not subject to external forces, the Hilbert space is infinite dimensional! Nevertheless, it is not hard to calculate in such spaces. The scalar product of two wave functions is the spatial integral of the product of the complex conjugate of the first function and the (unconjugated) second function. With this definition, all vector concepts (unit vectors, null vectors, basis vectors, etc.) can be meaningfully applied to wave functions.

In summary, for non-relativistic particles without spin effects, *the state or wave function of a quantum particle is a cloud, or a distributed wave, of rotating arrows.* This aspect of a quantum cloud is unusual. Since a quantum cloud is made of little arrows, every point of the cloud is described by a local density and a local orientation. This latter property does not occur in any cloud of everyday life.

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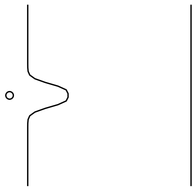
For many decades it was tacitly assumed that a wave function ψ cannot be visualized more simply than with a cloud of rotating arrows. Only the last years have shown that there are other visualization for such quantum clouds; one possible visualization is presented in the last volume of this series.

The state evolution – the Schrödinger equation

The description of the state of a non-relativistic quanton with negligible spin effects as a rotating cloud *completely* determines how the wave function evolves in time. Indeed, for such quantum particles the evolution follows from the total energy, the sum of kinetic and potential energy $T + V$, and the properties of matter waves:

▷ The local rate of change of the state arrow(s) $\psi(x)$, or simply ψ, is produced



by the local total energy, or Hamiltonian, $H = T + V$:

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi \ . \tag{33}$$

This famous equation is *Schrödinger's equation of motion*.* This *evolution equation* applies to all quantum systems and is one of the high points of modern physics.

Ref. 55 Ref. 56 In fact, Erwin Schrödinger had found his equation in two different ways. In his first paper, he deduced it from a variational principle. In his second paper, he deduced the evolution equation directly, by asking a simple question: how does the state evolve? He knew that the state of a quanton behaves both like a wave and like a particle. A wave is described by a field, which he denoted $\psi(t, x)$. If the state ψ behaves like a wave, then the corresponding wave function must be an amplitude W multiplied by a phase factor $\mathrm{e}^{i\boldsymbol{kx}-\omega t}$. The state can thus be written as

$$\psi(t, x) = W(t, x)\, \mathrm{e}^{i\boldsymbol{kx}-\omega t} \ . \tag{34}$$

The amplitude W is the *length* of the local arrow; the phase is the *orientation* of the local arrow. Equivalently, the amplitude is the local density of the cloud, and the phase is the local orientation of the cloud.

We know that the quantum wave must also behave like a particle of mass m. In particular, the non-relativistic relation between energy and momentum $E = \boldsymbol{p}^2/2m + V(\boldsymbol{x})$ – where $V(\boldsymbol{x})$ is the potential at position $\boldsymbol{x}$ – must be fulfilled for these waves. The two de Broglie relations (22) for matter wavelength and matter frequency then imply Page 76

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi = \frac{-\hbar^2}{2m} \nabla^2 \psi + V(\boldsymbol{x})\psi \ . \tag{35}$$

This is the complete form of Schrödinger's wave equation. ∇^2 is the Laplace operator, essentially the second derivative over space. It states how the arrow wave, the wave function ψ associated to a particle, evolves over time. In 1926, this *wave equation* for the complex field ψ became instantly famous when Schrödinger used it, by inserting the potential felt by an electron near a proton, to calculate the energy levels of the hydrogen atom. In a hydrogen atom, light is emitted by the single electron inside that atom; therefore a precise description of the motion of the electron in a hydrogen atom allows us to describe the light frequencies it can emit. (We will perform the calculation and the comparison with experiment below.) Page 181 First of all, the Schrödinger equation explained that only *discrete* colours are emitted by hydrogen. In addition, the frequencies of the emitted light were found to be in agreement with the prediction of the equation to five decimal places. Finally, the

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* Erwin Schrödinger (b. 1887 Vienna, d. 1961 Vienna) was famous for being a *physicien bohémien*, always living in a household with two women. In 1925 he discovered the equation that brought him international fame, and the Nobel Prize in Physics in 1933. He was also the first to show that the radiation discovered by Victor Hess in Vienna was indeed coming from the cosmos. He left Germany, and then again Austria, out of dislike for National Socialism, and was a professor in Dublin for many years. There he published his famous and influential book *What is life?*. In it, he came close to predicting the then-unknown nucleic acid DNA from theoretical insight alone.

FIGURE 45 Erwin Schrödinger (1887–1961)

size of atoms was predicted correctly. These were important results, especially if we keep in mind that classical physics cannot even explain the existence of atoms, let alone their light emission! In contrast, quantum physics explains all properties of atoms and their colours to high precision. In other words, the discovery of the quantum of action led the description of the motion of matter to a new high point.

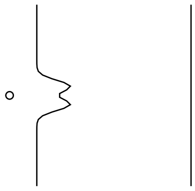
In fact, the exact description of matter quantons is only found when both *spin effects* and the *relativistic* energy–momentum relation are taken into account. We do this Page 188 below. No deviations between the full relativistic calculations and experiments have ever been found. And even today, predictions and measurements of atomic spectra remain the most precise and accurate in the whole study of nature: in the cases that experimental precision allows it, the calculated values agree with experiments to 13 decimal places.

Self-interference of quantons

Page 76 Waves interfere. All experiments, including the examples shown in Figure 38 and Figure 39, confirm that all quantum particles, and in particular all matter quantons, show interference. Interference is a direct consequence of the Schrödinger equation, as the Page 90 film of Figure 44 shows. The film illustrates the solution of the Schrödinger equation for a quantum particle moving through a double slit. The film visualizes how a double slit induces diffraction and interference for a matter particle.

It turns out that the Schrödinger equation completely reproduces and explains the observations of matter interference: also the interference of matter quantons is due to the evolution of clouds of rotating arrows. And like in all interference phenomena, the local intensity of the interference pattern turns out to be proportional to the square $|W|^2$ of the local wave amplitude. And the local wave amplitude results from the phase of the interfering wave trains. The analogy with light interefence is complete; even the formulae are the same.

We note that even though the wave function is spread out over the whole detection screen just before it hits the screen, it nevertheless yields only a localized spot on the Page 153 screen. This effect, the so-called *collapse of the wave function*, is explored in detail below.



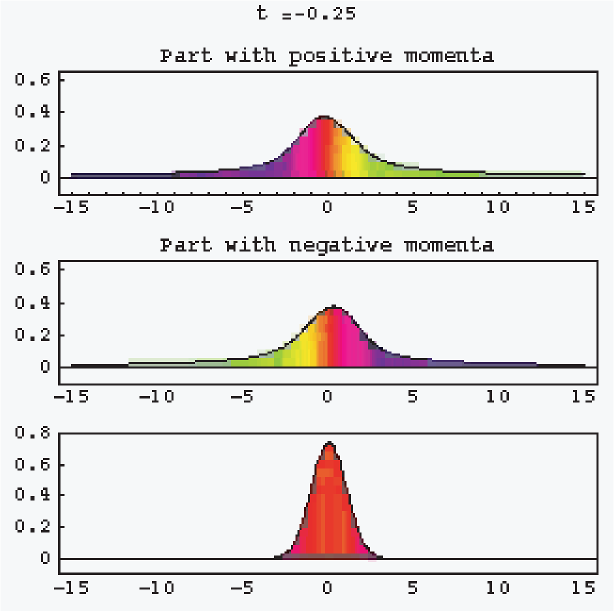


FIGURE 46 The evolution of a wave function (lowest curve) with zero momentum, and the motion of its parts with positive and negative momenta. Local phase is encoded in the colour. (QuickTime film © Bernd Thaller)

THE SPEED OF QUANTONS

Let us delve a little into the details of the description given by the Schrödinger equation (35). The equation expresses a simple connection: the classical speed of a matter particle is the *group velocity* of the wave function ψ. Seen from far away, the wave function thus moves like a *classical* particle would.

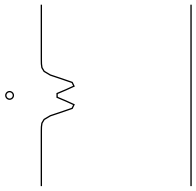
But we know from classical physics that the group velocity is not always well defined: in cases where the group dissolves into several peaks, the concept of group velocity is not of much use. These are also the cases in which quantum motion is very different from classical motion, as we will soon discover. But for well-behaved cases, such as free or almost free particles, we find that the wave function moves in the same way as a classical particle does.

Page 153

The Schrödinger equation makes another point: velocity and position of matter are not independent variables, and cannot be chosen at will. The initial condition of a system is given by the initial value of the wave function alone. No derivatives have to be (or can be) specified. Indeed, experiments confirm that quantum systems are described by a *first-order* evolution equation, in stark contrast to classical systems. The reason for this contrast is the quantum of action and the limit it poses on the possible state variables of a particle.

DISPERSION OF QUANTONS

For free quantum particles, the Schrödinger's evolution equation implies *dispersion*, as illustrated in Figure 46. Imagine a wave function that is localized around a given starting position. Such a wave function describes a quantum system at rest. When time passes,



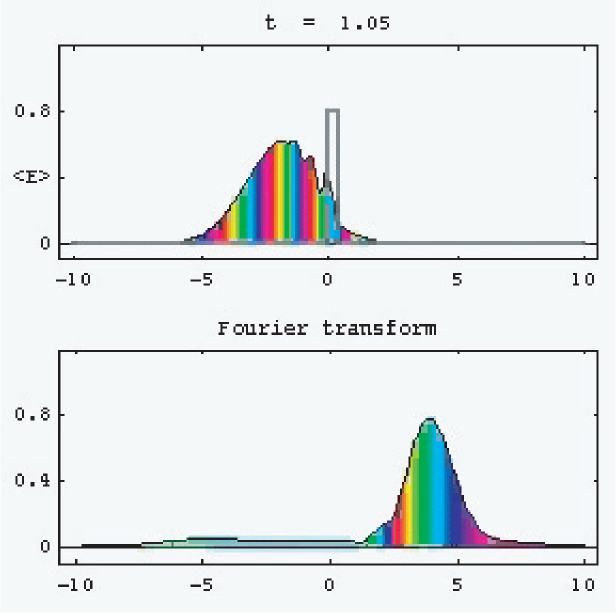


FIGURE 47 The tunnelling of a wave function through a potential hill (the rectangular column): most of the wave function is reflected, and part of the wave function passes to the other side. Local phase is encoded in the colour. (QuickTime film © Bernd Thaller)

Challenge 70 e

Challenge 71 ny

this wave function will *spread out* in space. Indeed, Schrödinger's evolution equation is similar, mathematically, to a diffusion equation. In the same way that a drop of ink in water spreads out, also the state of a localized quantum particle will spread out in space. True, the most probable position stays unchanged, but the probability to find the particle at large distances from the starting position increases over time. For quantum particles, this spreading effect is indeed observed by all experiments. The spread is a consequence of the wave aspect of matter, and thus of the quantum of action $\hbar$. It occurs for quantons at rest and therefore also for quantons in motion. For macroscopic objects, the spreading effect is not observed, however: cars rarely move away from parking spaces. Indeed, quantum theory predicts that for macroscopic systems, the effect of spreading is negligibly small. Can you show why?

In summary, the wave aspect of matter leads to the spreading of wave functions. Wave functions show dispersion.

TUNNELLING AND LIMITS ON MEMORY – DAMPING OF QUANTONS

'Common sense' says that a slow ball cannot roll over a high hill. More precisely, classical physics says that if the kinetic energy T is smaller than the potential energy V that the ball would have at the top of the hill, then the ball cannot reach the top of the hill. In contrast, according to quantum theory, there is a non-vanishing probability of passing the hill for *any* energy of the ball.

In quantum theory, hills and obstacles are described by potential barriers, and objects by wave functions. Any initial wave function will spread *beyond* any potential barrier of finite height and width. The wave function will also be non-vanishing *at* the location of the barrier. In short, any object can overcome any hill or barrier, as shown in Figure 48.

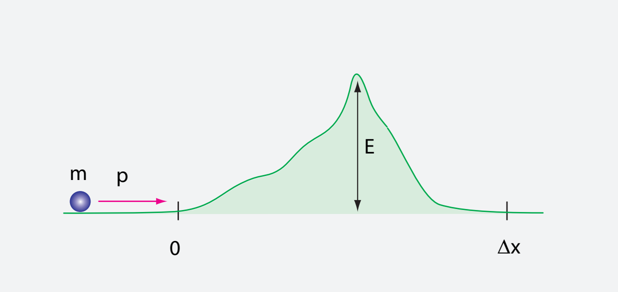


FIGURE 48 Climbing a hill.

This effect is called the *tunnelling effect*. It is in complete contrast to everyday experience – and to classical mechanics.

The tunnelling effect results from a new aspect contained in the quantum description of hills: in nature, any obstacle can be overcome with a *finite* effort. No obstacle is infinitely difficult to surmount. Indeed, only for a potential of infinite height would the wave function vanish and fail to spread to the other side. But such potentials exist only as approximations; in nature potentials are always of finite value.

Challenge 72 ny How large is the tunnelling effect? Calculation shows that the transmission probability P is given approximately by

$$P \approx \frac{16T(V-T)}{V^2}\, \mathrm{e}^{-\frac{2w}{\hbar}\sqrt{2m(V-T)}} \tag{36}$$

where w is the width of the hill, V its height, and m and T the mass and the kinetic energy of the particle. For a system of large number of particles, the probability is (at most) the product of the probabilities for the different particles.

Let us take the case of a car in a garage, and assume that the car is made of 10^{28} atoms at room temperature. A typical garage wall has a thickness of 0.1 m and a potential height of $V = 1\,\mathrm{keV} = 160\,\mathrm{aJ}$ for the passage of an atom. We get that the probability of finding the car outside the garage is

$$P \approx \left(10^{-(10^{12})}\right)^{(10^{28})} \approx 10^{-(10^{40})} \,. \tag{37}$$

Challenge 73 e The smallness of this value (just try to write it down, to be convinced) is the reason why it is never taken into account by the police when a car is reported missing. (Actually, the probability is even considerably smaller. Can you name at least one effect that has been forgotten in this simple calculation?) Challenge 74 s

Obviously, tunnelling can be important only for small systems, made of a few particles, and for thin barriers, with a thickness of the order of $\hbar/\sqrt{2m(V-T)}$. For example, tunnelling of single atoms is observed in solids at high temperature, but is not important in daily life. For electrons, the effect is more pronounced: the barrier width w

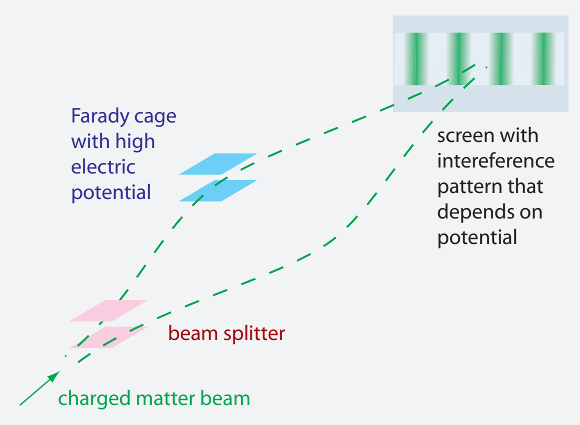


FIGURE 49 A localized electric potential in an interferometer leads to a shift of the interference pattern.

for an appreciable tunnelling effect is

$$w \approx \frac{0.5\,\mathrm{nm}\ \sqrt{\mathrm{aJ}}}{\sqrt{V - T}}\ . \qquad (38)$$

At room temperature, the kinetic energy T is of the order of 6 zJ; increasing the temperature obviously increases the tunnelling. As a result, electrons tunnel quite easily through barriers that are a few atoms in width. Indeed, every TV tube uses tunnelling at high temperature to generate the electron beam producing the picture. The necessary heating is the reason why in the past, television tubes took some time to switch on.

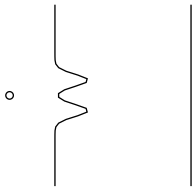
The tunnelling of electrons also limits the physical size of computer memories. Memory chips cannot be made arbitrary small. Silicon integrated circuits with one terabyte of random-access memory (RAM) will probably never exist. Can you imagine why? In fact, tunnelling limits the working of any type of memory, including that of our brain. Indeed, if we were much hotter than 37 °C, we could not remember anything! Challenge 75 s

Since light is made of particles, it can also tunnel through potential barriers. The best – or highest – potential barriers for light are mirrors; mirrors have barrier heights of the order of one attojoule. Tunnelling implies that light *can* be detected behind any mirror. These so-called *evanescent waves* have indeed been detected; they are used in various high-precision experiments and devices.

THE QUANTUM PHASE

We have seen that the *amplitude* of the wave function, the probability amplitude, shows the same effects as any wave: dispersion and damping. We now return to the *phase* of the wave function and explore it in more detail.

