Solution to problem 32

Differentiating gives $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{a(cx+d)-c(ax+b)}{(cx+d)^2} = \dfrac{ad-bc}{(cx+d)^2}$.

For the first integral, set $y = \dfrac{x+1}{x+3}$:

$$\int_0^1 \frac{1}{(x+3)^2}\ln\left(\frac{x+1}{x+3}\right)\mathrm{d}x = \frac{1}{2}\int_{1/3}^{1/2}\ln y\mathrm{d}y = \frac{1}{2}\Big[y\ln y - y\Big]_{1/3}^{1/2},$$

which gives the required answer. (The integral of $\ln y$ is a standard integral; it can be done by parts, first substituting $z = \ln y$, if you like.)

The second integral can be expressed as the sum of two integrals of the same form as the first integral, since

$$\ln\left(\frac{x^2+3x+2}{(x+3)^2}\right) = \ln\left(\frac{(x+1)(x+2)}{(x+3)^2}\right) = \ln\left(\frac{x+1}{x+3}\right) + \ln\left(\frac{x+2}{x+3}\right) .$$

We have already done the first of these integrals. Using the substitution $y = \dfrac{x+2}{x+3}$ in the second of these integrals gives

$$\int_{2/3}^{3/4} \ln y \ \mathrm{d}y = \frac{17}{12}\ln 3 - \frac{13}{6}\ln 2 - \frac{1}{12} .$$

For the final integral, note that

$$\ln\left(\frac{x+1}{x+2}\right) = \ln\left(\frac{x+1}{x+3}\right) - \ln\left(\frac{x+2}{x+3}\right) ,$$

so the required integral is the difference of the two integrals that we summed in the previous part, i.e.

$$-\frac{5}{4}\ln 3 + \frac{23}{12}\ln 2 .$$

Post-mortem

I hope you had time to try the original version of this question, given in the comments above. The last part is a nice puzzle.

For the first integral of the original version, the change of variable gives

$$\frac{1}{5}\int_{\frac{1}{3}}^{1} y\ln y \ \mathrm{d}y = \frac{1}{90}\ln 3 - \frac{2}{45} .$$

I checked this using Wolfram Alpha. For the second integral, you have to guess the substitution. There are plenty of clues, but the most obvious place to start is the log. The argument of the log can be written as

$$\frac{2\sqrt{z}-5}{\sqrt{z}} .$$

It is now a bit of a leap in the dark (too much of a leap, I thought), but if we decide to convert this exactly into the previous integral, we should take $z = (x+1)^2$. And, magic (or contrived STEP question), it works, giving the first integral almost exactly (note especially that the limits transform as they should). The only difference is a factor of 2 so the answer is twice the previous answer.

A change of variable of the form $(*)$ is called a *linear fractional* or *Möbius* transformation. It is of great importance in the theory of the geometry of the complex plane.

Problem 33: More curve sketching (✓✓)

The curve C has equation
$$y = \frac{x}{\sqrt{x^2 - 2x + a}}\ ,$$
where the square root is positive. Show that, if $a > 1$, then C has exactly one stationary point. Sketch C when (i) $a = 2$ and (ii) $a = 1$.

1999 Paper II

Comments

You have to be sure of the definition of $\sqrt{x^2 - 2x + a}$ to do this question. You might think that it is ambiguous, because the square root could be positive or negative, but by convention it means the *positive* square root. Thus $\sqrt{x^2} = |x|$ (not x).

For the sketches, you just need the position of any stationary points, any other interesting points that the graph passes through, behaviour as $x \to \pm\infty$ and any vertical asymptotes. It should not be necessary in such a simple case to establish the nature of the stationary point(s). The sketches are not particularly difficult, but they do need thought because they turn out to be rather different from the sketches of polynomials or trigonometric functions that you are probably used to.

You should at some stage think about the conditions given on a. In fact, I doubt if many of you will want to leave it there. It is clear that what the examiner would really have liked you to do is to sketch C in the different cases that arise according to the value of a but was told that this would be too long and/or difficult for the exam; but I am sure that this is what you will try to do (or maybe what you have already done by the time you read this).

Please do not use your calculator (or Wolfram Alpha or Geogebra or Desmos[23], etc) for these sketches, except perhaps to check your answers: there is no point.

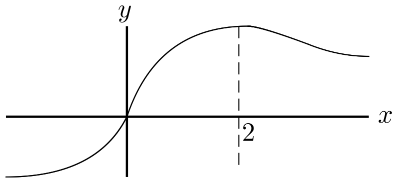
[23] https://www.desmos.com/calculator is brilliant: you can use a slider to change the value of a; but please don't.

Solution to problem 33

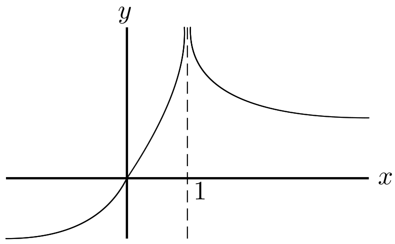
First differentiate to find the stationary points:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x^2-2x+a)-x(x-1)}{(x^2-2x+a)^{\frac{3}{2}}} = \frac{a-x}{(x^2-2x+a)^{\frac{3}{2}}}$$

which is only zero when $x = a$. I've used the quotient rule, but the product rule is just as good.



(i) When $a = 2$, the stationary point is at $(2, \sqrt{2})$ (which can be seen to be a maximum by evaluating the second derivative at $x = a$, though this is not necessary). The curve passes through $(0, 0)$. As $x \to \infty$, $y \to 1$; as $x \to -\infty$, $y \to -1$.



(ii) When $a = 1$, we can rewrite y as $\dfrac{x}{|x-1|}$. The modulus signs arise because the square root is taken to be positive (or zero — but that can be discounted here since it is in the denominator). The graph is as before, except that the maximum point has been stretched into a vertical asymptote (like a volcanic plug) at $x = 1$. Note that if $|x|$ is much larger than a, it doesn't really matter whether $a = 1$ or $a = 2$.

Post-mortem

The first thing to do after finishing the question is to try to understand the role of the parameter a. Part (ii) above is the key. Writing

$$\mathrm{f}(x) = x^2 - 2x + a = (x-1)^2 + a - 1\,,$$

we see that $\mathrm{f}(x) > 0$ for all x if $a > 1$ but if $a < 1$ there are values of x for which $\mathrm{f}(x) < 0$. The borderline value is $a = 1$, for which $\mathrm{f}(x) \geqslant 0$ and $\mathrm{f}(1) = 0$.

If $a < 1$, we find that $\mathrm{f}(x) = 0$ when $x = 1 \pm \sqrt{1-a}$ and $\mathrm{f}(x) < 0$ between these two values of x. The significance of this is that the square root in the denominator of the function we are trying to sketch is imaginary, so there is a gap in the graph between $x = 1 - \sqrt{1-a}$ and $x = 1 + \sqrt{1-a}$. There are vertical asymptotes at these values.

Is that the whole story? Well, no. We now have to decide whether the two significant points of our graph, namely $x = 0$ where the curve crosses the axis and $x = a$ where the curve has a maximum, lie inside or outside the forbidden zone $1 - \sqrt{1-a} \leqslant x \leqslant 1 + \sqrt{1-a}$. This turns out to depend on whether $a < 0$ or $a = 0$ or $0 < a < 1$. The graph will look qualitatively different in each case, so there is still quite a lot of work to do in this $a < 1$ case! We can see why it was excluded from the original question.

Problem 34: Trig. sum (✓✓✓)

Prove that

$$\sum_{k=0}^{n} \sin k\theta = \frac{\cos \frac{1}{2}\theta - \cos(n + \frac{1}{2})\theta}{2 \sin \frac{1}{2}\theta} \, . \tag{$*$}$$

(i) Deduce that, for $n \geqslant 1$,

$$\sum_{k=0}^{n} \sin\left(\frac{k\pi}{n}\right) = \cot\left(\frac{\pi}{2n}\right) \, .$$

(ii) By differentiating $(*)$ with respect to θ, or otherwise, show that, for $n \geqslant 1$,

$$\sum_{k=0}^{n} k \sin^2\left(\frac{k\pi}{2n}\right) = \frac{(n+1)^2}{4} + \frac{1}{4}\cot^2\left(\frac{\pi}{2n}\right) \, .$$

1999 Paper II

Comments

The very first part can be done by multiplying both sides of $(*)$ by $\sin \frac{1}{2}\theta$ and using the identity

$$2 \sin A \sin B = \cos(B - A) - \cos(B + A) \, .$$

It can also be done by induction (worth a try even if you do it by the above method) or by considering the imaginary part of $\sum \exp(ik\theta)$ (summing this as a geometric progression), if you know about de Moivre's theorem.

Before setting pen to paper for part (ii), it pays to think very hard about simplifications of the right-hand side of $(*)$ that might make the differentiation more tractable – perhaps bearing in mind the calculations required for part (i).

In the original question, you were asked to show that, for large n,

$$\sum_{k=0}^{n} \sin\left(\frac{k\pi}{n}\right) \approx \frac{2n}{\pi}$$

and

$$\sum_{k=0}^{n} k \sin^2\left(\frac{k\pi}{2n}\right) \approx \left(\frac{1}{4} + \frac{1}{\pi^2}\right) n^2 \, ,$$

using the approximations, valid for small θ, $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. Of course, these results follow quickly from the exact results; but if you only want approximate results you can save yourself a bit of work by approximating early to avoid doing some of the trigonometric calculations. I thought that the exact results were nicer than the approximate results so I changed the question a bit for this book (though I felt a bit guilty, because approximations are an important part of mathematics).

Solution to problem 34

For the first part, we will show that $\displaystyle\sum_{k=0}^{n} 2\sin k\theta \sin \tfrac12\theta = \cos\tfrac12\theta - \cos(n+\tfrac12)\theta$. Starting with the left hand side, we have:

$$\sum_{k=0}^{n} 2\sin k\theta \sin\tfrac12\theta = \sum_{k=0}^{n}\left[-\cos(k+\tfrac12)\theta + \cos(k-\tfrac12)\theta\right]$$

which gives the result immediately, since almost all the terms in the sum cancel. (Write out a few terms if you are not certain of this.)

(i) Let $\theta = \pi/n$. Then

$$\sum_{k=0}^{n} \sin\left(\frac{k\pi}{n}\right) = \frac{\cos\frac12(\pi/n) - \cos(\pi + \frac12(\pi/n))}{2\sin\frac12(\frac{\pi}{n})} = \frac{\cos\frac12(\pi/n) + \cos\frac12(\pi/n))}{2\sin\frac12(\pi/n)} = \cot\tfrac12(\pi/n)$$

as required.

(ii) Differentiating the left hand side of $(*)$ and using a double angle trig. formula gives

$$\sum_{k=0}^{n} k\cos k\theta = \sum_{k=0}^{n} k\left(1 - 2\sin^2\tfrac12 k\theta\right) = \tfrac12 n(n+1) - 2\sum_{k=0}^{n} k\sin^2\tfrac12 k\theta\ . \tag{$\dagger$}$$

Before attempting to differentiate the right hand side of $(*)$ it is a good idea to write it in a form that gets rid of some of the fractions. Omitting for the moment the factor $1/2$, we have

$$\text{RHS of } (*) \ = (1-\cos n\theta)\cot\tfrac12\theta + \sin n\theta$$

and differentiating gives $-\frac12(1-\cos n\theta)\,\text{cosec}^2\,\frac12\theta + n\sin n\theta\cot\frac12\theta + n\cos n\theta$. Now setting $\theta = \pi/n$ gives

$$-\,\text{cosec}^2\,\tfrac12(\pi/n) - n\ . \tag{$\ddagger$}$$

Setting $\theta = \pi/n$ in equation $(\dagger)$ and comparing with $(\ddagger)$ (remembering that there is a factor of $1/2$ missing) gives the required result.

Post-mortem

Now that I have had another go at this question it does not seem terribly interesting. At first, I thought I might ditch it. Then I thought that it perhaps was worthwhile: keeping cool under the pressure of the differentiation for part (ii) – it should just be a few careful lines – and getting it out is a good confidence booster. Anyway, now that I have slogged[24] through it, I am going to leave it in.

[Next day] I recall now that the reason that I included this question in the first place was because in its original form (with approximate answers as given in the comments section above), the result for part (ii) can be obtained very quickly from part (i) by differentiating both sides with respect to π and using a double angle formula. Try it!

Unfortunately, you'd have a hard time justifying this approach, and anyway it is not going to work for the exact result.[25]

---

[24] I should come clean at this point: I just differentiated $(*)$ on autopilot, without considering whether I could simplify the task by rewriting the formula as suggested in my hint on the previous page. It was hard work. Serves me right for not following my own general advice.

[25] For large n we are just adding up a lot of sin curves and it doesn't much matter what the exact value of π is (so we can take it to be a variable); for the exact result, the exact value of π does matter.

Problem 35: Roots of a cubic equation (✓✓✓)

Consider the cubic equation

$$x^3 - px^2 + qx - r = 0\ ,$$

where $p \neq 0$ and $r \neq 0$.

(i) If the three roots can be written in the form ak^{-1}, a and ak for some constants a and k, show that one root is q/p and that $q^3 - rp^3 = 0$.

(ii) If $r = q^3/p^3$, show that q/p is a root and that the product of the other two roots is $(q/p)^2$. Deduce that the roots are in geometric progression.

(iii) Find a necessary and sufficient condition involving p, q and r for the roots to be in arithmetic progression.

1999 Paper III

Comments

The Fundamental Theorem of Algebra says that a polynomial of degree n can be written as the product of n linear factors, so we can write

$$x^3 - px^2 + qx - r = (x-\alpha)(x-\beta)(x-\gamma)\,, \qquad (*)$$

where α , β and γ are the roots of the equation $x^3 - px^2 + qx - r = 0$. The basis of this question is the comparison between the left hand side of $(*)$ and the right hand side, multiplied out, of $(*)$. Some of the roots may not be real, but you don't have to worry about that here.

The 'necessary and sufficient' in part (iii) looks a bit forbidding, but if you just repeat the steps of parts (i) and (ii), it is straightforward.

Solution to problem 35

(i) We have

$$(x - ak^{-1})(x - a)(x - ak) \equiv x^3 - px^2 + qx - r$$

i.e.

$$x^3 - a(k^{-1} + 1 + k)x^2 + a^2(k^{-1} + 1 + k)x - a^3 = x^3 - px^2 + qx - r\ .$$

Thus $p = a(k^{-1} + 1 + k)$, $q = a^2(k^{-1} + 1 + k)$, and $r = a^3$. Dividing gives $q/p = a$, which is a root, and $q^3/p^3 = a^3 = r$ as required.

(ii) Set $r = q^3/p^3$. Substituting $x = q/p$ into the cubic shows that it is a root:

$$(q/p)^3 - p(q/p)^2 + q(q/p) - (q/p)^3 = 0\ .$$

Since q/p is one root, and the product of the three roots is q^3/p^3 ($= r$ in the original equation), the product of the other two roots must be q^2/p^2. The two roots can therefore be written in the form $k^{-1}(q/p)$ and $k(q/p)$ for some number k , which shows that the three roots are in geometric progression.

(iii) The three roots are in arithmetic progression if and only if they are of the form $(a-d)$, a and $(a+d)$. (I have followed the lead of part (i) in using this form rather than a, $a + d$, $a + 2d$.)

If the roots are in this form, then

$$\big(x - (a - d)\big)\big(x - a\big)\big(x - (a + d)\big) \equiv x^3 - px^2 + qx - r$$

i.e.

$$x^3 - 3ax^2 + (3a^2 - d^2)x - a(a^2 - d^2) \equiv x^3 - px^2 + qx - r\ .$$

Thus $p = 3a$, $q = 3a^2 - d^2$, $r = a(a^2 - d^2)$. A necessary condition is therefore $r = (p/3)(q - 2p^2/9)$. Note that one root is $p/3$.

Conversely, if $r = (p/3)(q - 2p^2/9)$, the equation is

$$x^3 - px^2 + qx - (p/3)(q - 2p^2/9) = 0\ .$$

We can verify that $p/3$ is a root by substitution. Since the roots sum to p and one of them is $p/3$, the others must be of the form $p/3 - d$ and $p/3 + d$ for some d. They are therefore in arithmetic progression.

A necessary and sufficient condition for the roots to be in arithmetic progression is therefore $r = (p/3)(q - 2p^2/9)$.

Post-mortem

As usual, it is a good idea to give a bit of thought to the conditions given, namely $p \neq 0$ and $r \neq 0$.

Clearly, if the roots are in geometric progression, we cannot have a zero root. We therefore need $r \neq 0$. However, we don't need the condition $p \neq 0$. If the roots are in geometric progression with $p = 0$, then $q = 0$, but there is no contradiction: the roots are $be^{-2\pi i/3}$, b and $be^{2\pi i/3}$ where $b^3 = r$, and these are certainly in geometric progression. So the necessary and sufficient conditions are $r \neq 0$ and $p^3 r = q^3$.

Neither of these conditions is required if the roots are in arithmetic progression.

Therefore, the condition $p \neq 0$ is only there as a convenience — one thing less for you to worry about. The condition $r \neq 0$ should really have been given only for the first part. It should be said that this sort of question is incredibly difficult to word, which is why the examiner was a bit heavy handed with the conditions.

Problem 36: Root counting ($\checkmark\checkmark\checkmark$)

(i) Let $\mathrm{f}(x) = (1+x^2)e^x - k$, where k is a given constant. Show that $\mathrm{f}'(x) \geqslant 0$ and sketch the graph of $\mathrm{f}(x)$ in the three cases $k<0$, $0<k<1$ and $k>1$.

Hence, or otherwise, show that the equation

$$(1+x^2)\mathrm{e}^x - k = 0$$

has exactly one real root if $k>0$ and no real roots if $k \leqslant 0$.

(ii) Determine the number of real roots of the equation

$$(\mathrm{e}^x - 1) - k\tan^{-1} x = 0$$

in the different cases that arise according to the value of the constant k.

Note: If $y = \tan^{-1} x$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{1+x^2}$.

1999 Paper III

Comments

In good STEP style, this question has two related parts. In this case, the first part not only gives you guidance for the second part, but also provides a result that is useful for the second part.

In part (ii) of the original question you were asked to determine the number of real roots in the cases $0 < k \leqslant 2/\pi$ and $2/\pi < k < 1$. I thought that you would like to work out all the different cases for yourself — but it makes the question considerably harder, especially if you think about the special values of k as well as the ranges of k.

For part (i) you need to know that $x\mathrm{e}^x \to 0$ as $x \to -\infty$. This is just a special case of the result that exponentials go to zero faster than any power of x.

For part (ii), it is not really necessary to do all the sketches, but I think you should (though you perhaps wouldn't under examination conditions) because it gives you a complete understanding of the way the function depends on the parameter k. For the graphs, you may want to consider the sign of $\mathrm{f}'(0)$; this will give you an important clue. Remember that, by definition, $-\frac{1}{2}\pi < \tan^{-1} x < \frac{1}{2}\pi$.

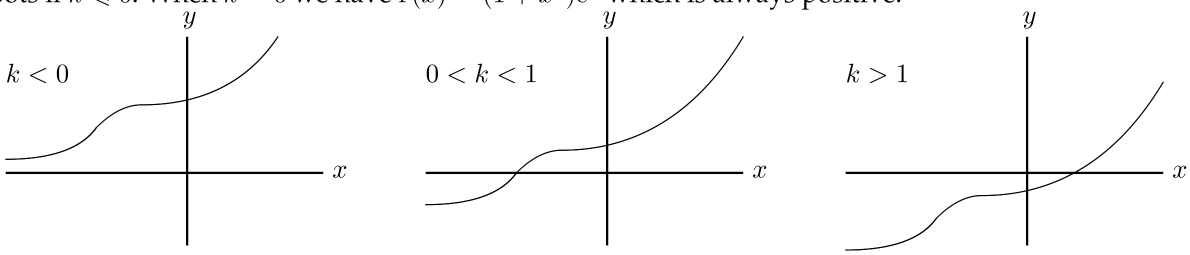
Solution to problem 36

(i) Differentiating gives $\mathrm{f}'(x) = 2x\mathrm{e}^x + (1+x^2)e^x = (1+x)^2\mathrm{e}^x$ which is non-negative because the square is non-negative and the exponential is positive.

There is one stationary point, at $x = -1$, which is a point of inflection, since the gradient on either side is positive.

Also, $\mathrm{f}(x) \to -k$ as $x \to -\infty$, $\mathrm{f}(0) = 1 - k$ and $\mathrm{f}(x) \to \infty$ as $x \to +\infty$. The graph is essentially exponential, with a hiccup at $x = -1$. The differences between the three cases are the positions of the horizontal asymptote ($x \to -\infty$) and the place where the graph cuts the y axis (above or below the x axis?).

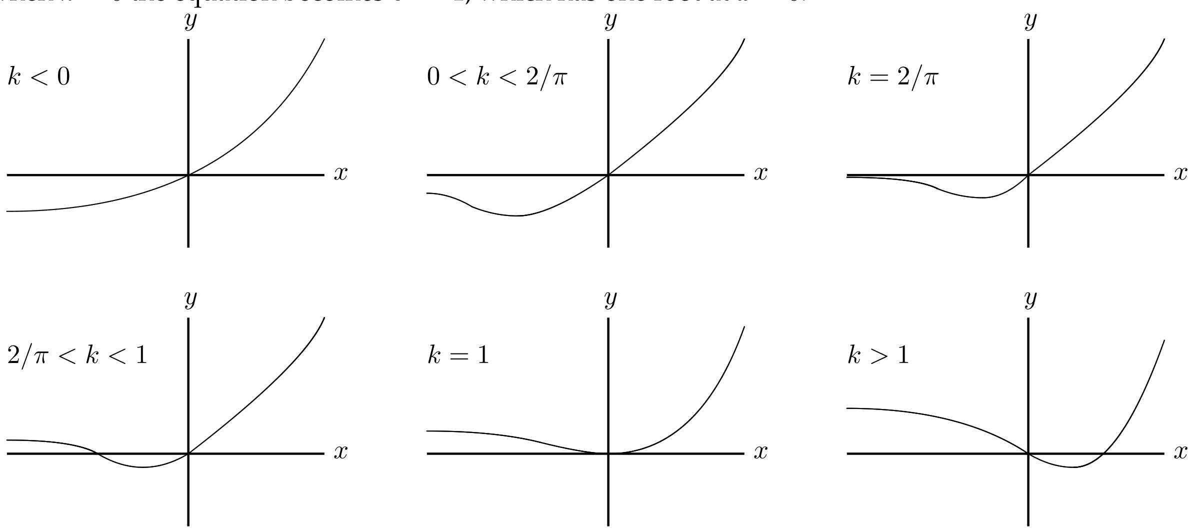
Since the graph is increasing and $0 < \mathrm{f}(x) < \infty$, the given equation has one real root if $k > 0$ and no real roots if $k \leqslant 0$. When $k = 0$ we have $\mathrm{f}(x) = (1+x^2)\mathrm{e}^x$ which is always positive.



(ii) First we collect up information required to sketch the graph, as in part (i). We have

$\mathrm{f}(0) = 0$, $\mathrm{f}(x) \to \frac{1}{2}k\pi - 1$ as $x \to -\infty$, $\mathrm{f}(x) \to \infty$ as $x \to \infty$, $\mathrm{f}'(x) = e^x - k(1+x^2)^{-1}$ (which we know from the first part can only be zero only if $k > 0$), $\mathrm{f}'(0) = 1 - k$.

The important values of k seem to be 0, $2/\pi$ and 1 so we will have to consider seven cases. Note that when $k = 0$ the equation becomes $\mathrm{e}^x = 1$, which has one root at $x = 0$.



As you see, the equation $(\mathrm{e}^x - 1) - k\tan^{-1} x = 0$ has one root if $k \leqslant \frac{2}{\pi}$ and two roots otherwise. The case $k = 1$ can be thought of as having one root at $x = 0$ or as having two roots, both at $x = 0$ (it is a double root).

Post-mortem

I think I made a bit of a meal of this; I got carried away with the sketches. But it was reassuring (compulsive, I found) to consider every case. My explanation is pretty compact, so I think you'll have to work it through on your own to convince yourself of the details.

Problem 37: Irrationality of e (✓✓)

For each positive integer n, let

$$a_n = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots,$$
$$b_n = \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots.$$

(i) Show that $b_n = 1/n$.

(ii) Deduce that $0 < a_n < 1/n$.

(iii) Show that $a_n = n!\,\mathrm{e} - [n!\,\mathrm{e}]$, where $[x]$ denotes the integer part of x.

(iv) Hence show that e is irrational.

1997 Paper III

Comments

Each part of this looks horrendously difficult, but it doesn't take much thought to see what is going on. If you haven't come across the concept of *the integer part* some examples should make it clear: $[21.25] = 21$, $[\pi] = 3$, $[1.999999] = 1$ and $[2] = 2$.

For part (iii), you need to know the series for the number e, which is $\sum\limits_{n=0}^{\infty} 1/n!$.

If you are stumped by the last part, just remember the definition of the word *irrational*: x is rational if and only if it can be expressed in the form p/q where p and q are integers; if x is not rational, it is irrational. Then look for a proof by contradiction ('Suppose that x is rational ...').

The first proof that e is irrational was given by Euler in 1736. He also named the number e, though he didn't 'invent' it; e is the basis for natural logarithms and as such was used implicitly by John Napier in 1614. Euler's proof involved the use of continuous fractions. He wrote e as:

$$2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{\cdots}}}}$$

the numbers in the denominators forming the infinite pattern $1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8 \ldots$. It is known that any such continuous fraction represents an irrational number.

In general, it is not easy to show that numbers are irrational. Johann Lambert showed that π is irrational in 1760, but the simple continued fraction for π is not known. It is still not known if $\pi + \mathrm{e}$ is irrational.

The proof that e is irrational given in this question is based on a proof by Fourier[26] from about 1815.

[26] Joseph Fourier (1768–1830) was a French mathematician and administrator. He is best known for his work on the theory of heat flow. He is generally credited with the discovery of the greenhouse effect. He spent some time in Egypt with Napoleon, and contributed much to the modern study of Egyptology.

Solution to problem 37

(i) Although the series for b_n does not at first sight look tractable, it is in fact just a geometric progression: the first term is $1/(n+1)$ and the common ratio is also $1/(n+1)$. Thus

$$b_n = \frac{1}{n+1}\left(\frac{1}{1-1/(n+1)}\right) = \frac{1}{n} \, .$$

(ii) Each term (after the first) of a_n is less than the corresponding term in b_n, so $a_n < b_n = 1/n$.

(iii) Multiplying the series for e by $n!$ gives

$$n!\mathrm{e} = n! + n! + \tfrac{1}{2}n! + \cdots + 1 + a_n$$

and the result follows because $a_n < 1$ and all the other terms on the right hand side of the above equation are integers.

