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Salma Alrasheed

Principles of Mechanics

Fundamental University Physics

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Preface

This book is aimed at taking the reader step by step through the beautiful concepts of mechanics in a clear and detailed manner. Mechanics is considered to be the core of physics and a deep understanding of the concepts is essential for all branches of physics. Many proofs and examples are included to help the reader grasp the fundamentals fully, paving the way to deal with more advanced topics. This book is useful for undergraduate students majoring in physics or other science and engineering disciplines. It can also be used as a reference for more advanced levels.

I would like to express my deep gratitude to my parents Abdulkareem Alrasheed and Mona Alzamil for their encouragement and support. I am grateful to all of those who have contributed to this book and made its completion possible. In particular, I would like to thank Khalid Alzamil, Dr. Laila Babsail, and Abbie Clifford for their efforts in revising the book. My sincere thanks are also extended to Ardel Flavier and Rodolfo Rodriguez for their assistance in creating the figures and illustrations. Finally to my daughter Layla, words can't express my appreciation to you.

Jeddah, Saudi Arabia
January 2019

Dr. Salma Alrasheed

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1 Units and Vectors

1.1 Introduction

Physics is an exciting adventure that is concerned with unraveling the secrets of nature based on observations and measurements and also on intuition and imagination. Its beauty lies in having few fundamental principles being able to reach out to incorporate many phenomena from the atomic to the cosmic scale. It is a science that depends heavily on mathematics to prove and express theories and laws and is considered to be the most fundamental of physical sciences. Astronomy, geology, and chemistry all involve applications of physics' principles and concepts. Physics doesn't only provide theories, but it also provides techniques that are used in every area of life. Modern physical techniques were the major contributors to the wealth of mankind's knowledge in the past century.

A simple law in physics can be used to explain a wide range of complex phenomena that may appear to be not related. When studying a complex physical system, a simplified model of the system is usually used, where the minor effects are neglected and the main features of the system are concentrated upon. For example, when dealing with an object falling near the earth's surface, air resistance can be neglected. In addition, the earth is usually assumed to be spherical and homogeneous. However, in reality, the earth is an ellipsoid and is not homogeneous. The difference between the calculations of these different models can be assumed to be insignificant.

Physics can be divided into two branches namely: classical physics and modern physics. This book focuses on mechanics, which is a branch of classical physics. Other branches of classical physics are: light and optics, sound, electromagnetism, and thermodynamics. Mechanics is the science of motion of objects and is the core of classical physics. On the other hand, modern branches of physics include theories that have been developed during the past twentieth century. Two main theories are the theory of relativity and the theory of quantum mechanics. Modern physics explains many physical phenomena that cannot be explained by classical physics.

1.2 The SI Units

A physical quantity is a quantitative description of a physical phenomenon. For a precise description, one has to measure the physical quantity and represent this measurement by a number. Such a measurement is made by comparing the quantity with a standard; this standard is called a unit. For example, mass is a physical quantity that refers to the quantity of matter contained in an object. The unit kilogram is one of the units used to measure mass and is defined as the mass of a specific platinum–iridium alloy cylinder, kept at the International Bureau of Weights and Measures. Therefore, when we say that a block's mass is 300 kg, we mean that it is 300 times the mass of the cylindrical platinum–iridium alloy. All units chosen should obey certain properties such as being accurate, accessible, and should remain stable under varied environmental conditions or time.

In 1960, the International System of units (SI) (formally known as the Metric System MKS) was established. The abbreviation is derived from the French phrase "System International". As shown in Table 1.1, the SI system consists of seven base fundamental units, each representing a quantity assumed to be naturally independent. The system also includes two supplementary units, the radian which is a unit of the plane angle, and the steradian which is a unit of the solid angle. All other quantities in physics are derived from these base quantities. For example, mechanical quantities such as force, velocity, volume, and energy can be derived from the fundamental quantities length, mass, and time. Furthermore, the powers of ten are used to represent the larger and smaller values for a certain physical quantity as listed in Table 1.2. The most recent definitions of the units of length, mass, and time in the SI system are as follows:

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Table 1.1 The SI system consists of seven base fundamental units, each representing a quantity assumed to be naturally independent

Quantity	Unit name	Unit symbol
Length	Meter	m
Mass	Kilogram	kg
Time	Second	s
Temperature	Kelvin	K
Electric Current	Ampere	A
Luminous Intensity	Candela	cd
Amount of Substance	mole	mol

Table 1.2 Prefixes for Powers of Ten

Factor	Prefix	Symbol
10^{-24}	yocto	y
10^{-21}	zepto	z
10^{-18}	atto	a
10^{-15}	femto	f
10^{-12}	pico	p
10^{-9}	nano	n
10^{-6}	micro	μ
10^{-3}	milli	m
10^{-2}	centi	c
10^{-1}	deci	d
10^{1}	deka	da
10^{2}	hecto	h
10^{3}	kilo	k
10^{6}	mega	M
10^{9}	giga	G
10^{12}	tera	T
10^{15}	peta	P
10^{18}	exa	E
10^{21}	zetta	Z

- The Meter: The distance that light travels in vacuum during a time of 1/299792458 s.
- The Kilogram: The mass of a specific platinum–iridium alloy cylinder, which is kept at the International Bureau of Weights and Measures.
- The Second: 9192631770 periods of the radiation from cesium-133 atoms.

1.3 Conversion Factors

There are two other major systems of units besides the SI units. The (CGS) system of units which uses the centimeter, gram and second as its base units, and the (FPS) system of units which uses the foot, pound, and second as its base units.The conversion factors between the SI units and other systems of units of length, mass, and time are

- $1\ \text{m} = 39.37\ \text{in} = 3.281\ \text{ft} = 6.214 \times 10^{-4}\ \text{mi}$
- $1\ \text{kg} = 10^3\ \text{g} = 0.0685\ \text{slug} = 6.02 \times 10^{26}\ \text{u}$
- $1\ \text{s} = 1.667 \times 10^{-2}\ \text{min} = 2.778 \times 10^{-4}\ \text{h} = 3.169 \times 10^{-8}\ \text{yr}$

Example 1.1 If a tree is measured to be 10 m long, what is its length in inches and in feet?

Solution 1.1

$$10\,\text{m} = (10\,\text{m})\left(\frac{39.37\ \text{in}}{1\ \text{m}}\right) = 393.7\ \text{in}$$

$$10\,\text{m} = (10\,\text{m})\left(\frac{3.281\ \text{ft}}{1\ \text{m}}\right) = 32.81\ \text{ft}$$

Example 1.2 If a volume of a room is 32 m^3, what is the volume in cubic inches?

Solution 1.2

$$32\ \text{m}^3 = (32\,\text{m}^3)\left(\frac{39.37\ \text{in}}{1\ \text{m}}\right)^3 = 1.95 \times 10^6\ \text{in}^3$$

1.4 Dimension Analysis

The symbols used to specify the dimensions of length, mass, and time are L, M and T, respectively. Dimension analysis is a method used to check the validity of an equation and to derive correct expressions. Only the same dimensions can be added or subtracted, i.e., they obey the rules of algebra. To check the validity of an equation, the terms on both sides must have the same dimension. The dimension of a physical quantity is denoted using brackets [ ]. For example, the dimension of the volume is $[V] = \text{L}^3$, and that of acceleration is $[\text{a}] = \text{L}/\text{T}^3$.

Example 1.3 Show that the expression $\text{v}^2 = 2ax$ is dimensionally consistent, where v represents the speed, x represent the displacement, and a represents the acceleration of the object.

Solution 1.3

$$[\text{v}^2] = \text{L}^2/\text{T}^2$$

$$[xa] = (\text{L}/\text{T}^2)(\text{L}) = \text{L}^2/\text{T}^2$$

Each term in the equation has the same dimension and therefore it is dimensionally correct.

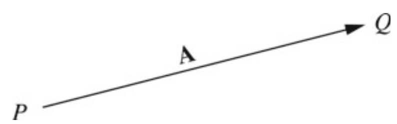


Fig. 1.1 A vector is represented geometrically by an arrow PQ 129 drawn to scale

1.5 Vectors

When exploring physical quantities in nature, it is found that some quantities can be completely described by giving a number along with its unit, such as the mass of an object or the time between two events. These quantities are called scalar quantities. It is also found that other quantities are fully described by giving a number along with its unit in addition to a specified direction, such as the force on an object. These quantities are called vector quantities.

Scalar quantities have magnitude but don't have a direction and obey the rules of ordinary arithmetic. Some examples are mass, volume, temperature, energy, pressure, and time intervals by a letter such as m, t, E..., etc. Vector quantities have both magnitude and direction and obey the rules of vector algebra. Examples are displacement, force, velocity, and acceleration. Analytically, a vector is specified by a bold face letter such as **A**. This notation (as used in this book) is usually used in printed material. In handwriting, the designation $\vec{A}$ is used. The magnitude of **A** is written as $|\mathbf{A}|$ or A in print or as $|\vec{A}|$ in handwriting.

A vector is represented geometrically by an arrow PQ drawn to scale as shown in Fig. 1.1. The length and direction of the arrow represent the magnitude and direction of the vector, respectively, and is independent of the choice of coordinate system. The point P is called the initial point (tail of **A**) and Q is called the terminal point (head of **A**).

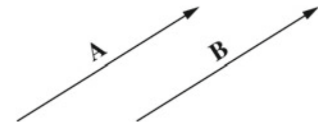
1.6 Vector Algebra

In this section, we will discuss how mathematical operations are applied to vectors.

1.6.1 Equality of Two Vectors

The two vectors **A** and **B** are said to be equal ($\mathbf{A} = \mathbf{B}$) only if they have the same magnitude and direction, whether or not their initial points are the same as shown in Fig. 1.2.

Fig. 1.2 The two vectors **A** and **B** are said to be equal (**A** = **B**) only if they have the same magnitude and direction



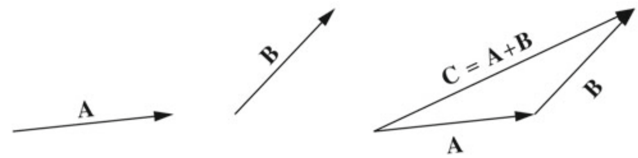


Fig. 1.3 To add two vectors **A** and **B** using the geometric method, place the head of **A** at the tail of **B** and draw a vector from the tail of **A** to the head of **B**

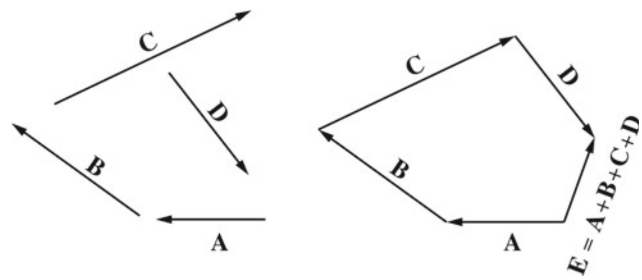


Fig. 1.4 Geometric method for summing more than two vectors

1.6.2 Addition

There are two ways to add vectors, geometrically and algebraically. Here, we will discuss the geometric method which is useful for solving problems without using a coordinate system. The algebraic method will be discussed later. To add two vectors **A** and **B** using the geometric method, place the head of **A** at the tail of **B** and draw a vector from the tail of **A** to the head of **B** as shown in Fig. 1.3. This method is known as the triangle method. An extension to sum up more than two vectors is shown in Fig. 1.4. An alternative procedure of vector addition using the geometric method is shown in Fig. 1.5. This is known as the parallelogram method, where **C** is the diagonal of a parallelogram with sides A and B. To find **C** analytically, Fig. 1.6 shows that

$$(DG)^2 = (DF)^2 + (FG)^2, \tag{1.1}$$

and that

$$DF = DE + EF = A + B\cos\theta,$$

Thus, Eq. 1.1 becomes

$$C^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2 = A^2 + B^2 + 2AB\cos\theta,$$

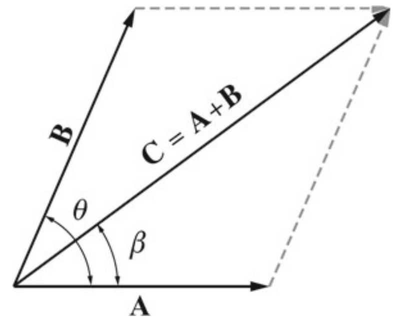


Fig. 1.5 The parallelogram method of adding two vectors

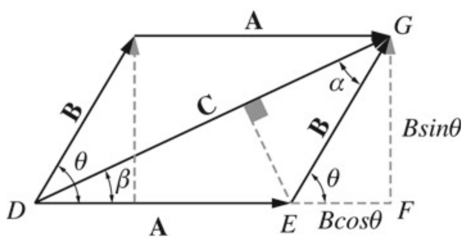


Fig. 1.6 Finding the magnitude and the direction of **C**

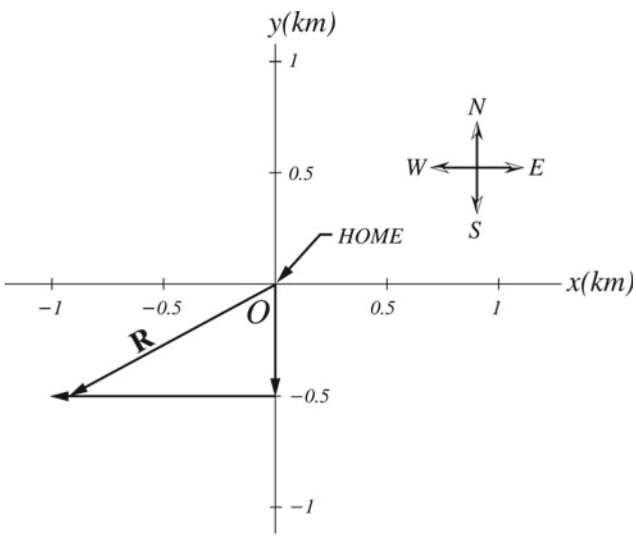


Fig. 1.7 The total displacement of the jogger is the vector **R**

or

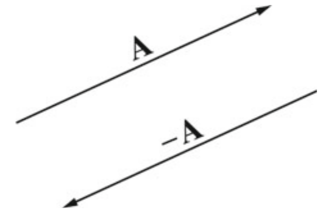
$$C = \sqrt{A^2 + B^2 + 2AB\cos\theta},$$

The direction of **C** is

$$\tan\beta = \frac{GF}{DF} = \frac{GF}{DE + EF} = \frac{B\sin\theta}{A + B\cos\theta},$$

Note that only when **A** and **B** are parallel, the magnitude of the resultant vector **C** is equal to $A + B$ (unlike the addition of scalar quantities, the magnitude of the resultant vector **C** is not necessarily equal to $A + B$).

Fig. 1.8 The negative vector of **A** is a vector of the same magnitude of **A** but in the opposite direction



Example 1.4 A jogger runs from her home a distance of 0.5 km due south and then 1 km to the west. Find the magnitude and direction of her resultant displacement.

Solution 1.4 From Fig. 1.7, we can see that the magnitude of the resultant displacement is given by

$$R = \sqrt{(0.5\ \text{km})^2 + (1\ \text{km})^2} = 1.1\ \text{m}$$

The direction of R is

$$\theta = \tan^{-1}\frac{(0.5\ \text{m})}{(1\ \text{m})} = 26.6^{\text{o}}$$

south of west.

1.6.3 Negative of a Vector

The negative vector of **A** is a vector of the same magnitude of **A** but in the opposite direction as shown in Fig. 1.8, and it is denoted by $-\mathbf{A}$.

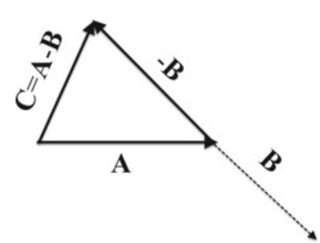
1.6.4 The Zero Vector

The zero vector is a vector of zero magnitude and has no defined direction. It may result from $\mathbf{A} = \mathbf{B} - \mathbf{B} = \mathbf{0}$ or from $\mathbf{A} = c\mathbf{B} = 0$ if $c = 0$.

1.6.5 Subtraction of Vectors

The vector $\mathbf{A} - \mathbf{B}$ is defined as the vector that when added to **B** gives us **A**. Equivalently, $\mathbf{A} - \mathbf{B}$ can be defined as the vector **A** added to vector $-\mathbf{B}$ ($\mathbf{A} + (-\mathbf{B})$) as shown in Fig. 1.9.

Fig. 1.9 Subtraction of two vectors



1.6.6 Multiplication of a Vector by a Scalar

The product of a vector $\mathbf{A}$ by a scalar q is a vector $q\mathbf{A}$ or $\mathbf{A}q$. Its magnitude is qA and its direction is the same as $\mathbf{A}$ if q is positive and opposite to $\mathbf{A}$ if q is negative, as shown in Fig. 1.10.

1.6.7 Some Properties

- $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$ (Commutative law of addition). This can be seen in Fig. 1.11.
- $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$, as seen from Fig. 1.12 (Associative law of addition).
- $\mathbf{A}+\mathbf{0}=\mathbf{A}$
- $\mathbf{A}+(-\mathbf{A})=\mathbf{0}$

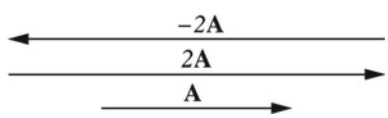


Fig. 1.10 The product of a vector by a scalar

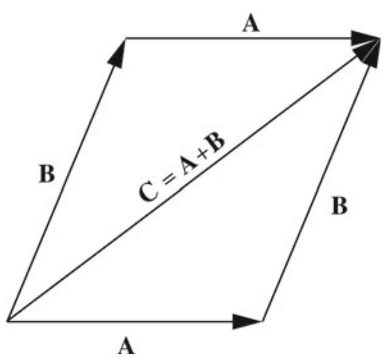


Fig. 1.11 Commutative law of addition

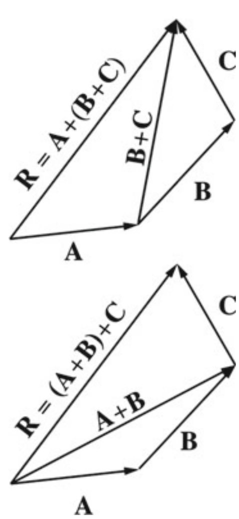


Fig. 1.12 Associative law of addition

- $p(q\mathbf{A})=(pq)\mathbf{A}=q(p\mathbf{A})$ (where p and q are scalars) (Associative law for multiplication).
- $(p+q)\mathbf{A}=p\mathbf{A}+q\mathbf{A}$ (Distributive law).
- $p(\mathbf{A}+\mathbf{B})=p\mathbf{A}+p\mathbf{B}$ (Distributive law).
- $1\mathbf{A}=\mathbf{A}$, $0\mathbf{A}=\mathbf{0}$ (Here, the zero vector has the same direction as $\mathbf{A}$, i.e., it can have any direction), $q\mathbf{0}=\mathbf{0}$

1.6.8 The Unit Vector

The unit vector is a vector of magnitude equal to 1, and with the same direction of $\mathbf{A}$. For every $\mathbf{A}\neq 0$, $\mathbf{a}=\mathbf{A}/|\mathbf{A}|$ is a unit vector.

1.6.9 The Scalar (Dot) Product

The scalar product is a scalar quantity defined as $\mathbf{A}\cdot\mathbf{B}=AB\cos\theta$, where θ is the smaller angle between $\mathbf{A}$ and $\mathbf{B}$ $(0\leq\theta\leq\pi)$ (see Fig. 1.13).

1.6.9.1 Some Properties of the Scalar Product

- $\mathbf{A}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{A}$ (Commutative law of scalar product).
- $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C})=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$ (Distributive law).
- $m(\mathbf{A}\cdot\mathbf{B})=(m\mathbf{A})\cdot\mathbf{B}=\mathbf{A}\cdot(m\mathbf{B})=(\mathbf{A}\cdot\mathbf{B})m$, where m is a scalar.

1.6.10 The Vector (Cross) Product

The vector product is a vector quantity defined as $\mathbf{C}=\mathbf{A}\times\mathbf{B}$ (read A cross B) with magnitude equal to $|\mathbf{A}\times\mathbf{B}|=AB\sin\theta$, $(0\leq\theta\leq\pi)$. The direction of $\mathbf{C}$ is found from the right-hand rule or of advance of a right-handed screw rotated from $\mathbf{A}$ to $\mathbf{B}$ as shown in Fig. 1.14. $\mathbf{C}$ is perpendicular to the plane formed by $\mathbf{A}$ and $\mathbf{B}$.

1.6.10.1 Some Properties

- $\mathbf{A}\cdot\mathbf{A}=A^2$, $\mathbf{0}\cdot\mathbf{A}=0$
- $\mathbf{A}\times\mathbf{B}=-\mathbf{B}\times\mathbf{A}$
- $\mathbf{A}\times(\mathbf{B}+\mathbf{C})=\mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$ (Distributive law).
- $(\mathbf{A}+\mathbf{B})\times\mathbf{C}=\mathbf{A}\times\mathbf{C}+\mathbf{B}\times\mathbf{C}$

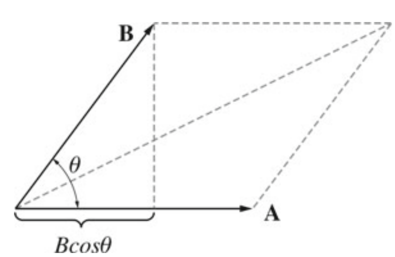


Fig. 1.13 The scalar product of two vectors

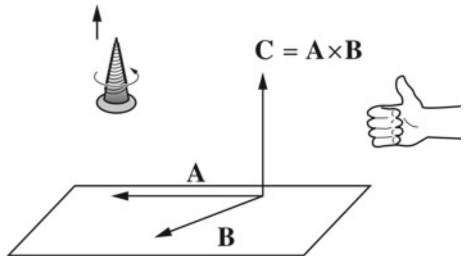


Fig. 1.14 The vector product of two vectors

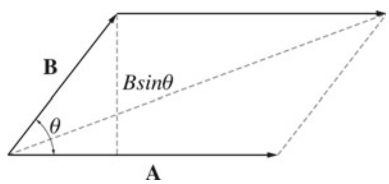


Fig. 1.15 The magnitude of the vector product $|\mathbf{A} \times \mathbf{B}| =$ is the area of a parallelogram with sides A and B

- $q(\mathbf{A} \times \mathbf{B}) = (q\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (q\mathbf{B}) = (\mathbf{A} \times \mathbf{B})q$, where q is a scalar.
- $|\mathbf{A} \times \mathbf{B}|$ = The area of a parallelogram that has sides A and B as shown in Fig. 1.15.

1.7 Coordinate Systems

To specify the location of a point in space, a coordinate system must be used. A coordinate system consists of a reference point called the origin O and a set of labeled axes. The positive direction of an axis is in the direction of increasing numbers, whereas the negative direction is opposite. Figures 1.16 and 1.17 show the rectangular (or Cartesian) coordinate system and the polar coordinates of a point, respectively The rectangular coordinates x and y are related to the polar coordinates r and θ by the following relations:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\tan\theta = y/x$$

$$r = \sqrt{x^2 + y^2}$$

In three dimensions, the cartesian coordinate system is shown in Fig. 1.18. Other used coordinate systems in three dimensions are the spherical and cylindrical coordinates (Figs. 1.19 and 1.20).

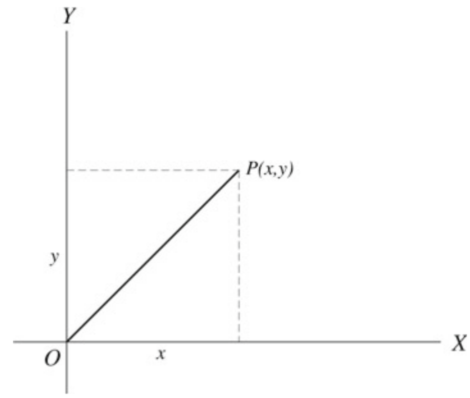


Fig. 1.16 The rectangular (cartesian) coordinate system

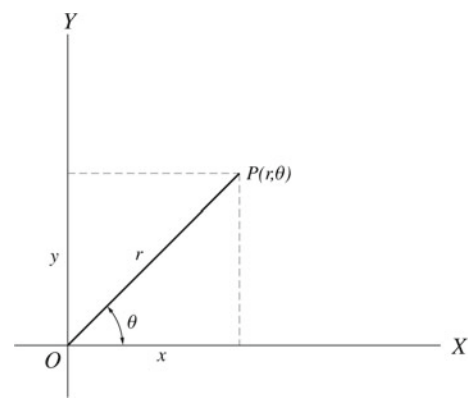


Fig. 1.17 The polar coordinate system

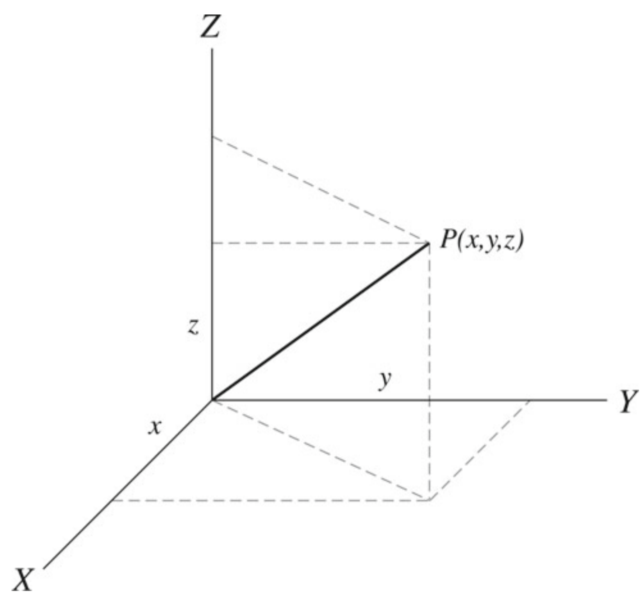


Fig. 1.18 The cartesian coordinate system in three dimensions

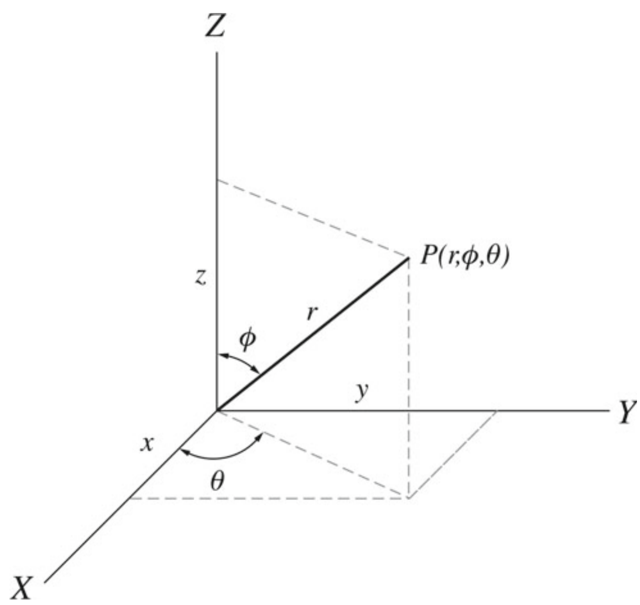


Fig. 1.19 The spherical coordinate system

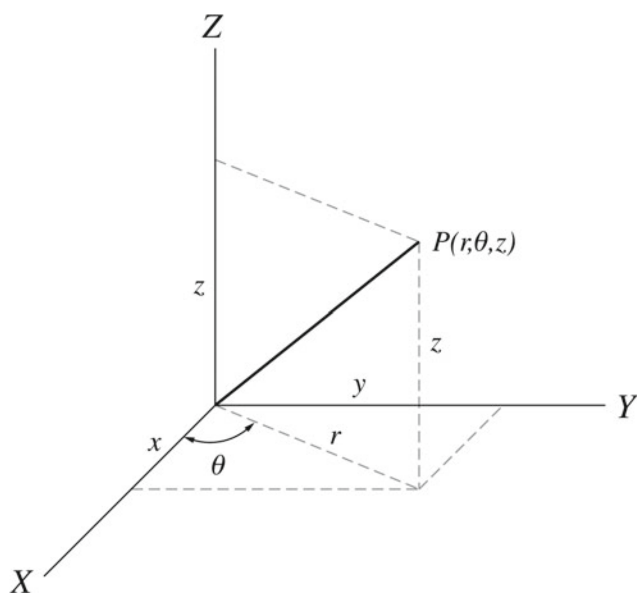


Fig. 1.20 The cylindrical coordinate system

1.8 Vectors in Terms of Components

In two dimensions, the vector **A** can be expressed as the sum of two other vectors $\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$, where $A_x = A\cos\theta$ and $A_y = A\sin\theta$ as shown in Fig. 1.21.

$\mathbf{A}_x$ and $\mathbf{A}_y$ are called the rectangular components, or simply components of **A** in the x and y directions respectively The magnitude and direction of **A** are related to its components through the expressions:

$$A = \sqrt{A_x^2 + A_y^2}$$

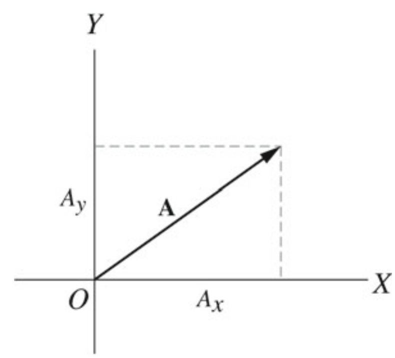


Fig. 1.21 In two dimensions, the vector **A** can be expressed as the sum of two other vectors $\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$, where $A_x = A\cos\theta$ and $A_y = A\sin\theta$

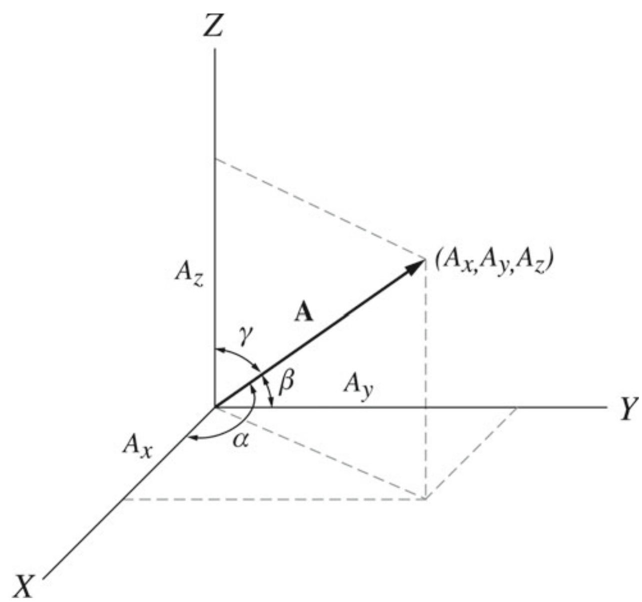


Fig. 1.22 In three dimensions the magnitude of A is $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

$$\tan\theta = A_y/A_x$$

In three dimensions (see Fig. 1.22), the magnitude of A is given by

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

with directions given by

$$\cos\alpha = A_x/A,\ \cos\beta = A_y/A,\ \cos\gamma = A_z/A$$

1.8.1 Rectangular Unit Vectors

The rectangular unit vectors **i**, **j**, and **k** are unit vectors defined to be in the direction of the positive x-, y-, and z-axes, respectively, of the rectangular coordinate system as shown in Fig. 1.23. Note that labeling the axes in this way forms a

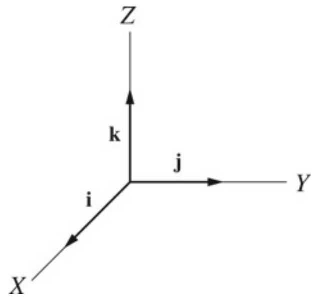


Fig. 1.23 The rectangular unit vectors **i**, **j** and **k** are unit vectors defined to be in the direction of the positive x, y, and z axes respectively

right-handed system. This name derives from the fact that a right- handed screw rotated through 90° from the x-axis into the y-axis will advance in the positive z-direction. (Note that throughout this book the right-handed coordinate system is used). In terms of unit vectors, vector A can be written as

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

1.8.2 Component Method

Suppose we have $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j}$ and $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j}$

1.8.2.1 Addition

The resultant vector **C** is given by

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} = C_x\mathbf{i} + C_y\mathbf{j}$$

$$C_x = A_x + B_x$$

$$C_y = A_y + B_y$$

Thus, the magnitude of **C** is

$$C = \sqrt{C_x^2 + C_y^2}$$

with a direction

$$\tan\theta = \frac{C_y}{C_x} = \frac{A_y + B_y}{A_x + B_x}$$

in three dimensions

$$\mathbf{C} = (A_x+B_x)\mathbf{i}+(A_y+B_y)\mathbf{j} = (A_z+B_z)\mathbf{k} = C_x\mathbf{i}+C_y\mathbf{j}+C_z\mathbf{k}$$

the magnitude of **C** is

$$C = \sqrt{C_x^2 + C_y^2 + C_z^2}$$

And the directions are

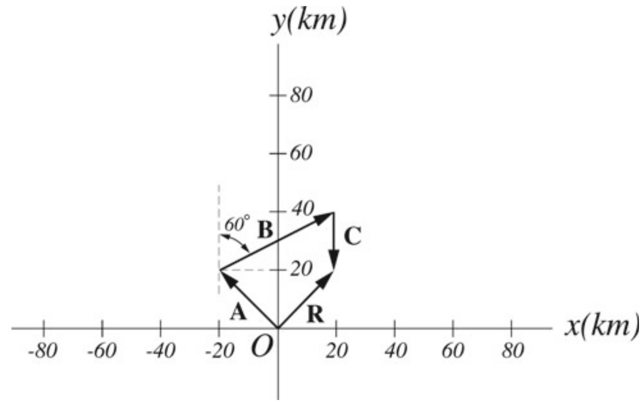


Fig. 1.24 The displacements are drawn to scale with the head of **A** placed at the tail of **B** and the head of **B** placed at the tail of **C**.The resultant vector **R** is the vector that extends from the tail of **A** to the head of **C**

$$\cos\alpha = C_x/C,\ \cos\beta = C_y/C,\ \cos\gamma = C_z/C$$

This component method is easy to use in adding any number of vectors.

Example 1.5 A truck travels northwest a distance of 30 km, and then 50 km at 30° north of east, and finally travels a distance of 20 km due south. Determine both graphically and analytically the magnitude and direction of the resultant displacement of the truck from its starting point.

Solution 1.5 Graphically, in Fig. 1.24 the displacements are drawn to scale with the head of **A** placed at the tail of **B** and the head of **B** placed at the tail of **C**.The resultant vector **R** is the vector that extends from the tail of **A** to the head of **C**. By using graph paper and a protractor, the magnitude of **R** is measured to have the value of 34.8 km and a direction of 49.8° from the positive x axis. Analytically, from Fig. 1.24, we have

$$A_x = A\cos 135^{\circ} = (30\text{ km})(-0.707) = -21.2\text{ km}$$

$$A_y = A\sin 135^{\circ} = (30\text{ km})(0.707) = 21.2\text{ km}$$

$$B_x = B\cos 30^{\circ} = (50\text{ km})(0.866) = 43.3\text{ km}$$

$$B_y = B\sin 30^{\circ} = (50\text{ km})(0.5) = 25\text{ km}$$

$$C_x = C\cos 270^{\circ} = (20\text{ km})(0) = 0$$

$$C_y = C\sin 270^{\circ} = (20\text{ km})(-1) = -20\text{ km}$$

$$\mathbf{R} = \mathbf{A}+\mathbf{B}+\mathbf{C} = (A_x+B_x+C_x)\mathbf{i}+(A_y+B_y+C_y)\mathbf{j}+(A_z+B_z+C_z)\mathbf{k} = 22.1\mathbf{i}+26.2\mathbf{j}$$

Thus, the magnitude of **R** is given by

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(221\ \text{km})^2 + (262\ \text{km}\ )^2} = 34.3\ \text{km}$$

and its direction is

$$\theta = \tan^{-1}\left(\frac{26.2\ \text{km}}{22.1\ \text{km}}\right) = 49.9^{\text{o}}$$

north of east.

1.8.2.2 Subtraction

$$\mathbf{C} = \mathbf{A} - \mathbf{B} = (A_x - B_x)\mathbf{i} + (A_y - B_y)\mathbf{j} + (A_z - B_z)\mathbf{k}$$

The magnitude and direction of **C** are as in the case of addition except that the plus sign is replaced by the minus sign.

1.8.2.3 Scalar Product

$$\mathbf{A} \cdot \mathbf{B} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \cdot (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$

Using the definition of scalar product and by applying the distributive law we get nine terms: since $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k}, \cdot$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0$, we get

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

The dot product of any vector (for example **A**) by itself is

$$\mathbf{A} \cdot \mathbf{A} = A^2 = A_x^2 + A_y^2 + A_z^2$$

1.8.2.4 The Angle Between Two Vectors

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta = A_x B_x + A_y B_y + A_z B_z$$

$$\cos\theta = \frac{A_x B_x + A_y B_y + A_z B_z}{AB}$$

Example 1.6 Two vectors **A** and B are given by $\mathbf{A} = \mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$ and $\mathbf{B} = 6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. Find the angle between them.

Solution 1.6

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\phi = A_x B_x + A_y B_y + A_z B_z$$

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{1 + 25 + 49} = 8.7$$

$$B = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{36 + 4 + 9} = 7$$

$$\cos\phi = \frac{A_x B_x + A_y B_y + A_z B_z}{AB} = \frac{6 - 10 - 21}{(8.7)(7)} = -0.4$$

$$\phi = 113.6^{\text{o}}$$

1.8.2.5 Perpendicular and Parallel Vectors

Nonzero vectors **A** and **B** are perpendicular if $\mathbf{A} \cdot \mathbf{B} = 0$ or $A_x B_x + A_y B_y + A_z B_z = 0$ and they are parallel if $\mathbf{A} \times \mathbf{B} = \mathbf{0}$. For any two parallel vectors **A** and **B**, we have $\mathbf{A} = q\mathbf{B}$, where they have the same direction if $q > 0$, and are in opposite direction if $q < 0$. Also we can write

$$\frac{\mathbf{A}}{\mathbf{B}} = q$$

or

$$\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z}$$

1.8.2.6 Vector Product

From the vector product definition, we can see that

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i},\ \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k},\ \mathbf{k} \times \mathbf{j} = -\mathbf{i},\ \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

If we write the unit vectors around a circle as shown in Fig. 1.25, then reading counterclockwise gives the positive products and reading clockwise gives the negative products. Note that these results are for a right-handed coordinate system. We have

$$\mathbf{A} \times \mathbf{B} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$

using the distributive law and the above relations of unit vectors we get

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} + (A_z B_x - A_x B_z)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

since a determinant of order 2 is defined as

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Then, the above expression can be written as

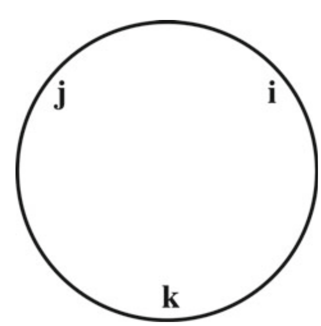


Fig. 1.25 If we write the unit vectors around a circle, then reading counter clockwise gives the positive products and reading clockwise gives the negative products

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix}\mathbf{i} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix}\mathbf{j} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}\mathbf{k}$$

A determinant of order 3 is

$$\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3$$

Hence, the cross product can be expressed as

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

Note that this is not a determinant since the elements in the first row are vectors and not scalars, but it is a convenient way to represent the cross product.

Example 1.7 Two vectors **A** and **B** are given by $\mathbf{A} = -\mathbf{i} + 3\mathbf{j}$ and $\mathbf{B} = 2\mathbf{i} + \mathbf{j}$. Find: (a) the sum of **A** and **B**, $\cdot$(**b**) $-$ **B** and 3**A**, $\cdot$(**c**)$\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{A}\times\mathbf{B}$.