Whereas the amplitude of a wave function is easy to picture – just think of the (square root of the) density of a real cloud – the phase takes more effort. As mentioned, states or wave functions are *clouds with a local phase*: they are clouds of rotating arrows, i.e., clouds of objects that rotate and can be rotated. In case of an everyday water cloud, a local rotation of droplets has no effect on the cloud. In contrast, in quantum theory, the local rotation of the cloud, thus the local change of its phase, does have a measurable



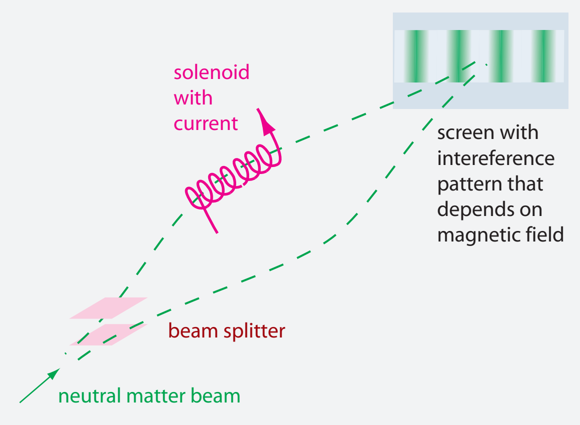


FIGURE 50 Magnetic fields change the phase of a spinning particle.

effect. Let us explore this point.

Page 56 The phase of free matter waves behaves like the phase of photons: it evolves with time, and thus increases along the path of a moving particle. The phase can be pictured by a small rotating arrow. The angular velocity with which the phase rotates is given by the famous relation $\omega = E/\hbar$. In short,

- ▷ We can picture the wave function of a free quantum particle as a moving cloud of arrows; the arrows rotate with constant frequency while the cloud disperses at the same time.

Above all, the phase is that aspect of the wave function that leads to interference effects. When two partial wave functions are separated and recombined after a relative phase change, the phase change will determine the interference pattern. This is the origin of the electron beam interference observations shown in Figure 38. Without the quantum phase, there would be no extinction and no interference.

The phase of a wave function can be influenced in many ways. The simplest way is the use of electric fields. If the wave function of a *charged* particle is split, and one part is led through a region with an electric field, a phase change will result. The arrangement is shown in Figure 49. A periodic change of the electric potential should yield a periodic shift of the interference pattern. This is indeed observed.

Another simple case of phase manipulation is shown in Figure 50: also a magnetic field changes the phase of a spinning neutral particle – if it contains charges – and thus influences the interference behaviour.

A famous experiment shows the importance of the phase in an even more surprising way: the *Aharonov–Bohm effect*. The effect is famous for two reasons: it is counter-intuitive and it was predicted before it was observed. Ref. 57 Look at the set-up shown in Figure 51. A matter wave of charged particles is split into two by a cylinder – positioned at a right angle to the matter's path – and the matter wave recombines behind it. Inside the cylinder there is a magnetic field; outside, there is none. (A simple way to realize such a cylinder is a long solenoid.) Quantum physics predicts that an interference pattern will be observed, and that the position of the stripes will depend on the value of the mag-

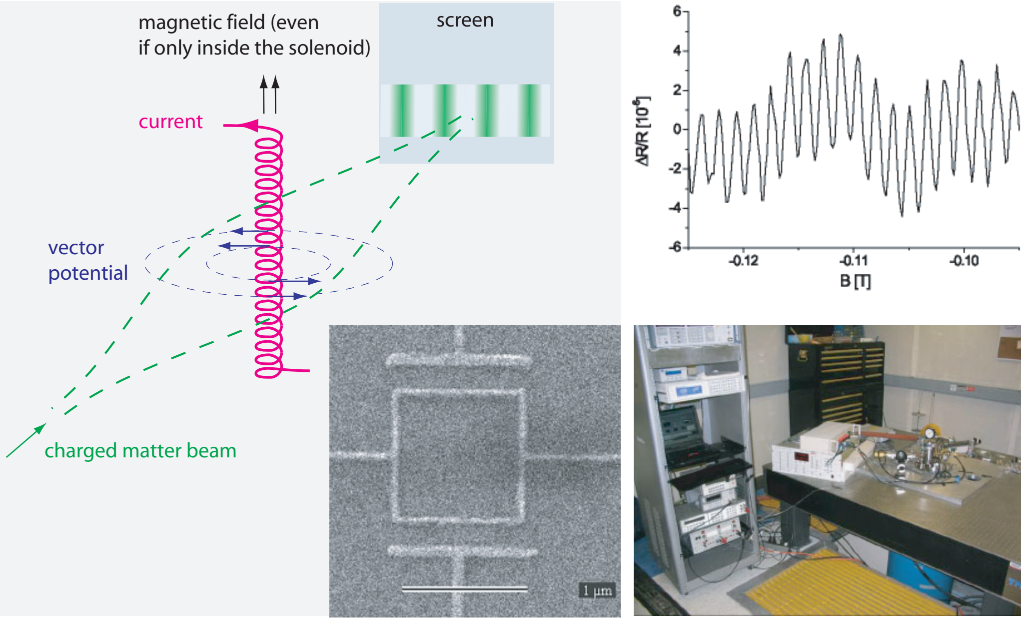


FIGURE 51 The Aharonov–Bohm effect: the influence of the magnetic vector potential on interference (left) and a measurement confirmation (right), using a microscopic sample that transports electrons in thin metal wires (© Doru Cuturela).

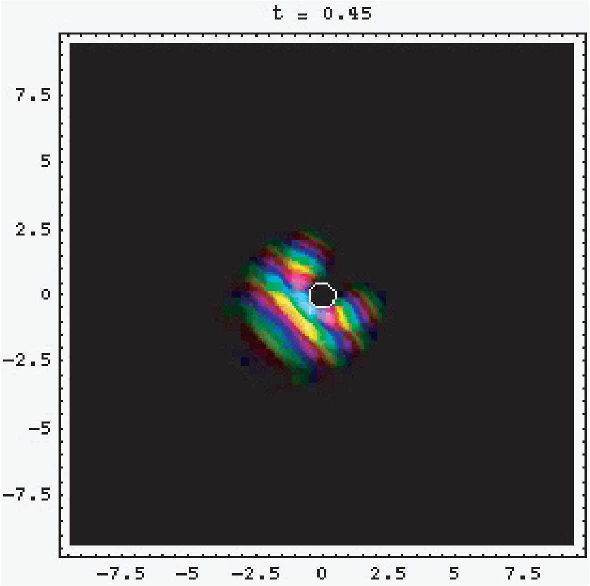


FIGURE 52 The motion of a wave function around a solenoid showing the Aharonov–Bohm effect. The density of the state is displayed by brightness, and the local phase is encoded in the colour. (QuickTime film © Bernd Thaller)

netic field. This happens even though the wave never enters the region with the field! The surprising effect has been observed in countless experiments.

The reason for the Aharonov–Bohm effect is simple: for a charged particle, the phase

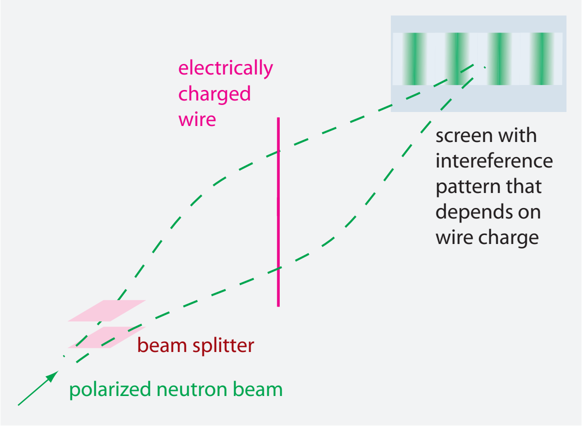


FIGURE 53 The Aharonov–Casher effect: the influence of charge on the phase leads to interference even for interfering neutrons.

of a wave function is determined by the vector potential $\boldsymbol{A}$, not by the magnetic field $\boldsymbol{B}$. The vector potential around a solenoid does not vanish – as we know from the section Vol. III, page 83 on electrodynamics – but circulates around the solenoid. This circulation distinguishes the two sides of the solenoid and leads to a phase shift – one that indeed depends on the magnetic field value – and thus produces interference, even though the particle never interacts with the magnetic field itself.

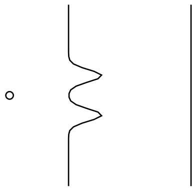
A further example for phase manipulation is the so-called *Aharonov–Casher effect*, which even occurs for neutral particles, as long as they have a magnetic moment, such as neutrons have. The phase of a *polarized* neutron will be influenced by an electric field, so that the arrangement shown in Figure 53 will show an interference pattern that depends on the applied electric potential.

Another case of phase manipulation will be presented later on: also gravitational fields can be used to rotate wave functions. Even the acceleration due to rotational motion can do so. In fact, it has been possible to measure the rotation of the Earth by observing the Ref. 58 change of neutron beam interference patterns.

Another important class of experiments that manipulate the phase of wave functions are possible with macroscopic quantum phenomena. In superconductivity and in superfluidity, the phase of the wave function is regularly manipulated with magnetic and electric fields. This possibility has many important technical applications. For example, the so-called *Josephson effect* is used to measure electric potential differences by measuring the frequency of emitted radio waves, and so-called *superconducting quantum interference devices*, or SQIDs, are used to measure tiny magnetic fields.

We note that all these experiments confirm that the *absolute* phase of a wave function *cannot* be measured. However, *relative phases* – phase differences or phase changes – *can* Challenge 76 e be measured. Can you confirm this?

All the phase shift effects just presented have been observed in numerous experiments. The phase is an essential aspect of the wave function: the phase leads to interference and is the main reason for calling it *wave* function in the first place. Like in any wave, the phase evolves over time and it can be influenced by various external influences. Above all, the experiments show that a localized quantum particle – thus when the spread of the wave function can be neglected – is best imagined as a rotating arrow;



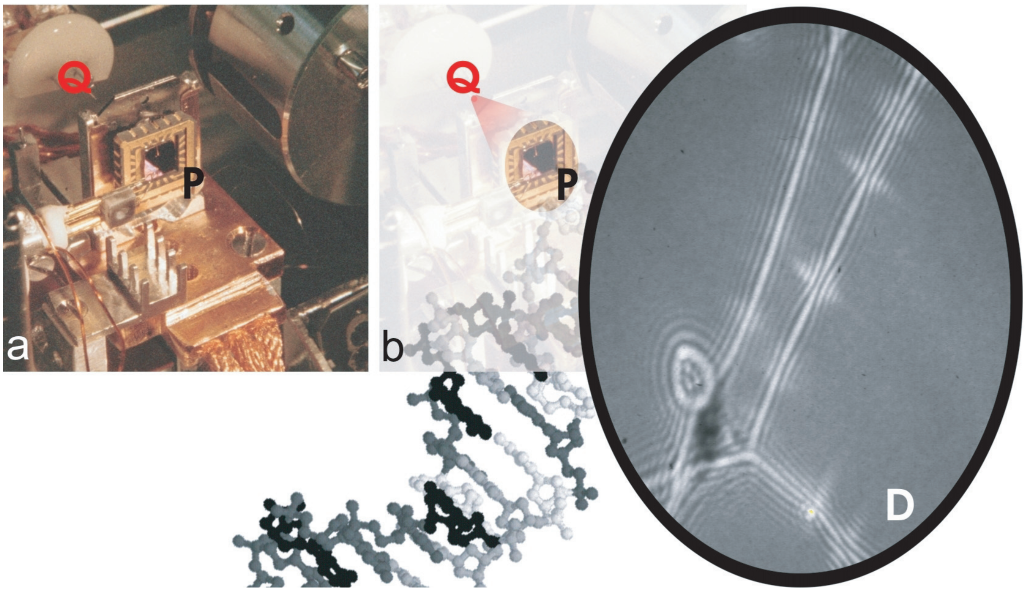


FIGURE 54 An electron hologram of DNA molecules (© Hans-Werner Fink/Wiley VCH).

in contrast, whenever the spread cannot be neglected, the wave function is best imagined as a wave of arrows rotating at each point in space.

CAN TWO ELECTRON BEAMS INTERFERE? ARE THERE COHERENT ELECTRON BEAMS?

Ref. 59 Do coherent electron sources exist? The question is tricky. Results in the literature, such as the one illustrated in Figure 54, state that is possible to make holograms with electron beams.* However, when one asks these authors about the meaning of coherence, they answer that electron coherence is only transversal, not longitudinal. Transversal coherence is determined by the possible size of wavefronts with a given phase. The upper limit of this size is given by the interactions such a state has with its environment. All this behaviour is as expected for actual coherence.

However, the concept of 'transversal coherence' is a misnomer. The ability to interfere with oneself, as implies in the term 'transversal coherence' is not the correct definition of coherence. Transversal coherence, be it for photons or for matter particles, only expresses the smallness of the particle source. Both small lamps (and lasers) can show interference when the beam is split and recombined with identical path length; this is not a proof of coherence of the light field. A similar reasoning shows that monochromaticity is not a proof for coherence either.

A state is called *coherent* if it possesses a well-defined phase throughout a given domain of space or time. The size of the spatial region or of the time interval defines the degree of coherence. This definition yields coherence lengths of the order of the source size for small 'incoherent' sources. Even for a small coherence length, the size of an interference pattern or the distance d between its maxima can be much larger than the

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Ref. 60 * In 2002, the first holograms have been produced that made use of neutron beams.

coherence length l or the source size s. In short, a large size (or a persistent duration in time) of an interference pattern alone is *not* a proof of coherence.

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Let us recall the situation for light. A light source is coherent if it produces an approximate sine wave over a certain length or time. Due to the indeterminacy relation, in any coherent beam of light, the photon number is undetermined. The same requirement applies to coherent electron beams: an undetermined electron number is needed for coherence. That is impossible, as electrons carry a conserved charge. Coherent electron beams do not exist.

In summary, even though an electron can interfere with itself, and even though it is possible to produce interference between two light sources, interference between two electron sources is impossible. Indeed, nobody has every managed to produce interference between two electron sources. There is no conventional concept of coherence for electron beams.

THE LEAST ACTION PRINCIPLE IN QUANTUM PHYSICS

Vol. I, page 253

In nature, motion happens in a way that minimizes change. Indeed, in classical physics, the principle of least action – or principle of cosmic lazyness – states: in nature, the motion of a particle happens along that particular path – out of all possible paths with the same end points – for which the action is minimal. This principle of cosmic laziness or cosmic efficiency was stated mathematically by saying that in nature, the *variation* δS of the action is zero. Action or change minimization explains all classical evolution equations. We now transfer this idea to the quantum domain.

For quantum systems, we need to redefine both the concept of action and the concept of variation: first of all, we have to find a description of action that is based on operators; secondly, we need to define the action variation without paths, as the concept of 'path' does not exist for quantum systems; thirdly, since there is a smallest action in nature, a vanishing variation is not a clearly defined concept, and we must overcome this hurdle. There are two main ways to achieve this goal: to describe the motion of quantum systems as a superposition of all possible paths, or to describe action with the help of wave functions. Both approaches are equivalent.

Page 56

In the first approach, the *path integral formulation*, the motion of a quantum particle is described as a democratic superposition of motions along all possible paths. (We called it the 'arrow model' above.) For each path, the evolution of the arrow is determined, and at the end point, the arrows from all paths are added. The action for each path is the number of turns that the arrow performs along the path. The result from this exercise is that the path for which the arrow makes the smallest number of turns is usually (but not always!) the most probable path. A more precise investigation shows that classical, macroscopic systems always follow only the path of smallest action, whereas quantum systems follow all paths.

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In the second approach to quantum physics, action is defined with the help of wave functions. In classical physics, we defined the action (or change) as the integral of the Lagrangian between the initial and final points in time, and the Lagrangian itself as the difference between kinetic and potential energy. In quantum physics, the simplest definition is the *quantum action* defined by Julian Schwinger. Let us call the initial and final states of the system ψ_i and ψ_f. The action S between these two states is defined as

$$S = \langle \psi_\mathrm{i} \,|\, \int L \,\mathrm{d}t \,|\, \psi_\mathrm{f} \rangle \ , \qquad (39)$$

where L is the Lagrangian (operator). The angle brackets represent the 'multiplication' of states and operators as defined in quantum theory.* In simple words, also in quantum theory, action – i.e., the change occurring in a system – is the integral of the Lagrangian. The Lagrangian operator L is defined in the same way as in classical physics: the Lagrangian $L = T - V$ is the difference between the kinetic energy T and the potential energy V operators. The only difference is that, in quantum theory, the momentum and position *variables* of classical physics are replaced by the corresponding *operators* of quantum physics.**

To transfer the concept of action variation δS to the quantum domain, Julian Schwinger introduced the straightforward expression

$$\delta S = \langle \psi_\mathrm{i} \,|\, \delta \int L \,\mathrm{d}t \,|\, \psi_\mathrm{f} \rangle \ . \qquad (40)$$

The concept of path is not needed in this expression, as the variation of the action is based on varying wave functions instead of varying particle paths.

The last classical requirement to be transferred to the quantum domain is that, because nature is lazy, the variation of the action must vanish. However, in the quantum domain, the variation of the action cannot be zero, as the smallest observable action is the quantum of action. As Julian Schwinger discovered, there is only one possible way to express the required minimality of action:

$$\delta S = \langle \psi_\mathrm{i} \,|\, \delta \int L \,\mathrm{d}t \,|\, \psi_\mathrm{f} \rangle = -i\hbar\, \delta \langle \psi_\mathrm{i} | \psi_\mathrm{f} \rangle \ . \qquad (41)$$

This so-called *quantum action principle* describes all motion in the quantum domain. Classically, the right-hand side is zero – since $\hbar$ is taken to be zero – and we then recover the minimum-action principle $\delta S = 0$ of classical physics. But in quantum theory, whenever we try to achieve small variations, we encounter the quantum of action and changes of (relative) phase. This is expressed by the right-hand side of the expression. The right side is the reason that the evolution equations for the wave function – Schrödinger's equation for the spinless non-relativistic case, or Dirac's equation for the spin 1/2 relativistic case – are valid in nature.