(iv) We use proof by contradiction. Suppose that there exist integers k and m such that $\mathrm{e} = k/m$. Then $m!\mathrm{e}$ is certainly an integer. But if $m!\mathrm{e}$ is an integer then $[m!\mathrm{e}] = m!\mathrm{e}$, which contradicts the result of part (iii) since we know that $a_m \neq 0$ (it is obvious from the definition that $a_m > 0$).

Post-mortem

Well, that was short!

There were, however, two key steps which were not difficult in themselves but were not easy to find in the context of this question.

The first was to recognise that b_n is, for each n, a simple geometric progression. If it had been presented in the form $b = r + r^2 + r^3 + \cdots$ it would have been immediately recognisable. Somehow, the fact that the ratio r is given in terms of n — or, even worse, $n+1$ — makes it difficult to spot; not least because n is fixed for each series instead of labelling the terms of an individual series.

The second key step, in part (iv), was to assume that $\mathrm{e} = k/m$ and show that $m!\,\mathrm{e}$, **not** $m\mathrm{e}$, cannot be an integer. It seems so wasteful when it would suffice to show that $m\mathrm{e}$ is not an integer. We seem to be making a great deal of extra work for ourselves and that profligacy, unusual in mathematics, might well have thrown you off the scent despite the clear signal from part (iii).

I don't regard the idea of using proof by contradiction as a difficult step in part (iv). An irrational number is by definition a number that is not rational, so the idea of a proof by contradiction should jump out at you. Indeed, the standard example, the proof of the irrationality of $\sqrt{2}$, is by contradiction.[27]

[27] Suppose $p\sqrt{2} = q$. Then $2p^2 = q^2$. Writing p and q as the product of primes gives the contradiction since there are an odd number of powers of 2 one side of this equation and an even number on the other.

Problem 38: Discontinuous integrands (✓✓)

For any number x, the largest integer less than or equal to x is denoted by $[x]$. For example, $[3.7] = 3$ and $[4] = 4$.

(i) Sketch the graph of $y = [x]$ for $0 \leqslant x < 5$ and evaluate

$$\int_0^5 [x] \, \mathrm{d}x \, .$$

(ii) Sketch the graph of $y = [\mathrm{e}^x]$ for $0 \leqslant x < \ln n$, where n is an integer, and show that

$$\int_0^{\ln n} [\mathrm{e}^x] \, \mathrm{d}x = n \ln n - \ln(n!) \, .$$

Hence show that $n! \geqslant n^n \mathrm{e}^{1-n}$.

2000 Paper I

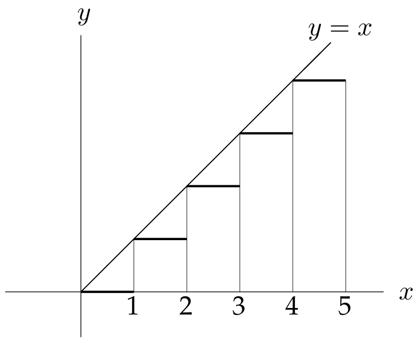
Comments

I had to add a bit to the original question because it was all dressed up with nowhere to go. The question is clearly about estimating $n!$ so I added in the last line, which makes the question a bit longer but not much more difficult. I could have added another part, but I thought that you would probably add it yourself: you will easily spot, once you have drawn the graphs, that a similar result for $n!$ with the inequality reversed can be obtained by considering rectangles the tops of which are above the graph of e^x instead of below it. You therefore end up with a nice sandwich inequality for $n!$.

Stirling (1692–1770) proved in 1730 that $n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} \mathrm{e}^{-n}$ for large n. This was a brilliant result — even reading his book, it is hard to see where he got $\sqrt{2\pi}$ from (especially as he wrote in Latin). Then he went on to obtain the approximation in terms of an infinite series; the expression above is just the first term.

Solution to problem 38

(i)

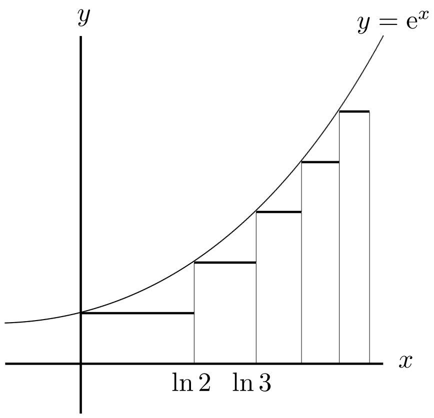


The graph of $[x]$ consists of the horizontal parts of an ascending staircase with 5 stairs, the lowest at height 0, each of width 1 unit and rising 1 unit.

The integral is the sum of the areas of the rectangles shown in the figure:

$$\text{area} = 0 + 1 + 2 + 3 + 4 = 10\,.$$

(ii)



The graph of $[\mathrm{e}^x]$ is also a staircase: the height of each stair is 1 unit and the width decreases as x increases because the gradient of e^x increases. It starts at height 1 and ends at height $(n-1)$.

The value of $[\mathrm{e}^x]$ changes from $k-1$ to k when $\mathrm{e}^x = k$ i.e. when $x = \ln k$. Thus $[\mathrm{e}^x] = k$ when $\ln k \leqslant x < \ln(k+1)$ and the area of the corresponding rectangle is $k\big(\ln(k+1) - \ln k\big)$.

The total area under the curve is therefore

$$1(\ln 2 - \ln 1) + 2(\ln 3 - \ln 2) + \cdots + (n-1)\big(\ln n - \ln(n-1)\big)$$

which you can rearrange to obtain $n \ln n - \ln(n!)$ as required.

For the last part, note that $[\mathrm{e}^x] \leqslant \mathrm{e}^x$ (this is clear from the definition) so

$$\int_0^{\ln n} [\mathrm{e}^x]\,\mathrm{d}x \leqslant \int_0^{\ln n} \mathrm{e}^x\,\mathrm{d}x = n-1 \quad \text{i.e.} \quad n\ln n - \ln(n!) \leqslant n-1\,.$$

Taking exponentials gives the required result.

Post-mortem

Usually when you draw a graph of a discontinuous function, you should specify at the jump whether the function takes the upper or lower value. For example $[x]$ takes the value 1 at $x = 1$, which is the upper value. This can be achieved by putting (say) a circle round the upper or lower point, as appropriate. I didn't bother on the above graphs, because it doesn't affect the value of the integral and I didn't want to clutter up the graphs.

Considering $[\mathrm{e}^x + 1]$ instead of $[\mathrm{e}^x]$ gives rectangles above the graph of $y = \mathrm{e}^x$ rather than below. The calculations are roughly the same, so you should easily arrive at $n! \leqslant n^{n+1}\mathrm{e}^{1-n}$. We have therefore proved that

$$n^n \mathrm{e}^{1-n} \leqslant n! \leqslant n^{n+1}\mathrm{e}^{1-n}\,.$$

This gives a pretty good (given the rather elementary method at our disposal) approximation for $n!$ for large n.

Problem 39: A difficult integral (✓✓✓)

Given that $\tan \frac{1}{4}\pi = 1$ show that $\tan \frac{1}{8}\pi = \sqrt{2} - 1$.
Let

$$I = \int_{-1}^{1} \frac{1}{\sqrt{1+x} + \sqrt{1-x} + 2}\, \mathrm{d}x\ .$$

Show, by using the change of variable $x = \sin 4t$, that

$$I = \int_{0}^{\frac{1}{8}\pi} \frac{2\cos 4t}{\cos^2 t}\, \mathrm{d}t\ .$$

Hence show that

$$I = 4\sqrt{2} - \pi - 2\ .$$

1996 Paper II

Comments

This tests trigonometric manipulation and integration skills. You will certainly need $\tan 2\theta$ in terms of $\tan\theta$, and $\cos 2\theta$ in terms of $\cos\theta$, and maybe other formulae.

Both parts of the question are *multistep*: there are half a dozen consecutive steps, each different in nature, with no guidance. This is unusual in school-level mathematics but normal in university mathematics.

I checked the answer on Wolfram Alpha, which turned out to be very good indeed at doing this sort of thing. I asked it to do the indefinite integral as well and, in less than a second, it came up with

$$\sqrt{1-x}\,\big[-1 - (\sqrt{x+1}+1)^{-1}\big] + [\sqrt{x+1}+1]^{-1} - 2\sin^{-1}\sqrt{\tfrac{1}{2}(x+1)}\,.$$

Not a pretty sight and not in its neatest form by a long way: for example, $2\sin^{-1}\sqrt{\frac{1}{2}(x+1)}$ reduces, after a bit of algebra, to $\frac{1}{2}\pi + \sin^{-1} x$. I also asked it to do the same integral with the 2 replaced by a parameter k and it took four seconds. The answer was about 20 times longer than the $k = 2$ result but it seemed to enjoy the problem, as far as I could tell.

Solution to problem 39

For the first part, to save writing, let $t = \tan \frac{\pi}{8}$. Then

$$\frac{2t}{1-t^2} = 1 \Rightarrow t^2 + 2t - 1 = 0 \Rightarrow t = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}\,.$$

We take the root with the $+$ sign since we know that t is positive.

Now the integral. Note first that the integrand has an obvious symmetry: it is unchanged when $x \leftrightarrow -x$. This means that we can do the integral over the half-range $x = 0$ to $x = 1$ and double the result. A glance at the required result suggests that this is a good idea.

Substituting $x = \sin 4t$ as instructed then gives

$$I = 2\int_0^{\frac{1}{8}\pi} \frac{4\cos 4t}{\sqrt{1+\sin 4t}\, + \sqrt{1-\sin 4t}\, + 2}\, \mathrm{d}t$$

so to obtain the given answer we need to show that

$$\sqrt{1+\sin 4t}\, + \sqrt{1-\sin 4t}\, + 2 = 4\cos^2 t\,,$$

i.e.

$$\sqrt{1+\sin 4t}\, + \sqrt{1-\sin 4t}\, = 2\cos 2t\,.$$

If we square both sides of this equation, noting that both sides are positive for the values of t in the integral so this is not dangerous, nice things happen:

$$(1+\sin 4t) + (1-\sin 4t) + 2\sqrt{1-\sin^2 4t}\, = 4\cos^2 2t \qquad \text{(RTP)}$$

i.e.

$$2 + 2\cos 4t = 4\cos^2 2t \qquad \text{(RTP)}$$

which is true by a standard trig. identity, so we have proved what we were required to prove.

For the last part, we have

$$\cos 4t = 2\cos^2 2t - 1 = 2(2\cos^2 t - 1)^2 - 1 = 8\cos^4 t - 8\cos^2 t + 1$$

so

$$I = 2\int_0^{\frac{1}{8}\pi} \left(8\cos^2 t - 8 + \sec^2 t\right) \mathrm{d}t = 2\int_0^{\frac{1}{8}\pi} \left(4\cos 2t - 4 + \sec^2 t\right) \mathrm{d}t = 2\Big[2\sin 2t - 4t + \tan t\Big]_0^{\frac{1}{8}\pi},$$

which gives the required result.

Post-mortem

Manipulating the integrand after the change of variable was really quite demanding. You could easily go down the wrong track and become mired in algebra. I did it by writing down what I was trying to prove and then showing that it was indeed true. This of course is hazardous, because if you are not careful you might assume the result in order to prove the result. I find it helps to write 'RTP' (Required To Prove) in the margin to indicate clearly to myself and others that I am not assuming it to be true.

The first thing we did with the integral, guided by the given answer, was to use the symmetry $x \leftrightarrow -x$ to reduce the range of integration to $0 \leqslant x \leqslant 1$, doubling the result. It is clear from a graph that this works but you could, if you were unsure, split the integral into two parts (integral from -1 to 0 plus integral from 0 to 1) then make the change of variable $y = -x$ in the lower integral to show that the two parts are equal.

Problem 40: Estimating the value of an integral (✓✓)

(i) Show that, for $0 \leqslant x \leqslant 1$, the largest value of $\dfrac{x^6}{(x^2+1)^4}$ is $\dfrac{1}{16}$.

What is the smallest value?

(ii) Find constants A, B, C and D such that, for all x,

$$\frac{1}{(x^2+1)^4} \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{Ax^5+Bx^3+Cx}{(x^2+1)^3}\right) + \frac{Dx^6}{(x^2+1)^4}\,.$$

(iii) Hence, or otherwise, prove that

$$\frac{11}{24} \leqslant \int_0^1 \frac{1}{(x^2+1)^4}\,\mathrm{d}x \leqslant \frac{11}{24} + \frac{1}{16}\,.$$

2000 Paper I

Comments

You should think about part (i) graphically, though it is not necessary to draw the graph: just set about it as if you were going to (starting point, finishing point, turning points, etc).

The equivalence sign in part (ii) indicates an equality that holds for all x — you are not being asked to solve the equation for x.

For part (iii), you need to know that inequalities can be integrated: this is 'obvious' if you think about integration in terms of area, though a formal proof requires a formal definition of integration and this is the sort of thing you would do in your first university course in mathematical analysis.

Solution to problem 40

(i) Let $\mathrm{f}(x) = \dfrac{x^6}{(x^2+1)^4}$. Then

$$\mathrm{f}'(x) = \frac{6x^5}{(x^2+1)^4} - \frac{8x^7}{(x^2+1)^5} = \frac{2(3-x^2)x^5}{(x^2+1)^5}$$

which is positive for $0 < x^2 < 3$. Therefore, $\mathrm{f}(x)$ increases in value from 0 at $x = 0$ to $\frac{1}{16}$ at $x = 1$.

(ii) Doing the differentiation gives

$$\frac{1}{(x^2+1)^4} \equiv \frac{5Ax^4+3Bx^2+C}{(x^2+1)^3} - \frac{6x(Ax^5+Bx^3+Cx)}{(x^2+1)^4} + \frac{Dx^6}{(x^2+1)^4}$$

and multiplying by $(x^2+1)^4$ gives the following identity:

$$1 \equiv (5Ax^4+3Bx^2+C)(x^2+1) - 6x(Ax^5+Bx^3+Cx) + Dx^6 \qquad (*)$$

i.e.

$$1 \equiv (D-A)x^6 + (5A-3B)x^4 + (3B-5C)x^2 + C\ .$$

Equating coefficients of the different powers of x on each side of the equivalence sign gives $1 = C$, $0 = 3B - 5C$, $0 = 5A - 3B$, $0 = D - A$, so $A = 1$, $B = \frac{5}{3}$, $C = 1$ and $D = 1$.

(iii) Using the results of parts (i) and (ii), we see that for $0 \leqslant x \leqslant 1$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^5+\frac{5}{3}x^3+x}{(x^2+1)^3}\right) \leqslant \frac{1}{(x^2+1)^4} \leqslant \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^5+\frac{5}{3}x^3+x}{(x^2+1)^3}\right) + \frac{1}{16}\ .$$

Inequalities can be integrated, so

$$\int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^5+\frac{5}{3}x^3+x}{(x^2+1)^3}\right)\mathrm{d}x \leqslant \int_0^1 \frac{1}{(x^2+1)^4}\,\mathrm{d}x \leqslant \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^5+\frac{5}{3}x^3+x}{(x^2+1)^3}\right)\mathrm{d}x + \int_0^1 \frac{1}{16}\,\mathrm{d}x$$

i.e.

$$\left[\frac{x^5+(5/3)x^3+x}{(x^2+1)^3}\right]_0^1 \leqslant \int_0^1 \frac{1}{(x^2+1)^4}\,\mathrm{d}x \leqslant \left[\frac{x^5+(5/3)x^3+x}{(x^2+1)^3}\right]_0^1 + \left[\frac{x}{16}\right]_0^1$$

from which the required result follows immediately.

Post-mortem

Instead of equating coefficients in $(*)$, you could obtain equations for A, B, C and D by putting four carefully chosen values of x into the equation. An obvious choice is $x = 0$, but (thinking flexibly!) you could try $x = \mathrm{i}$ to eliminate terms with factors of x^2+1. After this, it becomes more difficult to find good choices.

You might wonder why, in part (ii), the term inside the derivative has only odd powers of x. Would it not make it more general to include even powers as well? You could in fact include even powers but you would find that their coefficients would be zero: all the other terms in the equation are even in x, so the derivative has to be an even function which means that the function being differentiated must be odd.

Although the idea of this question is good, the final result is a bit feeble. It only gives the value of the integral to an accuracy of about $\frac{1}{16}/\frac{11}{24}$ which is about 15%. The actual value of the integral can be found fairly easily using the substitution $x = \tan t$ and is $\frac{11}{48} + \frac{5}{64}\pi$ so the inequalities can be used to give (rather bad) estimates for π, namely $2.933 \leqslant \pi \leqslant 3.733$.

Problem 41: Integrating the modulus function (✓✓)

Show that

$$\int_{-1}^{1} |\, x\mathrm{e}^{x} \,| \mathrm{d}x = -\int_{-1}^{0} x\mathrm{e}^{x} \mathrm{d}x + \int_{0}^{1} x\mathrm{e}^{x} \mathrm{d}x$$

and hence evaluate the integral.

Evaluate the following integrals:

(i) $\displaystyle\int_{0}^{4} |\, x^3 - 2x^2 - x + 2 \,| \, \mathrm{d}x\,;$

(ii) $\displaystyle\int_{-\pi}^{\pi} |\, \sin x + \cos x \,| \, \mathrm{d}x\,.$

2000 Paper I

Comments

The very first part shows you how to do this sort of integral (with mod signs in the integrand) by splitting up the range of integration at the points where the integrand changes sign. In parts (i) and (ii) you have to use the technique on different examples.

Solution to problem 41

For the first part, note that $|xe^x| = -xe^x$ if $x < 0$. Then integrate by parts to evaluate the integrals:

$$\int_{-1}^{0}(-xe^x)\,dx + \int_0^1 xe^x\,dx = -\left(\Big[xe^x\Big]_{-1}^{0} - \int_{-1}^{0} e^x\,dx\right) + \left(\Big[xe^x\Big]_0^1 - \int_0^1 e^x\,dx\right) = 2 - 2e^{-1}\,.$$

For the next parts, we have to find out where the integrand is positive and where it is negative.

(i) $x^3 - 2x^2 - x + 2 = (x-1)(x+1)(x-2)$ (spotting the factors), so the integrand is positive for $0 \leqslant x < 1$, negative for $1 < x < 2$ and positive for $2 < x < 4$ (a quick sketch will help with this). Splitting the range of integration into these ranges and integrating gives

$$\begin{aligned} &\int_0^4 \left| x^3 - 2x^2 - x + 2 \right| dx &(*)\\ &\quad = \int_0^1 (x^3 - 2x^2 - x + 2)\,dx - \int_1^2 (x^3 - 2x^2 - x + 2)\,dx + \int_2^4 (x^3 - 2x^2 - x + 2)\,dx\\ &\quad = 22\tfrac{1}{6}\,. \end{aligned}$$

(ii) $\sin x + \cos x$ changes sign when $\tan x = -1$, i.e. when $x = -\frac{1}{4}\pi$ and $x = \frac{3}{4}\pi$. Splitting the range of integration into these ranges and integrating gives

$$\begin{aligned} \int_{-\pi}^{\pi} \left| \sin x + \cos x \right| dx &= \int_{-\pi}^{-\frac{1}{4}\pi} -(\sin x + \cos x)\,dx + \int_{-\pi/4}^{\frac{3}{4}\pi} (\sin x + \cos x)\,dx + \int_{\frac{3}{4}\pi}^{\pi} -(\sin x + \cos x)\,dx\\ &= \Big[\cos x - \sin x\Big]_{-\pi}^{-\frac{1}{4}\pi} + \Big[-\cos x + \sin x\Big]_{-\frac{1}{4}\pi}^{\frac{3}{4}\pi} + \Big[+\cos x - \sin x\Big]_{\frac{3}{4}\pi}^{\pi}\\ &= 4\sqrt{2}\,. \end{aligned}$$

Alternatively, start by writing $\sin x + \cos x = \sqrt{2}\,\cos(x - \frac{1}{4}\pi)$ which makes the changes of sign easier to spot and the integrals easier to do.

Post-mortem

If you give a bit more thought to part (ii), you will see easier ways of doing it. Since the trigonometric functions are periodic with period 2π the integrand is also periodic. Write the integrand in the form $\sqrt{2}\left|\cos(x - \frac{1}{4}\pi)\right|$. Integrating this over any 2π interval gives the same result. Indeed, we may as well integrate $\sqrt{2}\left|\sin x\right|$ from 0 to 2π; or from 0 to π and double the answer.

After the first edition of this book appeared, a correspondent e-mailed to suggest that we should integrate $(*)$ by writing it in the form

$$\int_0^4 (x^3 - 2x^2 - x + 2)\,dx - 2\int_1^2 (x^3 - 2x^2 - x + 2)\,dx.$$

Yes, that's quite a good idea; it would have saved a bit of writing and reduced the risk of arithmetical errors.

Problem 42: Geometry (✓✓)

In the triangle ABC, angle $BAC = \alpha$ and angle $CBA = 2\alpha$, where 2α is acute, and $BC = x$. Show that $AB = (3 - 4\sin^2\alpha)x$.

The point D is the midpoint of AB and the point E is the foot of the perpendicular from C to AB. Find an expression for DE in terms of x.

The point F lies on the perpendicular bisector of AB and is a distance x from C. The points F and B lie on the same side of the line through A and C. Show that the line FC trisects the angle ACB.

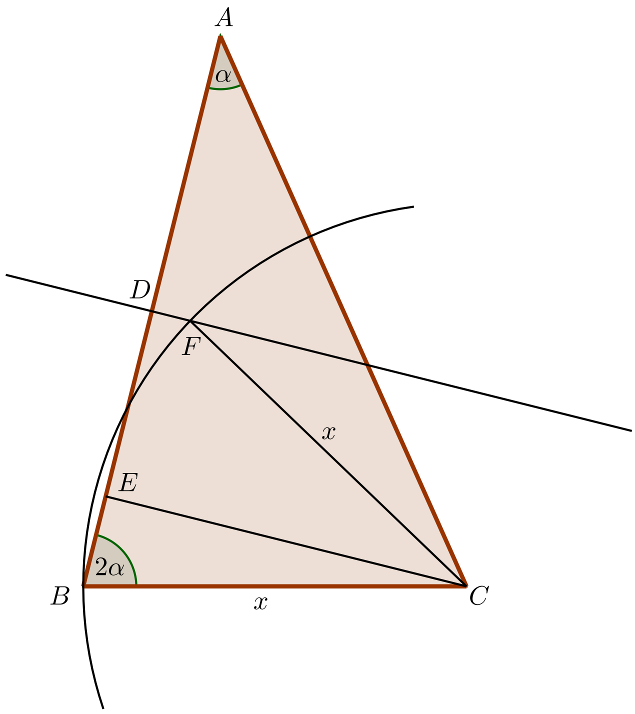
2015 Paper II

Comments

I thought it would be good to include at least one plane geometry question in this collection, so here it is. As always with geometry, the first thing to do is draw a BIG diagram. You will probably need to have quite a few tries at it.

Then you have the usual tools at your disposal: similar triangles, congruent triangles, angle-chasing, and — moving away from classical Euclidean geometry — Pythagoras, and sine and cosine rules. I leave it to you to decide what will be useful here.

Solution to problem 42



Here is a careful diagram, with a construction arc left in to show that $CB = CF = x$. It took me ages. But it is pretty much plain sailing now.

By the sine rule in $\triangle ABC$ (and calculating an expression for $\sin 3\alpha$ using double angle formulae)

$$AB = \frac{x\sin(180^\circ - 3\alpha)}{\sin\alpha} = \frac{x\sin(3\alpha)}{\sin\alpha} = \frac{x(3\cos^2\alpha\sin\alpha - \sin^3\alpha)}{\sin\alpha} = (3 - 4\sin^2\alpha)x$$

as required.

Then, since D is the mid-point of AB and $\angle BEC = 90^\circ$,

$$DE = \tfrac{1}{2}AB - BE = \tfrac{1}{2}x(3 - 4\sin^2\alpha) - x\cos 2\alpha = \tfrac{1}{2}x,$$

rather surprisingly.

For the last part, since $\angle BCA = 180^\circ - 3\alpha$, we need to show that $\angle FCA = 60^\circ - \alpha$ or, equivalently, $\angle FCB = 120^\circ - 2\alpha$. Now $\angle BCE = 90^\circ - 2\alpha$ so we are done if we can show that $\angle ECF = 30^\circ$. But that follows almost immediately from the $DE = \frac{1}{2}x$ and $FC = x$: draw the perpendicular from F to CE, which has length $\frac{1}{2}x$, and consider the right-angled triangle thus formed.

Post-mortem

I liked the mixture here of angle-chasing and use of the sine rule. Geometry purists, though, would be appalled; using the sine rule in a geometry problem is like using metal screws instead of wooden dowels on an antique wooden cabinet. As it happens, there is a beautiful proof of this result by pure geometry. You have to reflect the whole diagram about the perpendicular bisector of AB. For the details, see Ross Honsberger's excellent *Mathematical Chestnuts from around the World* (Cambridge: Cambridge University Press, 2001), section 20.

Problem 43: The t substitution (✓✓)

Show that

$$\sin\theta = \frac{2t}{1+t^2}\,, \qquad \cos\theta = \frac{1-t^2}{1+t^2}\,, \qquad \text{and} \qquad \frac{1+\cos\theta}{\sin\theta} = \tan(\tfrac12\pi - \tfrac12\theta)\,,$$

where $t = \tan(\frac12\theta)$.

Let

$$I = \int_0^{\frac12\pi} \frac{1}{1+\cos\alpha\sin\theta}\,\mathrm{d}\theta\,.$$

Use the substitution $t = \tan(\frac12\theta)$ to show that, for $0 < \alpha < \frac12\pi$,

$$I = \int_0^1 \frac{2}{(t+\cos\alpha)^2 + \sin^2\alpha}\,\mathrm{d}t\,.$$

By means of the further substitution $t + \cos\alpha = \sin\alpha\tan u$ show that

$$I = \frac{\alpha}{\sin\alpha}\,.$$

Deduce a similar result for

$$\int_0^{\frac12\pi} \frac{1}{1+\sin\alpha\cos\phi}\,\mathrm{d}\phi\,.$$

2000 Paper II

Comments

The first of the two substitutions is familiarly known as the 't substitution'. It would have been very standard fare 30 years ago, but it seems to have gone out of fashion now. The second of the two substitutions is the normal substitution for integrals with quadratic denominators.

For the last part, 'deduce' implies that you don't have to do any further integration. Note that the variable in the integral is ϕ instead of θ. Since it is a definite integral, it doesn't matter what the variable is called; it could equally well have been called θ as in the original integral I. The use of a different variable was just a kindness on the part of the examiner to indicate that you should be thinking about a change of variable.

The 'similar result' that you deduce should include the conditions under which it is true. It is worth thinking about why the condition $0 < \alpha < \frac12\pi$ is required — or, indeed, if it is required.

