EXAMPLE 4.14

Converting from One Base into Another

Convert $3{,}601_7$ into base 10.

Solution

In base 7, the place values are powers of 7. Since there are four digits, the highest power of 7 that is used is 3. This yields $3{,}601_7 = 3 \times 7^3 + 6 \times 7^2 + 0 \times 7^1 + 1 \times 7^0 = 3 \times 343 + 6 \times 49 + 0 \times 7 + 1 = 1{,}029 + 294 + 0 + 1 = 1{,}324.$

VIDEO

Convert Base 7 to Base 10 (https://openstax.org/r/Convert_Base_7_to_Base_10)

YOUR TURN 4.14

1. Convert 421_6 into base 10.

EXAMPLE 4.15

Converting from Base 14 to Base 10

Convert $4B7_{14}$ into base 10.

Solution

In base 14, the place values are powers of 14. Since there are three digits, the highest power of 14 is 2. Also recall that in base 14, 10 is represented by A, 11 is represented by B, 12 is represented by C, and 13 is represented by D. Using that, we convert to base 10:

$$4B7_{14} = 4 \times 14^2 + B \times 14^1 + 7 \times 14^0 = 4 \times 196 + 11 \times 14 + 7 \times 1 = 784 + 154 + 7 = 945.$$

YOUR TURN 4.15

1. Convert $A3C_{14}$ into base 10.

EXAMPLE 4.16

Converting from Base 12 to Base 10

Convert $A16_{12}$ into base 10.

Solution

In base 12, the place values are powers of 12. Since there are three digits, the highest power of 12 is 2. Also recall that in base 12, 10 is represented by A and 11 is represented by B. Using that, we convert to base 10:

$$A16_{14} = 10 \times 12^2 + 1 \times 12^1 + 6 \times 12^0 = 10 \times 144 + 1 \times 12 + 6 \times 1 = 1{,}440 + 12 + 6 = 1{,}458.$$

YOUR TURN 4.16

1. Convert $5AB_{12}$ into base 10.

EXAMPLE 4.17

Converting from Base 2 to Base 10

Convert 1011_2 into base 10.

Solution

In base 2, the place values are powers of 2. Since there are four digits, the highest power of 2 is 3. Using that, we convert to base 10:

$$1011_2 = 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 1 \times 8 + 0 \times 4 + 1 \times 2 + 1 \times 1 = 8 + 0 + 2 + 1 = 11.$$

YOUR TURN 4.17

1. Convert 11011_2 into base 10.

WHO KNEW?

Before Napoleon

Before Napoleon's France, which adopted the base 10 system, a modified base 12 system was often used in Europe. Twelve is easily divisible into groups of 2, 3, 4, and 6, which makes it easier to work with. Even our numbering system retains a bit of this. You have likely noticed that we use the words *thirteen*, *fourteen*, *fifteen*, and so on to indicate 10 and 3, 10 and 4, 10 and 5, and so one. Even the 20s reinforce this idea, as in twenty-one, and twenty-two. However, two numbers don't follow this pattern, namely 11 and 12. If they followed the same rules, they'd be one teen and two teen. We even have a special word for 12; that is, a dozen. However, etymologically speaking, the words *eleven* and *twelve* are likely derived by referencing the number 10. These two numbers may date back to the Old English words *endleofan* and *twelf*, which can be traced back further to *ain lif* and *twa lif*. The word *lif* here may be the base word for "to leave." This would suggest *ain lif* is one left after 10, and *twa lif* is two left after 10, or, 11 and 12.

EXAMPLE 4.18

Writing Numbers in Base Systems Other Than Base 10

Write the numbers in base 7 up to 100_7.

Solution

Step 1: Using the patterns we indicated earlier, we begin with the first seven digits.

0, 1, 2, 3, 4, 5, 6

Step 2: Since we've run out of digits, we start with 10, indicating we've run out of symbols once.

10, 11, 12, 13, 14, 15, 16

Step 3: Continuing in the same way, we get:

20, 21, 22, 23, 24, 25, 26

30, 31, 32, 33, 34, 35, 36

40, 41, 42, 43, 44, 45, 46

50, 51, 52, 53, 54, 55, 56

60, 61, 62, 63, 64, 65, 66

Now, all the digits have been used in the leading digits. Since the digits have all been used in that leading digit, we use 100, as in base 10.

100

YOUR TURN 4.18

1. Write the numbers of base 4 up to 100_4.

EXAMPLE 4.19

Writing Numbers in Bases with More Than 10 Symbols

Write the numbers in base 14 up to 100_{14}.

Solution

Step 1: Using the patterns we indicated earlier, we begin with the first 14 digits.

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D

Step 2: Since we've run out of digits, we start with 10, indicating we've run out of symbols once.

10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 1A, 1B, 1C, 1D

Step 3: Continuing in the same way, we get:

20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 2A, 2B, 2C, 2D

30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 3A, 3B, 3C, 3D

40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 4A, 4B, 4C, 4D

50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 5A, 5B, 5C, 5D

60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 6A, 6B, 6C, 6D

70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 7A, 7B, 7C, 7D

80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 8A, 8B, 8C, 8D

90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 9A, 9B, 9C, 9D

A0, A1, A2, A3, A4, A5, A6, A7, A8, A9, AA, AB, AC, AD

B0, B1, B2, B3, B4, B5, B6, B7, B8, B9, BA, BB, BC, BD

C0, C1, C2, C3, C4, C5, C6, C7, C8, C9, CA, CB, CC, CD

D0, D1, D2, D3, D4, D5, D6, D7, D8, D9, DA, DB, DC, DD

100

YOUR TURN 4.19

1. Write the numbers of base 12 up to 100_4.

Base 2 is important in the digital age, as it is the system used by computers. It is the simplest base to work with, but has the drawback that the numbers in base 2 may use many, many digits. In Addition and Subtraction in Base Systems and Multiplication and Division in Base Systems, we will look at base 2 in each situation.

EXAMPLE 4.20

Writing Numbers in Base 2

Write the numbers in base 2 up to 100_2.

Solution

Base 2 uses only two symbols: 0 and 1. Following the pattern established previously, the numbers in base 2 up to 100_2 are 0, 1, 10, 11, and 100.

> **YOUR TURN 4.20**
>
> **1.** Write the numbers in base 3 up to 100_3.

> **WHO KNEW?**
>
> Early Hawaiian Numeration System
>
> Before the British arrived in Hawaii, people there used a system that combined two different bases. Objects were initially grouped into collections of four, and a collection of four was referred to as *kauna*. A person could have two *kauna* and three "ones" (in Hindu-Arabic, 11). Or they could have eight *kauna* and one "ones" (In Hindu-Arabic, 33). However, sets of four were grouped in collections of 10. A set of 10 *kauna* was *ka'au*. The collections of *ka'au* were grouped by 10 also. Which meant that 10 *ka'au* (this is 40 in Hindu-Arabic) would be *lau* (or 400 in Hindu-Arabic). What this shows is that the Hawaiian culture developed a system that used base 4 combined with base 10.

Conversion of Base 10 into Another Base

Converting from base 10 into another base uses repeated division, recording the **remainder** at each step. Then, the number in the new base is the remainder starting from the last remainder found. To be accurate in what we're saying, we need to remind ourselves of some terminology associated with division. When integers are divided, the one being divided is the **dividend**, and the one that is dividing the dividend is the **divisor**. The **quotient** is the largest natural number that can be multiplied by the divisor where the product is less than the dividend.

When the integer n is divided by the integer d, n is called the **dividend** and d is the **divisor.**

To convert a base 10 number n into base d, we divide n by d, recording the remainder. Then we divide the quotient from that step by the base d, and record the remainder again. We continue this process until the quotient is 0. Then, the base d number has digits that start with the last remainder and use each remainder in reverse order.

EXAMPLE 4.21

Converting from Base 10 into a Lower Base
Convert 298 to base 6.

Solution
We divide 298 by 6, and record the remainder. Then we divide the quotient from that step by 6, and record the remainder again. We continue this process until the quotient is 0. Then, the base 6 number has digits that start with the last remainder and use each remainder in reverse order.

Step 1: When we divide 298 by 6, we get $6\overline{)298}^{\,49r4}$. The quotient is 49 and the remainder is 4.

Step 2: Now we divide the quotient, 49, by 6. This gives $6\overline{)49}^{\,8r1}$. The quotient is 8 and the remainder is 1.

Step 3: Repeating, we get $6\overline{)8}^{\,1r2}$. The quotient is 1 and the remainder is 2.

Step 4: Finally, we perform the operation on the quotient 1, $6\overline{)1}^{\,0r1}$ giving us a quotient of 0 and a remainder of 1.

Step 5: The base 6 number has digits equal to the remainders in reverse order, 1214_6. So, 298 in base 10 when converted to base 6 is 1214_6.

> **YOUR TURN 4.21**
>
> **1.** Convert 693 to base 7.

VIDEO

[Converting from Base 10 to Another Base (https://openstax.org/r/Base_10_to_Another_Base)]

EXAMPLE 4.22

Converting from Base 10 into a Higher Base
Convert 45,134 to base 13.

Solution
We divide 45,134 by 13, and record the remainder. Then we divide the quotient from that step by 13, and record the remainder again. We continue this process until the quotient is 0. Then, the base 13 number has digits that start with the last remainder and use each remainder in reverse order. It is at this step that we'll convert to the base 13 digits, which are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C.

Step 1: When we divide 45,134 by 13, we get $13\overline{)45{,}134}^{\;3{,}471r11}$. The quotient is 3,471 and the remainder is 11.

Step 2: Now we divide the quotient, 3,471, by 13. This gives $13\overline{)3{,}471}^{\;267r0}$. The quotient is 267 and the remainder is 0.

Step 3: Repeating, we get $13\overline{)267}^{\;20r7}$. The quotient is 20 and the remainder is 7.

Step 4: Again, and we get $13\overline{)20}^{\;1r7}$. The quotient is 1 and the remainder is 7.

Step 5: Finally, we get $13\overline{)1}^{\;0r1}$, with quotient 0 and a remainder 1.

Step 6: The base 13 number has digits equal to the remainders in reverse order, which were 1, 7, 7, 0, and 11. The 11 is written as B in base 13. So, 45,134 in base 10 when converted to base 13 is $1770B_{13}$.

YOUR TURN 4.22

1. Convert 9,275 to base 12.

EXAMPLE 4.23

Converting from Base 10 into Base 2
Convert 100 to base 2.

Solution
Following the pattern above:

Step 1: We divide 100 by 2, and record the remainder.

Step 2: Then we divide the quotient from that step by 2, and record the remainder again.

Step 3: We continue this process until the quotient is 0.

Step 4: Following this process, the remainders are, in order, 0, 0, 1, 0, 0, 1, 1. Writing those in reverse order gives the number in base 2, 1100100_2.

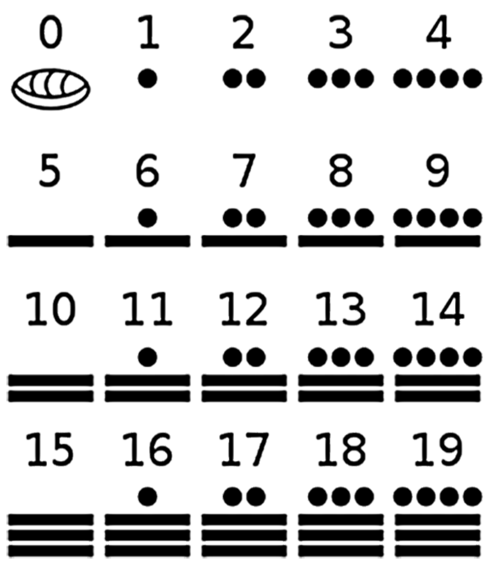
Notice that 100 in base 2 took seven digits.

YOUR TURN 4.23

1. Convert 137 to base 2.

Converting from Hindu-Arabic Numbers to Mayan Numbers

To convert from a Hindu-Arabic number to a Mayan number involves two distinct processes. First, the number must be converted to base 20, using the process described and demonstrated previously. Next, that base 20 number has to be written using Mayan numerals. For reference, the Mayan numerals and their values are below.



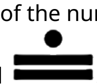
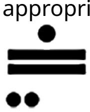
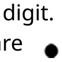
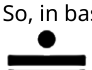
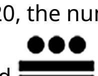
EXAMPLE 4.24

Converting from Base 10 into the Mayan System

Convert the following into Mayan numbers.

1. 51
2. 653

Solution

1. The Mayan system is base 20, so we must use 20 in the process from above. The first division has a quotient of 2 and remainder of 11. The 11 serves as the "ones" digit. Dividing that quotient, 2, by 20 has a quotient of 0 with a remainder of 2. The 2 becomes the "twenties" digit of the number. So, in base 20, the number would be 2 followed by 11. The Mayan symbols for 2 and 11 are  and . Writing these vertically, with the "ones" digit on top, as appropriate for Mayan numbers, results in:

2. The Mayan system is base 20, so we must use 20 in the process from above. The first division, 673 divided by 20, has a quotient of 32 and remainder of 13. Dividing that quotient, 32, by 20 has a quotient of 1 with a remainder of 12. Dividing that quotient, 1, by 20 has a quotient of 0 and a remainder of 1. Since there are three remainders here, this is a three-digit number. The 1 is the "20-squared" digit, the 12 is the "twenties" digit, and the 13 is the "ones" digit. So, in base 20, the number would be 1 followed by 12 followed by 13.. The Mayan symbols for 1, 12 and 13 are , , and . Writing these vertically, as appropriate for Mayan numbers, would result in:


YOUR TURN 4.24

Convert the following into Mayan numbers.

1. 137
2. 2,171

WHO KNEW?

Other Languages, Other Bases

There have been base systems that use bases other than 10. Some bases used were 20, 12, and 27! Visit this site to see more on the languages that used other bases (https://openstax.org/r/number-systems_other_languages).

Errors in Converting Between Bases

There are some common errors that are made when converting between bases. Often, it comes down to using an “illegal” symbol in the new base.

EXAMPLE 4.25

Detecting an Illegal Symbol When Converting Between Bases

A base 10 number is converted to base 7 and the result was 2081_7. Was an error committed? How do you know?

Solution

The result has the digit 8 in it. In base 7, 8 is an illegal symbol. Based on that, an error was committed.

YOUR TURN 4.25

1. A base 10 number is converted to base 4 and the result was 3702_4. Was an error committed? How do you know?

When converting from base 10 to another base, an illegal symbol will be used if a mistake was made in the division process used to find the number in the new base. Since the digits are based on the remainders, any remainder that is an illegal symbol would indicate an error.

EXAMPLE 4.26

Detecting an Error in Division When Converting Between Bases

When changing from base 10 to base 8, the division process resulted in the following remainders: 1, 0, 9, 2, 4. Was an error committed? How do you know?

Solution

The remainders include 9, which in base 8 is an illegal symbol.

YOUR TURN 4.26

1. When changing from base 10 to base 6, the division process resulted in the following remainders: 5, 0, 0, 10. Was an error committed? How do you know?

Another possible way to detect an error in converting between bases is to count the number of digits. When converting from a higher base to a lower base, the number of digits cannot get smaller. Similarly, when converting from a lower base to a higher base, the number of digits cannot get bigger. So, if a base 10 number is converted to a base 3 number, the number of digits in the new base 3 numbers cannot be less than the number of digits in the base 10 number. Similarly, if a base 7 number is converted to base 10, the number of digits in the base 10 number cannot be more than the number of digits in the original base 7 number.

EXAMPLE 4.27

Detecting an Error in Number of Digits When Converting Between Bases

A five-digit base 10 number is converted to a base 5 number. The base 5 number has four digits. Was an error committed? How do you know?

Solution

Since 10 is larger than 5, the base 5 number cannot have less digits than the base 10 number. Since it did, we know an error has been made.

YOUR TURN 4.27

1. A six-digit base 12 number is converted into a base 10 number. The base 10 number has five digits. Was an error committed? How do you know?

Check Your Understanding

14. In base 25, how many symbols would be necessary?
15. In base 18, what would the place value of the 4 be in the number 348_{18} be?
16. Convert 2304_5 into base 10.
17. When counting in base 9, what number would follow 38_9?
18. Convert 329 into base 8.
19. Convert ABC_{14} into base 10.
20. How do you know a mistake was made when converting from base 10 to base 4 and the result is 152_4?

SECTION 4.3 EXERCISES

1. What does it mean when we say a number is written in base 7?
2. What does it mean when we say a number is written in base 12?
3. How many symbols are there in a base 3 system? What are they?
4. How many symbols are there in a base 15 system? What are they?
5. List the numbers, up to 100, in the base 5 system.
6. List the numbers, up to 100, in the base 2 system.

For the following exercises, convert the number into a base 10 number.

7. 14_5
8. 21_6
9. 12_3
10. 34_5
11. 14_8
12. 78_9
13. $3B_{12}$
14. 241_6
15. 101_2
16. $4A7_{14}$

17. 804_9
18. 101001_2
19. 3223_6
20. 1436_7
21. $8A0BD_{15}$
22. 110202_3
23. $100A4_{12}$
24. 1010000101_2

For the following exercises, convert the base 10 number into the given base.

25. 12 into base 4
26. 25 into base 2
27. 43 into base 12
28. 153 into base 5
29. 203 into base 2
30. 431 into base 4
31. 543 into base 12
32. 1,023 into base 2
33. 2,876 into base 4
34. 1,765 into base 5
35. 1,993 into base 7
36. 2,000 into base 2
37. 4,368 into base 12
38. 12,562 into base 16 (Hint: Base 16 uses the symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F.)

39. When converting from base 10 to base 6, a student finds the following remainders: 3, 1, 3, 6. How do you know that a mistake was made?