In other words, all quantum motion – i.e., the quantum evolution of a state ψ or $|\psi\rangle$ – happens in such a way that the action variation is the same as $-i$ times the quantum of action $\hbar$ times the variation of the scalar product between initial and final states. In

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* We skip the details of notation and mathematics here; in the simplest description, states are wave functions, operators act on these functions, and the product of two different brackets is the integral of the function product over space.

** More precisely, there is also a condition governing the ordering of operators in a mixed product, so that the non-commutativity of operators is taken into account. We do not explore this issue here.

simple terms, in the actual motion, the intermediate states are fixed by the requirement that they must lead from the initial state to the final state with the smallest number of effective turns of the state phase. The factor $-i$ expresses the dependence of the action on the rotation of the wave function.

In summary, the least action principle is also valid in quantum physics, provided one takes into account that action values below $\hbar$ cannot be found in experiments. The least action principle governs the evolution of wave function. The least action principle thus explains the colour of all things, all other material science, all chemistry and all biology, as we will see in the following.

THE MOTION OF QUANTONS WITH SPIN

> “Everything turns.”
>
> Anonymous

What is the origin of the quantum phase? Classical physics helps to answer the question. Like everyday objects, also quantons can rotate around an axis: we speak of particle *spin*. Page 82 But if quantum particles can spin, they should possess angular momentum. And indeed, experiments confirm this deduction.

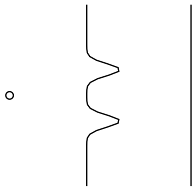
In particular, electrons have spin. The full details of electron spin were deduced from Ref. 61 experiments by two Dutch students, George Uhlenbeck and Samuel Goudsmit, in 1925. They had the guts to publish what Ralph Kronig had also suspected: that electrons rotate around an axis with a projected component of the angular momentum given by $\hbar/2$. In fact, this value – often called spin 1/2 for short – is valid for *all* elementary *matter* particles. (In contrast, all known elementary *radiation* particles have spin values of $\hbar$, or spin 1 for short.)

If a spinning particle has angular momentum, it must be possible to rearrange the axis by applying a torque, to observe precession, to transfer the spin in collisions, etc. All these effects are indeed observed; for example, the Stern–Gerlach experiment already allows Page 83 all these observations. The only difference between particle spin and classical angular Page 82 momentum is that particle spin is quantized, as we deduced above.

In other words, the *spin* $\boldsymbol{L}$ of a quantum particle has all the properties of a rotation around an axis. As a consequence, spinning *charged* quantum particles act as small dipole magnets, with the magnet oriented along the axis of rotation. The observed strength of the dipole magnet, the *magnetic moment*, is proportional to the spin and to the conversion factor $-e/2m_e$, as expected from classical physics. Therefore, the natural unit for the magnetic moment of the electron is the quantity $\mu_\mathrm{B} = e\hbar/2m_e$; it is called *Bohr's magneton*. It turns out that the magnetic moment $\boldsymbol{\mu}$ of quantons behaves differently from that of classical particles. The quantum effects of spin are described by the so-called *g-factor*, which is a pure number:

$$\boldsymbol{\mu} = g\frac{-e}{2m_e}\boldsymbol{L} = -g\mu_\mathrm{B}\frac{\boldsymbol{L}}{\hbar}\,, \quad \text{with} \quad \mu_\mathrm{B} = \frac{e\hbar}{2m_e}\,. \tag{42}$$

From the observed optical spectra, Uhlenbeck and Goudsmit deduced a g-factor of 2 for Page 107 the electron. Classically, one expects a value $g = 1$. The experimental value $g = 2$ was



Ref. 62 explained by Llewellyn Thomas as a relativistic effect a few months after its experimental discovery.

By 2004, experimental techniques had become so sensitive that the magnetic effect of a single electron spin attached to an impurity (in an otherwise non-magnetic material) could be detected. Researchers now hope to improve these so-called 'magnetic-resonance-force microscopes' until they reach atomic resolution.

In 1927, Wolfgang Pauli* discovered how to include spin 1/2 in a quantum-mechanical description: instead of a state function described by a single complex number, a state function with *two* complex components is needed. The reason for this expansion is simple. In general, the little rotating arrow that describes a quantum state does *not* rotate around a *fixed* axis, as is assumed by the Schrödinger equation; the *axis of rotation* has also to be specified at each position in space. This implies that two additional parameters are required at each space point, bringing the total number of parameters to four real numbers, or, equivalently, two complex numbers. Nowadays, Pauli's equation for quantum mechanics with spin is mainly of conceptual interest, because – like that of Schrödinger – it does not comply with special relativity.

In summary, the non-relativistic description of a quanton with spin implies the use of wave functions that specify *two* complex numbers at each point in space and time. The additional complex number describe the local rotation plane of the spin. The idea of including the local rotation plane was also used by Dirac when he introduced the relativistic description of the electron, and the idea is also used in all other wave equations for particles with spin.

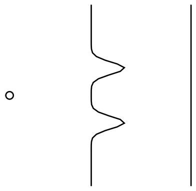
Relativistic wave equations

In 1899, Max Planck had discovered the quantum of action. In 1905, Albert Einstein published the theory of special relativity, which was based on the idea that the speed of light c is independent of the speed of the observer. The first question Planck asked himself was whether the value of the quantum of action would be independent of the speed of the observer. It was his interest in this question that led him to invite Einstein to Berlin. With this invitation, he made the patent-office clerk famous in the world of physics.

Experiments show that the quantum of action is indeed independent of the speed of the observer. All observers find the same minimum value. To include special relativity into quantum theory, we therefore need to find the correct quantum Hamiltonian H, i.e., the correct energy operator.

---

* Wolfgang Ernst Pauli (b. 1900 Vienna, d. 1958 Zürich), at the age of 21, wrote one of the best texts on special and general relativity. He was the first to calculate the energy levels of hydrogen using quantum theory, discovered the exclusion principle, incorporated spin into quantum theory, elucidated the relation between spin and statistics, proved the CPT theorem, and predicted the neutrino. He was admired for his intelligence, and feared for his biting criticisms, which led to his nickname, 'conscience of physics'. Despite this trait, he helped many people in their research, such as Heisenberg with quantum theory, without Ref. 63 claiming any credit for himself. He was seen by many, including Einstein, as the greatest and sharpest mind of twentieth-century physics. He was also famous for the 'Pauli effect', i.e., his ability to trigger disasters in laboratories, machines and his surroundings by his mere presence. As we will see shortly, one can argue that Pauli actually received the Nobel Prize in Physics in 1945 – officially 'for the discovery of the exclusion principle' – for finally settling the question of how many angels can dance on the tip of a pin.



For a free relativistic particle, the classical Hamiltonian function – that is, the energy of the particle – is given by

$$H = \pm\sqrt{c^4 m^2 + c^2 \boldsymbol{p}^2} \quad \text{with} \quad \boldsymbol{p} = \gamma m \boldsymbol{v} \,. \tag{43}$$

Thus we can ask: what is the corresponding Hamilton operator for the quantum world? The simplest answer was given, in 1949 by T.D. Newton and E.P. Wigner, and in 1950, by Ref. 64 L.L. Foldy and S.A. Wouthuysen. The operator is almost the same one:

$$H = \beta\sqrt{c^4 m^2 + c^2 \boldsymbol{p}^2} \quad \text{with} \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{44}$$

The signs appearing in the matrix operator β distinguish, as we will see, between particles and antiparticles. The numbers +1 and −1 appear twice, to take care of the two possible spin directions for each case.

With this relativistic Hamiltonian operator for spin 1/2 particles – and with all others – the wave function is described by *four* complex numbers, *two* for particles and *two* for antiparticles. Why? We saw above that a quantum particle with spin requires two Page 105 complex components for its state; this followed from the requirement to specify, at each Vol. II, page 72 point in space, the length of the arrow, its phase, and its plane of rotation. Earlier on we also found that relativity automatically introduces antimatter. (We will explore the issue Page 192 in more detail below.) Both matter and antimatter are thus part of any relativistic description of quantum effects. The wave function for a particle has vanishing antiparticle components, and vice versa. In total, the wave function for relativistic spin 1/2 particle has thus *four* complex components.

The Hamilton operator yields the velocity operator $\boldsymbol{v}$ through the same relation that is valid in classical physics:

$$\boldsymbol{v} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \beta \frac{\boldsymbol{p}}{\sqrt{c^4 m^2 + c^2 \boldsymbol{p}^2}} \,. \tag{45}$$

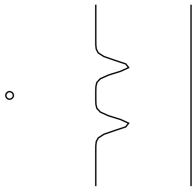
This velocity operator shows a continuum of eigenvalues, from minus to plus the speed of light. The velocity $\boldsymbol{v}$ is a constant of motion, as are the momentum $\boldsymbol{p}$ and the energy

$$E = \sqrt{c^4 m^2 + c^2 \boldsymbol{p}^2} \,. \tag{46}$$

Also the orbital angular momentum $\boldsymbol{L}$ is defined as in classical physics, through

$$\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p} \,. \tag{47}$$

Ref. 65 The orbital angular momentum $\boldsymbol{L}$ and the spin $\boldsymbol{\sigma}$ are separate constants of motion. A particle (or antiparticle) with positive (or negative) angular momentum component has



a wave function with only one non-vanishing component; the other three components vanish.

But alas, the representation of relativistic motion named after Foldy and Wouthuysen is not the simplest when it comes to take electromagnetic interactions into account. The simple identity between the classical and quantum-mechanical descriptions is lost when electromagnetism is included. We will solve this problem below, when we explore Dirac's evolution equation for relativistic wave functions. Page 189

Bound motion, or composite vs. elementary quantons

When is an object composite, and not elementary? Whenever it contains internal, or bound motion. When is this the case? Quantum theory gives several pragmatic answers.

Ref. 66 The first criterion for compositeness is somewhat strange: an object is composite when its gyromagnetic ratio is different from the one predicted by quantum electrodynamics. Page 189 The *gyromagnetic ratio* γ – not to be confused with the relativistic dilation factor – is defined as the ratio between the magnetic moment $\boldsymbol{M}$ and the angular momentum $\boldsymbol{L}$:

$$\boldsymbol{M} = \gamma \boldsymbol{L} \,. \tag{48}$$

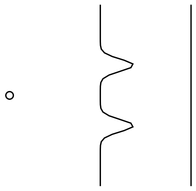
Challenge 77 e The gyromagnetic ratio γ is measured in units of $\mathrm{s^{-1}T^{-1}}$, i.e., C/kg, and determines the energy levels of magnetic spinning particles in magnetic fields; it will reappear later in Vol. V, page 162 the context of magnetic resonance imaging. All candidates for elementary particles have spin 1/2. The gyromagnetic ratio for spin-1/2 particles of magnetic moment M and mass m can be written as

$$\gamma = \frac{M}{\hbar/2} = g\frac{e}{2m} \,. \tag{49}$$

The criterion for being elementary can thus be reduced to a condition on the value of the dimensionless number g, the so-called *g-factor*. (The expression $e\hbar/2m$ is often called the *magneton* of the particle.) If the g-factor *differs* from the value predicted by quantum Page 189 electrodynamics for point particles – about 2.0 – the object is *composite*. For example, a $^4\mathrm{He}^+$ helium ion has spin 1/2 and a g value of $14.7 \cdot 10^3$. Indeed, the radius of the helium ion is $3 \cdot 10^{-11}$ m, obviously a finite value, and the ion is a composite entity. For the proton, one measures a g-factor of about 5.6. Indeed, experiments yield a finite proton radius of about 0.9 fm and show that it contains several constituents.

The neutron, which has a magnetic moment despite being electrically neutral, must therefore be composite. Indeed, its radius is approximately the same as that of the proton. Similarly, molecules, mountains, stars and people must be composite. According to this first criterion, the only elementary particles are *leptons* (i.e., electrons, muons, tauons and neutrinos), *quarks*, and *intermediate bosons* (i.e., photons, W-bosons, Z-bosons and Vol. V, page 162 gluons). More details on these particles will be revealed in the chapters on the nucleus.

Another simple criterion for compositeness has just been mentioned: *any object with a measurable size is composite*. This criterion yields the same list of elementary particles as the first. Indeed, the two criteria are related. The simplest model for composite structures



Ref. 67

predicts that the g-factor obeys

$$g - 2 = \frac{R}{\lambda_C} \tag{50}$$

Challenge 78 e

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where R is the radius and $\lambda_C = h/mc$ is the Compton wavelength of the system. This expression is surprisingly precise for helium-4 ions, helium-3, tritium ions and protons, as you may wish to check. The tables in Appendix B in the next volume make the same point. In short, the second criterion for compositeness is equivalent to the first.

A third criterion for compositeness is more general: *any object larger than its Compton length is composite*. The argument is simple. An object is composite if one can detect *internal* motion, i.e., motion of some components. Now the action of any part with mass m_{part} moving inside a composed system of size r obeys

$$S_{\text{part}} < 2\pi\, r\, m_{\text{part}}\, c < \pi\, r\, m\, c \tag{51}$$

where m is the mass of the *composite* object. On the other hand, following the principle of quantum theory, this action, to be observable, must be larger than $\hbar/2$. Inserting this condition, we find that for any composite object*

$$r > \frac{\hbar}{2\pi\, m\, c}\ . \tag{52}$$

The right-hand side differs only by a factor $4\pi^2$ from the so-called *Compton (wave)length*

$$\lambda = \frac{h}{m\, c} \tag{53}$$

Challenge 80 e

of an object. Thus any object *larger* than its own Compton wavelength is composite; and any object *smaller* than the right-hand side of expression (52) is elementary. Again, only leptons, quarks and intermediate bosons passed the test. (For the Higgs boson discovered in 2012, the test has yet to be performed, but it is expected to comply as well.) All other objects are composite. In short, this third criterion produces the same list as the previous ones. Can you explain why?

Vol. V, page 261

Challenge 81 s

A fourth criterion for compositeness is regularly cited by Steven Weinberg: a particle is elementary if it appears in the Lagrangian of the standard model of particle physics, i.e., in the description of the fundamental building blocks of nature. Can you show that this criterion follows from the previous ones?

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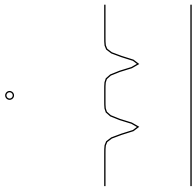
Interestingly, we are not yet finished with this topic. Even stranger statements about compositeness will appear when gravity is taken into account. Just be patient: it is worth it.

Challenge 79 ny

* Can you find the missing factor of 2? And is the assumption that the components must always be lighter than the composite a valid one?



CURIOSITIES AND FUN CHALLENGES ABOUT QUANTUM MOTION OF MATTER

> Die meisten Physiker sind sehr naiv, sie glauben immer noch an wirkliche Wellen oder Teilchen.*
>
> Anton Zeilinger

Take the sharpest knife edge or needle tip you can think of: the quantum of action implies that their boundaries are not sharp, but fuzzy, like the boundaries of clouds. Take the hardest or most solid object you can think of, such as diamond or a block of tungsten: the quantum of action implies that its surface is somewhat soft. All experiments confirm these statements. *Nothing in nature is really sharp or really solid.* Quantum physics thus disagrees with several ideas of the ancient Greek atomists.

* *

Do hydrogen atoms exist? Most types of atom have been imaged with microscopes, photographed under illumination, levitated one by one, and even moved with needles, one by one, as the picture on page 344 in volume I shows. Researchers have even moved single Ref. 68 atoms by using laser beams to push them. However, not a single one of these experiments has measured or imaged hydrogen atoms. Is that a reason to doubt the existence Challenge 82 s of hydrogen atoms? Taking this not-so-serious discussion seriously can be a lot of fun.

* *

Is the wave function ’real’? More precisely, is the wave function really a cloud? Some physicists still doubt this. This dying group of physicists, often born around the middle of the twentieth century, have heard so often – incorrectly and usually from questionable authorities – that a wave function has no reality that they stopped asking and answering Challenge 83 e the simplest questions. To dispel their doubts, ask them whether they have a non-zero height or whether they think that atoms are round. If they agree, they have admitted that wave functions have some sort of reality. All everyday objects are made of elementary particles that are so unmeasurably small that we can call them point-like. Therefore, the size, surface area and volume of all everyday objects are exclusively due to wave functions. Every length, area and volume is a proof that wave functions have some sort of reality.

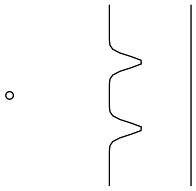
* *

Two observables can commute for two different reasons: either they are very *similar* – such as the coordinates x and x^2 – or they are very *different* – such as the coordinate x Challenge 84 d and the momentum p_y. Can you give an explanation for this?

* *

Space and time translations commute. Why then do the momentum operator and the Challenge 85 ny Hamiltonian not commute in general?

* *



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[*] ‘Most physicists are very naive; they still believe in real waves or real particles.’ Anton Zeilinger, physicist at the University of Vienna, is well-known for his experiments on quantum mechanics.