Solution to problem 43

The three identities just require use of $\cos\theta = \cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta$ and $\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$. If you divide each by $\cos^2\frac{1}{2}\theta + \sin^2\frac{1}{2}\theta$ (i.e. by 1) the first two identities drop out. Remember, for the last one, that $\cot x = \tan(\frac{1}{2}\pi - x)$.

For the first change of variable, we have $\mathrm{d}t = \frac{1}{2}\sec^2\frac{1}{2}\theta\,\mathrm{d}\theta = \frac{1}{2}(1+t^2)\mathrm{d}\theta$ and the new limits are 0 and 1, so

$$\begin{aligned} I = \int_0^{\frac{1}{2}\pi} \frac{1}{1+\cos\alpha\sin\theta}\,\mathrm{d}\theta &= \int_0^1 \frac{1}{1+\cos\alpha\frac{2t}{1+t^2}}\,\frac{2}{1+t^2}\,\mathrm{d}t \\ &= \int_0^1 \frac{2}{1+2t\cos\alpha+t^2}\,\mathrm{d}t = \int_0^1 \frac{2}{(t+\cos\alpha)^2+\sin^2\alpha}\,\mathrm{d}t\,. \end{aligned}$$

For the second change of variable, we have $\mathrm{d}t = \sin\alpha\sec^2 u\,\mathrm{d}u$. When $t=0$, $\tan u = \cot\alpha$ so $u = \frac{1}{2}\pi - \alpha$. When $t=1$, $\sin\alpha\tan u = 1+\cos\alpha$ so (after a bit of work with double-angle formulae) $u = \frac{1}{2}\pi - \frac{1}{2}\alpha$. Thus

$$I = \int_{\frac{1}{2}\pi-\alpha}^{\frac{1}{2}\pi-\frac{1}{2}\alpha} \frac{2}{\sin^2\alpha(1+\tan^2 u)}\sin\alpha\sec^2 u\mathrm{d}u = \int_{\frac{1}{2}\pi-\alpha}^{\frac{1}{2}\pi-\frac{1}{2}\alpha} \frac{2}{\sin\alpha}\,\mathrm{d}u = \frac{\alpha}{\sin\alpha}\,.$$

For the last part, we want to make a substitution that changes the cosine in the denominator to a sine. One possibility is to set $\phi = \frac{1}{2}\pi - \theta$. This will swap the limits but also introduces a minus sign since $\mathrm{d}\theta = -\mathrm{d}\phi$. Thus

$$\frac{\alpha}{\sin\alpha} = I = -\int_{\frac{\pi}{2}}^0 \frac{1}{1+\cos\alpha\cos\phi}\,\mathrm{d}\phi = \int_0^{\frac{\pi}{2}} \frac{1}{1+\cos\alpha\cos\phi}\,\mathrm{d}\phi\,.$$

This is almost the integral we want: we still need to replace $\cos\alpha$ in the denominator by $\sin\alpha$. Remembering what we did a couple of lines back, we just replace α by $\frac{1}{2}\pi - \alpha$ in the integral and in the answer, giving

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos\alpha\cos\phi}\,\mathrm{d}\phi = \frac{\frac{1}{2}\pi-\alpha}{\cos\alpha}\,.$$

If the original α satisfied $0<\alpha<\frac{1}{2}\pi$, the new α must satisfy $\frac{1}{2}\pi > \alpha > 0$, which is the same.

Post-mortem

One reason for the restriction on α might be to prevent the denominator of the integrand being zero for some value of θ ; a zero in the denominator usually means that the integral is undefined. However, the only value of α for which $\cos\alpha\sin\theta$ could possibly be as small as -1 (for $0\leqslant\theta\leqslant\frac{1}{2}\pi$) is π (and of course 3π, etc). From this point of view, we only need $\alpha\neq\pi$ (etc). There is a slight awkwardness in the answer when $\alpha=0$ but this can be overcome by taking limits:

$$\lim_{\alpha\to 0}\frac{\alpha}{\sin\alpha} = 1$$

which you can easily verify is the correct value of the integral when $\alpha = 0$.

You might therefore think that the restriction $0<\alpha<\frac{1}{2}\pi$ is superfluous. But here is a curious thing: increasing α by 2π does not change I but does change the answer! If you work through the solution with this in mind you see that it can make a difference only when you work out $\tan^{-1}u$, which by definition lies in the range $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, and this is why we must have $-\frac{1}{2}\pi<\alpha\leqslant\frac{1}{2}\pi$. We lose nothing by using instead $0\leqslant\alpha\leqslant\frac{1}{2}\pi$, since $\alpha\to-\alpha$ doesn't change the integral. The strict inequalities ($<$ rather than $\leqslant$) avoid trouble with the denominators when $\alpha=0$ in the first integral or $\alpha=\frac{1}{2}\pi$ in the second.

Problem 44: A differential-difference equation (✓✓)

A damped system with feedback is modelled by the equation

$$\mathrm{f}'(t) = -\mathrm{f}(t) + k\mathrm{f}(t-1)\,, \tag{1}$$

where k is a given non-zero constant.
Show that (non-zero) solutions for f of the form $\mathrm{f}(t) = A\mathrm{e}^{mt}$, where A and m are constants, are possible provided m satisfies

$$m + 1 = k\mathrm{e}^{-m}. \tag{2}$$

Show also, by means of a sketch or otherwise, that equation (2) can have 0, 1 or 2 real roots, depending on the value of k, and find the set of values of k for which such solutions exist. For what set of values of k do the corresponding solutions of (1) tend to zero as $t \to \infty$?

1990 Paper II

Comments

Do not be put off by the words at the very beginning of the question, which you do not need to understand. In any case, the question very soon morphs into an investigation of roots of an equation using curve sketching. However, their meaning can be gleaned from the equation itself (sometimes called a *differential-difference* equation). The left hand side of the equation is the rate of increase of the function f, which may measure the amplitude of some physical disturbance. According to the equation, this is equal to the sum of two terms. One is $-\mathrm{f}(t)$, which represents damping: this term alone would give exponentially decreasing solutions. The other is a positive term proportional to $\mathrm{f}(t-1)$: this is called a feedback term, because it depends on the value of f one year (say) previously. The feedback term could represent some seasonal effect, while the damping term may be caused by some resistance-to-growth factor.

The suggested method of solving the equation is similar to what you might have used for second order linear equations with constant coefficients: you guess a solution (e^{mt}) and then substitute into the equation to check that this is a possible form of solution and to find the values of m which will work. It is always possible to multiply the exponential by a constant factor, since any constant multiple of a solution is also a solution. (This is a consequence of the *linearity* of the equation: i.e. no terms involving $\mathrm{f}(t)^2$, $\mathrm{f}(t)^3$, etc.) We can also take linear combinations of solutions to produce a more general solution.

To find the set of values of k which give real solutions of (2), you need to investigate the borderline case, where the two curves $y = m+1$ and $y = k\mathrm{e}^{-m}$ just touch (and therefore have the same gradient). Remember that the sign of k is not restricted.

Unlike the second order differential equation case, the equation for m is not quadratic; in fact, it is not even polynomial, since it has an exponential term. This means, as we will find, that there may not be exactly 2 solutions; in general, there may be many solutions (e.g. the non-polynomial equation $\sin m = 0$ has solutions $m = 0, \pi, 2\pi, \ldots$) or no solutions (e.g. $\mathrm{e}^m = 0$ has no solutions). And unlike the differential equation case, we have no idea whether the method is going to work, because there may be solutions of a completely different form. The full analysis of equations such as (1) provides an extraordinarily rich field of study with many surprising results.

Solution to problem 44

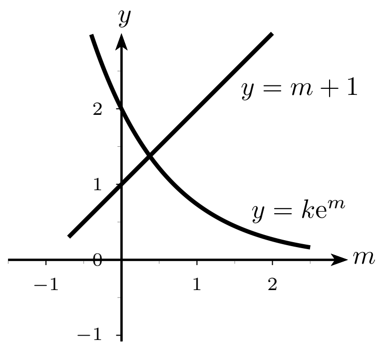
We do as we are told to start and substitute $A\mathrm{e}^{mt}$ into equation (1). After cancelling the overall factor A, we get

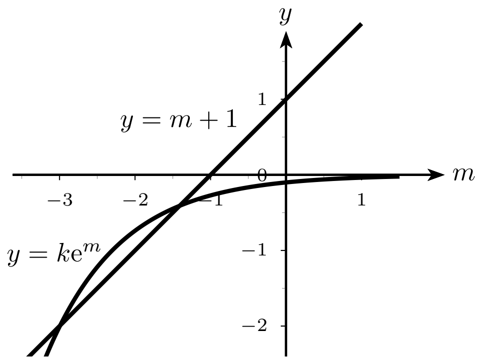
$$m\mathrm{e}^{mt} = -\mathrm{e}^{mt} + k\mathrm{e}^{m(t-1)}\,.$$

We can cancel the overall factor e^{mt}, but an exponential remains on account of the $\mathrm{f}\,(t-1)$ term:

$$m = -1 + k\mathrm{e}^{-m},$$

which is equivalent to equation (2) overleaf.





The above sketches show $y = m + 1$ and $y = k\mathrm{e}^{-m}$ on the same axes, for positive and negative k: in the first sketch, $k = 2$; in the second $k = -0.1$.

As can be seen from the sketches, equation (2) always has exactly one solution if $k > 0$. For $k < 0$, there may be two or zero solutions depending on whether the line and curve intersect, or just one solution if they touch. They will touch if there is a value of m such that

$$m + 1 = k\mathrm{e}^{-m} \;\text{ and }\; 1 = -k\mathrm{e}^{-m},$$

the second of these equations being the condition that the gradients (differentiate with respect to m to find the gradient) of the line and curve are the same at the common point. Solving the two equations gives $m = -2$ and $k = -\mathrm{e}^{-2}$.

If $0 > k > -\mathrm{e}^{-2}$, then the curve and line will intersect, so the set of values of k for which equation (2) has solutions is $k > 0$ and $-\mathrm{e}^{-2} \leqslant k < 0$.

The corresponding solutions to equation (1) tend to zero as $t \to \infty$ if and only if $m < 0$, because then they are exponentially decreasing rather than increasing.

If $k < 0$, we can see from the sketch that the solutions of (2), if there are any (i.e. if $-\mathrm{e}^{-2} \leqslant k < 0$), occur in the bottom left quadrant, and so $m < 0$. The corresponding solutions of (1) will tend to zero.

If $k > 0$, the intersection of the graph of $y = k\mathrm{e}^{-m}$ with the y-axis is at $y = k$ whereas the intersection of the graph of $y = m + 1$ with the y-axis is at $y = 1$. For $k \geqslant 1$, the solution of (2) occurs in the top right quadrant and has $m \geqslant 0$. For $0 < k < 1$, the solution occurs in the top left quadrant and has $m < 0$.

The range of k for which the solutions tend to zero as $t \to \infty$ is therefore $-\mathrm{e}^{-2} \leqslant k < 0$ and $0 < k < 1$.

Problem 45: Lagrange's identity (✓✓✓)

If $y = \mathrm{f}(x)$, the inverse of f is given by *Lagrange's identity*:

$$\mathrm{f}^{-1}(y) = y + \sum_{1}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}} \big[y - \mathrm{f}(y)\big]^{n},$$

when this series converges.
(i) Verify Lagrange's identity when $\mathrm{f}(x) = ax$.
(ii) Show that one root of the equation $x - \frac{1}{4}x^3 = \frac{3}{4}$ is

$$x = \sum_{0}^{\infty} \frac{3^{2n+1}\,(3n)!}{n!(2n+1)!\,4^{3n+1}}\,. \qquad (\dagger)$$

(iii) Find a solution for x, as a series in λ, of the equation $x = \mathrm{e}^{\lambda x}$.
[You may assume that the series in part (ii) converges and that the series in parts (i) and (iii) converge for suitable a and λ.]

1987 Paper II

Comments

This looks pretty frightening at first, because of the complicated and unfamiliar formula. However, its bark is worse than its bite. Once you have decided what you need to find the inverse of, you just substitute it into the formula and see what happens. Do not worry about the use of the word 'convergence'; this can be ignored. It is just included to satisfy the legal eagles who will point out that the series might not have a finite sum.

In part (ii) you can, as it happens, solve the cubic by normal means (find one root by inspection, factorise and use the usual formula to solve the resulting quadratic equation). The root found by Lagrange's equation is the one closest to zero. Equation ($\dagger$) turns out to be a very obscure way of writing a familiar quantity.[28]

Lagrange was one of the leading mathematicians of the 18th century; Napoleon referred to him as the 'lofty pyramid of the mathematical sciences'. He attacked a wide range of problems, from celestial mechanics to number theory. In the course of his investigation of the roots of polynomial equations, he discovered group theory (in particular, his eponymous theorem about the order of a subgroup dividing the order of the group), though the term 'group' and the systematic theory had to wait until Galois and Abel in the first part of the 19th century.

Lagrange's formula, produced before the advent of the theory of integration in the complex plane, which allows a relatively straightforward derivation, testifies to his remarkable mathematical ability. It is practically forgotten now, but in its day it had a great impact. The applications given above give an idea of how important it was, in the age before computers.

---

[28] The expansion sums to 1; I don't know how you can see that directly. I thought it would come from using the classical formula for the root of a cubic:

$$-\left[\tfrac{3}{2}\big(1 + \mathrm{i}\sqrt{\tfrac{13}{243}}\,\big)\right]^{\frac{1}{3}} - \left[\tfrac{3}{2}\big(1 - \mathrm{i}\sqrt{\tfrac{13}{243}}\,\big)\right]^{\frac{1}{3}},$$

expanding each bracket binomially but it doesn't seem to. The roots are obtained from this complicated expression by noticing that $-\left[\tfrac{3}{2}\big(1 \pm \mathrm{i}\sqrt{\tfrac{13}{243}}\,\big)\right]^{\frac{1}{3}} = \tfrac{1}{2}\big(1 \pm \mathrm{i}\sqrt{\tfrac{13}{3}}\,\big)$.

Solution to problem 45

(i) The inverse of $\mathrm{f}(x) = ax$ is given by $\mathrm{f}^{-1}(y) = y/a$. Substituting into $\mathrm{f}(y) = ay$ into Lagrange's identity gives

$$\mathrm{f}^{-1}(y) = y + \sum_1^\infty \frac{1}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}[y-ay]^n = y + \sum_1^\infty (1-a)^n \frac{1}{n!}\frac{\mathrm{d}^{n-1}y^n}{\mathrm{d}y^{n-1}}$$
$$= y + \sum_1^\infty (1-a)^n y = y + y\frac{1-a}{1-(1-a)}\,,$$

where the last equality follows from summing the geometric progression. This simplifies to y/a, thus verifying Lagrange's formula.

(ii) Let $\mathrm{f}(x) = x - \frac{1}{4}x^3$. Then the equation becomes $\mathrm{f}(x) = \frac{3}{4}$, so we must find $\mathrm{f}^{-1}(\frac{3}{4})$. Again, we just substitute into Lagrange's formula, leaving y arbitrary for the moment:

$$\mathrm{f}^{-1}(y) = y + \sum_1^\infty \frac{1}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}[y-(y-\tfrac{1}{4}y^3)]^n = y + \sum_1^\infty \frac{1}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}[\tfrac{1}{4}y^3]^n$$
$$= y + \sum_1^\infty \frac{1}{4^n n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}y^{3n}$$
$$= y + \sum_1^\infty \frac{1}{4^n n!}\frac{(3n)!}{(2n+1)!}y^{2n+1}.$$

This is a solution to the equation[29] $x - \frac{1}{4}x^3 = y$, so we just set $y = \frac{3}{4}$ to obtain the given result.

(iii) The obvious choice for f is $\mathrm{f}(x) = x - \mathrm{e}^{\lambda x}$, in which case the equation becomes $\mathrm{f}(x) = 0$ and we want $\mathrm{f}^{-1}(0)$. Again substituting into Lagrange's identity gives

$$\mathrm{f}^{-1}(y) = y + \sum_1^\infty \frac{1}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}\left[\mathrm{e}^{\lambda y}\right]^n = y + \sum_1^\infty \frac{1}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}\mathrm{e}^{ny\lambda} = y + \sum_1^\infty \frac{1}{n!}(n\lambda)^{n-1}\mathrm{e}^{ny\lambda}\,.$$

Setting $y = 0$ gives a series for the root:

$$\mathrm{f}^{-1}(0) = \sum_1^\infty \frac{n^{n-1}}{n!}\lambda^{n-1},$$

which cannot be further simplified.

Post-mortem

Regarding convergence in Lagrange's formula:

Part (i): we need, somewhat mysteriously, $0 < a < 2$ for the geometric progression to converge, but then the result is valid for any y.

Part (ii): Using the approximation $n! \approx (n/\mathrm{e})^n$, which is a simplified version of Stirling's formula, it can be seen that the series converges provided $|y| < 4/\sqrt{27}$. This is related to the condition for x to lie between the two turning points of $\mathrm{f}(x)$, which guarantees that $\mathrm{f}^{-1}(y)$ is well defined.

Part (iii): We can use Stirling's formula, as above, to show that series converges for $|\lambda| < \mathrm{e}^{-1}$. You might like to sketch the two functions x and $\mathrm{e}^{\lambda x}$; you should find that the range of values of λ for which the equation has a real root corresponds exactly to the range for which the series converges.

---

[29] Note that this equation cannot in general be solved by spotting roots. In fact, by translation and scaling, any cubic equation can be reduced to this form, so our series solution can be used to find a solution of any cubic equation.

Problem 46: Bernoulli polynomials (✓✓)

The Bernoulli polynomials, $\mathrm{B}_n(x)$ (where $n = 0,\ 1,\ 2, \ldots$), are defined by $\mathrm{B}_0(x) = 1$ and, for $n \geqslant 1$,

$$\frac{\mathrm{dB}_n}{\mathrm{d}x} = n\mathrm{B}_{n-1}(x) \tag{1}$$

and

$$\int_0^1 \mathrm{B}_n(x)\mathrm{d}x = 0\,. \tag{2}$$

(i) Show that $\mathrm{B}_4(x) = x^2(x-1)^2 + c$, where c is a constant (which you need not evaluate).

(ii) Show that, for $n \geqslant 2$, $\mathrm{B}_n(1) - \mathrm{B}_n(0) = 0$.

(iii) Show, by induction or otherwise, that

$$\mathrm{B}_n(x+1) - \mathrm{B}_n(x) = nx^{n-1} \qquad (n \geqslant 1). \tag{3}$$

(iv) Hence show that

$$n\sum_{m=0}^{k} m^{n-1} = \mathrm{B}_n(k+1) - \mathrm{B}_n(0)\,,$$

and deduce that $\sum\limits_{m=0}^{1000} m^3 = (500500)^2$.

1987 Paper III

Comments

The Swiss family Bernoulli included no fewer than eight mathematicians who were counted amongst the leading scholars of their day. They made major contributions to all branches of mathematics, especially differential calculus. There was great rivalry between some members of the family; between brothers Jakob (1654–1705) and Johann (1667–1748), in particular.

Johann once published an important result in the form of a Latin anagram, in order to retain the priority of discovery without giving the game away to his brother. The anagram was: $24a$, $6b$, $6c$, $8d$, $33e$, $5f$, $2g$, $4h$, $33i$, $6l$, $21m$, $26n$, $16o$, $8p$, $5q$, $17r$, $16s$, $25t$, $32u$, $4x$, $3y$, $+$, $-$, –, $\pm$, $=$, 4, 2, 1, $'$. The notation means that his important result contained, for instance, the letter a 24 times either in text or in equations. After waiting for a year for someone to solve it, Bernoulli weakened and published the solution himself. If you are trying to solve the anagram yourself, you might like to know that it is about the Riccati equation $y' = ay^2 + bx^n$, which can be solved (very cunningly, as it turns out) when n is of the form $-4m/(2m\pm 1)$ for any positive integer m. (Newton also published some work in the form of anagrams, during his conflict with Leibniz.)

The polynomials described above were discovered by Jakob Bernoulli. They are defined recursively; that is to say, the zeroth polynomial is given an explicit value, and the nth is determined from the $(n-1)$th. Here, B_{n-1} has to be integrated to obtain B_n, which means that B_n is a polynomial of degree n. The constant of integration is determined by the condition (2), so B_n is uniquely determined. We have to do this explicitly for part (i).

Solution to problem 46

(i) First we find $\mathrm{B}_1(x)$ by integrating $1 \times \mathrm{B}_0(x)$, using equation (1): $\mathrm{B}_1(x) = x + k$, where k is a constant. We find k by applying the condition $\int_0^1 \mathrm{B}_1(x)\mathrm{d}x = 0$, which gives $k = -\frac{1}{2}$. Next we find $\mathrm{B}_2(x)$ by similar means, giving $x^2 - x + \frac{1}{6}$, and similarly $\mathrm{B}_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ and $\mathrm{B}_4(x)$ is as given.

(ii) We are asked to prove a result involving $\mathrm{B}_n(x)$ evaluated at $x = 1$ and $x = 0$, i.e. at the limits of the integral (2). We therefore try the effect of integrating both sides of equation (1) between these limits:

$$\mathrm{B}_n(1) - \mathrm{B}_n(0) \equiv \int_0^1 \frac{\mathrm{dB}_n(x)}{\mathrm{d}x}\,\mathrm{d}x = n\int_0^1 \mathrm{B}_{n-1}(x)\,\mathrm{d}x = 0\,,$$

using property (2) with n replaced by $(n-1)$.

(iii) First the easy bit of the induction proof. For $n = 1$, we have $\mathrm{B}_1(x) = x - \frac{1}{2}$, so

$$\mathrm{B}_1(x+1) - \mathrm{B}_1(x) = (x + 1 - \tfrac{1}{2}) - (x - \tfrac{1}{2}) = 1 \equiv nx^{n-1},$$

so the formula holds.

Now suppose that it holds for $n = k$:

$$\mathrm{B}_k(x+1) - \mathrm{B}_k(x) - kx^{k-1} = 0 \tag{4}$$

and investigate

$$\mathrm{B}_{k+1}(x+1) - \mathrm{B}_{k+1}(x) - (k+1)x^k\,, \tag{5}$$

which we hope will also equal zero.

The only helpful thing we know about Bernoulli polynomials involves the derivatives. Therefore, let us see what happens when we differentiate the expression (5):

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{B}_{k+1}(x+1) - \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{B}_{k+1}(x) - (k+1)kx^{k-1}\,.$$

Now using (1) gives

$$(k+1)B_k(x+1) - (k+1)\mathrm{B}_k(x) - (k+1)kx^{k-1}.$$

Note that we have used the chain rule to differentiate $\mathrm{B}_{k+1}(x+1)$ with respect to x rather than with respect to $(x+1)$. Note also that there is a pleasing overall factor of $(k+1)$, which suggests that we are on the right track. In fact, taking out this factor gives exactly the left hand side of equation (4), which is zero.

Of course, we are not finished yet: we have only shown that the derivative of equation (5) is equal to zero; the expression (5) is therefore constant:

$$\mathrm{B}_{k+1}(x+1) - \mathrm{B}_{k+1}(x) - (k+1)x^k = A\,.$$

We must show that $A = 0$. Setting $x = 0$ gives $\mathrm{B}_{k+1}(1) - \mathrm{B}_{k+1}(0) = A$, which implies that $A = 0$ by part (ii).

(iv) Summing (3) from $x = 0$ to $x = k$ gives the first of these results immediately because nearly all the terms cancel in pairs. The evaluation of the sum follows by calculating $\mathrm{B}_4(1001) - \mathrm{B}_4(0)$ from the result given in part (i).

Problem 47: Vector geometry (✓✓)

The line ℓ has vector equation $\mathbf{r} = \lambda\mathbf{s}$, where

$$\mathbf{s} = (\cos\theta + \sqrt{3}\,)\,\mathbf{i} + \sqrt{2}\,\sin\theta\,\mathbf{j} + (\cos\theta - \sqrt{3}\,)\,\mathbf{k}$$

and λ is a scalar parameter. Find an expression for the angle between ℓ and the line $\mathbf{r} = \mu(a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k})$. Show that there is a line m through the origin such that, whatever the value of θ, the acute angle between ℓ and m is $\frac{1}{6}\pi$.

A plane has equation $x - z = 4\sqrt{3}$. The line ℓ meets this plane at P. Show that, as θ varies, P describes a circle, with its centre on m. Find the radius of this circle.

2000 Paper II

Comments

It is not easy to set vector questions at this level: they tend to become merely complicated and tedious, rather than difficult in an interesting way. In a good question, there is usually some underlying geometry and it pays to try to understand what this is. Here, the question is about the geometrical object traced out by ℓ as θ varies.

You will need to know about scalar products of vectors for this question, but otherwise it is really just coordinate geometry.

Vectors form an extremely important part of almost every branch of mathematics (maybe *every* branch of mathematics) and will probably be one of the first topics you tackle on your university course.

Solution to problem 47

Both ℓ and the line $\mathbf{r} = \mu(a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k})$ pass through the origin, so the angle α between the lines is given by the scalar product of the unit vectors, i.e. the scalar product between the given vectors divided by the product of the lengths of the two vectors:

$$\begin{aligned}\cos\alpha &= \frac{a(\cos\theta+\sqrt{3}) + b(\sqrt{2}\sin\theta) + c(\cos\theta-\sqrt{3})}{\sqrt{(\cos\theta+\sqrt{3})^2 + 2\sin^2\theta + (\cos\theta-\sqrt{3})^2}\ \sqrt{a^2+b^2+c^2}}\\ &= \frac{(a+c)\cos\theta + \sqrt{2}\,b\sin\theta + (a-c)\sqrt{3}}{2\sqrt{2}\ \sqrt{a^2+b^2+c^2}}\ . \qquad (*)\end{aligned}$$

Now we want to show that there is some choice of a, b and c such that $\cos\alpha$ does not depend on the value of θ. By inspection of equation $(*)$, we see that this requires $a = -c$ and $b = 0$.

Setting $a = -c$ in $(*)$ gives $\cos\alpha = \frac{1}{2}\sqrt{3}$ and $\alpha = \frac{1}{6}\pi$ as required. We can absorb the constant a into μ, so the equation of the line m becomes

$$\mathbf{r} = \mu(\mathbf{i} - \mathbf{k})\ . \qquad (**)$$

The coordinates of a general point on the line ℓ are

$$x = \lambda(\cos\theta + \sqrt{3})\,, \quad y = \lambda\sqrt{2}\sin\theta\,, \quad z = \lambda(\cos\theta - \sqrt{3})\ .$$

For a point which is also on the plane $x - z = 4\sqrt{3}$ we have

$$\lambda(\cos\theta + \sqrt{3}) - \lambda(\cos\theta - \sqrt{3}) = 4\sqrt{3}$$

so $\lambda = 2$. The point P at the intersection between the line and the plane therefore has coordinates

$$(2\cos\theta + 2\sqrt{3}\,,\ 2\sqrt{2}\sin\theta\ ,\ 2\cos\theta - 2\sqrt{3})\ . \qquad (***)$$

As θ varies, does P moves round a circle? That is not easy to see, but fortunately we gather from the question that the centre of the circle is on m. It must also lie in the plane of the circle, which is the plane $x - z = 4\sqrt{3}$. The line $(**)$ meets this plane at the point $(2\sqrt{3},\, 0\,, -2\sqrt{3})$; call it O. To verify that P describes a circle with centre O we must check that the distance from P, given by $(***)$, to O is independent of θ. We have

$$OP^2 = (2\cos\theta)^2 + (2\sqrt{2}\sin\theta)^2 + (2\cos\theta)^2 = 8\ ,$$

which is indeed independent of θ. The radius of the circle is therefore $2\sqrt{2}$.