40. When converting from base 10 to base 4, a student finds the following remainders: 0, 0, 3, 7, 2. How do you know a mistake was made?

41. Suppose a base 12 number is converted into a base 10 number, and one of the digits is A. Was an error committed? How do you know?

42. Suppose a base 10 number is converted into a base 5 number and one of the digits is 6. Was an error committed? How do you know?

43. Suppose a base 2 number is converted into a base 10 number, and the base 10 number has more digits that the base 2 number. Was an error committed? How do you know?

44. Suppose a base 16 number is converted to base 2, and the base 2 number has fewer digits than the base 16 number. Was an error committed? How do you know?

For the following exercises, convert the Hindu-Arabic number into a Mayan number.

45. 25
46. 71
47. 400
48. 723

The Babylonian system used base 60. To convert from Hindu Arabic numbers into Babylonian numbers, the process for converting from base 10 to a different base would be done first. Then, the results found in the conversion process would be changed to Babylonian numerals. This process is similar to the one for Mayan numbers.

The Babylonian system used base 60. To convert from Hindu-Arabic numbers into Babylonian numbers, the process for converting from base 10 to a different base would be done first. Then, the results found in the conversion process would be changed to Babylonian numerals. This process is similar to the one for Mayan numbers. For the following exercises, convert the Hindu-Arabic number to a Babylonian number.

49. 67
50. 135
51. 781
52. 10,952

4.4 Addition and Subtraction in Base Systems

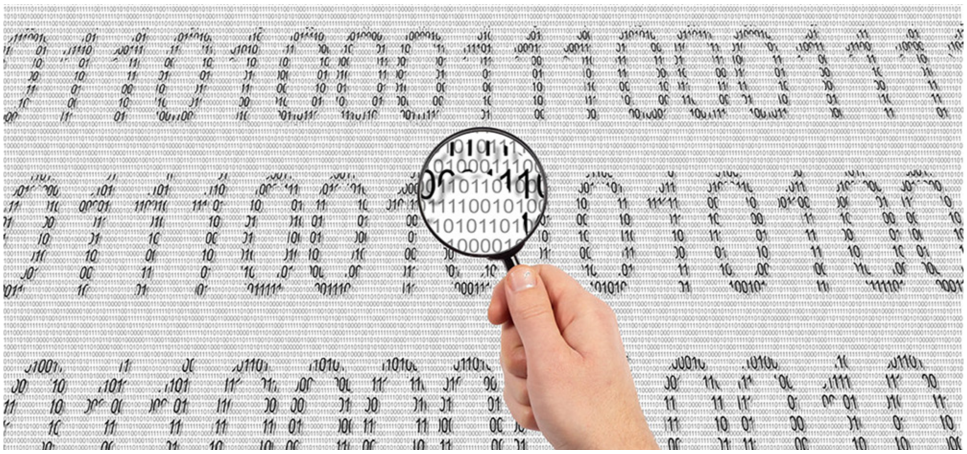


Figure 4.5 All information in computers is represented by 0's and 1's, including quantity, which means computers use Base 2 for arithmetic. (credit: modification of work "Magnifying glass and binary code" by Marco Verch Professional Photographer/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Add and subtract in bases 2–9 and 12.
2. Identify errors in adding and subtracting in bases 2–9 and 12.

Once we decide on a system for counting, we need to establish rules for combining the numbers we're using. This begins with the rules for addition and subtraction. We are familiar with base 10 arithmetic, such as $2 + 5 = 7$ or $3 \times 5 = 15$. How does that change if we instead use a different base? A larger base? A smaller one? In particular, computers use base 2 for all number representation. When your calculator adds or subtracts, multiplies or divides, it uses base 2. This is because the circuitry recognizes only two things, high current and low current, which means the system is uses only has two symbols. Which is what base 2 is.

In this section, we use addition and subtraction in bases other than 10 by referencing the processes of base 10, but applied to a new base system.

Addition in Bases Other Than Base 10

Now that we understand what it means for numbers to be expressed in a base other than 10, we can look at arithmetic using other bases, starting with addition. When you think back to when you first learned addition, it is very likely you learned the addition table. Once you knew the addition table, you moved on to addition of numbers with more than one digit. The same process holds for addition in other bases. We begin with an addition table, and then move on to adding numbers with two or more digits.

We worked with base 6 earlier, and have the numbers in base 6 up to 100_6. Using that table of values, we can create the base 6 addition table.

Here's the beginning of the base 6 addition table:

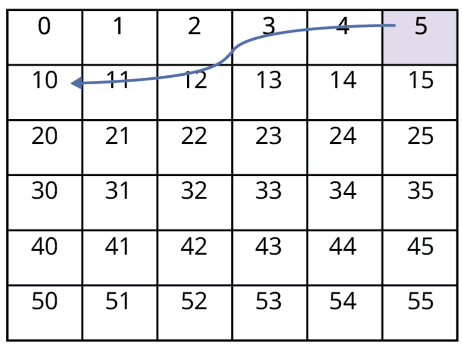
+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	?
2	2	3	4	5	?	?
3	3	4	5	?	?	?

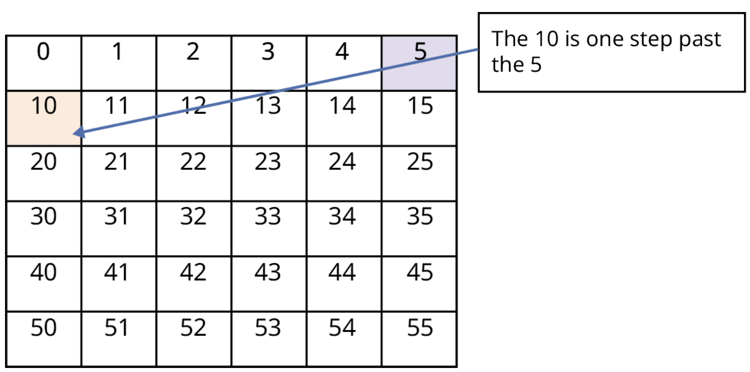
4	4	5	?	?	?	?
5	5	?	?	?	?	?

Many of the cells are not filled out. The ones filled in are values that never get past 5, which is the largest legal symbol in base 6, so they are acceptable symbols. But what do we do with 5 + 3 in base 6? We can't represent the answer as "8" since "8" is not a symbol available to us. Let's go back to the list of numbers we have for base 6.

0	1	2	3	4	5
10	11	12	13	14	15
20	21	22	23	24	25
30	31	32	33	34	35
40	41	42	43	44	45
50	51	52	53	54	55

So, what is 5 + 1 equal to in base 6? Well, start at the 5, and jump ahead one step. You land on 10.





This means that, in base 6, 5 + 1 = 10.

So, what is 5 + 2 in base 6? Well, 5 + 2 = 5 + 1 + 1, so 10 + 1…jump one more space and you land on 11. So, 5 + 2 = 11 in base 6.

0	1	2	3	4	5
10	11	12	13	14	15
20	21	22	23	24	25
30	31	32	33	34	35
40	41	42	43	44	45
50	51	52	53	54	55

And so it goes. Using that process, stepping one more along the list, we can fill in the remainder of the base 6 addition table (Table 4.4).

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	**10**
2	2	3	4	5	**10**	**11**
3	3	4	5	**10**	**11**	**12**
4	4	5	**10**	**11**	**12**	**13**
5	5	**10**	**11**	**12**	**13**	**14**

Table 4.4 Base 6 Addition Table

With this table, and with our understanding of "carrying the one," we can then use the addition table to do addition in base 6 for numbers with two or more digits, using the same processes you learned for addition when you did it by hand.

EXAMPLE 4.28

Adding in Base 6

Calculate $251_6 + 133_6$.

Solution

Step 1: Let's set up the addition using columns.

	2	5	1
+	1	3	3

Step 2: Let's do the one's place first. According to the base 6 addition table (Table 4.4), 1 + 3 = 4.

	2	5	1
+	1	3	3
			4

Step 3: Now, we do the "tens" place (it's really the sixes place). According to the base 6 addition table (Table 4.4), we have 5 + 3 = 12. So, like in base 10, we use the 2 and carry the 1.

	1		
	2	5	1
+	1	3	3
		2	4

Step 4: Now the "hundreds" place (really, thirty-sixes place). There, we have 1 + 2 + 1 = 3 + 1 = 4.

	1		
	2	5	1
+	1	3	3
	4	2	4

So, $251_6 + 133_6 = 424_6$.

As you can see, the process is the same as when you learned base 10 addition, just a different symbol set.

YOUR TURN 4.28

1. Calculate $453_6 + 345_6$.

EXAMPLE 4.29

Creating an Addition Table for a Base Lower Than 10

1. Create the addition table for base 7.
2. Create the addition table for base 2.

Solution

1. We begin with the table below.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	
2	2	3	4	5	6		
3	3	4	5	6			
4	4	5	6				

5	5	6					
6	6						

In base 7, the number that follows 6 is 10 (since we've run out of symbols!). So, $6_7 + 1_7 = 10_7$. Once that is established, $6_7 + 2_7$ will be two numbers past 6, which is 11 in base 7.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	**10**
2	2	3	4	5	6		**11**
3	3	4	5	6			
4	4	5	6				
5	5	6					
6	6	**10**	**11**				

Continuing, we can fill in the rows as we would in base 10, but being aware that we are working in base 7 (Table 4.5).

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	10
2	2	3	4	5	6	10	11
3	3	4	5	6	10	11	12
4	4	5	6	10	11	12	13
5	5	6	10	11	12	13	14
6	6	10	11	12	13	14	15

Table 4.5 Base 7 Addition Table

2. We revisit base 2 here. Begin with the table:

+	0	1
0	0	1
1	1	

Table 4.6 Base 2 Addition Table

In base 2, the number that follows 1 is 10 (since we've run out of symbols!). So, $1_2 + 1_2 = 10_2$. The complete table for base two then is below.

+	0	1
0	0	1
1	1	10

This demonstrates that the rules necessary for base 2 addition are as small as possible: four rules.

YOUR TURN 4.29

1. Create the addition table for base 4.

To summarize the creation of the addition tables for a given base, do the following.

Step 1: Set up the table.

Step 2: Fill in all the additions that use the "legal" symbols for the base. The diagonal that goes from upper left to lower right that is immediately next to the filled boxes all get the value 10, regardless of base.

Step 3: Enter the values that are in the "teens." This can all be done on one table without creating multiple copies of previously done work.

EXAMPLE 4.30

Adding in Base 7

Calculate $536_7 + 433_7$.

Solution

Step 1: Let's set up the addition using columns.

	5	3	6
+	4	3	3

Step 2: Let's do the one's place first. According to the base 7 addition table in the solution for Example 4.29, 6 + 3 = 12. We will carry the 1.

		1	
	5	3	6
+	4	3	3
			2

Step 3: Now, we do the "tens" place (it's really the sevens place). According to the base 7 addition table in the solution for Example 4.29, we have 1 + 3 + 3 = 10. So, like in base 10, we use the 0 and carry the 1.

	1		
	5	3	6
+	4	3	3
		0	2

Step 4: Now the "hundreds" place (really, forty-ninths place). There, we have 1 + 5 + 4 = 6 + 4 = 13.

	1		
	5	3	6
+	4	3	3
1	**3**	0	2

So, $536_7 + 333_7 = 1302_7$.

YOUR TURN 4.30

1. Calculate $461_7 + 142_7$.

As seen previously, when performing addition in another base, set up the problem exactly as you would for addition in base 10. At each step, check the addition table for the base. As in base 10 addition, move right to left, adding down the columns using the rules in the addition table. When necessary and just as in base 10, be sure to carry the 1.

EXAMPLE 4.31

Creating an Addition Table for a Base Higher Than 10

Create the addition table for base 12.

Solution

Step 1: Recall, in base 12, the symbol set is 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, and B. So, the addition table begins as shown below.

+	0	1	2	3	4	5	6	7	8	9	A	B
0	0	1	2	3	4	5	6	7	8	9	A	B
1	1	2	3	4	5	6	7	8	9	A	B	
2	2	3	4	5	6	7	8	9	A	B		
3	3	4	5	6	7	8	9	A	B			
4	4	5	6	7	8	9	A	B				
5	5	6	7	8	9	A	B					
6	6	7	8	9	A	B						
7	7	8	9	A	B							
8	8	9	A	B								
9	9	A	B									
A	A	B										
B	B											

Step 2: The diagonal immediately to the right of the filled in boxes is where the 10 goes for this base.

+	0	1	2	3	4	5	6	7	8	9	A	B
0	0	1	2	3	4	5	6	7	8	9	A	B
1	1	2	3	4	5	6	7	8	9	A	B	10
2	2	3	4	5	6	7	8	9	A	B	10	
3	3	4	5	6	7	8	9	A	B	10		
4	4	5	6	7	8	9	A	B	10			
5	5	6	7	8	9	A	B	10				
6	6	7	8	9	A	B	10					
7	7	8	9	A	B	10						
8	8	9	A	B	10							
9	9	A	B	10								

A	A	B	10									
B	B	10										

Step 3: Using the pattern we're familiar with, and counting in base 12, we can fill in the other cells.

+	0	1	2	3	4	5	6	7	8	9	A	B
0	0	1	2	3	4	5	6	7	8	9	A	B
1	1	2	3	4	5	6	7	8	9	A	B	10
2	2	3	4	5	6	7	8	9	A	B	10	11
3	3	4	5	6	7	8	9	A	B	10	11	12
4	4	5	6	7	8	9	A	B	10	11	12	13
5	5	6	7	8	9	A	B	10	11	12	13	14
6	6	7	8	9	A	B	10	11	12	13	14	15
7	7	8	9	A	B	10	11	12	13	14	15	16
8	8	9	A	B	10	11	12	13	14	15	16	17
9	9	A	B	10	11	12	13	14	15	16	17	18
A	A	B	10	11	12	13	14	15	16	17	18	19
B	B	10	11	12	13	14	15	16	17	18	19	1A

Table 4.7 Base 12 addition table

Notice that the lower-right entry is $1A_{12}$, as this is the number one past 19_{12}.

YOUR TURN 4.31

1. Create the addition table for base 14.

EXAMPLE 4.32

Adding in Base 12

Calculate $3A7_{12} + 9BA_{12}$.

Solution

Step 1: Using the process established in the earlier addition problem, set up the columns.

	3	A	7
+	9	B	A

Step 2: Using the rules from the base 12 addition table in the solution for Example 4.31, and being careful to carry the 1 when necessary, we get the following:

	1	1	
	3	A	7
+	9	B	A
1	**1**	A	5

The ones that were carried are located over the columns.

So, $3A7_{12}$ + $9BA_{12}$= $11A5_{12}$.

YOUR TURN 4.32

1. Calculate $4B3_{12}$ + $B06_{12}$.

EXAMPLE 4.33

Adding in Base 2

We again return to base 2, the base used by computers. Calculate 1001_2 + 11011_2.

Solution

Step 1: Using the process established in the earlier addition problem, set up the columns.

		1	0	0	1
+	1	1	0	1	1

Step 2: Using the rules from the base 2 addition table in the solution for Example 4.29, and being careful to carry the 1 when necessary (and shown at the top of the grid), we get the following:

	1		1	1	
		1	0	0	1
+	1	1	0	1	1
1	0	0	1	0	0

Step 3: Calculate 1001_2 + 11011_2 = 100100_2.

So, 1001_2 + 11011_2 = 100100_2.

YOUR TURN 4.33

1. Calculate $101111_2 + 1100011_2$.

Subtraction in Bases Other Than Base 10

Subtraction in bases other than base 10 follow the same processes as base 10 subtraction, but, as with addition, using the addition table for the base.

EXAMPLE 4.34

Subtracting in Base 6

Calculate $52_6 - 34_6$.

Solution

Step 1: Let's set up the subtraction using columns.

	5	2
−	3	4

Step 2: Just as we might do in base 10, we borrow a 1 from the 5 for the ones digit.

	~~5~~ 4	**12**
−	3	4

Step 3: Referring to the base 6 addition table (Table 4.4), we see that 4 + 4 = 12, so $12_6 - 4_6$ is 4_6.

	~~5~~ 4	12
−	3	4
		4

Step 4: Now we deal with the "tens" (really, sixes) digit, $4_6 - 3_6$, which equals 1_6 according to the base 6 addition table (Table 4.4).

	~~5~~ 4	12
−	3	4
	1	4

So, $52_6 - 34_6 = 14_6$.

YOUR TURN 4.34

1. Calculate $115_6 - 43_6$.

EXAMPLE 4.35

Subtracting in Base 12

Calculate $A17_{12} - 4B3_{12}$.

Solution

Step 1: Let's set up the subtraction using columns.

	A	1	7
–	4	B	3

Step 2: Even in base 12, $7_{12} - 3_{12} = 4_{12}$.

	A	1	7
–	4	B	3
			4

Step 3: Moving to the "tens" digit, we have $1_{12} - B_{12}$. Since 1 is less than B in base 12, we need to borrow a 1 from the A, just as we would for subtraction in base 10.

	~~A~~ **9**	**11**	7
–	4	B	3
			4

Step 4: According to the base 12 addition table in the solution for Example 4.31, $B_{12} + 2_{12} = 11_{12}$, so $11_{12} - B_{12} = 2_{12}$.

	~~A~~ 9	11	7
–	4	B	3
		2	4

Step 5: Finally, we deal with the "hundreds" digit. According to the base 12 addition table in the solution for Example 4.31, $4_{12} + 5_{12} = 9_{12}$, so $9_{12} - 4_{12} = 5_{12}$.

	~~A~~ 9	11	7
–	4	B	3
	5	2	4

So, $A17_{12} - 4B3_{12} = 524_{12}$.