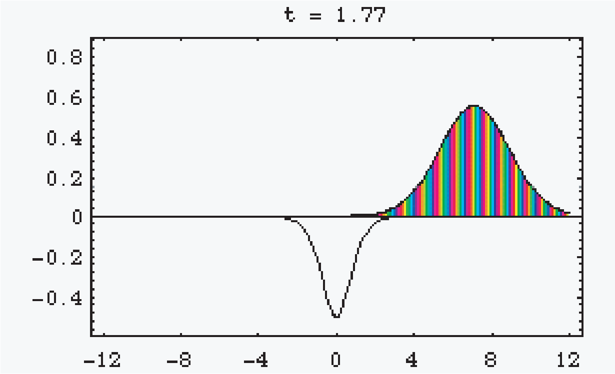


FIGURE 55 A special potential well that does *not* disturb a wave function. Colour indicates phase. (QuickTime film © Bernd Thaller)

There exist special potentials that have *no* influence on a wave function. Figure 55 shows an example. This potential has reflection coefficient zero for all energies; the scattered wave has no reflected part. The mathematical reason is fascinating. The potential well has the shape of a soliton of the Korteweg–de Vries equation; this equation is related to the Schrödinger equation.

* *

Any bound system in a non-relativistic state with no angular momentum obeys the Ref. 69 relation

$$\langle r^2\rangle\,\langle T\rangle \;\geqslant \frac{9\hbar^2}{8m}\,, \tag{54}$$

where m is the reduced mass and T the kinetic energy of the components, and r is the size of the system. Can you deduce this result, and check it for the ground state of hydrogen?

Challenge 86 s

* *

In high school, it often makes sense to visualize electron wave functions as a special type of fluid-like matter, called *electronium*, that has a negative charge density. In this visualization, an atom is a positive nucleus surrounded by an electronium cloud. Deforming the electronium cloud around a nucleus requires energy; this happens when a photon of the correct frequency is absorbed, for example. When atoms of the right kind approach each other, the electronium clouds often form stable bridges – chemical bonds.

* *

Quantum theory allows for many unusual bound states. Usually we think of bound states as states of low energy. But there are situations in which bound states arise due to external forcing with oscillating potentials. We encountered such a situation in classical physics: (Vol. I, page 319) the vertically driven, upside-down pendulum that remain vertical despite being unstable. Similar situations also occur in quantum physics. Examples are Paul traps, the helium Ref. 70 atom, negative ions, Trojan electrons and particle accelerators.

* *

One often reads that the universe might have been born from a quantum fluctuation. Can you explain why this statement make no sense?

Challenge 87 s

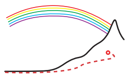
A SUMMARY ON MOTION OF MATTER QUANTONS

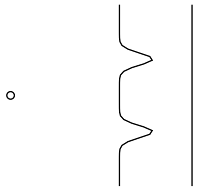
In summary, the motion of massive quantons, i.e., of quantum matter particles, can be described in two ways:

- At high magnification, quantum matter particles are described by wave functions that move like *advancing, rotating and precessing clouds of arrows.* The local cloud orientation, or local phase, follows a wobbling motion. The square of the wave function, i.e., the density of the cloud, is the probability for finding the particle at a given spot.
- Seen from far away, at low magnification, a moving massive quantum particle behaves as a single advancing, rotating and precessing arrow. The details of the rotation and precession of the arrow depend on the energy and momentum of the particle and the potential it is subjected to. The arrow is a probability amplitude: the squared length of the arrow is the probability to observe the particle. If a particle can get from a starting point to a final point in several ways, the probability amplitudes for each way add up.

The single rotating arrow results from a cloud average. The single arrow combines particle and wave properties. A full rotation of the arrow corresponds to the quantum of action $\hbar$. This central feature implies that a non-relativistic particle whose spin can be neglected follows the Schrödinger equation, and that a relativistic electron follows the Dirac equation. The Dirac equation agrees with all known experiments. In particular, the Dirac equation describes all of materials science, chemistry and biology, as we will find out.

To continue with the greatest efficiency on our path across quantum physics, we explore three important topics: the indistinguishability of particles of the same kind, the spin of quantum particles, and the meaning of probabilities.





Chapter 5

PERMUTATION OF PARTICLES – ARE PARTICLES LIKE GLOVES?

Why are we able to distinguish twins from each other? Why can we distinguish what looks alike, such as a copy from an original? Most of us are convinced that whenever we compare an original with a copy, we can find a difference. This conviction turns out to be correct also in the quantum domain, but the conclusion is not straightforward.

Think about any method that allows you to distinguish objects: you will find that it runs into trouble for point-like particles. Therefore, in the quantum domain something must change about our ability to distinguish particles and objects. Challenge 88 s

We could argue that differences between an original object and a copy can always be made to disappear: it should be sufficient to use the same number and type of atoms. In fact, the quantum of action shows that this is not sufficient, even though all atoms of the same type are indeed indistinguishable copies of each other! In the following we explore the most important consequences on motion of the indistinguishability of atoms and of the distinguishability of macroscopic objects.

Distinguishing macroscopic objects

A number of important properties of objects are highlighted by studying a combinatorial puzzle: the *glove problem*. It asks:

> How many surgical gloves (for the right hand) are necessary if m doctors need to operate w patients in a hygienic way, so that nobody gets in contact with the body fluids of anybody else?

The same problem also appears in other settings. For example, it also applies to computers, interfaces and computer viruses or to condoms, men and women – and is then called the *condom problem*. To be clear, the optimal number of gloves is *not* the product mw. In fact, the problem has three subcases. Ref. 71

- Challenge 89 s — The simple case $m = w = 2$ already provides the most important ideas needed. Are you able to find the optimal solution and procedure?
- Challenge 90 e — In the case $w = 1$ and m odd, the solution is $(m + 1)/2$ gloves. The corresponding expression $(w+1)/2$ holds for the case $m = 1$ and w odd. This is the optimal solution, as you can easily check yourself.
- Ref. 72 — A solution with a simple procedure for all other cases is given by $\lceil 2w/3+m/2 \rceil$ gloves, where $\lceil x \rceil$ means the smallest integer greater than or equal to x. For example, for two

doctors and three patients this gives only three gloves. (However, this formula does not always give the optimal solution; better values exist in certain subcases.)

Challenge 91 e

Enjoy working on the puzzle. You will find that three basic properties of gloves determine the solution. First, gloves have two sides, an interior and an exterior one, that can be distinguished from each other. Secondly, gloves turned inside out exchange left and right and can thus be distingusihed from gloves that are not reversed. Thirdly, gloves can be distinguished from each other.

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Now we come back to our original aim: Do the three basic properties of gloves also apply to quantum particles? We will explore the issue of double-sidedness of quantum particles in the last part of our mountain ascent. The question whether particles can be turned inside out will be of importance for their description and their motion. We will also explore the difference between right- and left-handed particles, though in the next part of our adventure. In the present chapter we concentrate on the third issue, namely whether objects and particles can always be distinguished from copies. We will find that *elementary* particles do not behave like gloves – but in a much more surprising manner.

In everyday life, distinction of macroscopic objects can be achieved in two ways. On the one hand, we are able to distinguish objects – or people – from each other because they differ in their *intrinsic properties*, such as their mass, colour, size or shape. On the other hand, we are able to distinguish objects even if they have the same intrinsic properties. Any game of billiard shows us that by following the path of each ball, we can distinguish it from the other balls. In short, we can distinguish objects with identical properties also using their *state*.

The state of a billiard ball is given by its position, its linear and its angular momentum. We are able to distinguish two identical billiard balls because the measurement error for the position of each ball is much smaller than the size of the ball itself. The different states of two billiard balls allow us to track each ball. However, in the microscopic domain, this is not possible! Let us take two atoms of the same type. Two such atoms have exactly the same intrinsic properties. To distinguish them in collisions, we would need to keep track of their motion. But due to the quantum of action and the ensuing indeterminacy relation, we have no chance to achieve this. In fact, a simple experiment from the nineteenth century showed that even nature itself is not able to do it! This profound result was discovered studying systems which incorporate a large number of colliding atoms of the same type: *gases*.

DISTINGUISHING ATOMS

Vol. I, page 402

What is the entropy of a gas? The calculation of the entropy S of a simple gas, made of N simple particles[*] of mass m moving in a volume V, gives

$$\frac{S}{kN} = \ln\left[\frac{V}{\Lambda^3}\right] + \frac{3}{2} + \frac{\ln\alpha}{N} \; . \qquad (55)$$

Here, k is the Boltzmann constant, ln the natural logarithm, T the temperature, and $\Lambda = \sqrt{2\pi\hbar^2/mkT}$ is the thermal wavelength (approximately the de Broglie wavelength of the

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[*] Particles are *simple* if they are fully described by their momentum and position; atoms are simple particles. Molecules are not simple, as they are describe also by their orientation.

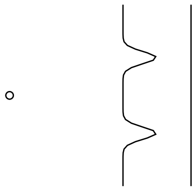

FIGURE 56 Willard Gibbs (1839–1903)

particles making up the gas). In this result, the pure number α is equal to 1 if the particles are distinguishable like billiard balls, and equal to $1/N!$ if they are not distinguishable at all. Measuring the entropy of a simple gas thus allows us to determine α and therefore to test experimentally whether particles are distinguishable. Challenge 92 e

It turns out that only the second case, $\alpha = 1/N!$, describes nature. We can easily check this without even performing the measurement: only in the second case does the entropy of two volumes of identical gas *add* up.* The result, often called *Gibbs' paradox*,** thus proves that the microscopic components of matter are *indistinguishable*: in a system of quantum particles – be they electrons, protons, atoms or small molecules – there is no way to say which particle is which. Challenge 93 e Ref. 73

Indistinguishability of particles is thus an experimental property of nature. It holds without exception. For example, when radioactivity was discovered, people thought that it contradicted the indistinguishability of atoms, because decay seems to single out certain atoms compared to others. But quantum theory then showed that this is not the case and that even atoms and molecules are indistinguishable.

Since $\hbar$ appears in the expression for the entropy, indistinguishability is a quantum effect. Indeed, indistinguishability plays no role if quantum effects are negligible, as is the case for billiard balls. Nevertheless, indistinguishability is important in everyday life. We will find out that the properties of everyday matter – plasma, gases, liquids and solids – would be completely different without indistinguishability. For example, we will discover that without it, knifes and swords would not cut. In addition, the soil would not carry us; we would fall right through it. To illuminate the issue in more detail, we explore the following question.

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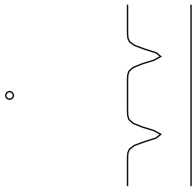
* Indeed, the entropy values observed by experiment, for a monoatomic gas, are given by the so-called Sackur–Tetrode formula Challenge 94 d

$$\frac{S}{kN} = \ln\left[\frac{V}{N\Lambda^3}\right] + \frac{5}{2} \qquad (56)$$

which follows when $\alpha = 1/N!$ is inserted above. It was deduced independently by the German physicist Otto Sackur (1880–1914) and the Dutch physicist Hugo Tetrode (1895–1931). Note that the essential parameter is the ratio between V/N, the classical volume per particle, and Λ^3, the de Broglie volume of a quantum particle.

** Josiah Willard Gibbs (1839–1903), US-American physicist who was, with Maxwell and Planck, one of the three founders of statistical mechanics and thermodynamics; he introduced the concept of *ensemble* and the term *thermodynamic phase*.



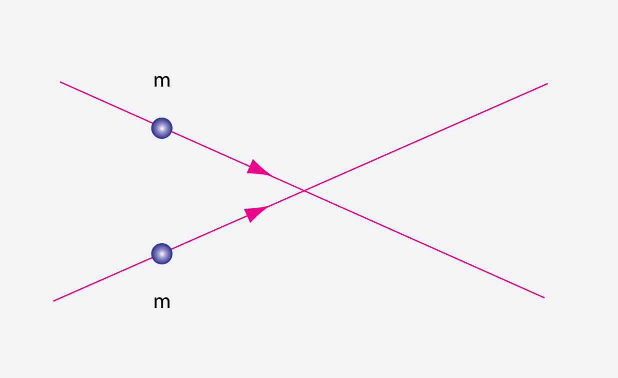


FIGURE 57 Identical objects with crossing paths.

Why does indistinguishability appear in nature?

Take two quantum particles with the same mass, the same composition and the same shape, such as two atoms of the same kind. Imagine that their paths cross, and that they approach each other to small distances at the crossing, as shown in Figure 57. In a gas, both a collision of atoms or a near miss are examples. Now, all experiments ever performed show that at small distances it is impossible to say whether the two quantons have switched roles or not.

▷ It is *impossible* in a gas to follow quantum particles moving around and to determine which one is which. Tracking colliding quantons is impossible.

The impossibility to distinguish nearby particles is a direct consequence of the quantum of action $\hbar$. For a path that brings two approaching particles very close to each other, a role switch requires only a small amount of change, i.e., only a small (physical) action. However, we know that there is a smallest observable action in nature. Keeping track of each quantum particle at small distances would require action values *smaller* than the quantum of action. The existence of the quantum of action thus makes it impossible to keep track of quantum particles when they come too near to each other. Any description of systems with several quantons must thus take into account that after a close encounter, it is impossible to say which quanton is which.

If we remember that quantum theory describes quantons as clouds, the indistinguishability appears even more natural. Whenever two clouds meet and depart again, it is impossible to say which cloud is which. On the other hand, if two particles are kept distant enough, one does have an effective distinguishability; indistinguishability thus appears only when the particles come close.

In short, indistinguishability is a natural, unavoidable consequence of the existence of a smallest action value in nature. This result leads us straight away to the next question:

Can quantum particles be counted?

In everyday life, we can count objects because we can distinguish them. Since quantum particles cannot always be distinguished, we need some care in determining how to count them. The first step in counting particles is the definition of what is meant by a situation without any particle at all. This seems an easy thing to do, but later on we will encounter situations where already this step runs into difficulties. In any case, the first step of counting is thus the *specification of the vacuum*. Any counting method requires that the situation without particles is clearly separated from situations with particles.

The second step necessary for counting is the specification of an observable useful for determining quantum particle number. The easiest way is to choose one of those conserved quantum numbers that add up under composition, such as electric charge. Counting itself is then performed by measuring the total charge and dividing by the unit charge.

In everyday life, the weight or mass is commonly used as observable. However, it cannot be used generally in the quantum domain, except for simple cases. For a large number of particles, the interaction energy will introduce errors. For very large particle numbers, the gravitational binding energy will do so as well. But above all, for transient phenomena, unstable particles or short measurement times, mass measurements reach their limits. In short, even though counting stable atoms through mass measurements works in everyday life, the method is not applicable in general; especially at high particle energies, it cannot be applied.

Counting with the help of conserved quantum numbers has several advantages. First of all, it works also for transient phenomena, unstable particles or short measurement times. Secondly, it is not important whether the particles are distinguishable or not; counting always works. Thirdly, virtual particles are not counted. This is a welcome state of affairs, as we will see, because for virtual particles, i.e., particles for which $E^2 \neq p^2c^2 + m^2c^4$, there is *no way* to define a particle number anyway. Using a conserved quantity is indeed the best particle counting method possible.

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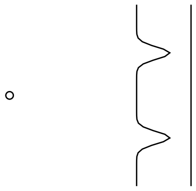
The side effect of counting with the help of quantum numbers is that antiparticles count negatively! Also this consequence is a result of the quantum of action. We saw above that the quantum of action implies that even in vacuum, particle–antiparticle pairs are observed at sufficiently high energies. As a result, an antiparticle must count as minus one particle. In other words, any way of counting quantum particles can produce an error due to this effect. In everyday life this limitation plays no role, as there is no antimatter around us. The issue does play a role at higher energies, however. It turns out that there is no general way to count the exact number of particles and antiparticles separately; only the sum can be defined. In short, quantum theory shows that particle counting is never perfect.

In summary, nature does provide a way to count quantum particles even if they cannot be distinguished, though only for everyday, low energy conditions; due to the quantum of action, antiparticles count negatively. Antiparticles thus provide a limit to the counting of particles at high energies, when the mass–energy equivalence becomes important.

What is permutation symmetry?

Since quantum particles are countable but indistinguishable, there exists a symmetry of nature for systems composed of several identical quantons. *Permutation symmetry*, also called *exchange symmetry*, is the property of nature that observations are unchanged under exchange of identical particles. Permutation symmetry forms one of the four pil-



lars of quantum theory, together with space-time symmetry, gauge symmetry and the not yet encountered renormalization symmetry. Permutation symmetry is a property of *composed* systems, i.e., of systems made of many (identical) subsystems. Only for such systems does indistinguishability play a role.

In other words, 'indistinguishable' is not the same as 'identical'. Two quantum particles of the same type are not the *same*; they are more like exact copies of each other. On the other hand, everyday life experience shows us that two copies can always be distinguished under close inspection, so that the term 'copy' is not fully appropriate either.

> ▷ Quantons, quantum particles, are countable and completely indistinguishable.* Quantum particles are *perfect copies* of each other.

Being perfect copies, not even nature can distinguish particles; as a result, permutation symmetry appears.

In the next chapter, we will discover that permutation is partial rotation. Permutation symmetry thus is a symmetry under partial rotations. Can you find out why?