Post-mortem

As mentioned in the comments section, it is helpful to understand the geometry of vector questions. Since the angle between the variable line ℓ and the fixed line m is constant ($\frac{1}{6}\pi$), the shape generated by ℓ as θ varies is the surface of a cone (actually a pair of cones).

The intersection of a plane with a cone is in general a *conic section*: an ellipse (of which a circle is a special case), hyperbola, parabola or pair of straight lines. Try to picture these possibilities. In this case, the normal to the plane, which is in direction $(1, 0, -1)$, is parallel to the axis of the cone (the line m), so the intersection is indeed a circle.

Problem 48: Solving a quartic (✓✓)

Given that

$$x^4 + px^2 + qx + r \equiv (x^2 - ax + b)(x^2 + ax + c)\ ,$$

express p, q and r in terms of a, b and c.
Show that a^2 is a root of the cubic equation

$$u^3 + 2pu^2 + (p^2 - 4r)u - q^2 = 0\ .$$

Verify that $u = 9$ is a root in the case $p = -1$, $q = -6$, $r = 15$.
Hence, or otherwise, solve the equation

$$y^4 - 8y^3 + 23y^2 - 34y + 39 = 0\ .$$

2000 Paper III

Comments

The long-sought solution of the general cubic was found, in 1535, by Niccolò Tartaglia (c. 1500–1557). He was persuaded to divulge his secret (in the form of a poem) by Girolamo Cardano (1501–1576), who promised not to publish it before Tartaglia did. However, Cardano discovered that it had previously been discovered by del Ferro (1465–1525/6) before 1515 so he published it himself in his algebra book *The Great Art*. There followed an acrimonious dispute between Tartaglia and Cardano, in which the latter was championed by his student Ferrari (1522–1565). The dispute culminated in a public mathematical duel between Ferrari and Tartaglia held in the church of Santa Maria in Milan in 1548, in which they attempted to solve each others' cubics. The duel ended in a shouting match with Tartaglia storming off. It seems Ferrari was the winner. Tartaglia was sacked from his job as lecturer and Ferrari made his fortune as a tax assessor before becoming a professor of mathematics at Bologna. He was poisoned by his sister, with arsenic, in 1565.

Ferrari found a way of reducing quartic equations to cubic equations; his method (roughly) is used in this question to solve a quartic which could probably be solved easier 'otherwise'. But it is the method that is interesting, not the solution.

The first step of the Ferrari method is to reduce the general quartic to a quartic equation with the cubic term missing by means of a linear transformation of the form $x \to x - a$. Then this reduced quartic is factorised (the first displayed equation in this question). The factorisation can be found by solving a cubic equation (the second displayed equation above) that must be satisfied by one of the coefficients in the factorised form.

The solutions of the quartic are all complex, but don't worry if you haven't come across complex numbers: you will be able to do everything except perhaps write down the last line.

You will no doubt be full of admiration for this clever method of solving quartic equations. One thing you are bound to ask yourself is how the other two roots of the cubic equation fit into the picture. In fact, the cubic equation gives (in general) 6 distinct values of a so there is quite a lot of explaining to do, given that the quartic has at most 4 distinct roots.

Solution to problem 48

We have

$$\begin{aligned}(x^2-ax+b)(x^2+ax+c) &= (x^2-ax)(x^2+ax)+b(x^2+ax)+c(x^2-ax)+bc\\ &= x^4+(b+c-a^2)x^2+a(b-c)x+bc\end{aligned}$$

so

$$p=b+c-a^2, \qquad q=a(b-c)\,, \qquad r=bc\,. \tag{$*$}$$

To obtain an equation for a in terms of p, q and r we eliminate b and c from $(*)$ using the identity $(b+c)^2=(b-c)^2+4bc$. This gives $(p+a^2)^2=(q/a)^2+4r$ which simplifies easily to the given cubic with u replaced by a^2.

We can easily verify by direct substitution that $u=9$ satisfies the given cubic.

To solve the quartic equation, the first task is to reduce it to the form $x^4+px^2+qx+r=0$, which has no term in x^3. This is done by means of a translation. Noting that $(y-a)^4=y^4-4ay^3+\cdots$, we set $x=y-2$. This gives

$$(x+2)^4-8(x+2)^3+23(x+2)^2-34(x+2)+39=0\,.$$

which (not surprisingly) boils down to $x^4-x^2-6x+15=0$, so that $p=-1$, $q=-6$ and $r=15$. We have already shown that one root of the cubic corresponding to these values is $u=9$. Thus we can achieve the factorisation of the quartic into two quadratic factors by setting $a=3$ (or $a=-3$; it doesn't matter which we use) and

$$\begin{aligned}b&=\frac{1}{2}\left(p+a^2+\frac{q}{a}\right)=3\\ c&=\frac{1}{2}\left(p+a^2-\frac{q}{a}\right)=5\,.\end{aligned}$$

Thus

$$x^4-x^2-6x+15=(x^2-3x+3)(x^2+3x+5)\,.$$

Setting each of the two quadratic factors equal to zero gives

$$x=\frac{3\pm \mathrm{i}\sqrt{3}}{2} \quad \text{and} \quad x=\frac{-3\pm \mathrm{i}\sqrt{11}}{2}\,,$$

so

$$y=\frac{7\pm \mathrm{i}\sqrt{3}}{2} \quad \text{or} \quad y=\frac{1\pm \mathrm{i}\sqrt{11}}{2}\,.$$

Post-mortem

Did you work out the relation between the six possible values of a (corresponding to given values of p, q and r) and the roots of the quartic? The point is that the quartic can be written as the product of four linear factors (by the fundamental theorem of algebra) and there are six ways of grouping the linear factors into two quadratic factors. Each way corresponds to a value of a.

Problem 49: Areas and volumes (✓✓)

The function f satisfies the condition $\mathrm{f}'(x) > 0$ for $a \leqslant x \leqslant b$, and g is the inverse of f.

(i) By making a suitable change of variable, prove that

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = b\beta - a\alpha - \int_\alpha^\beta \mathrm{g}(y)\,\mathrm{d}y\,, \tag{1}$$

where $\alpha = \mathrm{f}(a)$ and $\beta = \mathrm{f}(b)$. Interpret this formula geometrically, by means of a sketch, in the case where α and a are both positive.

Verify the result (1) for $\mathrm{f}(x) = e^{2x}$, $a = 0$, $b = 1$.

(ii) Prove similarly and interpret the formula

$$2\pi\int_a^b x\mathrm{f}(x)\,\mathrm{d}x = \pi(b^2\beta - a^2\alpha) - \pi\int_\alpha^\beta \big[\mathrm{g}(y)\big]^2\,\mathrm{d}y\,. \tag{2}$$

1987 Paper II

Comments

As is often the case, the required change of variable for part (i) can be worked out by inspection of the limits.

To find the inverse function (note: inverse, not reciprocal) of the function f, it is often best to try to think of the function g such that $\mathrm{g}\big(\mathrm{f}(x)\big) = x$, though making x the subject of $y = \mathrm{f}(x)$ is perhaps safer with an unfamiliar function.

The condition $\mathrm{f}'(x) > 0$ ensures that f has a unique inverse; a function such as sin which has maximum and minimum points has a unique inverse only on restricted ranges of its argument which do not contain the turning points. (This is obvious from a sketch). The condition $\mathrm{f}'(x) < 0$ would do equally well.

The geometrical interpretations of these formulae are exceptionally pleasing, though the second one needs some artistic skill to make it convincing.

Solution to problem 49

(i) The limits of the integral on the right hand side of equation (1) are $\mathrm{f}(a)$ and $\mathrm{f}(b)$, which suggests the change of variable $y = \mathrm{f}(x)$. Making this change, so that $\mathrm{d}y = \mathrm{f}'(x)\mathrm{d}x$, gives

$$\int_\alpha^\beta \mathrm{g}(y)\,\mathrm{d}y = \int_a^b \mathrm{g}\big(\mathrm{f}(x)\big)\,\mathrm{f}'(x)\,\mathrm{d}x = \int_a^b x\,\mathrm{f}'(x)\,\mathrm{d}x\,.$$

For the last equality, we have used the definition of g as the inverse of f, i.e. $\mathrm{g}\big(\mathrm{f}(x)\big) = x$. This last integral is begging to be integrated by parts:

$$\int_a^b x\,\mathrm{f}'(x)\,\mathrm{d}x = x\mathrm{f}(x)\Big|_a^b - \int_a^b \mathrm{f}(x)\,\mathrm{d}x\,,$$

which gives the required result after evaluating $x\mathrm{f}(x)$ at a and b. The first sketch below shows these areas: the area between the large and small rectangles is $(b\beta - a\alpha)$, which is split into the areas represented by the two integrals of equation (1), hatched vertically and horizontally, respectively.

Setting $\mathrm{f}(x) = \mathrm{e}^{2x}$ and $a = 0$, $b = 1$ in the left hand side of (1) gives $\displaystyle\int_0^1 \mathrm{e}^{2x}\,\mathrm{d}x = \tfrac{1}{2}\big(\mathrm{e}^2 - 1\big)$. For the right hand side of (1), we have $\alpha = 1$ and $\beta = \mathrm{e}^2$ so $b\beta - a\alpha = \mathrm{e}^2$. The inverse of e^{2x} is $\frac{1}{2}\ln y$ so the integral becomes

$$\int_1^{\mathrm{e}^2} \tfrac{1}{2}\ln y\,\mathrm{d}y = \tfrac{1}{2}(y\ln y - y)\Big|_1^{\mathrm{e}^2} = \tfrac{1}{2}(2\mathrm{e}^2 - \mathrm{e}^2) - \tfrac{1}{2}(0-1) = \tfrac{1}{2}\mathrm{e}^2 + \tfrac{1}{2}\,.$$

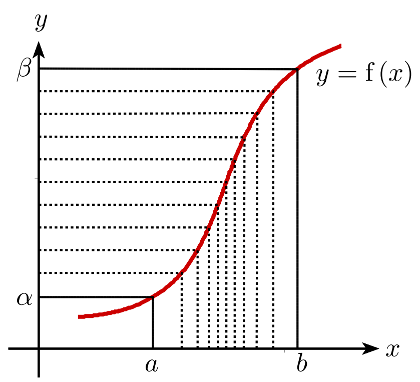
Thus the left hand side of (1) agrees with the right hand side.

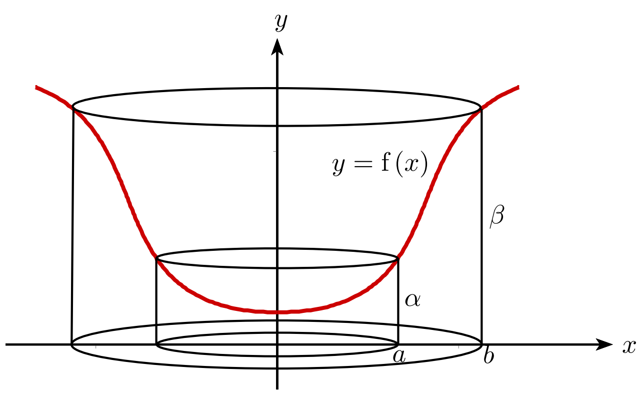
(ii) We can use the same method (change of variable followed by integration by parts):

$$\int_\alpha^\beta \big[\mathrm{g}(y)\big]^2\,\mathrm{d}y = \int_a^b x^2\mathrm{f}'(x)\,\mathrm{d}x = x^2\mathrm{f}(x)\Big|_a^b - \int_a^b 2x\mathrm{f}(x)\,\mathrm{d}x = (b^2\beta - a^2\alpha) - 2\int_a^b x\mathrm{f}(x)\,\mathrm{d}x\,,$$

which gives the required formula (2) on multiplication by π.

The first of the integrals in (2) is the volume of the solid body *under* the surface formed by rotating the curve $y = \mathrm{f}(x)$ round the y-axis; this volume is thought of as a set of concentric cylindrical shells of height $\mathrm{f}(x)$ with internal radius x ($a \leqslant x \leqslant b$), and thickness $\mathrm{d}x$. The second integral is the volume *inside* the surface formed by rotating the curve $y = \mathrm{f}(x)$ round the y-axis; this volume is thought of as a pile of infinitesimally thin discs of radius $\mathrm{g}(y)$ ($\alpha \leqslant y \leqslant \beta$) and thickness $\mathrm{d}y$. The sum of the two integrals is equal to the difference between the volumes of the two concentric cylinders (of radii a and b, heights α and β, respectively) as shown in the second sketch below.





Problem 50: More curve sketching (✓✓✓)

(i) The curve C_1 passes through the origin in the x–y plane and its gradient is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(1-x^2)\mathrm{e}^{-x^2}.$$

Show that C_1 has a minimum point at the origin and a maximum point at $\left(1, \frac{1}{2}\,\mathrm{e}^{-1}\right)$. Find the coordinates of the other stationary point. Give a rough sketch of C_1.

(ii) The curve C_2 passes through the origin and its gradient is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(1-x^2)\mathrm{e}^{-x^3}.$$

Show that C_2 has a minimum point at the origin and a maximum point at $(1, k)$, where $k > \frac{1}{2}\,\mathrm{e}^{-1}$. (You need not find k.)

2001 Paper II

Comments

No work is required to find the x coordinate of the stationary points, but you have to integrate the differential equation to find the y coordinate. For the second part, you cannot integrate the equation — other than numerically, or in terms of rather obscure special functions that you almost certainly haven't come across, such as the incomplete gamma function defined by

$$\Gamma(x,a) = \int_0^a t^{x-1}\mathrm{e}^{-t}\mathrm{d}t\,.$$

However, you can obtain an estimate, which is all that is required, by comparing the gradients of C_1 with C_2 and thinking of the graphs for $-1 \leqslant x \leqslant 1$. This is perhaps a bit tricky; an idea that you may well not alight on under examination conditions.

Solution to problem 50

C_1 has stationary points when $\dfrac{dy}{dx} = 0$, i.e. when $x = 0$, $x = +1$ or $x = -1$. To find the y coordinates of the stationary points, we integrate the differential equation, using integration by parts:

$$y = \int (1-x^2)xe^{-x^2}dx = (1-x^2)(-\tfrac{1}{2}e^{-x^2}) - \int xe^{-x^2}dx = -\tfrac{1}{2}(1-x^2)e^{-x^2} + \tfrac{1}{2}e^{-x^2} + \text{const} = \tfrac{1}{2}x^2e^{-x^2},$$

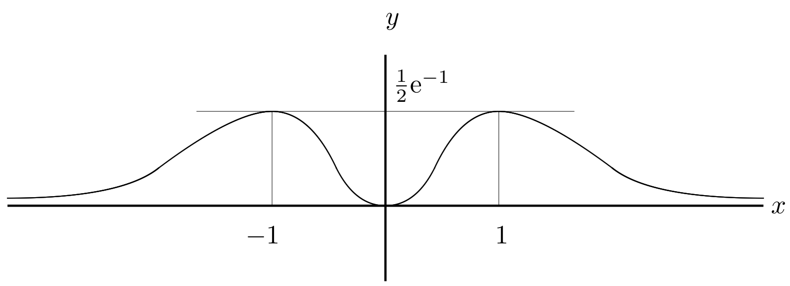
where we have used the fact that C_1 passes through the origin to evaluate the constant of integration. The coordinates of the stationary points are therefore $(1, \frac{1}{2}e^{-1})$, $(0, 0)$ and $(-1, \frac{1}{2}e^{-1})$.

One way of classifying the stationary points is to look at the second derivative:

$$\frac{dy}{dx} = (x - x^3)e^{-x^2} \Longrightarrow \frac{d^2y}{dx^2} = (1-3x^2)e^{-x^2} - 2x^2(1-x^2)e^{-x^2} = (1-5x^2+2x^4)e^{-x^2},$$

which is positive when $x = 0$ (indicating a minimum) and negative when $x = 1$ and $x = -1$ (indicating maxima).

Since $y \to 0$ as $x \to \pm\infty$, the sketch is:



For C_2, the stationary points are again at $x = 0$ and $x = \pm 1$. To classify the stationary points, we calculate the second derivative:

$$\frac{d^2y}{dx^2} = (1 - 3x^2 - 3x^3 + 3x^5)e^{-x^3},$$

which is positive at $x = 0$ (a minimum) and negative at $x = 1$ (a maximum).

Since we cannot integrate this differential equation explicitly, we must compare C_2 with C_1 to see whether the value of y at the maxima is indeed greater for C_2 than for C_1.

For $0 < x < 1$, $x^3 < x^2$, so $e^{x^3} < e^{x^2}$ and $e^{-x^3} > e^{-x^2}$. Thus the gradient of C_2 is greater than the gradient of C_1. Since both curves pass through the origin, we deduce that C_2 lies above C_1 for $0 < x \leqslant 1$ and therefore the maximum point $(1, k)$ on C_2 has $k > \frac{1}{2}e^{-1}$.

Post-mortem

You were probably surprised that the examiners didn't ask you to sketch C_2; they usually do. It looks as if it could be done, because you know that C_2 passes through the origin and you can relate its slope, roughly at least, to that of C_1, which you did sketch. The difficulty lies in determining its behaviour as $x \to \infty$; does $y \to 0$ as for C_1? In fact, it doesn't; no reason why it should. Instead, $y \to 0.153...$. We shouldn't have expected C_1 to asymptote to the x-axis either; the fact that it does so is due to a rather delicate balancing of the two terms that determine its gradient.

Problem 51: Spherical loaf (✓✓✓)

Note: In this question slices of bread are eaten in order, starting at one end of the loaf.

A spherical loaf of bread is cut into parallel slices of equal thickness. Show that, after any number of the slices have been eaten, the area of crust remaining is proportional to the number of slices remaining.

A European ruling decrees that a parallel-sliced spherical loaf can only be referred to as 'crusty' if the ratio of volume V (in cubic metres) of bread remaining to area A (in square metres) of crust remaining after any number of slices have been eaten satisfies $V < A$. Show that the radius of a crusty parallel-sliced spherical loaf must be less than $2\frac{2}{3}$ metres.

[*The area A and volume V formed by rotating a curve in the x–y plane round the x-axis from $x = a - t$ to $x = a$ are given by*

$$A = 2\pi \int_{a-t}^{a} y\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{\frac{1}{2}} \mathrm{d}x\ , \qquad V = \pi \int_{a-t}^{a} y^2\, \mathrm{d}x\ .\ ]$$

2001 Paper I

Comments

The first result (the mathematical result, I mean, not the European ruling which I invented) came as a bit of a surprise to me — though no doubt it is well known. I wondered if it was the only surface of revolution with this property. You might like to think about this.

Don't be distracted by the use of the word 'slices'; since the thickness of the slices is not given, it is clear that you are supposed to think in terms of the continuous distance along the loaf rather than the number of slices.

For the last part, you will need to minimise a ratio as a function of t (the 'length' of loaf remaining). To find the ratio you have to do a couple of integrals. It is this 'multi-stepping' that makes the problem difficult (and very different from typical school-level questions) rather than any individual step.

The note at the beginning of the question is new. Seventeen years after the question was set two people wrote, independently, to say the solution was wrong and the question was wrongish. If you eat the slices out of order you may eat the middle slice first which gives the weaker result $r < 3$ for the last part. The first paragraph is still OK, because each slice has the same area of crust.

Volumes of revolution are no longer in the A level Mathematics specifications so have been moved from the STEP 1 to the STEP 3 specification.

Solution to problem 51

The first thing we need is an equation for the surface of a spherical loaf. The obvious choice, especially given the hint at the bottom of the question, is the circle $x^2 + y^2 = a^2$ in the plane $z = 0$, rotated about the x-axis.

If the loaf is cut at a distance t from the end $x = a$, and the portion from $x = -a$ to $x = a - t$ is eaten, then the area remaining is

$$\begin{aligned} 2\pi \int_{a-t}^{a} y \left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{\frac{1}{2}} \mathrm{d}x &= 2\pi \int_{a-t}^{a} (a^2 - x^2)^{\frac{1}{2}} \left(1 + \left(\frac{-x}{(a^2 - x^2)^{\frac{1}{2}}}\right)^2\right)^{\frac{1}{2}} \mathrm{d}x \\ &= 2\pi \int_{a-t}^{a} a \,\mathrm{d}x \\ &= 2\pi a t\,. \end{aligned} \qquad (*)$$

This is proportional to the length t of remaining loaf, so proportional to the number of slices remaining (if the loaf is evenly sliced).

The remaining volume is

$$\begin{aligned} \pi \int_{a-t}^{a} y^2 \mathrm{d}x &= \pi \int_{a-t}^{a} (a^2 - x^2)\,\mathrm{d}x = \pi \Big[(a^2 x - \tfrac{1}{3}x^3)\Big]_{a-t}^{a} \\ &= \pi(at^2 - \tfrac{1}{3}t^3). \end{aligned}$$

As a quick check on the algebra, notice that this is zero when $t = 0$ and $\frac{4}{3}\pi a^3$ when $t = 2a$.

Thus $V/A = (3at - t^2)/(6a)$. This is a quadratic curve with zeros at $t = 0$ and $t = 3a$, so it has a maximum at $t = \frac{3}{2}a$ (by differentiating or otherwise), where $V/A = \frac{3}{8}a$. Since we require this ratio to be less than one, we must have $a < \frac{8}{3}$ metres.

Post-mortem

It was well worth studying the information given at the end of the question before plunging into it: it not only gave the necessary formulae for the surface area and volume, but also gave them in a form that suggested a way forward right at the start of the question.

Did you think about whether there are other shapes that would have the property proved for the sphere in the first paragraph of the question? Mathematically, it boils down to whether there are functions $y(x)$, other than our function $y = \sqrt{a^2 - x^2}$, that can satisfy $(*)$. If we differentiate $(*)$, we obtain

$$2\pi y \left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{\frac{1}{2}} = 2\pi a$$

which looks formidable, but in fact simplifies to an equation that you can integrate quite easily. The sphere is, it turns out, the only shape with the required property.

Problem 52: Snowploughing (✓✓✓)

Two identical snowploughs plough the same stretch of road. The first starts at a time t_1 seconds after it starts snowing, and the second starts from the same point $t_2 - t_1$ seconds later, going in the same direction. Snow falls so that the depth of snow increases at a constant rate of k ms^{-1}. The speed of each snowplough is ak/z ms^{-1} where z is the depth (in metres) of the snow it is ploughing and a is a constant. Each snowplough clears all the snow. Show that the time t at which the second snowplough has travelled a distance x metres satisfies the equation

$$a\frac{\mathrm{d}t}{\mathrm{d}x} = t - t_1 e^{x/a}. \tag{†}$$

Hence show that the snowploughs will collide when they have travelled $a(t_2/t_1 - 1)$ metres.

1987 Specimen Paper III

Comments

There is something exceptionally beautiful about this question, but it is hard to identify exactly what it is; seeing this question for the first time makes even hardened mathematicians smile with pleasure.

There is a modelling element to it: you have to set up equations from the information given in the text. The first equation you need is a simple first order differential equation to find the time taken by the first snowplough to travel a distance x. The corresponding equation for the second snowplough is a bit more complicated, because the depth of snow at any point depends on the time at which the first snowplough reached that point, clearing the snow.

The differential equation (†) can be solved using an integrating factor. However, the equation which arises naturally at this point is one involving $\dfrac{\mathrm{d}x}{\mathrm{d}t}$, which cannot (apparently) be solved by any means. It is the rather good trick of turning the equation upside down (regarding t as a function of x instead of x as a function of t) that allows the problem to be solved so neatly. Apologies if you haven't come across integrating factors for first order differential equations; but they are really not difficult—you can look online and find an easily understandable explanation.

You won't surprised to learn that there is a generalisation to n identical snowploughs . . . .

Solution to problem 52

Suppose that the first snowplough reaches a distance x at time T after it starts snowing. Then the depth of snow it encounters is kT and its speed is therefore $ak/(kT)$, i.e. a/T. The equation of motion of the first snowplough is

$$\frac{\mathrm{d}x}{\mathrm{d}T} = \frac{a}{T}.$$

Integrating both sides with respect to T gives

$$x = a\ln T + \text{constant of integration}.$$

We know that $T = t_1$ when $x = 0$ (the snowplough started t_1 seconds after the snow started), so

$$x = a\ln T - a\ln t_1.$$

This can be rewritten as

$$T = t_1 e^{x/a}.$$

When the second snowplough reaches x at time t, snow has been falling for a time $t - T$ since it was cleared by the first snowplough, so the depth at time t is $k(t - T)$ metres, i.e. $k(t - t_1 e^{x/a})$ metres. Thus the equation of motion of the second snowplough is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{ak}{k(t - t_1 e^{x/a})}.$$

Now we use the standard result (a special case of the chain rule)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1\Big/\frac{\mathrm{d}t}{\mathrm{d}x}$$

to obtain the required equation (†).

Multiplying by $e^{-x/a}$ (an *integrating factor*) and rearranging gives

$$e^{-x/a}\frac{\mathrm{d}t}{\mathrm{d}x} - \frac{e^{-x/a}t}{a} = -\frac{t_1}{a} \qquad \text{i.e.} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(te^{-x/a}\right) = -\frac{t_1}{a}$$

which integrates to

$$te^{-x/a} = -\frac{t_1}{a}x + \text{constant of integration}.$$

Since the second snowplough started ($x = 0$) at time t_2, the constant of integration is just t_2 and the solution is

$$t = (t_2 - t_1 x/a)e^{x/a}.$$

The snowploughs collide when they reach the same position at the same time. Let this position be $x = X$. Then

$$T = t \Longrightarrow t_1 e^{X/a} = (t_2 - t_1 X/a)e^{X/a},$$

so X is given by

$$t_1 = (t_2 - t_1 X/a).$$

This is equivalent to the given formula.

Problem 53: Tortoise and hare (✓✓)

A tortoise and a hare have a race to the vegetable patch, a distance X kilometres from the starting post, and back. The tortoise sets off immediately, at a steady v kilometres per hour. The hare goes to sleep for half an hour and then sets off at a steady speed V kilometres per hour. The hare overtakes the tortoise half a kilometre from the starting post, and continues on to the vegetable patch, where she has another half an hour's sleep before setting off for the return journey at her previous pace. One and quarter kilometres from the vegetable patch, she passes the tortoise, still plodding gallantly and steadily towards the vegetable patch. Show that

$$V = \frac{10}{4X - 9}$$

and find v in terms of X.