Solution 1.7 (a)

$$\mathbf{R} = \mathbf{A}+\mathbf{B} = (A_x+B_x)\mathbf{i}+(A_y+B_y)\mathbf{j} = (-1+2)\mathbf{i}+(3+1)\mathbf{j} = \mathbf{i}+4\mathbf{j}$$

$$R_x = 1$$

$$R_y = 4$$

(b)

$$-\mathbf{B} = -2\mathbf{i} - \mathbf{j}$$

$$3\mathbf{A} = -3\mathbf{i} + 9\mathbf{j}$$

(c)
$\mathbf{A}\cdot\mathbf{B} = (-\mathbf{i}+3\mathbf{j})(2\mathbf{i}+\mathbf{j}) = -\mathbf{i}\cdot 2\mathbf{i} - \mathbf{i}\cdot\mathbf{j} + 3\mathbf{j}\cdot 2\mathbf{i} + 3\mathbf{j}\cdot\mathbf{j} = -2 + 3 = 1$

$$\mathbf{A}\times\mathbf{B} = (-\mathbf{i}+3\mathbf{j})\times(2\mathbf{i}+\mathbf{j}) = -\mathbf{i}\times\mathbf{j} + 3\mathbf{j}\times 2\mathbf{i} = -\mathbf{k} - 6\mathbf{k} = -7\mathbf{k}$$

Example 1.8 Find a vector of magnitude 1 that is perpendicular to each of the vectors $\mathbf{A} = 5\mathbf{i}+\mathbf{j}-3\mathbf{k}$ and $\mathbf{B} = 3\mathbf{i}+7\mathbf{j}-2\mathbf{k}$.

Solution 1.8 By the definition of the unit vector, we have

$$\mathbf{c} = \frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|}$$

where c is a unit vector perpendicular to the plane formed by A and B. We have

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 1 & -3 \\ 3 & 7 & -2 \end{vmatrix} = 19\mathbf{i} + \mathbf{j} + 32\mathbf{k}$$

$$|\mathbf{A}\times\mathbf{B}| = \sqrt{(19)^2 + (1)^2 + (32)^2} = 37.23$$

$$\mathbf{C} = \frac{19\mathbf{i} + \mathbf{j} + 32\mathbf{k}}{37.23} = 0.5\mathbf{i} + 0.027\mathbf{j} + 0.86\mathbf{k}$$

Example 1.9 Given that $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$, $\mathbf{B} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{C} = \mathbf{j} - 4\mathbf{k}$, find (a) $\mathbf{A}\times\mathbf{B}$ (b)$(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ (c) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$.

Solution 1.9 (a)

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & -1 \\ 3 & -1 & 0 \end{vmatrix} = -\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$$

(b)

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -3 & 7 \\ 0 & 1 & -4 \end{vmatrix} = 5\mathbf{i} - 4\mathbf{j} - \mathbf{k}$$

(c)

$$\mathbf{B}\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 0 \\ 0 & 1 & -4 \end{vmatrix} = 4\mathbf{i} + 12\mathbf{j} + 3\mathbf{k}$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (2\mathbf{i}-3\mathbf{j}-\mathbf{k})\cdot(4\mathbf{i}+12\mathbf{j}+3\mathbf{k}) = 8-36-3 = -31$$

Example 1.10 Using vectors method, find the area of a triangle if the coordinates of its three vertices are A(2, 1, 3), B(2, 5, 7), C(−1, 4, 2).

Solution 1.10

$$\mathbf{AB} = (2-2)\mathbf{i} + (5-1)\mathbf{j} + (7-3)\mathbf{k} = 4\mathbf{j} + 4\mathbf{k}$$

$$\mathbf{AC} = (-1-2)\mathbf{i} + (4-1)\mathbf{j} + (2-3)\mathbf{k} = -3\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

Area

$$= \frac{1}{2}|\mathbf{AB}\times\mathbf{AC}| = \frac{1}{2}|(4\mathbf{j}+4\mathbf{k})\times(-3\mathbf{i}+3\mathbf{j}-\mathbf{k})| = \frac{1}{2}|4(-4\mathbf{i}-3\mathbf{j}+3\mathbf{k})|$$

$$= 2\sqrt{(-4)^2 + (-3)^2 + (3)^2} = 11.7$$

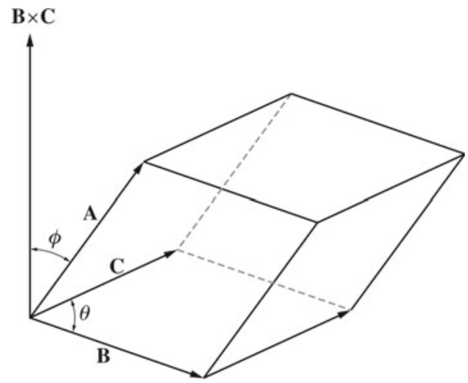


Fig. 1.26 The triple scalar product is equal to the volume of a parallepiped with sides **A**, **B**, and **C**

1.8.2.7 Triple Product

Scalar Triple Product

The triple scalar product is a scalar quantity defined as $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$. This quantity can be represented by a determinant that involves the components of the vectors,

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

where $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$, $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$, and $\mathbf{C} = C_x\mathbf{i}+C_y\mathbf{j}+C_z\mathbf{k}$. Furthermore, the triple scalar product is equal to the volume of a parallepiped with sides **A**, **B**, and **C** as shown in Fig. 1.26. Because any edges can be used, the triple scalar product can be written as $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ or as $\mathbf{A}\cdot(\mathbf{C}\times\mathbf{B})$. These products are positive and negative for a right-handed coordinate system respectively. Therefore, there are 6 equal triple scalar products or 12 if you include the terms of the form $(\mathbf{B}\times\mathbf{C})\cdot\mathbf{A}$. A. Three of these six products are positive and the rest are negative. By expanding the determinant, you can prove that

$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = -\mathbf{A}\cdot(\mathbf{C}\times\mathbf{B}) = -\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}) = -\mathbf{C}\cdot(\mathbf{B}\times\mathbf{A})$

Vector Triple Product

The triple vector product is a vector quantity defined as A × (B × C) . You can prove by expanding this equation that

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

Example 1.11 Given that $\mathbf{A} = A_x\mathbf{i}$, $\mathbf{B} = B_x\mathbf{i}+B_z\mathbf{k}$, and $\mathbf{C} = C_y\mathbf{j}$, show that the identity $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$ is correct.

Solution 1.11

$$(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_x & 0 & B_z \\ 0 & C_y & 0 \end{vmatrix} = -B_zC_y\mathbf{i} + B_xC_y\mathbf{k}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & 0 & 0 \\ -B_zC_y & 0 & B_xC_y \end{vmatrix} = -A_xB_xC_y\mathbf{j}$$

$$(\mathbf{A}\cdot\mathbf{C})\mathbf{B} = 0$$

$$-(\mathbf{A}\cdot\mathbf{B})\mathbf{C} = -(A_xB_x)\mathbf{C} = -A_xB_xC_y\mathbf{j}$$

Hence, the identity is valid.

1.9 Derivatives of Vectors

If **A**(t) is a vector function of t, where t is a scalar variable such as

$$\mathbf{A}(t) = A_x(t)\mathbf{i} + A_y(t)\mathbf{j} + A_z(t)\mathbf{k}$$

Then

$$\frac{d\mathbf{A}(t)}{dt} = \frac{dA_x(t)}{dt}\mathbf{i} + \frac{dA_y(t)}{dt}\mathbf{j} + \frac{dA_z(t)}{dt}\mathbf{k}$$

1.9.1 Some Rules

If **A**(t) and **B**(t) are vector functions and $\phi(t)$ is a scalar function then

$$\frac{d}{dt}(\phi\mathbf{A}) = \phi\frac{d\mathbf{A}}{dt} + \frac{d\phi}{dt}\mathbf{A}$$

$$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\cdot\mathbf{B}$$

$$\frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\times\mathbf{B}$$

Example 1.12 Two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are given by $\mathbf{r}_1 = 2t^2\mathbf{i} + \cos t\mathbf{j} + 4\mathbf{k}$ and $\mathbf{r}_2 = \sin t\mathbf{i} + \cos t\mathbf{k}$, find at $t = 0$ (a) $\frac{d^2\mathbf{r}_1}{dt^2}$ and (b) $\frac{d(\mathbf{r}_1\cdot\mathbf{r}_2)}{dt}$.

Solution 1.12 (a)

$$\frac{d\mathbf{r}_1}{dt} = 4t\mathbf{i} - \sin t\mathbf{j}$$

$$\frac{d^2\mathbf{r}_1}{dt^2} = 4\mathbf{i} - \cos t\mathbf{j}$$

At $t = 0$

$$\frac{d^2\mathbf{r}_1}{dt^2} = 4\mathbf{i} - \mathbf{j}$$

(b)

$$\frac{d(\mathbf{r}_1 \cdot \mathbf{r}_2)}{dt} = \frac{d\{(2t^2\mathbf{i} + \cos t\mathbf{j} + 4\mathbf{k})(\sin t\mathbf{i} + \cos t\mathbf{k})\}}{dt} =$$

$$\frac{d(2t^2 \sin t + 4\cos t)}{dt} = 4t\sin t + 2t^2\cos t - 4\sin t = 4(t-1)\sin t + 2t^2\cos t$$

At $t = 0$

$$\frac{d(\mathbf{r}_1 \cdot \mathbf{r}_2)}{dt} = 0.$$

1.9.2 Gradient, Divergence, and Curl

If $\mathbf{A} = \mathbf{A}(x,\ y,\ z)$ is a vector function of x, y, and z then $\mathbf{A}(x,\ y,\ z)$ is called a vector field. Similarly, the scalar function $\phi(x,\ y,\ z)$ is called a scalar field.

1.9.2.1 Del

The vector differential operator del is defined as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

1.9.2.2 Gradient

$$\nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$

The vector $\nabla\phi$ is called the gradient of ϕ (written gradϕ).

1.9.2.3 Divergence

$$\nabla \cdot \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$\nabla \cdot \mathbf{A}$ is called the divergence of A (written div$\mathbf{A}$).

1.9.2.4 Curl

$$\nabla \times \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k}$$

$\nabla \times \mathbf{A}$ is called the curl of $\mathbf{A}$ (written curl$\mathbf{A}$).

1.9.2.5 Some Identities

- divcurl$\mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$.
- curlgrad$\phi = \nabla \times (\nabla\phi) = \mathbf{0}$

.

Example 1.13 A vector field A and a scalar field B are given by $\mathbf{A} = 3xy\mathbf{i} + (2y^2 - x)\mathbf{j}$ and $B = 3x^2y$, Find at the point (−1,1)(a) $\nabla \cdot \mathbf{A}$ (b) $\nabla \times \mathbf{A}$ (c) ∇B.

Solution 1.13 (a)

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} = 3y + 4y = 7y$$

at $(-1, 1)$, $\nabla \cdot \mathbf{A} = 7$.

(b)

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xy & (2y^2 - x) & 0 \end{vmatrix} = (-3x - 1)\mathbf{k}$$

at $(-1, 1)$, $\nabla \times \mathbf{A} = 2\mathbf{k}$.

(c)

$$\nabla B = \frac{\partial B}{\partial x}\mathbf{i} + \frac{\partial B}{\partial y}\mathbf{j} + \frac{\partial B}{\partial z}\mathbf{k} = 6xy\mathbf{i} + 3x^2\mathbf{j}$$

at $(-1, 1)$, $\nabla B = -6\mathbf{i} + 3\mathbf{j}$.

1.10 Integrals of Vectors

If $\mathbf{A}(t) = A_x(t)\mathbf{i} + A_y(t)\mathbf{j} + A_z(t)\mathbf{k}$, where t is a scalar variable, the indefinite integral is defined as

$$\int \mathbf{A}(t)dt = \mathbf{i}\int A_x(t)dt + \mathbf{j}\int A_y(t)dt + \mathbf{k}\int A(t)dt$$

If $\mathbf{A}(t) = d\mathbf{B}(t)/dt$, then

$$\int \mathbf{A}(t)dt = \int \frac{d}{dt}\{\mathbf{B}(t)\}dt = \mathbf{B}(t) + \mathbf{C}$$

where $\mathbf{C}$ is an arbitrary constant vector. The definite integral between the limits $t = a$ and $t = b$ is defined as

$$\int_a^b \mathbf{A}(t)dt = \int_a^b \frac{d}{dt}\{\mathbf{B}(t)\}dt = \mathbf{B}(t) + \mathbf{C}|_a^b = \mathbf{B}(b) - \mathbf{B}(a)$$

1.10.1 Line Integrals

The line integral refers to an integral along a line or a curve. This curve may be open or closed. The line integral may

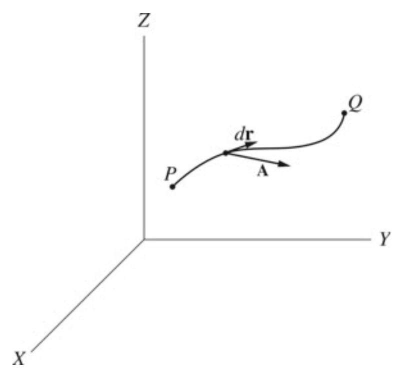


Fig. 1.27 The line integral

appear in three different forms shown by $\int_c \phi d\mathrm{r}$, $\int_c \mathbf{A}.\, \mathrm{dr}$, and $\int_c \mathbf{A} \times d\mathbf{r}$. The second is the most common one and it will be used throughout this book. Suppose the position vector of any point (x, y, z) on the curve C (see Fig. 1.27) that extends from $\mathbf{P}(x_1, y_1, z_1)$ at t_1 to $\mathbf{Q}(x_2, y_2, z_2)$ at t_2 is given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

where t is a scalar variable, and suppose that $\mathbf{A} = \mathbf{A}(x, y, z) = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ is a vector field, then the line integral of $\mathbf{A}$ is given by

$$\int_P^Q \mathbf{A} \cdot d\mathrm{r} = \int_C \mathbf{A} \cdot d\mathbf{r} = \int_C (A_x dx + A_y dy + Adz) \quad (1.2)$$

Note that $\mathbf{A} \cdot \mathbf{r}$ is the tangential component of $\mathbf{A}$ along C. If C is a simple closed curve (does not intersect with itself) then the line integral is written as

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \oint_C (A_x dx + A_y dy + Adz)$$

1.10.2 Independence of Path

The line integral in general depends on the path, but sometimes it does not. Instead, it depends only on the coordinates of the end points of the curve (path) but not on the curve itself. The line integral in Eq. 1.2 is independent of the path, joining the points P and Q if and only if $\mathbf{A} = \nabla\phi$, or equivalently $\nabla \times \mathbf{A} = \mathbf{0}$. The value of Eq. (1.2) is then given by

$$\int_P^Q \mathbf{A} \cdot d\mathbf{r} = \int_P^Q d\phi = \phi(P) - \phi(Q) = \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$$

Note that $\phi(x, y, z)$ has continuous partial derivatives. Furthermore, if the line integral of $\mathbf{A}$ is independent of the path then the line integral of $\mathbf{A}$ about any closed path is equal to zero:

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = 0$$

Example 1.14 A force field is given by $\mathbf{F} = (4xy^2 + z^2)\mathbf{i} + (4yx^2)\mathbf{j} + (2xz - 1)\mathbf{k}$

(a) Show that $\nabla \times \mathbf{F}$,

(b) Find a scalar function ϕ such that $\mathbf{F} = \nabla\phi$.

Solution 1.14 (a)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (4xy^2 + z^2) & (4yx^2) & (2xz - 1) \end{vmatrix} = (2z - 2z)\mathbf{j} + (8xy - 8xy)\mathbf{k} = \mathbf{0}$$

(b)

$$\mathbf{F} \cdot d\mathbf{r} = \nabla\phi \cdot d\mathbf{r} = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$$

$$d\phi = (4xy^2 + z^2)dx + (4yx^2)dy + (2xz - 1)dz$$

Hence

$$\phi = (2x^2y^2 + z^2x) + (2y^2x^2) + (z^2x - z)$$

Example 1.15 A vector $\mathbf{F}$ is given by $\mathbf{F} = 3x^2y\mathbf{i} - (4y + x)\mathbf{j}$. Compute $\int_c \mathbf{F} \cdot d\mathbf{r}$ along each of the following paths:

(a) The straight lines from (0, 0) to (0, 1) and then to (1, 1).

(b) Along the straight line $y = x$. (c) Along the curve $x = t, y = t^2$.

Solution 1.15 (a) Along the straight line from (0,0) to (0,1) we have $x = 0$, and $dx = 0$, therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C 3x^2ydx - (4y + x)dy = \int_{y=0}^1 -4ydy = -2y^2|_0^1 = -2$$

Along the straight line from (0, 1) to (1, 1) we have $y = 1, dy = 1$, hence

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{x=0}^1 3x^2dx = x^3|_0^1 = 1$$

Thus, we have for the total path

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -2 + 1 = -1$$

(b) Along the straight line $y = x$, we have $dy = dx$,

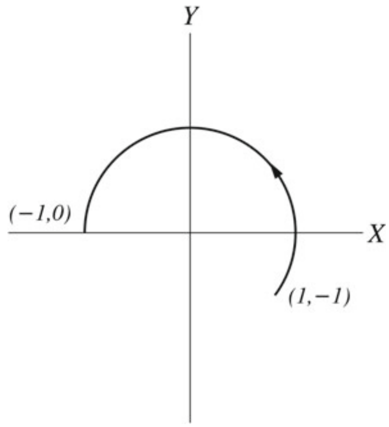


Fig. 1.28 The line integral along the curve using polar coordinates

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C 3x^2 y dx - (4y + x)dy = \int_{x=0}^{1} (3x^3 - 5x)dx$$

$$= 3/4x^4 - 5/2x^2|_0^1 = -3/2.$$

(c) Finally along the curve $x = t, y = t^2$, we have $dx = dt, dy = 2tdt$, furthermore the points (0, 0) and (1, 1) corresponds to $t = 0$ and $t = 1$, respectively. Hence

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C 3x^2 y dx - (4y+x)dy = \int_{t=0}^{1} 3t^4 dt - 2t(4t^2+t)dt$$

$$= 3/5t^5 - 2t^4 - 2/3t^3|_0^1 = -31/15.$$

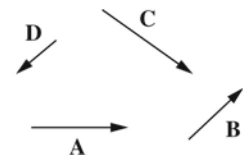
Example 1.16 If a vector **A** is given by $\mathbf{A} = xy\mathrm{i} - x^2\mathrm{j}$, find the line integral $\int_C \mathbf{A}\cdot d\mathbf{r}$ along the circular arc shown in Fig. 1.28.

Solution 1.16 By using the polar coordinates, we have $x = \cos\theta$ and $y = \sin\theta$ (since $r = 1$) , $dx = -\sin\theta d\theta$ and $dy = \cos\theta d\theta$, also $x^2 + y^2 = r^2 = 1$, therefore we have

$$\int_c \mathbf{A}\cdot d\mathbf{r} = \int_{\theta=\pi}^{-\pi/4} -\cos\theta\sin^2\theta d\theta - \cos^3 d\theta = \int_{\theta=\pi}^{-\pi/4} -\cos\theta(\sin^2\theta + \cos^2\theta)d\theta$$

$$= \int_{\theta=\pi}^{-\pi/4} -\cos\theta d\theta = -\sin\theta|_{\pi}^{-\pi/4} = 0.71$$

Fig. 1.29 Vectors **A**, **B**, **C** and **D**



Problems

1. Check if the relation $v = \sqrt{2GM_E/R_E}$ is dimensionally correct, where v represents the escape speed of a body, M_E and R_E are the mass and radius of the earth, respectively, and G is the universal gravitational constant.
2. If the speed of a car is 180 km/h, find its speed in m/s.
3. How many micrometers are there in an area of 3 km^2.
4. Figure 1.29 shows vectors **A**, **B**, **C**, and D. Find graphically the following vectors (a) $\mathbf{A} + 2\mathbf{B} - \mathbf{C}$(**b**)$2(\mathbf{A} - \mathbf{B}) + \mathbf{C} - 2\mathbf{D}$(**c**) show that $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.
5. A car travels a distance of 1 km due east and then a distance of 0.5 km north of east. Find the magnitude and direction of the resultant displacement of the car using the algebraic method.
6. Prove that $\mathbf{A}\cdot(\mathbf{B} + \mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$.
7. A parallelogram has sides **A** and **B**. Prove that its area is equal to $|\mathbf{A}\times\mathbf{B}|$.
8. If $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, find (a) $\mathbf{A} - 2\mathbf{B}$(b)$\mathbf{A}\times\mathbf{B}$ (c)$\mathbf{A}\cdot\mathbf{B}$ (d) the length of **A** and the length of **B**(e) the angle between **A** and **B**(f) the scalar projection of **A** on **B** and the scalar projection of **B** on **A**.
9. Show that **A** is perpendicular to **B** if $|\mathbf{A} + \mathbf{B}| = |\mathbf{A} - \mathbf{B}|$.
10. Given that $\mathbf{A} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathrm{B} = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ and $\mathbf{C} = 6\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$, determine which vectors are perpendicular and which are parallel.
11. Use the vectors $\mathbf{A} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ and $\mathbf{B} = \cos\phi\mathbf{i} - \sin\phi\mathbf{j}$ to prove that $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$.
12. If $\mathbf{A} = 5x^2y\mathbf{i} + yz\mathbf{j} - 3x^2z^2\mathbf{k}$, $\mathbf{B} = 7y^3z\mathbf{i} - 2zx\mathbf{j} + xz^2y\mathbf{k}$ and $\phi(x,\ y,\ z) = 2z^2y$, find at (−1,1,1)(a)$\partial(\phi\mathbf{A})/\partial x(b)\partial^2(\mathbf{A}\times\mathbf{B})/\partial z\partial y(c)\nabla\phi$(d)$\nabla\times(\phi\mathbf{A})$.
13. Evaluate $\nabla\times(r^2\mathbf{r})$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$ and $r = |\mathbf{r}|$.
14. If $\mathbf{r} = A\cos\omega t\mathbf{i} + A\sin\omega t\mathbf{j}$, show that $d^2\mathbf{r}/dt^2 + \omega^2\mathbf{r} = 0$.
15. A force field is given by $\mathbf{F} = -kx\mathbf{i} - ky\mathbf{j}$, find (a) $\nabla\times\mathbf{F}$ (b) a scalar field ϕ such that $\mathbf{F} = \nabla\phi$(c) Calculate the line integral along the straight lines from (0, 0) to (1, 0) to (1, 1) and from (0, 0) to (0, 1) to (1, 1). Is the line integral independent of path?

2 Kinematics

2.1 Introduction

Mechanics is the science that studies the motion of objects and can be divided into the following:

1. Kinematics: Describes how objects move in terms of space and time.
2. Dynamics: Describes the cause of the object's motion.
3. Statics: Deals with the conditions under which an object subjected to various forces is in equilibrium.

This chapter is considered with kinematics which answers many questions such as: How long it takes for an apple to reach the ground when it falls from a tree? What is the maximum height reached by a baseball when thrown into air? What is the distance it takes an airplane to take off?

In physics, there are three types of motion: translational, rotational, and vibrational. A block sliding on a surface is in translational motion, a (Merry-go-Round) is an example of rotational motion, and a mass–spring system when stretched and released is in vibrational motion. From here until Chap. 7, the object studied will be treated as a particle (i.e., a point mass with no size). This assumption is possible only if the object moves in translational motion without rotating and by neglecting any internal motions that might exist in the object.

That is, an object can be treated as a particle only if all of its parts move in exactly the same way.

For example, if a man jumps into a pool without rotating by doing a somersault (freezing his body), he can be treated as a particle since all particles in his body will move in exactly the same way. Another example of an object that can be treated as a particle is the Earth in its motion about the Sun. Since the dimensions of the Earth are small compared to the dimensions of its path, it can be considered as a particle. The motion of an object is described either by equations or by graphs. Both ways provide information about the motion; however, equations provide precise information while graphs give greater insight about the motion.

2.2 Displacement, Velocity, and Acceleration

This section will discuss the concepts of displacement, velocity, and acceleration in one dimension. These concepts are essential in analyzing the motion of an object.

2.2.1 Displacement

Consider a car that is treated as a particle moving along the straight-line path shown in Fig. 2.1. The x-axis of a coordinate system is used to describe the position of the car with respect to the origin O, where the points P and Q correspond to the positions x_i at t_i and x_f at t_f, respectively. The position–time graph of this motion is shown in Fig. 2.2. The displacement of the truck is a vector quantity defined as the change in its position during the time interval from t_i to t_f and is given by

$$\triangle x = x_f - x_i$$

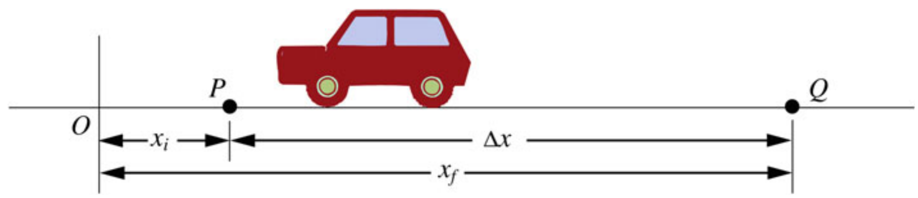
Hence displacement is a quantity that depends only on the initial and final positions of the object. The direction of the displacement in one dimension is specified by a plus or minus sign. It is positive if the particle is moving in the positive x direction and negative if the particle is moving in the negative x direction. In two or three dimensions, the displacement is represented by a vector. The SI unit of the displacement is the meter (m).

2.2.2 Average Speed

The average speed of an object is a scalar quantity defined as the total distance traveled divided by the total time:

S. Alrasheed, *Principles of Mechanics*, Advances in Science, Technology & Innovation, https://doi.org/10.1007/978-3-030-15195-9_2

Fig. 2.1 A car that is treated as a particle moving along the straight-line path



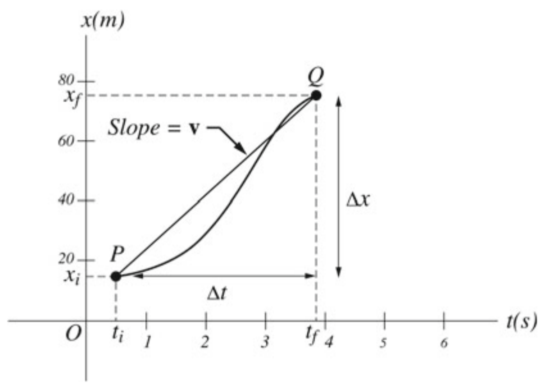


Fig. 2.2 The position time graph of the carõs motion

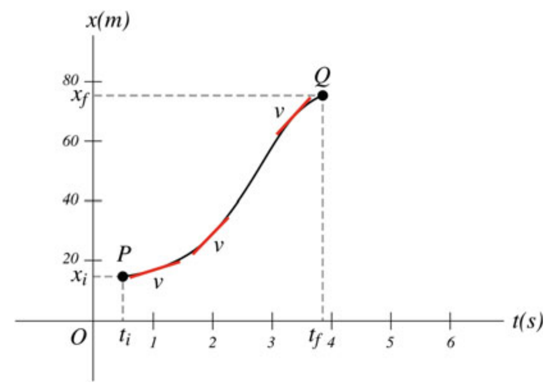


Fig. 2.3 Geometrically, the instantaneous velocity of a particle at a particular time on the position-time curve is the slope (the tangent) to the position-time curve at that point or instance

$$\text{Average speed} = \frac{\text{Total distance traveled}}{\text{Total time}}$$

The SI unit of the average speed is meter per second (m/s)
.

2.2.3 Velocity

The average velocity $\overline{v}$ of an object is a vector quantity defined in terms of displacement rather than the total distance traveled:

$$\overline{v} = \frac{\triangle x}{\triangle t}$$

$\overline{v}$ is positive if the motion is in the positive x-direction and negative if it is in the negative x-direction. On the position–time graph in Fig. 2.2, $\overline{v}$ is the slope of the straight line connecting the points P and Q. The average velocity helps in describing the overall motion of the particle in a certain time interval. To describe the motion in more detail, the instantaneous velocity is defined. This velocity corresponds to the velocity of a particle at a particular time. That involves allowing $\triangle t$ to approach zero:

$$v = \lim_{\triangle t \to \infty} \frac{\triangle x}{\triangle t} = \frac{dx}{dt}$$

Geometrically, the instantaneous velocity of a particle at a particular time on the position–time curve is the slope (the tangent) to the position–time curve at that point or instance (see Fig. 2.3). The SI unit of the velocity is m/s.

2.2.4 Speed

The speed of the particle is defined as the magnitude of its velocity. Note that speed and average speed are different since speed is defined in terms of displacement, whereas average speed is defined in terms of the total distance traveled.

2.2.5 Acceleration

If the particle's velocity changes with time, it is said to be accelerating. The average acceleration $\overline{a}$ of the particle is defined as the ratio of the change of its velocity $\triangle v$ to the time interval $\triangle t$:

$$\overline{a} = \frac{\triangle v}{\triangle t}$$

The SI unit of acceleration is m/s^2. The instantaneous acceleration is defined as

$$a = \lim_{\triangle t \to 0} \frac{\triangle v}{\triangle t} = \frac{dv}{dt}$$

The average acceleration is the slope of the line joining the points P and Q on the velocity–time graph, whereas the instantaneous acceleration is the slope of the curve at a particular point (see Fig. 2.4). Figure 2.5 shows the position, velocity, and acceleration for a particle simultaneously.

Example 2.1 A car travels along the path shown in Fig. 2.6, where it is located at $x_i = 3$ km at $t_i = 0$, and at $x_f = 19$ km

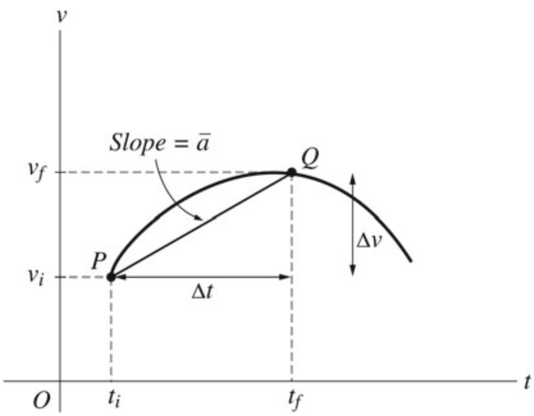


Fig. 2.4 The average acceleration is the slope of the line joining the points P and Q on the velocity-time graph, whereas the instantaneous acceleration is the slope of the curve at a particular point

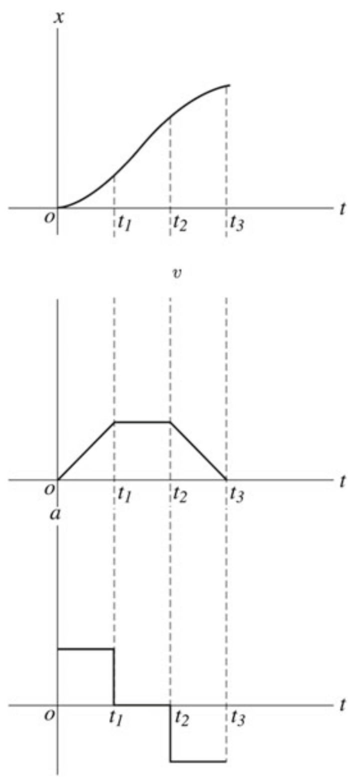


Fig. 2.5 This figure shows the position, velocity and acceleration as a function of time of a particle moving in one direction. The particle starts from rest, accelerates to a certain speed, is maintained at that speed for some time, then it decelerates back to rest

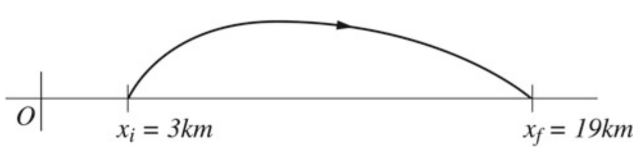


Fig. 2.6 A car moving along the curved path where it is located at $x_i = 3$ km at $t_i = 0$, and at $x_f = 19$ km at $t_f = 0.25$ hr

at $t_f = 0.25$ h. Find the displacement, average velocity, and average speed of the car during this time interval if the total distance traveled is 20 km.

Solution 2.1 The displacement of the car is

$$\triangle x = x_f - x_i = (19 \text{ km}) - (3 \text{ km}) = 16 \text{ km}$$

Its average velocity is

$$\bar{v} = \frac{\triangle x}{\triangle t} = \frac{x_f - x_i}{t_f - t_i} = \frac{(16 \text{ km})}{(0.25 \text{ h})} = 64 \text{ m/s}$$

$$\text{Average speed} = \frac{\text{Total distance traveled}}{\text{Total time}}$$

$$= \frac{(2.0 \text{ km})}{(025 \text{ h})} = 80 \text{ km/h}$$

Example 2.2 A particle moves along the x-axis according to the expression $x = 2t^2$.The plot of this equation is shown in Fig. 2.7. Find : (a) the displacement and average velocity of the particle during the time interval between $t = 1$ s and $t = 3$ s, ·(b) the instantaneous velocity of the particle as a function of time and at $t = 1$ s and $t = 3$ s.

Solution 2.2 (a)

$$x_i = 2t_i^2 = 2(1)^2 = 2 \text{ m}$$

$$x_f = 2t_f^2 = 2(3)^2 = 18 \text{ m}$$

The displacement of the particle is

$$\triangle x = x_f - x_i = (18 \text{ m}) - (2 \text{ m}) = 16 \text{ m}$$

The average velocity is

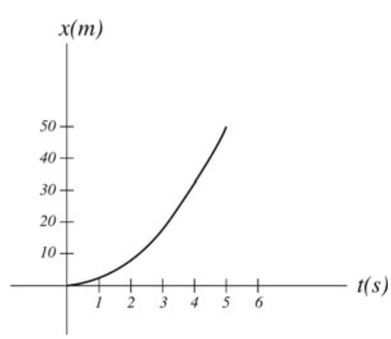


Fig. 2.7 A particle moves along the x-axis according to the expression $x = 2t^2$

$$\bar{v} = \frac{\triangle x}{\Delta t} = \frac{(16\,\text{m})}{(2\,\text{s})} = 8 \text{ m/s}$$

(b) The instantaneous velocity is given by

$$v = \frac{dx}{dt} = (4t) \text{ m/s}$$

at $t = 1$ s, $v = 2$ m/s, and at $t = 3$ s, $v = 12$ m/s.

Example 2.3 A particle is moving along the x-axis. The position–time graph of its motion is shown in Fig. 2.8. Find: (a) the average velocity between a and b, ·(b) the instantaneous velocity at the points a, c and d.

Solution 2.3 (a)

$$\bar{v}_{ab} = \frac{\triangle x}{\Delta t} = \frac{(2 \text{ m}) - (-1.8 \text{ m})}{(3 \text{ s}) - (1 \text{ s})} = 1.9 \text{ m/s}$$

(b)

$$v_a = \frac{\triangle x}{\triangle t} = \frac{0 - (-2.5 \text{ m})}{(3 \text{ s}) - 0} = 0.83 \text{ m/s}$$

$$v_c = 0$$

$$v_d = \frac{\triangle x}{\triangle t} = \frac{0 - (3 \text{ m})}{(8.5 \text{ s}) - (4 \text{ s})} = -0.67 \text{ m/s}$$

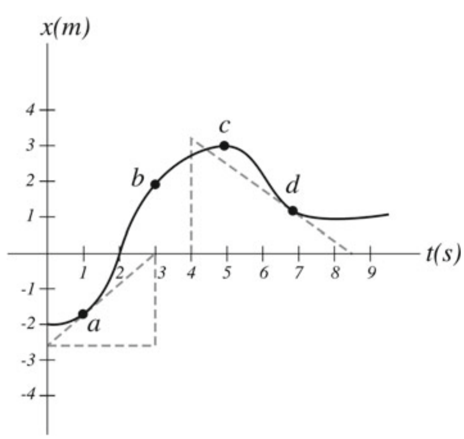


Fig. 2.8 The position-time graph of a particle moving along the x-axis

Example 2.4 The acceleration of an object is given by $a = (1-4t)$ m/s^2. If the object has an initial velocity of 3 m/s and an initial displacement of 2 m, determine (a) its velocity and displacement at any time; (b) the displacement of the object when it reaches its maximum speed.

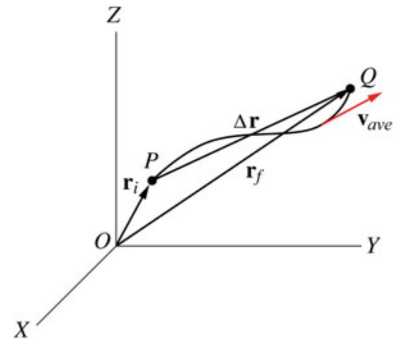


Fig. 2.9 A particle moving from point P to point Q along a path or curve C during a time interval $\triangle t = t_f - t_I$

Solution 2.4 (a)

$$v = \int a dt = \int (1 - 4t)dt = t - 2t^2 + c_1$$

At $t = 0$, $v = 3$ m/s and therefore $c_1 = 3$ m/s. Thus

$$v = (t - 2t^2 + 3) \text{ m/s}$$

$$x = \int v dt = \int (t - 2t^2 + 3)dt = 0.5t^2 - 0.66t^3 + 3t + c_2$$

At $t = 0$, $x = 2$ m and $c_2 = 2$ m. Therefore

$$x = (0.5t^2 - 0.66t^3 + 3t + 2) \text{ m} \tag{2.1}$$

(b) When the object reaches its maximum speed $\frac{dv}{dt} = 0$ and hence $1 - 4t = 0$, that gives $t = 0.25$ s. Substituting into Eq. 2.1 gives

$$x = 1/2(0.25 \text{ s})^2 - 2/3(0.25 \text{ s})^3 + 3(0.25 \text{ s}) + 2 = 2.8 \text{ m}$$

2.3 Motion in Three Dimensions

Consider the particle moving from point P to point Q along a path or curve C during a time interval $\triangle t = t_f - t_i$ as shown in Fig. 2.9. To locate the particle at any point the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is used. $\mathbf{r}_i$ and $\mathbf{r}_f$ corresponds to the position vectors of the particle at t_i and t_f respectively. A position vector should be drawn from a reference point (usually the origin of the coordinate system).

The displacement vector is then given by

$$\triangle \mathbf{r} = \mathbf{r}_f - \mathbf{r}_i$$

The average velocity is

$$\overline{\mathbf{v}} = \frac{\triangle \mathbf{r}}{\triangle t} = \frac{\mathbf{r}_f - \mathbf{r}_i}{t_f - t_i}$$

The instantaneous velocity at a particular time is defined as

$$\mathbf{v} = \lim_{\triangle t \to 0} \frac{\triangle \mathbf{r}}{\triangle t} = \frac{d\mathbf{r}}{dt}$$

As $\triangle t$ approaches zero, $\triangle \mathbf{r}$ becomes tangent to the path and it is replaced by $d\mathbf{r}$. The direction of y is in the direction of dr, hence, y is always tangent to the path at any point. In terms of components y is given by

$$\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

The magnitude of the instantaneous velocity is

$$|\mathbf{v}| = |\frac{d\mathbf{r}}{dt}| = v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \frac{ds}{dt}$$

where ds is the infinitesimal arc length along the path and comes from the fact that as $\triangle t$ approaches zero, the distance traveled by the particle along the path becomes equal to the vector displacement $|\triangle \mathbf{r}|$. Figure 2.10 shows the instantaneous velocities along the path. The average acceleration is

$$\overline{\mathbf{a}} = \frac{\triangle \mathbf{v}}{\triangle t} = \frac{\mathbf{v}_f - \mathbf{v}_i}{t_f - t_i}$$

The direction of $\overline{\mathbf{a}}$ is of the same direction as $\triangle \mathbf{v}$. The instantaneous acceleration is then

$$\mathbf{a} = \lim_{\triangle t \to 0} \frac{\triangle \mathbf{v}}{\triangle t} = \frac{d\mathbf{v}}{dt}$$

In terms of components

$$\mathbf{a} = \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

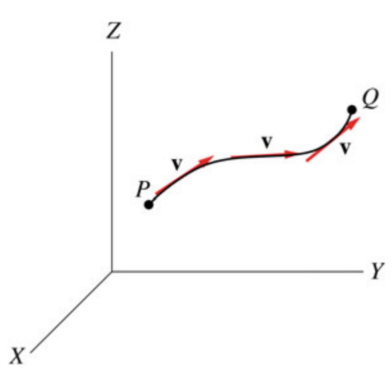


Fig. 2.10 The instantaneous velocity vectors along the path

Another way to describe motion in three dimensions is by using spherical or cylindrical coordinates. In this book, we will only use rectangular coordinates for three-dimensional motion.