Challenge 95 e

Indistinguishability and wave function symmetry

The indistinguishability of quantum particles leads to important conclusions about the description of their state of motion. This happens because it is impossible to formulate a description of motion that includes indistinguishability right from the start. (Are you able to confirm this?) We need to describe a n-particle state with a state $\Psi_{1...i...j...n}$ which assumes that distinction is possible, as expressed by the ordered indices in the notation, and we introduce the indistinguishability afterwards.

Challenge 96 s

Indistinguishability, or permutation symmetry, means that the exchange of any two quantum particles results in the same physical observations.** Now, two quantum states have the same physical properties if they differ at most by a phase factor; indistinguishability thus requires

$$\Psi_{1...i...j...n} = e^{i\alpha}\,\Psi_{1...j...i...n} \tag{57}$$

for some unknown angle α. Applying this expression twice, by exchanging the same couple of indices again, allows us to conclude that $e^{2i\alpha} = 1$. This implies that

$$\Psi_{1...i...j...n} = \pm\Psi_{1...j...i...n} \,, \tag{58}$$

in other words, a wave function is either *symmetric* or *antisymmetric* under exchange of indices. (We can also say that the eigenvalue for the exchange operator is either +1 or −1.)

> ▷ Quantum theory thus predicts that quantum particles can be indistinguish-

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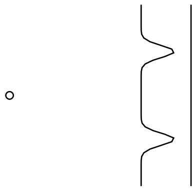
[*] The word 'indistinguishable' is so long that many physicists sloppily speak of 'identical' particles nevertheless. Take care.

[**] We therefore have the same situation that we encountered already several times: *an overspecification of the mathematical description*, here the explicit ordering of the indices, *implies a symmetry of this description*, which in our case is a symmetry under exchange of indices, i.e., under exchange of particles.



able in one of two distinct ways.*

- Particles corresponding to *symmetric* wave functions – those which transform under particle exchange with a '+' in equation (58) – are called** *bosons*.
- Particles corresponding to *antisymmetric* wave functions – those which transform under particle exchange with a '−' in equation (58) – are called*** *fermions*.

Experiments show that the exchange behaviour depends on the *type* of particle. Photons are found to be bosons. On the other hand, electrons, protons and neutrons are found to be fermions. Also about half of the atoms are found to behave as bosons (at moderate energies), the other half are fermions. To determine they type of atom, we need to take into account the spin of the electron and that of the nucleus.

In fact, a composite of an *even* number of fermions (at moderate energies) – or of any number of bosons (at any energy) – turns out to be a boson; a composite of an *odd* number of fermions is (always) a fermion. For example, ^{4}He is a boson, ^{3}He a fermion. Also the natural isotopes ^{23}Na, ^{41}K, ^{85}Rb, ^{87}Rb and ^{133}Cs are bosons, because they have odd numbers of electrons and of nucleons; in contrast, ^{40}K and ^{134}Cs are fermions (and, in this case, also radioactive).

THE BEHAVIOUR OF PHOTONS

A simple experiment, shown in Figure 58, allows observing an important aspect of photon behaviour. Take a source that emits two indistinguishable photons, i.e., two photons of identical frequency and polarization, at the same time. The photon pair is therefore in an entangled state. In the laboratory, such a source can be realized with a down-converter, a material that converts a photon of frequency $2f$ into two photons of frequency f. The two entangled photons, after having travelled exactly the same distance, are made to enter the two sides of an ideal beam splitter (for example, a half-silvered mirror). Two detectors are located at the two exits of the beam splitter. Experiments show Ref. 75 that both photons are always detected together on the *same* side, and never separately on

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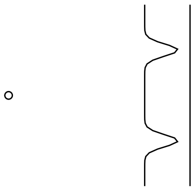
* This conclusion applies to three-dimensional space. In two dimensions there are more possibilities. Such possibilities have been and partly still are topic of research.

** 'Bosons' are named after the physicist Satyenra Nath Bose (b. 1894 Calcutta, d. 1974 Calcutta) who first Ref. 74 described the statistical properties of photons. The work was later expanded by Albert Einstein, so that one speaks of Bose–Einstein statistics.

*** The term 'fermion' is derived from the name of the physicist and Nobel Prize winner Enrico Fermi (b. 1901 Rome, d. 1954 Chicago) famous for his all-encompassing genius in theoretical and experimental physics. He mainly worked on nuclear and elementary particle physics, on spin and on statistics. For his experimental work he was called 'quantum engineer'. He is also famous for his lectures, which are still published in his own hand-writing, and his brilliant approach to physical problems. Nevertheless, his highly deserved Nobel Prize was one of the few cases in which the prize was given for a discovery which turned out to be incorrect. He left Italy because of the bad treatment his Jewish wife was suffering and emigrated to the USA. Fermi worked on the Manhattan project that built the first atomic bombs. After the Second World War, he organized one of the best physics department in the world, at the University of Chicago, where he was admired by everybody who worked with him.



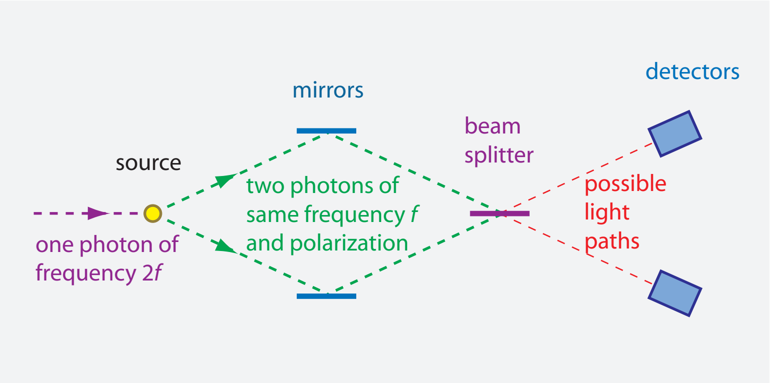


FIGURE 58 Two-photon emission and interference: two indistinguishable photons are always found arriving together, at the same detector.

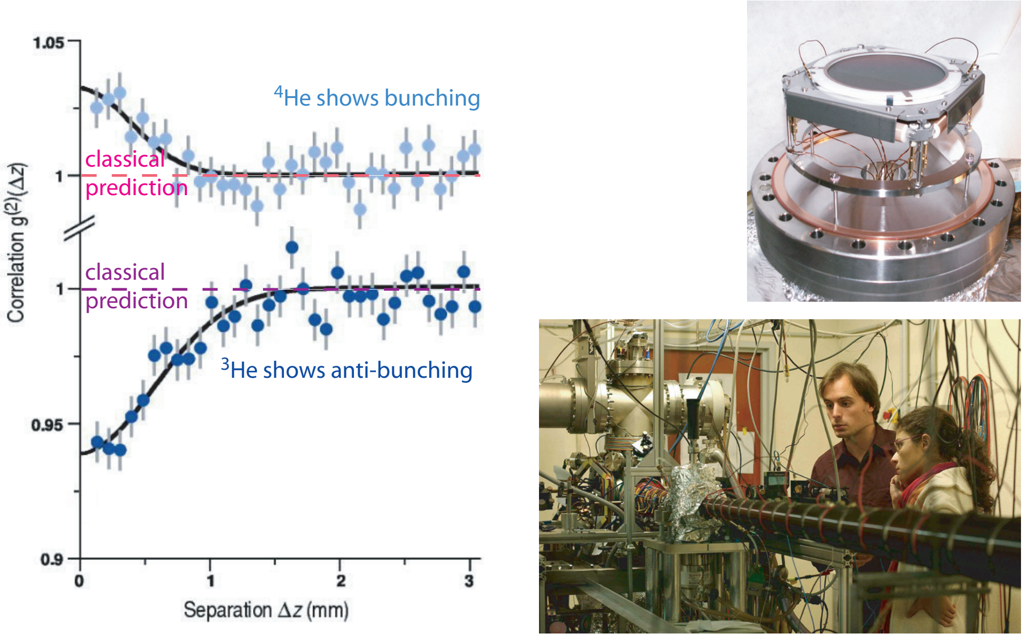
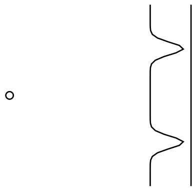


FIGURE 59 Bunching and antibunching of ^{3}He and ^{4}He helium!bunching atoms: the measurement result, the detector and the experiment (from atomoptic.iota.u-psud.fr/research/helium/helium.html, photo © Denis Boiron, Jerome Chatin).

opposite sides. This happens because the two options where one of the photons is transmitted and the other reflected interfere destructively. (The discussion mentioned above applies also here: despite two photons being involved, also in this case, when investigating the details, only one photon interferes with itself.) Page 59

The experiment shows that photons are *bosons*. Indeed, in the same experiment, fermions behave in exactly the opposite way; two fermions are always detected separately on *opposite* sides, never together on the same side. Ref. 76



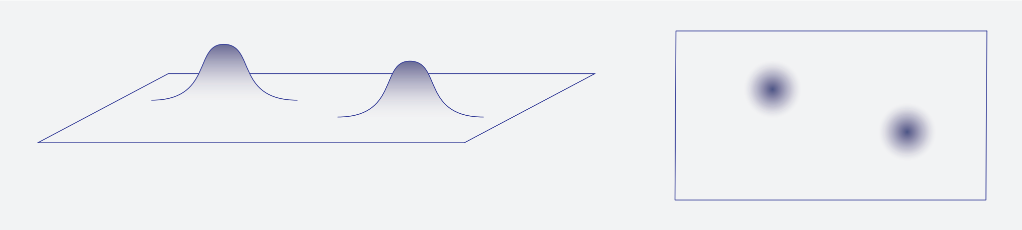


FIGURE 60 Picturing particles as localized excitations (left) or clouds (right).

Bunching and antibunching

Another way to test the exchange character of a particle is the Hanbury Brown–Twiss experiment described earlier on. First of all, this beautiful experiment shows that quantum particles behave differently than classical particles. In addition, compared to classical particles, fermions show antibunching – because of Pauli's exclusion principle – and bosons show bunching. Hanbury Brown and Twiss performed the experiment with photons, which are bosons.

Page 53

Ref. 77

In 2005, a French–Dutch research collaboration performed the experiment with atoms. By using an extremely cold helium gas at 500 nK and a clever detector principle, they were able to measure the correlation curves typical for the effect. The results, shown in Figure 59, confirm that ^{3}He is a fermion and ^{4}He is a boson, as predicted from the composition rule of quantum particles.

The energy dependence of permutation symmetry

If experiments force us to conclude that nobody, not even nature, can distinguish between two particles of the same type, we deduce that they do not form two separate entities, but some sort of unity. Our naive, classical sense of particle as a separate entity from the rest of the world is thus an incorrect description of the phenomenon of 'particle'. Indeed, no experiment can track particles with identical intrinsic properties in such a way that they can be distinguished with certainty. This impossibility has been checked experimentally with all elementary particles, with nuclei, with atoms and with numerous molecules.

How does this fit with everyday life, i.e., with classical physics? Photons do not worry us much here. Let us focus the discussion on matter particles. We know to be able to distinguish electrons by pointing to different wires in which they flow; also, we can distinguish our fridge, with its electrons and atoms, from that of our neighbour. While the quantum of action makes distinction impossible, everyday life does allow it.

The simplest explanation for both observations is to imagine a microscopic particle, especially an elementary one, as a *bulge*, i.e., as a localized excitation of the vacuum, or as a tiny cloud. Figure 60 shows two such bulges and two clouds representing particles. It is evident that if particles are too near to each other, it makes no sense to distinguish them; we cannot say any more which is which.

The bulge image shows that either for large distances or for high potential walls separating them, distinction of identical particles does become possible. In such situations, measurements allowing us to track particles independently do exist – as we know from everyday life. In other words, we can specify a limit energy at which permutation symmetry of objects or particles separated by a distance d becomes important. It is given by

$$E = \frac{c\,\hbar}{d} \, . \tag{59}$$

Challenge 97 e Are you able to confirm the expression? For example, at everyday temperatures we *can* distinguish atoms inside a solid from each other, since the energy so calculated is much higher than the thermal energy of atoms. To have fun, you might want to determine at Challenge 98 e what energy two truly identical human twins become indistinguishable. Estimating at what energies the statistical character of trees or fridges will become apparent is then straightforward.

To sum up, in daily life we are able to distinguish objects and thus people for two reasons: because they are made of *many* parts, and because we live in a *low energy* environment. The bulge image of particles purveys the idea that distinguishability exists for objects in everyday life but not for particles in the microscopic domain.

The energy issue immediately adds a new aspect to the discussion. How can we describe fermions and bosons in the presence of virtual particles and of antiparticles?

Indistinguishability in quantum field theory

Quantum field theory, as we will see in the next volume, simply puts the cloudy bulge idea of Figure 60 into mathematical language. A situation without any bulge is called *vacuum state*. Quantum field theory describes all particles of a given type as *excitations* of a single fundamental field. Particles are indistinguishable because each particle is an excitation of the same basic substrate and each excitation has the same properties. A situation with one particle is then described by a vacuum state acted upon by a *creation operator*. Adding a second particle is described by adding a second creation operator, and subtracting a particle by adding a *annihilation operator*; the latter turns out to be the adjoint of the former.

Quantum field theory studies how creation and annihilation operators must behave to describe observations.* It arrives at the following conclusions:

- Field operators for particles with half-integer spin are *fermions* and imply (local) anticommutation.
- Field operators for particles with with integer spin are *bosons* and imply (local) commutation.
- For all field operators, the commutator, respectively anticommutator, taken at two points with space-like separations, vanishes.

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* Whenever the relation

$$[b, b^\dagger] = bb^\dagger - b^\dagger b = 1 \tag{60}$$

holds between the creation operator $b^\dagger$ and the annihilation operator b, the operators describe a *boson*. The dagger can thus be seen as describing the operation of adjoining; a double dagger is equivalent to no dagger. If the operators for particle creation and annihilation anticommute

$$\{d, d^\dagger\} = dd^\dagger + d^\dagger d = 1 \tag{61}$$

they describe a *fermion*. The so defined bracket is called the *anticommutator bracket*.

- Antiparticles of fermions are fermions, and antiparticles of bosons are bosons.
- Virtual particles behave under exchange like their real counterparts.

These connections are at the basis of quantum field theory. They describe how quantons behave under permutation.

But why are quantum particles identical? Why are all electrons identical? Lead by experiment, quantum field theory describes electrons as identical excitations of the vacuum, and as such as identical by construction. Of course, this answer is not really satisfying. We will find a better one only in the final part of our mountain ascent.

How accurately is permutation symmetry verified?

Are electrons *perfect* fermions? In 1990, a simple but effective experiment testing their Ref. 78 fermion behaviour was carried out by Ramberg and Snow. They sent an electric current of 30 A through a copper wire for one month and looked for X-ray emission. They did not find any. They concluded that electrons are always in an antisymmetric state, with a symmetric component of less than

$$2 \cdot 10^{-26} \qquad (62)$$

of the total state. In short, electrons are always in an antisymmetric state: they are fermions.

The reasoning behind this elegant experiment is the following. If electrons would not always be fermions, every now and then an electron could fall into the lowest energy level of a copper atom, leading to X-ray emission. The lack of such X-rays implies that electrons are fermions to a very high accuracy. X-rays could be emitted only if they were bosons, at least part of the time. Indeed, two electrons, being fermions, cannot be in the same quantum state: this restriction is called the *Pauli exclusion principle*. It applies to all fermions and is the topic of the next chapter.

Copies, clones and gloves

Can classical systems be indistinguishable? They can: large molecules are examples – provided they are made of exactly the same isotopes. Can *large* classical systems, made of a mole or more particles be indistinguishable? This simple question effectively asks whether a *perfect* copy, or (physical) *clone*, of a physical system is possible.

It could be argued that any factory for mass-produced goods, such as one producing shirt buttons or paper clips, shows that copies are possible. But the appearance is deceiving. Seen under a microscope, there is usually some difference. Is this always the case? In 1982, the Dutch physicist Dennis Dieks and independently, the US-American physicists Ref. 79 Wootters and Zurek, published simple proofs that quantum systems cannot be copied. This is the famous *no-cloning theorem*.

A *copying machine* is a machine that takes an original, reads out its properties and produces a copy, leaving the original unchanged. This definition seems straightforward. However, we know that if we extract information from an original, we have to interact with it. As a result, the system will change at least by the quantum of action. We thus expect that due to quantum theory, copies and originals can never be identical.*

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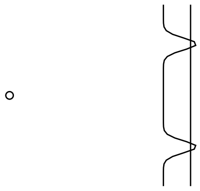
* This seems to provide a solution against banknote forgeries. In fact, Stephen Wiesner proposed to use



Quantum theory indeed shows that copying machines are impossible. A copying machine is described by an operator that maps the state of an original system to the state of the copy. In other words, a copying machine is linear. This linearity leads to a problem. Simply stated, if a copying machine were able to copy originals either in state $|A\rangle$ or in state $|B\rangle$, it could not work if the state of the original were a superposition $|A\rangle + |B\rangle$. Let us see why.