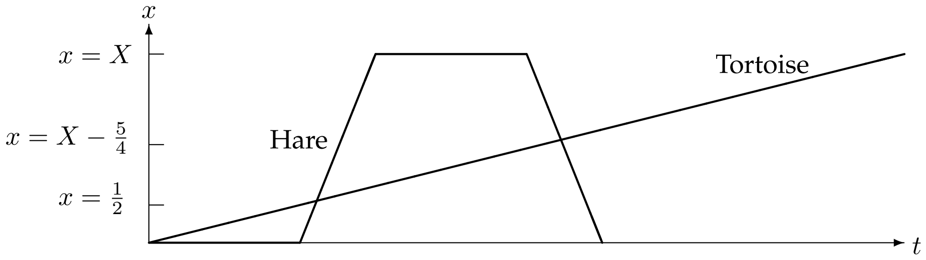
Find X if the hare arrives back at the starting post one and a half hours after the start of the race. How long does it take the tortoise to reach the vegetable patch?

1999 Paper I

Comments

The first thing to do is draw a distance–time diagram. You might also find it useful to let the times at which the two animals meet be T_1 and T_2.

Solution to problem 53



Let the times of the first and second meetings be T_1 and T_2. Then

$$vT_1 = \tfrac{1}{2}\,, \quad V(T_1 - \tfrac{1}{2}) = \tfrac{1}{2}\,, \quad vT_2 = (X - \tfrac{5}{4})\,, \quad V(T_2 - 1) = X + \tfrac{5}{4}.$$

The first pair and the second pair of equations give, respectively,

$$\frac{1}{v} - \frac{1}{V} = 1\,, \qquad \frac{X - \frac{5}{4}}{v} = \frac{X + \frac{5}{4}}{V} + 1\,,$$

and hence (first eliminating v):

$$V = \frac{10}{4X - 9}\,, \qquad v = \frac{10}{4X + 1}\,.$$

The total distance travelled by the hare in $\frac{1}{2}$ hour is $2X$, so

$$2X = \frac{10}{4X - 9} \times \frac{1}{2}, \quad \text{i.e.} \quad 8X^2 - 18X - 5 = 0\,.$$

Factorising the quadratic gives two roots $-\frac{1}{4}$ and $\frac{5}{2}$, and we obviously need the positive root.

The speed of the tortoise is given by

$$v = \frac{10}{4X + 1} = \frac{10}{11}$$

so the time taken to travel $\frac{5}{2}$ kilometres is $\frac{11}{4}$ hours.

Post-mortem

It is interesting to see how quickly a mechanics question can change from mechanics to something else: algebra, calculus or, as in this case, coordinate geometry. As soon as the picture is drawn, the equations of the lines can be written down and everything follows as if it were a geometry problem.

Sometimes the change happens even more quickly. There is a well-known problem about a monk ascending a hill one day and descending the next day starting at the same time on each day. The problem is to show that there is a time at which the monk was at the same height on both days. You are not even given the speed at which the monk climbs. But if you draw (or just imagine drawing) height-time graphs of the ascent and descent, and superimpose them, you are done.

Problem 54: How did the chicken cross the road? (✓✓)

A single stream of cars, each of width a and exactly in line, is passing along a straight road of breadth b with speed V. The distance between successive cars (i.e. the distance between the back of one car and the front of the following car) is c.

A chicken crosses the road in safety at a constant speed u in a straight line making an angle θ with the direction of traffic. Show that

$$u \geqslant \frac{Va}{c\sin\theta + a\cos\theta}\,. \tag{$*$}$$

Show also that if the chicken chooses θ and u so that she crosses the road at the least possible (constant) speed, she crosses in time

$$\frac{b}{V}\left(\frac{c}{a}+\frac{a}{c}\right).$$

1997 Paper I

Comments

I like this question because it relates to (an idealised version of) a situation we have probably all thought about. Once you have visualised it, there are no great difficulties. As usual, you have to be careful with the inequalities, though it turns out here that there is no danger of dividing by a negative quantity.

Solution to problem 54

The easiest way to think about this problem is to consider the cars to be stationary and the velocity of the chicken to be $(u\cos\theta - V, u\sin\theta)$. Then the diagrams are very easy to visualise.

Let t be the time taken to cross the distance a in which the chicken is at risk. Then $a = ut\sin\theta$.

For safety, the chicken must choose $ut\cos\theta + c \geqslant Vt$: equality here occurs when the chicken starts at the near-side rear of one car and just avoids being hit by the far-side front of the next car.

Eliminating t from these two equations gives the required inequality:

$$\begin{aligned}
ut\cos\theta &\geqslant Vt - c \\
\implies (u\cos\theta - V)t &\geqslant -c \\
\implies (u\cos\theta - V)\frac{a}{u\sin\theta} &\geqslant -c \\
\implies au\cos\theta - aV &\geqslant -cu\sin\theta \\
\implies u(c\sin\theta + a\cos\theta) &\geqslant aV
\end{aligned}$$

which is the required result.

For a given value of θ, the minimum speed satisfies

$$u(c\sin\theta + a\cos\theta) = aV.$$

The smallest value of this u is therefore obtained when $c\sin\theta + a\cos\theta$ is largest. This can be found by calculus (regard it as a function of θ and differentiate: the maximum occurs when $\tan\theta = c/a$) or by trigonometry:

$$c\sin\theta + a\cos\theta = \sqrt{a^2+c^2}\cos\left(\theta - \tan^{-1}(c/a)\right)$$

so the maximum value is $\sqrt{a^2+c^2}$ and it occurs when $\tan\theta = c/a$.

The time of crossing is

$$\frac{b}{u\sin\theta} = \frac{b(c\sin\theta + a\cos\theta)}{Va\sin\theta} = \frac{b(c + a\cot\theta)}{Va} = \frac{b(c + a^2/c)}{Va}.$$

Post-mortem

There is another inequality besides $(*)$ that you might have noticed. If $u\cos\theta > V$ (so the chicken moves faster than the cars — a bit unlikely unless the chicken is trying to cross the M25), the chicken should start her run at the *front* nearside of a car and must not collide with the car *ahead*. This requires $(u\cos\theta - V)t \leqslant c$, so

$$u(-c\sin\theta + a\cos\theta) \leqslant aV.$$

If $(-c\sin\theta + a\cos\theta) < 0$, this places no constraint on u. But if $(-c\sin\theta + a\cos\theta) > 0$, then

$$u \leqslant \frac{aV}{-c\sin\theta + a\cos\theta}.$$

In both cases, the inequality $(*)$ overleaf does not apply. This is clearly not the situation envisaged by the examiners, and probably not by any of the candidates either, but still it should have been catered for in the wording of the question.

Problem 55: Hank's gold mine (✓✓)

Hank's Gold Mine has a very long vertical shaft of height l. A light chain of length l passes over a small smooth light fixed pulley at the top of the shaft. To one end of the chain is attached a bucket A of negligible mass and to the other a bucket B of mass m.

The system is used to raise ore from the mine as follows. When bucket A is at the top it is filled with mass $2m$ of water and bucket B is filled with mass λm of ore, where $0 < \lambda < 1$. The buckets are then released, so that bucket A descends and bucket B ascends. When bucket B reaches the top both buckets are emptied and released, so that bucket B descends and bucket A ascends. The time to fill and empty the buckets is negligible. Find the time taken from the moment bucket A is released at the top until the first time it reaches the top again.

This process goes on for a very long time. Show that, if the greatest amount of ore is to be raised in that time, then λ must satisfy the condition $\mathrm{f}'(\lambda) = 0$ where

$$\mathrm{f}(\lambda) = \frac{\lambda(1-\lambda)^{1/2}}{(1-\lambda)^{1/2} + (3+\lambda)^{1/2}}.$$

1998 Paper I

Comments

One way of working out the acceleration of a system of two masses connected by a light string passing over a pulley is to write down the equation of motion of each mass, bearing in mind that the force due to tension will be the same for each mass (it cannot vary along the string, because then the acceleration of some portion of the *massless* string would be infinite). Then you eliminate the tension.

Alternatively, you can use the equation of motion of the system of two joined masses. The system has inertial mass equal to the sum of the masses (because both masses must accelerate equally) but gravitational mass equal to the difference of the masses (because the gravitational force on one mass cancels, partially, the gravitational force on the other), so the equation of motion is just (Newton's law of motion)

$$(m_1 + m_2)a = (m_1 - m_2)g.$$

Solution to problem 55

When bucket A ascends, the acceleration is g.

For bucket A's downward journey, at acceleration a, the equations of motion for bucket A and bucket B, respectively, are

$$-T + 2mg = 2ma, \quad T - (1+\lambda)mg = (1+\lambda)ma,$$

where T is the tension in the rope. Eliminating T gives so $a = \dfrac{1-\lambda}{3+\lambda}\,g$.

The time of descent (using $l = \frac{1}{2}at^2$) is $\sqrt{2l/a}$ and the time of ascent is $\sqrt{2l/g}$. The total time required for one complete cycle is therefore

$$\sqrt{\frac{2l}{g}}\left(1+\sqrt{\frac{3+\lambda}{1-\lambda}}\right).$$

Call this t. The number of round trips in a long time t_{long} is t_{long}/t so the amount of ore lifted in time t_{long} is $\lambda m t_{\text{long}}/t$.

To maximise this, we have to maximise λ/t with respect to λ, and λ/t is exactly the $\mathrm{f}(\lambda)$ given. Note that $\mathrm{f}(0) = 0$ (which makes sense because no ore is raised if $\lambda = 0$), and $\mathrm{f}(1) = 0$ (which also makes sense because the buckets don't move on the raising stage if $\lambda = 1$). That means the greatest value of $\mathrm{f}(\lambda)$ must occur at a value of λ in the range $0 < \lambda < 1$ at which $\mathrm{f}'(\lambda) = 0$.

Post-mortem

You may have wondered why, in the last part, the question says that the process goes on for a very long time. The reason for this is that when you maximise what I have called $\lambda m t_{\text{long}}/t$ the result may not correspond to a complete number of cycles. If you stop in mid-cycle, you raise no ore from that cycle so a calculus maximisation of a continuous function is not the right method. However, if the process continues for many cycles, the contribution from the last cycle becomes negligible, and a calculus maximisation becomes appropriate.

You may also wonder why you were not asked to find the maximising value of λ. If you are feeling exceptionally energetic, you could try to solve $\mathrm{f}'(\lambda) = 0$. This will eventually lead you to the rather discouraging quartic equation

$$\lambda^4 + 4\lambda^3 + 2\lambda^2 - 8\lambda + 3 = 0.$$

You will have done a lot of tedious work to obtain an equation that cannot be solved without either a formula for the solutions of a general quartic equation or a computer. There are in fact two real solutions of this equation, $\lambda = 0.528$ and $\lambda = 0.704$.

Problem 56: A chocolate orange (✓)

A chocolate orange consists of a sphere of delicious smooth uniform chocolate of mass M and radius a, sliced into segments by planes through a fixed axis. It stands on a horizontal table with this axis vertical and it is held together by a narrow ribbon round its equator. Show that the tension in the ribbon is at least $\frac{3}{32}Mg$.

[You may assume that the centre of mass of a segment of angle 2θ is at distance $\dfrac{3\pi a\sin\theta}{16\theta}$ from the axis.]

1987 Specimen Paper II

Comments

This question can be done by the usual methods (resolving forces and taking moments about a suitably chosen point). Since the chocolate is smooth, there is no friction. The ribbon may be elastic, so it could be tighter than is needed just to keep the orange together. At the lowest tension possible the orange is on the point of falling apart, so there are no forces between the faces of the segments, except at the point of contact with the table.

The ribbon is said to be thin, but actually this has to be interpreted as massless as well as thin; a ribbon with mass would simply drop off the equator of the orange.

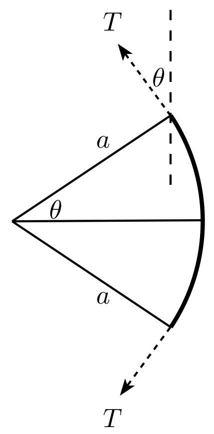
Notice that the question does not say that the segments are all the same size. That suggests that we should look at just one segment, expecting to find that the tension required for that segment is independent of the angle of the segment.

When I set this question originally on a 1985 examination paper (pre-dating STEP, which started in 1987), I gave the wrong formula for the distance of the centre of mass of the segment, and the answer for the tension was also incorrect, but consistent with the incorrect formula. Not surprisingly, no one pointed it out, either at the time or afterwards. With a bit of luck, it is correct now.

I sent a copy of the examination paper to a well-known manufacturer of high quality chocolate confections[30] and was rewarded with a puzzled letter and a parcel; not as large a parcel as I had hoped for (perhaps they spotted the incorrect formula), but better than nothing. It seemed worth including the question in this book in case they wanted an opportunity to make amends . . . .

[30] Originally, the firm made chocolate apples which, being apple-shaped, would not have worked well for this question (the centre of mass of an orange segment was hard enough for me). The chocolate lemon, introduced in 1979, would not have been good either but luckily it was discontinued rapidly after it turned out that no one wanted to eat it.

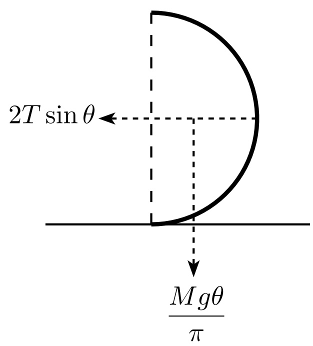
Solution to problem 56



The above diagram shows a horizontal cross-section of a segment of the orange.

Resolving the tension T in the ribbon as shown gives a horizontal component of force due to tension on the segment of $2T\sin\theta$ towards the left of the diagram.

The volume of the segment is a fraction $2\theta/(2\pi)$ of the volume of the sphere, so the segment has mass $M(2\theta)/(2\pi)$. The weight of the segment gives a force of $Mg\,\theta/\pi$ acting downwards through the centre of mass as shown in the diagram below, which is a vertical cross-section of a segment.



Taking moments about the point of contact of the table and the segment gives

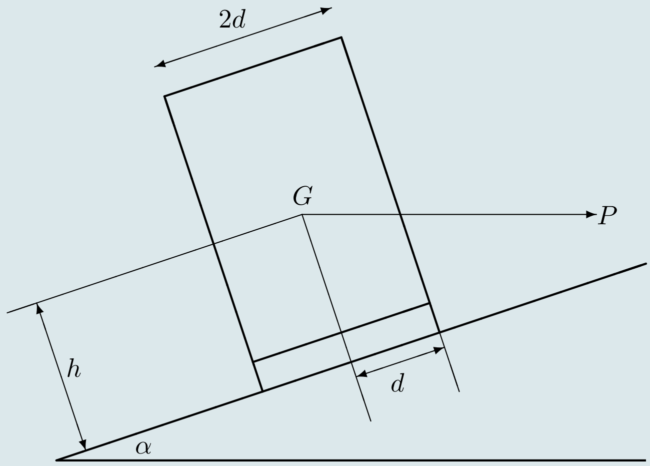
$$\frac{Mg\theta}{\pi}\times\frac{3\pi a\sin\theta}{16\theta}=2T\sin\theta\times a$$

which gives the required answer.

Post-mortem

Another way of tackling this sort of problem is to use the principle of virtual work, for which you imagine that the system relaxes a very small amount (in this case, by allowing the ribbon to stretch) and equate the work done against the constraints (here, tension times extension) to the change of potential energy of the system, to obtain a formula for the constraint force (here, the tension). In many cases, this method is simpler, but here it turns out to be very difficult: not recommended at all. The difficulty is that if the orange relaxes so that each segment is now inclined at a small angle α to the vertical, quite a lot of work is needed to calculate how far the centre of mass has fallen, even working to lowest order in α.

Problem 57: Lorry on bend (✓)



A lorry of weight W stands on a plane inclined at an angle α to the horizontal. Its wheels are a distance $2d$ apart, and its centre of gravity G is at a distance h from the plane, and halfway between the sides of the lorry. A horizontal force P acts on the lorry through G, as shown.

(i) If the normal reactions on the lower and higher wheels of the lorry are equal, show that the sum of the frictional forces between the wheels and the ground is zero.

(ii) If P is such that the lorry does not tip over (but the normal reactions on the lower and higher wheels of the lorry need not be equal), show that

$$W\tan(\alpha - \beta) \leqslant P \leqslant W\tan(\alpha + \beta)\ ,$$

where $\tan\beta = d/h$.

2002 Paper I

Comments

There is not much more to this than just resolving forces and taking moments about a suitable point.

You might think it a bit odd to have a force that acts horizontally through the centre of gravity of the lorry: it is supposed to be centrifugal. The first draft of the question was intended as a model of a lorry going round a bend on a cambered road. The idea was to relate the speed of the lorry to the angle of camber: the speed should be chosen so that there is no tendency to skid. However, the modelling part of the question was eventually abandoned.

You can do the second part algebraically (using equations derived from resolving forces and taking moments), but there is a more direct approach.

There was a slight inaccuracy in the wording of the question, which I have not corrected here. Maybe you will spot it in the course of your solution. The diagram is correct.

Solution to problem 57

Let the normal reactions at the lower and upper wheels be N_1 and N_2, respectively, and let the frictional forces at the lower and upper wheels be F_1 and F_2, respectively, (both up the plane).

(i) Taking moments about G, we have

$$(N_1 - N_2)d = (F_1 + F_2)h \tag{$*$}$$

so if $N_1 = N_2$ the sum of the frictional forces is zero.

(ii) The condition for the lorry not tip over *down* the plane is $N_2 \geqslant 0$, which is the same as saying that the total moment of W and P about the point of contact between the lower wheel and the plane in the clockwise sense is positive. This gives $P \leqslant W \tan(\alpha + \beta)$ after a bit of geometry. Actually, working out the shortest distance between the line of action of the forces and the wheels requires a bit of thought. You find that $\alpha + \beta$ is the angle between the vertical and the lines joining G to the wheels. This is obvious, at least in retrospect: if the plane were not tilted, the angle would be β and tilting just increases this by the angle of tilt.

The other inequality comes from a similar calculation using the other wheel.

Post-mortem

I expect that you found that really clear and BIG diagrams made the question much easier: that is nearly always the case with mechanics.

The slight inaccuracy in the question occurs because if the line of action of P passes between the wheels of the lorry (not, as shown in the diagram, higher than the higher wheel), then the lorry cannot tip over up the plane however large P is. The given inequality is not correct in this case, which corresponds to $\alpha + \beta > \frac{1}{2}\pi$ and therefore $\tan(\alpha + \beta) < 0$.

An alternative way of obtaining the inequalities of part (ii) is to work with the resultant of P and W. The direction of this resultant force must be such that its line of action passes between the wheels. The line of action makes an angle of $\tan^{-1}(P/W)$ with the vertical through G and the inequalities follow after the same geometry as in the answer to part (ii) above.

Problem 58: Fielding (✓✓)

In a game of cricket, a fielder is perfectly placed to catch a ball. She watches the ball in flight and takes the catch just in front of her eye. The angle between the horizontal and her line of sight at a time t after the ball is struck is θ. Show that $\dfrac{\mathrm{d}}{\mathrm{dt}}(\tan\theta)$ is constant during the flight.

The next ball is also struck in the direction of the fielder but at a different velocity. In order to be perfectly placed to catch the ball, the fielder runs at constant speed towards the batsman. Assuming that the ground is horizontal, show that again $\dfrac{\mathrm{d}}{\mathrm{dt}}(\tan\theta)$ is constant during the flight.

1998 Paper II

Comments

As with all the very best questions, nine-tenths of this question is submerged below the surface. It uses the deepest properties of Newtonian dynamics, and a good understanding of the subject makes the question completely transparent. However, it can still be done without too much trouble by a straightforward approach, in which case the difficulty lies only in setting it up for yourself.

The cleverness of this question lies in its use of two fundamental invariances of Newton's second law:

$$m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = m\mathbf{g}. \tag{†}$$

The first is invariance under *time reflection symmetry*, which arises because equation (†) is not affected by the transformation $t \to -t$. This means that any given solution can be replaced by one where the projectile goes back along the trajectory, i.e. time runs backwards.

The second is invariance under what are called *Galilean transformations*. Equation (†) is also invariant under the transformation $\mathbf{x} \to \mathbf{x} + \mathbf{v}t$, where $\mathbf{v}$ is an arbitrary constant velocity. This means that we can solve the equation in a frame that moves with constant speed.

Solution to problem 58

We take a straightforward approach. Let the height above the fielder's eye level at which the ball is struck be h. Let the speed at which the ball is struck be u and the angle which the trajectory of the ball initially makes with the horizontal be α.

Then, taking x to be the horizontal distance of the ball at time t from the point at which the ball was struck and y to be the height of the ball at time t above the fielder's eye level[31], we have

$$x = (u\cos\alpha)t\ , \quad y = h + (u\sin\alpha)t - \tfrac{1}{2}gt^2\ .$$

Let d be the horizontal distance of the fielder from the point at which the ball is struck, and let T be the time of flight of the ball. Then

$$d = (u\cos\alpha)T\ , \quad 0 = h + (u\sin\alpha)T - \tfrac{1}{2}gT^2 \tag{$*$}$$

and

$$\begin{aligned}
\tan\theta = \frac{y}{d-x} &= \frac{h + (u\sin\alpha)t - \frac{1}{2}gt^2}{d - (u\cos\alpha)t} \\
&= \frac{-(u\sin\alpha)T + \frac{1}{2}gT^2 + (u\sin\alpha)t - \frac{1}{2}gt^2}{(u\cos\alpha)T - (u\cos\alpha)t} && \text{(using $*$)} \\
&= \frac{-(u\sin\alpha)(T-t) + \frac{1}{2}g(T^2-t^2)}{(u\cos\alpha)(T-t)} \\
&= \frac{-u\sin\alpha + \frac{1}{2}g(T+t)}{u\cos\alpha}. && \text{(cancelling the factor $(T-t)$)}
\end{aligned}$$

This last expression is a polynomial of degree one in t, so its derivative is constant, as required.

For the second part, let l be the distance from the fielder's original position to the point at which she catches the ball. Then $l = vT$ and

$$\tan\theta = \frac{y}{(l-vt)+d-x} = \frac{y}{v(T-t)+d-x} = \frac{-u\sin\alpha + \frac{1}{2}g(T+t)}{v + u\cos\alpha}$$

cancelling the factor of $(T-t)$ as before. This again has constant derivative.

Post-mortem

The invariance mentioned in the comments section above can be used to answer the question almost without calculation.

Using time reflection symmetry to reverse the trajectory shows that the batsman is completely irrelevant: it only matters that the fielder caught a ball. We just think of the ball being projected from the fielder's hands (the time-reverse of a catch). Taking her hands as the origin of coordinates, and using u to denote the projection (i.e. the catching) speed of the ball and α to be the angle of projection (i.e. the final value of θ), we have $y = (u\sin\alpha)t - \frac{1}{2}gt^2$, $x = (u\cos\alpha)t$ and $\tan\theta = (u\sin\alpha - \frac{1}{2}gt)/u\cos\alpha$. The first derivative of this expression is constant, as before.

We use the Galilean transformation for the second part of the question. Instead of thinking of the fielder running with constant speed v towards the batsman, we can think of the fielder being stationary and the ball having an additional horizontal speed of v. The situation is therefore not changed from that of the first part of the question, except that $u\cos\theta$ should be replaced by $v + u\cos\theta$.

---

[31] Draw a diagram! I would, but there isn't enough room on the page.

Problem 59: Equilibrium of rod of non-uniform density (✓✓)

A rigid straight rod AB has length l and weight W. Its weight per unit length at a distance x from B is

$$\alpha W l^{-1}\left(\frac{x}{l}\right)^{\alpha-1},$$

where α is a constant greater than 1. Show that the centre of mass of the rod is at a distance $\dfrac{\alpha l}{\alpha+1}$ from B.

The rod is placed with the end A on a rough horizontal floor and the end B resting against a rough vertical wall. The rod is in a vertical plane at right angles to the plane of the wall and makes an angle of θ with the floor. The coefficient of friction between the floor and the rod is μ and the coefficient of friction between the wall and the rod is also μ. Show that, if the equilibrium is limiting at both A and B, then

$$\tan\theta = \frac{1-\alpha\mu^2}{(1+\alpha)\mu}\,.$$

Given that $\alpha = \frac{3}{2}$ and given also that the rod slides for any $\theta < \frac{1}{4}\pi$ find the greatest possible value of μ.

2002 Paper II

Comments

This is a pretty standard situation: a rod leaning against a wall, prevented from slipping by friction at both ends. The only slight variation is that the rod is not uniform; the only effect of this is to alter the position of the centre of gravity through which the weight of the rod acts.

The question is not general, since the coefficient of friction is the same at both ends of the rod. I think that it is a good idea (at least, if you are not working under examination conditions) to set out the general equations (with coefficients of friction μ_A and μ_B and frictional forces F_A and F_B) without at first assuming limiting friction. This will help understand the structure of the equations and highlight the symmetry between the ends of the rod.

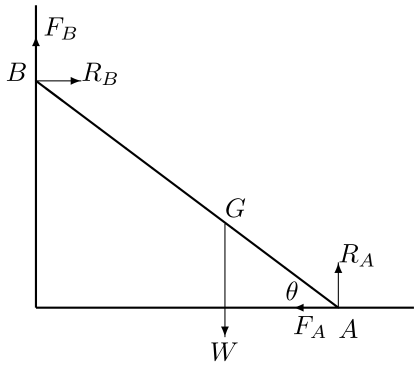
You might wonder why the weight per unit length is given in such a complicated way; why not simply kx^α? The reason is to keep the dimensions honest. The factor of W/l appears so that the dimension is clearly weight divided by length. Then x appears only divided by l to make a dimensionless ratio. Finally, the factor of α is required for the total weight of the rod to be W.

Solution to problem 59

The distance, $\bar{x}$, of the centre of mass from the end B of the rod is given by

$$\bar{x} = \frac{\int_0^l \alpha W l^{-\alpha} x^\alpha \, dx}{\int_0^l \alpha W l^{-\alpha} x^{\alpha-1} \, dx} = \frac{\int_0^l x^\alpha \, dx}{\int_0^l x^{\alpha-1} \, dx} = \frac{(\alpha+1)^{-1} l^{\alpha+1}}{(\alpha)^{-1} l^\alpha} = \frac{\alpha l}{\alpha+1}.$$

The response to the rest of this question should be preceded by an annotated diagram showing all relevant forces on the rod and with G nearer to A than B. This will help to clarify ideas.