YOUR TURN 4.35

1. Calculate $716_{12} - 4AB_{12}$.

Errors When Adding and Subtracting in Bases Other Than Base 10

Errors when computing in bases other than 10 often involve applying base 10 rules or symbols to an arithmetic problem in a base other than base 10. The first type of error is using a symbol that is not in the symbol set for the base. For instance, if a 9 shows up when working in base 7, you know an error has happened because 9 is not a legal symbol in base 7.

EXAMPLE 4.36

Identifying an Illegal Symbol in Arithmetic in a Base Other Than Base 10

Explain the error in the following calculation:

$$15_6 + 34_6 = 49_6$$

Solution

Since the problem is in base 6, the symbol set available is 0, 1, 2, 3, 4 and 5. The 9 in the answer is clearly not a legal symbol for base 6. Looking back to the base 6 addition table (Table 4.4), we see that $5_6 + 4_6 = 13_6$. Correcting the error, we see the sum is $15_6 + 34_6 = 53_6$.

YOUR TURN 4.36

1. Explain the error in the following calculation and correct the problem:
$133_4 + 112_4 = 245_4$

The second type of error is using a base 10 rule when the numbers are not in base 10. For instance, if you are working in base 13, then $9_{13} + 9_{13}$ is not 18_{13}, even though 18 is the correct answer in base 10.

EXAMPLE 4.37

Identifying an Arithmetic Error in a Base Other Than Base 10

Explain the error in the following calculation, and correct the error:

$$89_{12} + 76_{12} = 165_{12}$$

Solution

If this problem was a base 10 problem, this would be the correct answer. However, in base 12, 9 + 6 is not 15, but is instead 13. To correct this error, carefully use the addition table for base 12. If properly used, the correct answer would be 143_{12}, as seen below:

		8	9
+		7	6
	1	4	3

YOUR TURN 4.37

1. Explain the error in the following calculation, and correct the error:

$149_{14} + 19_{14} = 168_{14}$

Check Your Understanding

21. Determine the addition table for base 8.

22. Compute $24_6 + 53_6$.

23. Compute $35_8 - 26_8$.

24. Compute $3B_{14} + 45_{14}$.

25. Compute $A4_{12} - 9B_{12}$.

26. How do you know an error has occurred in a base 8 addition question if the answer obtained was 28_8?

27. What is one common error made in calculating in base 14?

SECTION 4.4 EXERCISES

For the following exercises, create the addition table for the given base.

1. base 5
2. base 3
3. base 16 (Hint: Use the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, and F.)
4. base 2

For the following exercises, perform the indicated base 6 operation.

5. $4_6 + 3_6$
6. $14_6 + 25_6$
7. $31_6 + 3_6$
8. $43_6 + 34_6$
9. $532_6 + 23_6$
10. $254_6 + 143_6$
11. $20_6 - 3_6$
12. $23_6 - 5_6$

For the following exercises, perform the indicated base 12 operation.

13. $5_{12} + 6_{12}$
14. $3_{12} + A_{12}$
15. $34_{12} + 7_{12}$
16. $76_{12} + B_{12}$
17. $59_{12} + 1A_{12}$
18. $A1_{12} + 36_{12}$
19. $53_{12} - 9_{12}$
20. $2B_{12} - 7_{12}$

21. Explain two ways to detect an error in arithmetic in bases other than base 10.

22. Explain the error in the following calculation: $28_{13} + 47_{13} = 75_{13}$.

23. Explain the error in the following calculation: $36_7 + 23_7 = 59_7$.

24. In base 10 addition, there are 100 addition rules plus a rule for carrying a 1. How many addition rules are there for base 6?

25. In base 10 addition, there are 100 addition rules plus a rule for carrying a 1. How many addition rules are there for base 14?

26. In base 10 addition, there are 100 addition rules plus a rule for carrying a 1. How many addition rules are there for base 2?

For the following exercises, use the addition table that you created from Exercise 4 to perform the indicated base 2 operations.

27. $101_2 + 111_2$
28. $1011_2 + 10011_2$
29. $11111_2 + 11111_2$
30. $1010101_2 + 1010101_2$

For the following exercises, use the addition table that you created from Exercise 3 to perform the indicated base 16 operations.

31. $29_{16} + 38_{16}$
32. $4D_{16} + 89_{16}$
33. $927_{16} + 438_{16}$
34. $BFA_{16} - 78E_{16}$

For the following exercises, tell how you know an error was committed without performing the operation in the given base.

35. $43_5 + 32_5 = 75_5$
36. $15_{14} + 19_{14} = 34_{14}$

4.5 Multiplication and Division in Base Systems

Figure 4.6 The processes for multiplication and division are the same for arithmetic in any bases. (credit: modification of work “NCTR Intern Claire Boyle” by Danny Tucker/U.S. Food and Drug Administration, Public Domain)

Learning Objectives

After completing this section, you should be able to:

1. Multiply and divide in bases other than 10.
2. Identify errors in multiplying and dividing in bases other than 10.

Just as in Addition and Subtraction in Base Systems, once we decide on a system for counting, we need to establish rules for combining the numbers we’re using. This includes the rules for multiplication and division. We are familiar with those operations in base 10. How do they change if we instead use a different base? A larger base? A smaller one?

In this section, we use **multiplication and division in bases** other than 10 by referencing the processes of base 10, but applied to a new base system.

Multiplication in Bases Other Than 10

Multiplication is a way of representing repeated additions, regardless of what base is being used. However, different bases have different addition rules. In order to create the multiplication tables for a base other than 10, we need to rely on addition and the addition table for the base. So let’s look at multiplication in base 6.

Multiplication still has the same meaning as it does in base 10, in that 4×6 is 4 added to itself six times, $4 \times 6 = 4 + 4 + 4 + 4 + 4 + 4$.

So, let’s apply that to base 6. It should be clear that 0 multiplied by anything, regardless of base, will give 0, and that 1 multiplied by anything, regardless of base, will be the value of “anything.”

Step 1: So, we start with the table below:

*	0	1	2	3	4	5
0	0	0	0	0	0	0

1	0	1	2	3	4	5
2	0	2	4			
3	0	3				
4	0	4				
5	0	5				

Step 2: Notice $2 \times 2 = 4$ is there. But we didn't hit a problematic number there (4 works fine in both base 10 and base 6). But what is 2×3? If we use the repeated addition concept, $2 \times 3 = 2 + 2 + 2 = 4 + 2$. According to the base 6 addition table (Table 4.4), $4 + 2 = 10$. So, we add that to our table:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	10		
3	0	3	10			
4	0	4				
5	0	5				

Step 3: Next, we need to fill in 2×4. Using repeated addition, $2 \times 4 = 2 + 2 + 2 + 2 = 10 + 2 = 12$ (if we use our base 6 addition rules). So, we add that to our table:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	10	12	
3	0	3	10			
4	0	4	12			
5	0	5				

Step 4: Finally, $2 \times 5 = 2 + 2 + 2 + 2 + 2 = 12 + 2 = 14$. And so we add that to our table:

*	0	1	2	3	4	5
0	0	0	0	0	0	0

1	0	1	2	3	4	5
2	0	2	4	10	12	14
3	0	3	10			
4	0	4	12			
5	0	5	14			

Step 5: A similar analysis will give us the remainder of the entries. Here is 4×5 demonstrated: $4 \times 5 = 4 + 4 + 4 + 4 + 4 = 12 + 12 + 4 = 24 + 4 = 32$.

This is done by using the addition rules from Addition and Subtraction in Base Systems, namely that $4 + 4 = 12$, and then applying the addition processes we've always known, but with the base 6 table (Table 4.4). In the end, our multiplication table is as follows:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	10	12	14
3	0	3	10	13	20	23
4	0	4	12	20	24	32
5	0	5	14	23	32	41

Table 4.8 Base 6 Multiplication Table

Notice anything about that bottom line? Is that similar to what happens in base 10?

To summarize the creation of a multiplication in a base other than base 10, you need the addition table of the base with which you are working. Create the table, and calculate the entries of the multiplication table by performing repeated addition in that base. The table needs to be drawn only the one time.

EXAMPLE 4.38

Creating a Multiplication Table for a Base Lower Than 10

Create the multiplication table for base 7.

Solution

Step 1: Let's apply the process demonstrated and outlined above to find the base 7 multiplication table. It should be clear that 0 multiplied by anything, regardless of base, will give 0, and that 1 multiplied by anything, regardless of base, will be the value of "anything." So, we start with the table below:

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6

2	0	2	4	6			
3	0	3	6				
4	0	4					
5	0	5					
6	0	6					

Step 2: Notice $2 \times 2 = 4$ is there. But we didn't hit a problematic number there (4 works fine in both base 10 and base 6). The same is true for 2×3 and 3×2, which equal 6. But what is 2×4? If we use the repeated addition concept, $2 \times 4 = 2 + 2 + 2 + 2 = 6 + 2$. According to the base 7 addition table in the solution for Example 4.29, $6 + 2 = 11$. So, we add that to our table:

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	**11**		
3	0	3	6				
4	0	4	**11**				
5	0	5					
6	0	6					

Step 3: Next, we need to fill in 2×5. Using repeated addition, $2 \times 5 = 2 + 2 + 2 + 2 + 2 = 11 + 2 = 13$ if we use our base 7 addition rules. So, we add that to our table:

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	**13**	
3	0	3	6				
4	0	4	11				
5	0	5	**13**				
6	0	6					

Step 4: Finally, $2 \times 6 = 2 + 2 + 2 + 2 + 2 + 2 = 13 + 2 = 15$. And so we add that to our table:

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	**15**
3	0	3	6				
4	0	4	11				
5	0	5	13				
6	0	6	**15**				

Step 5: A similar analysis will give us the remainder of the entries. Here is $4_7 \times 5_7$ demonstrated:

$$4_7 \times 5_7 = 4_7 + 4_7 + 4_7 + 4_7 + 4_7 = 11_7 + 11_7 + 4_7 = 22_7 + 4_7 = 26_7$$

This is done by using the addition rules from Addition and Subtraction in Base Systems, namely that $4_7 + 4_7 = 11_7$ and then applying the addition processes we've always known, but with the base 7 table in the solution for Example 4.29. Using those addition rules, the rest of the table is given below:

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0	4	11	15	22	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

YOUR TURN 4.38

1. Create the multiplication table for base 4.

EXAMPLE 4.39

Creating a Multiplication Table for a Base Higher Than 10

Create the multiplication table for base 12.

Solution

Let's apply the repeated addition to base 12. Here is $7_{12} \times 9_{12}$ demonstrated:

$$7_{12} \times 9_{12} = 7_{12} + 7_{12} + 7_{12} + 7_{12} + 7_{12} + 7_{12} + 7_{12} + 7_{12} + 7_{12} = 12_{12} + 12_{12} + 12_{12} + 12_{12} + 7_{12} = 48_{12} + 7_{12} = 53_{12}$$

This is done by using the addition rules from Addition and Subtraction in Base Systems, namely that $7_{12} + 7_{12} = 12_{12}$ and then applying the addition processes we've always known, but with the base 12 table in the solution for Example 4.31. Using those addition rules, the rest of the table is given below:

*	0	1	2	3	4	5	6	7	8	9	A	B
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	A	B
2	0	2	4	6	8	A	10	12	14	16	18	1A
3	0	3	6	9	10	13	16	19	20	23	26	29
4	0	4	8	10	14	18	20	24	28	30	34	38
5	0	5	A	13	18	21	26	2B	34	39	42	47
6	0	6	10	16	20	26	30	36	40	46	50	56
7	0	7	12	19	24	2B	36	41	48	53	5A	65
8	0	8	14	20	28	34	40	48	54	60	68	74
9	0	9	16	23	30	39	46	53	60	69	76	83
A	0	A	18	26	34	42	50	5A	68	76	84	92
B	0	B	1A	29	38	47	56	65	74	83	92	A1

Table 4.9 Base 12 Multiplication Table

YOUR TURN 4.39

1. Create the multiplication table for base 14.

The multiplication table in base 2 below is as minimal as the addition table in the solution for Table 4.6. Since the product of 1 with anything is itself, the following multiplication table is found.

*	0	1
0	0	0
1	0	1

Table 4.10 Base 2 Multiplication Table

As with the addition table, we can use the multiplication tables and the addition tables to perform multiplication of two numbers in bases other than base 10. The process is the same, with the same carry rules and placeholder rules.

EXAMPLE 4.40

Multiplying in a Base Lower Than 10

1. Calculate $45_6 \times 24_6$.
2. Calculate $101_2 \times 110_2$.

Solution

1. **Step 1:** Use the base 6 multiplication table (Table 4.8) and, when necessary, the base 6 addition table (Table 4.4).

   Set up this calculation using columns:

	4	5
x	2	4

Step 2: Multiply the 1s digits, 5 and 4, using the base 6 multiplication table (Table 4.8). There we see the result is 32_6. So, we enter the 2 and carry the 3.

	3	
	4	5
x	2	4
		2

Step 3: So, now we multiply the 4 and the 4, then add the 3 (just as you would do if multiplying two base 10 numbers!). $4_6 \times 4_6 = 24_6$ (from the base 6 table [Table 4.8]), then $24_6 + 3_6 = 31_6$. So, we enter the 31.

	3	
	4	5
x	2	4
3	**1**	2

Step 4: Now we move on to the 2 in the "tens" place in the bottom value. We multiply the 2_6 and the 5_6, and we get 14_6. So, we enter the 4 and carry the 1.

	1	
	4	5
×	2	4
3	1	2
	4	0

Remember, on line two, you place a 0 in that far right spot, as a placeholder.

Step 5: Next up, we multiply the 2 and the 4, and then add 1. This gives us $12_6 + 1_6 = 13_6$. We enter those on that second line.

		1	
		4	5
	x	2	4
	3	1	2
1	**3**	4	0

Step 6: Now we add down the columns.

		1	
		4	5
	x	2	4
1	3	1	2
1	3	4	0
2	0	5	2

Step 7: The 3 and the 3 add to 10 in base 6, so we enter the 0 and carry the 1. We now have the result:

$45_6 \times 24_6 = 2052_6$.

2. **Step 1:** Use the base 2 multiplication table (Table 4.10) and, when necessary, the base 2 addition table in the solution for Example 4.29. Set up this calculation using columns:

	1	0	1
x	1	1	0

Step 2: Using the pattern established above, and the processes from multiplication from base 10, we find the following:

			1	0	1
x			1	1	0
			0	0	0
		1	0	1	
	1	0	1		

Step 3: Adding down the columns results in the following:

			1	0	1
x			1	1	0
			0	0	0
		1	0	1	
	1	0	1		
	1	1	1	1	0

So, $101_2 \times 110_2 = 11110_2$.

YOUR TURN 4.40

1. Calculate $43_6 \times 52_6$.
2. Calculate $11101_2 \times 11_2$.

Summarizing the process of multiplying two numbers in different bases, the multiplication table is referenced. Using that table, the multiplication is carried out in the same manner as it is in base 10. The addition rules for the base will also be referenced when carrying a 1 or when adding the results for each digit's multiplication line.

EXAMPLE 4.41

Multiplying in a Base Higher Than 10

Calculate $3A_{12} \times 74_{12}$.

Solution

Step 1: Use the base 12 multiplication table in the solution for Example 4.39 and, when necessary, the base 12 addition table in the solution for Table 4.7. Set up this calculation using columns:

	3	A
×	7	4

Step 2: First, the 4 is multiplied by 3A, resulting in the first line.

	3	A
×	7	4
1	**3**	**4**

Step 3: Now we move on to the 7 in the "tens" place in the bottom value.

		5	
		3	A
	x	7	4
	1	3	4
2	**2**	**A**	**0**

Step 4: Now we add down the columns.

		3	A
	x	7	4
	1	3	4
2	2	A	0
2	**4**	**1**	**4**

Step 5: The 3 and the A add to 11 in base 12, so we enter the 1 and carry the 1.

We now have the result: $3A_{12} \times 74_{12} = 2414_{12}$.

YOUR TURN 4.41

1. Calculate $B3_{12} \times 47_{12}$.

Division in Bases Other Than 10

Just as with the other operations, **division in a base** other than 10, the process of division in a base other than 10 is the same as the process when working in base 10. For instance, $72 \div 9 = 8$ because, we know that $9 \times 8 = 72$. So, for many division problems, we are simply looking to the multiplication table to identify the appropriate multiplication rule.

EXAMPLE 4.42

Dividing with a Base Other Than 10

1. Calculate $14_6 \div 5_6$.
2. Calculate $5A_{12} \div 7_{12}$

Solution

1. Looking at the multiplication table for base 6 (Table 4.8), we see that $5_6 \times 2_6 = 12_6$. Using that, we know that $14_6 \div 5_6 = 2_6$.
2. Looking at the multiplication table for base 12 in the solution for Example 4.39, we see that $7_{12} \times A_{12} = 5A_{12}$. Using that, we know that $5A_{12} \div 7_{12} = A_{12}$.

YOUR TURN 4.42

1. Calculate $10_6 \div 3_6$

2. Calculate $50_{12} \div A_{12}$.

Errors in Multiplying and Dividing in Bases Other Than Base 10

The types of errors encountered when multiplying and dividing in bases other than base 10 are the same as when adding and subtracting. They often involve applying base 10 rules or symbols to an arithmetic problem in a base other than base 10. The first type of error is using a symbol that is not in the symbol set for the base.

EXAMPLE 4.43

Identifying an Illegal Symbol in a Base Other Than Base 10

Explain the error in the following calculation, and determine the correct answer:

$$4_6 \times 2_6 = 8_6$$

Solution

Since the problem is in base 6, the symbol set available is 0, 1, 2, 3, 4, and 5. The 8 in the answer is clearly not a legal symbol for base 6. Looking back to the base 6 multiplication table (Table 4.8), we see that $4_6 \times 2_6 = 12_6$.