2.3.1 Normal and Tangential Components of Acceleration

The acceleration describes the change in both the magnitude and direction of the velocity. That is, the acceleration is not necessarily produced due to the change in the magnitude of the velocity only. Sometimes, it is produced due to the change in the direction of the velocity even if its magnitude is unchanged, and sometimes due to the change in both the magnitude and direction. Furthermore, the direction of **a** is not necessarily in the direction of **v**. If **v** is changed in magnitude only (motion along a straight line) then **a** is parallel to **v** if **v** is increasing, and antiparallel if **v** is decreasing. If **v** is changed in direction only (motion along a curved path with constant speed), then **a** is always perpendicular to **v** at any point (see Fig. 2.11). Finally, if **v** is changed in both magnitude and direction then **a** will be directed at some angle to **v** as in Fig. 2.12.

In this case, the acceleration can be resolved into parallel and perpendicular components. The parallel component corresponds to the change in the magnitude of **v**, while the perpendicular component corresponds to the change in the direction of **v**. These components can be viewed to be directed along a rectangular coordinate system that moves with the particle (as it moves in space), where the particle is located at the origin of this coordinate system. The parallel (or tangential) component of the acceleration is always tangent to the path while the perpendicular (or normal) component is normal to the path at each point as shown in Fig. 2.13.

Figure 2.14 shows the direction of the acceleration of a car moving down a ramp under the influence of gravity.

In terms of unit vectors, let **T** be the unit vector along the tangent axis, **N** is the unit vector along the normal axis (also called the principal unit normal vector) and **B** a third unit vector called the binormal vector defined by $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. These unit vectors form a frame called the TNB frame, where it moves with the particle (see Fig. 2.15). Since **v** is always tangent to the path we may write

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{d\mathbf{r}/dt}{|d\mathbf{r}/_{dt}|} = \frac{d\mathbf{r}/dt}{ds/dt}$$

Because **T** is a unit vector we have $\mathbf{T}\cdot\mathbf{T} = 1$, differentiating this with respect to s gives

$$\mathbf{T} \cdot \frac{d\mathbf{T}}{ds} + \frac{d\mathbf{T}}{ds} \cdot \mathbf{T} = 2\mathbf{T} \cdot \frac{d\mathbf{T}}{ds} = 0$$

or

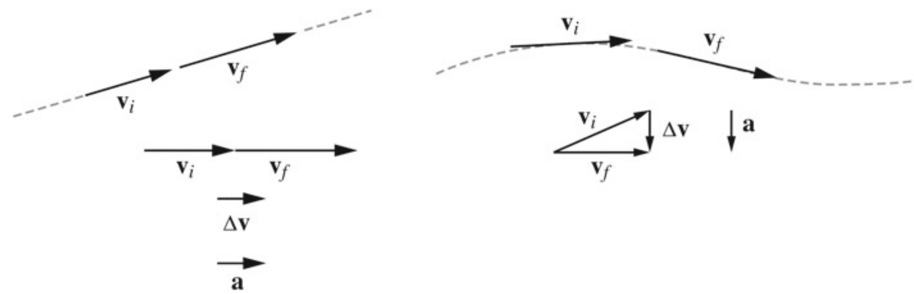


Fig. 2.11 If **v** is changed in magnitude only (motion along a straight line) then **a** is parallel to **v** if **v** is increasing, and antiparallel if **v** is decreasing. If **v** is changed in direction only (motion along a curved path with constant speed) then **a** is always perpendicular to **v** at any point

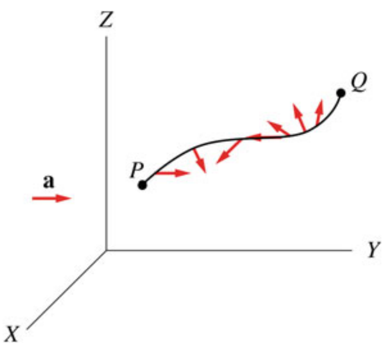


Fig. 2.12 If **v** is changed in both magnitude and direction then **a** will be directed at some angle to **v**

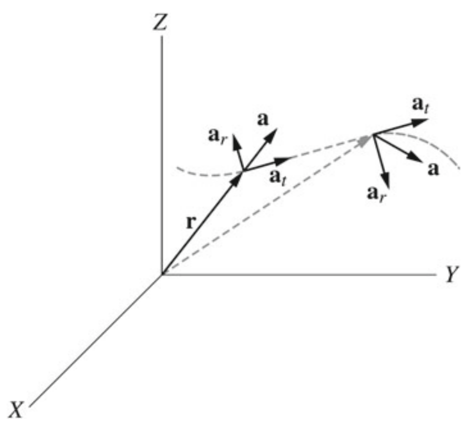


Fig. 2.13 The parallel (or tangential) component of the acceleration is always tangent to the path while the perpendicular (or normal) component is normal to the path at each point

$$\mathbf{T}\cdot\frac{d\mathbf{T}}{ds}=0$$

Hence, **T** is perpendicular to $d\mathbf{T}/ds$. Since **N** is also perpendicular to **T**, then we have

$$\mathbf{N}=\frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|}=\frac{1}{k}\frac{d\mathbf{T}}{ds}$$

k is called the curvature of C at a certain point and it has the value $k=|d\mathbf{T}/ds|$. The quantity $R=1/k$ is the radius of curvature at that point. Thus, $\mathbf{N}=R(d\mathbf{T}/ds)$. The total

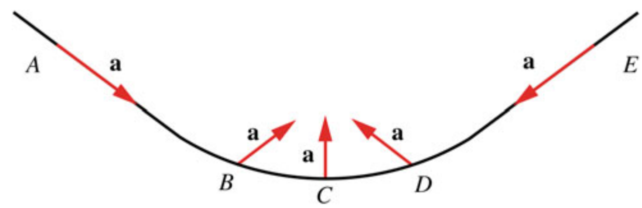


Fig. 2.14 At A the acceleration of a car is in the same direction of the velocity since the latter changes only in magnitude. As it moves its velocity is changed in both magnitude and direction. Therefore at B the direction of the acceleration is at some angle to the velocity. At C the speed reaches a maximum and therefore the instantaneous change of speed is zero at this point and the acceleration has only a perpendicular component. As the car moves up its velocity decreases and changes in direction also, thus the acceleration has both parallel and perpendicular components. Finally at E, the acceleration is in the opposite direction of the velocity since the velocity is decreasing but its direction is the same

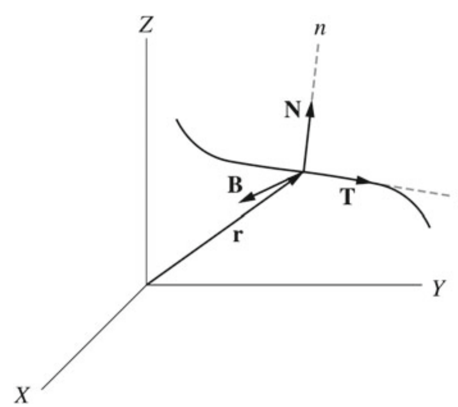


Fig. 2.15 The TNB frame moves with the particle

acceleration of the particle in terms of the unit tangent **T** vector and the principal unit normal vector **N** can be written as

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d}{dt}(v\mathbf{T})=\frac{dv}{dt}\mathbf{T}+v\frac{d\mathbf{T}}{dt} \tag{2.2}$$

Furthermore,

$$\frac{d\mathbf{T}}{dt}=\frac{d\mathbf{T}}{ds}\frac{ds}{dt}=\frac{\mathbf{N}}{R}\frac{ds}{dt}=\frac{v\mathbf{N}}{R} \tag{2.3}$$

Substituting Eq. 2.2 into Eq. 2.3 gives

$$\mathbf{a} = \frac{dv}{dt}\mathbf{T} + \frac{v^2}{R}\mathbf{N}$$

Therefore, $a_n = v^2/R$ and $a_t = dv/dt$. Note that unlike $d|\mathbf{v}|/dt$, $|d\mathbf{v}/dt|$ corresponds to the change in the magnitude of the velocity or in its direction or in both (as it represents the magnitude of the total acceleration vector), whereas $d|\mathbf{v}|/dt$ corresponds to the change in the magnitude only.

Example 2.5 A particle is moving in space according to the expression

$$\mathbf{r} = (5\cos t\mathbf{i} + 5\sin t\mathbf{j} + 7t\mathbf{k})\ \text{m}$$

Find the radius of curvature at any point on the space curve.

Solution 2.5

$$\frac{d\mathbf{r}}{dt} = (-5\sin t\mathbf{i} + 5\cos t\mathbf{j} + 7\mathbf{k})\ \text{m/s}$$

$$\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{(-5\sin t)^2 + (5\cos t)^2 + (7)^2} = 10\ \text{m/s}$$

Hence

$$\mathbf{T} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{(-5\sin t\mathbf{i} + 5\cos t\mathbf{j} + 7\mathbf{k})}{10} = -0.5\sin t\mathbf{i} + 0.5\cos t\mathbf{j} + 0.7\mathbf{k}$$

The radius of curvature is

$$R = \frac{1}{k} = \frac{1}{|{}^{d\mathbf{T}}/_{ds}|}$$

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\frac{dt}{ds} = \frac{d\mathbf{T}/dt}{ds/dt} = \frac{-0.5\cos t\mathbf{i} - 0.5\sin t\mathbf{j}}{10} = -0.05\cos t\mathbf{i} - 0.05\sin t\mathbf{j}$$

$$\left|\frac{d\mathbf{T}}{ds}\right| = \sqrt{(-005\cos t)^2 + (-005\sin t)^2} = 0.07$$

$$R = \frac{1}{0.07} = 14.3\ \text{m}$$

Example 2.6 A car moves with constant tangential acceleration down a ramp as shown in Fig. 2.16. If it starts from rest at A and reaches B after 4 s with a speed of 10 m/s, find the radius of curvature at B if the total acceleration of the car at that point is 3.2 m/s^2.

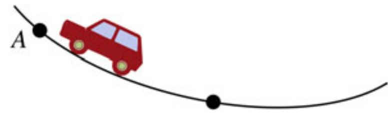


Fig. 2.16 A car moving with a constant tangential acceleration down a ramp

Solution 2.6 Since the tangential acceleration of the car is constant, it can be found from

$$a_t = \frac{v_B - v_A}{t} = \frac{(10\ \text{m/s}) - 0}{4\ \text{s}} = 2.5\ \text{m/s}^2$$

Since the total acceleration of the car at B is 2 m/s^2 then the normal acceleration is

$$a_n^2 = a^2 - a_t^2 = (3.2\ \text{m/s}^2)^2 - (2.5\ \text{m/s}^2)^2 = 4\ (\text{m/s}^2)^2$$

$$a_n = 2\ \text{m/s}^2$$

The radius of curvature is

$$R = \frac{v^2}{a_n} = \frac{(10\ \text{m/s})^2}{(2\ \text{m/s}^2)} = 50\ \text{m}$$

2.4 Some Applications

2.4.1 One-Dimensional Motion with Constant Acceleration

An acceleration that does not change with time is said to be a constant or uniform acceleration. In that case, the average and instantaneous accelerations are equal. This type of motion is more easily analyzed than when the acceleration is varied. Since the motion is in one dimension, it follows that the y and z components are zero. That is,

$$\mathbf{r} = x\mathbf{i}$$

$$\triangle\mathbf{r} = (x_f - x_i)\mathbf{i}$$

Hence, as we've mentioned earlier, the direction of the displacement can be specified with a plus or minus sign, as well as the directions of the velocity and acceleration. Let us assume that $t_i = 0, t_f = t, v_{xf} = v, v_{xi} = v_0, x_i = x_0$ and $x_f = x$. Since the acceleration is constant, the velocity will vary linearly with time, and thus the average velocity can be expressed as

$$\overline{v} = \frac{v_0 + v}{2}$$

$$a = - = \frac{v_f - v_i}{t_f - t_i} = \frac{v - v_0}{t}$$

$$v = v_0 + at \tag{2.4}$$

$$v = \frac{\Delta x}{\Delta t} = \frac{(v + v_0)}{2}$$

$$x - x_0 = \frac{1}{2}(v + v_0)t \tag{2.5}$$

Furthermore,

$$x - x_0 = \frac{1}{2}(v + v_0)t = \frac{1}{2}(v_0 + v_0 + at)t$$

$$x - x_0 = v_0 t + \frac{1}{2}at^2 \tag{2.6}$$

Finally,

$$x - x_0 = \frac{1}{2}(v + v_0)t = \frac{1}{2}(v + v_0)\left(\frac{v - v_0}{a}\right)$$

$$v^2 = v_0^2 + 2a(x - x_0) \tag{2.7}$$

Equations 2.4, 2.5, 2.6, and 2.7 are called the kinematic equations for motion in a straight line under constant acceleration. The motion graphs for an object moving with constant acceleration in the positive x-direction are shown in Fig. 2.17.

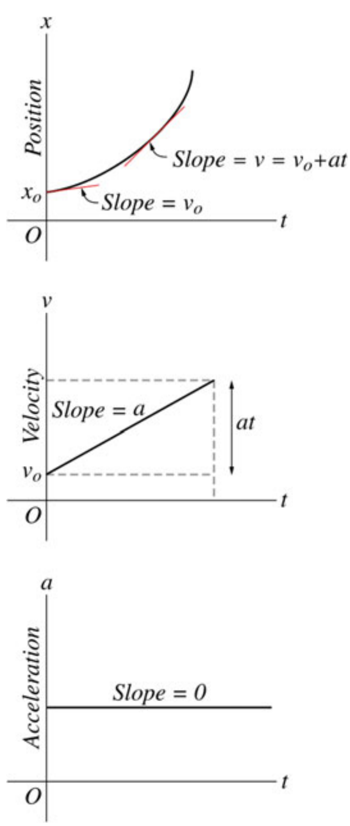


Fig. 2.17 The motion graphs for an object moving with constant acceleration in the positive x-direction

Example 2.7 A train accelerates uniformly from rest and travels a distance of 200 m in the first 8 s. Determine: (a) the acceleration of the train; (b) the time it takes the train to reach a velocity of 70 m/s,(c) the distance traveled during that time; (d) the velocity of the train 5 s later from the time calculated in (b).

Solution 2.7 (a)

$$x - x_0 = v_0 t - \frac{1}{2}at^2$$

Since $v_0 = 0$, we have

$$a = \frac{2(x - x_0)}{t^2} = \frac{2(200\ \text{m})}{(8\ \text{s})^2} = 6.25\ \text{m/s}^2$$

(b)

$$v = v_0 + at$$

$v_0 = 0$ and therefore

$$t = \frac{v}{a} = \frac{(70\ \text{m/s})}{(6.25\ \text{m/s}^2)} = 11.2\ \text{s}$$

(c)

$$x - x_0 = \frac{1}{2}at^2 = \frac{1}{2}(6.25)(11.2)^2 = 392\ \text{m}$$

(d)

$$v = v_0 + at = (70\ \text{m/s}) + (6.25\ \text{m/s}^2)(5\ \text{s}) = 101.25\ \text{m/s}$$

Example 2.8 An airplane accelerates uniformly from rest at a rate of 3 m/s^2 before taking off. If it is to take off at a speed of 100 m/s : (a) how much time is required for it to take off; (b) what distance will it have traveled before taking off?

Solution 2.8 (a)

$$v = v_0 + at$$

We have $v_0 = 0$, this gives

$$t = \frac{v}{a} = \frac{(100\ \text{m/s})}{(3\ \text{m/s}^2)} = 33.3\ \text{s}$$

(b)

$$x = \frac{1}{2}at^2 = \frac{1}{2}(3\ \text{m/s}^2)(33.3\ \text{s})^2 = 1.7 \times 10^3\ \text{m}$$

Example 2.9 A car moving at a constant velocity of 140 km/h passed a police car moving at a constant velocity of 80 km/h.5 s after the car had passed the police car, the police vehicle begins to accelerate toward the car at a constant rate of $1.4 \times 10^4\ \text{km/h}^2$ (a) How much time will it take the police car to catch the other car? (b) What is the distance traveled by both during that time? (c) How much time has passed from where the car passed the police car to where it was caught?

Solution 2.9 Let's assume that $x = 0$ at where the car passed the police car and that $t = 0$ at the instant the police car begins to accelerate. The velocity of the car is equal to 38.9 m/s, and the initial velocity and acceleration of the police car are 22.2 m/s and 1.1 m/s^2, respectively The police will catch the car when both their displacements from $x = 0$ are equal. (a) From the expression $x = x_0 + v_0 t + \frac{1}{2}at^2$, the displacement of the car at any time is

$$x_c = x_{0c} + v_{0c}t = (194.5\ \text{m}) + (38.9\ \text{m/s})t$$

The displacement of the police car at any time is

$$x_p = x_{0p} + v_{0p}t + \frac{1}{2}a_p t^2 = (111\ \text{m}) + (22.2\ \text{m/s})t + \frac{1}{2}(1.1\ \text{m/s}^2)t^2$$

The police will catch the car when $x_c = x_p$, and therefore if $(194.5\ \text{m}) + (38.9\ \text{m/s})t = (111\ \text{m}) + (22.2\ \text{m/s})t + \frac{1}{2}(1.1\ \text{m/s}^2)t^2$ or

$$t^2 - 30.4t - 151.8 = 0$$

Thus

$$t = \frac{(30.4) \pm \sqrt{(304)^2 + (4)(1518)}}{2}$$

That gives $t = 34.8$ s.

(b)

$x_p = x_c = (111\ \text{m}) + (22.2\ \text{m/s})(34.8\ \text{s}) + \frac{1}{2}(1.1\ \text{m/s}^2)(34.8\ \text{s})^2 = 1.55 \times 10^3$ m

(c)

$$t = (5\ \text{s}) + (34.8\ \text{s}) = 39.8\ \text{s}$$

2.4.2 Free-Falling Objects

Galileo Galilei (1564–1642) was an Italian scientist, who studied and experimented the acceleration of falling objects. By dropping various objects from the Leaning Tower of Pisa (or by releasing objects from inclined planes according to another story), Galileo discovered that when air resistance is neglected then all objects would fall with the same constant acceleration regardless of their mass or size. This acceleration, denoted by g, is known as the free-fall acceleration since air resistance is neglected and the object is assumed to be moving freely under gravity alone. The direction of the vector **g** is downwards toward the earth's center. However, g varies with altitude as well as other factors which will be discussed in Chap. 9.

In solving problems involving objects falling near the surface of the earth, g can be assumed to be constant with a value of 9.8 m/s^2 and air resistance can be neglected. A free-falling motion is a motion along a straight line (for example along the y-axis) where objects may move upwards or downwards. The kinematics equations of the free-falling motion with constant acceleration can be found from Eqs. (2.4), (2.5), (2.6), and (2.7) by simply replacing x with y and a with g. If the positive direction of y is chosen to be upwards, then the acceleration is negative (downwards) and is given by $(a = -g)$. These substitutions give

$$v = v_0 - gt$$

$$y - y_0 = \frac{1}{2}(v + v_0)t$$

$$y - y_0 = v_0 t - \frac{1}{2}gt^2$$

$$v^2 = v_0^2 - 2g(y - y_0)$$

The displacement and velocity graphs are shown in Fig. 2.18. Note that it does not matter whether the object is falling or moving upward, it will experience the same acceleration g which is directed downwards. Figure 2.19 shows the important features of a free-falling object that is dropped from rest.

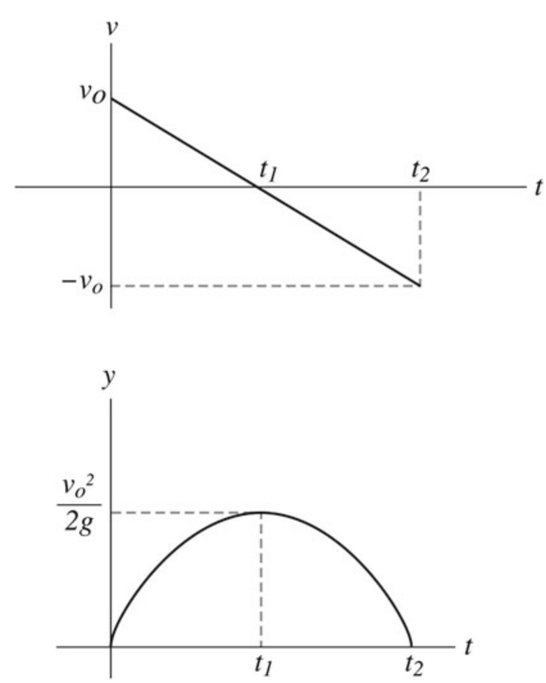


Fig. 2.18 The displacement and velocity graph for a free-falling object

t(s)	y(m)	v(m/s)	a(m/s²)	The velocity vector	The acceleration vector
0	0	0	−9.8		↓
1	−4.9	−9.8	−9.8	↓	↓
2	−19.6	−19.6	−9.8	↓	↓
3	−44.1	−29.4	−9.8	↓	↓

Fig. 2.19 The important features of a free falling object that is dropped from rest

Example 2.10 A ball is thrown directly upwards with an initial velocity of 15 m/s. On its way down, it was caught at a distance of lm below the point from where it was thrown. Determine (a) the maximum height reached by the ball; (b) the time it takes the ball to reach that height; (c) the velocity of the ball when it is caught; (d) the total time elapsed from where the ball was thrown to where it was caught.

Solution 2.10 (a) First we take $y = 0$ at the position where the ball is thrown and positive y to be upwards. At the maximum height the velocity of the ball is zero,

$$v^2 = v_0^2 - 2g(y - y_0)$$

$$0 = (15\ \text{m/s})^2 - 2(9.8\ \text{m/s}^2)h_{\text{max}}$$

$$h_{\text{max}} = 11.5\ \text{m}$$

(b) Using the expression $v = v_0 - gt$ we have

$$0 = (15\ \text{m/s}) - (9.8\ \text{m/s}^2)t$$

$$t = 1.5\ \text{s}$$

(c) When the ball is caught its position is $y = -1$ m,

$$v^2 = v_0^2 - 2g(y - y_0)$$

taking the initial position of the ball at $y = 0$, we get

$$v^2 = (15\ \text{m/s})^2 - 2(9.8\ \text{m/s}^2)((-1\ \text{m}) - 0)$$

and

$$v = -15.6\ \text{m/s}$$

or if we take the initial position at $y = 11.5$ m we have

$$v^2 = 0 - 2(9.8\ \text{m/s}^2)((-1\ \text{m}) - (11.5\ \text{m}))$$

and

$$v = -15.6\ \text{m/s}.$$

(d) $v = v_0 - gt$, substituting for v and v_0 we have

$$(-15.6\ \text{m/s}) = (15\ \text{m/s}) - (9.8\ \text{m/s}^2)t$$

$$t = 3.1\ \text{s}$$

Example 2.11 A tennis ball is dropped from a building that is 30 m high. Find (a) its position and velocity 2 s later; (b) the total time it takes the ball to fall to the ground; (c) its velocity just before it hits the ground.

Solution 2.11 (a) Taking $y_0 = 0$ and $v_0 = 0$ at $t = 0$ we have

$$y - y_0 = v_0 t - \frac{1}{2}gt^2$$

at $t = 2$ s

$$y - 0 = 0 - \frac{1}{2}(9.8\ \text{m/s}^2)(2\text{s})^2 = -19.6\ \text{m}$$

$$v = v_0 - gt = 0 - (9.8\ \text{m/s}^2)(2\ \text{s}) = -19.6\ \text{m/s}$$

(b)

$$y - y_0 = v_0 t - \frac{1}{2}gt^2$$

$$(-30\ \text{m}) - 0 = 0 - \frac{1}{2}(9.8\ \text{m/s}^2)t^2$$

$$t = 2.5\ \text{s}$$

(c)

$$v = v_0 - gt = 0 - (9.8\ \text{m/s}^2)(2.5\ \text{s})$$

$$v = -24.5\ \text{m/s}$$

Example 2.12 A ball is thrown vertically downwards from a 100 m high building with an initial speed of 1 m/s.3 s later a second ball is thrown. What initial speed must the second ball have so that the two balls hit the ground at the same time?

Solution 2.12 The time it takes the first ball to hit the ground is found from

$$y - y_0 = v_0 t - \frac{1}{2} g t^2$$

$$0 - (100\text{ m}) = (-1\text{ m/s})t_1 - \frac{1}{2}(9.8\text{ m/s}^2)t_1^2$$

$$t_1 = 6.4\text{ s}$$

The second ball must fall the same distance during a time of

$$t_1 - (3\text{ s}) = (6.4\text{ s}) - (3\text{ s}) = 3.4\text{ s}$$

and therefore

$$y - y_0 = v_0 t - \frac{1}{2} g t^2$$

$$0 - (100\text{m}) = v_0(3.4\text{ s}) - \frac{1}{2}(9.8\text{ m/s}^2)(3.4\text{ s})^2$$

$$v_0 = -12.6\text{ m/s}$$

2.4.3 Motion in Two Dimensions with Constant Acceleration

The position vector can be written as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$$

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j}$$

Because a is a constant both a_x and a_y are constants. Therefore, the kinematic in Sect. 2.4.1 applies in each direction:

$$v_x = v_{0x} + a_x t \tag{2.8}$$

$$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2 \tag{2.9}$$

$$v_y = v_{0y} + a_y t \tag{2.10}$$

$$y = y_0 + v_{0y} t + \frac{1}{2} a y t^2 \tag{2.11}$$

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = (x_0 + v_{0x} t + \frac{1}{2} a_x t^2)\mathbf{i} + (y_0 + v_{0y} t + \frac{1}{2} a_y t^2)\mathbf{j}$$

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a} t^2 \tag{2.12}$$

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} = (v_{0x} + a_x t)\mathbf{i} + (v_{0y} + a_y t)\mathbf{j}$$

$$= (v_{0x}\mathbf{i} + v_{0y}\mathbf{j}) + (a_x\mathbf{i} + a_y\mathbf{j})t$$

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a} t \tag{2.13}$$

Example 2.13 If the motion of a particle in a plane is described by $v_y = (-8t)$ m/s and $x = (5 - 2t^2)$ m : (a) plot the y component of the particle as a function of time if at $t = 0$, $y = 0$, ·(b) find the total speed and magnitude of the acceleration of the particle at $t = 2$ s.

Solution 2.13 (a)The y-component of position is

$$y = \int v_y dt = \int (-8t) dt = -4t^2 + c$$

since at $t = 0$, $y = 0$, then

$$y = (-4t^2)\text{ m}$$

The plot of y against t is shown in Fig. 2.20.

(b) The x-components of velocity and acceleration is

$$v_x = \frac{dx}{dt} = \frac{d(5 - 2t^2)}{dt}$$

$$v_x = (-4t)\text{ m/s}$$

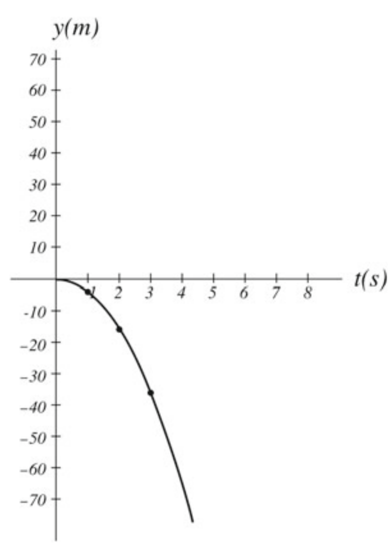


Fig. 2.20 The y component of the particle as a function of time

$$a_x = \frac{dv_x}{dt} = \frac{d(-4t)}{dt}$$

$$a_x = -4 \text{ m/s}^2$$

The y-component of acceleration is

$$a_y = \frac{dv_y}{dt} = \frac{d(-8t)}{dt}$$

or

$$a_y = (-8) \text{ m/s}^2$$

at $t = 2$ s, $v_x = -8$ m/s, $v_y = -16$ m/s and the velocity is

$$v = \sqrt{v_x + v_y} = \sqrt{(-8 \text{ m/s})^2 + (-16 \text{ m/s})^2} = 17.9 \text{ m/s}$$

$$a_x = -4 \text{ m/s}^2$$

and

$$a_y = (-8) \text{ m/s}^2$$

Therefore, the acceleration of the particle is constant at any time and is given by

$$a = \sqrt{a_x + a_y} = \sqrt{(-4 \text{ m/s}^2)^2 + (-8 \text{ m/s}^2)^2} = 8.9 \text{ m/s}^2$$

2.4.4 Projectile Motion

Projectile motion is the motion of an object thrown (projected) into the air at some angle with respect to the surface of the earth, such as the motion of a baseball thrown into the air or an object dropped from a moving airplane. In the simplified model where air resistance as well as other factors such as the Earth's curvature and rotation are neglected, and if the free-fall acceleration **g** is assumed constant in magnitude and direction throughout the motion of the object, then the path of the projectile is always a parabola that depends on the magnitude and direction of its initial velocity. Therefore, the projectile can be considered as a combination of a vertical motion with a constant acceleration directed downwards and a horizontal motion with zero acceleration (constant velocity). We can see from Fig. 2.21 that

$$\cos\theta_0 = v_{0x}/v_o$$

$$\sin\theta_0 = v_{0y}/v_o$$

At $t = 0$, we have $x_0 = y_0 = 0$ and $v_i = v_0$. Because $a_y = -g$ and $a_x = 0$ and by substituting in Eqs. 2.8, 2.9, 2.10, and 2.11 gives

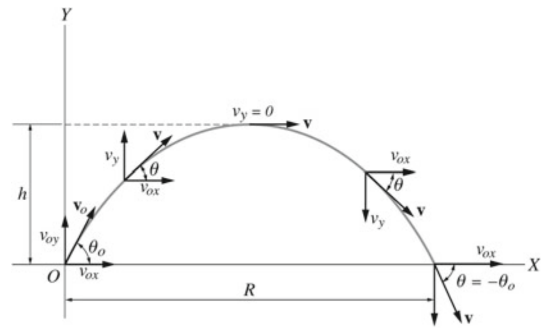


Fig. 2.21 The projectile motion

$$v_x = v_{0x} = v_0 \cos\theta_0 = \text{constant} \tag{2.14}$$

$$v_y = v_{y0} - gt = v_0 \sin\theta_0 - gt \tag{2.15}$$

$$x = v_{x0}t = (v_0 \cos\theta_0)t \tag{2.16}$$

$$y = v_{y0}t - \frac{1}{2}gt^2 = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2 \tag{2.17}$$

Combining and eliminating t from Eqs. 2.16 and 2.17 we find that

$$y = (\tan\theta_0)x - \left(\frac{g}{2v_0^2\cos^2\theta_0}\right)x^2$$

$$(0 < \theta_0 < \frac{\pi}{2})$$

This equation which is of the form $y = ax–bx2$ (a and b are constants), is the equation of a parabola. Therefore, when air resistance is neglected (when using the simplified model of the system), the trajectory of the projectile is always a parabola. At any instant, the velocity of the object is tangent to its trajectory Its magnitude and direction with respect to the positive x-direction are given by

$$v = \sqrt{v_x^2 + v_y^2}$$

and

$$\theta = \tan^{-1}(v_y/v_x)$$

respectively The maximum height h of the projectile, as in Fig. 2.22 , is found at $t = t_1$ by noting that at the peak h, $v_y = 0$. Substituting this in Eq. 2.15 gives

$$v_0 \sin\theta_0 = gt_1$$

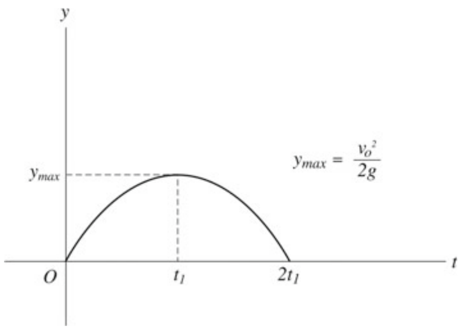


Fig. 2.22 The maximum height of a projectile

$$t_1 = \frac{v_0 \sin \theta_0}{g}$$

Substituting t_1 into Eq. 2.17 we get

$$y_{\max} = h = (v_0 \sin \theta_0) t_1 - \frac{1}{2} g t_1^2$$

$$h = (v_0 \sin \theta_0) \left(\frac{v_0 \sin \theta_0}{g} \right) - \frac{1}{2} g \left(\frac{v_0 \sin \theta_0}{g} \right)^2$$

$$h = \frac{v_0^2 \sin^2 \theta_0}{2g}$$

The maximum range R is at $t = 2t_1$. Substituting t into Eq. 2.16 gives

$$x = R = (v_0 \cos \theta_0) 2t_1 = (v_0 \cos \theta_0) \frac{2v_0 \sin \theta_0}{g} = \frac{2v_0^2 \sin \theta_0 \cos \theta_0}{g}$$

$$R = \frac{v_0^2 \sin 2\theta_0}{g}$$

Example 2.14 A baseball is thrown at angle of 35° to the horizontal with an initial speed of 20 m/s. Neglecting air resistance, find: (a) the maximum height reached by the ball; (b) the time it takes the ball to hit the ground; (c) the range; and (d) the speed of the ball just before it strikes the ground.

Solution 2.14 (a) The maximum height reached by the ball is

$$h = \frac{v_0^2 \sin^2 \theta_0}{2g} = \frac{(20 \text{ m/s})^2 \sin^2(35^o)}{2(9.8 \text{ m/s}^2)} = 6.7 \text{ m}$$

(b) The time it takes the ball to hit the ground is

$$t = 2t_1 = \frac{2v_0 \sin \theta_0}{g} = \frac{2(20 \text{ m/s}) \sin(35^o)}{(9.8 \text{ m/s}^2)} = 2.34 \text{ s}$$

(c) The range is

$$R = \frac{v_0^2 \sin 2\theta_0}{g} = \frac{(20 \text{ m/s})^2 \sin(70^o)}{(9.8 \text{ m/s}^2)} = 38.4 \text{ m}$$

(d) The x-component of the velocity of the ball just before it hits the ground is

$$v_x = v_{0x} = v_0 \cos \theta_0 = (20 \text{ m/s}) \cos(35^o) = 16.4 \text{ m/s}$$

The y-component is

$$v_y = v_{0y} - gt = v_0 \sin \theta_0 - gt = (20 \text{ m/s}) \sin(35^o) - (9.8 \text{ m/s}^2)(2.34 \text{ s}) = -11.5 \text{ m/s}$$

Hence, the speed is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(164 \text{ m/s})^2 + (-11.5 \text{ m/s})^2} = 20 \text{ m/s}$$

Example 2.15 A boy throws a ball with a constant horizontal velocity of 1 m/s at an altitude of 0.6 m. Find the horizontal distance between the releasing point to the point where the ball hits the ground.

Solution 2.15 Let the origin of the reference frame be the releasing point. Since $v_{0y} = 0$ we have

$$y = -\frac{1}{2} g t^2$$

and

$$x = v_{0x} t$$

Hence, when the ball reaches the ground, the elapsed time is

$$t = \sqrt{\frac{-2y}{g}} = \sqrt{\frac{-2(0.6 \text{ m})}{(-9.8 \text{ m/s}^2)}} = 0.34 \text{ s}$$

and

$$x = (1 \text{ m/s})(0.34 \text{ s}) = 0.34 \text{ m}$$

2.4.5 Uniform Circular Motion

A particle moving in a circular path with constant speed is said to be in uniform circular motion. The motion of the moon about earth, and the motion of clothes in a washing machine are examples of uniform circular motion. In this motion, the direction of the velocity of the particle is continuously changing but its magnitude is constant. As we have mentioned in Sect. 2.3.1, when only the direction of the velocity changes, the acceleration is then always perpendicular to the velocity

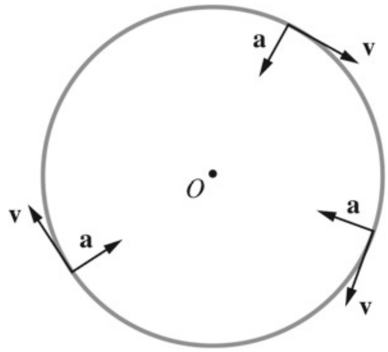


Fig. 2.23 The directions of y and a change continuously with time but their magnitudes are constant

at any time. Therefore, we have only the normal component of the acceleration $a_n = v^2/R$, and the tangential component of the acceleration $a_t = dv/dt$ is zero. In the case of the circular path the radius of curvature R is constant, denoted by r, and the normal acceleration is directed along the radius of the circle

$$a_{rad} = \frac{v^2}{r}$$

The subscript *rad* is for radial. Thus, this radial or centripetal acceleration a_{rad} is always directed toward the center of the circle. Therefore, the directions of **v** and a change continuously with time but their magnitudes are constant (see Fig. 2.23). The time required for the particle to complete one revolution around the circle is called the period of revolution and is given by

$$T = \frac{2\pi r}{v}$$

Thus

$$a_{rad} = \frac{4\pi^2 r}{T^2}$$

Example 2.16 In a fun fair ride, the passengers rotate in a circle with a constant speed of 3 m/s. If the period of revolution is 1.5 s, find the total acceleration of the passenger.

Solution 2.16 Since the speed of the passenger is constant, it follows that the passenger's total acceleration is just the centripetal acceleration given by

$$a_{rad} = \frac{v^2}{r}$$

The radius of the circular path is

$$r = \frac{vT}{2\pi} = \frac{(3\ \text{m/s})(1.5\ \text{s})}{2(3.14)} = 0.7\ \text{m}$$

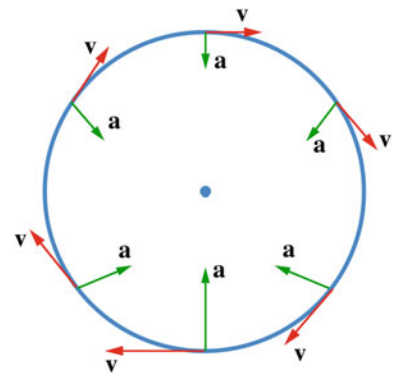


Fig. 2.24 The velocity and total acceleration vectors of a particle moving in a circular path with increasing speed (clockwise) until it reaches the maximum speed at the bottom, and then slows down as it goes back up. An example of this motion is in a roller coaster ride in a vertical circle

$$a_{rad} = \frac{v^2}{r} = \frac{(3\ \text{m/s})^2}{(0.7\ \text{m})} = 12.86\ \text{m/s}^2$$

2.4.6 Nonuniform Circular Motion

In nonuniform circular motion, the velocity of the particle varies in both magnitude and direction. As mentioned in Sect. 2.3.1, when both the magnitude and direction of the particle's velocity change then its acceleration is directed at some angle to **v**. Thus, in addition to the normal acceleration in uniform circular motion that corresponds to the change in the direction of **v**, there is a tangential component that corresponds to the change in the magnitude of **v**. Furthermore a_{rad} is not constant since **v** changes with time. Therefore, the resultant acceleration is

$$\mathbf{a} = \mathbf{a}_n + \mathbf{a}_t = \frac{v^2}{r}\mathbf{N} + \frac{d|\mathbf{v}|}{dt}\mathbf{T}$$

In Chap. 8, the concepts of angular velocity and acceleration and their vector relationship with the normal and tangential accelerations are introduced. Figure 2.24 shows the velocity and total acceleration vectors of a particle moving in a circular path with increasing speed (clockwise) until it reaches the maximum speed at the bottom, and then slows down as it goes back up. An example of this motion is in a roller coaster ride in a vertical circle.