The equations that determine the equilibrium are

$$F_B + R_A = W \qquad \text{(vertical forces)}$$
$$F_A - R_B = 0 \qquad \text{(horizontal forces)}$$
$$R_B \bar{x} \sin\theta + F_B \bar{x} \cos\theta = R_A(l - \bar{x})\cos\theta - F_A(l - \bar{x})\sin\theta \qquad \text{(clockwise moments about } G\text{)}$$

If the rod is about to slip then the frictional and normal forces at A can be specified as $F_A = \mu R_A$ and $F_B = \mu R_B$. Substituting in the first two of the above equations gives

$$R_A = \frac{W}{\mu^2+1}, \quad R_B = \frac{\mu W}{\mu^2+1}, \quad F_A = \frac{\mu W}{\mu^2+1}, \quad F_B = \frac{\mu^2 W}{\mu^2+1}.$$

Substituting into the third equation gives the required result.

For the final part, setting $\alpha = \frac{3}{2}$ gives

$$\tan\theta = \frac{2 - 3\mu^2}{5\mu}$$

If the rod slips for any angle less than $\theta = \frac{1}{4}\pi$, then the angle at which limiting friction occurs at both ends must be at least $\frac{1}{4}\pi$. Therefore

$$\frac{2-3\mu^2}{5\mu} \geqslant 1$$

i.e. $3\mu^2 + 5\mu - 2 \leqslant 0$, or $(3\mu - 1)(\mu + 2) \leqslant 0$. The greatest possible value of μ is $\frac{1}{3}$.

Post-mortem

In the original version, the question spoke about limiting friction without saying that it was limiting at both ends. We then wondered whether the friction could be limiting at one end only. The answer is yes. The three equations that govern the system do not determine uniquely the four unknown forces (normal reaction and frictional forces at each end of the rod); extra information is required. If the angle is also to be determined, two extra pieces of information are required; in this case the information is that the friction is limiting at both ends.

Problem 60: Newton's cradle (✓✓)

N particles P_1, P_2, P_3, ..., P_N with masses m, qm, q^2m, ... , $q^{N-1}m$, respectively, are at rest at distinct points along a straight line in gravity-free space. The particle P_1 is set in motion towards P_2 with velocity V and in every subsequent impact the coefficient of restitution is e, where $0 < e < 1$.

(i) Show that after the first impact the velocities of P_1 and P_2 are

$$\left(\frac{1-eq}{1+q}\right)V \quad \text{and} \quad \left(\frac{1+e}{1+q}\right)V,$$

respectively.

(ii) Show that, if $q \leqslant e$, then there are exactly $N - 1$ impacts.

(iii) Show further that, if $q = e$, then the total loss of kinetic energy after all impacts have occurred is

$$\tfrac{1}{2}me\big(1 - e^{N-1}\big)V^2.$$

1999 Paper II

Comments

This situation models the toy called 'Newton's Cradle' which consists of four or more heavy metal balls suspended from a frame so that they can swing. At rest, they are in contact in a line. When the first ball is raised and let swing, there follows a rather pleasing pattern of impacts. In this case, the coefficient of restitution is nearly 1 and the balls all have the same mass, so, as the first displayed formula shows, the impacting ball is reduced to rest by the impact. At the first swing of the ball, nothing happens except that the first ball is reduced to rest and the last ball swings away. Note that this is consistent with the balls being separated by a very small amount; what actually happens is that the ball undergoes a small elastic deformation at the impact, and the impulse takes a small amount of time to be transmitted across the ball to the next ball.

Solution to problem 60

(i) Let V_1 and V_2 be the velocities of P_1 and P_2 after the first collision. Using conservation of momentum and Newton's law of impact at the first collision results in the equations

$$mV = mV_1 + mqV_2\,, \quad V_1 - V_2 = -eV\,,$$

so

$$V_1 = \left(\frac{1-eq}{1+q}\right)V \quad \text{and} \quad V_2 = \left(\frac{1+e}{1+q}\right)V\,.$$

(ii) Note that V_1 is positive since $eq \leqslant e^2 < 1$. Conserving momentum and using Newton's law (and doing a bit of algebra) shows that the speeds $\widehat{V}_2$ and V_3 of P_2 and P_3 after the next collision are given by

$$\widehat{V}_2 = \left(\frac{1-eq}{1+q}\right)V_2 = \left(\frac{1-eq}{1+q}\right)\left(\frac{1+e}{1+q}\right)V \quad \text{and} \quad V_3 = \left(\frac{1+e}{1+q}\right)V_2 = \left(\frac{1+e}{1+q}\right)^2 V. \qquad (*)$$

Note that $\widehat{V}_2 \geqslant V_1$ so there is no further collision between P_1 and P_2. Applying this argument at each collision shows that there are exactly $N-1$ collisions: P_1 with P_2; P_2 with P_3; etc.

(iii) The speed of P_k after it has hit P_{k+1} is (by extending $(*)$)

$$\left(\frac{1-eq}{1+q}\right)\left(\frac{1+e}{1+q}\right)^{k-1} V,$$

and the speed of P_{k+1} after this collision and before it hits P_{k+2} is

$$\left(\frac{1+e}{1+q}\right)^{k} V.$$

If $q = e$,

$$V_1 = \left(\frac{1-e^2}{1+e}\right) V = (1-e)V$$

and similarly the final speeds of $P_2, \ldots, P_{N-1}$ are all $(1-e)V$. The final speed of P_N is V. Thus the final total kinetic energy is

$$\begin{aligned} \tfrac{1}{2}(m + mq + \cdots + mq^{N-2})[(1-e)V]^2 + \tfrac{1}{2}mq^{N-1}V^2 &= \tfrac{1}{2}m\,\frac{1-e^{N-1}}{1-e}\,(1-e)^2V^2 + \tfrac{1}{2}me^{N-1}V^2 \\ &= \tfrac{1}{2}m(1-e+e^N)V^2, \end{aligned}$$

(replacing all the q's with e's and summing the geometric progression). Thus the loss of kinetic energy is

$$\tfrac{1}{2}mV^2 - \tfrac{1}{2}m(1-e+e^N)V^2,$$

as required.

Post-mortem

There are two tricky aspects to these multiple collision questions. First there is the matter of notation. Above, I have used a hat for the second collision of a particle ($\widehat{V}_2$), retaining the subscript for labelling particles. That works, but if P_2 undergoes another collision, entailing a double hat, it starts getting messy. You could use a different letter, but that gets confusing. The only good method is to use a double subscript: $V_{m,n}$ is the velocity of the mth particle after the nth collision. But that is a sledgehammer for this nut of a question.

The other tricky aspect is getting the signs right. I always think of velocity (not speed), and it is always positive for particles travelling to the right. I also use common sense to check each equation! As usual, a good diagram of each collision is essential.

Problem 61: Kinematics of rotating target (✓✓)

An automated mobile dummy target for gunnery practice is moving anti-clockwise around the circumference of a large circle of radius R in a horizontal plane at a constant angular speed ω. A shell is fired from O, the centre of this circle, with initial speed V and angle of elevation α.

(i) Show that if $V^2 < gR$, then no matter what the value of α, or what vertical plane the shell is fired in, the shell cannot hit the target.

(ii) Assume now that $V^2 > gR$ and that the shell hits the target, and let β be the (positive) angle through which the target rotates between the time at which the shell is fired and the time of impact. Show that β satisfies the equation

$$g^2\beta^4 - 4\omega^2V^2\beta^2 + 4R^2\omega^4 = 0.$$

Deduce that there are exactly two possible values of β.

(iii) Let β_1 and β_2 be the possible values of β and let P_1 and P_2 be the corresponding points of impact. By considering the quantities $(\beta_1^2 + \beta_2^2)$ and $\beta_1^2\beta_2^2$, or otherwise, show that the linear distance between P_1 and P_2 is

$$2R\sin\left(\frac{\omega}{g}\sqrt{V^2 - Rg}\right).$$

1999 Paper II

Comments

The rotation of the target is irrelevant for the first part, which contravenes the setters' rule of not introducing information before it is required. In this case, it seemed better to describe the set-up immediately — especially as you are asked for a familiar result.

Remember, when you are considering the roots of the quartic (which is really a quadratic in β^2), that the question gives $\beta > 0$.

The hint in the last paragraph ('by considering ...') is supposed to direct you towards the relation between the coefficients in a quadratic equation and the sum and product of the roots; otherwise, you get into some pretty heavy algebra.

Solution to problem 61

(i) This part is about the range of the gun. If the shell lands at distance x, then

$$x = (V\cos\alpha)t, \qquad 0 = (V\sin\alpha)t - \tfrac{1}{2}gt^2.$$

Eliminating t gives $xg = V^2\sin 2\alpha$. The range r is the largest value of x, which is given by $rg = V^2$. The shell cannot reach the target (for any angle of elevation) if $R > r$, so the shell cannot hit its target if $V^2 < Rg$.

(ii) The time of flight is $R/(V\cos\alpha)$, and this must also equal the time for the target to rotate through β, i.e. β/ω. Thus $\cos\alpha = R\omega/(\beta V)$. Substituting this into the range equation $Rg = 2V^2\sin\alpha\cos\alpha$ gives

$$Rg = 2V^2\frac{R\omega}{\beta V}\left(1 - \frac{R^2\omega^2}{\beta^2V^2}\right)^{\frac{1}{2}}.$$

Squaring both sides of the equation, and simplifying, leads to the given quadratic in β^2. Solving the quadratic using the quadratic formula gives

$$g^2\beta^2 = 2\omega^2\left(V^2 \pm \sqrt{V^4 - g^2R^2}\right).$$

Since $V^2 > gR$, both the values of β^2 roots are real and positive, but they lead to only two relevant values of β since we only want positive values.

(iii) The linear distance between the two points of impact is $2R\sin\frac{1}{2}(\beta_2 - \beta_1)$. Now $\beta_1^2 + \beta_2^2 = 4\omega^2V^2/g^2$ and $\beta_1^2\beta_2^2 = 4R^2\omega^4/g^2$ (sum and product of the roots of the quadratic), so

$$(\beta_2 - \beta_1)^2 = \beta_2^2 + \beta_1^2 - 2\beta_1\beta_2 = \frac{4\omega^2V^2}{g^2} - \frac{4R\omega^2}{g},$$

which leads to the given result.

Post-mortem

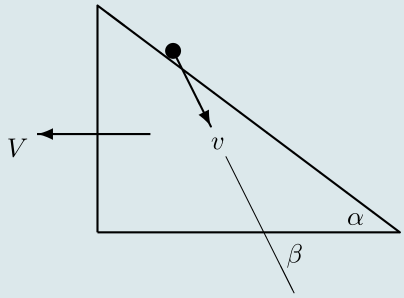
Why do we get two values for β? In order to hit the large circle, if it is in range, we can fix any speed of projection and choose the angle of projection appropriately. In general, there are two possible angles of projection, one above $45°$ and one below $45°$ (satisfying $\sin 2\alpha = Rg/V^2$).

However, in this question, we want to hit the circle essentially at a given time (when the target is at the landing place), which means that $V\cos\alpha$ is fixed, though of course it is a little more complicated because the position of the target is fixed but unknown. But still, the two values of β, corresponding to two different angles of elevation, emerge in the same way.

Did you notice the slight inaccuracy in the question? The final 'distance' could be negative (if for example $\pi g < \omega\sqrt{V^2 - Rg} < 2\pi g$). Mod signs are needed.

Problem 62: Particle on wedge (✓✓✓)

A wedge of mass M rests on a smooth horizontal surface. The face of the wedge is a smooth plane inclined at an angle α to the horizontal. A particle of mass m slides down the face of the wedge, starting from rest. At a later time t, the speed V of the wedge, the speed v of the particle and the angle β of the velocity of the particle below the horizontal are as shown in the diagram.



Let y be the vertical distance descended by the particle. Derive the following results, stating in (ii) and (iii) the mechanical principles you use:

(i) $V \sin\alpha = v \sin(\beta - \alpha)$;

(ii) $\tan\beta = (1+k)\tan\alpha$, where $k = m/M$;

(iii) $2gy = v^2(1 + k\cos^2\beta)$.

Hence show that $2g'y = \left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2$, where $g' = \dfrac{g\tan^2\beta}{\tan^2\beta + K} = \dfrac{gK\tan^2\alpha}{K\tan^2\alpha + 1}$ and $K = 1 + k$. Deduce that $y = \frac{1}{2}g't^2$.

1998 Paper II

Comments

I was surprised at the difficulty of this problem, compared with a particle on a fixed wedge. There is an easier method (just using forces and Newton's second law) but the route suggested here uses basic principles, the outcomes of which are the numbered parts of the question. For a fixed wedge, the horizontal component of momentum is not conserved, because of the force required to hold the wedge; but (i) and (iii) are just what you would use in the fixed case.

Part (ii) is used only (apart from the very last result) to show that the angle β is constant. At first sight, this is a bit surprising: it means that the particle moves in a straight line.

Most of the important intermediate results are given to you, but it is still very good discipline to check that they hold in special cases: for example, $k = 0$ corresponding to a massless particle or equivalently a fixed wedge; and $\alpha = 0$ or $\alpha = \frac{1}{2}\pi$ corresponding to a horizontal or vertical wedge face. You should check that you understand what should happen in these special cases and that your understanding is consistent with the formulae.

Solution to problem 62

(i) This equation follows immediately from the constraint placed on the particle: it remains in contact with the plane so components of the velocities of the particle and the wedge perpendicular to the face must be equal.

(ii) Horizontal momentum is conserved, so $MV = mv\cos\beta$. Substitution for V/v using (i) gives

$$k\cos\beta = \frac{V}{v} = \frac{\sin(\beta-\alpha)}{\sin\alpha} = \frac{\sin\beta\cos\alpha - \sin\alpha\cos\beta}{\sin\alpha} = \frac{\sin\beta}{\tan\alpha} - \cos\beta$$

which leads immediately to the required result. Note that the angle β remains constant in the motion.

(iii) Conservation of energy gives

$$mgy = \tfrac{1}{2}MV^2 + \tfrac{1}{2}mv^2, \quad \text{i.e.} \quad kgy = \tfrac{1}{2}(kv\cos\beta)^2 + \tfrac{1}{2}kv^2.$$

Thus $2gy = (k\cos^2\beta + 1)v^2$.

The next step is to write v in terms of y to obtain the differential equation. The vertical component of velocity is $v\sin\beta$, so

$$v\sin\beta = \frac{\mathrm{d}y}{\mathrm{d}t}\,.$$

Now using the result of (iii) gives

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v\sin\beta = \sqrt{\frac{2gy}{1+k\cos^2\beta}}\,\sin\beta = \sqrt{\frac{2gy\sin^2\beta}{1+(K-1)\cos^2\beta}} = (2g')^{\frac{1}{2}}y^{\frac{1}{2}}\,. \qquad (*)$$

Squaring gives the required result, and we can use part (ii) to obtain the expression for g' in terms of α and K.

Now, at last, we use the hard-won result that β, and hence g', is constant. The equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 2g'y$$

is exactly the same as the equation of conservation of energy for a particle falling vertically in a gravitational field of strength g'. For such a particle, we have $y = \frac{1}{2}g't^2$, as required.

Post-mortem

The basic principles behind this solution are conservation of momentum and energy, together with the constraint that the particle moves on the surface of the face of the wedge. The benefit of using conservation of energy is that you do not have to worry about the normal reaction force, because it does no work.

Of course, we could obtain the final result by integrating the differential equation $(*)$, which takes only a few lines.

Problem 63: Sphere on step (✓)

A uniform solid sphere of radius a and mass m is drawn very slowly and without slipping from horizontal ground onto a step of height $\frac{1}{2}a$ by a horizontal force of magnitude F which is always applied to the highest point of the sphere and is always perpendicular to the vertical plane which forms the face of the step. Find the maximum value of F in the motion, and prove that the coefficient of friction between the sphere and the edge of the step must exceed $1/\sqrt{3}$.

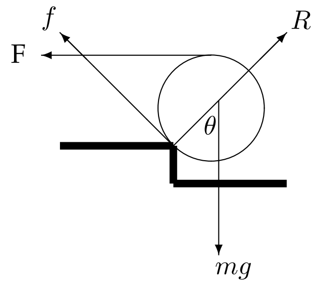
1997 Paper II

Comments

This is quite straightforward once you have realised that *very slowly* means so slowly that the sphere can be considered to be static at each position. You just have to solve a statics problem with the usual tools: resolving forces and taking moments about suitably chosen points.

It seems odd to have a force that always acts at the highest point of the sphere. The idea is that there is a string wrapped round the sphere and it is being pulled horizontally to get the sphere up the step.

Solution to problem 63



Let the angle between the radius to the point of contact with the step and the downward vertical be θ, as shown. At the point of contact, let the frictional force, which is tangent to the sphere, be f and the reaction R (along the radius of the sphere).

To investigate F, we take moments about a point cunningly chosen to eliminate other unknown forces. Since the lines of action both R and f pass through the point of contact of the sphere with the step, we can eliminate both forces by taking moments about this point:

$$Fa(1+\cos\theta) = mga\sin\theta\,,$$

so

$$F = mg\frac{\sin\theta}{1+\cos\theta} = mg\tan\tfrac{1}{2}\theta\,.$$

This takes its maximum value when θ is largest, i.e. when the sphere just touches the horizontal ground. At this position, $\cos\theta = \frac{1}{2}$ and $\sin\theta = \frac{1}{2}\sqrt{3}$ and $F_{\rm max} = mg/\sqrt{3}$.

Taking moments about the centre of the sphere (to eliminate the normal reaction and weight) gives $F = f$, and resolving forces parallel to the radius at the point of contact gives

$$R = F\sin\theta + mg\cos\theta.$$

Now

$$F\sin\theta = mg\frac{\sin^2\theta}{1+\cos\theta} = mg\frac{1-\cos^2\theta}{1+\cos\theta} = mg(1-\cos\theta)$$

so $R = mg$. We therefore need $\mu mg > F_{\rm max}$, as required.

Post-mortem

Should we have known that $R = mg$ (in the last line of the solution)? It doesn't look obvious from the diagram. However, if we follow the principle, used twice already, of taking moments about a point that eliminates unwanted forces, the result drops out. To relate mg and R directly, we need to eliminate F and f. We therefore need to take moments about the intersection of the lines of action of these two forces, which is shown in the diagram (call it P). Moments must balance about *any* point: it doesn't matter whether the point is inside the body, or outside it as in the case of P. It is easy to see from the geometry (f and F act along tangents to the sphere) that the distances from P to the lines of action of mg and R are equal, which means that the two forces are equal.

To be precise about the meaning of *slow* in this context, you have to compare the dynamic forces connected with the motion of the sphere with the static forces. The motion of the sphere is rotation about the fixed point of contact with the step. If we assume that the centre moves with constant speed, the only extra force due to the motion is an additional reaction at the point of contact with the step. This reaction is centrifugal in nature and so is roughly of the form mv^2/a, which is to be small compared with R. The static approximation ('very slowly') is therefore valid if $v^2 \ll ga$.

Problem 64: Elastic band on cylinder (✓)

A smooth cylinder with circular cross-section of radius a is held with its axis horizontal. A light elastic band of unstretched length $2\pi a$ and modulus of elasticity λ is wrapped round the circumference of the cylinder, so that it forms a circle in a plane perpendicular to the axis of the cylinder. A particle of mass m is then attached to the rubber band at its lowest point and released from rest.

(i) Given that the particle falls to a distance $2a$ below the axis of the cylinder, but no further, show that

$$\lambda = \frac{9\pi mg}{(3\sqrt{3}-\pi)^2}\ .$$

(ii) Given instead that the particle reaches its maximum speed at a distance $2a$ below the axis of the cylinder, find a similar expression for λ.

2001 Paper I

Comments

This question uses the most basic ideas in mechanics, such as conservation of energy. The only two things you need to know about stretched strings are, for a stretched string of natural (i.e. unstretched) length l and extended length $l+x$ with modulus of elasticity λ:

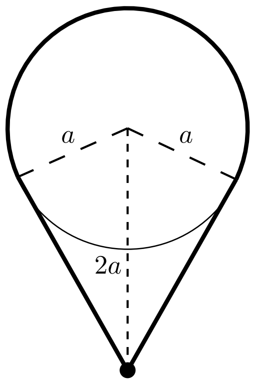
(i) the potential energy stored in the stretched string is $\dfrac{\lambda x^2}{2l}$;

(ii) the tension in the stretched string, by Hooke's law, is $\dfrac{\lambda x}{l}$.

Stretched springs and strings are now included in the STEP 2 specification; between 2003 and 2019 they were in STEP 3 only,

Solution to problem 64

The diagram below shows the system when the particle has fallen a distance a from its initial position.



A bit of geometry (including Pythagoras) on the above diagram shows that the length of the extended band is $2\pi a - 2a\cos^{-1}\frac{1}{2} + 2\sqrt{3}\,a$ so the extension of the band is

$$-2a\cos^{-1}\left(\tfrac{1}{2}\right) + 2\sqrt{3}\,a\,, \quad \text{i.e.} \quad 2a(\sqrt{3} - \tfrac{1}{3}\pi)\,.$$

(i) At the lowest point, the speed is zero. That suggests using an energy equation, which will involve speed and displacement (for the potential energy) but not acceleration.

Conserving energy (taking the initial potential energy to be zero), we find that when the particle has fallen a distance a and has speed v,

$$0 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\,\lambda\frac{[2a(\sqrt{3} - \frac{1}{3}\pi)]^2}{2\pi a} - mga\,.$$

At the lowest point, $v = 0$, which gives the required answer.

(ii) Now we are interested in the point where the speed is greatest, i.e. where the acceleration is zero, so this time we should use the equation of motion of the particle.

The component of the tension T in the band in the vertical direction, acting upwards on the particle, when the particle has fallen a distance y is $2T\cos\theta$ where $\sin\theta = a/(a+y)$. Applying Newton's second law to the motion of the particle gives

$$m\ddot{y} = 2T\cos\theta - mg\,, \tag{$*$}$$

where, by Hooke's law, $T = \lambda \times$ extension$/2\pi a$. At the maximum speed, $\ddot{y} = 0$. This occurs at $y = a$, so $(*)$ becomes

$$0 = 2\lambda\frac{2a(\sqrt{3} - \frac{1}{3}\pi)}{2\pi a}\,\frac{\sqrt{3}}{2} - mg\,.$$

In this case $\lambda = \dfrac{\sqrt{3}\,\pi mg}{3\sqrt{3} - \pi}$.

Post-mortem

The question was relatively simple because you only needed to evaluate energy or tension at very special points: maximum extension or maximum speed. You could of course have worked from the general equation of motion $(*)$, with $\cos\theta = \sqrt{2ay + y^2}/(a+y)$ and extension $2\sqrt{2ay + y^2} - 2a\sin^{-1}\big(a/(a+y)\big)$, but you can't solve the differential equation to find the general motion. It would be possible for small oscillations.

Problem 65: A knock-out tournament (✓)

A tennis tournament is arranged for 2^n players. It is organised as a knockout tournament, so that only the winners in any given round proceed to the next round. Opponents in each round except the final are drawn at random, and in any match either player has a probability $\frac{1}{2}$ of winning. Two players are chosen at random before the start of the first round. Find the probabilities that they play each other:

(i) in the first round;

(ii) in the final round;

(iii) in the tournament.

1987 Specimen Paper II

Comments

Note that the set-up is not the usual one for a tennis tournament, where the only random element is in the first round line-up. Two players cannot then meet in the final if they are in the same half of the draw.

Part (i) is straightforward, but parts (ii) and (iii) need a bit of thought. There is a short way and a long way of tackling these parts, and both have merits.

It is a good plan to check your answers, if possible, by reference to simple special cases where you can see what the answers should be; $n = 1$ or $n = 2$, for example.

Interestingly, the answers are independent of the probability that the players have of winning a match; the 2s in the answers represent the number of players in each match rather than (the reciprocal of) the probability that each player has of winning a match. It also does not matter how the draw for each round is made. This is clear if you use the short method mentioned above.

Solution to problem 65

Call the two players P_1 and P_2.

(i) Once P_1 has been given a slot, there are $2^n - 1$ slots for P_2, in only one of which will he or she play P_1. The probability of P_1 playing P_2 is therefore

$$\frac{1}{2^n - 1}.$$

Note that this works for $n = 1$ and $n = 2$.

(ii) **Long way**. To meet in the final, P_1 and P_2 must each win every round before the final, and must also not meet before the final. The probability that P_1 and P_2 do not meet in the first round and that they both win their first round matches, is

$$\left(1 - \frac{1}{2^n - 1}\right)\frac{1}{2^2} \qquad \text{i.e.} \qquad \frac{1}{2}\left(\frac{2^{n-1} - 1}{2^n - 1}\right).$$

The probability that they win each round and do not meet before the final (i.e. for $n - 1$ rounds) is

$$\frac{1}{2}\left(\frac{2^{n-1} - 1}{2^n - 1}\right) \times \frac{1}{2}\left(\frac{2^{n-2} - 1}{2^{n-1} - 1}\right) \times \cdots \times \frac{1}{2}\left(\frac{2^1 - 1}{2^2 - 1}\right) \qquad \text{i.e.} \qquad \frac{1}{2^{n-1}}\frac{1}{2^n - 1}.$$

(ii) **Short way**. Since all processes are random here, the probability that any one pair contests the final is the same as that for any other pair. There are a total of $\frac{1}{2} \times 2^n(2^n - 1)$ different pairs, so the probability for any given pair is $1/[2^{n-1}(2^n - 1)]$.

(iii) **Long way**. We need to add the probabilities that P_1 and P_2 meet in each round. The probability that they meet in the kth round is the probability that they reach the kth round times the probability that they meet in the kth round given that they reach it, the latter (conditional) probability being $1/(2^{n-k+1} - 1)$, as can be inferred from part (i). As in part (ii), the probability that they reach the kth round is

$$\frac{1}{2^{k-1}}\frac{2^{n-k+1} - 1}{2^n - 1},$$

so the probability that they meet in the kth round is

$$\frac{1}{2^{k-1}}\frac{2^{n-k+1} - 1}{2^n - 1} \times \frac{1}{2^{n-k+1} - 1} = \frac{1}{2^{k-1}}\frac{1}{2^n - 1}.$$

Summing this as a geometric progression from $k = 1$ to n gives $1/2^{n-1}$.

(iii) **Short way**. By the same short argument as in part (ii), the probability of a given pair meeting in any given match (not necessarily the final) is $1/[2^{n-1}(2^n - 1)]$. Since the total number of matches is $2^n - 1$ (because one match is needed to knock out each player, and all players except one get knocked out), the probability of a given pair playing is

$$\frac{2^n - 1}{2^{n-1}(2^n - 1)} = \frac{1}{2^{n-1}}.$$

Post-mortem

If (like me) you plodded through this question the long way, you might be wondering how you were supposed to think of the short way. Instead of working out what happens to individual players as they progress through the tournament, you think about the space of all possible outcomes (the *sample space*), and attach a probability to each. That way, you can use the symmetry between all the players to help you.