YOUR TURN 4.43

1. Explain the error in the following calculation and determine the correct answer: $13_4 \times 21_4 = 54_4$

The second type of error is using a base 10 rule when the numbers are not in base 10. For instance, in base 17, $6_{17} \times 9_{17} = 54_{17}$ would be incorrect, even though in base 10, $6 \times 9 = 54$. That rule doesn't apply in base 17.

EXAMPLE 4.44

Identifying an Error in Arithmetic in a Base Other Than Base 10

Explain the error in the following calculation. Determine the correct answer:

$$18_{12} \times 7_{12} = 126_{12}$$

Solution

If this problem was a base 10 problem, this would be the correct answer. However, in base 12, $8_{12} \times 7_{12}$ is not 56, but is instead 48. To correct this error, carefully use the multiplication table for base 12 (Table 4.9). If properly used, the correct answer would be $18_{12} \times 7_{12} = B8_{12}$.

YOUR TURN 4.44

1. Explain the error in the following calculation. Determine the correct answer: $49_{14} \times 9_{14} = 441_{14}$

Check Your Understanding

28. To create the multiplication table for a given base, what should be used?
29. What are the differences between multiplying in base 10 and multiplying in a different base?
30. When dividing in a base other than base 10, what table is referenced?
31. Compute $24_6 \times 53_6$.
32. Compute $32_{14} \div 4_{14}$.
33. How do you know an error has occurred in a base 5 multiplication question if the answer obtained was 28_5?
34. What are two common ways to determine an error is committed when computing in s base other than base 10?

SECTION 4.5 EXERCISES

For the following exercises, create the multiplication table for the given base.

1. base 5
2. base 3
3. base 16 (Hint: Use the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F.)
4. base 2

For the following exercises, perform the indicated base 6 operation.

5. $4_6 \times 3_6$
6. $14_6 \times 5_6$
7. $31_6 \times 3_6$
8. $43_6 \times 34_6$
9. $532_6 \times 23_6$
10. $254_6 \times 143_6$
11. $20_6 \div 3_6$
12. $23_5 \div 5_6$

For the following exercises, perform the indicated base 12 operation.

13. $5_{12} \times 6_{12}$
14. $3_{12} \times \mathrm{A}_{12}$
15. $34_{12} \times 7_{12}$
16. $76_{12} \times \mathrm{B}_{12}$
17. $59_{12} \times 1\mathrm{A}_{12}$
18. $\mathrm{A}1_{12} \times 36_{12}$
19. $53_{12} \div 9_{12}$
20. $2\mathrm{B}_{12} \div 7_{12}$

21. Explain two ways to detect an error in arithmetic in bases other than base 10.

22. Explain the error in the following calculation: $15_{12} \times 7_{12} = 105_{12}$

23. Explain the error in the following calculation: $45_8 \times 6_8 = 94_8$.

24. In base 10 multiplication, there are 100 multiplication rules plus a rule for carrying a number. How many multiplication rules are there for base 6?

25. In base 10 multiplication, there are 100 multiplication rules plus a rule for carrying a number. How many multiplication rules are there for base 14?

26. In base 10 multiplication, there are 100 multiplication rules plus a rule for carrying a number. How many multiplication rules are there for base 2?

27. Consider the answers from Exercise 24 and Exercise 26. Which base do you think would be more efficient: base 10, base 6, or base 2?

For the following exercises, use the multiplication table that you created from Exercise 4 to perform the indicated base 2 operations.

28. $11_2 \times 11_2$
29. $101_2 \times 10_2$
30. $11011_2 \times 1011_2$
31. $1011_2 \times 1010101_2$

32. Convert 1011_2 and 1010101_2 to base 10. Then multiply those base 10 numbers. Next, convert the answer you got for Exercise 31 to base 10. Do these numbers match?

For the following exercises, use the multiplication table that you created from Exercise 3 to perform the indicated base 16 operations.

33. $19_{16} \times 5_{16}$
34. $3\mathrm{B}_{16} \times \mathrm{A}_{16}$
35. $25_{16} \times 16_{16}$
36. $\mathrm{C}_{16} \div 4_{16}$

For the following exercises, explain how you know an error was committed without performing the operation in the given base.

37. $43_5 \times 32_5 = 126_5$

38. $5_{14} \times 9_{14} = 45_{14}$

Chapter Summary

Key Terms

4.1 Hindu-Arabic Positional System

- numeral
- number
- exponential expression
- base
- exponent
- place value
- base 10 system
- Hindu-Arabic numeration system
- expanded form

4.2 Early Numeration Systems

- additive system of numbers
- positional system of numbers
- Babylonian system of numbers
- Mayan system of numbers
- Roman system of numbers

4.3 Converting with Base Systems

- base 10
- remainder
- dividend
- divisor
- quotient

Key Concepts

4.1 Hindu-Arabic Positional System

- Exponents are used to represent repeated multiplication of a base.
- In arithmetic, exponents are computed before multiplication, division, addition, and subtraction. Computing an exponent is done by multiplying the base by itself the number of times equal to the exponent.
- The system of numbers currently used is the Hindu-Arabic system. Digits in this system take on values based on their place in the number. The place values are determined by multiplying the digit by 10 raised to the appropriate power.
- The expanded form of a Hindu-Arabic number is the sum of each digit times 10 raised to the exponent for that place value.

4.2 Early Numeration Systems

- Historically, there have been many systems for numbering. One system is an additive system, in which symbols are repeated to express larger numbers. Another system is a positional system, in which the digits and their positions determine the quantity being represented.
- The Babylonian system was a combination of a positional and additive system. It used 60 as its base. Using that in the positional system makes it possible to convert between Babylonian and Hindu-Arabic numbers.
- The Mayan system was a combination of a positional and additive system. It used 20 as its base. Using that in the positional system makes it possible to convert between Mayan and Hindu-Arabic numbers.
- The Roman system was an additive system. Knowing what each symbol represents makes it possible to convert between Roman and Hindu-Arabic numbers.

4.3 Converting with Base Systems

- The system we use is the base 10 system. Base 10 is not the only base that can be used. To use another base, one could start with a list of numbers in that base.
- To indicate that a number is written in a base other than 10, a subscript is appended to the end of the number. That subscript indicates the base for the number.
- Numbers written in a base smaller than 10 use the same symbols as base 10. However, when using bases larger

than 10, the symbols A, B, C, ... are used to represent digits larger than 9.
- To convert from a number written in a base other than 10 into a base 10 number, the number is written in expanded form and then that expression is computed.
- To convert a number from base 10 into another base, the base 10 number is repeatedly divided by the new base. The remainders when performing these divisions become the digits for the number in the new base.
- Common errors can be detected when performing base conversions.

4.4 Addition and Subtraction in Base Systems

- Addition tables for bases other than 10 can be built using the same processes that are used in base 10, including using a number line.
- Addition in bases other than base 10 use the same processes as addition in base 10, but use the addition table for that base.
- Subtraction in bases other than base 10 use the same processes as subtraction in base 10, but use the addition table for that base.

4.5 Multiplication and Division in Base Systems

- Multiplication tables for bases other than 10 can be built using the same processes that are used in base 10, including using repeated addition and the addition table for the base.
- Multiplication in bases other than base 10 use the same processes as multiplication in base 10, but use the multiplication table for that base.
- Basic division in bases other than base 10 use the same processes as basic division in base 10, where the missing factor process is used.

Videos

4.1 Hindu-Arabic Positional System

- Exponential Notation (https://openstax.org/r/Exponential_Notation)

4.2 Early Numeration Systems

- Converting Between Babylonian and Hindu-Arabic numbers (https://openstax.org/r/Babylonian_to_Hindu-Arabic_Numbers)
- Converting Mayan Numbers to Hindu-Arabic Numbers (https://openstax.org /r/Mayan_to_Hindu-Arabic_Numbers)
- Converting From Roman Numbers to Hindu-Arabic Numbers (https://openstax.org/r/Roman_to_Hindu-Arabic_Numbers)
- Converting From Hindu-Arabic Numbers to Roman Numbers (https://openstax.org/r/Hindu-Arabic_to_Roman_Numbers)

4.3 Converting with Base Systems

- Convert Base 7 to Base 10 (https://openstax.org/r/Convert_Base_7_to_Base_10)
- Converting from Base 10 to Another Base (https://openstax.org/r/Base_10_to_Another_Base)

Projects

Additive Systems

Go online. Google "additive number systems."What system comes up?

- Describe the additive system you found.

Using Google, identify three more additive systems of numbers.

- Compare and contrast the systems you found. For instance, how many times can a symbol be used before a new symbol is used.
- Identify three situations where additive systems are still used.

Computers and Bases

Use Google to determine what base computers use.

Were other bases attempted for use in computers?

Determine why the base used in computers is appropriate.

Determine how the base used in computers is related to the circuitry in computers.

Determine how Boolean logic and the base used in computers are related, and might be identical.

There is research into using quibits in computers. Find out what quibits are and how can they improve computing speed.

Cultures Using Base Systems Other Than 10

Using Google, find three cultures, other than Babylonian or Mayan, that use base systems other than 10.

- Tell what base is used for each system.
- If possible, determine why the culture used that base system.
- Choose one of those systems. Explain that base system. Be sure to address whether the system is additive, place-value based, a blend of the two, and if it employs a zero.

History of Zero

Using any resources available to you, determine the history of 0 in at least three different numbering systems. Address at least when and why such a development occurred and why a 0 is vital to the use of a positional system.

Numbering Systems from Other Global Regions

Using any resources available to you, find at least three numbering systems from sub-Saharan Africa, Australia, China, or the Pacific Islands. Explore if they are positional or additive systems (or combinations!), the terminology of the system, if they used a 0, and what base they employed (if positional).

Chapter Review

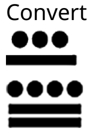
Hindu-Arabic Positional System

1. What is the base of 5^7?
2. What is the exponent of 5^7?
3. Compute $4^5 + 3^2$.
4. Convert the Hindu-Arabic number into expanded form: 4,201.
5. Convert the expression to a Hindu-Arabic numeral: $6 \times 10^3 + 3 \times 10^2 + 7 \times 10^1 + 5 \times 10^0$.

Early Numeration Systems

6. Which systems—Hindu-Arabic, Roman, Mayan, or Babylonian—are additive systems?
7. Which systems—Hindu-Arabic, Roman, Mayan, or Babylonian—are positional systems?
8. Which systems—Hindu-Arabic, Roman, Mayan, or Babylonian—use a 0?
9. In the Babylonian system, what are the place values based on?
10. In the Mayan system, what are the place values based on?
11. Convert the Babylonian numeral to a Hindu-Arabic numeral.

12. Convert the Mayan numeral to a Hindu-Arabic numeral.



13. Convert the Roman numeral MMCDXLVII into a Hindu-Arabic numeral.
14. Convert the Hindu-Arabic numeral 394 to a Roman numeral.

Converting with Base Systems

15. List the numbers from 0 to 100 in base 5.
16. In base 8, what is the place value of the 3 in the number 638_8?
17. How many symbols are needed for a base 17 system?
18. What does it mean for a number to be in base 6?
19. What symbols are used in a base 12 system?
20. When converting from a base 10 number to a base 2 number, would the number of digits decrease?
21. Convert 311_5 to base 10.
22. Convert 45_{12} to base 10.
23. Convert 1001_2 to base 10.
24. Convert 459 to base 8.
25. Convert 1198 to base 12.
26. Convert 38 to base 2.
27. When converting from base 10 to base 4, the result obtained was 142_4. How can you tell an error was made?

Addition and Subtraction in Base Systems

28. Create the addition table for base 5.
29. How many addition rules are there for a base 7 system?
30. Calculate $34_5 + 44_5$.
31. Calculate $A7_{12} + 88_{12}$.

32. Calculate $541_6 - 233_6$.

33. Calculate $5B_{12} - 1A_{12}$.

34. When adding in base 8, the result 911_8 is found. How do we know a mistake was made?

Multiplication and Division in Base Systems

35. What is the process for creating the multiplication table for a base other than 10?

36. Create the multiplication table for base 7.

37. How many multiplication rules are there in a base 3 system?

38. Calculate $14_7 \times 25_7$.

39. Calculate $67_{12} \times 3B_{12}$.

40. Calculate $42_7 \div 5_7$.

41. Calculate $38_{12} \div B_{12}$.

42. When multiplying $14_{12} \times 10_{12}$, the result 140 is found. How do we know a mistake was made?

Chapter Test

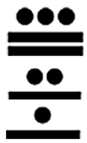
1. Expand the Hindu-Arabic numeral 5,789.

2. Evaluate the expression $4 \times 7^3 + 5 \times 2^5$.

3. Rewrite $6 \times 10^5 + 0 \times 10^4 + 8 \times 10^3 + 0 \times 10^2 + 1 \times 10^1 + 9 \times 10^0$ in Hindu-Arabic form.

4. Convert the Babylonian numeral to a Hindu-Arabic numeral.

5. Convert the Mayan number to a Hindu Arabic numeral.



6. What base system did the Babylonians use?

7. Which system—Roman, Babylonian, Mayan—used place values?

8. Convert the Roman numeral MDXLVII to a Hindu-Arabic numeral.

9. How many symbols are needed for a base 9 system?

10. For a system in a base larger than 10, what symbols are used as digits representing more than 10?

11. Convert 132_8 to a base 10 number.

12. List the numbers in base 4 up to 100_4.

13. Convert 74 to a base 12 number.

14. Create the addition table for base 4.

15. Calculate $314_6 + 453_6$.

16. Calculate $4B1_{12} - 2A6_{12}$.

17. When calculating $23_6 + 53_6$, a student obtains 76_6 . How do you know an error was made?

18. Create the multiplication table for base 4.

19. Calculate $323_4 \times 132_4$.

20. Calculate $21_4 \div 3_4$.

5 ALGEBRA

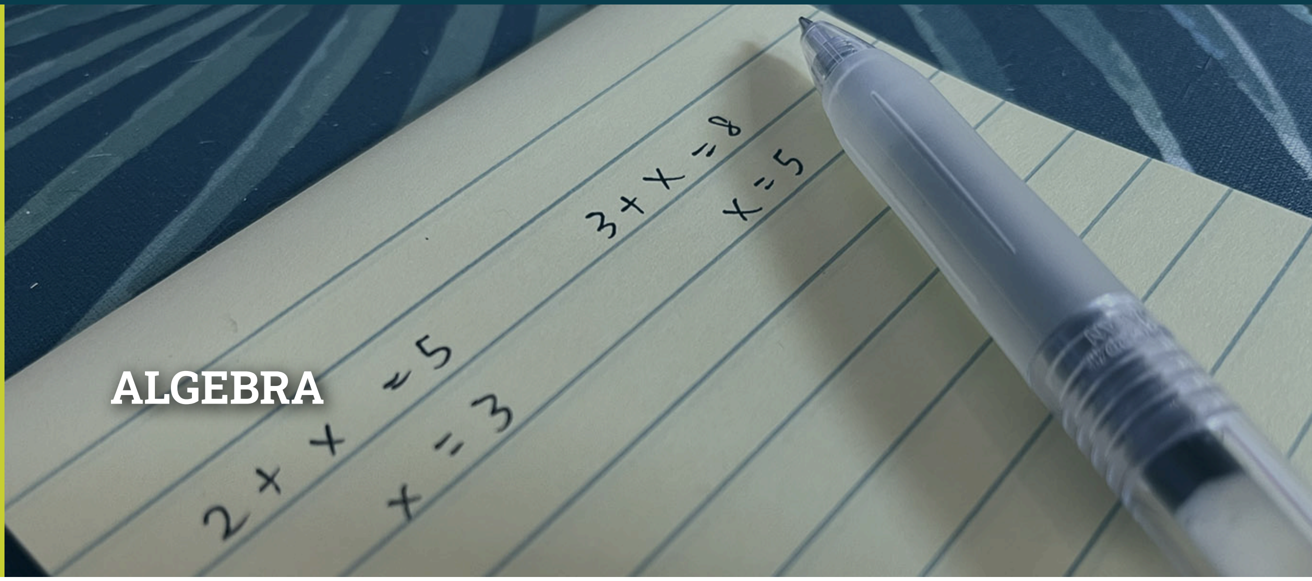


Figure 5.1 In these algebraic equations, the x represents different numbers. (credit: Larissa Chu, CC BY 4.0)

Chapter Outline

Introduction

The jump from arithmetic to algebra can be a difficult one for many students. Many students struggle with the idea that mathematics can include situations that aren't static and do change. In elementary arithmetic, a situation such as:

$5 + 3 =$ _____

is a static situation and will yield the answer of 8 every time. However, a situation such as: $5x + 3 =$ _____

can yield many different answers because the answer depends on what amount (number) that x represents. Since the value of x can vary (represent different values), it is known as a variable.

Algebra is useful to better model real life situations. In the first equation shown, $5 + 3 =$ _____ can only model situations where you add those two numbers together. For example, if your uncle gives you five dollars and your aunt gives you three dollars, then you will always receive eight dollars. The second equation $5x + 3 =$ _____ can model more complex situations. For example, you wish to buy a game that costs $38 but you only have three dollars. Your uncle will pay you five dollars an hour to work for him. If you've worked five hours, have you earned enough money? If not, how many hours will you have to work?

Algebra and algebraic thinking open up a world of possibilities that arithmetic alone cannot do.

5.1 Algebraic Expressions

Figure 5.2 Two college graduates! (credit: modification of work UC Davis College of Engineering/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Convert between written and symbolic algebraic expressions and equations.
2. Simplify and evaluate algebraic expressions.
3. Add and subtract algebraic expressions.
4. Multiply and divide algebraic expressions.

Algebraic expressions are the building blocks of algebra. While a numerical expression (also known as an arithmetic expression) like $5 + 3$ can represent only a single number, an algebraic expression such as $5x + 3$ can represent many different numbers. This section will introduce you to algebraic expressions, how to create them, simplify them, and perform arithmetic operations on them.