Example 2.17 A car moving on a circular track of a 20 m radius accelerates uniformly from a speed of 30 km/h to a speed of 50 km/h in 3 s. Find the total acceleration of the car at the instant its speed is 40 km/s.

Solution 2.17 Since both the direction and the magnitude of the car's velocity change, its total acceleration is the vector sum of its tangential and radial accelerations. The tangential acceleration is

$$a_t = \frac{v - v_0}{t} = \frac{(13.8 \text{ m/s}) - (8.3 \text{ m/s})}{(3 \text{ s})} = 1.83 \text{ m/s}^2$$

When $v = 40$ km/h $= 11.1$ m/s the radial acceleration is

$$a_{rad} = \frac{v^2}{r} = \frac{(11.1 \text{ m/s})^2}{(20 \text{ m})} = 6.2 \text{ m/s}^2$$

And the total acceleration is

$$a = \sqrt{(1.83 \text{ m/s}^2)^2 + (6.2 \text{ m/s}^2)^2} = 6.5 \text{ m/s}^2$$

2.5 Relative Velocity

In this section, we will see how observers moving relative to each other obtain different results when measuring the velocity of a moving body. Suppose two cars are moving besides each other at the same speed of 120 km/h with respect to earth. In this case, any of the two cars is at rest relative to the other. According to an observer who is stationary with respect to earth, each car is moving with a speed of 120 km/s. A second observer, in any of the cars, will see the stationary observer moving backwards at a speed of 120 km/h. In addition, if a third car is moving ahead of the two cars at a speed of 140 km/h relative to earth, then its speed relative to an observer in any of the two cars is 20 km/s. Thus, the displacement and velocities may have different values when measured relative to different observers. Therefore, the description of motion depends on the observer. By attaching a coordinate system to an observer together with an appropriate time scale, he or she are then said to be in a reference frame. In measuring quantities, it is essential to specify the reference frame. In most situations, the earth (the lab) is used as our frame of reference. To understand this, consider a particle moving in one dimension in the positive x-direction. Suppose two observers want to describe its motion, one is observer S who is stationary relative to the ground, and the other is observer S′, who is moving in the positive x-direction with a constant velocity relative to the ground (see Fig. 2.25). At any instant, the position of the particle relative to S is x_{PS}, and its position relative to S′ is $x_{PS'}$. The relation between these two observations is

$$x_{PS} = x_{PS'} + x_{S'S} \tag{2.18}$$

Therefore, the position of P relative to O_S is equal to the position of P relative to $O_{S'}$ plus the distance between O_S and $O_{S'}$. Differentiating Eq. 2.18 with respect to time we get

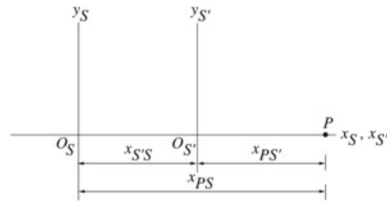


Fig. 2.25 Observer S is stationary relative to the ground, and observer S′ is moving in the positive x-direction with a constant velocity relative to the ground

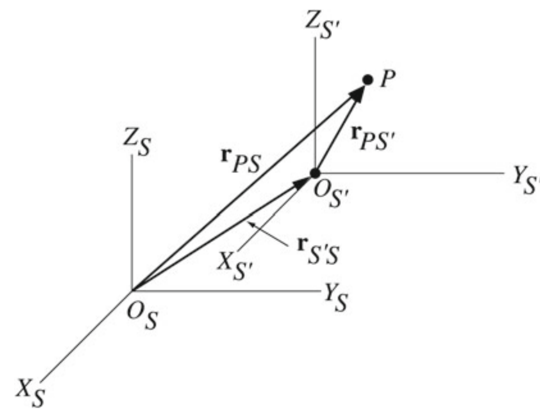


Fig. 2.26 The velocity of S′ with respect to S($v_{S'S}$) is constant in both magnitude and direction

$$\frac{dx_{PS}}{dt} = \frac{dx_{PS'}}{dt} + \frac{dx_{S'S}}{dt}$$

or

$$v_{PS} = v_{PS'} + v_{S'S}$$

We will extend this to three dimensions in the case where the velocity of S′ with respect to S($v_{S'S}$) is constant in both magnitude and direction (see Fig. 2.26). The position vector of the particle P relative to S is given by

$$\mathbf{r}_{PS} = \mathbf{r}_{PS'} + \mathbf{r}_{S'S} \tag{2.19}$$

Differentiating this with respect to time gives

$$\mathbf{v}_{PS} = \mathbf{v}_{PS'} + \mathbf{v}_{S'S} \tag{2.20}$$

Equations 2.19 and 2.20 are called the Galilean transformation equations. In addition, for any two frames of reference S and S we have

$$\mathbf{v}_{SS'} = -\mathbf{v}_{S'S}$$

Example 2.18 Two motor cyclists A and B are driving along the same road (See Fig. 2.27) with speeds 90 km/h and 50 km/s, respectively. Determine: (a) the velocity of motorcyclist A relative to B and of B relative to A?, · and (b) if the two motor cyclists approach each other along two parallel roads, (See Fig. 2.28), A moving at 80 km/s, and B moving at 60 km/s, what is the velocity of motorcyclist A relative to B and of B relative to A.

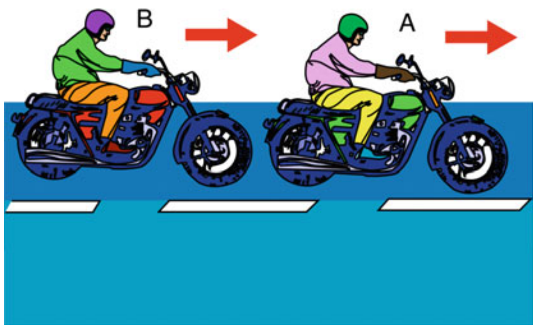


Fig. 2.27 Two motor cyclists A and B driving with speeds 90 km/h and 50 km/s respectively

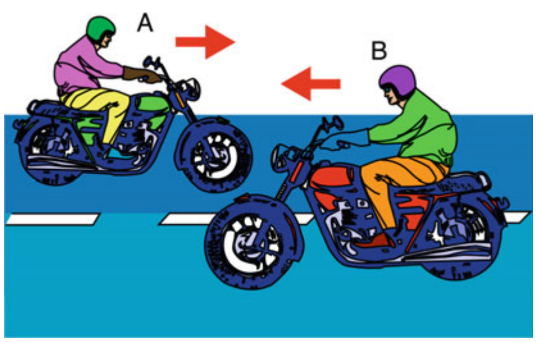


Fig. 2.28 A is moving at 80 km/s, and B moving at 60 km/s

Solution 2.18 Using the above discussion, consider S as the Earth's frame of reference denoted E, S′ as the frame of reference of motorcyclist B and the point P as the motor cyclist A

(a) The velocity of A relative to B is found from

$$v_{AB} = v_{AE} - v_{BE} = (90\text{ km/h}) - (50\text{ km/h}) = 40\text{ km/h}$$

The velocity of B relative to A is

$$v_{BA} = -40\text{ km/h}$$

(b)

$$v_{AB} = v_{AE} - v_{BE} = (80\text{ km}) - (-60\text{ km/h}) = 140\text{ km/h}$$

$$v_{BA} = -140\text{ km/h}$$

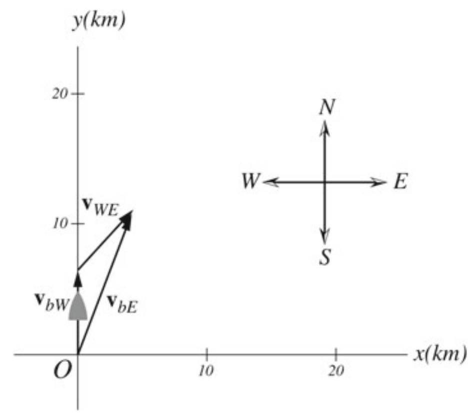


Fig. 2.29 A boat is traveling at 8 km/h north relative to the sea's waves, and the waves are traveling northeast relative to the earth at a constant speed of 4 km/h

Example 2.19 A boat is traveling at sea at 8 km/h north relative to the sea's waves, and the waves are traveling northeast relative to the earth at a constant speed of 4 km/h. What is the velocity of the boat relative to the earth?

Solution 2.19 Using Fig. 2.26, consider the Earth as S (denoted E), the waves as S′, and the boat as the point P. As we can see from Fig. 2.29, the velocity of the boat relative to the earth is given by $\mathbf{v}_{bE} = \mathbf{v}_{bw} + \mathbf{v}_{wE}$, where $\mathbf{v}_{bw}$ and $\mathbf{v}_{wE}$ are the velocities of the boat relative to the waves and the velocity of the waves relative to the earth respectively With the east as the direction of the positive x-axis we get

$$v_{(bE)y} = v_{(bw)y} + v_{(wE)y} = (8\text{ km/h}) + (4\text{ km/h})\sin 45^\circ = 10.83\text{ km/h}$$

$$v_{(bE)x} = v_{(wE)x} = (4\text{ km/h})\cos 45^\circ = 2.83\text{ km/h}$$

Hence

$$v_{bE} = \sqrt{(v_{(bE)x})^2 + (v_{(bE)y})^2} = \sqrt{(10.83\text{ km/h})^2 + (2.83\text{ km/h})^2} = 11.2\text{ km/h}$$

The direction of $\mathbf{v}_{bE}$ is

$$\theta = \tan^{-1}\frac{(v_{bE})_y}{(v_{bE})_x} = \tan^{-1}\frac{(10.83\text{ km/h})}{(2.83\text{ km/h})} = 75.35^{\text{o}}$$

2.6 Motion in a Plane Using Polar Coordinates

Consider a particle moving in the x–y plane. A useful way to describe the position, velocity, and acceleration of the particle is by using its polar coordinates (r, θ) . The relationship between the polar and rectangular coordinates is

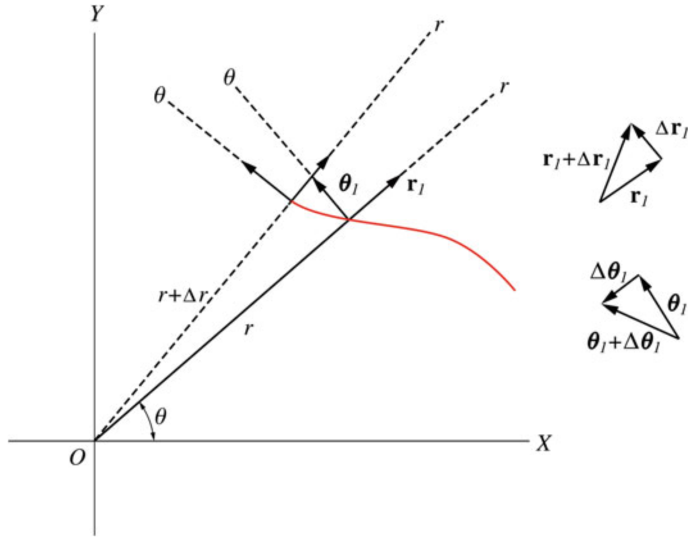


Fig. 2.30 r_1 is a unit vector along the increasing r direction and θ_1 is a unit vector in the direction of increasing θ (anticlockwise direction)

$$x = r\cos\theta$$

$$y = r\sin\theta$$

where θ is measured from the positive x- axis. Suppose a particle is located at (r, θ) . If the particle moves in a straight line along the r direction, then θ is constant through the motion of the particle. If the particle moves in a circle, then r is constant. Let r_1 be a unit vector along the increasing r direction and θ_1 to be a unit vector in the direction of increasing θ (anticlockwise direction). From Fig. 2.30, we have

$$\mathbf{r}_1 = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$

and

$$\boldsymbol{\theta}_1 = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$

Unlike the rectangular unit vectors, the polar unit vectors are not fixed in direction. Their direction changes as the particle moves along some path. Therefore, when finding the velocity and acceleration of a particle the derivatives of the polar unit vectors must be considered. The position vector of the particle is given by

$$\mathbf{r} = r\mathbf{r}_1$$

To find the velocity in terms of the polar unit vectors let us differentiate $\mathbf{r}_1$ and $\boldsymbol{\theta}_1$ with respect to time. That gives

$$\dot{\mathbf{r}}_1 = \frac{d\mathbf{r}_1}{dt} = -\sin\theta\frac{d\theta}{dt}\mathbf{i} + \cos\theta\frac{d\theta}{dt}\mathbf{j}$$

$$= \boldsymbol{\theta}_1\frac{d\theta}{dt} = \dot{\theta}\boldsymbol{\theta}_1$$

$$\dot{\boldsymbol{\theta}}_1 = \frac{d\boldsymbol{\theta}_1}{dt} = -\cos\theta\frac{d\theta}{dt}\mathbf{i} - \sin\theta\frac{d\theta}{dt}\mathbf{j}$$

$$= -\mathbf{r}_1\frac{d\theta}{dt} = -\dot{\theta}\mathbf{r}_1$$

The velocity of the particle is given by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\mathbf{r}_1) = \frac{dr}{dt}\mathbf{r}_1 + r\frac{d\mathbf{r}_1}{dt}$$

$$= \dot{\mathrm{r}}\mathbf{r}_1 + r\dot{\mathbf{r}}_1 = \dot{\mathrm{r}}\mathbf{r}_1 + r\dot{\theta}\boldsymbol{\theta}_1$$

Hence, the velocity is (Fig. 2.31)

$$\mathbf{v} = \dot{\mathrm{r}}\mathbf{r}_1 + r\dot{\theta}\boldsymbol{\theta}_1 \tag{2.21}$$

We may write

$$\mathbf{v} = v_r\mathbf{r}_1 + v_\theta\boldsymbol{\theta}_1$$

where $v_r = \dot{r}$ and $v_\theta = r\dot{\theta}$ and $v = \sqrt{v_r^2 + v_\theta^2}$. The total acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\dot{\mathrm{r}}\mathbf{r}_1 + r\dot{\theta}\boldsymbol{\theta}_1) = \ddot{r}\mathbf{r}_1 + \dot{\mathrm{r}}\dot{\mathbf{r}}_1 + \dot{\mathrm{r}}\dot{\theta}\boldsymbol{\theta}_1 + r\ddot{\theta}\boldsymbol{\theta}_1 + r\dot{\theta}\dot{\boldsymbol{\theta}}_1$$

$$= \ddot{r}\mathbf{r}_1 + \dot{\mathrm{r}}(\dot{\theta}\boldsymbol{\theta}_1) + \dot{\mathrm{r}}\dot{\theta}\boldsymbol{\theta}_1 + r\ddot{\theta}\boldsymbol{\theta}_1 + r\dot{\theta}(-\dot{\theta}\mathbf{r}_1)$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{r}_1 + (r\ddot{\theta} + 2\dot{\mathrm{r}}\dot{\theta})\boldsymbol{\theta}_1 \tag{2.22}$$

or

$$\mathbf{a} = a_r\mathbf{r}_1 + a_\theta\boldsymbol{\theta}_1$$

where

$$a_r = (\ddot{r} - r\dot{\theta}^2)$$

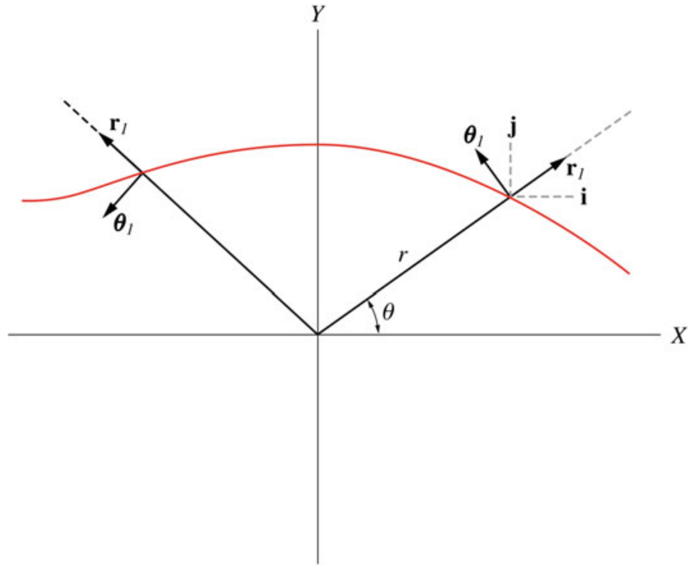


Fig. 2.31 Unlike the rectangular unit vectors, the polar unit vectors are not fixed in direction. Their direction changes as the particle moves along some path

and

$$a_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

and

$$a = \sqrt{a_r^2 + a_\theta^2}$$

Example 2.20 If a particle moves in a plane according to the expressions $\theta = 0.3t + 0.2t^2$ and $r = 0.5t + 0.4t^2$. Find its velocity and acceleration at $t = 2$ s

Solution 2.20 At $t = 2$ s, $\theta = 0.3t + 0.2t^2 = 1.4$ rad, $\dot{\theta} = 0.3 + 0.4t = 1.1$ rad/s and $\ddot{\theta} = 0.4$ rad/s^2. Also $r = 0.5t + 0.4t^2 = 2.6$ m, $\dot{r} = 0.5 + 0.8t = 2.1$ m/s and $\ddot{r} = 0.8$ m/s^2. Therefore

$$v_r = \mathrm{r} = 2.1 \text{ m/s}$$

$$v_\theta = r\dot{\theta} = (2.6 \text{ m})(1.1\text{rad/s}) = 2.9 \text{ m/s}$$

$$v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{(2.1 \text{ m/s})^2 + (2.9 \text{ m/s})^2} = 3.6 \text{ m/s}$$

and

$$a_r = \ddot{r} - r\dot{\theta}^2 = (0.8 \text{ m/s}^2) - (2.6 \text{ m})(1.1\text{rad/s})^2 = -2.35 \text{ m/s}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = (2.6 \text{ m})(0.4 \text{ rad/s}^2) + 2(2.1 \text{ m/s})(1.1 \text{ rad/s}) = 5.7 \text{ m/s}^2$$

$$a = \sqrt{a_r^2 + a_\theta^2} = \sqrt{(-2.35 \text{ m/s}^2)^2 + (5.7 \text{ m/s}^2)^2} = 6.2 \text{ m/s}^2$$

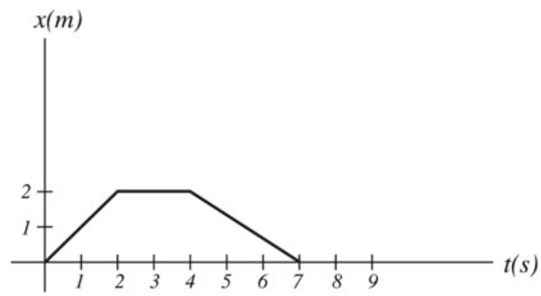


Fig. 2.32 An object moving in one dimension along the x-axis

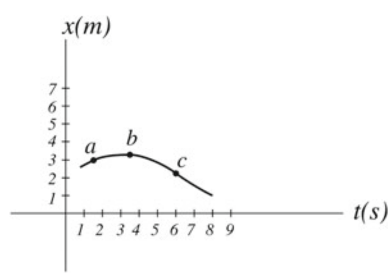


Fig. 2.33 The position-time graph of a particle moving along the x-axis

Problems

1. A sports car moves around a circular track of radius of 100 m. If the car makes one round in 75 s, find the car's (a) average speed (b) average velocity.
2. An object is moving in one dimension along the x-axis according to Fig. 2.32. Describe the motion of the object.

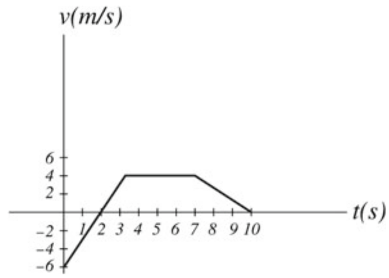
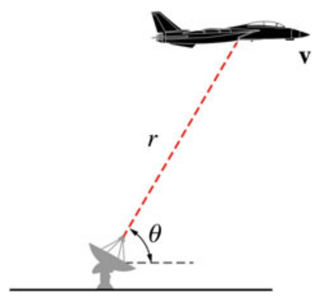


Fig. 2.34 The speed of a motorcyclist varying with time

Fig. 2.35 A car moves at a constant speed of 40 km/h along curved path

Fig. 2.36 An aircraft tracked by a radar coordinates



3. The position–time graph of a particle moving along the x-axis is shown in Fig. 2.33. Find (a) the average velocity between a and b(b) the instantaneous velocity at a, b, and c.
4. A motorist drives along a straight-line road. His speed varies with time according to Fig. 2.34. Sketch the position versus time and acceleration versus time graphs of the motorist.
5. A particle moves along the curve defined by $x = 5e^{-t}$ and $y = \sin 5t$. Find the position, velocity and acceleration of the particle at any time.
6. A car moves at constant speed of 40 km/h along the road shown in Fig. 2.35. If the radius of curvature at A is 350 m and the total acceleration of the car at B is $1\,\text{m/s}^2$, find (a) the total acceleration of the car at A and C(b) the radius of curvature at B.(Hint: the radius of curvature at C is infinite).
7. A body with initial speed of 15 m/s undergoes a uniform acceleration of $-2\,\text{m/s}^2$. Find the elapsed time and the distance it traveled when it reaches a speed of 3 m/s.
8. A stone is thrown downwards from a height of 10 m. Find its initial speed if it reaches the ground after 1 s.
9. A block is thrown horizontally from the top of a cliff that is 30 m high with a speed of 10 m/s. Find (a) the block's magnitude of displacement from the origin and its velocity after 1.5 s, (b) the horizontal distance from the releasing point to where the block hits the ground.(Hint: the magnitude of displacement from the origin is $d = \sqrt{x^2 + y^2}$).
10. A river has a uniform speed of 0.5 m/s due east. If a boat travels east at a speed of 3 m/s relative to the water, find the time it takes the boat to travel a distance of 1100 km and return to its starting point.
11. An aircraft is tracked by a radar (see Fig. 2.36). If at a certain instant the radar measurements give $r = 7 \times 10^4$ m, r = 1000 m/s, $\ddot{r} = 7\,\text{m/s}^2$, $\theta = 45^{\text{o}}$, $\dot{\theta} = 0.6$ deg/s, and $\ddot{\theta} = 0.02$ deg/s^2. Find the velocity and acceleration of the airplane at that instant.

3 Newton's Laws

3.1 Introduction

In this chapter, dynamics which is a branch of mechanics will be discussed. Dynamics is concerned with the cause behind the motion of objects and answers questions such as: Why does a skydiver float in air? What makes an apple fall from a tree? Why a block connected to a spring oscillates when the spring is stretched? We will find that these motions occur when objects interact with each other, i.e., the apple is interacting with earth, the skydiver is interacting with air, and the block is interacting with the spring.

3.1.1 The Concept of Force

The interaction between one object and another or between the object and its environment defines a quantity called force. A force is a pull or a push in a certain direction that may cause the object to move or deform. However, motion does not always occur if the force is not large enough to overcome other forces such as friction or gravity But whether or not an object moves due to a force, there is always some deformation. In this book, it is assumed that objects remain undeformed under the influence of any forces. Experimentally, a force is found to be a vector quantity The net external force acting on an object (the vector sum of all forces acting on the object) causes the object to accelerate where the direction of the acceleration is in the direction of that force.

Hence, acceleration is a measure of force. If the net force equals zero, the acceleration of the object is zero, and the velocity of the object remains unchanged (constant). Forces in nature are one of two:

1. *Contact forces* resulting from direct contact between two objects (e.g., kicking a ball or punching a bag);
2. *Field forces* that can act through empty space and in which physical contact is not necessary (e.g., gravitational force between two objects and the electric force between two electric charges).

3.1.2 The Fundamental Forces in Nature

The following fundamental forces are all field forces:

1. The gravitational force between any two objects;
2. The electromagnetic force between two electric charges;
3. The strong nuclear force between subatomic particles which is responsible for the stability of the nuclei;
4. The weak nuclear force which produces certain kinds of radioactive decay and is responsible for the instability of some nuclei.

The first two fundamental forces are examples of long-range forces, which act over a great distance. The second two are examples of short-range forces, which are forces that act over a very short distance. Note that contact forces are fundamentally electromagnetic since they involve electromagnetic forces between the atoms of the surfaces in contact.

3.2 Newton's Laws

Sir Isaac Newton (1642–1727) formulated his three famous laws of motion describing the relationship between the force acting on an object and the acceleration of that object. Newtonian or classical mechanics which is based mainly on Newton's three laws of motion, deals only with objects that are

- Large compared to the size of an atom ($\approx 10^{-10}$ m).
- Moving at speeds much less than the speed of light ($\approx 3 \times 10^8$ m/s).

Einstein's special theory of relativity replaces Newtonian mechanics when an object's speed approaches the speed of light. On the other hand, quantum mechanics replaces Newtonian mechanics when the object's dimensions are close to atomic scale.

S. Alrasheed, *Principles of Mechanics*, Advances in Science, Technology & Innovation, https://doi.org/10.1007/978-3-030-15195-9_3

3.2.1 Newton's First Law

It was believed long ago that a force is necessary to keep an object moving and that any object's natural state is to be at rest. Later, these statements were proved to be incorrect. To understand this, suppose a block resting on a surface is given a push and is released. As a result, the block will slide for sometime before coming to rest. The time elapsed between pushing the block until it comes to rest will increase as the surface gets smoother. If the surface becomes so smooth, such that friction is almost negligible, the block will continue to move along a straight line with constant speed for a greater distance before coming to rest.

An example of frictionless motion is the motion of the puck in the air-hockey table. The puck floats on a thin column of air that is used as the lubricant. In situations where there is no friction at all, the object will continue to move along a straight line with constant speed without requiring any force to keep it moving. However, a force is required to initiate motion. This concept was formulated by Newton and became his first law of motion: *An object at rest remains at rest and an object in motion will continue in motion with constant velocity* (*constant speed in a straight line*) *unless acted upon by a net external force*. That is, if

$$\Sigma \mathbf{F} = \mathbf{0}$$

$$\mathbf{a} = \mathbf{0}$$

A body's tendency to stay at rest or maintain uniform motion in a straight line is called inertia. Thus, Newton's first law is often referred to as the law of inertia, where it defines specific kinds of reference frames called inertial reference frames. An inertial reference frame is a frame in which Newton's first law is valid. That is, in an inertial frame of reference, an object has no acceleration if there is no net force acting upon it. Any reference frame moving with constant velocity relative to an inertial frame is also inertial. Observers in different inertial frames measure the same acceleration for a moving object. To prove this, consider the two inertial reference frames S and S′ mentioned in Sect. 2.5, where S is stationary and S′ is moving with constant velocity relative to S. By differentiating Eq. 2.20, we have

$$\frac{d\mathbf{v}_{PS}}{dt} = \frac{d\mathbf{v}_{PS'}}{dt} + \frac{d\mathbf{v}_{S'S}}{dt}$$

Because $\mathbf{v}_{S'S}$ is constant we have

$$\mathbf{a}_{PS} = \mathbf{a}_{PS'}$$

That is, the acceleration of the particle P measured from both inertial reference frames S and S′ is the same. To show that Newton's first law is only valid when applied with respect to an inertial frame of reference; consider a girl named Mia that is at rest while watching her friend Lea driving a car moving at constant velocity. Lea has her seatbelt fastened and put her suitcase in the seat right next to her without restraining it. Now, suppose that Lea steps on the brakes, which would cause her vehicle to decelerate, her suitcase will start to move forward. According to Lea, who is in an accelerated frame, the suitcase moved from rest even though there was no apparent net external force acting on it. Therefore, in Lea's frame, Newton's first law seems to be incorrect.

The situation, however, is different to Mia, who is in an inertial frame of reference. In her perspective, the suitcase was initially moving with constant velocity and the net force on it was zero. When the car started to decelerate, the net force on the suitcase is still equal to zero and thus the suitcase must continue to move forward with constant velocity and stop by friction or impact with the inside of the car. Therefore, it is apparent to Mia that Newton's first law is valid. From the previous example, we conclude that Newton's first law (and in general Newton's laws) is not valid in all kinds of reference frames; it is only valid when applied with respect to inertial frames. That is, Lea must not apply Newton's first law in her reference frame.

The same situation would be observed by Lea if she were to turn her car while moving. When the car turns, the suitcase will start to move in the direction opposite to the turn. Once again, Lea observes that the suitcase has moved from rest without any apparent force acting on it which contradicts Newton's first law in her opinion. Mia sees no contradiction with Newton's first law because when the car turns, the suitcase tends to continue its initial uniform straight line motion, and thus it moves toward the direction opposite to the turn. Therefore Newton's laws are obeyed by objects when observed from inertial frames of reference (see Fig. 3.1).

To apply classical (Newtonian) mechanics with respect to a noninertial reference frame, new forces named as pseudo forces are introduced. In this book, only inertial frames are used, and all laws are stated with respect to those frames. One convenient inertial frame of reference, used throughout this book, is the surface of the earth. The earth can be considered as an inertial frame since its motion about its axis and about the sun has a small effect on calculations and thus can be neglected.

3.2.2 The Principle of Invariance

Some quantities such as mass, force, time, and acceleration are invariant, which means that they have the same numerical values when measured in different inertial frames of reference. Other quantities such as velocity, kinetic energy, and work have different values in different inertial frames. How-

Fig. 3.1 The boy is throwing the water out by pitching the bucket forward. If he stops, the water will continue its motion along a straight line. However, because of the force of gravity, it follows a parabolic path

ever, the laws of physics have the same form in all inertial frames of reference. This is called the principle of invariance.

3.2.3 Mass

As mentioned earlier, the tendency of an object to resist any change in its motion (i.e., to remain at rest or maintain uniform motion along a straight line) is called inertia. From experiments and everyday experience, it is observed that a certain force produces different accelerations when applied to different bodies. This variation in the produced acceleration depends upon the quantity of matter contained in the body Such quantity is known as the mass of the body Therefore, mass is a measure of inertia. Objects with large masses have less acceleration when exposed to the same force. Thus, mass is a quantity that relates the acceleration of the body to the force acting on it. The SI unit of mass is the kilogram (kg). Experimentally, it is found that the ratio of the masses of any two bodies (say m_1 and m_2) is equal to the inverse of the ratio of the magnitudes of their accelerations if both are acted upon by the same force. That is, we have

$$\frac{m_1}{m_2} = \frac{a_2}{a_1}$$

The mass of any body can be found by comparing its acceleration to the acceleration of a 1 kg mass when both bodies are acted upon by the same force. This leads to the conclusion that mass is independent of force; it is an inherent characteristic of matter. Furthermore, it has been experimentally proved that when two masses m_1 and m_2 are attached together, the combined body behaves as a single body of mass $m_1 + m_2$. Thus, mass is a scalar quantity and obeys the rules of ordinary arithmetic.

3.2.4 Newton's Second Law

Unlike Newton's first law, Newton's second law describes the situation in which the net force acting on an object is not zero. It was found that when different forces act on an object, the object undergoes different accelerations. The magnitude of the acceleration is directly proportional to the magnitude of the applied force, and its direction is in the direction of that force. Furthermore, this acceleration is inversely proportional to the mass of the object for a certain applied force. These observations are summarized in Newton's second law of motion: *The acceleration of an object produced by a net external force is directly proportional to the force in a direction parallel to that force and is inversely proportional to its mass* That is,

$$\Sigma \mathbf{F} = m\mathbf{a}$$

In terms of components, the vector equation $\Sigma \mathrm{F} = m\mathrm{a}$ can be written as

$$\Sigma F_x = ma_x$$

$$\Sigma F_y = ma_y$$

$$\Sigma F_z = ma_z$$

where $a_x = \ddot{x}$, $a_y = \ddot{y}$ and $a_z = \ddot{z}$ In terms of normal and tangential coordinates, the net normal force is

$$\Sigma F_n = ma_n$$

and the net tangential force is

$$\Sigma F_t = ma_t$$

where $a_n = v^2/R$ and $a_t = dv/dt$. Finally, in terms of polar coordinates we have

$$\Sigma F_r = ma_r$$

and

$$\Sigma F_\theta = ma_\theta$$

where $a_r = (\ddot{r} - r\dot{\theta}^2)$ and $a_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta})$. The unit of force in the SI system is the Newton (N). One Newton (1N)

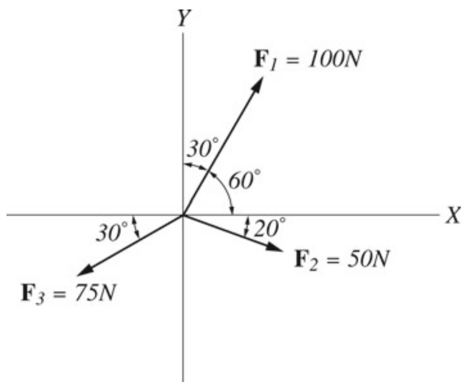


Fig. 3.2 A body is exposed to three forces acting in different directions

is defined as the force that gives a 1 kg mass an acceleration of $1\,\mathrm{m/s^2}$.

$$1\,\mathrm{N} = 1\,\mathrm{kg.m/s^2}$$

Example 3.1 A body is exposed to three forces acting in different directions as shown in Fig. 3.2. Find the magnitude and direction of the resultant force acting on the body and the corresponding acceleration.

Solution 3.1 The net force in the x-direction is

$$\sum F_x = F_{1x} + F_{2x} + F_{3x} = F_1 \cos 60^o + F_2 \cos 20^o - F_3 \cos 30^o$$

$$= (100\,\mathrm{N})(0.5) + (50\,\mathrm{N})(0.94) - (75\,\mathrm{N})(0.866) = 32.1\,\mathrm{N}$$

The net force in the y-direction is

$$\sum F_y = F_{1y} + F_{2y} + F_{3y} = F_1 \sin 60^o - F_2 \sin 20^o - F_3 \sin 30^o$$

$$= (100\,\mathrm{N})(0.866) - (50\,\mathrm{N})(0.34) - (75\,\mathrm{N})(0.5) = 32.1\,\mathrm{N}$$

The magnitude of the net force is

$$F_{net} = \sqrt{(32.1)^2 + (32.1)^2} = 45.4\,\mathrm{N}$$

The direction of F_{net} relative to the x-axis is

$$\theta = \tan^{-1} \frac{F_{nety}}{F_{netx}} = \tan^{-1} \frac{(32.1\mathrm{N})}{(32.1\,\mathrm{N})} = 45^o$$

The acceleration of the body is, therefore,

$$a = \frac{F}{m} = \frac{(45.4\,\mathrm{N})}{(50\,\mathrm{kg})} = 0.91\,\mathrm{m/s^2}$$

and its direction is the same as that of the force.

Example 3.2 If a man pushes a 60 kg box with a constant horizontal force of 100 N : (a) how far will the container be moved when its speed reaches a value of 3 m/s; (b) how far will the container be moved when its speed reaches a value of 3 m/s if the same force is applied at 30° to the horizontal; (c) find the normal force acting on the block in (a) and (b).

Solution 3.2 (a) The acceleration of the container is given by

$$a = \frac{F}{m} = \frac{(100\,\mathrm{N})}{(60\,\mathrm{kg})} = 1.67\,\mathrm{m/s^2}$$

the distance moved when its speed reaches 3 m/s is found from

$$v^2 - v_0^2 = 2a(x - x_0)$$

By taking $x_0 = 0$ at the starting point we have

$$(3\,\mathrm{m/s})^2 - 0 = 2(1.67\,\mathrm{m/s^2})(x - 0)$$

That gives $x = 2.7\,\mathrm{m}$.

(b) If the force is at 30° to the horizontal, the acceleration is

$$a = \frac{F}{m} = \frac{(100\,\mathrm{N})\cos 30^o}{(60\,\mathrm{kg})} = 1.45\,\mathrm{m/s^2}$$

$$(3\,\mathrm{m/s})^2 - 0 = 2(1.45\,\mathrm{m/s^2})(x - 0)$$

and $x = 3.1$ m.

(c) In situation (a) we have

$$\sum F_y = n - mg = 0$$

and

$$n = mg = (60\,\mathrm{kg})(9.8\,\mathrm{m/s^2}) = 588\,\mathrm{N}$$

In (b) we have

$$\sum F_y = n + F \sin\theta - mg = 0$$

and

$$n = (588\,\mathrm{N}) - (100\,\mathrm{N}) \sin 30^o = 538\,\mathrm{N}$$

Example 3.3 A particle of mass 0.5 kg is moving along the curve given by $\mathbf{r} = (1/3t^4\mathbf{i} - t^3\mathbf{j})$ m where t is time. Determine the force acting on the particle.

Solution 3.3 The force acting on the particle is

$$\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2} = m\frac{d}{dt}(4/3t^3\mathbf{i} - 3t^2\mathbf{j}) = m(4t^2\mathbf{i} - 6t\mathbf{j})$$

$$= (0.5\,\mathrm{kg})(4t^2\mathbf{i} - 6t\mathbf{j}) = (2t^2\mathbf{i} - 3t\mathbf{j})\,\mathrm{N}$$

Example 3.4 A particle of mass of 1 kg is moving under the influence of a force given by

$$\mathbf{F} = ((12t^2 - 5t)\mathbf{i} + (9t - 1)\mathbf{j} - 3t^2\mathbf{k})\,\mathrm{N}$$

If at $t = 0$, $\mathbf{r}_0 = \mathbf{0}$ and $\mathbf{v}_0 = \mathbf{0}$, find the velocity and the position of the particle at any time.

Solution 3.4

$$\mathbf{F} = m\mathbf{a}$$

$$\mathbf{a} = \frac{\mathbf{F}}{m} = ((12t^2 - 5t)\mathbf{i} + (9t - 1)\mathbf{j} - 3t^2\mathbf{k})\,\mathrm{m/s^2}$$

$$\mathbf{v} = \int \mathbf{a}dt = \left(4t^3 - \frac{5}{2}t^2\right)\mathbf{i} + \left(\frac{9}{2}t^2 - t\right)\mathbf{j} - t^3\mathbf{k} + \mathbf{c}_1$$

Since at $t = 0$, $\mathbf{v}_0 = \mathbf{0}$, then $\mathbf{c}_1 = \mathbf{0}$ and the velocity at any time is

$$\mathbf{v} = \left(\left(4t^3 - \frac{5}{2}t^2\right)\mathbf{i} + \left(\frac{9}{2}t^2 - t\right)\mathbf{j} - t^3\mathbf{k}\right)\,\mathrm{m/s}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$$\mathbf{r} = \int \mathbf{v}dt = \int \left[\left(4t^3 - \frac{5}{2}t^2\right)\mathbf{i} + \left(\frac{9}{2}t^2 - t\right)\mathbf{j} - t^3\mathbf{k}\right]dt$$

$$\mathbf{r} = \left(\left(t^4 - \frac{5}{6}t^3\right)\mathbf{i} + \left(\frac{3}{2}t^3 - \frac{1}{2}t^2\right)\mathbf{j} - \frac{1}{4}t^4\mathbf{k}\right)\,\mathrm{m}$$

Example 3.5 If a man weighs himself on an elevator that is accelerating upwards at a rate a relative to an observer outside the elevator (in an inertial frame) as shown in Fig. 3.3, what reading will he get for the normal force acting on him by the floor? what is the force if the elevator is accelerating downwards?

Solution 3.5 The normal force is n = m(a + g) for upward acceleration and n = m(g − a) for downward acceleration. Since a weighing scale measures the normal force and calculates the mass from it, the downward journey might be a more pleasant one!