A copy machine is a device described by an operator U that changes the starting state $|s\rangle_\mathrm{c}$ of the copy in the following way:

— If the original is in state $|A\rangle$, a copier acts on the copy $|s\rangle_\mathrm{c}$ as

$$U|A\rangle|s\rangle_\mathrm{c} = |A\rangle|A\rangle_\mathrm{c} \; . \tag{63}$$

— If the original is in state $|B\rangle$, a copier acts on the copy $|s\rangle_\mathrm{c}$ as

$$U|B\rangle|s\rangle_\mathrm{c} = |B\rangle|B\rangle_\mathrm{c} \; . \tag{64}$$

As a result of these two requirements, an original in the state $|A + B\rangle$ is treated by the copier as

$$U|A + B\rangle|s\rangle_\mathrm{c} = |A\rangle|A\rangle_\mathrm{c} + |B\rangle|B\rangle_\mathrm{c} \; . \tag{65}$$

This is in contrast to what we want, which would be

$$U_\mathrm{wanted}|A + B\rangle|s\rangle_\mathrm{c} = (|A\rangle + |B\rangle)(|A\rangle_\mathrm{c} + |B\rangle_\mathrm{c}) \; . \tag{66}$$

In other words, a copy machine cannot copy a state completely.* This is the so-called *no-cloning theorem*.

The impossibility of copying is implicit in quantum theory. If we were able to clone systems, we could measure a variable of a system and a second variable on its copy. We would be thus able to beat the indeterminacy relation in both copies. This is impossible. In short, copies are always imperfect.

The lack of quantum mechanical copying machines is disappointing. Such science fiction machines could be fed with two different inputs, such as a lion and a goat, and produce a superposition: a chimaera. Quantum theory shows that all these imaginary beings or situations cannot be realized.

Other researchers then explored how near to perfection a copy can be, especially in the
Ref. 81 case of classical systems. To make a long story short, these investigations show that also the copying or cloning of macroscopic systems is impossible. In simple words, *copying machines do not exist.* Copies can always be distinguished from originals if observations

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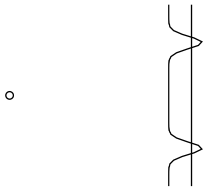
Ref. 80 quantum theory already in 1970; he imagined to use polarizations of stored single photons as bits of serial
Challenge 99 s numbers. Can you explain why this cannot work?

* The no-cloning theorem puts severe limitations on quantum computers, as computations often need copies of intermediate results. The theorem also shows that faster-than-light communication is impossible in EPR experiments. In compensation, *quantum cryptography* becomes possible – at least in the laboratory. Indeed, the no-cloning theorem shows that nobody can copy a quantum message without being noticed. The specific ways to use this result in cryptography are the 1984 Bennett–Brassard protocol and the 1991 Ekert protocol.



are made with sufficient care. In particular, this is the case for biological clones; biological clones are identical twins born following separate pregnancies. They differ in their finger prints, iris scans, physical and emotional memories, brain structures, and in many other Challenge 100 s aspects. (Can you specify a few more?) In short, biological clones, like identical twins, are not copies of each other.

In summary, everyday life objects such as photocopies, billiard balls or twins are always distinguishable. There are two reasons: first, quantum effects play no role in everyday life, so that there is no danger of unobservable exchange; secondly, perfect clones of classical systems do not exist anyway, so that there always are tiny differences between any two objects, even if they look identical at first sight. Gloves, being classical systems, can thus always be distinguished.

Summary

As a consequence of the quantum of action $\hbar$, quantum particles are *indistinguishable*. This happens in one of two ways: they are either bosons or fermions. Not even nature is able to distinguish between identical quantum particles.

Despite the indistinguishability of quantons, the state of a physical system cannot be copied to a second system with the same particle content. Therefore, perfect clones do not exist in nature.

CHAPTER 6

ROTATIONS AND STATISTICS – VISUALIZING SPIN

Spin is the observation that matter beams can be *polarized*: rays can be rotated. Spin thus describes how particles behave under rotations. Particles are thus not simply point-like: quantum particles can rotate around an axis. This proper rotation is called *spin*; like macroscopic rotation, spin is described by an angular momentum. Page 104

In the following, we recall that the spin of quantons is quantized in units of $\hbar/2$. Then we show a deep result: the value of spin determines whether a quantum particle, and any general quantum system, is a boson or a fermion. And we will show that spin is the rotation of quantons.

QUANTUM PARTICLES AND SYMMETRY

Ref. 82 The general background for the appearance of spin was clarified by Eugene Wigner in 1939.[**] He started by recapitulating that any quantum particle, if *elementary*, must behave like an *irreducible representation* of the set of all viewpoint changes. This set of viewpoint changes forms the symmetry group of flat space-time, the so-called *inhomogeneous Lorentz group*. Why?

Vol. I, page 266 We have seen in the chapter on symmetry, in the first volume of this adventure, that the symmetry of any composite system leads to certain requirements for the components of the system. If the components do not follow these requirements, they cannot build a symmetric composite.

We know from everyday life and precision experiments that all physical systems are symmetric under translation in time and space, under rotation in space, under boosts, and – in many cases – under mirror reflection, matter–antimatter exchange and motion reversal. We know these symmetries from everyday life; for example, the usefulness of what we call ‘experience’ in everyday life is simply a consequence of time translation symmetry. The set of all these common symmetries, more precisely, of all these symmetry transformations, is called the *inhomogeneous Lorentz group*.

These *symmetries*, i.e., these changes of viewpoints, lead to certain requirements for the components of physical systems, i.e., for the elementary quantum particles. In mathematical language, the requirement is expressed by saying that *elementary particles* must be *irreducible representations* of the symmetry group. Vol. I, page 266

[**] Eugene Wigner (b. 1902 Budapest, d. 1995 Princeton), theoretical physicist, received the Nobel Prize in Physics in 1963. He wrote over 500 papers, many about various aspects of symmetry in nature. He was also famous for being the most polite physicist in the world.

Every textbook on quantum theory carries out this reasoning in systematic detail. Starting with the Lorentz group, one obtains a list of all possible irreducible representations. In other words, on eobtains a list of all possible ways that elementary particles can behave. * Cataloguing the possibilities, one finds first of all that every elementary particle is described by *four-momentum* – no news so far – by an internal angular momentum, the *spin*, and by a set of *parities*.

- Four-momentum results from the translation symmetry of nature. The momentum value describes how a particle behaves under translation, i.e., under position and time shift of viewpoints. The magnitude of four-momentum is an invariant property, given by the mass, whereas its orientation in space-time is free.
- Spin results from the rotation symmetry of nature. The spin value describes how an object behaves under rotations in three dimensions, i.e., under orientation change of viewpoints.** The magnitude of spin is an invariant property, and its orientation has various possibilities with respect to the direction of motion. In particular, the spin of massive quantum particles behaves differently from that of massless quantum particles.

  For *massive* quantum particles, the inhomogeneous Lorentz group implies that the invariant magnitude of spin is $\sqrt{J(J+1)}\,\hbar$, often written, by oversimplification, as J. It is thus customary to say and write ‘spin J’ instead of the cumbersome ‘spin $\sqrt{J(J+1)}\,\hbar$’. Since the value of the quantum number J specifies the magnitude of the angular momentum, it gives the representation under rotations of a given particle type. The exploration shows that the spin quantum number J can be any multiple of $1/2$, i.e., it can take the values $0, 1/2, 1, 3/2, 2, 5/2$, etc. As summarized in Table 4, experiments show that electrons, protons and neutrons have spin $1/2$, the W and Z particles spin 1 and helium atoms spin 0. In addition, the representation of spin J is $2J+1$ dimensional, meaning that the spatial orientation of the spin has $2J+1$ possible values. For electrons, with $J = 1/2$, there are thus two possibilities; they are usually called ‘up’ and ‘down’. Spin thus only takes *discrete* values. This is in contrast with linear momentum, whose representations are infinite dimensional and whose possible values form a *continuous* range.

  Also *massless* quantum particles are characterized by the value of their spin. It can take the same values as in the massive case. For example, photons and gluons have spin 1. For massless particles, the representations are one-dimensional, so that massless particles are completely described by their *helicity*, defined as the projection of the spin onto the direction of motion. Massless particles can have positive or negative helicity, often also called right-handed and left-handed polarization. There is no other freedom for the orientation of spin in the massless case.
- To complete the list of particle properties, the remaining, discrete symmetries of the inhomogeneous Lorentz group must be included. Since *motion inversion*, *spatial parity* and *charge inversion* are parities, each elementary particle has to be described by three additional numbers, called T, P and C, each of which can only take the values

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* To be of physical relevance for quantum theory, representations have to be *unitary*. The full list of *irreducible* and *unitary* representations of viewpoint changes thus provides the range of possibilities for any particle that wants to be *elementary*.

** The group of physical rotations is also called SO(3), since mathematically it is described by the group of Special Orthogonal 3 by 3 matrices.

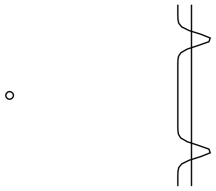


TABLE 4 Particle spin as representation of the rotation group.

Spin	System	Massive examples		Massless examples
[$\hbar$]	unchanged after rotation by	elementary	composite	elementary
0	any angle	Higgs boson	mesons, nuclei, atoms	none [a]
1/2	2 turns	e, μ, τ, q, ν_e, ν_μ, ν_τ	nuclei, atoms, molecules, radicals	none, as neutrinos have a tiny mass
1	1 turn	W, Z	mesons, nuclei, atoms, molecules, toasters	photon γ, gluon g
3/2	2/3 turn	none [a]	baryons, nuclei, atoms	none [a]
2	1/2 turn	none	nuclei	'graviton' [b]
5/2	2/5 turn	none	nuclei	none
3	1/3 turn	none	nuclei [c]	none
etc.[c]	etc.[c]	etc.[c]	etc.[c]	none possible

a. Supersymmetry, a symmetry conjectured in the twentieth century, predicts elementary particles in these and other boxes.
b. The graviton has not yet been observed.
c. Nuclei exist with spins values up to at least 101/2 and 51 (in units of $\hbar$). Ref. 83

+1 or −1. Being parities, these numbers must be *multiplied* to yield the value for a composed system.

In short, the symmetries nature lead to the classification of all elementary quantum particles by their mass, their momentum, their spin and their P, C and T parities.

Types of quantum particles

The spin values observed for all quantum particles in nature are given in Table 4. The parities and all known intrinsic properties of the elementary particles are given in Table 5. Spin and parities together are called *quantum numbers*. All other intrinsic properties of quantons are related to interactions, such as mass, electric charge or isospin, and we will explore them in the next volume. Vol. V, page 162

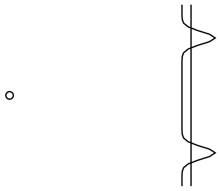


TABLE 5 Elementary particle properties.

Particle	Mass m[a]	Lifetime τ or energy width,[b] main decay modes	Isospin I, spin J,[c] parity P, charge parity C	Charge, isospin, strangeness,[c] charm, beauty,[d] topness: $QISCBT$	Lepton & baryon[e] numbers LB
Elementary radiation (bosons)					
photon γ	0 $(<10^{-53}$ kg)	stable	$I(J^{PC}) = 0, 1(1^{--})$	000000	0, 0
$W^{\pm}$	80.398(25) GeV$/c^2$	2.124(41) GeV 67.60(27) % hadrons, 32.12(36) % $l^{+}\nu$	$J = 1$	±100000	0, 0
Z	91.1876(21) GeV$/c^2$	$2.65(2) \cdot 10^{-25}$ s or 2.4952(23) GeV$/c^2$ 69.91(6) % hadrons, 10.0974(69) % $l^{+}l^{-}$	$J = 1$	000000	0, 0
gluon	0	stable	$I(J^P) = 0(1^-)$	000000	0, 0
Elementary matter (fermions): leptons					
electron e	9.109 382 15(45) $\cdot$ 10^{-31} kg = 81.871 0438(41) pJ$/c^2$ = 0.510 998 910(13) MeV$/c^2$ = 0.000 548 579 909 43(23) u gyromagnetic ratio $\mu_e/\mu_B = -1.001\,159\,652\,1811(7)$	$> 13 \cdot 10^{30}$ s	$J = \frac{1}{2}$	−100 000	1, 0
muon μ	0.188 353 130(11) yg = 105.658 3668(38) MeV$/c^2$ = 0.113 428 9256(29) u gyromagnetic ratio $\mu_\mu/(e\hbar/2m_\mu) = -1.001\,165\,9208(6)$	2.197 03(4) μs 99 % $e^{-}\bar{\nu}_e\nu_\mu$	$J = \frac{1}{2}$	−100000	1, 0
tau τ	1.776 84(17) GeV$/c^2$	290.6(1.0) fs	$J = \frac{1}{2}$	−100000	1, 0
el. neutrino ν_e	< 2 eV$/c^2$		$J = \frac{1}{2}$		1, 0
muon neutrino ν_μ	< 2 eV$/c^2$		$J = \frac{1}{2}$		1, 0
tau neutrino ν_τ	< 2 eV$/c^2$		$J = \frac{1}{2}$		1, 0
Elementary matter (fermions): quarks[f]					
up u	1.5 to 3.3 MeV$/c^2$	see proton	$I(J^P) = \frac{1}{2}(\frac{1}{2}^{+})$	$+\frac{2}{3}+\frac{1}{2}0000$	$0, \frac{1}{3}$
down d	3.5 to 6 MeV$/c^2$	see proton	$I(J^P) = \frac{1}{2}(\frac{1}{2}^{+})$	$-\frac{1}{3}-\frac{1}{2}0000$	$0, \frac{1}{3}$
strange s	70 to 130 MeV$/c^2$		$I(J^P) = 0(\frac{1}{2}^{+})$	$-\frac{1}{3}0-1000$	$0, \frac{1}{3}$
charm c	1.27(11) GeV$/c^2$		$I(J^P) = 0(\frac{1}{2}^{+})$	$+\frac{2}{3}00+100$	$0, \frac{1}{3}$

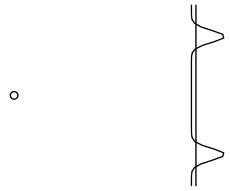


TABLE 5 (Continued) Elementary particle properties.

PARTICLE	MASS m [a]	LIFETIME τ OR ENERGY WIDTH, [b] MAIN DECAY MODES	ISOSPIN I, SPIN J, [c] PARITY P, CHARGE PARITY C	CHARGE, ISOSPIN, STRANGENESS, [c] CHARM, BEAUTY, [d] TOPNESS: $QISCBT$	LEPTON & BARYON [e] NUMBERS LB
bottom b	$4.20(17)\,\mathrm{GeV}/c^2$	$\tau = 1.33(11)\,\mathrm{ps}$	$I(J^P) = 0(\frac{1}{2}^+)$	$-\frac{1}{3}000{-}10$	$0, \frac{1}{3}$
top t	$171.2(2.1)\,\mathrm{GeV}/c^2$		$I(J^P) = 0(\frac{1}{2}^+)$	$+\frac{2}{3}0000{+}1$	$0, \frac{1}{3}$
Observed elementary boson					
Higgs boson	$126\,\mathrm{GeV}/c^2$		$J = 0$		

Notes:

a. See also the table of SI prefixes on page 206. About the eV/c^2 mass unit, see page 210.

b. The *energy width* Γ of a particle is related to its lifetime τ by the indeterminacy relation $\Gamma\tau = \hbar$. There is a difference between the *half-life* $t_{1/2}$ and the lifetime τ of a particle: they are related by $t_{1/2} = \tau \ln 2$, where $\ln 2 \approx 0.693\,147\,18$; the half-life is thus shorter than the lifetime. The unified *atomic mass unit* u is defined as 1/12 of the mass of a carbon 12 atom at rest and in its ground state. One has $1\,\mathrm{u} = \frac{1}{12}m(^{12}\mathrm{C}) = 1.660\,5402(10)\,\mathrm{yg}$.

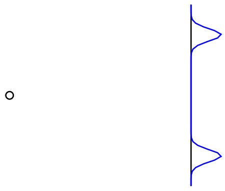
c. To keep the table short, its header does not explicitly mention *colour*, the – confusingly named – charge of the strong interactions. It has to be added to the list of basic object properties. Quantum numbers containing the word ‘parity’ are multiplicative; all others are additive. Parity P and charge parity C are written as + or −. Time parity T (not to be confused with topness T), better called motion inversion parity, is equal to CP in all known particles. The isospin I (or I_Z) appears twice in the table; it is defined only for up and down quarks and their composites, such as the proton and the neutron. In the literature one also sees references to the so-called G-parity, defined as $G = (-1)^{IC}$.

The table header also does not mention the *weak charge* of the particles. The details on weak charge g, or, more precisely, on the *weak isospin*, a quantum number assigned to all left-handed fermions (and right-handed anti-fermions), but to no right-handed fermion (and no left-handed antifermion), are given in the section on the weak interactions. Vol. V, page 245

d. ‘Beauty’ is now commonly called *bottomness*; similarly, ‘truth’ is now commonly called *topness*. The signs of the quantum numbers S, I, C, B, T can be defined in different ways. In the standard assignment shown here, the sign of each of the non-vanishing quantum numbers is given by the sign of the charge of the corresponding quark.

e. If supersymmetry existed, R-parity would have to be added to this column. R-parity is a multiplicative quantum number related to the lepton number L, the baryon number B and the spin J through the definition $R = (-1)^{3B+L+2J}$. All particles from the standard model are R-even, whereas their conjectured supersymmetric partner particles would be R-odd. However, supersymmetry is now known to be in contrast with experiment.

f. For the precise definition and meaning of quark masses, see page 233 in volume V.