Algebraic Expressions and Equations

Xavier and Yasenia have the same birthday, but they were born in different years. This year Xavier is 20 years old and Yasenia is 23, so Yasenia is three years older than Xavier. When Xavier was 15, Yasenia was 18. When Xavier will be 33, Yasenia will be 36. No matter what Xavier's age is, Yasenia's age will always be 3 years more.

In the language of algebra, we say that Xavier's age and Yasenia's age are variable and the 3 is a constant. The ages change, or vary, so age is a **variable**. The 3 years between them always stays the same or has the same value, so the age difference is the **constant**. In algebra, letters of the alphabet are used to represent variables. The letters most often used for variables are x, y, z, a, b, and c. Suppose we call Xavier's age x. Then we could use $x + 3$ to represent Yasenia's age, as shown in the table below.

Xavier's Age	Yasenia's Age
15	18
20	23
33	36
x	$x + 3$

To write algebraically, we need some symbols as well as numbers and variables. The symbols for the four basic arithmetic operations: addition, subtraction, multiplication, and division are summarized in Table 5.1, along with words we use for the operations and the result.

Operation	Notation	Say:	The result is...
Addition	$a + b$	a plus b	The sum of a and b
Subtraction	$a - b$	a minus b	The difference of a and b
Multiplication	$a \cdot b$, $(a)(b)$, $(a)b$, $a(b)$, ab, ba	a times b	The product of a and b
Division	$a \div b$, a/b	a divided by b	The quotient of a and b

Table 5.1 Symbols for Operations

In algebra, the cross symbol (x) is normally not used to show multiplication because that symbol could cause confusion. For example, does $3xy$ mean $3 \times y$ (three times y) or $3 \bullet x \bullet y$ (three times x times y)? To make it clear, use ● or parentheses for multiplication.

We perform these operations on two numbers. When translating from symbolic form to words, or from words to symbolic form, pay attention to the words *of* or *and* to help you find the numbers.

- The *sum* ***of*** 5 ***and*** 3 means add 5 plus 3, which we write as $5 + 3$.
- The *difference* ***of*** 9 ***and*** 2 means subtract 9 minus 2, which we write as $9 - 2$.
- The *product* ***of*** 4 ***and*** 8 means multiply 4 times 8, which we can write as $4 \bullet 8$.
- The *quotient* ***of*** 20 ***and*** 5 means divide 20 by 5, which we can write as $20 \div 5$.

EXAMPLE 5.1

Translating from Algebra to Words

Translate the following algebraic expressions from algebra into words.

1. $12 + 14$
2. $(30)(5)$
3. $64 \div 8$
4. $x - y$

Solution

1. According to Table 5.1, this could be translated as 12 plus 14 OR the sum of 12 and 14.
2. According to Table 5.1, this could be translated as 30 times 5 OR the product of 30 and 5.
3. According to Table 5.1, this could be translated as 64 divided by 8 OR the quotient of 64 and 8.
4. According to Table 5.1, this could be translated as x minus y OR the difference of x and y.

YOUR TURN 5.1

Translate the following algebraic expressions from algebra into words.

1. $18 + 11$
2. $(27)(9)$
3. $84 \div 7$
4. $p - q$

EXAMPLE 5.2

Translating from Words to Algebra

Translate the following phrases from words into algebraic expressions.

1. The difference of 47 and 19

2. 72 divided by 9
3. The sum of m and n
4. 13 times 7

Solution

1. According to Table 5.1, these words could be translated as $47 - 19$.
2. According to Table 5.1, these words could be translated as $72 \div 9$.
3. According to Table 5.1, these words could be translated as $m + n$.
4. According to Table 5.1, these words could be translated as $(13)(7)$.

YOUR TURN 5.2

Translate the following phrases from words into algebraic expressions.

1. 43 plus 67
2. The product of 45 and 3
3. The quotient of 45 and 3
4. 89 minus 42

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. “Running very fast” is a phrase, but “The football player was running very fast” is a sentence. A sentence has a subject and a verb. In algebra, we have **expressions** and equations. Example 5.1 and Example 5.2 used expressions. An expression is like an English phrase. Notice that the English phrases do not form a complete sentence because the phrase does not have a verb. The following table has examples of expressions, which are numbers, variables, or combinations of numbers and variables using operation symbols.

Expression	Words	English Phrase
$3 + 5$	3 plus 5	The sum of three and five
$n - 1$	n minus one	The difference of n and one
$6 \bullet 7$	6 times 7	The product of six and seven
$x \div y$	x divided by y	The quotient of x and y

EXAMPLE 5.3

Translating from an English Phrase to an Expression

Translate the following phrases from words into algebraic expressions.

1. Seven more than a number n.
2. A number n times itself.
3. Six times a number n, plus two more.
4. The cost of postage is a flat rate of 10 cents for every parcel, plus 34 cents per ounce x.

Solution

1. $n + 7$
2. $n \bullet n$ or n^2
3. $6n + 2$
4. $10 + 34x$

YOUR TURN 5.3

Translate the following phrases from words into algebraic expressions.

1. Twenty less than a number n. (Hint: you have a number n and you want 20 less than it.)
2. Add two to a number n, then multiply it by six.
3. A number n to the third power minus five.
4. A plumber charges \$60 per hour h, plus a \$40 flat fee for every job.

An **equation** is two expressions linked with an **equal sign** (the symbol =). When two quantities have the same value, we say they are equal and connect them with an equal sign. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb. So, $a = b$ is read "a is equal to b." The following table has some examples of equations.

Equation	English Sentence
$3 + 5 = 8$	The sum of three and five is equal to eight.
$n - 1 = 14$	n minus one equals fourteen.
$6 \bullet 7 = 42$	The product of six and seven is equal to forty-two.
$x = 53$	x is equal to fifty-three.
$y + 9 = 2y - 3$	y plus nine is equal to two times y minus three.

EXAMPLE 5.4

Translating from an English Sentence to an Equation

Translate the following sentences from words into algebraic equations.

1. Two times x is 6.
2. n plus 2 is equal to n times 3.
3. The quotient of 35 and 7 is 5.
4. Sixty-seven minus x is 56.

Solution

1. $2x = 6$
2. $n + 2 = 3n$
3. $35 \div 7 = 5$
4. $67 - x = 56$

YOUR TURN 5.4

Translate the following sentences from words into algebraic equations.

1. Five times y is 50.
2. Half of a number n is 30.
3. The difference of three times a number n and 7 is 2.
4. Two times x plus 7 is 21.

WHO KNEW?

The Use of Variables

French philosopher and mathematician René Descartes (1596–1650) is usually given credit for the use of the letters x, y, and z to represent unknown quantities in algebra. He introduced these ideas in his publication of *La Geometrie*, which was printed in 1637. In this publication, he also used the letters a, b, and c to represent known quantities. There is a (possibly fictitious) story that, when the book was being printed for the first time, the printer began to run short of the last three letters of the alphabet. So the printer asked Descartes if it mattered which of x, y, or z were used for the mathematical equations in the book. Descartes decided it made no difference to him; so the printer decided to use x predominantly for the mathematics in the book, because the letters y and z would occur more often in the body of the text (written in French) than the letter x would! This might explain why the letter x is still used today as the most common variable to represent unknown quantities in algebra.

Simplifying and Evaluating Algebraic Expressions

To **simplify an expression** means to do all the math possible. For example, to simplify $4 \bullet 2 + 1$ we would first multiply $4 \bullet 2$ to get 8 and then add 1 to get 9. We have introduced most of the symbols and notation used in algebra, but now we need to clarify the order of operations. Otherwise, expressions may have different meanings, and they may result in different values. Consider $2 + 7 \bullet 3$. Do you add first or multiply first? Do you get different answers?

Add first: $9 \bullet 3 = 27$	Multiply first: $2 + 21 = 23$	Which one is correct?

Early on, mathematicians realized the need to establish some guidelines when performing arithmetic operations to ensure that everyone would get the same answer. Those guidelines are called the order of operations and are listed in the table below.

Step 1: Parentheses and Other Grouping Symbols	Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
Step 2: Exponents	Simplify all expressions with exponents.
Step 3: Multiplication and Division	Perform all multiplication and division in order from left to right. These operations have equal priority.
Step 4: Addition and Subtraction	Perform all addition and subtraction in order from left to right. These operations have equal priority.

You may have heard about Please Excuse My Dear Aunt Sally or PEMDAS. Be careful to notice in Steps 3 and 4 in the table above that multiplication and division, as well as addition and subtraction, happen in order from LEFT to RIGHT. It is possible, for example, to have PEDMAS or PEMDSA. The PEMDAS trick can be misleading if not fully understood!

EXAMPLE 5.5

Making a Numerical Equation True Using the Order of Operations

Use parentheses to make the following statements true.

1. $17 - 10 + 3 = 10$
2. $2 \bullet 26 - 7 = 38$
3. $8 + 12 \div 5 - 3 = 14$
4. $5 + 2^3 \bullet 7 = 91$

Solution

1. Add the parentheses around the $17 - 10$. Then you have $(17 - 10) + 3 = 7 + 3 = 10$.
2. Add the parentheses around the $26 - 7$. Then you have $2 \bullet (26 - 7) = 2 \bullet 19 = 38$.
3. Add the parentheses around the $5 - 3$. Then you have $8 + 12 \div (5 - 3) = 8 + 12 \div 2 = 8 + 6 = 14$.
4. Add the parentheses around the $5 + 2^3$. Then you have $\left(5 + 2^3\right) \bullet 7 = (5 + 8) \bullet 7 = 13 \bullet 7 = 91$.

YOUR TURN 5.5

Use parentheses and the order of operations to make each equation true.

1. $24 - 17 - 6 = 13$
2. $3 \bullet 6 + 13 = 31$
3. $12 - 6 \div 5 - 3 = 3$
4. $5 \bullet 3^2 + 5 = 70$

In the last example, we simplified expressions using the order of operations. Now we'll evaluate some expressions—again following the order of operations. To evaluate an expression means to find the value of the expression when the variable is replaced by a given number.

EXAMPLE 5.6

Evaluating and Simplifying an Expression

1. Evaluate $3x + 5$ when $x = 2$.
2. Evaluate $x^2 + 3x + 1$ when $x = 2$.

Solution

1. To evaluate, let $x = 2$ in the expression, and then simplify: $3(2) + 5 = 6 + 5 = 11$.
2. To evaluate, let $x = 2$ in the expression, and then simplify: $2^2 + 3(2) + 1 = 4 + 6 + 1 = 11$.

YOUR TURN 5.6

1. Evaluate $5x - 6$ when $x = 3$.
2. Evaluate $x^2 - 6x + 3$ when $x = 3$.

Operations of Algebraic Expressions

Algebraic expressions are made up of **terms**. A term is a constant or the product of a constant and one or more variables. Examples of terms are 7, y, $5x^2$, $9a$, and b^5. The constant that multiplies the variable is called the **coefficient**. Think of the coefficient as the number in front of the variable. Consider the algebraic expressions $5x^2$, which has a coefficient of 5, and $9a$, which has a coefficient of 9. If there is no number listed in front of the variable, then the coefficient is 1 since $x = 1 \bullet x$.

Some terms share common traits. When two terms are constants or have the same variable and exponent, we say they are **like terms**. If there are like terms in an expression, you can simplify the expression by combining the like terms. We add the coefficients and keep the same variable.

EXAMPLE 5.7

Adding Algebraic Expressions

Add $\left(x^2 + 4x - 9\right) + \left(3x^2 - x + 12\right)$.

Solution

Step 1: Add the terms in any order and get the same result (think: $2 + 3 = 3 + 2$) and drop the parentheses:

$$x^2 + 4x - 9 + 3x^2 - x + 12$$

Step 2: Group like terms together:

$$x^2 + 3x^2 + 4x - x - 9 + 12$$

Step 3: Combine the like terms:

$$4x^2 + 3x + 3$$

YOUR TURN 5.7

1. Add $\left(2x^2 - 4x + 5\right) + \left(3x^2 + x - 12\right)$.

EXAMPLE 5.8

Subtracting Algebraic Expressions

Subtract $\left(5x^2 + 4x - 9\right) - \left(3x^2 - x + 12\right)$.

Solution

Step 1: Distribute the negative inside the parentheses (think: $2 - (3 - 4) = 2 - 3 + 4 = -1 + 4 = 3$, which is the correct answer). You cannot just drop the parentheses (for example, $2 - 3 - 4 = -1 - 4 = -5$, which is not correct as we have already verified the answer is 3):

$$5x^2 + 4x - 9 - 3x^2 + x - 12$$

Step 2: Group like terms together:

$$5x^2 - 3x^2 + 4x + x - 9 - 12$$

Step 3: Combine the like terms:

$$2x^2 + x - 21$$

YOUR TURN 5.8

1. Subtract $\left(x^2 - 4x + 8\right) - \left(x^2 + 5x + 12\right)$.

Before looking at multiplying algebraic expressions we look at the **Distributive Property**, which says that to multiply a sum, first you multiply each term in the sum and then you add the products. For example, $5\,(4 + 3) = 5\,(4) + 5\,(3) = 20 + 15 = 35$ can also be solved as $5\,(4 + 3) = 5\,(7) = 35$. If we use a variable, then $5\,(x + 3) = 5x + 15$.

We can extended this example to $(5 + 2)\,(4 + 3) = (5)\,(4) + (5)\,(3) + (2)\,(4) + (2)\,(3) = 20 + 15 + 8 + 6 = 49$, which can also be solved as $(5 + 2)\,(4 + 3) = (7)\,(7) = 49$. If we use variables, then $(x + 5)\,(x + 4) = (x)\,(x) + (x)\,(4) + (5)\,(x) + (5)\,(4) = x^2 + 4x + 5x + 20 = x^2 + 9x + 20$.

FORMULA

Distributive Property: $a\,(b + c) = ab + ac$

EXAMPLE 5.9

Simplifying an Expression Using the Order of Operations

Simplify each expression.

1. $(x - 3)\,5$
2. $(-3)\,(x + y - 2)$
3. $5^2\,(7 + 3)\,(x)$
4. $4 + x \bullet 5$
5. $(4 + x) \bullet 5$

Solution

1. $5x - 3 \bullet 5 = 5x - 15$
2. $(-3) \bullet x + (-3) \bullet y - (-3) \bullet 2 = -3x - 3y + 6$
3. $25\,(7 + 3)\,(x) = 25\,(10)\,(x) = 250x$
4. $4 + 5x$
5. $(4) \bullet (5) + (x) \bullet (5) = 20 + 5x$

YOUR TURN 5.9

Simplify each expression.

1. $2\,(y + 5)$
2. $(-2)\,(a + b - 4)$
3. $4^2\,(47 - 40 + x)$
4. $(18 \div 3)\,(x + 7 - 4)$
5. $2\,(3a + 5) + (-3)\,(a + 2)$

EXAMPLE 5.10

Multiplying Algebraic Expressions

Multiply $(4x - 9)\,(x + 2)$.

Solution

Step 1: Use the Distributive Property:

$$(4x)\,(x) + (4x)\,(2) - (9)\,(x) - (9)\,(2)$$

Step 2: Multiply:

$$4x^2 + 8x - 9x - 18$$

Step 3: Combine the like terms:

$$4x^2 - x - 18$$

YOUR TURN 5.10

1. Multiply $(x - 4)\,(2x - 3)$.

You may have heard the term FOIL which stands for: First, Outer, Inner, Last. FOIL essentially describes a way to use the Distributive Property if you multiply a two-term expression by another two-term expression, but FOIL only works in that specific situation. For example, suppose you have a two-term expression multiplied by a three-term expression, such as $(x + 2)\,(x + y - 5)$. What terms qualify as inner terms and what terms qualify as outer terms? In *this particular situation, FOIL cannot possibly work; the multiplication of* $(x+2)(x+y-5)$ *should yield six terms, where FOIL is designed to only give you four! The Distributive Property works regardless of how many terms there are. FOIL can be misleading and applied inappropriately if not fully understood!*

EXAMPLE 5.11

Dividing Algebraic Expressions

Divide $\left(8x^2+4x-16\right)\div(4x)$.

Solution

Divide EACH term by $4x$:

$$\left(8x^2\div 4x\right)+(4x\div 4x)-(16\div 4x)=2x+1-\frac{4}{x}$$

YOUR TURN 5.11

1. Divide $\left(16x^2+4x-8\right)\div(4)$.

Be careful how you divide! Sometimes students incorrectly divide only one term on top by the bottom term. For example, $\frac{8x^2+6x-3}{2x}$ *might turn into* $4x+3x-3=7x-3$ *if done incorrectly. When we divide expressions, EACH term is divided by the divisor. So,* $\frac{8x^2+6x-3}{2x}=\frac{8x^2}{2x}+\frac{6x}{2x}-\frac{3}{2x}=4x+3-\frac{3}{2x}$*. If you forget, it is always a good idea to check these rules by creating an example using numerical expressions. For example,* $\frac{9+6+3}{3}=\frac{18}{3}=6$*. Dividing each term on top by 3 would yield* $\frac{9+6+3}{3}=\frac{9}{3}+\frac{6}{3}+\frac{3}{3}=3+2+1=6$*, which is the correct answer. However, if you just divided the 9 on top by the 3 on the bottom, getting* $\frac{9+6+3}{3}=3+6+3=12$*, this does not result in the correct answer.*

PEOPLE IN MATHEMATICS

Al-Khwarizmi

Figure 5.3 Al-Khwarizmi

Abu Ja'far Muhammad ibn Musa Al-Khwarizmi was born around 780 AD, probably in or around the region of Khwarizm, which is now part of modern-day Uzbekistan. For most of his adult life, he worked as a scholar at the House of Wisdom in Baghdad, Iraq. He wrote many mathematical works during his life, but is probably most famous for his book *Al-kitab al-muhtasar fi hisab al-jabr w'al'muqabalah*, which translates to *The Condensed Book on the Calculation of al-Jabr* (*completion*) *and al'muqabalah* (*balancing*). The word *al-jabr* would eventually become the word we use to describe the topic that he was writing about in this book: *algebra*. From another book of his, with the Latin title *Algoritmi de numero Indorum* (*Al-Khwarizmi on the Hindu Art of Reckoning*), our word *algorithm* is derived. In addition to writing on mathematics, Al-Khwarizmi wrote works on astronomy, geography, the sundial, and the calendar.