3.2.5 Newton's Third Law

A force acting on an object is always due to another object in the surrounding environment. Newton's third law shows that: if body 1 exerts a force $\mathbf{F}_{21}$ on body 2 then body 2 will exert an equal and opposite force $\mathbf{F}_{12}$ on body 1. That is

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

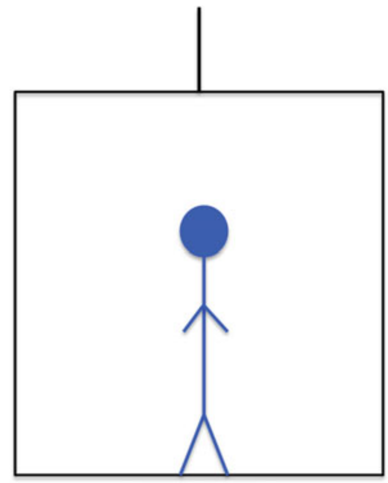


Fig. 3.3 A man weighing himself in an elevator

Any of these forces can be called an action force. When one of these forces is called an action force the other force is called a reaction (see Fig. 3.4). This law is sometimes stated *as "To every action there is an equal and opposite reaction*." Note that *the action and reaction forces always act on different objects*, i.e., they can't cancel each other out. This law also shows that forces come in pairs and that there is no such thing as a single isolated force.

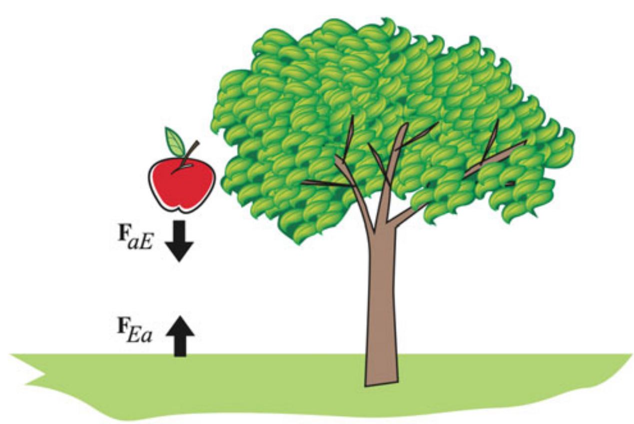


Fig. 3.4 The gravitational force exerted by the Earth on the apple and that exerted by the apple to the Earth form an action-reaction pair

Example 3.6 Three blocks of masses m_1, m_2, and m_3 are placed on a frictionless surface and pushed by a horizontal force F as in Fig. 3.5. Determine (a) the acceleration of the system; (b) the contact forces between m_1 and m_2 and between m_2 and m_3.

Solution 3.6 The free-body diagram of each block is shown in Fig. 3.5, where F_{21} is the force exerted on m_2 by m_1. (a) Applying Newton's second law for the ,

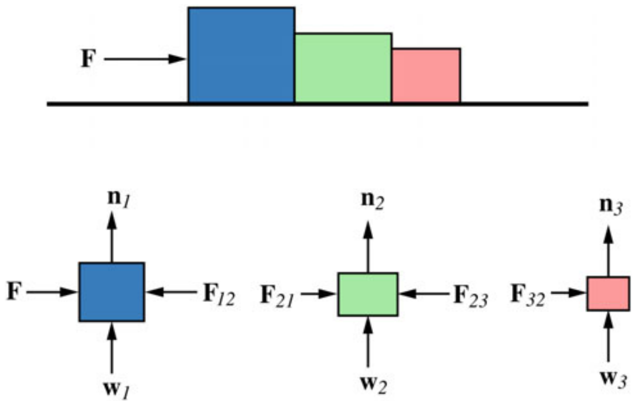


Fig. 3.5 Three blocks of masses m_1, m_2, and m_3 are placed on a frictionless surface and pushed by a horizontal force F

$$F = (m_1 + m_2 + m_3)a$$

$$a = \frac{F}{(m_1 + m_2 + m_3)}$$

(b) Applying Newton's second law for each block, we have $F - F_{12} = m_1 a$, $F_{21} - F_{23} = m_2 a$ and $F_{32} = m_3 a$. From Newton's law of action and reaction we have $F_{12} = F_{21}$, and $F_{32} = F_{23}$, and therefore

$$F_{12} = m_2 a + F_{23} = (m_2 + m_3)a = \frac{(m_2 + m_3)}{(m_1 + m_2 + m_3)}F$$

$$F_{32} = \frac{m_3 F}{(m_1 + m_2 + m_3)}$$

3.3 Some Particular Forces

3.3.1 Weight

In Sect. 2.4.2, we've seen that an object in free fall near the surface of the earth has a gravitational acceleration of magnitude $9.8\,\mathrm{m/s^2}$ that is directed toward the center of earth. Using Newton's second law, we can calculate the force that caused this acceleration. If an object has a mass m then the gravitational force is given by $m\mathbf{g}$, and is denoted by w, i.e., $\mathbf{w} = m\mathbf{g}$. **w** is known as the weight of an object and is defined as the gravitational force exerted on it by earth (or any other astronomical body, where g is different than that of earth). In Chap. 9, we will see that a gravitational force exists between any two bodies. When one of the bodies is an astronomical body, such as the earth or moon, and the other body is relatively smaller in size and mass the gravitational force is then called the weight of the body We will also see that the gravitational force varies with the distance between objects, and that the value of **g** becomes less at greater altitudes. Thus, weight is not an intrinsic property of an object. In everyday life, it is common to use the word weight when measuring the mass of a body mass and weight represent different quantities but they are proportional for a given value of **g**. For two masses at the same location, the ratio of their weights is equal to the ratio of their masses.

3.3.2 The Normal Force

If an object is in contact with a surface, either at rest or moving on it, the surface exerts a supporting force **n** on the object that is always perpendicular to the surface of contact. This force is called the normal force.

3.3.3 Tension

The tension force **T** is the force that a cord, rope, cable, or any other similar object exerts on an object attached to it. This force is directed along the rope away from the object at the point where the rope is attached. In solving problems, ropes are usually assumed to be massless (referred to as light ropes) and unstretchable. For any light rope, the magnitude of the tension force T is the same at all points along the rope.

3.3.4 Friction

Imagine that everything around you is coated with an extremely good lubricant. Simple activities such as walking, sitting, driving a car, or holding objects would become extremely difficult or impossible. Therefore, friction plays a very important role in our everyday life. The frictional force is due to the interaction between the surface atoms of any two bodies in contact. The direction of this force is always parallel to the surface of contact, opposing the motion or the planned motion of one object relative to the other. Hence, the normal and frictional forces are both contact forces and they are always perpendicular to each other.

Consider a block resting on a table. If the block is pushed with a horizontal force **F** and remains stationary, it is because that the applied force is balanced by an equal and opposite force. This opposing force is known as the statistical frictional force $\mathbf{f}_s$ and it has the value $f_s = F$. The statistical frictional force increases with increasing **F** (see Fig. 3.6). The name statistical comes from the fact that the block remains stationary.

However, if F is increased to a certain maximum value, the block will eventually accelerate (see Fig. 3.7). This maximum value is equal to the maximum frictional force $f_{s\,\max} = F$ and it represents the applied force when the block is at the verge of slipping, i.e., it is the minimum force necessary to initiate

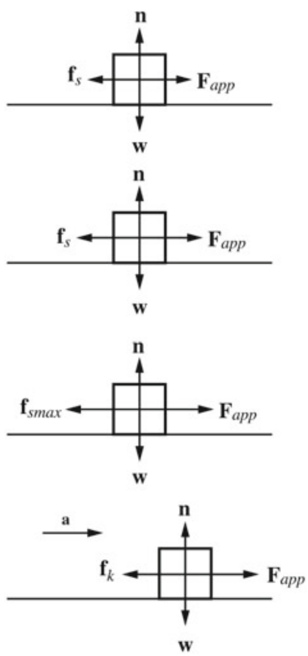


Fig. 3.6 The opposing force is known as the statistical frictional force $\mathbf{f}_s$ and it has the value $f_s = F$. Its maximum value $f_{s\,\max} = F$ represents the applied force when the block is at the verge of slipping i.e. it is the minimum force necessary to initiate motion. When the block moves, the retarding frictional force is then called the kinetic frictional force $\mathbf{f}_k$

motion. When the block moves, the retarding frictional force is then called the kinetic frictional force $\mathbf{f}_k$ and is usually less than $f_{s\,\max}$. If $f_k = F$, the block will move with a constant speed. If $F < f_s$ or if $\mathbf{F}$ is removed, the block will decelerate and will eventually be brought to rest. Experiments show that $f_{s\,\max}$ and f_k have the following properties:

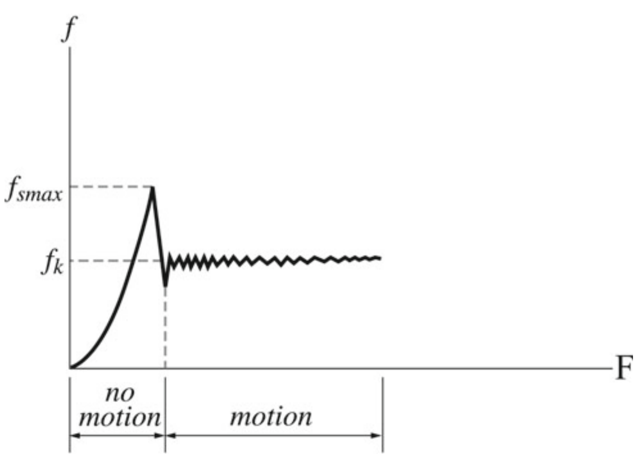


Fig. 3.7 A graph of the friction force versus the applied force

1. $f_{s\,\max} = \mu_s n$, where μ_s is the coefficient of static friction and n is the normal force acting on the block. As long as the block is at rest $f_s = F$, where $f_s \le \mu_s n$.
2. $f_k = \mu_k n$ where μ_k is the coefficient of the kinetic friction.
3. The directions of $\mathbf{f}_s$ and $\mathbf{f}_k$ are always parallel to the surface. $\mathbf{f}_s$ is opposite to the component of the applied force that is parallel to the surface and $\mathbf{f}_k$ is opposite to the instantaneous velocity of the body relative to the surface.

The dimensionless coefficients μ_s and μ_k depend on the nature of the surfaces in contact and are independent of the area of contact between these surfaces. μ_k is generally less than μ_s. Table 3.1 lists μ_s and μ_k for some materials. Note that μ_k may vary with speed but such variations are not included here. Friction is a very complex phenomenon. One reason behind this is that the actual area of contact viewed from a microscopic level is much less than the area of contact viewed from a macroscopic level, as in Fig. 3.8, even for very smooth surfaces. For our purposes here, the detailed friction mechanism will not be discussed.

Table 3.1 Coefficients of Friction

Materials	μ_s (static)	μ_k (kinetic)
Steel on steel	0.74	0.57
Aluminum on steel	0.61	0.47
Copper on steel	0.53	0.36
Wood on wood	0.25–0.5	0.2
Rubber on concrete	1.0	0.8
Glass on glass	0.94	0.4
Copper on glass	0.68	0.53
Teflon on teflon	0.04	0.04

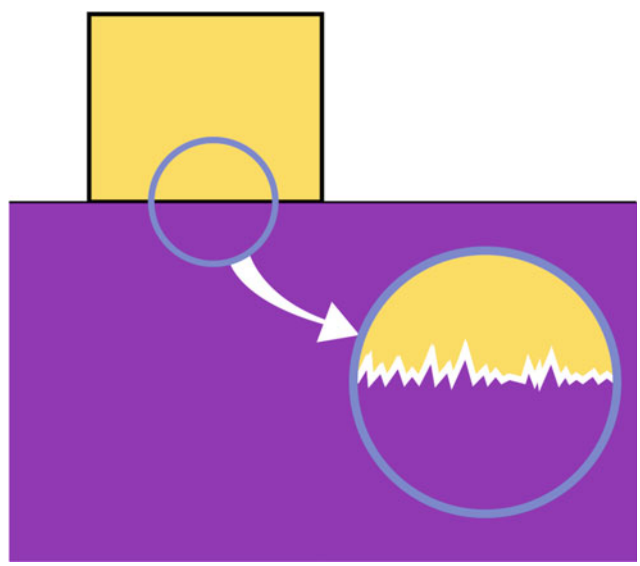


Fig. 3.8 The actual area of contact viewed from a microscopic level is much less than the area of contact viewed from a macroscopic level

3.3.5 The Drag Force

If a fluid (such as gas or liquid) and a body are in relative motion, the body will experience a resistive force opposing the relative motion called the drag force D. The direction of **D** is the direction in which the fluid is flowing relative to the body. The drag force depends on some factors, such as the speed of the object. We will consider the situation in which a relatively large object is moving through air at high speed (such as a skydiver or an airplane). In this case, the drag force D is proportional to the square of the body's speed and is given by

$$D = \frac{1}{2} c\rho A v^2$$

where ρ is the air density, A is the effective cross-sectional area of the body taken in a plane perpendicular to its velocity, and c is the drag coefficient. c is a dimensionless constant that has a value ranging from 0.4 to 2. The value of c may vary with v, but such variations will be ignored.

As an example, consider an object falling through air far from the earth's surface. The forces acting on the object are the drag force and the gravitational force as in Fig. 3.9. As the object falls, its speed increases and thus **D** increases from zero until it eventually becomes equal to the object's weight ($\mathbf{D} = -m\mathbf{g}$). The net force on the object when ($\mathbf{D} = -m\mathbf{g}$) will be equal to zero, and hence the object's acceleration will become zero ($\mathbf{a} = \mathbf{0}$). As a result the body will fall at a constant speed called the terminal speed v_t:

$$\Sigma \mathbf{F} = \mathbf{0}$$

$$\frac{1}{2} c\rho A v_t^2 - mg = 0$$

$$\frac{1}{2} c\rho A v_t^2 = mg$$

$$v_t = \sqrt{\frac{2mg}{c\rho A}}$$

where m is the mass of the body.

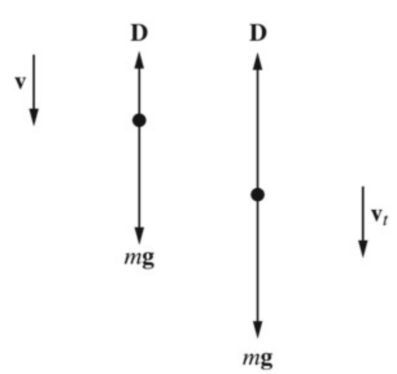


Fig. 3.9 An object falling through air experiences a drag force **D** and a gravitational force $m\mathbf{g}$

3.4 Applying Newton's Laws

It is necessary to follow some steps when solving problems using Newton's second law. These steps can be summarized in the following:

1. Draw a simple diagram of the objects in the system analyzed;
2. Draw a free-body diagram for each object in the system. In a free-body diagram, the body is represented by a dot, and all external forces (represented by vectors) acting on the body are shown. The forces exerted by the body on other bodies in the system are not included in its free-body diagram;
3. A coordinate system should be drawn in a free-body diagram with the body at its origin. Newton's second law is then applied along each axis using the components of each force. The coordinate system must be oriented in such a way that simplifies the analysis, i.e., some forces should be directed along the axes;
4. Solve obtained equations for the unknowns.

Note that from here until Chap. 5, any object is assumed to behave as a particle, i.e., all of its parts move in exactly the same way When applying Newton's second law, a particle is represented by a dot on the free-body diagram. Furthermore, the mass or friction of any rope or pulley is neglected.

Example 3.7 A 25 kg block is released from rest at the top of a rough 40° inclined surface. It then accelerates at a constant rate of 0.1 m/s^2. Find: (a) the coefficient of kinetic friction between the box and the surface; (b) the maximum angle the box would be at the verge of slipping if the angle of the incline is changeable.

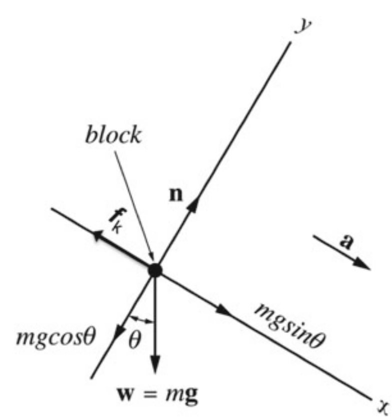


Fig. 3.10 The free-body diagram of a block accelerating down an incline

Solution 3.7 (a) The free-body diagram is shown in Fig. 3.10. Applying Newton's second law to the box gives

$$\sum F_x = mg\sin\theta - f_k = ma$$

$$\sum F_y = n - mg\cos\theta = 0$$

$$f_k = mg\sin\theta - ma = (25\,\text{kg})(9.8\,\text{m/s}^2)\sin 40^\circ - (25\,\text{kg})(0.1\,\text{m/s}^2)$$

$$= 155\,\text{N}$$

$$n = mg\cos\theta = (25\,\text{kg})(9.8\,\text{m/s}^2)\cos 40^\circ = 187.7\,\text{N}$$

The coefficient of kinetic friction is

$$\mu_k = \frac{f_k}{n}$$

$$\mu_k = \frac{(120\,\text{N})}{(212.2\,\text{N})} = 0.57$$

(c) At the verge of slipping the force of static friction is maximum:

$$f_{s\,\max} = \mu_s n$$

Applying Newton's second law we get

$$\sum F_x = mg\sin\theta - f_{s\,\max} = 0$$

$$f_{s\,\max} = mg\sin\theta$$

Also we have $n = mg\cos\theta$, therefore

$$\mu_s = \frac{mg\sin\theta}{mg\cos\theta} = \tan\theta$$

Since $\mu_s = 0.74$ from Table 3.1, we have

$$\theta = \tan^{-1} 0.74 = 36.5^\text{o}$$

Example 3.8 Two masses $m_1 = 2\,\text{kg}$ and $m_2 = 5\,\text{kg}$ are connected by a massless cord that passes over a massless and frictionless pulley (Atwood's machine) as shown in Fig. 3.11. Find the acceleration of the system and the tension in the cord.

Solution 3.8 The free-body diagram of each mass is shown in Fig. 3.11. Applying Newton's second law to each block (taking positive y to be upwards) gives

$$\sum F_{1y} = T - m_1 g = m_1 a$$

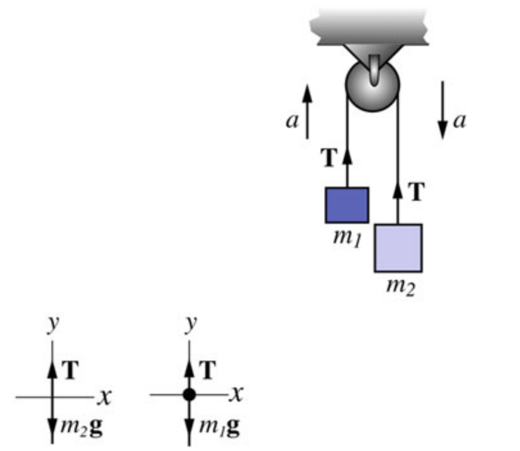


Fig. 3.11 The free-body diagram of an Atwoodõs machine

$$\sum F_{2y} = T - m_2 g = -m_2 a$$

adding the two equations we have

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) g = \left(\frac{3\,\text{kg}}{7\,\text{kg}}\right)(9.8\,\text{m/s}^2) = 4.2\,\text{m/s}^2$$

Substituting a in any of the two equations gives

$$T = \left(\frac{2m_1 m_2}{m_1 + m_2}\right) g = \left(\frac{2(10\,\text{kg})}{(7\,\text{kg})}\right)(9.8\,\text{m/s}^2) = 28\,\text{N}$$

Example 3.9 Two blocks of masses $m_1 = 1.5\,\text{kg}$ and $m_2 = 3.2\,\text{kg}$ are connected by a light string that passes over a massless frictionless pulley as shown in Fig. 3.12. If the surface is frictionless: (a) what is the distance that m_2 will drop during the first 0.6 s? (b) if a third block is attached to m_1 using strong glue, what must its mass be such that the system moves with constant speed?

Solution 3.9 (a) The acceleration value is the same for both masses since they are connected by a string. Figure 3.12 shows the free-body diagram for each mass. Applying Newton's law to m_1 and m_2 in the direction of motion we have for m_2

$$\sum F_{2y} = T - m_2 g = -m_2 a$$

and for m_1

$$\sum F_{1x} = m_1 g\sin\theta - T = -m_1 a$$

from this we have

$$T = m_2(g - a)$$

and

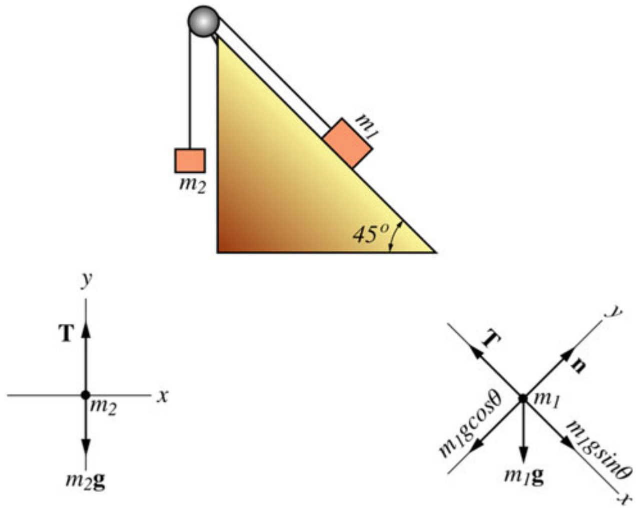


Fig. 3.12 The free-body diagram showing the forces on each block

$$m_1 g \sin\theta - m_2(g - a) = -m_1 a$$

and therefore

$$a = \frac{(m_2 - m_1\sin\theta)g}{m_1 + m_2} = \frac{((3.2\,\text{kg}) - (1.5\,\text{kg})\sin 45^{\text{o}})(9.8\,\text{m/s}^2)}{4.7\,\text{kg}}$$
$$= 4.5\,\text{m/s}^2$$

After 0. 6s, the distance that m_2 falls is

$$h = \frac{1}{2}at^2 = \frac{1}{2}(-4.5\,\text{m/s}^2)(0.6\text{ s})^2 = -0.81\,\text{m}$$

(b) If the system moves with constant speed, its acceleration is zero

$$\sum F_{1x} = (m_1 + m_3)g\sin\theta - T = 0$$

and

$$\sum F_{2y} = T - m_2 g = 0$$

that gives

$$T = m_2 g$$

and

$$(m_1 + m_3)g\sin\theta - m_2 g = 0$$

hence

$$m_3 = \frac{m_2 - m_1\sin\theta}{\sin\theta} = \frac{((3.2\,\text{kg}) - (1.5\,\text{kg})\sin 45^{\text{o}})}{\sin 45°} = 3.03\,\text{kg}$$

Example 3.10 A 3 kg block is hanged from the ceiling as in Fig. 3.13. Find the magnitude of T_1, T_2, and T_3.

Solution 3.10 The free-body diagrams of the block and the knot are shown in Fig. 3.14. From Newton's second law T_3 is equal to the weight of the block, i.e.,

Fig. 3.13 A block hanged from the ceiling

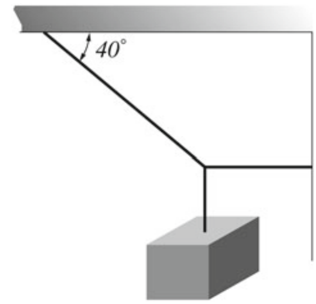
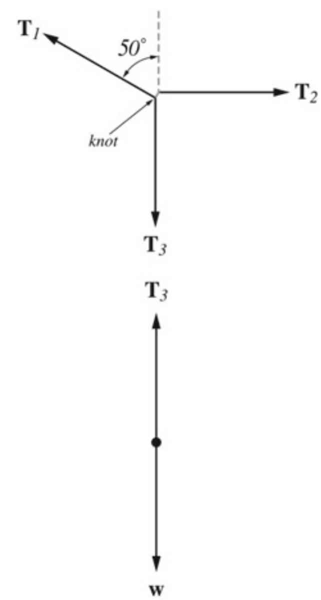


Fig. 3.14 The free-body diagram of the block



$$T_3 = w = mg = (3\,\text{kg})(9.8\,\text{m/s}^2) = 29.4\,\text{N}$$

For the knot, we have

$$\sum F_x = T_2 - T_1\sin 50^{\text{o}} = 0 \qquad (3.1)$$

$$\sum F_y = T_1\cos 50^{\text{o}} - T_3 = 0 \qquad (3.2)$$

Solving for T_1 from Eq. 3.2 gives

$$T_1 = \frac{T_3}{\cos 50^{\text{o}}} = \frac{(29.4\,\text{N})}{(0.64)} = 45.7\,\text{N}$$

Substituting this result into Eq. 3.1 we get

$$T_2 = T_1\sin 50° = (45.7\,\text{N})(0.76) = 35\,\text{N}$$

Example 3.11 Figure 3.15 shows a weight of 200 N that is lifted with a constant speed. Find the tension in each part of the rope and the force of lift.

Solution 3.11 Since the pulleys are massless and frictionless we have $2T_1 = T_2$ (T_1 is the tension in each rope)and $T_2 =$

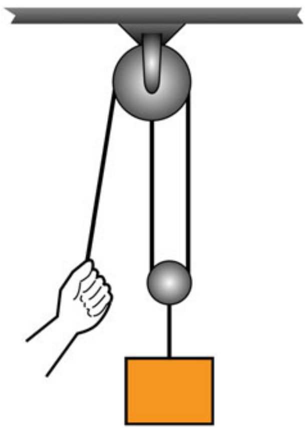


Fig. 3.15 Using two pulleys to reduce the force necessary to lift a weight

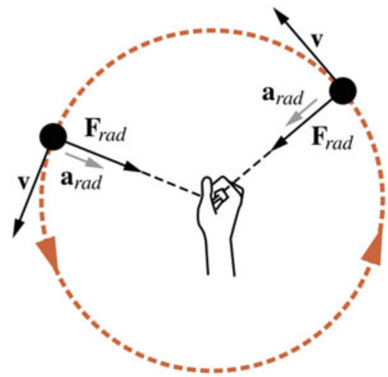


Fig. 3.16 An object attached to a string in uniform circular motion

$w = 200$N, thus

$$T_1 = \frac{T_2}{2} = \frac{(200\,\mathrm{N})}{2} = 100\,\mathrm{N}$$

we also have

$$T_1 = F = 100\,\mathrm{N}$$

3.4.1 Uniform Circular Motion

In Sect. 2.4.5, it was found that a particle moving in a circle with a constant linear speed v (uniform circular motion) has a centripetal acceleration directed towards the center of the circle. Its magnitude is given by

$$a_{rad} = \frac{v^2}{r}$$

where r is the radius of the circle. Figure 3.16 shows an object attached to a string in uniform circular motion (the plane of motion is parallel to the Earth's surface). From Newton's second law, the centripetal acceleration is caused by a force or net force directed towards the center of the circle. Therefore, as a_{rad}, the centripetal (or radial) force F_{rad} has a constant magnitude but its direction changes continuously The magnitude of this centripetal force is given by

$$|\Sigma\mathbf{F}| = F_{rad} = ma_{rad} = m\frac{v^2}{r}$$

If at some instant the radial force becomes zero, the object would then move along a straight line path tangent to the circle. Hence, the centripetal force is necessary to keep the object in its circular path. The centripetal force may be any kind of force such as friction, gravity, or tension.

Example 3.12 A conical pendulum consists of a bob of mass m attached to a light string rotating in a horizontal circle as in Fig. 3.17. If the bob rotates with a constant speed and if θ and m are known, find: (a) the tension in the string; (b) the speed of the bob; (c) the period of motion.

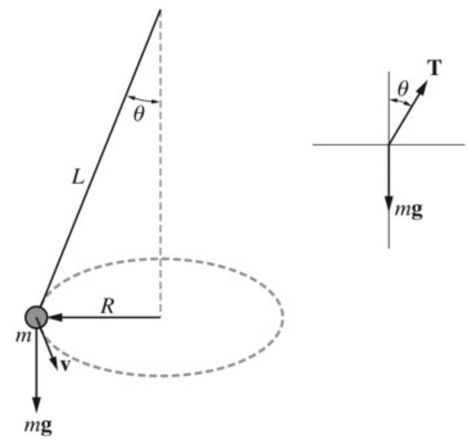


Fig. 3.17 A conical pendulum consisting of a bob of mass m attached to a light string rotating in a horizontal circle

Solution 3.12 The horizontal component of the tension force supplies the required centripetal force to keep the bob in its circular path while the vertical component balances the weight of the bob. (a) Applying Newton's second law in both the x- and y-directions we have

$$T\cos\theta - mg = 0$$

$$T = \frac{mg}{\cos\theta} \tag{3.3}$$

and

$$T\sin\theta = \frac{mv^2}{R} \tag{3.4}$$

(b) Dividing Eq. 3.4 by Eq. 3.3 gives

$$v = \sqrt{gR\tan\theta}$$

(c) The period of motion is given by

$$\tau = \frac{2\pi R}{v} = 2\pi \sqrt{\frac{R}{g \tan\theta}}$$

Since $R = L\sin\theta$, we have

$$\tau = 2\pi \sqrt{\frac{L\cos\theta}{g}}$$

Example 3.13 (a) A car needs to turn on a level road without skidding as in Fig. 3.18. Find the maximum speed for which the car can take the curved path of the level road safely (b) If the road is banked, i.e., the outer edge is raised relative to the inner edge as in Fig. 3.19, find the maximum speed for which the car can take the curved path of the level road safely without depending on friction.

Solution 3.13 (a) The centripetal force required for the car to remain in its circular path, is in this case, is the force of static friction. The maximum speed for which the car can take the curve without skidding is when the static frictional force is a maximum. That is,

$$f_{s\,\max} = \mu m g = \frac{m v_{\max}^2}{r}$$

Hence

$$v_{\max} = \sqrt{\mu r g}$$

(b) If the road is banked the car can take the turn without depending on friction as the required centripetal force. In that case, the horizontal component of the normal force supplies

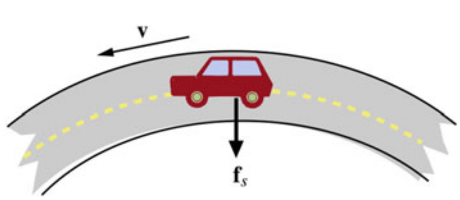


Fig. 3.18 A car turning without skidding

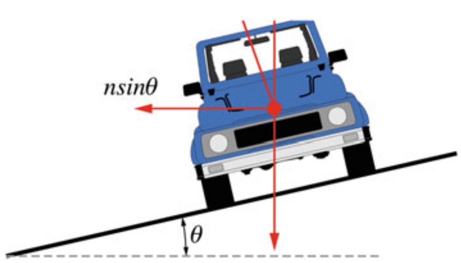


Fig. 3.19 A car turning on a banked road

the necessary centripetal force. Thus we have

$$n\sin\theta = \frac{mv^2}{r}$$

and

$$n\cos\theta = mg$$

where v is the speed of the car. Dividing these two equations gives

$$\tan\theta = \frac{v^2}{rg}$$

If the angle θ and the curvature r are known, then the safe speed limit can be found. If the car moves at a speed lower or higher than that speed then the frictional force must supply the additional centripetal force for the car to stay in its circular path.

3.4.2 Nonuniform Circular Motion

In Sect. 2.4.6, we saw that an object in nonuniform circular motion has both perpendicular (centripetal) and parallel components of acceleration given by

$$a_{rad} = \frac{v^2}{r}$$

and

$$a_t = \frac{d|\mathbf{v}|}{dt}$$

The total acceleration is

$$\mathbf{a} = \mathbf{a}_{rad} + \mathbf{a}_t$$

These radial and tangential accelerations are caused by radial and tangential forces respectively (see Fig. 3.20):

$$F_{rad} = m a_{rad} = m\frac{v^2}{r}$$

$$F_t = m a_t = m\frac{d|\mathbf{v}|}{dt}$$

The net force is

$$\mathbf{F} = \mathbf{F}_{rad} + \mathbf{F}_t$$

Example 3.14 An object attached to a light string is rotating in a vertical circle of radius r (see Fig. 3.21). Find: (a) the tension in the cord at the lowest and highest points; (b) the minimum speed at the highest point such that the object remains in its circular path.

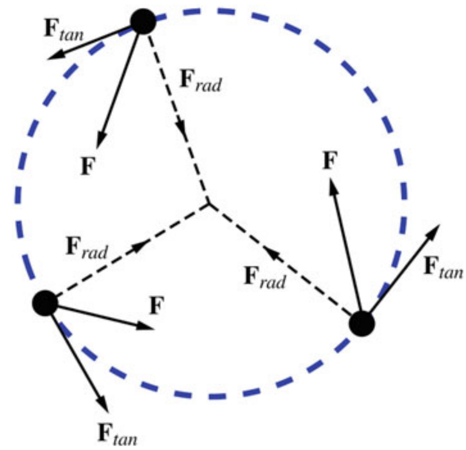


Fig. 3.20 The radial and tangential forces in nonuniform circular motion

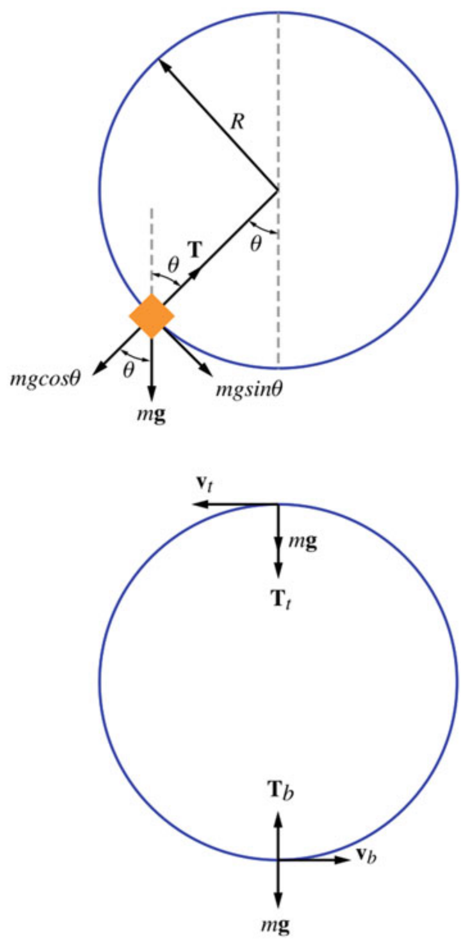


Fig. 3.21 An object attached to a string rotating in a vertical circle

Solution 3.14 (a) Applying Newton's second law in both the tangential and radial directions gives

$$mg\sin\theta = ma_t$$

and

$$T - mg\cos\theta = \frac{mv^2}{r}$$

At the bottom $\theta = 0$ and therefore $a_t = 0$ and

$$T_b = m\left(\frac{v_b^2}{r} + g\right)$$

At the top $\theta = 180^0$ and $a_t = 0$ and

$$T_t = m\left(\frac{v_t^2}{r} - g\right)$$

(b) For the object to remain in its circular path, the string must remain taut, i.e., T_t must be positive ($T_t > 0$). If $T_t = 0$ then $v_t = \sqrt{gr}$. Hence, the velocity must satisfy $v_t > \sqrt{gr}$.

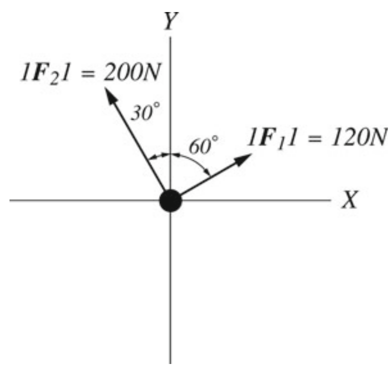


Fig. 3.22 An object subjected to two forces acting in different directions

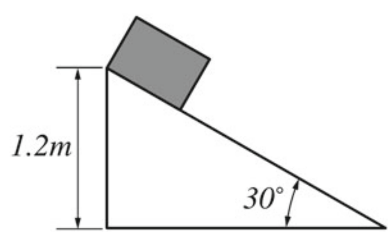


Fig. 3.23 A block released from the top of an incline

Fig. 3.24 Two masses connected by a light string over a frictionless pulley of negligible mass

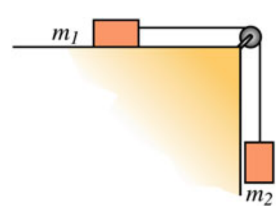


Fig. 3.25 A block hanged from the ceiling

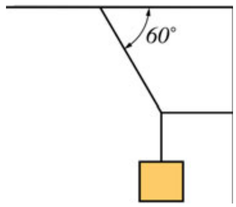
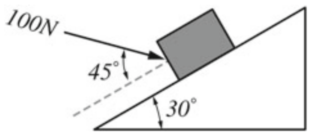


Fig. 3.26 A block held in equilibrium



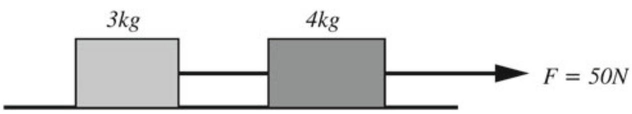


Fig. 3.27 Two blocks connected by a light rope and pulled by a force

Problems

1. A 4 kg object is exposed to two forces (see Fig. 3.22). Find the magnitude and direction of the acceleration of the object.
2. A 0.2 kg block is released from the top of an inclined plane of angle 30° as in Fig. 3.23. Find the speed of the block just as it reaches the bottom.
3. Two masses are connected by a light string that is connected to a frictionless pulley of negligible mass as in Fig. 3.24. Find the magnitude of the acceleration of the masses and the tension in the string.
4. A block is pushed up along a smooth inclined plane of angle of 45° where it is given an initial velocity of 8 m/s. Determine the time it takes the block to return to its initial position.
5. Find the tension in each string in the system shown in Fig. 3.25.
6. A 5 kg block is held in equilibrium as in Fig. 3.26. Find the normal force acting on the block.
7. Find the normal force exerted on a 70 kg man standing inside an elevator that is accelerating upwards at a rate of $2\,\text{m/s}^2$.
8. Find the acceleration of the system shown in Fig. 3.27 and the tension in the string if $m_1 = 3$ kg and $m_2 = 4$ kg (assume massless string and frictionless surface).

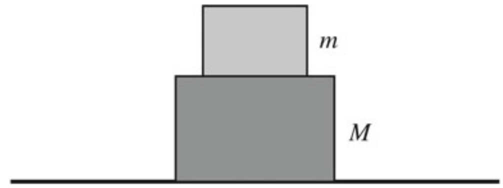


Fig. 3.28 Two blocks placed on top of each other, where a horizontal force is applied to the lower block

Fig. 3.29 A car moving on a curved path

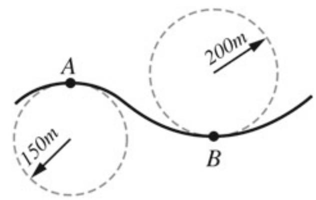
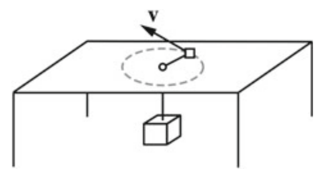


Fig. 3.30 A block of mass *m* on a frictionless table is attached to light string that passes through the center of the table and is connected to a larger block of mass *M*



9. Two blocks of masses 3 and 5 kg are placed on top of each other as in Fig. 3.28. If the coefficient of static friction between the blocks is 0.2 and assuming there is no friction between the lower block and the surface on which it rests, find the maximum horizontal force that can be applied to the lower block such that the blocks move together.
10. A 1000 kg car move along the track shown in Fig. 3.29. Find (a) the maximum speed the car can have at point A such that it does not leave the track (b) the normal force exerted on the car at B if its speed there is 15 m/s.
11. A block of mass m on a frictionless table is attached to light string that passes through the center of the table and is connected to a larger block of mass M (see Fig. 3.30). If m moves in uniform circular motion of radius r and speed v, find v such that M remains at rest when released.
12. A 1 kg particle moves in the force field given by $\mathbf{F} = 2t\mathbf{i} + (5t - 1)\mathbf{j} - 6t^2\mathbf{k}$. Find the position of the particle at any time if at $t = 0$, $\mathbf{r}_0 = \mathbf{0}$, and $\mathbf{v}_0 = \mathbf{0}$.

4 Work and Energy

4.1 Introduction

Energy is a very important concept that is heavily used in everyday life. Everything around us, including ourselves, needs energy to function. For example, electricity provides home appliances with the energy they require, food gives us energy to survive, and the sun provides earth with the energy needed for the existence of life!