Problem 66: Harry the calculating horse (✓✓)

Harry the Calculating Horse will do any mathematical problem I set him, providing the answer is 1, 2, 3 or 4. When I set him a problem, he places a hoof on a large grid consisting of unit squares and his answer is the number of squares partly covered by his hoof. Harry has circular hoofs, of radius $\frac{1}{4}$ unit.

After many years of collaboration, I suspect that Harry no longer bothers to do the calculations, instead merely placing his hoof on the grid completely at random. I often ask him to divide 4 by 4, but only about $\frac{1}{4}$ of his answers are right; I often ask him to add 2 and 2, but disappointingly only about $\frac{1}{16}\pi$ of his answers are right. Is this consistent with my suspicions?

I decide to investigate further by setting Harry many problems, the answers to which are 1, 2, 3, or 4 with equal frequency. If Harry is placing his hoof at random, find the expected value of his answers. The average of Harry's answers turns out to be 2. Should I get a new horse?

1997 Paper II

Comments

Hans von Osten, a horse, lived in Berlin around the turn of the last century. He was known far and wide for his ability to solve complex arithmetical problems. Distinguished scientists travelled to Berlin to examine Hans and test his marvellous ability. They would write an equation on a chalkboard and Hans would respond by pawing the ground with his hoof. When Hans reached the answer he would stop. Though he sometimes made errors, his success rate was far higher than would be expected if his answers were random. The accepted verdict was that Hans could do arithmetic.

Hans's reputation as a calculating horse nosedived when an astute scientist simply made sure that neither the person asking the questions nor the audience knew the answers. Hans became an instant failure. His success was based on his ability to sense any change in the audience: a lifted eyebrow, a sigh, a nodding head or the tensing of muscles was enough to stop him from pawing the ground. Anyone who knew the answer was likely to give almost imperceptible clues to the horse. But we shouldn't overlook Hans's talents: at least he had terrific examination technique.

We are investigating the situation when Harry places his hoof at random, so that the probability that the centre of his hoof lands in any given region is proportional to the area of the region. We therefore move swiftly from a question about probability to a question about areas. You just have to divide a given square into regions, each determined by the number of squares that will be partially covered by Harry's hoof if its centre lands in the region under consideration. Remember that the areas must add to one, so the most difficult area calculation can be left until last and deduced from the others.

To answer the last part properly, you really need to set out a hypothesis testing argument: you will accept the null hypothesis (random hoof placing) if, using the distribution implied by the null hypothesis, the probability of obtaining the given result is greater than some pre-decided figure. Obviously, nothing so elaborate was intended here, since it is the last demand of an already long question: just one line would do.

Solution to problem 66

The diagram shows one square of the grid.

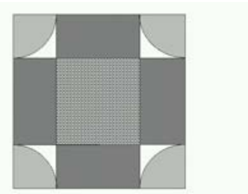
Harry's hoof will land completely within exactly one square if he places the centre of his hoof in a square of side $\frac{1}{2}$ centred on the centre of any square, shown with horizontal and vertical lines in the diagram. The area of any such square is $\frac{1}{4}$ square units.

His hoof will partially cover exactly four squares if he places the centre in a circle of radius $\frac{1}{4}$ centred on any intersection of grid lines. The total area of any one such circle is $\frac{1}{16}\pi$ square units, which may be thought of as four quarter circles, one in each corner of any given square, shown lightly shaded in the diagram.

His hoof will partially cover exactly two squares if he places the centre in any one of four $\frac{1}{2}$ by $\frac{1}{4}$ rectangles, of total area $4 \times \frac{1}{8}$ in any given square, shown darkly shaded in the diagram.

Otherwise, his hoof will partially cover three squares; the area of this remaining region, shown with no shading in the diagram, is

$$1 - \frac{1}{4} - \frac{\pi}{16} - \frac{1}{2}\,.$$



If Harry placed his hoof at random, the probabilities of the different outcomes would be equal to the corresponding area calculated above (divided by the total area of the square, which is 1). Thus the data given in the question are exactly consistent with random placement.

The expected value given random placements is

$$1 \times \frac{1}{4} + 2 \times \frac{1}{2} + 3 \times \left(1 - \frac{1}{4} - \frac{1}{2} - \frac{\pi}{16}\right) + 4 \times \frac{\pi}{16} = 2 + \frac{\pi}{16}$$

The expected value (given that Harry gets all questions right) is $(1+2+3+4)/4 = 5/2$. Harry has a less accurate expected value even than the random expected value. He is clearly hopeless and should go.

Post-mortem

The explanations in the first paragraph of the solution are very concise. To understand them, you should draw yourself lots of diagrams.

The proportion of Harry's answers that are correct, in the case of the $2 + 2$ calculation, is given as 'about $\frac{1}{16}\pi$'. That should have given you pause for thought. Why not $\frac{1}{5}$? Clearly, it was supposed to be helpful, so it should have been no surprise that the probability of getting the answer 4 under the random assumption turned out to be exactly $\frac{1}{16}\pi$. Of course, these questions shouldn't turn into a game of spotting hidden clues, but it is always worth looking for unusual features of a question — or of any mathematical problem.

Problem 67: PIN guessing (✓)

In order to get money from a cash dispenser I have to punch in a Personal Identification Number. I have forgotten my PIN, but I do know that it is equally likely to be any one of the integers 1, 2, ..., n. I plan to punch in integers in ascending order until I get the right one. I can do this at the rate of r integers per minute. As soon as I punch in the first wrong number, the police will be alerted. The probability that they will arrive within a time t minutes is $1 - e^{-\lambda t}$, where λ is a positive constant. If I follow my plan, show that the probability of the police arriving before I get my money is

$$\sum_{k=1}^{n} \frac{1 - e^{-\lambda(k-1)/r}}{n} \, .$$

Simplify the sum.

On past experience, I know that I will be so flustered that I will just punch in possible integers at random, without noticing which I have already tried. Show that the probability of the police arriving before I get my money is

$$1 - \frac{1}{n - (n-1)e^{-\lambda/r}} \, .$$

2002 Paper I

Comments

This was originally about getting money from a cash dispenser using a stolen card, but it was decided that STEP questions should not be immoral. Hence the rather more improbable scenario.

The *exponential distribution*, here governing the police arrival time, is often used for failure of equipment (light-bulbs, etc). It has the useful property (not used here) that the reliability for a light bulb (here the probability of the police not coming within a certain time period of duration t) doesn't depend on which time period you choose; i.e. given that the light bulb has survived to time T, the probability of it surviving until time $T + t$ is independent of T (which doesn't seem very suitable for light bulbs). This is referred to as the *memoryless property*.

To simplify the sums, you have to recognise a geometric series where the common ratio of terms is an exponential; not difficult, but easy to miss the first time you see it.

Solution to problem 67

The probability that I get the right number on the kth go is $1/n$. The time taken to key in k integers[32] is $(k-1)/r$. The probability that the police arrive before this is $1-e^{-\lambda(k-1)/r}$ so the total probability that the police arrive before I get my money is

$$\sum_{k=1}^{n} \frac{1}{n} \times \left(1-e^{-\lambda(k-1)/r}\right) .$$

We have

$$\sum_{k=1}^{n} \frac{1}{n} \times \left(1-e^{-\lambda(k-1)/r}\right) = 1 - \sum_{k=1}^{n} \frac{e^{-\lambda(k-1)/r}}{n} = 1 - \frac{1-e^{-\lambda n/r}}{n(1-e^{-\lambda/r})} ,$$

since the last part of the sum is a geometric progression with common ratio $e^{-\lambda/r}$.

This time, the probability of getting my money on the kth go is $1/n$ times the probability of not having punched in the correct integer in the preceding $k-1$ turns:

$$\mathrm{P}(\text{money on } k\text{th go}) = \frac{1}{n} \times \left(\frac{n-1}{n}\right)^{k-1} .$$

Thus the probability of the police arriving before I get my money is

$$\sum_{k=1}^{\infty} \frac{1}{n} \times \left(\frac{n-1}{n}\right)^{k-1} \times \left(1-e^{-\lambda(k-1)/r}\right) = \frac{1}{n} \times \frac{1}{1-(n-1)/n} - \frac{1}{n} \times \frac{1}{1-e^{-\lambda/r}(n-1)/n} ,$$

which is easily seen to be the given answer.

Post-mortem

For each part, you have to calculate the probability of the first success occurring on the kth go. You have to remember to multiply the probability of success on this go by the probability of failure on the preceding $k-1$ goes — a very typical idea in this sort of probability question. After that, the question is really just algebra. STEP questions on this sort of material nearly always involve significant algebra or calculus.

Car batteries as well as light bulbs often crop up as examples of objects to which exponential reliability can be applied. Suppose your car battery has a guarantee of three years, and your car is completely destroyed after two years. Your insurance company offers to give you one third of the price of the battery. What do you think they would say to your counterclaim for the full price, based on the fact that, given it had lasted two years, the exponential model says that it would have had an expected further three years of life in it?

[32] Think of fences and posts: k posts here but only $k-1$ fences.

Problem 68: Breaking plates (✓)

> Four students, one of whom is a mathematician, take turns at washing up over a long period of time. The number of plates broken by any student in this time obeys a Poisson distribution, the probability of any given student breaking n plates being $e^{-\lambda}\lambda^n/n!$ for some fixed constant λ, independent of the number of breakages by other students. Given that five plates are broken, find the probability that three or more were broken by the mathematician.

2001 Paper I

Comments

The way this is set up, it is largely a counting exercise (but see the post-mortem). To start with, you work out the probability of 5 breakages, then follow that with the probability that the mathematician broke 3 or more plates. You need to calculate the number of ways that 5 plates can be shared amongst four students, for which you have to consider each *partition* of the number 5 and the number of different ways it can arise.

Solution to problem 68

First we work out the probability of 5 breakages.

Let $\mathrm{P}(5,0,0,0)$ denote the probability that student A breaks 5 plates and students B, C and D break 0 plates. Then

$$\mathrm{P}(5,0,0,0) = (\text{Prob. of breaking 5}) \times (\text{Prob. of breaking 0})^3 = \frac{\mathrm{e}^{-\lambda}\lambda^5}{5!}\left(\mathrm{e}^{-\lambda}\right)^3 = \frac{\lambda^5\mathrm{e}^{-4\lambda}}{5!}.$$

The probability that one student breaks all the plates is $4 \times \lambda^5\mathrm{e}^{-4\lambda}/5!$, the factor 4 because it could be any one of the four students.

Let $\mathrm{P}(4,1,0,0)$ denote the probability that student A breaks 4 plates, student B breaks 1 plate and students C and D break 0 plates. Then

$$\begin{aligned}\mathrm{P}(4,1,0,0) &= (\text{Prob. of breaking 4}) \times (\text{Prob. of breaking 1}) \times (\text{Prob. of breaking 0})^2 \\ &= \frac{\mathrm{e}^{-\lambda}\lambda^4}{4!}\,\frac{\mathrm{e}^{-\lambda}\lambda}{1!}\left(\mathrm{e}^{-\lambda}\right)^2 = \frac{\lambda^5\mathrm{e}^{-4\lambda}}{4!}\,.\end{aligned}$$

The probability that any one student breaks 4 plates and any other student breaks 1 plate is therefore $4 \times 3 \times \lambda^5\mathrm{e}^{-4\lambda}/4!\,$.

Considering $\mathrm{P}(3,2,0,0)$, $\mathrm{P}(3,1,1,0)$, $\mathrm{P}(2,2,1,0)$ and $\mathrm{P}(2,1,1,1)$ in turn, together with the number of different ways these probabilities can occur, shows that the probability of 5 breakages is

$$\lambda^5\mathrm{e}^{-4\lambda}\left(\frac{4}{5!}+\frac{12}{4!\,1!}+\frac{12}{3!\,2!}+\frac{12}{3!\,1!\,1!}+\frac{12}{2!\,2!\,1!}+\frac{4}{2!\,1!\,1!\,1!}\right) = \lambda^5\mathrm{e}^{-4\lambda}\left(\frac{1024}{5!}\right)\,.$$

The probability that the mathematician (student A, say) breaks 3 or more is $\mathrm{P}(5,0,0,0) + 3\mathrm{P}(4,1,0,0) + 3\mathrm{P}(3,2,0,0) + 3\mathrm{P}(3,1,1,0)$, i.e.

$$\lambda^5\mathrm{e}^{-4\lambda}\left(\frac{4}{5!}+\frac{3}{4!\,1!}+\frac{3}{3!\,2!}+\frac{4}{3!\,1!\,1!}\right) = \lambda^5\mathrm{e}^{-4\lambda}\left(\frac{106}{5!}\right)\,.$$

The probability that the mathematician breaks 3 or more, given that 5 are broken, is therefore $106/1024\,$.

Post-mortem

If you did the question by the method suggested above, you will probably be wondering about the answer: why is it independent of λ; and why is the denominator 4^5? You will quickly decide that the Poisson distribution was a red herring (though I promise you that it was not an intentional herring), since there is no trace of the distribution in the answer.

The denominator is pretty suggestive. A completely different approach is as follows, It is clear that the probability that the mathematician breaks any given plate is $\frac{1}{4}$. The probability that he or she breaks k plates is therefore binomial:

$$\binom{5}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{5-k}$$

and adding the cases $k=3$, $k=4$ and $k=5$ gives rather rapidly

$$\frac{90+15+1}{1024}\,.$$

For the method given in the solution, we could substitute P_k for $\mathrm{e}^{-\lambda}\lambda^k/k!$ (as the probability of breaking k plates) and the calculation would work just the same, with all the P_ks disappearing; try it.

Problem 69: Lottery (✓✓)

The national lottery of Ruritania is based on the positive integers from 1 to N, where N is very large and fixed. Tickets cost £1 each. For each ticket purchased, the punter (i.e. the purchaser) chooses a number from 1 to N. The winning number is chosen at random, and the jackpot is shared equally amongst those punters who chose the winning number.

A syndicate decides to buy N tickets, choosing every number once to be sure of winning a share of the jackpot. The total number of tickets purchased in this draw is $3.8N$ and the jackpot is £W. Assuming that the non-syndicate punters choose their numbers independently and at random, find the most probable number of winning tickets and show that the expected net loss of the syndicate is approximately

$$N - \frac{5\big(1-e^{-2.8}\big)}{14}\,W\,.$$

2001 Paper II

Comments

This is a Binomial distribution problem: the probability that n out of m punters choose the winning ticket is

$$\binom{m}{n} p^n q^{m-n}$$

where here $m = 2.8N$, $p = 1/N$ and $q = 1 - p$. It is clear that an approximation to the Binomial distribution is expected (for example, the question uses the word 'approximately' and you have to think about how an approximation might arise); and the presence of the exponential in the given result gives a pretty broad hint that it should be a Poisson distribution — the use of which has to be justified. One can expect the Poisson approximation to work when the number of trials (call it m) is large (e.g. $m > 150$) and when $np \approx npq$, i.e. the mean is roughly equal to the variance (since these are equal for the Poisson distribution) — so $q \approx 1$.

You can't find the most probable number of winning tickets by differentiation (unless you fancy differentiating the factorial x!); instead, you must look at the ratios of consecutive terms and see when these turn from being greater than one to less than one.

Solution to problem 69

Let X be the random variable whose value is the number of winning tickets out of the $2.8N$ tickets purchased by the non-syndicate punters. Then $X \sim \text{Poisson}(2.8)$.

Let $p_j = \mathrm{P}(X = j)$. Then

$$p_j = \frac{(2.8)^j \mathrm{e}^{-2.8}}{j!} \Rightarrow \frac{p_{j+1}}{p_j} = \frac{2.8}{j+1}\,.$$

This fraction is greater than 1 if $j < 1.8$, so the most probable number of winning tickets by non-syndicate punters is 2. Overall (including the ticket bought by the syndicate), the most probable number of winning tickets is 3, which is very plausible.

The expected winnings of the syndicate is

$$\begin{aligned}\left(Wp_0 + \frac{1}{2}Wp_1 + \frac{1}{3}Wp_2 + \cdots\right) &= \mathrm{e}^{-2.8}W\left(1 + \frac{1}{2} \times \frac{1}{1!}(2.8) + \frac{1}{3} \times \frac{1}{2!}(2.8)^2 + \cdots\right)\\ &= \mathrm{e}^{-2.8}W\frac{\mathrm{e}^{2.8} - 1}{2.8}\,,\end{aligned}$$

so the expected loss is as given (note that $2.8 = \frac{14}{5}$).

Post-mortem

This is rather interesting. You might perhaps have considered whether it would be worth borrowing money to buy every single lottery combination, in order to win a share of the jackpot. Clearly, the people who run the lottery have to think about this sort of thing.

Suppose the number of tickets sold, excluding the N that we plan to buy, is expected to be kN. Suppose also that a fraction α of the total is paid out in the jackpot. Then, setting $2.8 = k$ and $W = \alpha(k+1)N$, the expected loss formula given in the question becomes

$$N\left(1 - \frac{1 - e^{-k}}{k}\alpha(k+1)\right).$$

If k is very small (take $k = 0$), we lose $N(1-\alpha)$ (obviously). If k is large, $k = 3$ say, the exponential can be ignored and we lose $N(1 - \alpha(k+1)/k)$. If $k \gg 1$, this becomes $N(1-\alpha)$ again. In between, there is a value of k that, for each fixed α, gives a minimum loss (which may be a gain if α is close to 1).

Note how informative it is to have k rather than 2.8; the numerical value was chosen in the question to model roughly that lottery system in the UK.

Having gone back to this solution after a break, I am now wondering about the use of the Poisson approximation. Of course, the set-up (large m, small p) begs us to approximate, but did we need to? Certainly not for the first result, since we can just as well look at the ratio of two terms of the Binomial distribution as at two terms of the Poisson distribution. Try it; the result is of course the same.

The second part is more difficult. What we want is the expectation of $1/(n+1)$, and this turns out to be a difficult sum using the Binomial distribution (in fact, it can only be expressed in terms of a hypergeometric function, which would then have to be approximated to get a less obscure answer).

It seems to me that the solution above is therefore a bit unsatisfactory. It would surely have been better to work with the exact distribution until it was necessary to approximate, even though one knows that the approximation is so good that the answers would be the same. We are, after all, mathematicians and not engineers.

Problem 70: Bodies in the fridge (✓✓)

My two friends, who shall remain nameless, but whom I shall refer to as P and Q, both told me this afternoon that there is a body in my fridge. I'm not sure what to make of this, because P tells the truth with a probability of p, while Q (independently) tells the truth with a probability of only q. I haven't looked in the fridge for some time, so if you had asked me this morning, I would have said that there was just as likely to be a body in the fridge as not. Clearly, in view of what my friends have told me, I must revise this estimate. Explain carefully why my new estimate of the probability of there being a body in the fridge should be

$$\frac{pq}{1-p-q+2pq}.$$

I have now been to look in the fridge and there is indeed a body in it; perhaps more than one. It seems to me that only my enemy E_1 or my other enemy E_2 or (with a bit of luck) both E_1 and E_2 could be in my fridge, and this evening I would have judged these three possibilities equally likely. But tonight I asked P and Q separately whether E_1 was in the fridge, and they each said that she was. What should be my new estimate of the probability that both E_1 and E_2 are in my fridge?

Of course, I always tell the truth.

1987 Paper II

Comments

The most difficult part of this problem is unravelling the narrative! The first paragraph says essentially 'what is the probability that there is a body in the fridge, given that P and Q both say there is?'. It can therefore be tackled by the usual methods of conditional probability: tree diagrams, for example, or Bayes' theorem. All the other words in the first paragraph are there to tell you about the *a priori* probabilities [33] of the events, without knowledge of which the question above would be meaningless.

In the second paragraph, the situation becomes more complicated, but the method used for the first paragraph will still work.

In case you want to use it, here is the statement of Bayes' theorem, in its simplest form:

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(B) \times \mathrm{P}(A|B)}{\mathrm{P}(A)}.$$

It is perhaps interesting to note that in 2002 it was decided that STEP questions should not be immoral (see Problem 67); but this seemed to be less of an issue in 1987.

[33] An *a priori* probability is a probability calculated theoretically (for example by using equally likely outcomes) or from existing information regarding a situation. It can then be revised using additional information, when it becomes an *a posteriori* probability.

Solution to problem 70

This problem can be solved using tree diagrams. A more sophisticated, but not necessarily better, method is to use Bayes' theorem.

Here, we take the events A and B to be

$$\begin{aligned} A &= P \text{ and } Q \text{ both say that there is a body in the fridge} \\ B &= \text{there is a body in the fridge} \end{aligned}$$

From the information given in the question, $P(B) = \frac{1}{2}$, so

$$\mathrm{P}(B|A) = \frac{\frac{1}{2} \times pq}{\mathrm{P}(A)}.$$

Now

$$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}(\text{there is a body}) \times \mathrm{P}(P \text{ and } Q \text{ both say there is}) \\ &\qquad + \mathrm{P}(\text{there is not a body}) \times \mathrm{P}(\text{they both say there is}) \\ &= \tfrac{1}{2} \times pq + \tfrac{1}{2} \times (1-p)(1-q) \end{aligned}$$

which gives the required answer.

For the second paragraph, let

$$\begin{aligned} X &= P \text{ and } Q \text{ both say that } E_1 \text{ is in the fridge} \\ Y &= E_1 \text{ and } E_2 \text{ are in the fridge} \end{aligned}$$

There are three possibilities (since we know that there is at least one body in the fridge): only E_1 is in the fridge; only E_2 is in the fridge; and both E_1 and E_2 are in the fridge. These are given as equally likely, so the *a priori* probabilities are each $\frac{1}{3}$.

We want $\mathrm{P}(Y|X)$, which by Bayes' theorem is

$$\frac{\mathrm{P}(Y) \times \mathrm{P}(X|Y)}{\mathrm{P}(X)} = \frac{\frac{1}{3} \times pq}{\mathrm{P}(X)}.$$

Now

$$\begin{aligned} \mathrm{P}(X) &= \mathrm{P}(\text{only } E_1 \text{ is in the fridge}) \times \mathrm{P}(P \text{ and } Q \text{ told the truth}) \\ &\qquad + \mathrm{P}(\text{both } E_1 \text{ and } E_2 \text{ are in the fridge}) \times \mathrm{P}(P \text{ and } Q \text{ both told the truth }) \\ &\qquad + \mathrm{P}(\text{only } E_2 \text{ is in the fridge}) \times \mathrm{P}(P \text{ and } Q \text{ both lied }) \\ &= \tfrac{1}{3} \times pq + \tfrac{1}{3} \times pq + \tfrac{1}{3} \times (1-p)(1-q) \end{aligned}$$

so the answer is $\dfrac{pq}{1-p-q+3pq}$.

Post-mortem

Note how much more difficult it is when the answer is not given; when the question was originally set, most candidates arrived at the given answer to the first part but were not sufficiently confident to extend their method to the second paragraph: they received $\frac{8}{20}$ for their efforts.

The last line of the question is not entirely frivolous; if I may have lied about what my friends answered when I asked them if there is a body in the fridge, the problem becomes difficult. However, my claim to be truthful is vacuous (it tells you nothing) because I may be lying. Contrast with the statement 'I am lying', which is inconsistent.

Problem 71: Choosing keys (✓)

I have k different keys on my key ring. When I come home at night I try one key after another until I find the key that fits my front door. What is the probability that I find the correct key, for the first time, on the nth attempt in each of the following three cases?

(i) At each attempt, I choose a key that I have not tried before, each choice being equally likely.

(ii) At each attempt, I choose a key from all my keys, each of the k choices being equally likely.

(iii) At the first attempt, I choose from all my keys, each of the k choices being equally likely. At each subsequent attempt, I choose from the keys that I did not try at the previous attempt, each of the $k - 1$ choices being equally likely.

2000 Paper II

Comments

This is very easy, and would really be far too easy if the answers were given.

You should set out your argument clearly and concisely, because if you come up with the wrong answer and an inadequate explanation you will not get many marks. Even if you write down the correct answer you may not get the marks if your explanation is inadequate.

You should of course run the usual checks on your answers. Do they lie in the range $0 \leqslant p \leqslant 1$? If you sum over all outcomes, do you get 1? (The latter is a good and not difficult exercise; I insist that you try it.)

You will probably be struck by the simplicity of the answer to part (i), after simplification. Is there an easy way of thinking about it (or, if you did it the easy way, is there a hard way)?

Solution to problem 71

(i)

P(finding correct key, for the first time, on nth attempt)

$= \text{P(fail first, fail second, ... , fail } (n-1)\text{th, succeed } n\text{th)}$

$$= \begin{cases} \dfrac{1}{k} & \text{for} \quad n = 1; \\ \dfrac{k-1}{k} \times \dfrac{k-2}{k-1} \times \cdots \times \dfrac{k-(n-1)}{k-(n-2)} \times \dfrac{1}{k-(n-1)} = \dfrac{1}{k} & \text{for} \quad 2 \leqslant n \leqslant k; \\ 0 & \text{for} \quad n > k. \end{cases}$$

(ii)

P(find correct key, for the first time, on nth attempt)

$$= \text{P(fail first, fail second, ... , succeed } n\text{th)} = \left(\frac{k-1}{k}\right)^{n-1} \frac{1}{k}.$$

(iii)

P(find correct key, for the first time, on nth attempt)

$= \text{P(fail first, fail second, ... , succeed } n\text{th)}$

$$= \begin{cases} \dfrac{1}{k} & \text{for} \quad n = 1; \\ \left(\dfrac{k-1}{k}\right)\left(\dfrac{k-2}{k-1}\right)^{n-2} \dfrac{1}{k-1} & \text{for} \quad n \geqslant 2. \end{cases}$$

Post-mortem

The above solution for part (i) is the 'hard way'. The easy way is to consider the keys all laid out in a row, instead of being picked sequentially. Exactly one of the keys is the correct one, which is equally likely to be any of the keys in the row, so has a probability of $1/k$ of being in any given position. Clearly, this corresponds exactly to picking them one by one.

I gave the hard solution above, because I think most people will be drawn into doing it this way by the phrasing of the question: keys are picked one by one and tried before going on to the next.

Did you check that your probabilities sum to 1? It is obvious for part (i), but you have to sum geometric progressions for parts (ii) and (iii).

Problem 72: Commuting by train (✓✓)

Tabulated values of $\Phi(\cdot)$, the cumulative distribution function of a standard normal variable, should not be used in this question.

Henry the commuter lives in Cambridge and his working day starts at his office in London at 0900. He catches the 0715 train to King's Cross with probability p, or the 0720 to Liverpool Street with probability $1-p$. Measured in minutes, journey times for the first train are $N(55, 25)$ and for the second are $N(65, 16)$. Journey times from King's Cross and Liverpool Street to his office are $N(30, 144)$ and $N(25, 9)$, respectively. Show that Henry is more likely to be late for work if he catches the first train.