In 2012, Andrew Hacker wrote an opinion piece in the *New York Times Magazine* suggesting that teaching algebra in high school was a waste of time. Keith Devlin, a British mathematician, was asked to comment on Hacker's article by his students in his Stanford University Continuing Studies course "Mathematics: Making the Invisible Visible" (https://openstax.org/r/Making_the_Invisible_Visible) on iTunes University. Devlin concludes that Hacker was displaying his ignorance of what algebra is.

VIDEO

Q&A: Why We Teach Algebra (https://openstax.org/r/Teach_Algebra)

Check Your Understanding

1. Juliette is 2 inches taller than her friend Vivian. Which algebraic equations represent their height? Use J for Juliette's height and V for Vivian's height.
   $J = V + 2$
   $V = J - 2$
   $J + 2 = V$
   $J = V - 2$

2. Which options represent algebraic expressions?

$2x^2 + 3x - 1 = 0$
$5x + 8$
$2n + 3m$
$5x - 7 = 3x + 1$

3. Which expression equals $10x$?
$(8x + 12x) \div 4x - 2x$
$8x + (12x \div 4x) - 2x$
$8x + 12x \div (4x - 2x)$
$(8x + 12x) \div (4x - 2x)$

4. Using the expression $3x^2 - 7x + 2$, when a certain number is put in for x, the result is 50. What is the value of x?
-2
-3
2
3

5. Which expression equals $(x - y)(x - y)$? Hint: Use the Distributive Property.
$x^2 - y^2$
$x^2 + y^2$
$x^2 - 2xy - y^2$
$x^2 - 2xy + y^2$

6. Given the expression $9x^3 + 3x^2 - 6x$, the Distributive Property allows it to be rewritten as:
$3x\left(3x^2 + x - 2\right)$
$3x^2 + x - 2$
$27x^5 - 54x^4$
$27x^6 - 54x^3$

7. Given the two algebraic expressions $(x + 2)$ and $(x + y - 5)$, the solution is $x^2 + xy - 3x + 2y - 10$. What mathematical operation was performed on the two algebraic expressions?

8. Given the two algebraic expressions $8x^2 - 9x + 6$ and $6x$, the solution is $3x - 1.5 + \frac{1}{x}$. What mathematical operation was performed on the two algebraic expressions?

SECTION 5.1 EXERCISES

For the following exercises, translate from algebra to words.

1. $50 - 15$
2. (10)(x)
3. $2a - b$
4. $100 \div 33$
5. $3x + 5$

For the following exercises, translate from words to algebra.

6. 15 divided by 3.
7. The sum of 13 and 13.
8. 120 minus 12.
9. The product of 5 and 4.
10. The sum of double x and 5.

For the following exercises, translate from an English phrase to an expression.

11. Three times y minus 7.
12. a divided by 2; then add 4.
13. x squared minus 3.
14. A rental car company charges $0.15 per mile m, plus a $40 flat fee for the rental.
15. A parking garage in New York City charges $20 for the first hour, then $5 per hour h.

For the following exercises, use parentheses to make the statements true.

16. $16 \div 4 \bullet 2 + 5 = 13$
17. $2^2 - 5 + 3 \bullet 2 = 5$
18. $x - 3 \bullet x - 2 = x^2 - 5x + 6$

19. $20x \div 5 - 1 - 5x = 0$
20. $5x + 3x \div 3 - 7x + 1 \bullet x = 0$

For the following exercises, evaluate and simplify the expression.

21. x^2 when $x = 9$
22. $2x + 5$ when $x = 3$
23. $(3x + 1)(4x - 6)$ when $x = 2$
24. $x^2 + 3x + 8$ when $x = 3$
25. $\left(x^2 + 5x - 4\right)(2x)$ when $x = 4$
26. $4a + 5 - 2a - 8$ when $a = 6$
27. $8a^2 + 4a + 9 - a^2 - 1$ when $a = 5$
28. Yasenia is 3 years older than Xavier. How old is Yasenia when Xavier is 18 years old?
29. A rental car company charges $0.15 per mile m, plus a $40 flat fee for the rental. What is the cost of the car rental if one drives 100 miles?
30. A parking garage in New York City charges $20 for the first hour, then $5 per hour h. What is the cost of parking for 10 hours?

For the following exercises, perform the indicated operation for the expressions.

31. Add $(4x - 9) + (x + 12)$.
32. Add $\left(3x^2 + 2x + 1\right) + \left(x^2 - 2x + 2\right)$.
33. Subtract $(4x - 9) - (-x + 2)$.
34. Subtract $\left(3x^2 + 5x\right) - \left(x^2 - 3x + 11\right)$.
35. Multiply $4(x + 2)$.
36. Multiply $2\left(3x^2 - 2x + 1\right)$.
37. Multiply $(3x)(x - 1)$.
38. Multiply $(2x - 1)(x + 3)$.
39. $\left(125x^2 + 35x - 5\right) \div (5)$.
40. $\left(9x^2 + 18x - 27\right) \div (3x)$.

5.2 Linear Equations in One Variable with Applications

Figure 5.4 Most gyms have a monthly membership fee. (credit: modification of work "Morning PT after the Holidays 2021" by Fort Drum & 10th Mountain Division (LI)/Flickr, Public Domain Mark 1.0)

Learning Objectives

After completing this section, you should be able to:

1. Solve linear equations in one variable using properties of equations.
2. Construct a linear equation to solve applications.
3. Determine equations with no solution or infinitely many solutions.
4. Solve a formula for a given variable.

In this section, we will study linear equations in one variable. There are several real-world scenarios that can be represented by linear equations: taxi rentals with a flat fee and a rate per mile; cell phone bills that charge a monthly fee plus a separate rate per text; gym memberships with a monthly fee plus a rate per class taken; etc. For example, if you join your local gym at $10 per month and pay $5 per class, how many classes can you take if your gym budget is $75 per month?

Linear Equations and Applications

Solving any equation is like discovering the answer to a puzzle. The purpose of solving an equation is to find the value or values of the variable that makes the equation a true statement. Any value of the variable that makes the equation true is called a solution to the equation. It is the answer to the puzzle! There are many types of equations that we will learn to solve. In this section, we will focus on a **linear equation**, which is an equation in one variable that can be written as

$$ax + b = 0$$

where a and b are real numbers and $a \neq 0$, such that a is the coefficient of x and b is the constant.

To solve a linear equation, it is a good idea to have an overall strategy that can be used to solve any linear equation. In the Example 5.12, we will give the steps of a general strategy for solving any linear equation. Simplifying each side of the equation as much as possible first makes the rest of the steps easier.

EXAMPLE 5.12

Solving a Linear Equation Using a General Strategy

Solve $7(n-3) - 8 = -15$

Solution

Step 1: Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is now simplified as much as possible.	$7(n-3)-8=-15$ $7n-21-8=-15$ $7n-29=-15$
Step 2: Collect all variable terms on one side of the equation.	Nothing to do; all n-terms are on the left side.	$7n-29=-15$
Step 3: Collect constant terms on the other side of the equation.	To get constants only on the right, add 29 to each side. Simplify.	$7n-\mathbf{29}+29=-15+\mathbf{29}$ $7n=14$
Step 4: Make the coefficient of the variable term equal to 1.	Divide each side by 7. Simplify.	$\frac{7n}{7}=\frac{14}{7}$ $n=2$
Step 5: Check the solution.	Let $n=2$ Subtract.	Check: $7(n-3)-8=-15$ $7(2-3)-8\stackrel{?}{=}-15$ $7(-1)-8\stackrel{?}{=}-15$ $-7-8\stackrel{?}{=}-15$ $-15=-15$✓

YOUR TURN 5.12

1. Solve $2(x+1)-3=5$

In Example 5.12, we used both the addition and division property of equations. All the properties of equations are summarized in table below. Basically, what you do to one side of the equation, you must do to the other side of the equation to preserve equality.

Operation	Property	Example
Addition	If $a=b$ Then $a+c=b+c$	$2=2$ $2+3=2+3$ $5=5$
Subtraction	If $a=b$ Then $a-c=b-c$	$5=5$ $5-2=5-2$ $3=3$

Operation	Property	Example
Multiplication	If $a = b$ Then $a \bullet c = b \bullet c$	$3 = 3$ $3 \bullet 4 = 3 \bullet 4$ $12 = 12$
Division	If $a = b$ Then $a \div c = b \div c$ for $c \neq 0$	$8 = 8$ $8 \div 2 = 8 \div 2$ $4 = 4$

Be careful to multiply and divide every term on each side of the equation. For example, $2 + x = \frac{x}{3}$ *is solved by multiplying BOTH sides of the equation by* 3 *to get* $3\,(2 + x) = 3\left(\frac{x}{3}\right)$ *which gives* $6 + 3x = x$. *Using parentheses will help you remember to use the distributive property! A division example, such as* $3\,(x + 2) = 6x + 9$, *can be solved by dividing BOTH sides of the equation by* 3 *to get* $\frac{3(x+2)}{3} = \frac{6x+9}{3}$, *which then will lead to* $x + 2 = 2x + 3$.

EXAMPLE 5.13

Solving a Linear Equation Using Properties of Equations

Solve $9\,(y - 2) - y = 16 + 7y$.

Solution

Step 1: Simplify each side.

$$\begin{aligned} 9\,(y-2) - y &= 16 + 7y \\ 9y - 18 - y &= 16 + 7y \\ 8y - 18 &= 16 + 7y \end{aligned}$$

Step 2: Collect all variables on one side.

$$\begin{aligned} 8y - 18 - \mathbf{7y} &= 16 + 7y - \mathbf{7y} \\ y - 18 &= 16 \end{aligned}$$

Step 3: Collect constant terms on one side.

$$\begin{aligned} y - 18 + \mathbf{18} &= 16 + \mathbf{18} \\ y &= 34 \end{aligned}$$

Step 4: Make the coefficient of the variable 1. Already done!

Step 5: Check.

$$\begin{aligned} 9\,(34) - 18 - (34) &\stackrel{?}{=} 16 + 7\,(34) \\ 306 - 18 - 34 &\stackrel{?}{=} 16 + 238 \\ 288 - 34 &\stackrel{?}{=} 254 \\ 254 &= 254 ✓ \end{aligned}$$

YOUR TURN 5.13

1. Solve $6\,(y - 2) - 5y = 4\,(y + 3) - 4\,(y - 1)$.

WHO KNEW?

Who Invented the Symbol for Equals ?

Before the creation of a symbol for equality, it was usually expressed with a word that meant equals, such as *aequales* (Latin), *esgale* (French), or *gleich* (German). Welsh mathematician and physician Robert Recorde is given credit for inventing the modern sign. It first appears in writing in *The Whetstone of Witte*, a book Recorde wrote about algebra, which was published in 1557. In this book, Recorde states, "I will set as I do often in work use, a pair of parallels, or Gemowe (twin) lines of one length, thus: ===, because no two things can be more equal." Although his version of the sign was a bit longer than the one we use today, his idea stuck and "=" is used throughout the world to indicate equality in mathematics.

In Algebraic Expressions, you translated an English sentence into an equation. In this section, we take that one step further and translate an English paragraph into an equation, and then we solve the equation. We can go back to the opening question in this section: *If you join your local gym at $10 per month and pay $5 per class, how many classes can you take if your gym budget is $75 per month?* We can create an equation for this scenario and then solve the equation (see Example 5.15).

EXAMPLE 5.14

Constructing a Linear Equation to Solve an Application

The Beaudrie family has two cats, Basil and Max. Together, they weigh 23 pounds. Basil weighs 16 pounds. How much does Max weigh?

Solution

Let b = Basil's weight and m = Max's weight.

$$b + m = 23$$

We also know that Basil weighs 16 pounds so:

Steps 1 and 2: $16 + m = 23$

Since both sides are simplified, the variable is on one side of the equation, we start in Step 3 and collect the constants on one side:

Step 3:

$$\begin{aligned} 16 + m - \mathbf{16} &= 23 - \mathbf{16} \\ m &= 7 \end{aligned}$$

Step 4: is already done so we go to Step 5:

Step 5:

$$\begin{aligned} 16 + 7 &\stackrel{?}{=} 23 \\ 23 &= 23 \checkmark \end{aligned}$$

Basil weighs 16 pounds and Max weighs 7 pounds.

YOUR TURN 5.14

1. Sam and Henry are roommates. Together, they have 68 books. Sam has 26 books. How many books does Henry have?

EXAMPLE 5.15

Constructing a Linear Equation to Solve Another Application

If you join your local gym at $10 per month and pay $5 per class, how many classes can you take if your gym budget is $75 per month?

Solution

If we let x = number of classes, the expression $5x + 10$ would represent what you pay per month if each class is $5 and there's a $10 monthly fee per class. $10 is your constant. If you want to know how many classes you can take if you have a $75 monthly gym budget, set the equation equal to 75. Then solve the equation $5x + 10 = 75$ for x.

Steps 1 and 2:

$$5x + 10 = 75$$

Step 3:

$$\begin{aligned} 5x + \mathbf{10} - 10 &= 75 - \mathbf{10} \\ 5x &= 65 \end{aligned}$$

Step 4:

$$\begin{aligned} \frac{5x}{\mathbf{5}} &= \frac{65}{\mathbf{5}} \\ x &= 13 \end{aligned}$$

Step 5:

$$\begin{aligned} 5\,(13) + 10 &\stackrel{?}{=} 75 \\ 65 + 10 &\stackrel{?}{=} 75 \\ 75 &= 75✓ \end{aligned}$$

The solution is 13 classes. You can take 13 classes on a $75 monthly gym budget.

YOUR TURN 5.15

1. On June 7, 2021, the national average price for regular gasoline was $3.053 per gallon. If Aiko fills up his car with 16 gallons, how much is the total cost? Round to the nearest cent.

EXAMPLE 5.16

Constructing an Application from a Linear Equation

Write an application that can be solved using the equation $50x + 35 = 185$. Then solve your application.

Solution

Answers will vary. Let's say you want to rent a snowblower for a huge winter storm coming up. If x = the number of days you rent a snowblower, then the expression $50x + 35$ represents what you pay if, for each day, it costs $50 to rent the snowblower and there is a $35 flat rental fee. $35 is the constant. To find out how many days you can rent a snowblower for $185, set the expression equal to 185. Then solve the equation $50x + 35 = 185$ for x.

Steps 1 and 2:

$$50x + 35 = 185$$

Step 3:

$$50x + 35 \mathbf{- 35} = 185 \mathbf{- 35}$$
$$50x = 150$$

Step 4:

$$\frac{50x}{\mathbf{50}} = \frac{150}{\mathbf{50}}$$
$$x = 3$$

Step 5:

$$50\,(3) + 35 \stackrel{?}{=} 185$$
$$150 + 35 \stackrel{?}{=} 185$$
$$185 = 185\checkmark$$

The equation is $50x + 35 = 185$ and the solution is 3 days. You can rent a snowblower for 3 days on a $185 budget.

YOUR TURN 5.16

1. Write an application that can be solved using the equation $25x + 75 = 200$. Then solve your application.

Linear Equations with No Solutions or Infinitely Many Solutions

Every linear equation we have solved thus far has given us one numerical solution. Now we'll look at linear equations for which there are no solutions or infinitely many solutions.

EXAMPLE 5.17

Solving a Linear Equation with No Solution

Solve $3\,(x + 4) = 4x + 8 - x$.

Solution

Step 1: Simplify each side. $3\,(x + 4) = 4x + 8 - x$

$$3x + 12 \mathbf{- 3x} = 3x + 8 \mathbf{- 3x}$$

Step 2: Collect all variables to one side. $3x + 12 \mathbf{- 3x} = 3x + 8 \mathbf{- 3x}$

$$12 = 8$$

The variable x disappeared! When this happens, you need to examine what remains. In this particular case, we have $12 = 8$, which is not a true statement. When you have a false statement, then you know the equation has no solution; there does not exist a value for x that can be put into the equation that will make it true.

YOUR TURN 5.17

1. Solve $2\,(x + 6) = 3x + 4 - (x + 5)$.

EXAMPLE 5.18

Solving a Linear Equation with Infinitely Many Solutions

Solve $2(x+5) = 4(x+3) - 2x - 2$.

Solution

Step 1:

$$\begin{aligned} 2(x+5) &= 4(x+3) - 2x - 2 \\ 2x + 10 &= 4x + 12 - 2x - 2 \\ 2x + 10 &= 2x + 10 \end{aligned}$$

Step 2:

$$\begin{aligned} 2x + 10 - \mathbf{2x} &= 2x + 10 - \mathbf{2x} \\ 10 &= 10 \end{aligned}$$

As with the previous example, the variable disappeared. In this case, however, we have a true statement ($10 = 10$). When this occurs we say there are infinitely many solutions; any value for x will make this statement true.

YOUR TURN 5.18

1. Solve $3x - 7 - (x+5) = 2(x-6)$.

Solving a Formula for a Given Variable

You are probably familiar with some geometry formulas. A **formula** is a mathematical description of the relationship between variables. Formulas are also used in the sciences, such as chemistry, physics, and biology. In medicine they are used for calculations for dispensing medicine or determining body mass index. Spreadsheet programs rely on formulas to make calculations. It is important to be able to manipulate formulas and solve for specific variables.