Experiments show that energy is a scalar quantity related to the state of an object. Energy may exist in various forms: mechanical, chemical, gravitational, electromagnetic, nuclear, and thermal. Furthermore, energy cannot be created or destroyed; it can only be transformed from one form to another. In other words, if energy were to be exchanged between objects inside a system, then the total amount of energy (the sum of all forms of energy) in the system will remain constant.

A transformation of energy occurs due to the action of a force known as work or due to heat exchange between objects (or between an object and its environment). If energy is transferred due to work then it may be defined as the capacity of doing work. This book is concerned with mechanical energy which involves kinetic energy (associated with the object's motion) and potential energy (associated with the position of the object in space).

4.2 Work

Work may have many meanings. Sometimes, work is said to be done when a muscular activity is performed. Work may also refer to mental activity (mental work). In physics, the definition of work is different. Work is said to be done if a force is applied to an object while it is moving, i.e., if there is no resulting displacement, no work is done. Suppose that a person holds a heavy box for sometime and then starts to feel tired. The reason he/she feels tired is because chemical energy in his/her body is converted into internal microscopic motions of the muscles. Since the energy is not transferred to the box being carried (the box did not move), the work done on the box is equal to zero.

4.2.1 Work Done by a Constant Force

Consider an object exposed to a constant force F (see Fig. 4.1). If the object is displaced through a displacement s, then the work done on the object is a scalar quantity defined as

$$W = Fs\cos\theta = \mathbf{F}\cdot\mathbf{s}$$

where θ is the smaller angle between $\mathbf{F}$ and $\mathbf{s}$. The component of $\mathbf{F}$ in the direction of $\mathbf{s}$ ($F\cos\theta$) is the only effective component that produces motion. The work done represents energy transferred to or from the object via that force. If ($\theta = 0$), the work done on the object is positive, i.e. energy is transferred to the object. If ($\theta = 180^{o}$), the work done is negative, i.e., energy is transferred from the object. The SI unit of work is Newton meter (N.m) also named as the Joule.

$$1\text{ Joul } = 1\text{ J} = 1\text{ kg.m}^2/\text{s}^2$$

Note that energy and work have the same units.

4.2.2 Work Done by Several Forces

Consider an object exposed to several forces as in Fig. 4.2. The work done by all of these forces is the sum of the individual amounts of work done by each force:

$$W = \mathbf{F}_1\cdot\mathbf{s} + \mathbf{F}_2\cdot\mathbf{s} + \mathbf{F}_3\cdot\mathbf{s} + \cdots$$

$$W = W_1 + W_2 + W_3 +$$

S. Alrasheed, *Principles of Mechanics*, Advances in Science, Technology & Innovation, https://doi.org/10.1007/978-3-030-15195-9_4

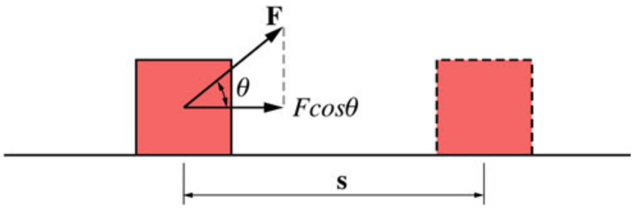


Fig. 4.1 An object exposed to a constant force **F** and undergoes a displacement of **s**

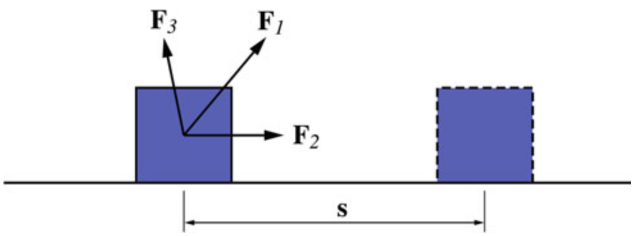


Fig. 4.2 An object exposed to several forces undergoes a displacement of **s**

Another method to find the work is by considering the resultant of these forces:

$$W = \left(\sum \mathbf{F}\right) \cdot \mathbf{s}$$

where

$$\Sigma\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 +$$

Example 4.1 A lady pulls an 80 kg block horizontally on a rough surface by a constant force of 400 N that is at 20^o to the horizontal. If the block is pulled a distance of 6 m and if the opposing force of friction has a magnitude of 118 N : (a) determine the work done on the block by each of the applied force, the frictional force, the normal force, and the force of gravity; (b) find the total work done on the block; (c) determine if it is easier for the lady to pull the block at an angle larger than 20°.

Solution 4.1 (a) The work done by the applied force is

$$W_{app} = \mathbf{F}\cdot\mathbf{s} = Fs\cos\theta = (400\,\text{N})(6\,\text{m})\cos 20^\circ = 2255.3\,\text{J}$$

The work done by the frictional force is

$$W_f = Fs\cos\theta = (118\,\text{N})(6\,\text{m})\cos 180^\circ = -708\,\text{J}$$

The work done by the normal force and the force of gravity are both zero since each force is perpendicular to the displacement.

(b) The total work done is

$$W_{tot} = W_{app} + W_f = (2255.3\,\text{J}) - (708\,\text{J}) = 1547.3\,\text{J}$$

The total work done can also be found by computing the net force acting on the block and calculating its work.

(c) For $(0 \leq \theta \leq 90^o)$, If $\theta_2 > \theta_1$, then $\cos\theta_2 < \cos\theta_1$ and therefore $W_{app2} < W_{app1}$, i.e., it is easier for the man to pull at an angle larger than 20^o.

Example 4.2 A delivery man wants to push a crate up a ramp of length s: (a) find the minimum work the man must do to lift the crate to the top of the ramp; (b) determine if a ramp with a steeper incline would be more difficult for the man to push the crate.

Solution 4.2 (a) The minimum work that the delivery man must do is the work done against gravity The work done on the crate by the force of gravity is

$$W_g = -mgs\sin\theta$$

Hence the minimum work W_w that the delivery man must do is equal to $+mgs\sin\theta$.

(b) For angles between 0 and 90^o, if $\theta_2 > \theta_1$, then $\sin\theta_2 > \sin\theta_1$. Hence $W_{w2} \geq W_{w1}$, i.e., the more inclined the ramp is the more difficult it is to move the crate.

4.2.3 Work Done by a Varying Force

Previously, the work done in the special case of a force that is constant in both magnitude and direction was discussed. The object there moved along a straight line. In many situations, the force may vary in magnitude or in direction or in both, and the object may move along a curved path. To find the work done in this case, consider a particle moving along the curved path shown in Fig. 4.3. While it is moving, a force **F** that varies in both magnitude and direction with the position of the particle acts on it. Let us divide the path into a large number n of very small displacements where each is tangent to the path. For each displacement, the force can be approximated to be constant in both magnitude and direction. The total work done as the particle moves from P to Q is the sum of the individual amounts of work done along each displacement, that is

$$W = \mathbf{F}_1 \cdot \triangle\mathbf{r}_1 + \mathbf{F}_2 \cdot \triangle\mathbf{r}_2 + \mathbf{F}_3 \cdot \triangle\mathbf{r}_2 + \cdots \mathbf{F}_n \cdot \triangle\mathbf{r}_n$$

$$W = \sum_{i=1}^{n} \mathbf{F}_i \cdot \triangle\mathbf{r}_i$$

By dividing the path into more displacements we have

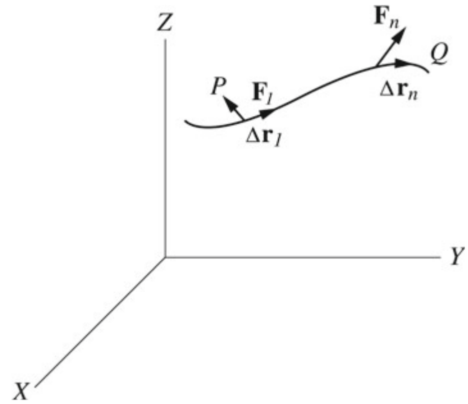


Fig. 4.3 A particle moving along a curved path. While itõs moving, a force **F** that varies in both magnitude and direction with the position of the particle acts on it

$$W = \lim_{\Delta\mathbf{r}_i \to 0} \sum_{i=1}^{n} \mathbf{F}_i \cdot \Delta\mathbf{r}_i$$

or

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_P^Q \mathbf{F} \cdot d\mathbf{r}$$

As mentioned in Sect. 1.10.1, this integral is called the line integral. Each component of $\mathbf{F}(F_x,\ F_y$ or $F_z)$ may be a function of x, y, and z, and the curve can be determined by its equations that relates x, y, and z to each other. The component form of the above equation is

$$W = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r} = \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy + \int_{z_i}^{z_f} F_z dz \tag{4.1}$$

Now consider the case in which the particle moves along a straight line (for example the positive x-axis) and in which the force acting on the particle has a constant direction along the x-axis and a magnitude that varies with x. Equation 4.1 is then reduced to

$$W = \int_{x_i}^{x_f} F_x(x) dx \tag{4.2}$$

This equation represents the area under the curve in Fig. 4.4. If $F(x)$ is constant then we have

$$W = \int_{x_i}^{x_f} F_x(x) dx = F \int_{x_i}^{x_f} dx = F(x_f - x_i) = Fs$$

The work is then equal to the rectangular area shown in Fig. 4.5.

Example 4.3 In Example 3.3, find the work done by the force in moving the particle during the time interval from $t = 0$ to $t = 1$ s.

Solution 4.3 The work done from $t = 0$ to $t = 1$ s is

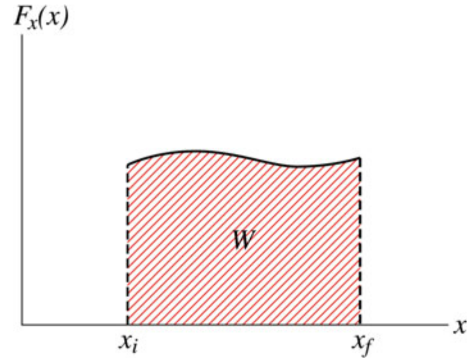


Fig. 4.4 The area under the curve represents the work

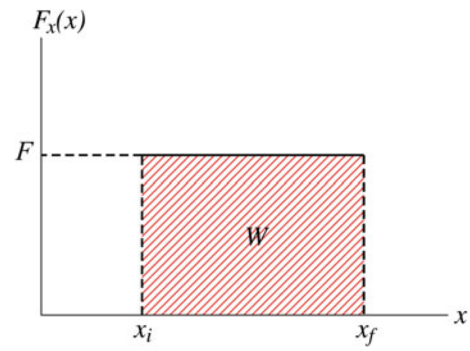


Fig. 4.5 The work is equal to the rectangular area

$$W = \int_{t=0}^{t=1} \mathbf{F} \cdot \mathbf{dr} = \int_{t=0}^{t=1} (2t^2\mathbf{i} - 3t\mathbf{j}).(1/3t^4\mathbf{i} - t^3\mathbf{j})dt$$

$$= \int_{t=0}^{t=1} (0.66t^6 + 3t^4)dt$$

$$= (0.1t^7 + 0.6t^5)|_{t=0}^{t=1} = 0.7\,\text{J}$$

Example 4.4 A force acting on a particle is a function of position according to Fig. 4.6. Find the work done by this force as the particle moves from $x_i = 0$ to $x_f = 9$ m.

Solution 4.4 The work done is equal to the area of the triangle under the curve between $x_i = 0$ to $x_f = 9$ m, i.e.

$$W = \frac{1}{2}(9\,\text{m})(4\,\text{N}) = 18\,\text{J}$$

Example 4.5 A ball that is suspended from a ceiling by a light rope is displaced a small distance to the position shown in Fig. 4.7. If it is released from rest at B, find the work done by the tension force and the force of gravity as the ball moves from B to A.

Solution 4.5 Because the tension force is always perpendicular to the displacement, the work done by it is zero at all times. The only component of the gravitational force that does work is its tangential component. Therefore,

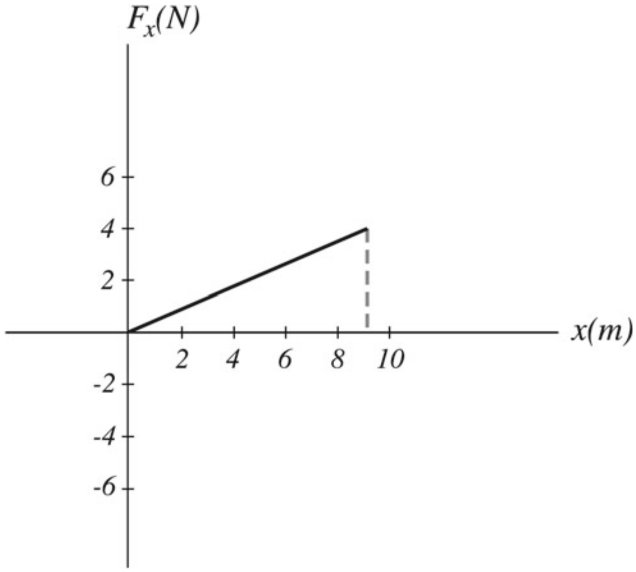


Fig. 4.6 A force acting on a particle is a function of position

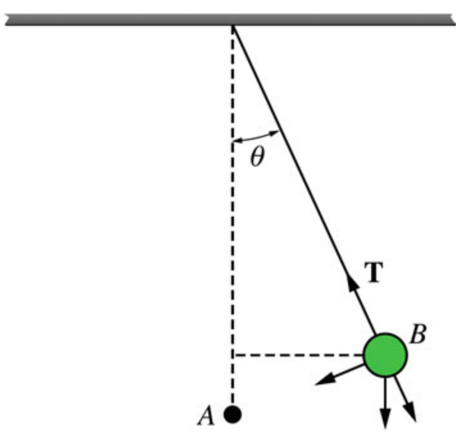


Fig. 4.7 A ball suspended by a light rope and displaced a small distance from the position of equilibrium

$$W = \int_0^{\theta_0} \mathbf{F} \cdot d\mathbf{s} = \int_0^{\theta_0} mg \sin\theta \cos(0)\mathrm{ds}$$

Since $s = R\theta$, then $ds = Rd\theta$, and we have

$$W = mgR \int_0^{\theta_0} \sin\theta d\theta = -mgR\cos\theta|_0^{\theta_0} = mgR(1 - \cos\theta_0)$$

4.3 Kinetic Energy (KE) and the Work–Energy Theorem

Consider a particle that is exposed to a net field force and is moving along a curve in space. Suppose that the particle is at P at t_i and at Q at t_f and that its velocity at P and Q is $\mathbf{v}_i$ and $\mathbf{v}_f$, respectively. The net work done on the particle as it moves from P to Q is then given by

$$W = \int_P^Q \mathbf{F} \cdot d\mathbf{r} = \int_{t_i}^{t_f} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_{t_i}^{t_f} \mathbf{F} \cdot \mathbf{v} dt = \int_{t_i} m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} dt$$

$$= m\int_{\mathbf{v}_i}^{\mathbf{v}_f} \mathbf{v} \cdot d\mathbf{v} = \frac{1}{2}m\int_{\mathbf{v}_i}^{\mathbf{v}_f} d(\mathbf{v} \cdot \mathbf{v}) = \frac{1}{2}m(\mathbf{v} \cdot \mathbf{v})|_{\mathbf{v}_i}^{\mathbf{v}_f}$$

$$= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

The quantity $\frac{1}{2}mv^2$ is the energy associated with the motion of the particle called the kinetic energy (KE). Thus, if a particle of constant mass m is moving with a speed v, its KE is a scalar quantity defined as

$$K = \frac{1}{2}mv^2$$

It also can be written as $K = \frac{1}{2}m(\mathbf{v} \cdot \mathbf{v})$. Hence, the total work done by the net force in displacing the particle is equal to the change in the KE of the particle

$$W_{net} = K_f - K_i = \triangle K$$

Similar to work, the SI unit of kinetic energy is the Joul. Note that the work–energy theorem is applied only if the object is treated as a particle (all of its parts move in exactly the same way). As an example of how the theorem is applied only for particle-like objects consider a man standing on a skateboard on a horizontal surface (see Fig. 4.8). If the man

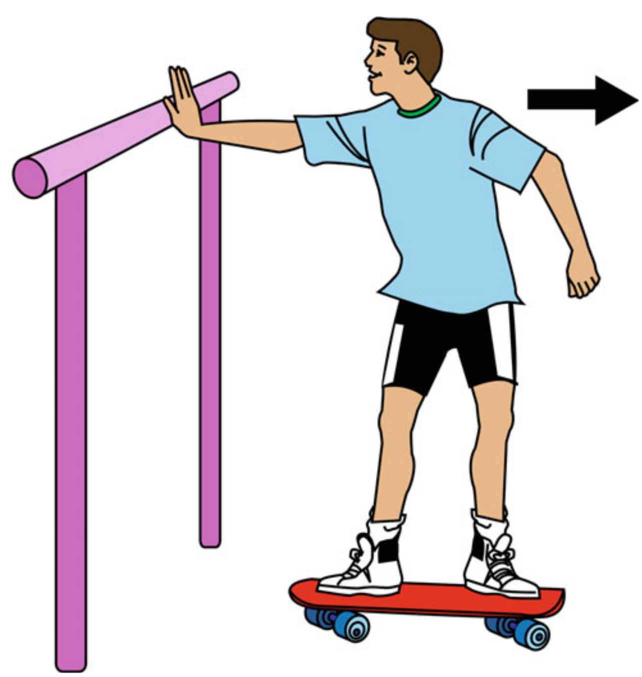


Fig. 4.8 The center of mass of the system (man+skateboard) moves and the work-energy theorem can be applied to that point

pushes the bar then that would move him backwards along with his skateboard. This motion is due to the reaction force **F** exerted on him by the bar. The work done by **n** or **w** is equal to zero since each force is perpendicular to the displacement. Because the point of application of **F** did not move it follows that the work done by that force is zero. Thus, from the work–energy theorem the man should not move. The question is why did he move?

The fact here is that it is incorrect to treat the man as a particle, since different parts of his body move in different ways as he pushes the bar. Therefore, the work–energy theorem does not hold. The man must be treated as a system of particles. In Chap. 6, we will see that the motion of a system of particles can be represented by the motion of its center of mass. The center of mass behaves as if all of the mass of the object (or system) is concentrated there and as if the net external force is applied there. In the case of the skateboarder, the center of mass of the system (man + skateboard) moves and the work–energy theorem can be applied to that point.

The work–energy theorem is an alternative method for describing motion without using Newton's laws. It is especially useful in problems involving a varying force. Note that the work and the kinetic energy are not invariant quantities; they have different values when measured in different inertial frames of reference. However, from the principle of invariance, the equation $W_{net} = \triangle K$ still holds for any inertial frame.

Example 4.6 A 5 kg block resting on a surface is given an initial velocity of 5 m/s. If the coefficient of kinetic friction of the surface is $\mu_k = 0.2$, find the distance the block would move before it stops.

Solution 4.6 As we will see later in Sect. 4.3.1, the change in the kinetic energy of the block due to friction is $\triangle K = -f_k s$, where s is the displacement of the block.

$$W_f = \triangle K = -f_k d = -\mu_k mgd = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

$$= -(0.2)(5\,\text{kg})(9.8\,\text{m/s}^2)d = 0 - \frac{1}{2}(5\,\text{kg})(5\,\text{m/s})^2$$

$$d = 6.4\,\text{m}$$

Example 4.7 A 10 kg block is pushed on a frictionless horizontal surface by a constant force of magnitude of 100 N and that is at 30° below the horizontal. If the block starts from rest, find its final speed after it has moved a distance of 3 m using work–energy theorem.

Solution 4.7

$$W = \text{F} \cdot \text{s} = Fs\cos\theta = (100\,\text{N})(3\,\text{m})\cos(-30^\circ) = 259.8\,\text{J}$$

From the work–energy theorem, we have

$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

since $v_i = 0$ we get

$$v_f^2 = \frac{2W}{m} = \frac{2(259.8\,\text{J})}{(10\,\text{kg})} = 52\,\text{m}^2/\text{s}^2$$

$$v_f = 7.2\,\text{m/s}$$

4.3.1 Work Done by a Spring Force

Consider a block attached to a light spring fixed at the other end on a frictionless horizontal surface as in Fig. 4.9. Suppose an external force $\mathbf{F}_{ext}$ is applied to the block by either stretching or compressing it through a small displacement from its equilibrium (relaxed) position taken at $x = 0$. The spring will then exert a restoring force $\mathbf{F}_s$ on the block that opposes the applied force and restores the block to its equilibrium position. For many kinds of springs and in the case of small displacements, the spring force varies linearly with the displacement x of the block (or any other object) from its equilibrium position ($x = 0$). That is

$$F_s = -kx$$

where k is a constant called the force or spring constant. k measures the stiffness of the spring. The stiffer the spring the larger is k. This equation is known as Hook's law. The minus sign indicates that the spring force is always acting in a direction opposing the displacement. The work done by the spring force in moving the block from an initial position x_i to a final position x_f is:

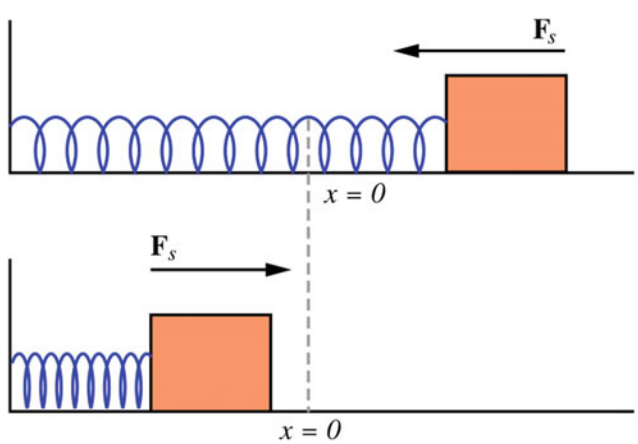


Fig. 4.9 A block attached to a light spring on a frictionless surface

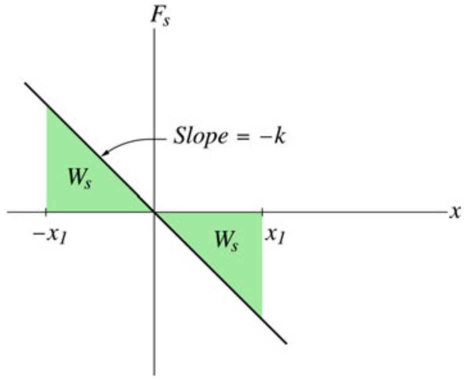


Fig. 4.10 A plot of F_s versus x for the mass-spring system

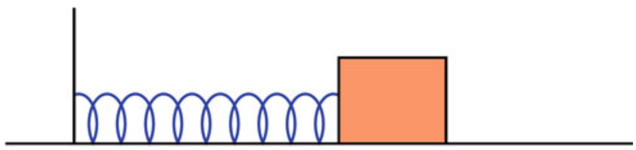


Fig. 4.11 A 2 kg block attached to a light spring of force constant 300 N/m on a horizontal smooth surface

$$W_s = \int_{x_i}^{x_f} F_x dx = \int_{x_i}^{x_f} (-kx)dx = -k\int_{x_i}^{x_f} x dx$$

$$W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$$

The work done on the block by the spring as it moves from an initial position $x_i = x$ to a final position $x_f = 0$ is

$$W_s = \frac{1}{2}kx^2$$

Figure 4.10 shows a plot of F_s versus x for the mass–spring system.

Example 4.8 A 2 kg block is attached to a light spring of force constant 300 N/m on a horizontal smooth surface as shown in Fig. 4.11. If the system is initially at rest at the position of equilibrium and is then stretched a distance of 3 cm, find the work done by the spring on the block as it moves from $x_i = 0$ to $x_f = 3$ cm.

Solution 4.8

$$W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2 = 0 - \frac{1}{2}(300\,\text{N/m})(0.03\,\text{m})^2 = -0.135\,\text{J}$$

4.3.2 Work Done by the Gravitational Force (Weight)

If a particle-like object of mass m is moving vertically upward or downward near the surface of the earth where g is assumed to be constant (see Fig. 4.12), and if air resistance is neglected, then the only force that does work on the object is the gravitational force mg. By taking the y-axis along the line of motion (positive upwards) with $y = 0$ at the earth's surface, the work done by the gravitational force is

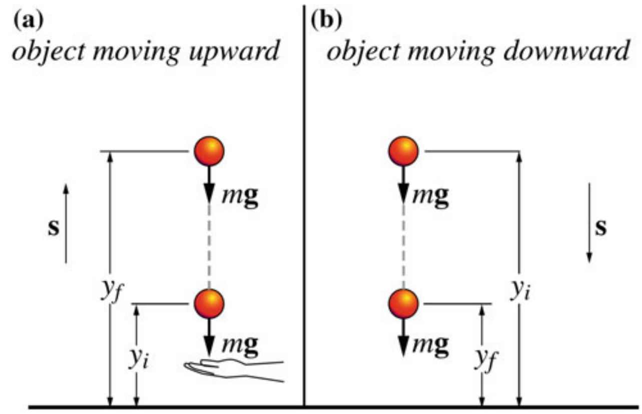


Fig. 4.12 By taking $y = 0$ at the hand level, in the work done by gravity **a** is $-mgy_f$ and in **b** is $+mgy_i$

$$W_g = \int_{y_i}^{y_f} F_y dy = -mg\int_{y_i}^{y_f} dy$$

$$W_g = mgy_i - mgy_f$$

Note that unlike the spring force the reference point y_i may be chosen anywhere. If the object moved downwards from $y_i = y$ to $y_f = 0$, the work done by the gravitational force is

$$W_g = mgy$$

Now suppose the object moves along a curved path from P to Q as in Fig. 4.13. The work done by the gravitational force is

$$W = \int_P^Q \mathbf{mg} \cdot d\mathbf{s} = -\int_P^Q mg\mathbf{i} \cdot d(dx\mathbf{i} + dy\mathbf{j}) = -\int_{y_i}^{y_f} mgdy = mgy_i - mgy_f$$

This result is the same as if the object has followed a straight vertical path. Therefore, the work done by the gravitational force depends only on the initial and final positions of the object.

Example 4.9 A man lifts a 300 kg weight a distance of 2 m above the ground. Find the work done by the force of gravity on the weight.

Solution 4.9

$$W = mgy_i - mgy_f = 0 - (300\,\text{kg})(9.8\,\text{m/s}^2)(2\,\text{m}) = -5880\,\text{J}$$

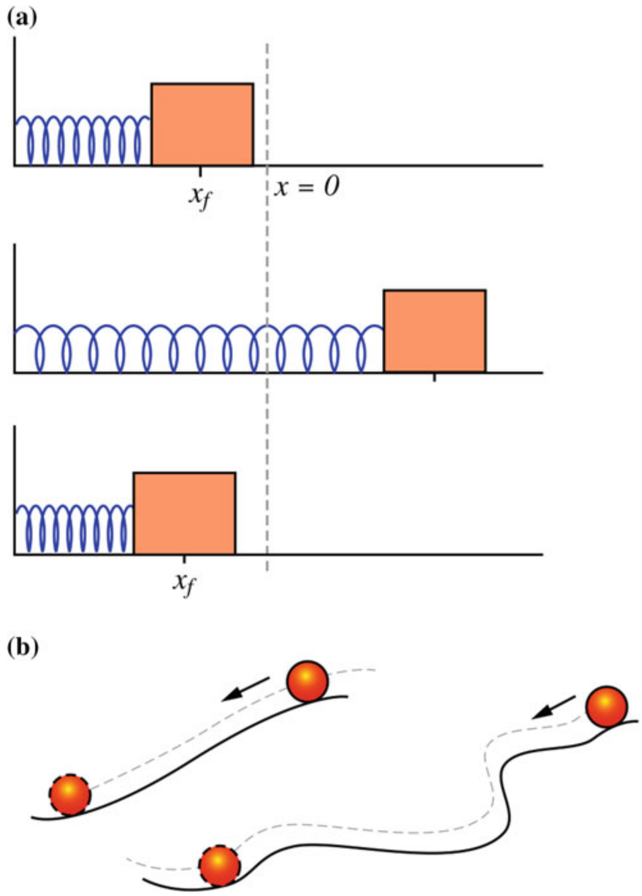


Fig. 4.13 **a** The total work done by the spring force on the block is zero since $x_i = x_f$. **b** Along any path the work done by the gravitational force is the same since the initial and final positions are the same

4.3.3 Power

Power is a quantity that defines how much work is done over a period of time, i.e., power is the time rate of doing work, or more generally, it is the time rate of energy transfer. If an external force F does work W on an object for a time interval $\triangle t$, then the average power during that time is

$$\overline{P} = \frac{W}{\triangle t}$$

The instantaneous power is

$$P = \lim_{\triangle t \to 0} \frac{W}{\triangle t} = \frac{dW}{dt}$$

Our concern in this book is the mechanical power since it involves mechanical work. If y is the velocity of the object, we have

$$P = \frac{dW}{dt} = \mathbf{F} \cdot \frac{d\mathbf{s}}{dt}$$

for a constant force, or

$$P = \mathbf{F} \cdot \mathbf{v}$$

The SI unit of power is joules per second (J/s) and is called the watt (W).

$$1\ \mathrm{W} = 1\ \mathrm{J/s} = 1\ \mathrm{kg.m^2/s^3}$$

Another unit of power that is often used is the horsepower:

$$1\ \mathrm{hp} = 746\ \mathrm{W}$$

4.4 Conservative and Nonconservative Forces

In nature, there are two kinds of forces: conservative and nonconservative forces. A conservative force is a force that conserves the energy of a system when acting upon it. The action of this force results in changing the kinetic energy of any object in the system. This change will be stored in the system in the form of potential energy. For every conservative force, there is a certain potential energy that is associated with it. Such potential energy can be retransformed into kinetic energy Thus, the total energy of the system would not be dissipated, instead it would be conserved. A force that does not act in this way is said to be a nonconservative force. Properties of a conservative force are given as follows:

1. The net work done by a conservative force on a particle moving from one point to another is independent of the path taken by the particle;
2. The net work done by a conservative force in moving a particle through any closed path is equal to zero.

A force not meeting these conditions is a nonconservative force. As mentioned in Sect. 1.10.2, property 2 of a conservative force can be obtained from property 1 (if **A** is a vector field and the line integral of **A** between any two points is independent of path, then $\oint_C \mathbf{A} \cdot \mathbf{r} = 0$). That is, these two properties are equivalent. Examples of conservative forces in mechanics are the gravitational and spring forces. To show this let us go back to Sects. 4.3.1 and 4.3.2, where the work done by the gravitational force or the spring force was calculated. We have seen that the work done in each case depends only on the initial and final positions of the object. Therefore, the work done by any of these forces is independent of the path joining the initial and final positions. Furthermore, if $(x_i = x_f)$ in the case of the spring or $(y_i = y_f)$ in the case of the gravitational force the net work done is zero. Hence, these forces are conservative.

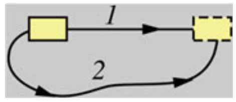


Fig. 4.14 The longer the path the more interaction between the block and the surface and the more the force of friction will act and do work on the block

The force of friction is an example of a nonconservative force. To show that, consider a block sliding on a rough surface. Figure 4.14 shows two possible paths connecting two points. The longer the path the more interaction between the block and the surface and the more the force of friction will act and do work on the block. Thus, the work depends on the path taken between the two points and therefore the frictional force is a nonconservative force.

4.4.1 Potential Energy

For a system consisting of two or more objects, the potential energy U of the system is the energy associated with the configuration of the system. That is, the potential energy is the energy associated with the position of objects in the system relative to each other. If the configuration of the system is changed, then the potential energy of the system also changes. Such energy is defined only in terms of a conservative force because if such a force acts on a system then it can transform the kinetic energy of any object in the system into potential energy of the system and vice versa. The potential energy means that the system has potential to do work.

In Sect. 1.10.2 it has been proven that the line integral in Eq. (1.2) is independent of the path joining the points P and Q if and only if $\mathbf{A} = \nabla\phi$, or equivalently $\nabla \times \mathbf{A} = 0$. Where $\phi(x, y, z)$ is some scalar that has continuous partial derivatives. Therefore, for a conservative force field $\mathbf{F}(x, y, z)$, there always exist a scalar field $U = U(x, y, z)$ (called the potential energy) such that

$$\mathbf{F} = -\nabla U = -\left(\frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k}\right)$$

Furthermore

$$\nabla \times \mathbf{F} = \mathbf{0}$$

Thus, the total work done by a conservative force in moving a particle from P_i to P_f (see Fig. 4.15) is

$$W = \int_{P_i}^{P_f} \mathbf{F}\cdot d\mathbf{s} = \int_{P_i}^{P_f} -\nabla U \cdot d\mathbf{s} = \int_{P_i}^{P_f} -dU = U_i - U_f = -\triangle U$$

or

$$\triangle U = -\int_{P_i}^{P_f} \mathrm{F}\cdot ds$$

where $U = U(x, y, z)$. Because only the change in the potential energy is significant, it does not matter where the reference point (U) is chosen. This is because if U_i is changed U_f will be also changed but $\triangle U$ will remain constant.

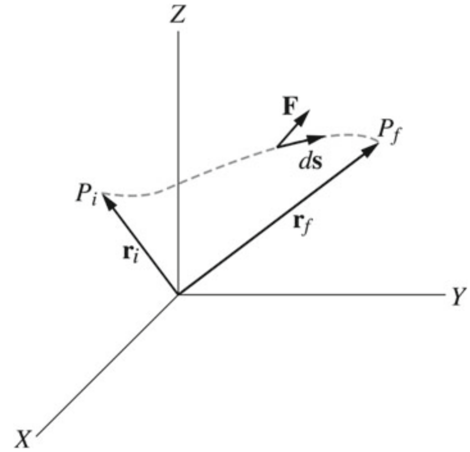


Fig. 4.15 The total work done by a conservative force in moving a particle from P_i to P_f

Example 4.10 A force acting on a particle is given by $\mathbf{F} = -k\mathbf{r}$. Determine: (a) whether or not the force is conservative; (b) the potential energy associated with the force if it is conservative.

Solution 4.10 (a)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -kx & -ky & -kz \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-kz) - \frac{\partial}{\partial z}(-ky)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(-kx) - \frac{\partial}{\partial x}(-kz)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(-ky) - \frac{\partial}{\partial y}(-kx)\right]\mathbf{k} = \mathbf{0}$$

Therefore, the force is conservative.
(b)

$$U = -\int \mathbf{F}\cdot d\mathbf{r} = -\int -k\mathbf{r}\cdot d\mathbf{r} = \int krdr = \frac{1}{2}kr^2 = \frac{1}{2}k(x^2 + y^2 + z^2)$$

Example 4.11 If a force acting on a particle is given by $\mathbf{F} = ay\mathbf{j}$, where a is a positive constant: (a) find the work done in moving the particle along the closed path shown in Fig. 4.16; (b) determine if the force is conservative.

Solution 4.11 (a) Along path 1 we have $y = 1$ and $dy = 0$ and along path 3 we have $y = 2$ and $dy = 0$.

$$W = \oint_c \mathbf{F}\cdot d\mathbf{r} = \int_1 \mathbf{F}\cdot d\mathbf{r} + \int_2 \mathbf{F}\cdot d\mathbf{r} + \int_3 \mathbf{F}\cdot d\mathbf{r} + \int_4 \mathbf{F}\cdot d\mathbf{r} = 0 + \int_{y=1}^{2} ay\,dy + 0 + \int_{y=2}^{1} ay\,dy = 0$$

(b) Since the total work done through the closed path is zero, the force is conservative.

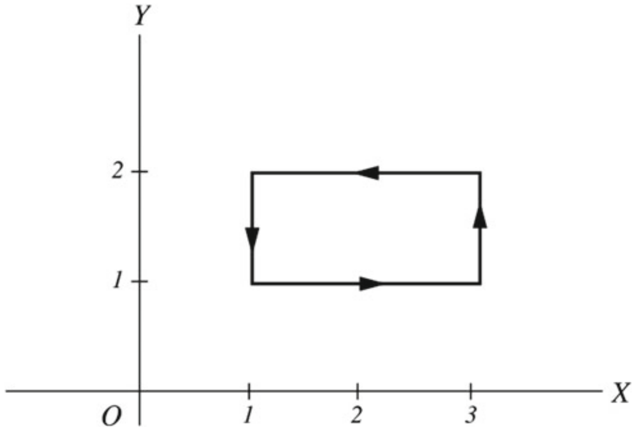


Fig. 4.16 The work done in moving the particle along a closed path

Example 4.12 Find the force acting on a particle if the potential energy associated with it is $U = 5y^2 - 3z$.

Solution 4.12

$$F_y = -\frac{\partial U}{\partial y} = -10y$$

$$F_z = -\frac{\partial U}{\partial z} = 3$$

and therefore $\mathbf{F} = -10y\mathbf{j} + 3\mathbf{k}$.

4.4.1.1 The Gravitational Potential Energy

As we have mentioned in Sect. (4.1.7), the work done by the gravitational force in displacing a particle near the earth's surface from y_i to y_f is

$$W_g = mgy_i - mgy_f$$

Since $W_g = -\triangle U_g = U_{gi} - U_{gf}$, we have

$$U_{gi} - U_{gf} = mgy_i - mgy_f$$

If $y_f = 0$ and $y_i = y$, the gravitational potential energy of the object–earth system may be written as

$$U_g = mgy$$

The force of gravity near the surface of the earth can be found from the gravitational potential energy In general we have $\mathbf{F} = -\nabla U$ here, since the motion is in one direction we have

$$\mathbf{F}_g = -\frac{dU}{dy}\mathbf{j} = -\frac{d}{dy}(mgy)\mathbf{j} = -mg\mathbf{j}$$

4.4.1.2 The Elastic Potential Energy

It was found in Sect. (4.1.6) that the work done by the spring force when moving a block from x_i to x_f (when it is stretched or compressed) is

$$W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$$

Since $W_s = -\triangle U_s = U_{si} - U_{sf}$, we have

$$U_{si} - U_{sf} = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$$

If $x_i = 0$ and $x_f = x$, the elastic potential energy of the block-spring system can be written as

$$U_s = \frac{1}{2}kx^2$$

The spring force can be found from the elastic potential energy

$$\mathbf{F} = -\frac{dU}{dx}\mathbf{i} = -\frac{d}{dx}\left(\frac{1}{2}kx^2\right)\mathbf{i} = -kx\mathbf{i}$$

4.5 Conservation of Mechanical Energy

The total mechanical energy of a system is defined as the sum of all of the kinetic energies of the objects within the system plus all of the potential energies of the system.

$$E_{tot} = K_{tot} + U_{tot}$$

Now, consider an isolated system in which there are no external forces acting on it, or the net external force is zero. The only forces acting on the system will be the internal forces within the system. These forces may be conservative or nonconservative. If only internal conservative forces exist, then the work done by any of these forces on an object in the system will transform its kinetic energy into potential energy (associated with that force), or vice versa. The internal conservative force can also transform one form of potential energy into another. The work done by such a force on an object in the system is

$$W = \triangle K$$

The change in potential energy due to this work is

$$W = -\triangle U$$

Thus,

$$\triangle K = -\triangle U$$

or

$$\triangle K + \triangle U = 0$$

or

$$K_i + U_i = K_f + U_f$$

If more than one conservative force acts, there will be a potential energy associated with each force. That is

$$K_i + \sum U_i = K_f + \sum U_f$$

Therefore we have

$$E_i = E_f$$

or

$$\triangle E = 0$$

From the previous discussion, we conclude that for an isolated system in which only conservative forces act, the total mechanical energy of the system remains constant (conserved). Figure 4.17 shows the changes of energy of a ball thrown upwards. Now suppose that the system is not isolated and that the external forces acting on the system are conservative. The change in the kinetic energy of the system is then equal to the work done on the system by an internal conservative force plus the amount of kinetic energy changed due to an external conservative force, that is,

$$\triangle K = W_{\text{intc}} + \triangle K_{ext}$$

or

$$\triangle K = -\triangle U - \triangle U_{ext}$$

Hence

$$\triangle K + \triangle U + \triangle U_{ext} = 0$$

Therefore, the total mechanical energy of the system remains constant under both external and internal conservative forces. If external nonconservative forces act on the system, or if there is heat transfer, or if internal nonconservative forces act, then the total mechanical energy may change and is no longer conserved.