Henry makes M journeys, where M is large. Writing A for $1 - \Phi(\frac{20}{13})$ and B for $1 - \Phi(2)$, find, in terms of A, B, M and p, the expected number, L, of times that Henry will be late and show that, for all possible values of p,

$$BM \leqslant L \leqslant AM.$$

Henry noted that in $\frac{3}{5}$ of the occasions when he was late, he had caught the King's Cross train. Obtain an estimate of p in terms of A and B.

Note: A random variable is said to be $N(\mu, \sigma^2)$ if it has a normal distribution with mean μ and variance σ^2.

2000 Paper II

Comments

This is impossible unless you know the following result:

If X_1 and X_2 are independent and normally distributed according to $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$, then $X_1 + X_2$ is also normally distributed and $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Even if you didn't know this result, you do now and it should not be difficult to complete the first parts of the question.

For the last part, you need to know something about conditional probability, namely that

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(B \cap A)}{\mathrm{P}(B)}$$

which makes sense intuitively and can easily be understood in terms of Venn diagrams. The denominator is essentially a normalising constant. The formula may be taken as the definition of conditional probability on the left-hand side.

Solution to problem 72

Let T_1 be the random variable representing the total journey time via Kings Cross, so that

$$T_1 \sim N(55 + 30, 25 + 144) = N(85, 169),$$

using the result mentioned on the previous page, and and let T_2 be the random variable representing the total journey time via Liverpool Street, so that

$$T_2 \sim N(65 + 25, 16 + 9) = N(90, 25).$$

Then the probabilities of being late are, respectively, $\mathrm{P}(T_1 > 105)$ and $\mathrm{P}(T_2 > 100)$, i.e. $1 - \Phi(\frac{20}{13})$ and $1 - \Phi(2)$. Note that $\Phi(\frac{20}{13}) < \Phi(2)$.

We have

$$L = [pA + (1-p)B]M = [B + (A-B)p]M.$$

L increases as p increases, since $A > B$, hence the given inequalities corresponding to $p = 0$ and $p = 1$.

We have

$$\text{P(Kings Cross given late)} = \frac{\text{P(Late and Kings Cross)}}{\text{P(Late)}}$$

An estimate for p (call it $\tilde{p}$) is therefore given by

$$\frac{3}{5} = \frac{A\tilde{p}}{A\tilde{p} + B(1-\tilde{p})},$$

so $\tilde{p} = \dfrac{3B}{2A + 3B}$.

Post-mortem

The result mentioned in the comment on the previous page is just the sort of thing you should try to prove yourself rather than take on trust. Unfortunately, such results in probability tend to be pretty hard to prove. You can also prove it fairly easily using generating functions (which are not on the STEP specifications any longer). You can also prove it from first principles. That would be difficult because of the awkward integrals involved, but not impossible.

Problem 73: Collecting voles (✓)

A group of biologists attempts to estimate the magnitude, N, of an island population of voles (*Microtus agrestis*). Accordingly, the biologists capture a random sample of 200 voles, mark them and release them. A second random sample of 200 voles is then taken of which 11 are found to be marked. Show that the probability, p_N, of this occurrence is given by

$$p_N = k\frac{\big((N-200)!\big)^2}{N!(N-389)!}\,,$$

where k is independent of N.

The biologists then estimate N by calculating the value of N for which p_N is a maximum. Find this estimate.

All unmarked voles in the second sample are marked and then the entire sample is released. Subsequently a third random sample of 200 voles is taken. Using your estimate for N, write down the probability that this sample contains exactly j marked voles, leaving your answer in terms of binomial coefficients.

Deduce that

$$\sum_{j=0}^{200}\binom{389}{j}\binom{3247}{200-j}=\binom{3636}{200}.$$

2000 Paper II

Comments

This is really just an exercise in combinations. (Recall that a permutation is a reordering of a set of objects, and a combination is a selection of a subset from a set.) You assume that you are equally likely to choose any given subset of the same size, so that the probability of a set of specific composition is the number of ways of choosing a set of that composition divided by the total number of ways of choosing any set of the same size. Of course, you are assuming that the voles are indistinguishable, except for the marks made by the biologists.

Maximising a discrete (not a continuous) function of N came up on one of the previous questions: you have to compare adjacent terms.

The numbers look rather bad, though they turn out OK. My instinct would be to do it algebraically first: replace 200 by a and 11 by b, then substitute back at the end. I am sure it will lead to a better understanding of what is going on.

Solution to problem 73

For the second sample, 200 out of N voles are already marked, so p_N is just the number of ways of choosing 11 from 200 and 189 from $N-200$ divided by the number of ways of choosing 200 from N:

$$p_N \;=\; \frac{\dbinom{200}{11}\dbinom{N-200}{189}}{\dbinom{N}{200}} \;=\; \frac{\dfrac{200!}{11!\,189!}\;\dfrac{(N-200)!}{189!\,(N-389)!}}{\dfrac{N!}{200!\,(N-200)!}}i\,,$$

so

$$k = \frac{(200!)^2}{11!\,(189!)^2}\,.$$

At the maximum value, $p_N \approx p_{N-1}$, i.e.

$$\frac{(N-200)^2}{N(N-389)} \approx 1\,,$$

which gives $N \approx 200^2/11 \approx 3636$ (just divide 40000 by 11).

At the third sample, there are 389 marked voles and an estimated $3636 - 389 = 3247$ unmarked voles. Hence

$$\mathrm{P}(\text{exactly } j \text{ marked voles}) = \frac{\begin{pmatrix}389\\ j\end{pmatrix}\begin{pmatrix}3247\\ 200-j\end{pmatrix}}{\begin{pmatrix}3636\\ 200\end{pmatrix}}\,.$$

The final part follows immediately, using

$$\sum_{j=0}^{200} \mathrm{P}(\text{exactly } j \text{ marked voles}) = 1\,.$$

Post-mortem

Adding the Latin name of the species was a nice touch, I thought (not my idea); it adds an air of verisimilitude to the problem.

I suppose that this might be the basis of a method of estimating the population of voles — rather clever really. I don't know how well it works in practice though. The assumption mentioned earlier, that picking one set of voles of size 200 is just as likely as picking any other set, surely relies on perfect mixing of the marked voles, which would be rather difficult to achieve (especially as female voles can be highly territorial).

You might be asking yourself why it was OK to find the maximum by setting $p_N \approx p_{N-1}$. This is a standard method, but of course it only works if the distribution is one-humped, like a normal distribution. An alternative approach would have been to approximate the distributions using Stirling's approximation, which at its most basic is

$$\ln N! \approx N \ln N - N\,.$$

This gives exactly the same equation as the $p_N \approx p_{N-1}$ method, and shows that the distribution is indeed one-humped.

Problem 74: Breaking a stick (✓✓)

A stick is broken at a point, chosen at random, along its length. Find the probability that the ratio, R, of the length of the shorter piece to the length of the longer piece is less than r, where r is a given positive number.

Find the probability density function for R, and calculate the mean and variance of R.

1999 Paper II

Comments

Continuous probability distributions often seem harder than discrete distributions, for no good reason: the concepts are the same, and in fact integrals are normally easier than sums.

Here, part of the difficulty is that you have to set up the problem yourself. You have a random variable with a known distribution (corresponding to the point at which the stick is broken), but you are interested in another random variable (the ratio of lengths) derived from the first. As always, it is best to work with the cumulative distribution functions rather than the probability density functions when deriving the distribution of the second random variable.

Solution to problem 74

Let the length of the stick be 2ℓ and let X be the length of the shorter piece of stick, so that $X \leqslant \ell$.

The random variable X is uniformly distributed on the interval $0 \leqslant x \leqslant \ell$, so

$$\mathrm{P}(0 \leqslant X \leqslant x) = \frac{x}{\ell}\,.$$

Now $R = \dfrac{X}{2\ell - X}$, by definition, so

$$X = \frac{2\ell R}{1+R}\,.$$

The cumulative distribution function for R is given by

$$\mathrm{P}(R \leqslant r) = \mathrm{P}\left(\frac{X}{2\ell - X} \leqslant r\right) = \mathrm{P}\left(X \leqslant \frac{2\ell r}{1+r}\right) = \frac{2\ell r}{(1+r)\ell} = \frac{2r}{1+r}\,.$$

Let's check that this satisfies the conditions for a cumulative distribution function: it should increase from 0 to 1 as r goes from its smallest value, which is 0, to its greatest value, which is 1. And it does, so that's OK.

The probability density function is the derivative of the cumulative distribution function:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{2r}{1+r}\right) = \frac{2}{(r+1)^2}\,.$$

Integrating to find $\mathrm{E}(R)$ and $\mathrm{E}(R^2)$ gives

$$\mathrm{E}(R) = \int_0^1 \frac{2r}{(1+r)^2}\mathrm{d}r = 2\ln 2 - 1\,; \quad \mathrm{E}(R^2) = \int_0^1 \frac{2r^2}{(1+r)^2}\mathrm{d}r = 3 - 4\ln 2\,,$$

and $\mathrm{Var}\,(R) = (3 - 4\ln 2) - (2\ln 2 - 1)^2 = 2 - 4(\ln 2)^2$.

Post-mortem

Note the check we made to verify that the function we found for the cumulative distribution could actually be a cumulative distribution function, which is equivalent to checking that the probability density function integrates to 1.

We should also check that the final answers make sense. Putting numbers into a calculator gives $2\ln 2 - 1 = 0.23$ for the expected value, which seems reasonable; at least it is less than 1. If you sketch the density function, you see that this could easily be its average value (do it!).

For the variance, I got 0.08, which means that the ratio is likely to be in the plausible range 0.15 to 0.31.

There are a variety of other stick-breaking problems, including a rather pleasing one about breaking the stick in two places and finding the probability that the pieces form a triangle. In this case, it matters how you break the stick: you could break it once and then break one of the pieces; or you could choose two points on the stick randomly at which to break it. For an interesting discussion, see the excellent cut-the-knot website: http://www.cut-the-knot.org/Curriculum/Probability/TriProbability.shtml

Problem 75: Random quadratics (✓✓✓)

The random variable B is normally distributed with mean zero and unit variance. Find the probability that the quadratic equation

$$X^2 + 2BX + 1 = 0$$

has real roots.
Given that the two roots X_1 and X_2 are real, find, giving your answers to three significant figures:

(i) the probability that both X_1 and X_2 are greater than $\frac{1}{5}$;

(ii) the expected value of $|X_1 + X_2|$.

1988 Paper II

Comments

It is quite difficult to find statistics questions at this level that are not too difficult and are also not simple applications of standard methods. For example, χ^2 tests are not really suitable, because the theory is sophisticated while the applications are usually rather straightforward. Most questions in the probability/statistics area tend therefore to concentrate on probability, and many of these have a bit of pure mathematics thrown in.

Here, the random variable is the coefficient of a quadratic equation, which is rather pleasing. But you have to handle the inequalities carefully. The difficulty is increased by the conditional element: for parts (i) and (ii) you are only interested in the case of real roots.

If you don't have statistical tables handy, don't bother to find some: just leave the answers in terms of the probability, $\Phi(z)$, that a standard ($\mu = 0$, $\sigma = 1$) normally distributed random variable Z satisfies $Z \leqslant z$.

Solution to problem 75

The solution of the quadratic is

$$X = -B \pm \sqrt{B^2 - 1}$$

which has real roots if $|B| \geqslant 1$. Let $\Phi(z)$ be the probability that a standard normally distributed variable Z satisfies $Z \leqslant z$. Then the probability that $|B| \geqslant 1$ is (taking the two tails of the normal distribution)

$$\Phi(-1) + \big(1 - \Phi(1)\big) = 2 - 2\,\Phi(1) = 0.3174.$$

(i) We need the smaller root to be greater than $\frac{1}{5}$. The smaller root is $-B - \sqrt{B^2-1}$. Now provided $\sqrt{B^2-1}$ is real, we have

$$\begin{aligned}
-B - \sqrt{B^2-1} > \tfrac{1}{5} \quad &\Leftrightarrow \quad B + \tfrac{1}{5} < -\sqrt{B^2-1} \\
&\Leftrightarrow \quad (B+\tfrac{1}{5})^2 > B^2 - 1 \qquad \text{and} \quad (B+\tfrac{1}{5}) < 0 \\
&\Leftrightarrow \quad \tfrac{2}{5}B + \tfrac{1}{25} > -1 \qquad \text{and} \quad B < -\tfrac{1}{5} \\
&\Leftrightarrow \quad -\tfrac{1}{5} > B > -\tfrac{13}{5}.
\end{aligned}$$

However, if $B < -\frac{1}{5}$, then the condition that $\sqrt{B^2-1}$ is real, i.e. $|B| \geqslant 1$, implies the stronger condition $B \leqslant -1$. The condition that both roots are real *and* greater than $\frac{1}{5}$ is therefore

$$-\tfrac{13}{5} < B \leqslant -1$$

and the probability that both roots are real and greater than $\frac{1}{5}$ is

$$\Phi(-1) - \Phi(-2.6) = \Phi(2.6) - \Phi(1) = 0.9953 - 0.8413 = 0.1540.$$

The conditional probability that both roots are greater than $\frac{1}{5}$ *given* that they are real is

$$\begin{aligned}
\mathrm{P}\big(\text{both roots} > \tfrac{1}{5} \;\big|\; \text{both real}\big) \quad &= \quad \frac{\mathrm{P}\big(\text{ both roots} > \tfrac{1}{5} \textit{ and } \text{both roots real}\big)}{\mathrm{P}\big(\text{ both roots real}\big)} \\
&= \quad \frac{0.1540}{0.3174} = 0.485.
\end{aligned}$$

(ii) The sum of the roots is $-2B$, so we want the expectation of $|2B|$ given that $|B| \geqslant 1$, which is

$$\frac{\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{-1} (-2x)e^{-\frac{1}{2}x^2}\,\mathrm{d}x + \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{1}^{\infty} 2xe^{-\frac{1}{2}x^2}\,\mathrm{d}x}{\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{-1} e^{-\frac{1}{2}x^2}\,\mathrm{d}x + \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{1}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x} = \frac{2 \times \dfrac{1}{\sqrt{2\pi}} \times 2e^{-\frac{1}{2}}}{2\big(1 - \Phi(1)\big)} = 3.05.$$

Syllabus

The mathematical requirements for this book are based on the pre-2019 syllabus for STEP Mathematics I and II, which is given below. The STEP specifications have been revised alongside the 2017-18 A-level reforms, and you can find the current specifications from the website www.admissionstesting.org/for-test-takers/step/about-step/

PURE MATHEMATICS

Specification	Notes
General	
Mathematical vocabulary and notation	including: equivalent to; necessary and sufficient; if and only if; $\Rightarrow$; $\Leftrightarrow$; $\equiv$.
Methods of proof	including proof by contradiction and disproof by counterexample; proof by induction.
Algebra	
Indices and surds	including rationalising denominators.
Quadratics	including proving positivity by completing a square.
The expansion for $(a+b)^n$	including knowledge of the general term; notation: $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$.
Algebraic operations on polynomials and rational functions	including factorisation, the factor theorem, the remainder theorem; including understanding that, for example, if $$x^3+bx^2+cx+d \equiv (x-\alpha)(x-\beta)(x-\gamma),$$ then $d=-\alpha\beta\gamma$.
Partial fractions	including denominators with a repeated or quadratic factor.
Sequences and series	including use of, for example, $a_{n+1} = \mathrm{f}(a_n)$ or $a_{n+1} = \mathrm{f}(a_n, a_{n-1})$; including understanding of the terms convergent, divergent and periodic in simple cases; including use of $\sum_{k=1}^{n} k$ to obtain related sums.
The binomial series for $(1+x)^k$, where k is a rational number	including understanding of the condition $\lvert x\rvert < 1$.
Arithmetic and geometric series	including sums to infinity and conditions for convergence, where appropriate.

Inequalities	including solution of, eg, $\frac{1}{a-x} > \frac{x}{x-b}$; including simple inequalities involving the modulus function; including the solution of simultaneous inequalities by graphical means.
Functions	
Domain, range, composition, inverse	including use of functional notation such as $y = \mathrm{f}(ax+b)$, $x = \mathrm{f}^{-1}(y)$ and $z = \mathrm{f}\big(\mathrm{g}(x)\big)$.
Increasing and decreasing functions	the precise definition of these terms will be included in the question.
Exponentials and logarithms	including $x = a^y \Leftrightarrow y = \log_a x$, $x = \mathrm{e}^y \Leftrightarrow y = \ln x$; including the exponential series
The effect of simple transformations	such as $y = a\mathrm{f}(bx+c)+d$.
The modulus function.	
Location of roots of $\mathrm{f}(x) = 0$ by considering changes of sign of $\mathrm{f}(x)$	
Approximate solution of equations using simple iterative methods	
Curve sketching	
General curve sketching	including use of symmetry, transformations, behaviour as $x \to \pm\infty$, points or regions where the function is undefined, turning points, asymptotes parallel to the axes.
Trigonometry	
Radian measure, arc length of a circle, area of a segment	
Trigonometric functions	including knowledge of standard values, such as $\tan(\frac{1}{4}\pi)$, $\sin 30^\circ$; including identities such as $\sec^2\phi - \tan^2\phi = 1$; including application to geometric problems in two and three dimensions.
Double angle formulae	including their use in calculating, eg, $\tan(\frac{1}{8}\pi)$.
Formulae for $\sin(A \pm B)$ and $\cos(A \pm B)$	including their use in solving equations such as $a\cos\theta + b\sin\theta = c$.
Inverse trigonometric functions	definitions including domains and ranges; notation: either $\arctan\theta$ or $\tan^{-1}\theta$, etc.
Coordinate geometry	
Straight lines in two-dimensions	including the equation of a line through two given points, or through a given point and parallel to a given line or through a given point and perpendicular to a given line; including finding a point which divides a segment in a given ratio.
Circles	using the general form $(x-a)^2 + (y-b)^2 = R^2$; including points of intersection of circles and lines.

Cartesian and parametric equations of curves and conversion between the two forms.	
Calculus	
Interpretation of a derivative as a limit and as a rate of change	including knowledge of both notations $\mathrm{f}'(x)$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$.
Differentiation of standard functions	including algebraic expressions, trigonometric and inverse trigonometric functions, exponential and log functions.
Differentiation of composite functions, products and quotients and functions defined implicitly.	
Higher derivatives	including knowledge of both notations $\mathrm{f}''(x)$ and $\frac{d^2y}{\mathrm{d}x^2}$; including knowledge of the notation $\frac{d^ny}{\mathrm{d}x^n}$.
Applications of differentiation to gradients, tangents and normals, stationary points, increasing and decreasing functions	including finding maxima and minima which are not stationary points; including classification of stationary points using the second derivative.
Integration as reverse of differentiation	
Integral as area under a curve	including area between two curves; including approximation of integral by the rectangle and trapezium rules.
Volume within a surface of revolution	rotation about either z or y axes.
Knowledge and use of standard integrals	including the forms $\int \mathrm{f}'(\mathrm{g}(x))\mathrm{g}'(x)\,\mathrm{d}x$ and $\int \mathrm{f}'(x)/\mathrm{f}(x)\,\mathrm{d}x$; including transformation of an integrand into standard (or some given) form; including use of partial fractions; not including knowledge of integrals involving inverse trigonometric functions.
Definite integrals	including calculation, without justification, of simple improper integrals such as $\int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x$ and $\int_0^1 x^{-\frac{1}{2}}\,\mathrm{d}x$ (if required, information such as the behaviour of $x\mathrm{e}^{-x}$ as $x \to \infty$ or of $x \ln x$ as $x \to 0$ will be given).
Integration by parts and by substitution	including understanding their relationship with differentiation of product and of a composite function; including application to (e.g.) $\int \ln x\,\mathrm{d}x$.
Formulation and solution of differential equations	formulation of first order equations; solution in the case of a separable equation or by some other method given in the question.
Vectors	
Vectors in two and three dimensions	including use of column vector and **i**, **j**, **k** notation.
Magnitude of a vector	including the idea of a unit vector.
Vector addition and multiplication by scalars	including geometrical interpretations.
Position vectors	including application to geometrical problems.

The distance between two points	
Vector equations of lines	including the finding the intersection of two lines; understanding the notion of skew lines (knowledge of shortest distance between skew lines is not required).
The scalar product	including its use for calculating the angle between two vectors.

MECHANICS

Questions on mechanics may involve any of the material in the Pure Mathematics syllabus.

Specification	**Notes**
Force as a vector	including resultant of several forces acting at a point and the triangle or polygon of forces; including equilibrium of a particle; forces include weight, reaction, tension and friction.
Centre of mass	including obtaining the centre of mass of a system of particles, of a simple uniform rigid body (possibly composite) and, in simple cases, of non-uniform body by integration.
Equilibrium of a rigid body or several rigid bodies in contact	including use of moment of a force; for example, a ladder leaning against a wall or on a movable cylinder; including investigation of whether equilibrium is broken by sliding, toppling or rolling; including use of Newton's third law; excluding questions involving frameworks.
Kinematics of a particle in a plane	including the case when velocity or acceleration depends on time (but excluding knowledge of acceleration in the form $v\frac{\mathrm{d}v}{\mathrm{d}x}$); questions may involve the distance between two moving particles, but detailed knowledge of relative velocity is not required.
Energy (kinetic and potential), work and power	including application of the principle of conservation of energy.
Collisions of particles	including conservation of momentum, conservation of energy (when appropriate); coefficient of restitution, Newton's experimental law; including simple cases of oblique impact (on a plane, for example); including knowledge of the terms *perfectly elastic* ($e = 1$) and *inelastic* ($e = 0$); questions involving successive impacts may be set.
Newton's first and second laws of motion	including motion of a particle in two and three dimensions and motion of connected particles, such as trains, or particles connected by means of pulleys.

Motion of a projectile under gravity	including manipulation of the equation $$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha},$$ viewed, possibly, as a quadratic in $\tan\alpha$; not including projectiles on inclined planes.

PROBABILITY AND STATISTICS

The emphasis is towards probability and formal proofs, and away from data analysis and use of standard statistical tests. Questions may involve use of any of the material in the Pure Mathematics syllabus.

Specification	**Notes**
Probability	
Permutations, combinations and arrangements	including sampling with and without replacement.
Exclusive and complementary events	including understanding of $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$, though not necessarily in this form.
Conditional probability	informal applications, such as tree diagrams.
Distributions	
Discrete and continuous probability density functions and cumulative distribution functions	including calculation of mean, variance, median, mode and expectations by explicit summation or integration for a given (possibly unfamiliar) distribution (eg exponential or geometric or something similarly straightforward); notation: $\mathrm{f}(x) = \mathrm{F}'(x)$.
Binomial distribution	including explicit calculation of mean.
Poisson distribution	including explicit calculation of mean; including use as approximation to binomial distribution where appropriate.
Normal distribution	including conversion to the standard normal distribution by translation and scaling; including use as approximation to the binomial or Poisson distributions where appropriate; notation: $X \sim \mathrm{N}(\mu, \sigma^2)$.
Hypothesis testing	
Basic concepts in the case of a simple null hypothesis and simple or compound alternative	including knowledge of the terminology *null hypothesis* and *alternative hypothesis*, *one* and *two tailed tests*.

This book need not end here...

Share

All our books—including the one you have just read—are free to access online so that students, researchers and members of the public who cant afford a printed edition will have access to the same ideas. This title will be accessed online by hundreds of readers each month across the globe: why not share the link so that someone you know is one of them?

This book and additional content is available at: https://doi.org/10.11647/OBP.0181

Customise

Personalise your copy of this book or design new books using OBP and third-party material. Take chapters or whole books from our published list and make a special edition, a new anthology or an illuminating coursepack. Each customised edition will be produced as a paperback and a downloadable PDF.

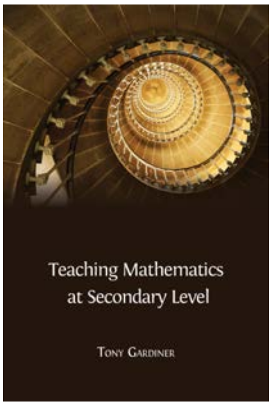
Find out more at: https://www.openbookpublishers.com/section/59/1

OpenBook Publishers

Knowledge is for sharing

You may also be interested in...

http://www.openbookpublishers.com/product/340



Teaching Mathematics is nothing less than a mathematical manifesto. Arising in response to a limited National Curriculum, and engaged with secondary schooling for those aged 11-14 (Key Stage 3) in particular, this handbook for teachers will help them broaden and enrich their students' mathematical education. It avoids specifying how to teach, and focuses instead on the central principles and concepts that need to be borne in mind by all teachers and textbook authors but which are little appreciated in the UK at present.

This study is aimed at anyone who would like to think more deeply about the discipline of 'elementary mathematics', in England and Wales and anywhere else. By analysing and supplementing the current curriculum, *Teaching Mathematics* provides food for thought for all those involved in school mathematics, whether as aspiring teachers or as experienced professionals. It challenges us all to reflect upon what it is that makes secondary school mathematics educationally, culturally, and socially important.

Tony Gardiner, former Reader in Mathematics and Mathematics Education at the University of Birmingham, was responsible for the foundation of the United Kingdom Mathematics Trust in 1996, one of the UK's largest mathematics enrichment programs. In 1997 Gardiner served as President of the Mathematical Association, and in 2011 was elected Education Secretary of the London Mathematical Society.

Advanced Problems in Mathematics

Preparing for University

Stephen Siklos

This new and expanded edition is intended to help candidates prepare for entrance examinations in mathematics and scientific subjects, including STEP (Sixth Term Examination Paper). STEP is an examination used by Cambridge Colleges for conditional offers in mathematics. They are also used by some other UK universities and many mathematics departments recommend that their applicants practice on the past papers even if they do not take the examination.

Advanced Problems in Mathematics bridges the gap between school and university mathematics, and prepares students for an undergraduate mathematics course. The questions analysed in this book are all based on past STEP questions and each question is followed by a comment and a full solution. The comments direct the reader's attention to key points and put the question in its true mathematical context. The solutions point students to the methodology required to address advanced mathematical problems critically and independently.

This book is a must read for any student wishing to apply to scientific subjects at university level and for anyone interested in advanced mathematics.

***Stephen Siklos** was a fellow of Jesus College, Cambridge. He worked in Cambridge University's Faculty of Mathematics, where his main area of research was Einstein's theory of general relativity and where he acted as Admissions Officer. He had a particular interest in the transition from school to university mathematics and for over thirty years presided over the STEP examinations, used as a qualification for undergraduate admissions by some of the top UK university mathematics departments.*

As with all Open Book publications, this entire book is available to read for free on the publisher's website. Printed and digital editions, together with supplementary digital material, can also be found at www.openbookpublishers.com

Cover image: Photograph by Logan Troxell on Unsplash at https://unsplash.com/photos/DsQmBlbywJ8
Cover design: Anna Gatti

ebook
ebook and OA editions also available
OPEN ACCESS

OpenBook Publishers