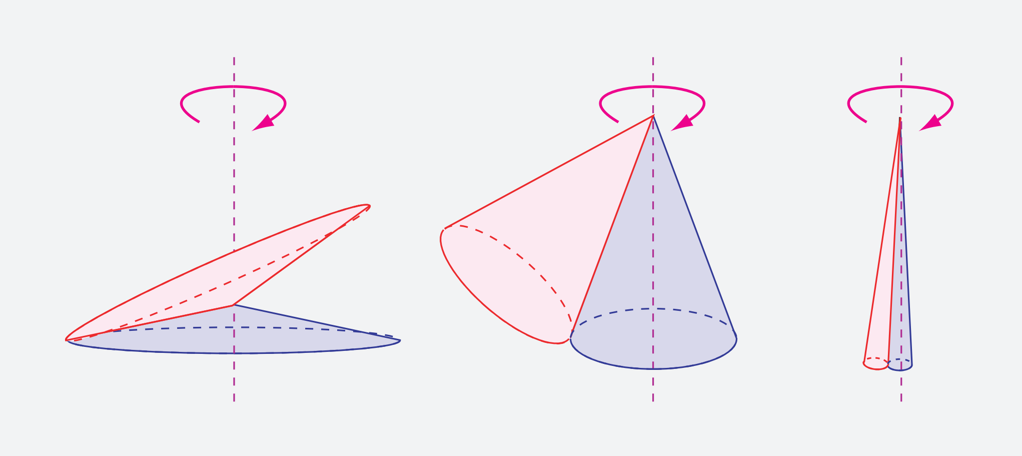


FIGURE 61 Illustrating an argument showing why rotations by 4π are equivalent to no rotation at all (see text).

Spin 1/2 and tethered objects

A central result of quantum theory is that spin 1/2 is a possibility in nature, even though this value does not appear in everyday life. For a system to have spin 1/2 means that for such a system only a rotation by *two turns* is equivalent to *none* at all, while one by *one turn* is not. No *simple* systems with this property exist in everyday life, but such systems do exist in microscopic systems: electrons, neutrinos, silver atoms and molecular radicals all have spin 1/2. Table 4 gives a more extensive list.

Vol. I, page 49 The mathematician Hermann Weyl used a simple image to explain that the rotation by two turns is equivalent to zero turns, whereas one turns differs. Take two cones, touching each other at their tips as well as along a line, as shown in Figure 61. Hold one cone and roll the other around it. When the rolling cone, after a full turn around the other cone, i.e., around the vertical axis, has come back to the original position, it has rotated by some angle. If the cones are wide, as shown on the left, the final rotation angle is small. The limit of extremely wide cones gives no rotation at all. If the cones are very thin, like needles, the moving cone has rotated by (almost) 720 degrees; this situation is like a coin rolling around a second coin of the same size, both lying on a table. The rolling coins rotates by two turns, thus by 720 degrees. Also in this case, the final rotation angle is small. The result for 0 degrees and for 720 degrees is the same. If we imagine the cone angle to vary continuously, this visualization shows that a 0 degree rotation can be continuously changed into a 720 degree rotation. In contrast, a 360 degree rotation *cannot* be 'undone' in this way.

Challenge 101 e

Challenge 102 e

There are systems in everyday life that behave like spin 1/2, but they are not simple: all such systems are *tethered*. The most well-known system is the belt. Figure 62 and Figure 63 show that a rotation by 4π of a belt buckle is equivalent to no rotation at all: this is easily achieved by moving the belt around. You may want to repeat the process by yourself, using a real belt or a strip of paper, in order to get a feeling for it. The untangling process is often called the *belt trick*, but also *scissor trick*, *plate trick*, *string trick*, *Philippine wine dance* or *Balinese candle dance*. It is sometimes incorrectly attributed to Dirac,

Challenge 103 e

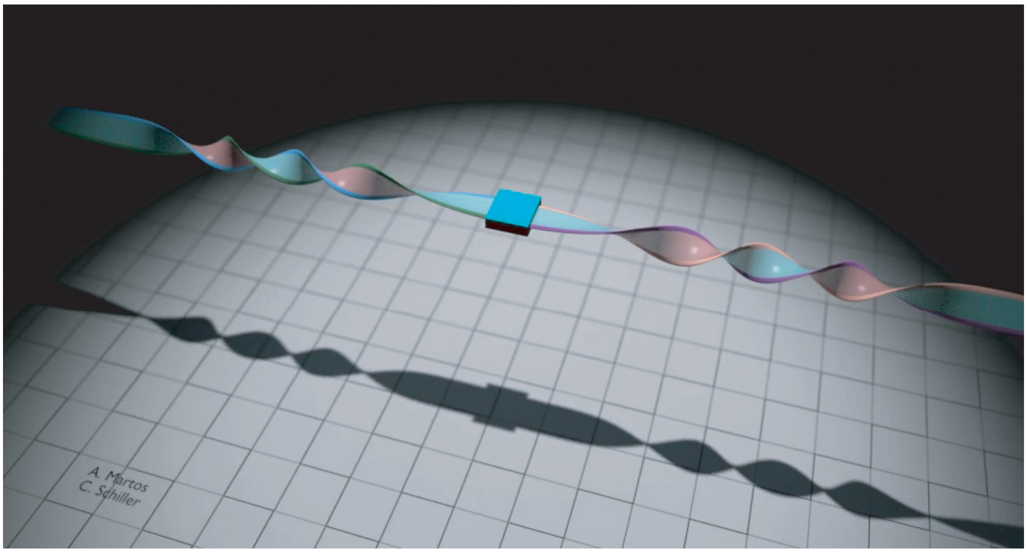


FIGURE 62 Assume that the belt cannot be observed, but the square object can, and that it represents a particle. The animation then shows that such a particle (the square object) can return to the starting position after rotation by 4π (and not after 2π). Such a 'belted' particle thus fulfils the defining property of a *spin 1/2 particle*: rotating it by 4π is equivalent to no rotation at all. The belt thus represents the spinor wave function; for example, a 2π rotation leads to a twist; this means a change of the sign of the wave function. A 4π rotation has no influence on the wave function. You can repeat the trick at home, with a paper strip. The equivalence is shown here with two attached belts, but the trick works with any positive number of belts! (QuickTime film © Antonio Martos)

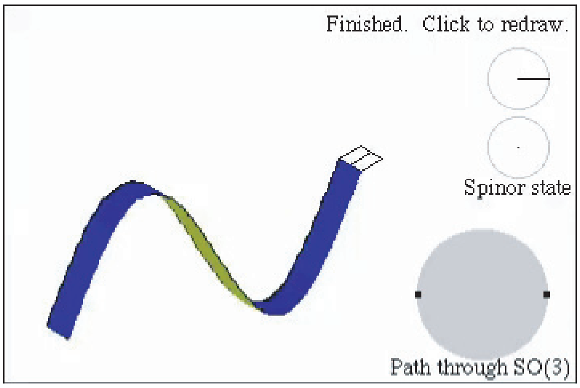
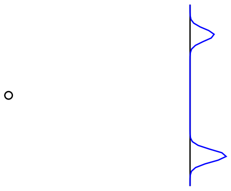


FIGURE 63 The belt trick with a simple belt: a double rotation of the belt buckle is equivalent to no rotation. (QuickTime film © Greg Egan)

because he used it extensively in his lectures.

The human body has such a belt built in: the *arm*. Just take your hand, put an object on it for clarity, such as a cup, and turn the hand and object by 2π by twisting the arm. After a second rotation the whole system will be untangled again, as shown in Figure 64. The trick is even more impressive when *many* arms are used. You can put your two hands (if you chose the correct starting position) under the cup or you can take a friend or two

Challenge 104 e



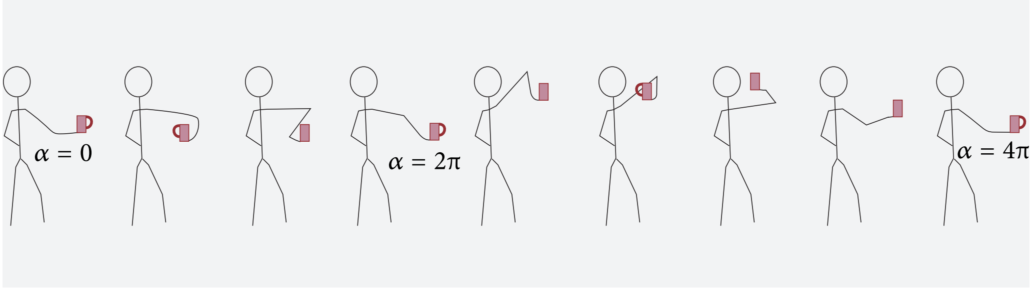


FIGURE 64 The human arm as spin 1/2 model.

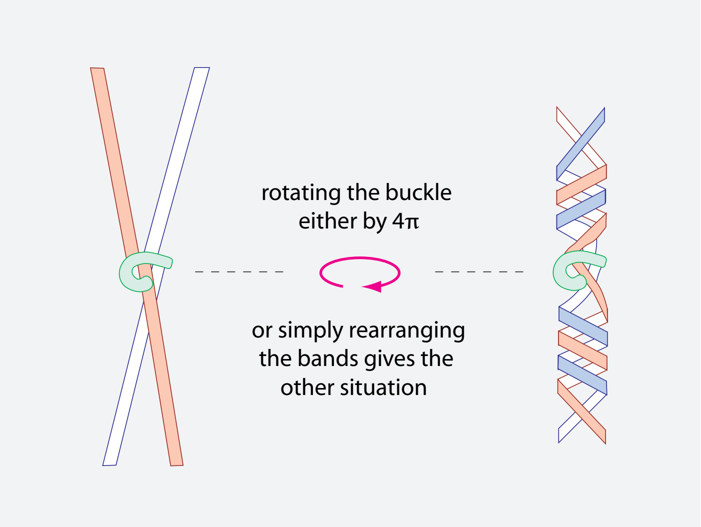


FIGURE 65 The generalized belt trick, modelling the rotation behaviour of a spin 1/2 particle: independently of the number of bands or tubes or strings attached, the two situations can be transformed into each other, either by rotating the central object by 4π or by keeping the central object fixed and moving the bands around it.

who each keep a hand attached to the cup together with you. The belt trick can still be performed, and the whole system untangles after two full turns. Challenge 105 e

This leads us to the most general way to show the connection between tethering and spin 1/2. Just glue any number of threads, belts or tubes, say half a metre long, to some object, as shown in Figure 65. (With many such tails, is not appropriate any more to call it a belt buckle.) Each band is supposed to go to spatial infinity and be attached there. Instead of being attached at spatial infinity, we can also imagine the belts attached to a distant, fixed object, like the arms are attached to a human body. If the object, which represents the particle, is rotated by 2π, twists appear in its tails. If the object is rotated by an additional turn, to a total of 4π, all twists and tangles can be made to disappear, without moving or turning the object. You really have to experience this in order to believe it. And the process really works with *any* number of bands glued to the object. The website www.evl.uic.edu/hypercomplex/html/dirac.html provides a animation showing this process with four attached belts.

In short, all these animations show that belt buckles, and in fact all (sufficiently) tethered systems, return to their original state only after rotations by 4π, and *not* after rotations by 2π only. *Tethered objects behave like spin 1/2 particles.* In fact, tethered ob-

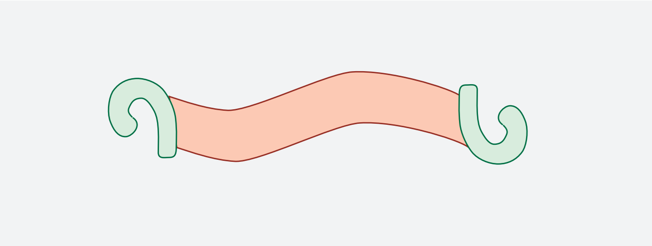


FIGURE 66 Two belt buckles connected by a belt, one way of visualizing two spin 1/2 particles.

Challenge 106 e jects, such as belt buckles, are the *only* systems that reproduce spin 1/2 properties. In the last part of our adventure we will discover the deep underlying reason for the equivalence between spin 1/2 particles and tethered systems.

Exploring the symmetries of wave functions, quantum theory shows that rotations *require* the existence of spin for all quantum particles. An investigation of the wave function shows that wave functions of elementary matter particles behave under rotation like tethered objects. For example, a wave function whose tethered equivalent is tangled acquires a negative sign.

In summary, quantum theory implies the existence of the slightly counter-intuitive spin 1/2 value. In particular, it appears for elementary matter particles.

THE EXTENSION OF THE BELT TRICK

But why do experiments show that all fermions have half-integer spin and that all bosons
Page 135 have integer spin? In particular, why do electrons obey the Pauli exclusion principle? At first sight, it is not clear what the spin value has to do with the statistical properties of a particle. In fact, there are several ways to show that rotations and statistics are connected.
Ref. 86 The first proof, due to Wolfgang Pauli, used the details of quantum field theory and was so complicated that its essential ingredients were hidden. It took several decades to convince
Ref. 87 everybody that a further observation about belts was the central part of the proof.

Page 120 Starting from the bulge model of quantum particles shown in Figure 60, we can imagine a tube connecting two particles, similar to a belt connecting two belt buckles, as shown in Figure 66. The buckles represent the particles. The tube keeps track of their relative orientation. If one particle/buckle is rotated by 2π along any axis, a twist is inserted into the belt. As just shown, if the same buckle is rotated by another 2π, bringing the total to 4π, the ensuing double twist can easily be undone without moving or rotating the buckles.

Now we look again at Figure 66. If we take the two buckles and simply *swap* their positions, a twist is introduced into the belt. If we swap them again, the twist will disappear. In short, two connected belt buckles return to their original state only after a double exchange, and not after a single exchange.

In other words, if we take each buckle to represent a particle and a twist to mean a factor -1, the belt exactly describes the phase behaviour of spin $1/2$ wave functions, both under *rotation* and under *exchange*. In particular, we see that rotation and exchange behaviour are related.

Similarly, also the belt trick itself can be extended to exchange. Take two buckles that

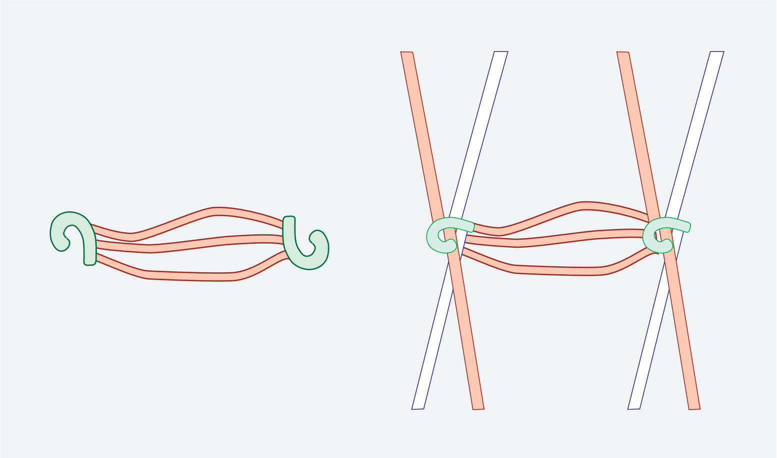


FIGURE 67 Extended belt models for two spin 1/2 particles.

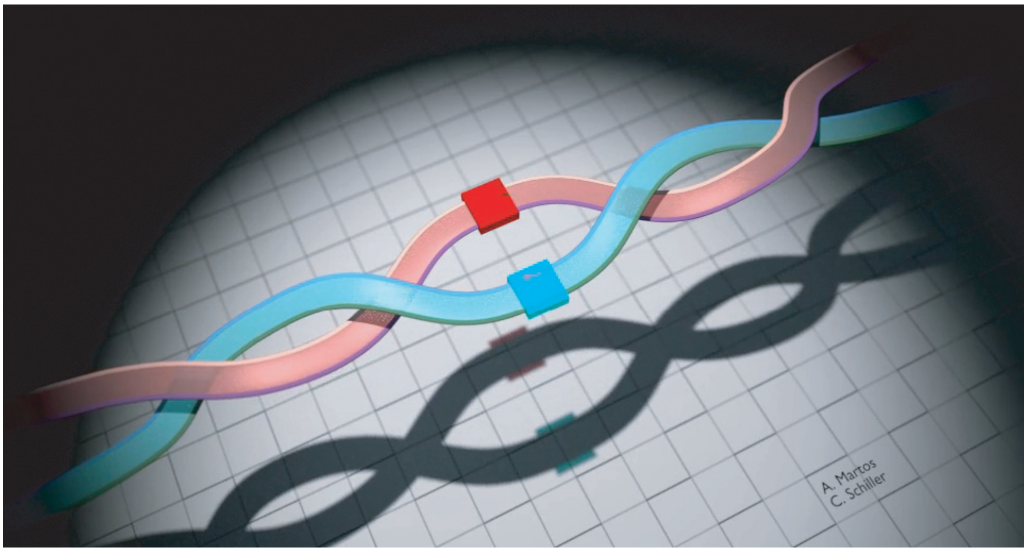


FIGURE 68 Assume that the belts cannot be observed, but the square objects can, and that they represent particles. We know from above that belted buckles behave as spin 1/2 particles. The animation shows that two such particles return to the original situation if they are switched in position twice (but not once). Such particles thus fulfil the defining property of *fermions*. (For the opposite case, that of bosons, a simple exchange would lead to the identical situation.) You can repeat the trick at home using paper strips. The equivalence is shown here with two belts per particle, but the trick works with any positive number of belts attached to each buckle. This animation is the essential part of the proof that spin 1/2 particles are fermions. This is called the *spin–statistics theorem*. (QuickTime film © Antonio Martos)

are connected with *many* bands or threads, like in Figure 67 or in Figure 68. The band can connect the particles, or go to spatial infinity, or both. An exchange of the two buckles produces quite a messy tangle. But almost incredibly, in all cases, a second exchange leads back to the original situation, if the belts are properly rearranged. You might want to test

Challenge 107 e

yourself that the behaviour is also valid if additional particles are involved, as long as you always exchange the same two particles twice.