4.5.1 Changes of the Mechanical Energy of a System due to External Nonconservative Forces

External nonconservative forces may act on a system if it is not isolated. Consider a system that is not isolated in which only internal conservative forces act. The change in the kinetic energy of the system is then equal to the work done on the system by an internal conservative force plus the amount of kinetic energy changed due to an external nonconservative force. This can be expressed as

$$\triangle K = W_{\text{intc}} + \triangle K_{ext}$$

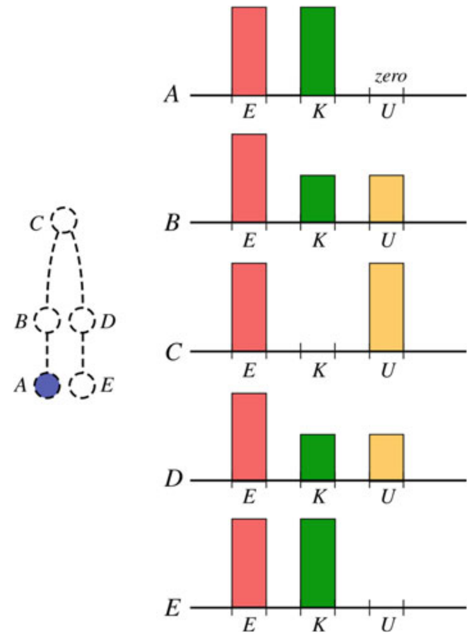


Fig. 4.17 Changes in the kinetic and potential energies of a ball thrown vertically upwards

or

$$\triangle K = -\triangle U + \triangle K_{ext}$$

Thus

$$\triangle E = \triangle K_{ext} \tag{4.3}$$

This implies that the total mechanical energy has changed by an amount of $\triangle K_{ext}$. Not that the work done by a nonconservative force cannot be calculated generally but the change in the kinetic energy can be observed.

4.5.2 Friction

Friction is a nonconservative force as seen in Sect. 4.2. If this force is applied externally to a system in which only internal conservative forces act, it will decrease (dissipate) the kinetic energy of the system by transforming it into thermal energy The change in the mechanical energy of the system is

$$\triangle E = \triangle K_{ext}$$

The work done by friction or any other nonconservative force cannot be calculated. In other words, the work done by friction is not simply $-f_k s$, where s is the displacement of the object in the system. The reason behind not being able to calculate the work done by friction is that at a microscopic level the frictional force is not a single force that acts at one point. Rather, it is a combination of forces acting at different points in the object. However, the loss in kinetic energy of the object can be calculated as shown below: Consider a block sliding on a rough surface. Let's choose the block only to be our system. From the equation of motion, we have

$$v_f^2 - v_i^2 = 2as$$

Newton's second law gives

$$-f_k = ma$$

Thus

$$-fs = mas$$

or

$$-f_s s = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

Therefore

$$\triangle K_{ext} = -f_k s$$

This quantity represents the magnitude of the loss in the kinetic energy of the block due to friction. This loss of energy appears as thermal energy of the block and of the surface on which it slides.

4.5.3 Changes in Mechanical Energy due to Internal Nonconservative Forces

In solving problems you are free to choose the system. If we considered the block plus the surface as our system, then friction will be an internal nonconservative force and we may write

$$\triangle E = -\triangle E_{\text{int}} = \triangle K_{in,nc} = -f_k s$$

where $\triangle K_{in,nc}$ is the change in the kinetic energy of the system due to an internal nonconservative force. Another example of a nonconservative force is the force that you exert on your body by your muscles. This force transfers the chemical energy of your body into kinetic energy In Sect. (4.1.5), we have seen that the motion of the skateboarder can be explained using the concept of the center of mass. Another way to explain the motion of the skateboarder is that the internal chemical energy of the man is transformed into kinetic energy, and we may write

$$\triangle E = \triangle K = -\triangle E_{\text{int}}$$

Since $\triangle U = 0$ in his case. An additional example of nonconservative forces is the forces that different parts in an object exert on each other when the object is deformed. These forces transform the kinetic energy of the object into internal energy. In all cases, even though energy can transfer from one object to another or to the environment, the total amount of energy in the universe is constant. That is, energy gained by a system is lost by another system. In other words, *energy cannot be created or destroyed it can only be transformed from one form to another and the total energy of an isolated system is conserved* (*constant*). This statement is known as the law of conservation of energy The law of conservation of energy is also valid in relativity and quantum mechanics.

4.5.4 Changes in Mechanical Energy due to All Forces

Consider a system in which there are both internal and external conservative and nonconservative forces acting on it. In this case, the change in the total mechanical energy of the system can be written as

$$\triangle E = \triangle K + \triangle U + \triangle U_{ext} = \triangle K_{ext} - \triangle E_{\text{int}}$$

Example 4.13 A 0.2 kg apple falls from a tree at a distance of 3 m above the ground. Find: (a) the velocity of the apple at an altitude of 2 m and at the instance just before it hits the ground; (b) the altitude of the apple when its velocity is 4 m/s.

Solution 4.13 (a) Consider the system to be the earth + the apple. By neglecting air resistance (the apple is in free-fall), the only internal force that acts within the earth–apple system is the force of gravity. Because the gravitational force is a conservative force, the total mechanical energy of the system is conserved. Therefore as the apple falls its gravitational potential energy is converted into kinetic energy such that at any instant the total mechanical energy of the system is constant. Applying the law of conservation of energy to the system and by taking $y = 0$ at the earth's surface and the gravitational potential energy to be zero at $y = 0$, we have

$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}mv_f^2 + mgy = 0 + mgh$$

where h is its initial altitude. That gives

$$v_f = \sqrt{2g(h-y)}$$

At $y = 2\,\text{m}$,

$$v_f = \sqrt{2(9.8\,\text{m/s}^2)(1\,\text{m})} = 4.43\,\text{m/s}$$

At $y = 0$

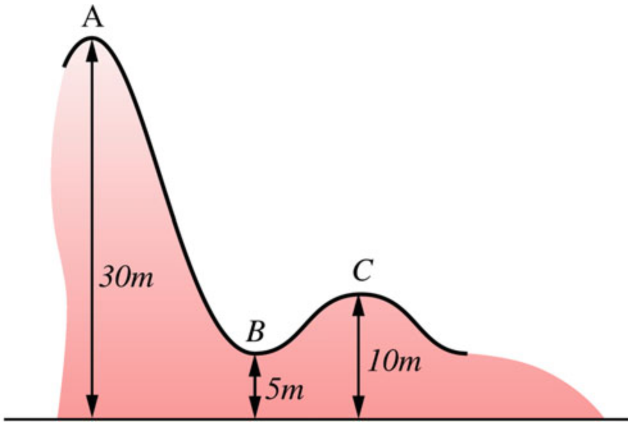


Fig. 4.18 By ignoring friction, the total energy of the roller coaster can be considered to be conserved

$$v_f = \sqrt{2(9.8\,\mathrm{m/s^2})(3\,\mathrm{m})} = 7.7\,\mathrm{m/s}$$

(b)

$$y = h - \frac{v_f^2}{2g} = (3\,\mathrm{m}) - \frac{(4\,\mathrm{m/s})^2}{2(9.8\,\mathrm{m/s^2})} = 2.2\,\mathrm{m}$$

Example 4.14 A roller coaster of mass 500 kg starts from rest at point A, and rolls down the track as shown in Fig. 4.18. Ignoring friction, determine: (a) the roller coaster speed at B and C; (b) the work done by gravity as the rollercoaster moves from A to B.

Solution 4.14 (a) Consider the system to consist of the roller-coaster + the track + the earth. Taking the gravitational potential energy to be zero at the earth's surface and from the conservation of energy we have

$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}mv_B^2 + mgh_B = 0 + mgh_A$$

Therefore,

$$v_B = \sqrt{2g(h_A - h_B)} = \sqrt{2(9.8\,\mathrm{m/s^2})(25\,\mathrm{m})} = 22.13\,\mathrm{m/s}$$

Similarly, the velocity at C is

$$v_C = \sqrt{2g(h_A - h_C)} = \sqrt{2(9.8\,\mathrm{m/s^2})(20\,\mathrm{m})} = 19.8\,\mathrm{m/s}$$

You may also calculate the velocity at C by taking B as the initial point.

(b) As the car moves from A to B the work done by gravity is

$$W_g = -\triangle U$$

Fig. 4.19 A block released from rest on top of an incline

$$W_g = -(mgh_b - mgh_a) = 1.22 \times 10^5\,\mathrm{J}$$

Example 4.15 A block of mass 5 kg is released from rest at the top of a 45° incline that is 0.5 m long as shown in Fig. 4.19. It then slides on a horizontal surface that is 0.7 m long and goes up again on a second ramp that is at 30° to the horizontal. If the coefficient of kinetic friction between the block and all three surfaces is 0.2, find the maximum distance that the block would move up the second ramp?

Solution 4.15 First, we divide the path into three parts. Let us consider the system as the block only Along the first part the change in the total mechanical energy of the system is equal to the energy dissipated by friction. Thus,

$$\triangle E = \triangle K_{ext}$$

$$K_f + U_f = K_i + U_i + \triangle K_{ext}$$

$$\frac{1}{2}mv_{f1}^2 + 0 = 0 + mgh - f_{k1}s_1$$

the force of kinetic friction is

$$f_{k1} = \mu_k n = \mu_k mg\cos\theta_1 = (0.2)(5\,\mathrm{kg})(9.8\,\mathrm{m/s^2})\cos 45^o = 6.93\,\mathrm{N}$$

That gives

$$\frac{1}{2}mv_{f1}^2 = mgs_1\sin\theta_1 - f_{k1}s_1 = (5\,\mathrm{kg})(9.8\,\mathrm{m/s^2})(0.5\,\mathrm{m})\sin 45^o - (6.93\,\mathrm{N})(0.5\,\mathrm{m}) = 13.9\,\mathrm{J}$$

$v_{f1} = 2.35\,\mathrm{m/s}$. Along the second path we have again

$$K_f + U_f = K_i + U_i + \triangle K_{ext}$$

$$\frac{1}{2}mv_{f2}^2 + 0 = \frac{1}{2}mv_{i2}^2 + 0 - f_{k2}s_2$$

The force of kinetic friction is given by

$$f_{k2} = \mu_k mg = (0.2)(5\,\mathrm{kg})(9.8\,\mathrm{m/s^2}) = 9.8\,\mathrm{N}$$

and therefore

$$\frac{1}{2}mv_{f2}^2 = \frac{1}{2}(5\,\text{kg})(2.35)^2 - (9.8\,\text{N})(0.7\,\text{m}) = 6.95\,\text{J}$$

$$v_{f2} = \sqrt{2\frac{(6.94\,\text{J})}{(5\,\text{kg})}} = 1.7\,\text{m/s}$$

Finally, along the third path, we also have

$$K_f + U_f = K_i + U_i + \triangle K_{ext}$$

and

$$0 + mgs_3 \sin 30^o = \frac{1}{2}mv_{i3}^2 + 0 - f_{k3}s_3$$

but we have

$$f_{k3} = \mu_k mg \cos 30^o = (0.2)(5\text{kg})(9.8\,\text{m/s}^2)(0.866) = 8.5\,\text{N}$$

and thus

$$(5\,\text{kg})(9.8\,\text{m/s}^2)s_3(0.5) = \frac{1}{2}(5\,\text{kg})(1.7\,\text{m/s})^2 - (8.5\,\text{N})s_3$$

That gives $s_3 = 0.2\,\text{m}$.

Example 4.16 Two masses $m_1 = 5$ kg and $m_2 = 9$ kg are connected by a light rope that passes over a massless frictionless pulley as in Fig. 4.20. If the system is released from rest when m_2 is at 0. 5 m above the ground, use the principle of conservation of energy to determine the speed with which m_2 will hit the ground.

Solution 4.16 If air resistance is neglected, the only force acting in the masses-earth system is the gravitational force between them and hence the total mechanical energy of the system is conserved, i.e.,

$$K_f + U_f = K_i + U_i$$

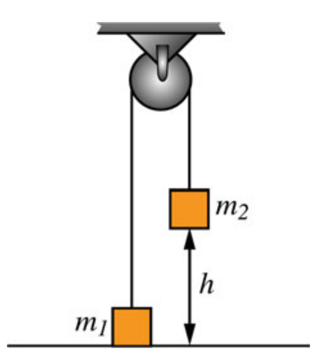


Fig. 4.20 Two masses connected by a light rope that passes over a massless frictionless pulley

Because the two masses are connected by a rope, they have the same speed at any instant. If m_2 descends a distance h, m_1 will rise through the same distance and we have

$$\frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 + m_1gh = m_2gh$$

$$\frac{1}{2}(m_1 + m_2)v^2 = g(m_2 - m_1)h$$

and therefore

$$v = \sqrt{\frac{2gh(m_2 - m_1)}{(m_1 + m_2)}} = \sqrt{\frac{2(0.5\,\text{m})(9.8\,\text{m/s}^2)(4\,\text{kg})}{(14\,\text{kg})}} = 1.7\,\text{m/s}$$

Example 4.17 A 0.25 kg ball is attached to alight string of length $L = 0.5$ m as in Fig. 4.21. Find (a) the tension in the string at B($\theta = 10°$) if the ball is given an initial velocity $v_a = 0.5\,\text{m/s}$ at its lowest position; (b) the velocity of the ball at A if the ball is released from rest at B.

Solution 4.17 (a) At point B some of the kinetic energy of the ball is converted into potential energy By taking the origin of the x-y coordinates at the lowest point A, we have

$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}mv_b^2 + mgL(1 - \cos\theta) = \frac{1}{2}mv_a^2 + 0$$

and therefore we get

$$v_b^2 = v_a^2 - 2gL(1 - \cos\theta) \qquad (4.4)$$

Applying Newton's second law along the radial direction to the ball at B we have

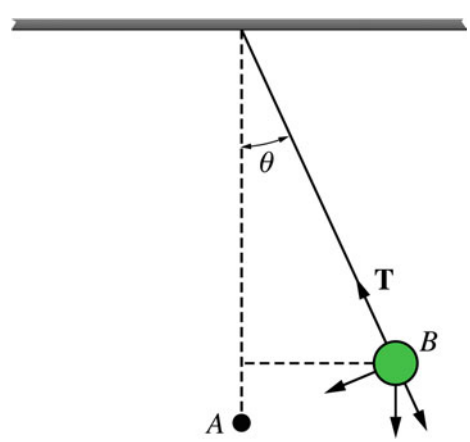


Fig. 4.21 A ball attached to a light string

$$T - mg\cos\theta = \frac{mv_b^2}{L} \tag{4.5}$$

Substituting Eq. 4.4 into Eq. 4.5 gives

$$T = mg\cos\theta + \frac{m}{L}(v_a^2 - 2gL(1-\cos\theta))$$

thus

$$T = m\left(g\cos\theta + \frac{v_a^2}{L} - 2g + 2g\cos\theta\right)$$

and hence

$$T = m\left(3g\cos\theta - 2g + \frac{v_a^2}{L}\right)$$

Substituting the values of θ and v_a gives $T = 2.5$ N.

(b) If $v_b = 0$, we have from $K_f + U_f = K_i + U_i$

$$\frac{1}{2}mv_a^2 + 0 = 0 + mgL(1-\cos\theta)$$

hence

$$v_a = \sqrt{2gL(1-\cos\theta)} = \sqrt{2(9.8\,\text{m/s}^2)(0.5\,\text{m})(1-\cos 10^o)} = 0.4\,\text{m/s}$$

Example 4.18 A 3 kg block compresses a spring of negligible mass a distance of 0.1 m from its equilibrium position as in Fig. 4.22. If the surface is frictionless and the force constant of the spring is 200 N/m, and the block is free to move, find: (a) the speed of the block just as it leaves the spring; (b) the maximum height that the block will reach; (c) suppose that a part of the horizontal track is rough with a length of 0.05 m, find the coefficient of kinetic friction if the block reaches a maximum height of 0.014 m.

Solution 4.18 (a) The only force acting inside the spring–mass–earth system is the spring force that acts on the block. This force is conservative and therefore the total mechanical energy of the system is conserved. The potential energy of the spring is transformed into kinetic energy of the block,

$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}mv_f^2 + 0 = 0 + \frac{1}{2}kx^2$$

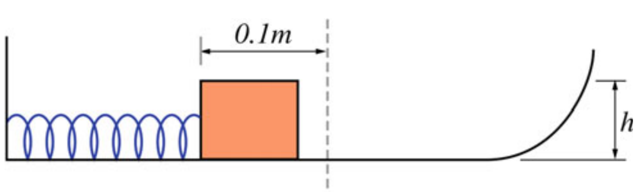


Fig. 4.22 A 3 kg block compresses a spring of negligible mass a distance of 0.1 m from its equilibrium position

and therefore

$$v_f^2 = \frac{k}{m}x^2 = \frac{(200\,\text{N/m})}{(3\,\text{kg})}(-0.1\,\text{m})^2$$

this gives $v_f = 0.82$ m/s.

(b)

$$K_f + U_f = K_i + U_i$$

$$0 + mgh = \frac{1}{2}mv_i^2 + 0$$

and hence

$$h = \frac{v_i^2}{2g} = \frac{(0.8.2\,\text{m/s})^2}{2(9.8\,\text{m/s}^2)} = 0.034\,\text{m}$$

We can also take the initial position before the block is released.

(c)

$$K_f + U_f = K_i + U_i + \Delta K_{in,nc}$$

$$0 + mgh = 0 + \frac{1}{2}kx^2 - f_k d$$

along the rough surface $f_k = \mu_k mgd$, and therefore

$$\mu_k mgd = \frac{1}{2}kx^2 - mgh$$

thus

$$\mu_k(3\,\text{kg})(9.8\,\text{m/s}^2)(0.05\,\text{m}) = \frac{1}{2}(200\,\text{N/m})(0.1)^2 - (3\,\text{kg})(9.8\,\text{m/s}^2)(0.014)$$

That gives $\mu_k = 0.2$

Example 4.19 A small stone of mass 0.1 kg is released from rest inside a large hemispherical bowl of radius $R = 0.2$ m. It then slides along the surface as in Fig. 4.23. (a) Find the speed of the stone at point B and C; (b) If the surface of the bowl is not frictionless, how much energy is dissipated by friction as the stone moves from A to B if the speed at B is 1.7 m/s?

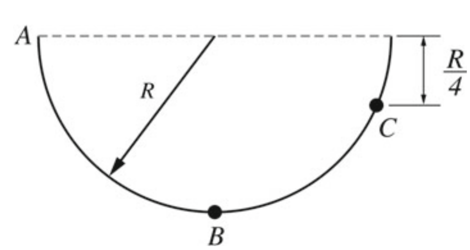


Fig. 4.23 A small stone of mass 0.1 kg is released from rest inside a large hemispherical bowl of radius $R = 0.2$m

Solution 4.19 (a)

$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}mv_B^2 + 0 = 0 + mgR$$

thus

$$v_B = \sqrt{2gR} = \sqrt{2(9.8\,\text{m/s}^2)(0.2\,\text{m})} = 2\,\text{m/s}$$

At point C some of the kinetic energy at B is converted into potential energy and we have

$$\frac{1}{2}mv_C^2 + mg\left(R - \frac{R}{4}\right) = \frac{1}{2}mv_B^2 + 0$$

$$v_C^2 = v_B^2 - \frac{3}{2}gR = (2\,\text{m/s})^2 - \frac{3}{2}(9.8\,\text{m/s}^2)(0.2\,\text{m})$$

and therefore $v_C = 1\,\text{m/s}$.

(b) If a force of kinetic friction exists between the stone and the bowl, the total mechanical energy at point B is given by

$$E_f = E_i + \triangle K_{ext}$$

where the stone is considered as the system, therefore

$$K_f + U_f = K_i + U_i + \triangle K_{ext}$$

$$\frac{1}{2}mv_b^2 + 0 = 0 + mgR + \triangle K_{ext}$$

hence the energy dissipated by friction is

$$\triangle K_{ext} = \frac{1}{2}mv_b^2 - mgR = (0.1\,\text{kg})\left(\frac{1}{2}(1.7\,\text{m/s})^2 - (9.8\,\text{m/s}^2)(0.2\,\text{m})\right) = -0.05\,\text{J}$$

Example 4.20 A skier starts at the top of a frictionless incline as in Fig. 4.24. Find the velocity with which he will leave the second incline.

Solution 4.20 From the conservation of energy the velocity when he leaves the track is

$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}mv^2 + mgh_2 = mgh_1$$

$$v = \sqrt{2g(h_1 - h_2)} = \sqrt{2(98\,\text{m/s}^2)((20\,\text{m}) - (10\,\text{m}))}$$

That gives $v = 14\,\text{m/s}$.

Fig. 4.24 A skier slides from rest on top of an incline

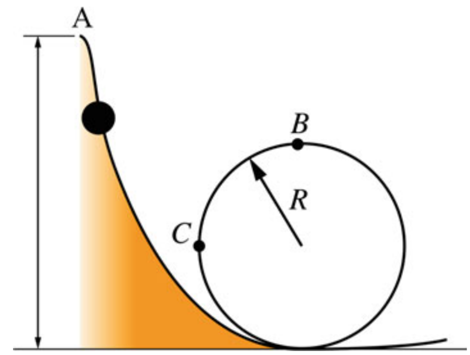


Fig. 4.25 A 0.4 kg stone is released from rest at point A where $h_A = 2$m

Example 4.21 A 0.4 kg stone is released from rest at point A where $h_A = 2\,\text{m}$ (see Fig. 4.25). It then slides without friction along the track shown where $R = 0.5\,\text{m}$. Determine: (a) the speed of the stone at B; (b) the normal force exerted on the stone at B; (c) the magnitude of the total acceleration of the stone at C; (d) the minimum height in which the stone must be released such that it does not fall off the track.

Solution 4.21 (a) From the conservation of energy, we have

$$mgh_A = \frac{1}{2}mv_B^2 + 2mgR$$

$$v_B = \sqrt{2g(h_A - 2R)} = \sqrt{2(9.8\,\text{m/s}^2)((2\,\text{m}) - 2(0.5\,\text{m}))} = 4.43\,\text{m/s}$$

(b) From Newton's second law, we have

$$n + mg = m\frac{v_B^2}{R}$$

$$n = m\frac{v_B^2}{R} - mg = (0.4\,\text{kg})\left[\frac{(4.43\,\text{m/s})^2}{(0.5\,\text{m})} - (9.8\,\text{m/s}^2)\right] = 11.78\,\text{N}$$

(c) The velocity of the stone at C is

$$v_C = \sqrt{2g(h_A - R)} = \sqrt{2(9.8\,\text{m/s}^2)((2\,\text{m}) - (0.5\,\text{m}))} = 5.42\,\text{m/s}$$

Therefore, the radial acceleration at C is

$$a_r = \frac{v_C^2}{R} = \frac{(5.42\,\text{m/s})^2}{(0.5\,\text{m})} = 58.8\,\text{m/s}^2$$

The tangential force exerted on the stone at C is its weight $F_t = -mg$, hence the tangential acceleration of the stone at

C is $a_t = -g$ and the magnitude of the total acceleration is

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{(58.8\,\text{m/s}^2)^2 + (-9.8\,\text{m/s}^2)^2} = 59.6\,\text{m/s}^2$$

(d) When the stone is at the verge of falling at B, then the only force acting on it is the force of gravity and we have $mg = mv_B^2/R$, $v_B^2 = gR$. From conservation of energy

$$v_B = \sqrt{2g(h_{A\,\text{min}} - 2R)}$$

or

$$2g(h_{A\,\text{min}} - 2R) = gR$$

and

$$h_{A\,\text{min}} = \frac{R}{2} + 2R = (0.25\,\text{m}) + (1\,\text{m}) = 1.25\,\text{m}$$

4.5.5 Power

Expanding on the definition of power, power is the rate of energy transfer due to a force. If $\triangle E$ is the amount of energy transferred in an amount of time $\triangle t$, the average power is

$$\overline{P} = \frac{\triangle E}{\triangle t}$$

The instantaneous power is then

$$P = \lim_{\triangle t \to 0} \frac{\triangle E}{\triangle t} = \frac{dE}{dt}$$

4.5.6 Energy Diagrams

Consider a particle that is a part of an isolated system where only internal conservative forces act. Suppose this particle is moving along the x-axis while a conservative force that depends only on the position of the particle acts on it. For simplicity, we will assume that is the only force acting on the system and that it does work only on that particle. The potential energy of the system as a function of the particle's position (x) is shown in Fig. 4.26. At any point $F(x)$ is given by

$$F(x) = -\frac{dU(x)}{dx}$$

That is, it is the negative of the slope of the curve at that point. Because this force is conservative it follows that the total mechanical energy of the system is conserved. Therefore the kinetic energy of the particle as a function of position is given by

$$K(x) = E - U(x)$$

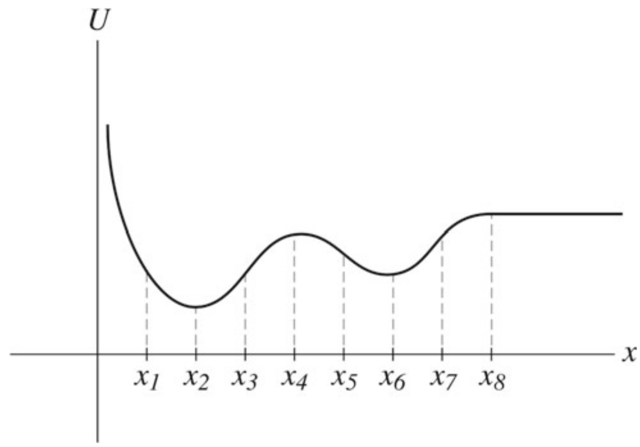


Fig. 4.26 The potential energy of the system as a function of the particle's position (x)

On the U versus x curve, the kinetic energy at any point can be found by subtracting the value of U (at that certain point) from E.

4.5.7 Turning Points

A turning point is a point in which the particle changes its direction of motion. The points x_1, x_3, x_5 and x_7 are all turning points.

4.5.8 Equilibrium Points

Equilibrium points occur in general when $\nabla U = 0$. In the case of one dimensional motion it occurs when $dU(x)/dx = 0$, i.e. when $F(x) = 0$.

4.5.9 Positions of Stable Equilibrium

If at an equilibrium point $d^2U(x)/dx^2 > 0$, then $U(x)$ is a minimum at that point. The point is then said to be a position of stable equilibrium, i.e., any minimum on the $U(x)$ curve is a position of stable equilibrium. Another method to find the position of stable equilibrium is to find the sign of $F(x)$ at each side of the point. As an example, consider the point x_2.

This point is a position of stable equilibrium since if the particle is displaced slightly to the right of x_2 then $dU(x)/dx$ is positive which leads to $F(x)$ being negative and the particle will accelerate back towards x_2. On the other hand, if the particle is displaced slightly to the left of x_2, then $dU(x)/dx$ is negative and thus $F(x)$ is positive and the particle will also accelerates back to x_2. Therefore, because $F(x)$ tends to restore the particle back to that position when the particle is

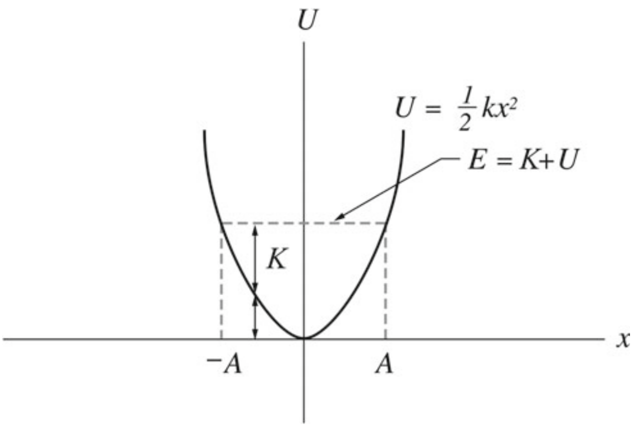


Fig. 4.27 The potential energy of a mass-spring system as a function of x

displaced in either direction, it is called a position of stable equilibrium. x_6 is also a position of stable equilibrium.

4.5.10 Positions of Unstable Equilibrium

If at an equilibrium point $d^2U(x)/dx^2 < 0$, then $U(x)$ is maximum at that point, and the point is called a position of unstable equilibrium. In Fig. 4.26, x_4 is a position of unstable equilibrium since if the particle is slightly displaced to the right of x_4, $F(x)$ is positive and the particle will accelerate away from x_4. If the particle is displaced to the left of x_4, $F(x)$ is negative and the particle will also accelerate away from that position. Therefore, because $F(x)$ tends to repel the particle away from that position, it is called a position of unstable equilibrium. In general this force tends to move the particle towards the minimum value of $U(x)$. Figure 4.27 shows the potential energy of a mass–spring system as a function of x.

4.5.11 Positions of Neutral Equilibrium

Any point in a region where $U(x)$ is constant and $F(x) = 0$ is called a position of neutral equilibrium. x_8 is a position of neutral equilibrium. If the particle is slightly displaced to the right or left of x_8, no restoring or repelling forces will act on the particle and it will remain stationary The position of the particle as a function of time can be obtained from

$$U(x) + K(x) = E$$

$$U(x) + \frac{1}{2}mv^2 = E$$

$$v = \pm\sqrt{\frac{2}{m}[E - U(x)]}$$

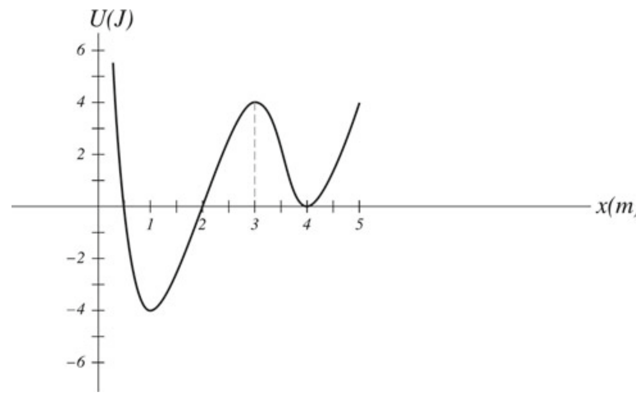


Fig. 4.28 The potential energy of a particle as a function of its displacement

or

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}[E - U(x)]}$$

hence

$$t = \int_{x_i}^{x} \frac{dx}{\pm\sqrt{\frac{2}{m}[E - U(x)]}}$$

By evaluating this integral, we would obtain the time as a function of the position, then by solving for x we get the position as a function of time.

Example 4.22 Figure 4.28 shows the potential energy of a particle as a function of its displacement. Find: (a) the values of x where the particle is in stable or unstable equilibrium; (b) the direction of the force acting on the particle at 0.5 m.

Solution 4.22 (a) We have $x = 1$ m and $x = 4$ m are positions of stable equilibrium, $x = 3$ m is a position of unstable equilibrium.

(b) At 0.5 m, $dU(x)/dx$ is negative and hence $F(x)$ is positive which means that the particle will accelerate in the positive x-direction.

Example 4.23 Consider a block attached to a light spring and released from rest at $x = A$. Find the position of the block as a function of time using energy methods.

Solution 4.23

$$t = \int_{x_i=A}^{x_f=x} \frac{dx}{\pm\sqrt{\frac{2}{m}[E - U(x)]}} = \int_{x_i=A}^{x_f=x} \frac{dx}{\pm\sqrt{\frac{2}{m}[(1/2)kA^2 - (1/2)kx^2]}}$$

$$= \int_{x_i=A}^{x_f=x} \frac{dx}{\pm\sqrt{\frac{k}{m}[A^2 - x^2]}} = \pm\sqrt{\frac{m}{k}} \int_{x_i=A}^{x_f=x} \frac{dx}{\sqrt{[A^2 - x^2]}}$$

$$= \pm\sqrt{\frac{m}{k}}\left[-\cos^{-1}\left(\frac{x}{A}\right)\right]_{x_i=A}^{x_f=x} = \pm\sqrt{\frac{m}{k}}\left[-\cos^{-1}(\frac{x}{A})\right]$$

$$-\cos\left(\pm\sqrt{\frac{k}{m}}t\right) = \cos\left(\cos^{-1}\left(\frac{x}{A}\right)\right)$$

or

$$x = A\cos\sqrt{\frac{k}{m}}t$$

Since $\cos(\pm\theta) = \theta$ and $-\cos\theta = \cos\theta$. In Chap. 10, we will see that this equation represents the equation of a simple harmonic motion.

Problems

1. A force acting on a particle varies with position as in Fig. 4.29. Find the work done by the force as the particle moves from $x = 0$ to $x = 8$ m.
2. A force $\mathbf{F} = (3\mathbf{i} + \mathbf{j} - 5\mathbf{k})$ N acts on a particle that undergoes a displacement $\mathbf{r} = (-2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ m. Find the work done by the force on the particle.
3. A 5 kg block is pulled from rest on a rough surface by a constant force of 10 N that is at 30° to the horizontal. If the coefficient of kinetic friction between the block and the surface is 0.15, find the final speed of the block as it moves through a displacement of 2 m using the work–energy theorem.
4. Calculate the work done against gravity in moving a 30 kg box through a height of 6 m.
5. A 1600 kg car accelerates from rest at a rate of $1\,\text{m/s}^2$. Find the average power delivered to the car during the first 5 s.
6. Determine whether or not the force $\mathbf{F} = -m\omega^2(x\mathbf{i} + y\mathbf{j})$ is conservative, where ω is constant and m is the mass of the particle. If the force is conservative determine the potential energy associated with it.

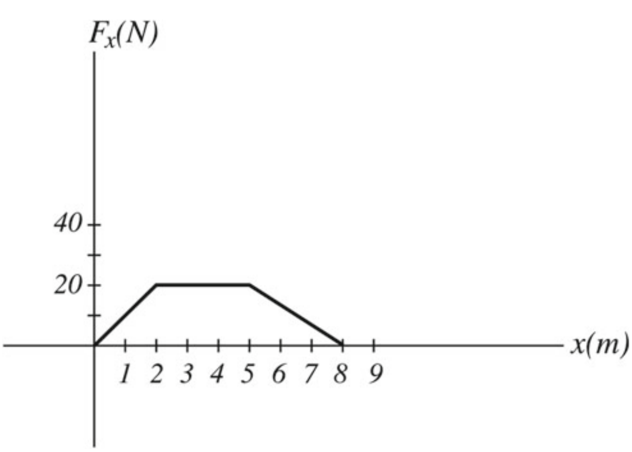


Fig. 4.29 A force acting on a particle varies with position

7. A 5 kg block slides down an inclined plane of angle 50° (see Fig. 4.30). Using energy methods, find the speed of the block just as it reaches the bottom if the coefficient of kinetic friction is $\mu_k = 0.2$.
8. A block of mass of 2 kg is pressed against a light spring of force constant 400 N/m (see Fig. 4.31). If the compression of the spring is 10 cm, find the maximum height the block will reach when it is released.
9. A force acting on a particle is given by $\mathbf{F} = -\beta y^2\mathbf{j}$. Find the work done in moving the particle along the path shown in Fig. 4.32.

Fig. 4.30 A block slides down an inclined plane

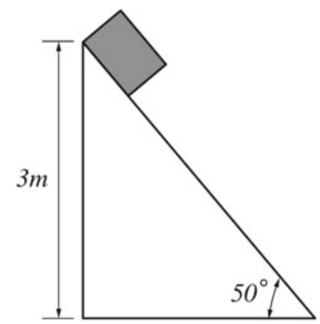
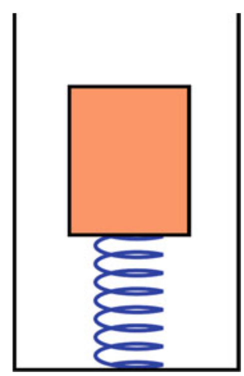


Fig. 4.31 A block pressed against a light spring and released



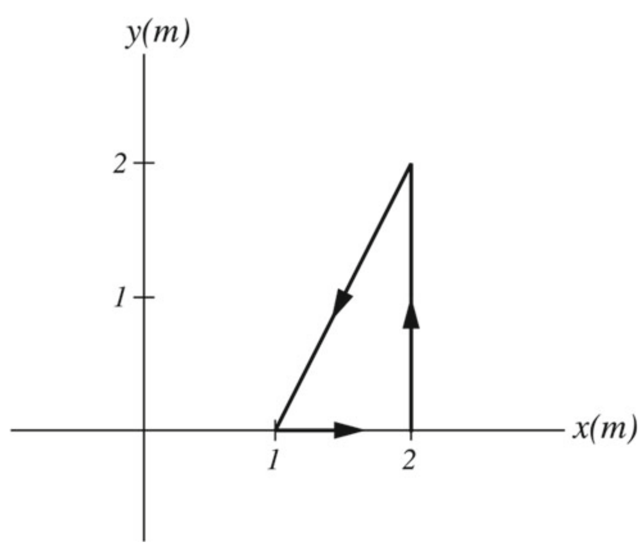


Fig. 4.32 The work done in moving the particle along a closed path

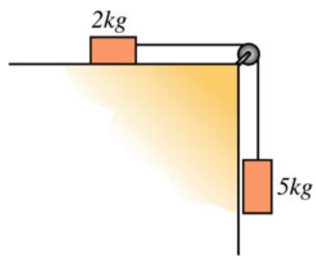


Fig. 4.33 Two blocks connected by a light rope that passes over a massless frictionless pulley

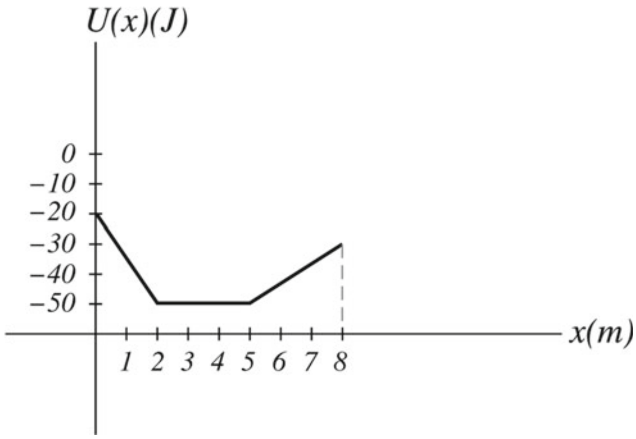


Fig. 4.34 The potential energy versus displacement of a particle

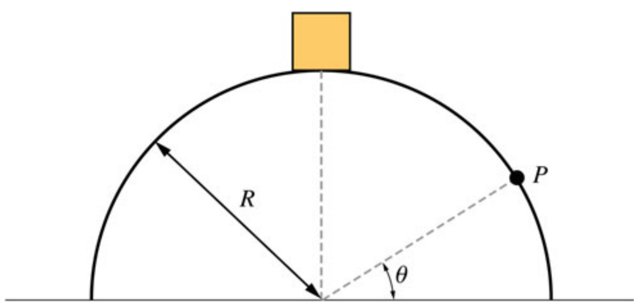


Fig. 4.35 A block of mass m resting on a hemispherical mound of ice

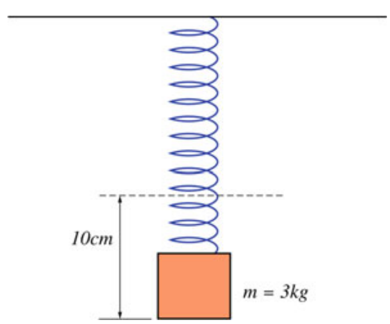


Fig. 4.36 A block hanging from a spring

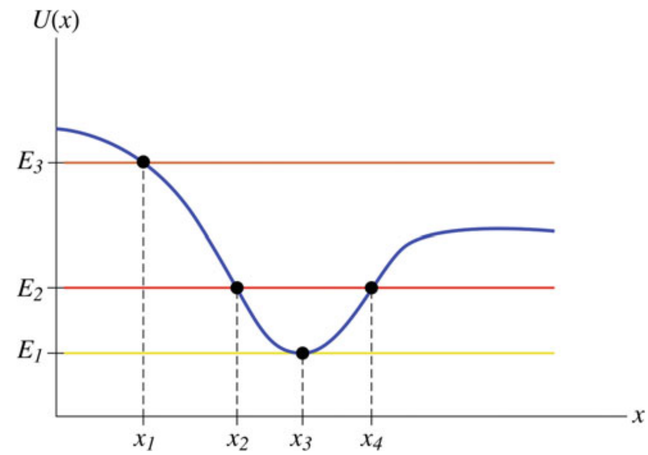


Fig. 4.37 The potential energy versus position of a particle

10. Two blocks are connected by a light rope that passes over a massless frictionless pulley (see Fig. 4.33). If the system is released from rest, find the total kinetic energy of the blocks when the 5 kg block descends a distance of 0.5 m assuming that the surface is frictionless.
11. A particle of mass 1.5 kg moves along the x-axis where its potential energy varies as in Fig. 4.34. Plot the force $F_x(x)$ versus x from $x = 0$ to $x = 8$ m.
12. A block of mass m rests on a hemispherical mound of ice as shown in Fig. 4.35. If it is given a very small push and start sliding, find the height of the point in which the block will lose contact with the mound.
13. A 3 kg block hangs from a spring as in Fig. 4.36. If the spring stretches a distance of 10 cm, find (a) the force constant of the spring (b) the work done in expanding the spring a distance of 5 cm without accelerating it.
14. In Fig. 4.37, determine the Turning points and the positions of stable and unstable equilibrium.