To solve a formula for a specific variable means to isolate that variable on one side of the equal sign with a coefficient of 1. All other variables and constants are on the other side of the equal sign. To see how to solve a formula for a specific variable, we will start with the distance, rate, and time formula.

EXAMPLE 5.19

Solving for a Given Variable with Distance, Rate, and Time

Solve the formula $d = rt$ for t. This is the distance formula where d = distance, r = rate, and t = time.

Solution

Divide both sides by r: $d/\mathbf{r} = rt/\mathbf{r}$

$d/r = t$

YOUR TURN 5.19

1. Solve the formula $I = Prt$ for t. This formula is used to calculate simple interest I, for a principal P, invested at a rate r, for t years.

VIDEO

[Solving for a Variable in an Equation (https://openstax.org/r/Solving_for_a_variable)](https://openstax.org/r/Solving_for_a_variable)

EXAMPLE 5.20

Solving for a Given Variable in the Area Formula for a Triangle

Solve the formula $A = ½\ bh$ for h. This is the area formula of a triangle where A = area, b = base, and h = height.

Solution

Step 1: Multiply both sides by 2.

$$2A = 2(½\ bh)$$
$$2A = bh$$

Step 2: Divide both sides by b.

$$\frac{2A}{b} = \frac{bh}{b}$$
$$\frac{2A}{b} = h$$
$$h = \frac{2A}{b}$$

YOUR TURN 5.20

1. Solve the formula $V = \frac{1}{3}\pi r^2 h$ for h. This formula is used to calculate the volume V of a right circular cone with radius r and height h.

WORK IT OUT

Using Algebra to Understand Card Tricks

You will need to perform this card trick with another person. Before you begin, the two people must first decide which of the two will be the *Dealer* and which will be the *Partner*, as each will do something different. Once you have decided upon that, follow the steps here:

Step 1: Dealer and Partner: Take a regular deck of 52 cards, and remove the face cards and the 10s.

Step 2: Dealer and Partner: Shuffle the remaining cards

Step 3: Dealer and Partner: Select one card each, but keep them face down and don't look at them yet.

Step 4: Dealer: Look at your card (just the Dealer!). Multiply its value by 2 (Aces = 1).

Step 5: Dealer: Add 2 to this result.

Step 6: Dealer: Multiply your answer by 5.

Step 7: Partner: Look at your card.

Step 8: Partner: Calculate: 10 - your card, and tell this information to the dealer.

Step 9: Dealer: Subtract the value the Partner tells you from your total to get a final answer.

Step 10: Dealer: verbally state the final answer.

Step 11: Dealer and Partner: Turn over your cards. Now, answer the following questions

1. Did the trick work? How do you know?
2. Why did this occur? In other words, how does this trick work?

Check Your Understanding

9. Is the solution strategy used in solving the linear equation correct? If it is correct, show the final step (check the solution). If it is not correct, explain why.

$$\begin{aligned} 8(x-2) &= 6(x+10) \\ 8x-16 &= 6x+60 \\ 8x-16-6x &= 6x+60-6x \\ 2x-16+\mathbf{16} &= 60+\mathbf{16} \\ 2x &= 76 \\ x &= 38 \end{aligned}$$

10. Is the solution strategy used in solving the linear equation correct? If it is correct, show the final step (check the solution). If it is not correct, explain why.

$$\begin{aligned} 7+4(2+5x) &= 3(6x+7)-(13x+36) \\ 7+8+20x &= 18x+21-13x-36 \\ 15+20x &= 5x-15 \\ 15+20x-\mathbf{5}x &= 5x-15-\mathbf{5}x \\ 15+15x-\mathbf{15} &= -15-\mathbf{15} \\ 15x &= -30 \\ x &= -2 \end{aligned}$$

11. Is the solution strategy used in solving the linear equation correct? If it is correct, show the final step (check the solution). If it is not correct, explain why.

$$\begin{aligned} 8x+7-(2x-9) &= 22-(4x-4) \\ 8x+7-2x-9 &= 22-4x-4 \\ 6x-2 &= 18-4x \\ 6x-2+\mathbf{4}x &= 18-4x+\mathbf{4}x \\ 10x-2+\mathbf{2} &= 18+\mathbf{2} \\ 10x &= 20 \\ x &= 2 \end{aligned}$$

For the following exercises, use this scenario: The Nice Cab Company charges a flat rate of $3.00 for each fare, plus $1.70 per mile. A competing taxi service, the Enjoyable Cab Company, charges a flat rate of $5.00 for each fare, plus $1.60 per mile.

12. Using the variable x for number of miles, write the equation that would allow you to find the total fare (T) using the Nice Cab Company.

13. It is 22 miles from the airport to your hotel. What would be your total fare using the Nice Cab Company?

14. Using the variable y for number of miles, write the equation that would allow you to find the total fare (T) using the Enjoyable Cab Company.

15. Using the same 22-mile trip from the airport to the hotel, how much would the total fare be for using the Enjoyable Cab Company?

16. Based on the cost of each cab ride, which cab company should you use for the trip from the airport to the hotel? Why?

17. After solving the linear equation $3(2x-3) = 12(x-3) - 3(2x-9)$, Nancy says there is no solution. Luis believes there are infinitely many solutions. Who is right?

18. The conversion formula between the Fahrenheit temperature scale and the Celsius temperature scale is given by this formula: $C = \frac{5}{9}(F-32)$, where C is the temperature in degrees Celsius and F is the temperature in degrees Fahrenheit. What is the correct formula when solved for F?

a. $F = \frac{5}{9}C - 32$
b. $F = \frac{9}{5}C - 32$
c. $F = \frac{5}{9}C + 32$
d. $F = \frac{9}{5}C + 32$

19. To find a temperature on the Kelvin temperature scale, add 273 degrees to the temperature in Celsius. Which formula illustrates this?

a. $C = K + 273$
b. $K = C + 273$
c. $K = C - 273$

d. $C = K - 273$

20. Using the information from exercise 18 and exercise 19, which conversion formula would you use to find degrees Kelvin when given degrees Fahrenheit?

a. $K = \frac{5}{9}(F - 32) + 273$

b. $K = \frac{5}{9}F + 241$

c. $K = \frac{9}{5}(F - 32) + 273$

d. $K = \frac{9}{5}F + 241$

21. There is a fourth temperature scale, although it is not used much today. The Rankin temperature scale varies from the Fahrenheit scale by about 460 degrees. So given a temperature in Fahrenheit, add 460 degrees to get the temperature in Rankin. Which formula represents a formula to find degrees Rankin when given degrees Celsius?

a. $R = \frac{5}{9}C - 492$

b. $R = \frac{9}{5}C + 492$

c. $R = C + 492$

d. $R = \frac{5}{9}(C - 492)$

SECTION 5.2 EXERCISES

For the following exercises, solve the linear equations using a general strategy.

1. $-(t - 19) = 28$

2. $51 + 5(4 - q) = 56$

3. $-6 + 6(5 - k) = 15$

4. $3(10 - 2x) + 54 = 0$

5. $-2(11 - 7x) + 54 = 4$

For the following exercises, solve the linear equations using properties of equations.

6. $-12 + 8(x - 5) = -4 + 3(5x - 2)$

7. $4(p - 4) - (p + 7) = 5(p - 3)$

8. $3(a - 2) - (a + 6) = 4(a - 1)$

9. $4[5 - 8(4c - 3)] = 12(1 - 13c) - 8$

10. $5[9 - 2(6d - 1)] = 11(4 - 10d) - 139$

For the following exercises, construct a linear equation to solve an application.

11. It costs \$0.55 to mail one first class letter. Construct a linear equation and solve to find how much it costs to mail 13 letters.

12. Normal yearly snowfall at the local ski resort is 12 inches more than twice the amount it received last season. The normal yearly snowfall is 62 inches. Construct a linear equation and solve to find what the snowfall was last season.

13. Guillermo bought textbooks and notebooks at the bookstore. The number of textbooks was three more than twice the number of notebooks. He bought seven textbooks. Construct a linear equation and solve to find how many notebooks he bought.

14. Gerry worked Sudoku puzzles and crossword puzzles this week. The number of Sudoku puzzles he completed is eight more than twice the number of crossword puzzles. He completed 22 Sudoku puzzles. Construct a linear equation and solve to find how many crossword puzzles he did.

15. Laurie has \$46,000 invested in stocks. The amount invested in stocks is \$8,000 less than three times the amount invested in bonds. Construct a linear equation and solve to find how much Laurie invested in bonds.

For the following exercises, construct an application from a linear equation.

16. $1,000x + 2,500 = 16,500.$

17. $0.36t$ for $t = 333.$

18. $150n + 120 = 570.$

19. $4c + 2c$ for $c = 5.$

20. $2s + 10 = 24.$

For the following exercises, state whether each equation has exactly one solution, no solution, or infinitely many solutions.

21. $23z + 19 = 3(5z - 9) + 8z + 46$

22. $18(5j - 11) = 47j + 17$
23. $22(3m + 4) = 17(4m - 6)$
24. $7v + 42 = 11(3v + 8) - 2(13v - 1)$
25. $45(3y - 2) = 9(15y - 6)$
26. $9(14d + 9) + 4d = 13(10d + 6) + 3$

For the following exercises, solve the given formula for the specified variable.

27. Solve the formula $C = \pi d$ for d.
28. Solve the formula $V = LWH$ for L.
29. Solve the formula $A = (1/2)\,bh$ for b.
30. Solve the formula $A = (1/2)\,d_1 d_2$ for d_1.
31. Solve the formula $A = (1/2)\,h(b_1 + b_2)$ for b_1.
32. Solve the formula $h = 54t + (1/2)\,a$ for a.
33. Solve $180 = a + b + c$ for a.
34. Solve the formula $A = (1/2)\,pl + B$ for p.
35. Solve the formula: $P = 2L + 2W$ for L.

5.3 Linear Inequalities in One Variable with Applications

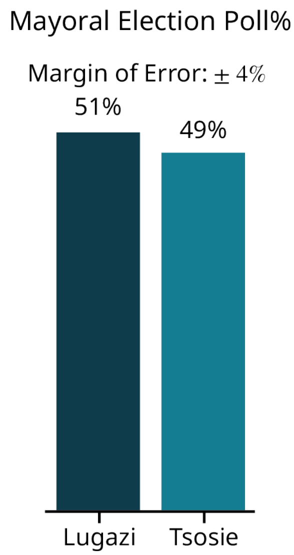


Figure 5.5 These poll results, showing a margin of error at 4 percent, are an example of a real-world scenario that can be represented by linear inequalities.

Learning Objectives

After completing this section, you should be able to:

1. Graph inequalities in one variable.
2. Solve linear inequalities in one variable.
3. Construct a linear inequality to solve applications.

In this section, we will study linear inequalities in one variable. Inequalities can be used when the possible values (answers) in a certain situation are numerous, not just a few, or when the exact value (answer) is not known but it is known to be within a range of possible values. There are many real-world scenarios that can be represented by linear inequalities. For example, consider the survey of the mayoral election in Figure 5.5 Surveys and polls are usually conducted with only a small group of people. The margin of error indicates a range of how the actual group of voters would vote given the results of the survey. This range can be expressed using inequalities.

Another example involves college tuition. Say a local community college charges $113 per credit hour. You budget $1,500 for tuition this fall semester. What are the number of credit hours that you could take this fall? Since this answer could be many different values, it can be expressed as an inequality.

Graphing Inequalities on the Number Line

In Algebraic Expressions, we introduced equality and the $=$ symbol. In this section, we look at inequality and the symbols $<$, $>$, $\leq$, and $\geq$. The table below summarizes the symbols and their meaning.

Symbol	Meaning
$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to

Suppose you had the inequality statement $x > 3$. What possible number or numbers would make the inequality $x > 3$ true? If you are thinking, "x could be 4," that's correct, but x could also be 5, 6, 37, 1 million, or even 3.001. The number of solutions is infinite; any number greater than 3 is a solution to the inequality $x > 3$.

Rather than trying to list all possible solutions, we show all the solutions to the inequality $x > 3$ on the number line. All the numbers to the right of 3 on the number line are shaded, to show that all numbers greater than 3 are solutions. At the number 3 itself, an open parenthesis is drawn, since the number 3 is not part of the solutions of $x > 3$.

We can also represent inequalities using interval notation. There is no upper end to the solution to this inequality. In interval notation, we express $x > 3$ as $(3, \infty)$. The symbol ∞ is read as "infinity." Infinity is not an actual number. Figure 5.6 shows both the number line and the interval notation for $x > 3$.

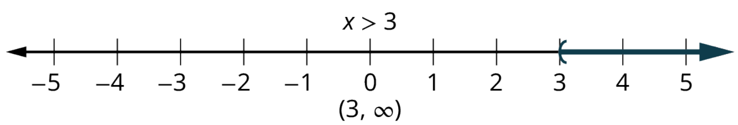


Figure 5.6 The inequality $x > 3$ is graphed on this number line and written in interval notation.

We used the left parenthesis symbol to show that the endpoint of the inequality is not included. Parentheses are used when the endpoints are not included as a possible answer to the inequality. The notation for inequalities on a number line and in interval notation use the same symbols to express the endpoints of intervals.

The inequality $x \leq 1$ means all numbers less than or equal to 1. To illustrate that solution on a number line, we first put a bracket at $x = 1$; brackets are used when the endpoint is included. We then shade in all the numbers to the left of 1, to show that all numbers less than one are solutions. There is no lower end to those numbers. We write $x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as "negative infinity." Figure 5.7 shows both the number line and interval notation for $x = 1$.

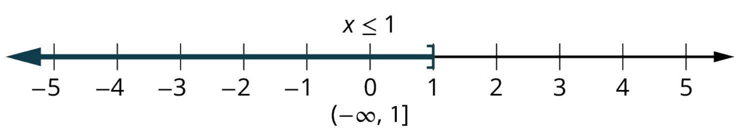


Figure 5.7 The inequality $x \leq 1$ is graphed on this number line and written in interval notation.

Figure 5.8 summarizes the general representations in both number line form and interval notation of solutions for $x > a$, $x < a$, $x \geq a$, and $x \leq a$.

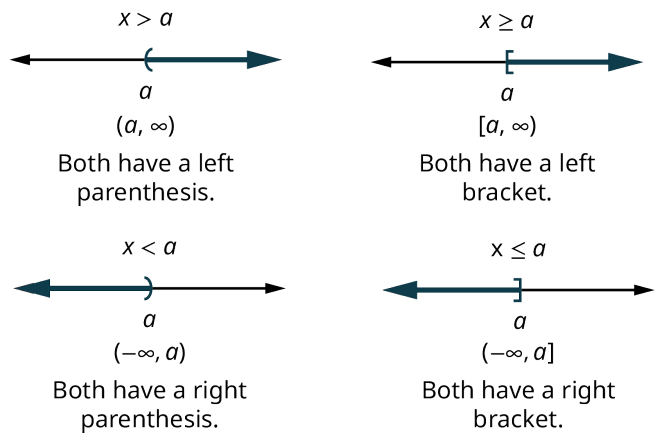


Figure 5.8 Summary of representations in number line form and interval notation.

EXAMPLE 5.21

Graphing an Inequality

Graph the inequality $x \geq -3$ and write the solution in interval notation.

Solution

Shade to the right of -3 to show all the numbers greater than -3, and put a bracket at -3 to show that the numbers are greater than or equal to -3 (Figure 5.9)

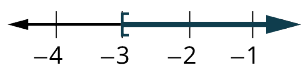


Figure 5.9

Write in interval notation starting at -3 with a bracket to show that -3 is included in the solution and then infinity because the solution includes all the numbers greater than or equal to -3:

$$[-3, \infty)$$

YOUR TURN 5.21

1. Graph the inequality $x < 2.5$ and write the solution in interval notation.

EXAMPLE 5.22

Graphing a Compound Inequality

Graph the inequality $x > -3$ and $x < 4$ and write the solution in interval notation.

Solution

Step 1: Graph $x > -3$ (Figure 5.10).

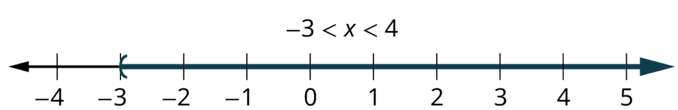


Figure 5.10

Step 2: Graph $x < 4$ (Figure 5.11).

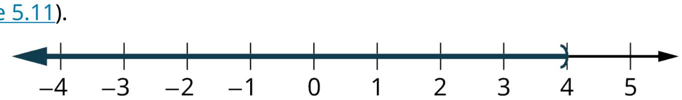


Figure 5.11

Step 3: Graph both on the same number line and think of where the solutions are to BOTH inequalities Figure 5.12. This will be where BOTH are shaded.

Figure 5.12

Step 4: Write the solution in interval notation:

$$(-3, 4)$$

YOUR TURN 5.22

1. Graph the inequality $x \geq 0$ and $x \leq 2.5$ and write the solution in interval notation.

WHO KNEW?

Where Did the Inequality Symbols Come From?

The first use of the $<$ symbol to represent "less than" and $>$ to represent "greater than" appeared in a mathematics book written by Englishman Thomas Harriot that was published in 1631. However, Harriot did not invent the symbols…the editor of the book did! Harriot used triangular symbols to represent less than and greater than; the editor, for reasons unknown, changed to symbols that are similar to the ones we use today. The symbols used to represent less than or equal to, and greater than or equal to ($\leq$ and $\geq$) were first used in 1731 by French hydrologist and surveyor Pierre Bouguer. Interestingly, English mathematician John Wallis had used similar symbols as early as 1670, but he put the bar above the less than and greater than symbols instead of below them.