We conclude that tethered objects behave like fermions under exchange. These observations together form the spin–statistics theorem for spin 1/2 particles: *spin and exchange behaviour are related*. Indeed, these almost 'experimental' arguments can be put Ref. 88 into exact mathematical language by studying the behaviour of the configuration space of particles. These investigations result in the following statements:

▷ Objects of spin 1/2 are fermions.[*]

▷ Exchange and rotation of spin 1/2 particles are similar processes.

In short, objects that behave like spin 1/2 particles under rotations also behave like fermions under exchange. And vice versa. The exchange behaviour of particles determines their statistical properties; the rotation behaviour determines their spin. By extending the belt trick to several buckles, each with several belts, we thus visualized the spin–statistics theorem for fermions.

Note that all these arguments require three dimensions of space, because there are no tangles (or knots) in fewer or more dimensions.[**] And indeed, spin exists only in three spatial dimensions.

The belt trick leads to interesting puzzles. We saw that a spin 1/2 object can be modelled by imagining that a belt leading to spatial infinity is attached to it. If we want to model the spin behaviour with attached one-dimensional *strings* instead of *bands*, what Challenge 109 s is the minimum number of strings we need? More difficult is the following puzzle: Can the belt trick be performed if the buckle is glued into a mattress, thus with the mattress Challenge 110 d acting like 'infinitely many' belts?

ANGELS, PAULI'S EXCLUSION PRINCIPLE AND THE HARDNESS OF MATTER

Why are we able to knock on a door? Why can stones not fly through tree trunks? How does the mountain we are walking on carry us? Why can't we walk across walls? In classical physics, we avoided this issue, by taking solidity as a defining property of matter. But we cannot do so any more: we have seen that matter consists mainly of low density electron clouds. The quantum of action thus forces us to *explain* the quantum of matter.

The explanation of the impenetrability of matter is so important that it led to a Nobel prize in physics. The interpenetration of bodies is made impossible by *Pauli's exclusion principle* among the electrons inside atoms. Pauli's exclusion principle states:

▷ Two fermions cannot occupy the same quantum state.

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[*] A mathematical observable behaving like a spin 1/2 particle is neither a vector nor a tensor, as you may Challenge 108 e want to check. An additional concept is necessary; such an observable is called a *spinor*. We will introduce Page 189 it in detail later on.

[**] Of course, knots and tangles do exist in higher dimensions. Instead of considering knotted one-dimensional lines, one can consider knotted planes or knotted higher-dimensional hyperplanes. For example, deformable planes can be knotted in four dimensions and deformable 3-spaces in five dimensions. However, the effective dimensions that produce the knot are always three.

All experiments known confirm the statement.

Why do electrons and other fermions obey Pauli's exclusion principle? The answer can be given with a beautifully simple argument. We know that exchanging two fermions produces a minus sign in the total wave function. Imagine these two fermions being, as a classical physicist would say, located at the same spot, or as a quantum physicist would say, in the same state. If that could be possible, an exchange would change nothing in the system. But an exchange of fermions must produce a minus sign for the total state. Both possibilities – no change at all as well as a minus sign – cannot be realized at the same time. There is only one way out: two fermions must avoid to ever be in the same state. This is Pauli's exclusion principle.

Ref. 89

The exclusion principle is the reason that two pieces of matter in everyday life cannot penetrate each other, but have to *repel* each other. For example, take a bell. A bell would not work if the colliding pieces that produce the sound would interpenetrate. But in any example of two interpenetrating pieces, the electrons from different atoms would have to be at the same spot: they would have to be in the same states. This is impossible. Pauli's exclusion principle forbids interpenetration of matter. Bells only work because of the exclusion principle.

Why don't we fall through the floor, even though gravity pulls us down, but remain standing on its surface? Again, the reason is Pauli's exclusion principle. Why does the floor itself not fall? It does not fall, because the matter of the Earth cannot interpenetrate and the atoms cannot made to approach each other than a certain minimal distance. In other words, Pauli's exclusion principle implies that atomic matter cannot be compressed indefinitely. At a certain stage an effective *Pauli pressure* appears, so that a compression limit ensues. For this reason for example, planets made of atomic matter – or neutron stars made of neutrons, which also have spin 1/2 and thus also obey the exclusion principle – do not collapse under their own gravity.

The exclusion principle is the reason that atoms are *extended* electron clouds and that different atoms have different sizes. In fact, the exclusion principle forces the electrons in atoms to form *shells*. When electrons are added around a nucleus and when one shell is filled, a new shell is started. This is the origin of the periodic systems of the elements.

The size of any atom is the size of its last shell. Without the exclusion principle, atoms would be as small as a hydrogen atom. In fact, most atoms are considerably larger. The same argument applies to nuclei: their size is given by the last nucleon shell. Without the exclusion principle, nuclei would be as small as a single proton. In fact, they are usually about 100 000 times larger.

The exclusion principle also settles an old question: How many *angels* can dance on the top of a pin? (Note that angels, if at all, must be made of fermions, as you might want to deduce from the information known about them, and that the top of a pin is a single point in space.) Both theory and experiment confirm the answer already given by Thomas Aquinas in the Middle Ages: Only *one* angel! The fermion exclusion principle could also be called 'angel exclusion principle'. To stay in the topic, the principle also shows that *ghosts* cannot be objects, as ghosts are supposed to be able to traverse walls.

Challenge 111 s

Ref. 90

Let us sum up. Simplifying somewhat, the exclusion principle keeps things around us in shape. Without the exclusion principle, there would be no three-dimensional objects. Only the exclusion principle fixes the diameter of atomic clouds, keeps these clouds from merging, and holds them apart. This repulsion is the origin for the size of soap, planets and neutron stars. All shapes of solids and fluids are a direct consequence of the exclusion principle. In other words, when we knock on a table or on a door, we prove experimentally that these objects and our hands are made of fermions.

So far, we have only considered fermions of spin 1/2. We will not talk much about particles with odd spin of higher value, such as 3/2 or 5/2. Such particles can all be seen as being composed of spin 1/2 entities. Can you confirm this? Challenge 112 e

We did not talk about lower spins than 1/2 either. A famous theorem states that a spin value between 0 and 1/2 is impossible in three dimensions. Ref. 82 Smaller spins are impossible because the largest rotation angle that can be distinguished and measured in three dimensions is 4π. There is no way to measure a larger angle; the quantum of action makes this impossible. Thus there cannot be any spin value between 0 and 1/2 in nature.

IS SPIN A ROTATION ABOUT AN AXIS?

The spin of a particle behaves experimentally like an intrinsic angular momentum, adds up like angular momentum, is conserved as part of angular momentum, is described like angular momentum and has a name synonymous with angular momentum. Despite all this, for many decades a strange and false myth was spread in many physics courses and textbooks around the world: "Spin 1/2, despite its name, is *not* a rotation about an axis." It is time to finish with this example of incorrect thinking.

Electrons do have spin 1/2 and are charged. Electrons and all other charged particles with spin 1/2 do have a magnetic moment.[*] A magnetic moment is expected for any rotating charge. In other words, spin 1/2 does behave like rotation. However, assuming that a particle consists of a *continuous* charge distribution in rotational motion gives the wrong value for the magnetic moment. In the early days of the twentieth century, when physicists were still thinking in classical terms, they concluded that charged spin 1/2 particles thus cannot be rotating. This myth has survived through many textbooks. The correct deduction, however, is that the assumption of continuous charge distribution is wrong. Indeed, charge is quantized; nobody expects that elementary charge is continuously spread over space, as that would contradict its quantization.

The other reason for the false myth is rotation itself. The myth is based on classical thinking and maintains that any rotating object must have *integer* spin. Since half integer spin is not possible in classical physics, it is argued that such spin is not due to rotation. But let us recall what rotation is. Both the belt trick for spin 1/2 as well as the integer spin case remind us: a *rotation* of one body around another is a fraction or a multiple of an exchange. What we call a rotating body in everyday life is a body continuously exchanging the positions of its parts – and vice versa.

▷ Rotation and exchange are the same process.

Now, we just found that spin is exchange behaviour. Since rotation is exchange and spin is exchange, it follows that

▷ Spin *is* rotation.

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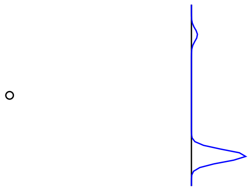
[*] This magnetic moment can easily be measured in an experiment; however, not one of the Stern–Gerlach type. Why not? Challenge 113 ny



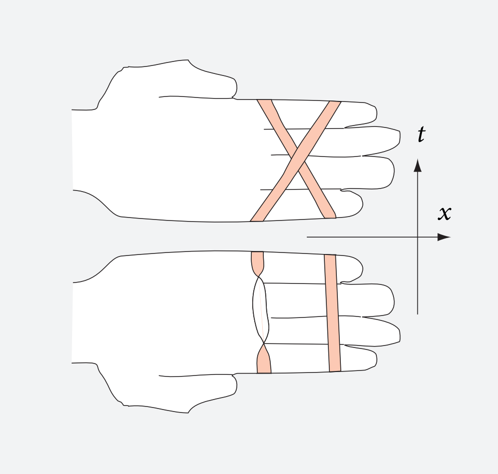


FIGURE 69 Equivalence of exchange and rotation in space-time.

Since we deduced spin, like Wigner, from rotation invariance, this conclusion is not a surprise. In addition, the belt model of a spin 1/2 particle tells us that such a particle can rotate continuously without any hindrance. Also the magnetic moment then gets its correct value. In short, we are allowed to maintain that spin is rotation about an axis, without any contradiction to observations, even for spin 1/2.

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In summary, the belt model shows that also spin 1/2 is rotation, as long as we assume that only the buckle can be observed, not the belt(s), and that elementary charge is not continuously distributed in space.*

Ref. 91

Since permutation properties and spin properties of fermions are so well described by the belt model, we could be led to the conclusion that these properties might really be consequence of such belt-like connections between particles and the outside world. Maybe for some reason we only observe the belt buckles, not the belts themselves. In the final part of this walk we will discover whether this idea is correct.

ROTATION REQUIRES ANTIPARTICLES

The connection between rotation and antiparticles may be the most astonishing conclusion from the experiments showing the existence of spin. So far, we have seen that rotation requires the existence of spin, that spin appears when relativity is introduced into quantum theory, and that relativity requires antimatter. Taking these three statements together, the conclusion of the title is not surprising any more: rotation requires antiparticles. Interestingly, there is a simple argument making the same point with the belt model, if it is extended from space alone to full *space-time*.

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To learn how to think in space-time, let us take a particle and reduce it to two short tails, so that the particle is a short line segment. When moving in a 2+1 dimensional

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* Obviously, the exact structure of the electron still remains *unclear* at this point. Any angular momentum S is given classically by $S = \Theta\omega$; however, neither the moment of inertia Θ, connected to the rotation radius and electron mass, nor the angular velocity ω are known at this point. We have to wait quite a while, until the final part of our adventure, to find out more.

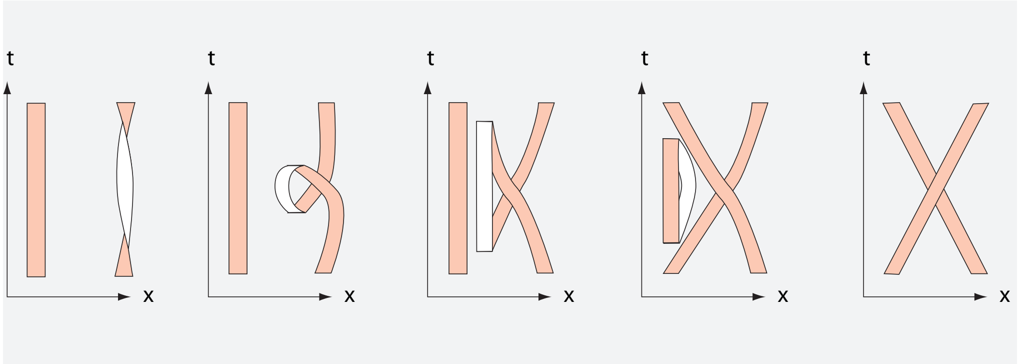

FIGURE 70 Belts in space-time: rotation and antiparticles.

Challenge 114 ny

space-time, the particle is described by a ribbon. Playing around with ribbons in space-time, instead of belts in space, provides many interesting conclusions. For example, Figure 69 shows that wrapping a rubber ribbon around the fingers can show, again, that a rotation of a body by 2π in presence of a second one is the same as exchanging the positions of the two bodies.* Both sides of the hand transform the same initial condition, at one edge of the hand, to the same final condition at the other edge. We have thus successfully extended a known result from space to space-time: rotation and exchange are equivalent.

If you think that Figure 69 is not a satisfying explanation, you are right. A more satisfying explanation must include a smooth sequence of steps realizing the equivalence between rotation and exchange. This is shown in Figure 70. We assume that each particle is described by a segment; in the figure, the two segments lie horizontally. The leftmost diagram shows two particles: one at rest and one being rotated by 2π. The deformation of the ribbons shows that this process is equivalent to the exchange in position of two particles, which is shown in the rightmost diagram.

But the essential point is made by the intermediate diagrams. We note that the sequence showing the equivalence between rotation and exchange requires the use of a loop. But such a loop in space-time describes the appearance of a particle–antiparticle pair! In other words, without antiparticles, the equivalence of rotation and exchange would not hold. In short, rotation in space-time requires the existence of antiparticles.

WHY IS FENCING WITH LASER BEAMS IMPOSSIBLE?

When a sword is approaching dangerously, we can stop it with a second sword. Many old films use such scenes. When a laser beam is approaching, it is impossible to fend it off with a second beam, despite all science fiction films showing so. Banging two laser beams against each other is impossible. The above explanation of the spin–statistics theorem shows why.

The electrons in the swords are fermions and obey the Pauli exclusion principle. Fermions make matter impenetrable. On the other hand, the photons in laser beams are

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* Obviously, the full argument would need to check the full spin 1/2 model of Figure 65 in four-dimensional space-time. But doing this is not an easy task; there is no good visualization yet.

Challenge 115 ny

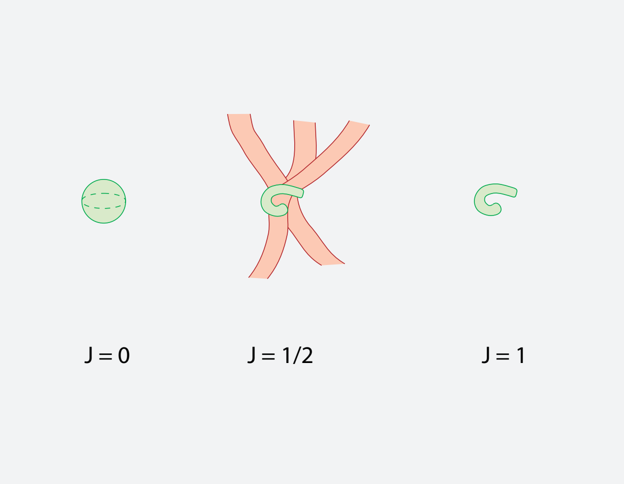

FIGURE 71 Some visualizations of spin representations.

bosons. Two bosons *can* be in the same state; bosons allow interpenetration. Matter is impenetrable because at the fundamental level *it is composed of fermions.* Radiation is composed of bosons; light beams can cross each other. The distinction between fermions and bosons thus explains why objects can be touched while images cannot. In the first part of our mountain ascent we started by noting this difference; now we know its origin. Vol. I, page 98

SPIN, STATISTICS AND COMPOSITION

Under rotations, integer spin particles behave differently from half-integer particles. Integer spin particles do not show the strange sign changes under rotations by 2π. In the belt imagery, integer spin particles need no attached strings. In particular, a spin 0 particle obviously corresponds to a sphere. Models for other important spin values are shown in Figure 71. Exploring their properties in the same way as above, we arrive at the full *spin–statistics theorem*:

> ▷ Exchange and rotation of objects are similar processes.
>
> ▷ Objects of half-integer spin are fermions. They obey the Pauli exclusion principle.
>
> ▷ Objects of integer spin are bosons.

Challenge 116 e You might prove by yourself that this suffices to show the following rule:

> ▷ Composites of bosons, as well as composites of an even number of fermions (at low energy), are bosons; composites of an uneven number of fermions are fermions.*

Challenge 117 s * This rule implies that spin 1 and higher can also be achieved *with* tails; can you find such a representation?

Note that composite fermions can be bosons only up to that energy at which the composition breaks down. Otherwise, by packing fermions into bosons, we could have fermions in the same state.