5 Impulse, Momentum, and Collisions

5.1 Linear Momentum and Collisions

When two billiard balls collide, in which direction would they travel after the collision? If a meteorite hits the earth, why does the earth remain in its orbit? When two cars collide with each other, why is one of the cars more damaged than the other? We will find that to answer such questions, new concepts must be introduced.

Consider the situation where two bodies collide with each other. During the collision, each body exerts a force on the other. This force is called an impulsive force, because it acts for a short period of time compared to the whole motion of the objects, and its value is usually large. To solve collision problems by using Newton's second law, it is required to know the exact form of the impulsive forces. Because these forces are complex functions of the collision time, it is difficult to find their exact form and would make it difficult to use Newton's second law to solve such problems. Thus, new concepts known as momentum and impulse were introduced. These concepts enable us to analyze problems that involve collisions, as well as many other problems.

The law of conservation of momentum is especially used in analyzing collisions and is applied immediately before and immediately after the collision. Therefore, it is not necessary to know the exact form of the impulsive forces, which makes the problem easy to analyze. Next, we will discuss and verify the concepts of momentum and impulse, and the law of conservation of momentum. The linear momentum (or quantity of motion as was called by Newton) of a particle of mass m is a vector quantity defined as

$$\mathbf{p} = m\mathbf{v}$$

where y is the velocity of the particle. A fast moving car has more momentum than a slow moving car of the same mass. Another example is that a bowling ball has more momentum than a basketball moving at the same speed. The SI unit of linear momentum is kg.m/s. In terms of components, we may write $p_x = mv_x$, $p_y = mv_y$, and $p_z = mv_z$. Newton's second law can be expressed in terms of momentum for a particle-like object of constant mass as

$$\Sigma\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d(m\mathbf{v})}{dt}$$

or

$$\Sigma\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

That is, the rate of change of the linear momentum of an object is equal to the resultant force acting on the object and is in the same direction as that force.

5.2 Conservation of Linear Momentum

The law of conservation of linear momentum states that if the net external force acting on a system equals zero (isolated) and if there is no mass exchange with the surroundings of the system (closed), then the total linear momentum of the system remains constant. To show that, consider an isolated system consisting of two particles where the only forces that act in the system are internal forces (see Fig. 5.1). The total linear momentum of the system at any particular time is given by

$$\mathbf{p}_{tot} = \mathbf{p}_1 + \mathbf{p}_2 \tag{5.1}$$

If the net force exerted on particle 2 by particle 1 is $\mathbf{F}_{21}$, then from Newton's third law, the net force exerted on particle 1 by particle 2 is $\mathbf{F}_{12}$, That is

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

Differentiating Eq. 5.1 with respect to time and by using Newton's second law, we have

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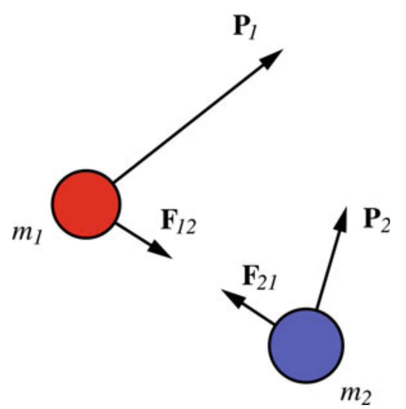

Fig. 5.1 An isolated system consisting of two particles where the only forces that act in the system are internal forces

$$\frac{d\mathbf{p}_{tot}}{dt} = \frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} = \mathbf{F}_{12} + \mathbf{F}_{21} = \mathbf{F}_{12} - \mathbf{F}_{12} = 0$$

That is,

$$\mathbf{p}_{tot} = \text{constant}$$

or

$$\mathbf{p}_i = \mathbf{p}_f$$

That is, the linear momentum of each particle may change, but the total linear momentum of the system is the same at all times. This statement is known as the law of conservation of linear momentum: *If the net external force on a system is zero, the total linear momentum of the system remains unchanged* (*constant*). In terms of components, we have $p_{ix} = p_{fx}$, $p_{iy} = p_{fy}$, and $p_{iz} = p_{fz}$. In solving problems involving collisions, p_i and p_f refers to the total momentum of the system immediately before and immediately after the collision, respectively. For a two-particle system, we have

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f}$$

From the principle of invariance, the law of conservation of momentum is valid with respect to any inertial frame of reference. Furthermore, as the law of conservation of energy, the law of conservation of momentum is valid in relativity and quantum mechanics.

5.3 Impulse and Momentum

Impulse is a quantity that defines how a certain force acting on a particle changes the linear momentum of that particle. Now, consider a time-dependent force acting on a particle. From Newton's second law ($\mathbf{F} = d\mathbf{p}/dt$), we have

$$d\mathbf{p} = \mathbf{F}dt$$

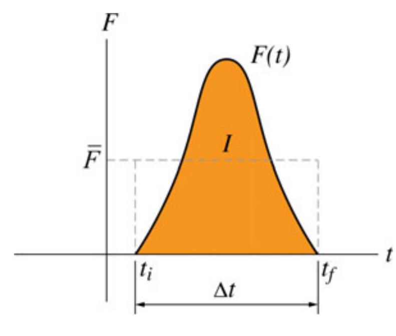

Fig. 5.2 One example of the variation of F over time

$$\int_{p_i}^{p_f} d\mathbf{p} = \int_{t_i} \mathbf{F}dt$$

$$\mathbf{p}_f - \mathbf{p}_i = \triangle\mathbf{p} = \int_{t_i}^{t_f} \mathbf{F}dt$$

The right side of the equation is a vector quantity known as the impulse I

$$\mathbf{I} = \int_{t_i}^{t_f} \mathbf{F}dt$$

Hence,

$$\mathbf{I} = \triangle\mathbf{P}$$

Which is known as the impulse–momentum theorem. In component form, we have $I_x = \triangle p_x$, $I_y = \triangle p_y$, and $I_z = \triangle p_z$. That is, *the impulse of a force that acts on a particle during a time interval is equal to the change in the momentum of the particle during that interval*. The direction of the impulse is in the same direction as the change of momentum. If F has a constant direction, the variation of its magnitude with time may be of the form as shown in Fig. 5.2. The average of F is given by

$$\overline{\mathbf{F}} = \frac{1}{\triangle t}\int_{t_i}^{t_f} \mathbf{F}dt$$

And thus, I can be written as

$$\mathbf{I} = \triangle\mathbf{p} = \overline{\mathbf{F}}\triangle t$$

That is, $\overline{\mathbf{F}}$ is a constant force that gives the same impulse as F. In the case of a collision between two bodies, the variation of the impulsive force that each body exerts on the other during the collision time takes the form as shown in Fig. 5.2.

5.4 Collisions

As discussed previously, when two bodies collide, they exert large forces on one another (during the time of the collision) called impulsive forces. These forces are very large such that any other forces (e.g., friction or gravity) present during the short time of the collision can be neglected. This approximation is known as the impulse approximation. For example, if a golf ball was hit by a golf club, the change in the momentum of the ball can be assumed to be only due to the impulsive force exerted on it by the club. The change in its momentum due to any other force present during the collision can be neglected. That is, the force in the expression $I = \triangle p = \overline{F}\triangle t$ can be assumed to be the impulsive force only The neglected forces present during the collision time are external to the two-body system, whereas the impulsive forces are internal. The two-body system can therefore be considered to be isolated during the short time of the collision (which is in the order of a few milliseconds). Hence, the total linear momentum of the system is conserved during the collision, which enables us to apply the law of conservation of momentum immediately before and immediately after the collision. In general, for any type of collision, the total linear momentum is conserved during the time of the collision. That is, $p_i = p_f$., where p_i and p_f are the momenta immediately before and after the collision. In the next sections, we will define various types of two- body collisions, depending on whether or not the kinetic energy of the system is conserved.

Example 5.1 A 50 g golf ball initially at rest is struck by a golf club. The golf club exerts a force on the ball that varies during a very short time interval from zero before impact, to a maximum value and back to zero when the ball is no longer in contact with the club. If the ball is given a speed of 25 m/s, and if the club is in contact with the ball for $7 \times 10^{-}4$ s, find the average force exerted by the club on the ball.

Solution 5.1 The impulse of the force is

$$I = \triangle p = mv_f - 0 = (0.05\,\text{kg})(25\,\text{m/s}) = 1.25\,\text{kg}\cdot\text{m/s}$$

the average force exerted on the ball by the club is then

$$\overline{F} = \frac{I}{\triangle t} = \frac{(1.25\,\text{kgm/s})}{(7\times 10^{-4}\,\text{s})} = 1785.7\,\text{N}$$

Example 5.2 A canon placed on a carriage fires a 250 kg ball to the horizontal with a speed of 50 m/s. If the mass of the canon and the carriage is 4000 kg, find the recoil speed of the canon.

Solution 5.2 Because there are no external horizontal forces acting on the cannon-carriage-ball system, then the total momentum of the system is constant (conserved) in the x-direction

$$p_{fx} = p_{ix}$$

$$m_1 v_{1f} + m_2 v_{2f} = 0$$

therefore,

$$v_{2f} = \frac{-m_1}{m_2} v_{1f} = -\frac{(250\,\text{kg})}{(4000\,\text{kg})}(50\,\text{m/s}) = -3.1\,\text{m/s}$$

i.e., the cannon recoils in the negative x-direction.

Example 5.3 A hockey puck of mass 0.16 kg traveling on a smooth ice surface collides with the court's edge. If its initial and final velocities are $\mathbf{v_i} = -2\ \mathbf{i}$m/s and $\mathbf{v_f} = 1\ \mathbf{i}$m/s and if the hockey puck is in contact with the wall for 2 ms, find the impulse delivered to the puck and the average force exerted on it by the wall.

Solution 5.3

$$\mathbf{I} = \triangle\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = m\mathbf{v}_f - m\mathbf{v}_i = (0.16\,\text{kg})((1\,\text{m/s}) - (-2\,\text{m/s}))\mathbf{i} = 0.48\mathbf{i}\,\text{kg}\cdot\text{m/s}$$

$$\overline{\mathbf{F}} = \frac{\mathbf{I}}{\triangle t} = \frac{(0.48\ \mathbf{i}\,\text{kg}\cdot\text{m/s})}{(0.002\,\text{s})} = 240\ \mathbf{i}\,\text{N}$$

Example 5.4 A 0.5 kg hockey puck is initially moving in the negative y-direction as shown in Fig. 5.3, with a speed of 7 m/s. If a hockey player hits the puck giving it a velocity of magnitude 12 m/s in a direction of 60° to the vertical, and if the collision lasts for 0.008 s, find the impulse due to the collision and the average force exerted on the puck.

Solution 5.4 Along the x-direction, we have

$$p_{ix} = mv_{ix} = 0$$

and

$$p_{fx} = mv_{fx} = (0.5\,\text{kg})(12\text{m/s})\cos 30^o = 5.2\,\text{kg}\cdot\text{m/s}$$

along the y-direction, we have

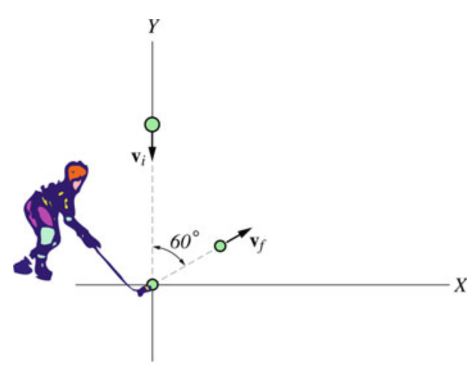

Fig. 5.3 A hockey player changing the momentum of the puck

$$p_{iy} = mv_{iy} = (0.5\,\text{kg})(-7\,\text{m/s}) = -3.5\,\text{kg}\cdot\text{m/s}$$

and

$$p_{fy} = mv_{fy} = (0.5\,\text{kg})(12\,\text{m/s})\sin 30^\circ = 3\,\text{kg}\cdot\text{m/s}$$

Therefore, the impulse of the force in each direction is

$$I_x = p_{fx} - p_{ix} = (5.2\text{ kg}\cdot\text{m/s}) - 0 = 5.2\,\text{kg}\cdot\text{m/s}$$

and

$$I_y = p_{fy} - p_{iy} = (3\,\text{kg}\cdot\text{m/s}) - (-3.5\,\text{kg}\cdot\text{m/s}) = 6.5\,\text{kg}\cdot\text{m/s}$$

$$\mathbf{I} = (5.2\,\mathbf{i} + 6.5\,\mathbf{j})\,\mathbf{k}\text{g}\cdot\text{m/s}$$

$$I = \sqrt{(5.2\,\text{kgm/s})^2 + (6.5\,\text{kgm/s})^2} = 8.3\,\text{kg}\cdot\text{m/s}$$

The direction of the impulse is

$$\tan\theta = \frac{I_y}{I_x} = \frac{(6.5\,\text{kg}\cdot\text{m/s})}{(5.2\,\text{kg}\cdot\text{m/s})} = 1.25$$

$$\theta = 51.3^{\text{o}}$$

where θ is measured from the positive x-axis. The average force acting on the puck is

$$\overline{F} = \frac{I}{\triangle t} = \frac{(8.3\,\text{kg}\cdot\text{m/s})}{(0.008\,\text{s})} = 1037.5\,\text{N}$$

Example 5.5 Two ice skaters of masses $m_1 = 50\,\text{kg}$ and $m_1 = 62\,\text{kg}$ standing face to face push each other on a frictionless horizontal surface. If skater (1) recoils with a speed of $5\,\text{m/s}$, find the recoil speed of the other skater.

Solution 5.5 For the two-skater system, the sum of the vertical forces are zero (weight and normal forces) and the forces exerted by one skater on the other is internal to the system. That is, there are no external forces acting on the system and the total momentum is conserved. Because the motion takes place in a straight line, we have

$$p_{1i} + p_{2i} = p_{1f} + p_{2f}$$

$$0 = m_1 v_{1f} + m_2 v_{2f}$$

and hence,

$$v_{2f} = \frac{-m_1}{m_2} v_{1f} = \frac{-(50\,\text{kg})}{(62\,\text{kg})}(5\,\text{m/s}) = -4.03\,\text{m/s}$$

Example 5.6 A particle is moving in space under the influence of a force. If its momentum as a function of time is

$$\mathbf{p} = ((4t^2 + t)\mathbf{i} - (3t - 1)\mathbf{j} + (5t^3 + 2t)\mathbf{k})\ \mathbf{k}\text{g. m/s}$$

(a) Find the force acting on the particle at any time; (b) Find the impulse of the force from $t = 0$ to $t = 1$ s.

Solution 5.6 (a)

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = ((8t + 1)\mathbf{i} - 3\mathbf{j} + (15t^2 + 2)\mathbf{k})\,\text{N}$$

(b)

$$\mathbf{I} = \triangle\mathbf{p} = (5\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}) - \mathbf{j} = (5\mathbf{i} - 3\mathbf{j} + 7\mathbf{k})\,\text{kg.m/s}$$

5.4.1 Elastic Collisions

An elastic collision is one in which the total kinetic energy, as well as momentum, of the two-colliding-body system is conserved. These collisions exist when the impulsive force exerted by one body on the other is conservative. Such force converts the kinetic energy of the body into elastic potential energy when the two bodies are in contact. It then reconverts the elastic potential energy into kinetic energy when there is no more contact. After collision, each body may have a different velocity and therefore a different kinetic energy. However, the total energy as well as the total momentum of the system is constant during the time of the collision. An example of such collisions is those between billiard balls.

5.4.2 Inelastic Collisions

An inelastic collision is one in which the total kinetic energy of the two-colliding-body system is not conserved, although momentum is conserved. In such a collision, some of the kinetic energy of the system is lost due to deformation and appear as internal or thermal energy. In other words, the (internal) impulsive forces are not conservative. Therefore, the kinetic energy of the system before the collision is less than that after the collision. If the two colliding objects stick together, the collision is said to be perfectly inelastic. There are some types of collisions in which the total kinetic energy after the collision occurs is greater than that before it occurs. This type of collision is called an explosive collision.

Fig. 5.4 Two particles of masses m_1 and m_2 experiencing an elastic head-on collision

5.4.3 Elastic Collision in One Dimension

When a collision takes place in one dimension, it is referred to as a head-on collision. Consider two particles of masses m_1 and m_2 experiencing an elastic head-on collision as in Fig. 5.4. Applying the law of conservation of energy and the law of conservation of linear momentum gives

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f}$$

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$

Solving these equations for v_{1f} and v_{2f}, we get

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i} \tag{5.2}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_{2i} \tag{5.3}$$

5.4.3.1 Special Cases

1. If $m_1 = m_2$, it follows from Eqs. 5.2 and 5.3 that $v_{1f} = v_{2i}$ and $v_{2f} = v_{1i}$. In other words, if the particles have equal masses they exchange velocities.

2. If m_2 is stationary ($v_{2i} = 0$) , then from Eqs. 5.2 and 5.3, we have

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} \tag{5.4}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} \tag{5.5}$$

In that case m_2 is called the target and m_1 is called the projectile. Furthermore, if $m_1 \gg m_2$, then from Eqs. 5.4 and 5.5, we find that $v_{1f} \approx v_{1i}$ and $v_{2f} \approx 2v_{1i}$. While if $m_2 \gg m_1$, then from Eqs. 5.4 and 5.5, we see that $v_{1f} \approx -v_{1i}$, and $v_{2f} \approx v_{2i} = 0$.

5.4.4 Inelastic Collision in One Dimension

Figure 5.5 shows a one-dimensional (head-on) perfectly inelastic collision between two particles of mass m_1 and m_2. Here, the kinetic energy of the system is not conserved, but the law of conservation of linear momentum still holds

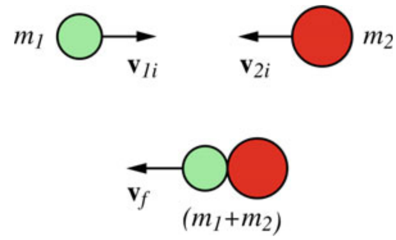

Fig. 5.5 A one dimensional (head-on) perfectly inelastic collision between two particles of mass m_1 and m_2

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = (m_1 + m_2)\mathbf{v}_f$$

$$\mathbf{v}_f = \frac{m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i}}{m_1 + m_2}$$

5.4.5 Coefficient of Restitution

For any collision between two bodies in one dimension, the coefficient of restitution is defined as

$$e = \frac{v_{2f} - v_{1f}}{v_{1i} - v_{2i}}$$

where v_{1i} and v_{2i} are velocities before the collision. v_{1f} and v_{2f} are velocities after the collision. $|v_{1i} - v_{2i}|$ is called the relative speed of approach and $|v_{2f} - v_{1f}|$ is the relative speed of recession.

- If $e = 1$ the collision is perfectly elastic.
- If $e < 1$ the collision is inelastic.
- If $e = 0$ the collision is perfectly inelastic (the two bodies stick together).

Example 5.7 Two marble balls of masses $m_1 = 7\,\text{kg}$ and $m_2 = 3\,\text{kg}$ are sliding toward each other on a straight frictionless track. If they experience a head-on elastic collision and if the initial velocities of m_1 and m_2 are $0.5\,\text{m/s}$ to the right and $2\,\text{m/s}$ to the left, respectively, find the final velocities of m_1 and m_2.

Solution 5.7 For an elastic head-on collision, we have

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i} = (0.4)(0.5\,\text{m/s}) + (0.6)(-2\,\text{m/s}) = -1\,\text{m/s}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_{2i} = (1.4)(0.5\,\text{m/s}) + (-0.4)(-2\,\text{m/s}) = 1.5\,\text{m/s}$$

Example 5.8 The ballistic pendulum consists of a large wooden block suspended by a light wire (see Fig. 5.6). The system is used to measure the speed of a bullet where the bullet is fired horizontally into the block. The collision is perfectly inelastic and the system (bullet+block) swings up a height h.

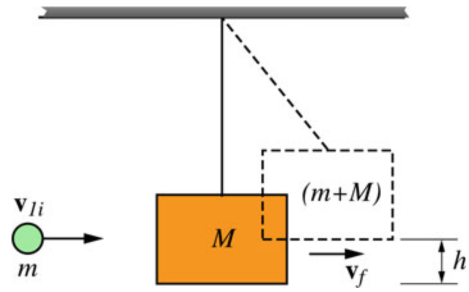

Fig. 5.6 The ballistic pendulum consists of a large wooden block suspended by a light wire

If $M = 3\,\text{kg}$, $m = 5\,\text{g}$ and $h = 5\,\text{cm}$, find (a) the initial speed of the bullet; (b) the mechanical energy lost due to the collision.

Solution 5.8 (a) Using the impulse approximation, the law of conservation of momentum gives the velocities just before and after the collision when the string is still nearly vertical. For a perfectly inelastic collision, the total momentum is conserved but the total kinetic energy is not conserved during the collision. Thus, we have

$$mv_{1i} = (m + M)v_f$$

$$v_{1i} = \frac{(m + M)}{m}v_f$$

After the collision, the energy of the (bullet +block+earth) system is conserved since the gravitational force is the only force acting in the system.

$$E_i = E_f$$

$$\frac{1}{2}(m + M)v_f^2 = (m + M)gh$$

$$v_f = \sqrt{2gh}$$

That gives

$$v_{1i} = \frac{(m + M)}{m}\sqrt{2gh} = \frac{(3.005\text{ kg})}{(0.005\text{ kg})}\sqrt{2(9.8\,\text{m/s}^2)(0.05\,\text{m})} = 595\text{ m/s}$$

(b) The kinetic energy of the bullet before collision is

$$\frac{1}{2}mv_{1i}^2 = \frac{1}{2}0.005\text{ kg})(595\,\text{m/s})^2 = 885\text{ J}$$

After collision, the kinetic energy of the (bullet+block) is

$$\frac{1}{2}(m + M)v_f^2 = (m + M)\,(gh) = (3.005\,\text{kg})(9.8\,\text{m/s}^2)(0.05\,\text{m}) = 1.5\text{ J}$$

therefore,

$$\triangle E = (885\text{ J}) - (1.5\text{ J}) = 883.5\text{ J}$$

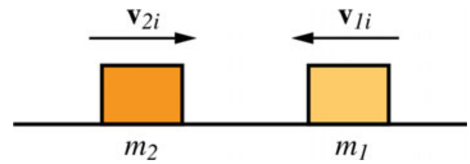

Fig. 5.7 Two blocks colliding head-on on a frictionless surface

That is, nearly, all the mechanical energy is dissipated and converted into internal (thermal) energy of the (block+bullet) system.

Example 5.9 Two masses $m_1 = 0.8\,\text{kg}$ and $m_2 = 0.5\,\text{kg}$ are heading toward each other with speeds of $0.25\,\text{m/s}$ and $-0.5\,\text{m/s}$, respectively. If they have a perfectly inelastic collision, find the final velocity of the system just after the collision.

Solution 5.9

$$v_f = \frac{m_1v_{1i} + m_2v_{2i}}{(m_1 + m_2)} = \frac{(0.8\,\text{kg})(0.25\,\text{m/s}) - (0.5\,\text{kg})(0.5\,\text{m/s})}{(1.3\,\text{kg})} = -0.04\,\text{m/s}$$

Example 5.10 Two blocks $m_1 = 2\,\text{kg}$ and $m_2 = 1\,\text{kg}$ collide head-on with each other on a frictionless surface (see Fig. 5.7. If $v_{1i} = -10\,\text{m/s}$ and $v_{2i} = 15\,\text{m/s}$ and the coefficient of restitution is $e = 1/4$, determine the final velocities of the masses just after the collision.

Solution 5.10

$$e = \frac{v_{2f} - v_{1f}}{v_{1i} - v_{2i}}$$

$$\frac{1}{4} = \frac{v_{2f} - v_{1f}}{(-25\text{m/s})}$$

$$v_{2f} - v_{1f} = -6.25\,\text{m/s} \tag{5.6}$$

From the conservation of momentum, we have

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}$$

$$(2\,\text{kg})(-10\,\text{m/s}) + (1\,\text{kg})(15\,\text{m/s}) = (2\,\text{kg})v_{1f} + (1\,\text{kg})v_{2f}$$

That gives

$$v_{2f} + (2\,\text{kg})v_{1f} = -5\,\text{m/s} \tag{5.7}$$

Solving Eqs. 5.6 and 5.7 gives $v_{1f} = 0.42\,\text{m/s}$ and $v_{2f} = -5.83\,\text{m/s}$.

Example 5.11 A $m_1 = 5\,\text{g}$ bullet is fired horizontally at the center of a wooden block with a mass of $m_2 = 2\,\text{kg}$. The bullet embeds itself in the block and the two slides a distance of 0.5 m on a rough surface ($\mu_k = 0.2$) before coming to rest. Find the initial speed of the bullet.

Solution 5.11 Applying the law of conservation of momentum immediately before and after the collision gives

$$p_{ix} = p_{fx}$$

$$m_1 v_{1i} + 0 = (m_1 + m_2) v_f$$

$$v_{1i} = \frac{(2.005\,\text{kg})}{(0.005\,\text{kg})} v_f = (401) v_f$$

by taking the (block+bullet) as the system after the collision until it comes to rest, we have

$$K_f + U_f = K_i + U_i + \triangle K_{ext}$$

that gives

$$0 = \frac{1}{2}(m_1 + m_2) v_f^2 - \mu_k (m_1 + m_2) g d$$

$$v_f = \sqrt{2\mu_k g d} = \sqrt{2(0.2)(9.8\,\text{m/s}^2)(0.5\,\text{m})} = 1.4\,\text{m/s}$$

Hence,

$$v_{1i} = (401)(1.4\,\text{m/s}) = 561.4\,\text{m/s}$$

5.4.6 Collision in Two Dimension

When a collision takes place in space, the total linear momentum is conserved along each of the $x-$, y-, and z-directions. That is, $p_{ix} = p_{fx}$, $p_{iy} = p_{fy}$, and $p_{iz} = p_{fz}$. Here, we will analyze a two-dimensional elastic collision between two particles where one particle is moving and the other is at rest as shown in Fig. 5.8. This type of collision is known as a glancing collision. Since the collision is elastic, it follows that the total linear momentum as well as the kinetic energy of the system are conserved. Applying these laws immediately before and immediately after the collision, we have $p_{ix} = p_{fx}$ and $p_{iy} = p_{fy}$ or

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

and

$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

From Fig. 5.8, we have

$$m_1 v_{1i} = m_1 v_{1f} \cos\alpha_1 + m_2 v_{2f} \cos\alpha_2$$

and

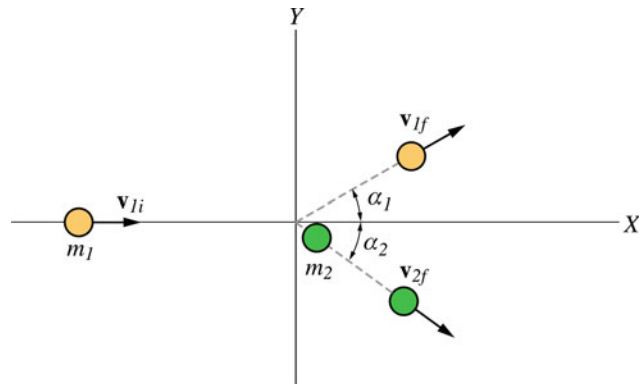

Fig. 5.8 A two dimensional elastic collision between two particles where one particle is moving and the other is at rest

$$0 = m_1 v_{1f} \sin\alpha_1 + m_2 v_{2f} \sin\alpha_2$$

Furthermore,

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

Therefore, we have three equations and seven unknown quantities. By knowing any four of these quantities, the three equations for the three variables can be solved.

Example 5.12 A ball of mass of 2 kg is sliding along a horizontal frictionless surface at a speed of 3 m/s. It then collides with a second ball of mass of 5 kg that is initially at rest. After the collision, the second ball is deflected with a speed of 1 m/s at an angle of 30° below the horizontal as shown in Fig. 5.9. (a) Find the final velocity of the first ball; (b) show that the collision is inelastic; (c) suppose that the two balls have equal masses and the collision is perfectly elastic, show that $\theta_1 + \theta_2 = 90^\circ$.

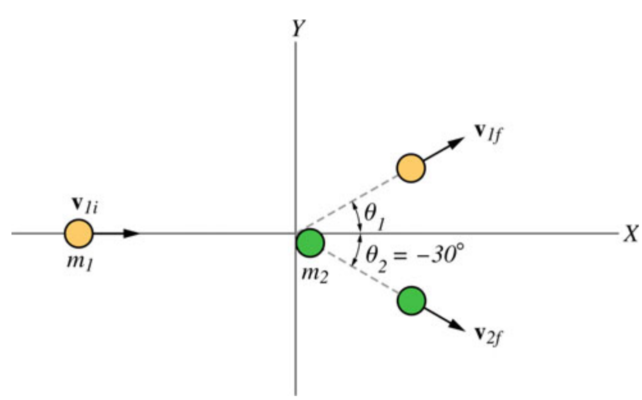

Fig. 5.9 A ball sliding along a horizontal frictionless surface collides with a second ball that is initially at rest

Solution 5.12 Applying the law of conservation of momentum immediately before and after the collision in each direction gives $p_{ix} = p_{fx}$ and $p_{iy} = p_{fy}$. Thus,

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

$$v_{1fx} = \frac{m_1 v_{1ix} + m_2 v_{2ix} - m_2 v_{2fx}}{m_1} = \frac{(2\,\text{kg})(3\,\text{m/s}) + 0 - ((5\,\text{kg})(1\,\text{m/s})\cos(-30))}{(2\,\text{kg})}$$

$$v_{1fx} = 0.84\,\text{m/s}$$

Along the y-direction, we have

$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

$$v_{1fy} = \frac{m_1 v_{1iy} + m_2 v_{2iy} - m_2 v_{2fy}}{m_1} = \frac{0 - ((5\,\text{kg})(1\,\text{m/s})\sin(-30^{\text{o}}))}{(2\,\text{kg})}$$

$$v_{1fy} = 1.25\,\text{m/s}$$

Thus, the final velocity of the first ball is

$$v_{1f} = \sqrt{v_{1fx}^2 + v_{1fy}^2} = \sqrt{(0.84\,\text{m/s})^2 + (1.25\,\text{m/s})^2} = 1.5\,\text{m/s}$$

The direction of the velocity is

$$\tan\theta_1 = \frac{v_{1fy}}{v_{1fx}} = \frac{(1.25\,\text{m/s})}{(0.84\,\text{m/s})} = 1.5$$

$$\theta_1 = 56^{\text{o}}$$

(b) The total kinetic energy before the collision is

$$K_i = \frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2}(2\,\text{kg})(3\,\text{m/s})^2 = 9\,\text{J}$$

The total kinetic energy after the collision is

$$K_f = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 = \frac{1}{2}(2\,\text{kg})(1.5\,\text{m/s})^2 + \frac{1}{2}(5\,\text{kg})(1\,\text{m/s})^2 = 4.75\,\text{J}$$

That is, some of the energy of the system is lost and thus the collision is inelastic.

(c) In a perfectly elastic collision, both the total momentum and the total mechanical energy of the system are conserved. That is

$$p_{ix} = p_{fx}$$

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

$$v_{1i} = v_{1f}\cos\theta_1 + v_{2f}\cos\theta_2 \qquad (5.8)$$

$$p_{iy} = p_{fy}$$

$$0 = v_{1f}\sin\theta_1 - v_{2f}\sin\theta_2$$

$$v_{1f}\sin\theta_1 = v_{2f}\sin\theta_2 \qquad (5.9)$$

From the conservation of kinetic energy, we have

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

or

$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2 \qquad (5.10)$$

Substituting Eq. 5.8 into Eq. 5.9 gives

$$v_{1i} = v_{2f}\frac{\sin\theta_2}{\sin\theta_1}\cos\theta_1 + v_{2f}\cos\theta_2$$

or

$$v_{1i} = \frac{v_{2f}\sin(\theta_1 + \theta_2)}{\sin\theta_1} \qquad (5.11)$$

Substituting Eq. 5.11 into Eq. 5.10 gives

$$\frac{v_{2f}^2 \sin^2(\theta_1 + \theta_2)}{\sin^2\theta_1} = \frac{v_{2f}^2 \sin^2\theta_2}{\sin^2\theta_1} + v_{2f}^2$$

Therefore,

$$\sin^2(\theta_1 + \theta_2) = \sin^2\theta_1 + \sin^2\theta_2$$

This is satisfied only if $\theta_1 + \theta_2 = 90^{\text{o}}$.

Example 5.13 A 1200 kg car traveling east at a speed of 18 m/s collides with another car of mass of 2500 kg that is traveling north at a speed of 23 m/s as shown in Fig. 5.10. If the collision is perfectly inelastic, how much mechanical energy is lost due to the collision?

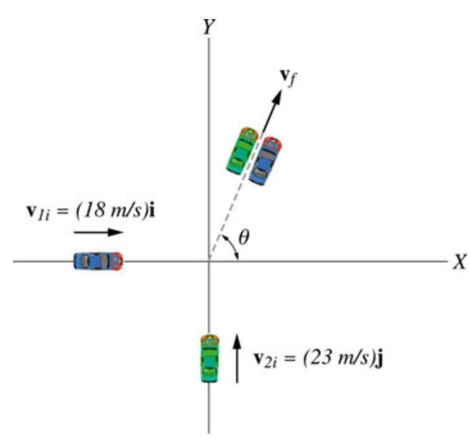

Fig. 5.10 A 1200 kg car traveling east at a speed of 18 m/s collides with another car of mass of 2500 kg that is traveling north at a speed of 23 m/s

Solution 5.13

$$p_{ix} = p_{fx}$$

$$m_1 v_{1ix} = (m_1 + m_2) v_{fx}$$

$$v_{fx} = \frac{m_1 v_{1ix}}{(m_1 + m_2)} = \frac{(1200\,\text{kg})(18\,\text{m/s})}{(3700\,\text{kg})} = 5.8\,\text{m/s}$$

$$p_{iy} = p_{fy}$$

$$m_2 v_{2iy} = (m_1 + m_2) v_{fy}$$

$$v_{fy} = \frac{m_2 v_{2iy}}{(m_1 + m_2)} = \frac{(2500\,\text{kg})(23\,\text{m/s})}{(3700\,\text{kg})} = 15.5\,\text{m/s}$$

$$v_f = \sqrt{v_{fx}^2 + v_{fy}^2} = \sqrt{(5.8\,\text{m/s})^2 + (15.5\,\text{m/s})^2} = 16.5\,\text{m/s}$$

The direction of v_f is

$$\theta = \tan^{-1}\frac{v_{fy}}{v_{fx}} = \tan^{-1}\frac{(15.5\,\text{m/s})}{(5.8\,\text{m/s})} = 69.5^{\text{o}}$$

from the positive x-axis. The change in the kinetic energy of the system is

$$\triangle K = K_f - K_i = \frac{1}{2}(m_1 + m_2)v_f^2 - \left(\frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2\right)$$

$$= \frac{1}{2}(3700\,\text{kg})(16.5\,\text{m/s})^2 - \left(\frac{1}{2}(1200\,\text{kg})(18\,\text{m/s})^2 + \frac{1}{2}(2500\,\text{kg})(23\,\text{m/s})^2\right)$$

$$\triangle K = -3.5 \times 10^5\ \text{J}$$

5.5 Torque

Consider a force F acting on a particle that has a position vector r with respect to some origin O that is in an inertial frame. The torque is a vector quantity that measures the tendency of that force to rotate the particle about O and is defined as

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

The direction of $\boldsymbol{\tau}$ is perpendicular to the plane formed by **r** and **F** and its sense is given by the right-hand rule or of advance of a right-handed screw rotating from **r** to **F**. From the vector product definition, this quantity has a magnitude given by

$$\tau = rF\sin\phi$$

where ϕ is the smaller angle between r and F, τ is positive if the force tends to rotate the particle counterclockwise and negative if it tends to rotate it clockwise. If $\phi = 0$ or 180°, the force is radial and thus it has no rotating tendency. In component form, we may write

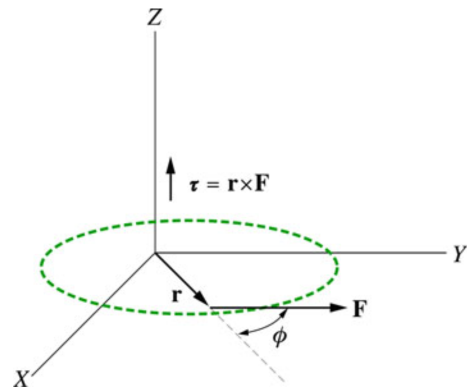

Fig. 5.11 A particle in the x-y plane exposed to a force that lies in that plane. The resulting torque is then perpendicular to the x-y plane parallel to the z-axis

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})$$

$$= (yF_z - zF_y)\mathbf{i} + (zF_x - xF_z)\mathbf{j} + (xF_y - yF_x)\mathbf{k}$$

Let us consider a particle in the x–y plane exposed to a force that lies in that plane (see Fig. 5.11). The resulting torque is then perpendicular to the x–y plane parallel to the z-axis. τ can also be written as

$$\tau = Fd$$

where $d = r\sin\phi$ is called the moment arm of F where it represents the perpendicular distance from the axis of rotation to the line of action of F as shown in Fig. 5.12. Note that because τ depends on r, it follows that τ depends on the choice of the origin O. The force F can be resolved into two components $F_t = F\sin\phi$ and $F_r = F\cos\phi$. Since the line of action of F_r passes through O, it has no rotating effect. Hence, F_t is the only component of F that causes rotation. The SI unit of torque is the Newton-metre (N m). This unit is the same unit of work, but they are different quantities and the torque should never be expressed in joules.

Example 5.14 A force $\mathbf{F} = (-2t\mathbf{i} - (t^2 - 3)\mathbf{j} + 4t^5\mathbf{k})$ N acts on a particle that has a position vector $\mathbf{r} = \left(-6\mathbf{i} + 5t\mathbf{j} + (\frac{t}{2} - 1)\,\mathbf{k}\right)$ m find the torque of the particle about the origin at $t = 1$ s.

Solution 5.14

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & 5t & (\frac{t}{2} - 1) \\ -2t & -(t^2 - 3) & 4t^5 \end{vmatrix}$$

Evaluating this at $t = 1$ s gives