Solving Linear Inequalities

A **linear inequality** is much like a linear equation—but the equal sign is replaced with an inequality sign. A linear inequality is an inequality in one variable that can be written in one of the forms $ax + b < c$, $ax + b \leq c$, $ax + b \geq c$, or $ax + b > c$, where a, b, and c are all real numbers.

When we solved linear equations, we were able to use the properties of equality to add, subtract, multiply, or divide both sides and still keep the equality. Similar properties hold true for inequalities. We can add or subtract the same quantity from both sides of an inequality and still keep the inequality. For example, we know that 2 is less than 4, i.e., $2 < 4$. If we add 6 to both sides of this inequality, we still have a true statement:

$$2 + 6 < 4 + 6$$
$$8 < 10$$

The same would happen if we subtracted 6 from both sides of the inequality; the statement would stay true:

$$2 - 6 < 4 - 6$$
$$-4 < -2$$

Notice that the inequality signs stayed the same. This leads us to the Addition and Subtraction Properties of Inequality.

FORMULA

For any numbers a, b, and c, if $a < b$, then $a + c < b + c$ and $a - c < b - c$.

For any numbers a, b, and c, if $a > b$, then $a + c > b + c$ and $a - c > b - c$.

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality the same. But what happens to an inequality when we divide or multiply both sides by a number? Let's first multiply and divide both sides by a positive number, starting with an inequality we know is true, $10 < 15$. We will multiply and divide this inequality by 5:

$$\begin{array}{cc} 10 < 15 & 10 < 15 \\ 10(5)\ ?\ 15(5) & \frac{10}{5}\ ?\ \frac{15}{5} \\ 50\ ?\ 75 & 2\ ?\ 3 \\ 50 < 75\,(\text{true}) & 2 < 3\,(\text{true}) \end{array}$$

The inequality signs stayed the same. Does the inequality stay the same when we divide or multiply by a negative number? Let's use our inequality $10 < 15$ to find out, multiplying it and dividing it by -5:

$$\begin{array}{cc} 10 < 15 & 10 < 15 \\ 10(-5)\ ?\ 15(-5) & \frac{10}{-5}\ ?\ \frac{15}{-5} \\ -50\ ?\ -75 & -2\ ?\ -3 \\ -50 > -75\,(\text{true}) & -2 > -3\,(\text{true}) \end{array}$$

Notice that when we filled in the inequality signs, the inequality signs reversed their direction in order to make it true! To summarize, when we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we divide or multiply an inequality by a negative number, the inequality sign reverses. This gives us the Multiplication and Division Property of Inequality.

FORMULA

For any numbers a, b, and c,

multiply or divide by a positive:	if $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$
	if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$
multiply or divide by a negative:	if $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$
	if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$

To summarize, when we divide or multiply an inequality by:

- a positive number, the inequality sign stays the same.
- a negative number, the inequality sign reverses.

Be careful to only reverse the inequality sign when you are multiplying and dividing by a negative. You do NOT reverse the inequality sign when you add or subtract a negative. For example, $2x < -4$ is solved by dividing both sides of the inequality by 2 to get $x < -2$. You do NOT reverse the inequality sign because there is a negative 4. As another example, $-2x + 5 < 3x$ is solved by adding $-2x$ to both sides to get $5 < 5x$. This does not reverse the inequality sign because we were not multiplying or diving by a negative. We then divide both sides by 5 and get $1 < x$.

EXAMPLE 5.23

Solving a Linear Inequality Using One Operation

Solve $9y < 54$, graph the solution on the number line, and write the solution in interval notation.

Solution

$$9y < 54$$
$$\frac{9y}{9} < \frac{54}{9}$$
$$y < 6$$

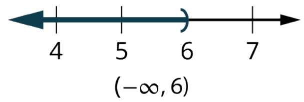


Figure 5.13

YOUR TURN 5.23

1. Solve $-13m \geq 65$, graph the solution on the number line, and write the solution in interval notation.

EXAMPLE 5.24

Solving a Linear Inequality Using Multiple Operations

Solve the inequality $6y \leq 11y + 17$, graph the solution on the number line, and write the solution in interval notation.

Solution

$$\begin{aligned} 6y &\leq 11y + 17 \\ 6y - \mathbf{11y} &\leq \mathbf{11y} + 17 - 11y \\ -5y &\leq 17 \\ \frac{-5y}{-5} &\geq -\frac{17}{5} \\ y &\geq -\frac{17}{5} \end{aligned}$$

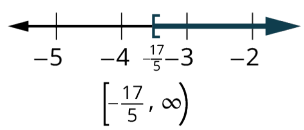


Figure 5.14

YOUR TURN 5.24

1. Solve the inequality $8p + 3(p - 12) \geq 7p - 28$, graph the solution on the number line, and write the solution in interval notation.

Solving Applications with Linear Inequalities

Many real-life situations require us to solve inequalities. The method we will use to solve applications with **linear inequalities** is very much like the one we used when we solved applications with equations. We will read the problem and make sure all the words are understood. Next, we will identify what we are looking for and assign a variable to represent it. We will restate the problem in one sentence to make it easy to translate into an inequality. Then, we will solve the inequality.

Sometimes an application requires the solution to be a whole number, but the algebraic solution to the inequality is not a whole number. In that case, we must round the algebraic solution to a whole number. The context of the application will determine whether we round up or down.

EXAMPLE 5.25

Constructing a Linear Inequality to Solve an Application with Tablet Computers

A teacher won a mini grant of $4,000 to buy tablet computers for their classroom. The tablets they would like to buy cost $254.12 each, including tax and delivery. What is the maximum number of tablets the teacher can buy?

Solution

Let $t =$ the number of tablets.

t times \$254.12 has to be less than \$4,000, so $254.12t \leq 4,000$.

Solve for t:

$$\begin{aligned} \frac{254.12t}{254.12} &\leq \frac{4,000}{254.12} \\ t &\leq 15.74 \end{aligned}$$

The teacher can buy 15 tablets and stay under \$4,000.

YOUR TURN 5.25

1. Taleisha's phone plan costs her \$28.80 per month plus \$0.20 per text message. How many text messages can she send/receive and keep her monthly phone bill no more than \$50?

EXAMPLE 5.26

Constructing a Linear Inequality to Solve a Tuition Application

The local community college charges \$113 per credit hour. Your budget is \$1,500 for tuition this fall semester. What number of credit hours could you take this fall?

Solution

Let $c =$ the number of credit hours you could take.

c times \$113 has to be less than \$1,500, so $113c \leq 1,500$.

Solve for c:

$$\begin{aligned} \frac{113c}{113} &\leq \frac{1500}{113} \\ c &\leq 13.27 \end{aligned}$$

You can take up to 13 credits and stay under \$1,500.

YOUR TURN 5.26

1. You are awarded a \$500 scholarship! In addition to the \$1,500 you have saved for tuition, you now have an additional \$500 to spend on credit hours for fall semester. Now, how many credit hours could you take this fall semester? Assume the cost is still \$113 per credit hour.

EXAMPLE 5.27

Constructing a Linear Inequality to Solve an Application with Travel Costs

Brenda's best friend is having a destination wedding and the event will last 3 days and 3 nights. Brenda has \$500 in savings and can earn \$15 an hour babysitting. She expects to pay \$350 for airfare, \$375 for food and entertainment, and \$60 a night for her share of a hotel room. How many hours must she babysit to have enough money to pay for the trip?

Solution

Let $b =$ number of babysitting hours.

b times \$15 plus \$500 has to be more than $\$350 + \$375 + \$60/\text{night}$, so $15b + 500 \geq 350 + 375 + 60\,(3)$.

Solve for b:

$$
\begin{aligned}
15b + 500 - \mathbf{500} &\geq 905 - \mathbf{500} \\
15b &\geq 405 \\
\frac{15b}{15} &\geq \frac{405}{15} \\
b &\geq 27
\end{aligned}
$$

Brenda must babysit at least 27 hours.

YOUR TURN 5.27

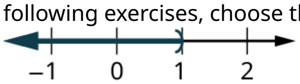
1. Malik is planning a 6-day summer vacation trip. He has $840 in savings, and he earns $45 per hour for tutoring. The trip will cost him $525 for airfare, $780 for food and sightseeing, and $95 per night for the hotel. How many hours must he tutor to have enough money to pay for the trip?

TECH CHECK

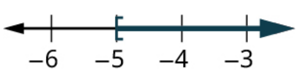
The Desmos activities called "Inequalities on a Number Line" (https://openstax.org/r/Inequalities_on_a) and "Compound Inequalities on a Number Line" (https://openstax.org/r/Compound_Inequalities) are ways for students to develop and deepen their understanding of inequalities. Teachers will need a Desmos account to assign the activity for student use. Once they have assigned the activity to their students, teachers need to share the code for the activity with their students. Students will input the code (https://openstax.org/r/will_input) to work on the activity.

Check Your Understanding

For the following exercises, choose the correct interval notation for the graph.

22. 

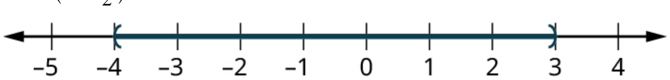
a. $[-1, \infty)$
b. $(-1, 1)$
c. $(\infty, 1)$
d. $(-\infty, 1)$
e. $(-\infty, -1)$

23. 

a. $(-5, \infty)$
b. $[-5, \infty)$
c. $[-5, \infty)$
d. $[-5, -3)$
e. $[-5, -3]$

24. 

a. $(1, \infty)$
b. $[1, \infty)$
c. $\left[\frac{3}{2}, \infty\right)$
d. $\left(\frac{3}{2}, \infty\right)$
e. $\left(\infty, \frac{3}{2}\right)$

25. 

a. $(-4, 3)$
b. $(3, -4)$
c. $[-\infty, \infty)$

d. $[-4, 3]$
e. $[3, -4]$

26. $[4, \infty)$ is the solution for which inequality?
a. $4x \geq 0$
b. $4x \leq 0$
c. $6x < 24$
d. $6x > 24$
e. $6x \geq 24$

27. $(-\infty, -3)$ is the solution for which inequality?
a. $-6x < 18$
b. $-6x > 18$
c. $-6x \leq 18$
d. $-6x \geq 18$
e. $-6x \leq -18$

28. $(-2, \infty)$ is the solution for which inequality?
a. $-4x > -8$
b. $4x + 3 > -11$
c. $4x - 3 > -11$
d. $-4x \leq -8$
e. $-4x + 3 \leq 5$

29. $(-\infty, 9)$ is the solution for which inequality?
a. $9x < 0$
b. $-3x \geq 27$
c. $-3x + 14 > -13$
d. $-3x - 14 > -13$
e. $-3x \geq 27$

For the following exercises, choose the equation that best models the situation.

30. Renaldo is hauling boxes of lawn chairs. Each box is the same size, 8 cubic feet. Renaldo's truck has a capacity of 764 cubic feet. How many boxes of lawn chairs can Renaldo put in his truck?
a. $8 < 764x$
b. $8x < 764$
c. $8 > 764x$
d. $8x > 764$
e. None of these

31. Bernadette babysits the neighbor's kids, making on average $50 a night. How many nights will she have to babysit in order to earn enough money to buy a used car, whose cost is $8,120?
a. $50 < 8{,}120\text{x}$
b. $50x < 8{,}120$
c. $50 < 8{,}120\text{x}$
d. $50x > 8{,}120$
e. None of these

SECTION 5.3 EXERCISES

For the following exercises, graph the inequality on a number line and write the interval notation.

1. $x > 3$
2. $x \leq -0.5$
3. $x \geq \frac{1}{3}$
4. $x < -\frac{7}{3}$
5. $-2 < x < 0$
6. $-5 \leq x < -3$

7. $0 \leq x \leq 3.5$
8. $-4 < x < 2$
9. $-5 < x \leq -2$
10. $-3.75 \leq x \leq 0$

For the following exercises, solve the inequality, graph the solution on the number line, and write the solution in interval notation.

11. $6y < 48$
12. $40 < \frac{5}{8}k$
13. $7s < -28$
14. $\frac{9}{4}g \leq 36$
15. $-8v \leq 96$
16. $\frac{b}{-10} \geq 30$
17. $-7d > 105$
18. $-18 < \frac{q}{-6}$
19. $5u \leq 8u - 21$
20. $9p > 14p - 18$
21. $9y + 5(y + 3) < 4y - 35$
22. $4k - (k - 2) \geq 7k - 26$

For the following exercises, construct a linear inequality to solve the application.

23. The elevator in Yehire's apartment building has a sign that says the maximum weight is 2,100 pounds. If the average weight of one person is 150 pounds, how many people can safely ride the elevator?
24. Arleen got a $20 gift card for the coffee shop. Her favorite iced drink costs $3.79. What is the maximum number of drinks she can buy with the gift card?
25. Ryan charges his neighbors $17.50 to wash their car. How many cars must he wash next summer if his goal is to earn at least $1,500?
26. Kimuyen needs to earn $4,150 per month in order to pay all her expenses. Her sales job pays her $3,475 per month plus 4 percent of her total sales. What is the minimum Kimuyen's total sales must be in order for her to pay all her expenses?
27. Nataly is considering two job offers. The first job would pay her $83,000 per year. The second would pay her $66,500 plus 15 percent of her total sales. What would her total sales need to be for her salary on the second offer to be higher than the first?
28. Kiyoshi's phone plan costs $17.50 per month plus $0.15 per text message. What is the maximum number of text messages Kiyoshi can use so the phone bill is no more than $56.60?
29. Kellen wants to rent a banquet room in a restaurant for her cousin's baby shower. The restaurant charges $350 for the banquet room plus $32.50 per person for lunch. How many people can Kellen have at the shower if she wants the maximum cost to be $1,500?
30. Noe installs and configures software on home computers. He charges $125 per job. His monthly expenses are $1,600. How many jobs must he work in order to make a profit of at least $2,400?

5.4 Ratios and Proportions

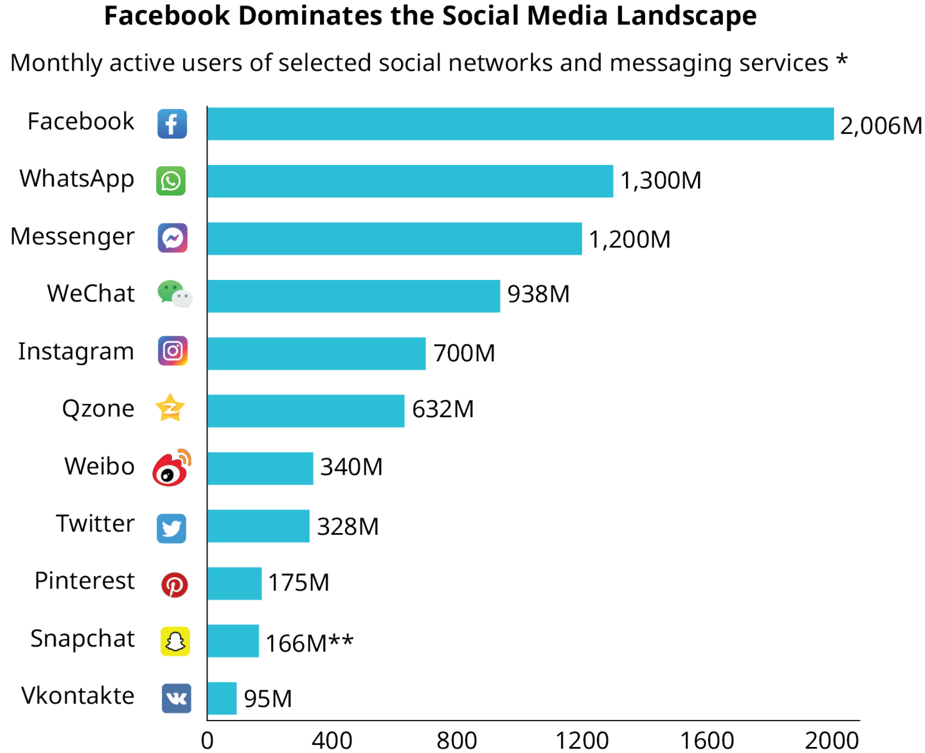


Figure 5.15 This bar graph shows popular social media app usage. (Source (https://openstax.org /r/media_apps_chart))

Learning Objectives

After completing this section, you should be able to:

1. Construct ratios to express comparison of two quantities.
2. Use and apply proportional relationships to solve problems.
3. Determine and apply a constant of proportionality.
4. Use proportions to solve scaling problems.

Ratios and proportions are used in a wide variety of situations to make comparisons. For example, using the information from Figure 5.15, we can see that the number of Facebook users compared to the number of Twitter users is 2,006 M to 328 M. Note that the "M" stands for million, so 2,006 million is actually 2,006,000,000 and 328 million is 328,000,000. Similarly, the number of Qzone users compared to the number of Pinterest users is in a ratio of 632 million to 175 million. These types of comparisons are ratios.

Constructing Ratios to Express Comparison of Two Quantities

Note there are three different ways to write a **ratio**, which is a comparison of two numbers that can be written as: a to b OR $a : b$ OR the fraction a/b. Which method you use often depends upon the situation. For the most part, we will want to write our ratios using the fraction notation. Note that, while all ratios are fractions, not all fractions are ratios. Ratios make part to part, part to whole, and whole to part comparisons. Fractions make part to whole comparisons only.

EXAMPLE 5.28

Expressing the Relationship between Two Currencies as a Ratio

The Euro (€) is the most common currency used in Europe. Twenty-two nations, including Italy, France, Germany, Spain, Portugal, and the Netherlands use it. On June 9, 2021, 1 U.S. dollar was worth 0.82 Euros. Write this comparison as a ratio.

Solution

Using the definition of ratio, let $a = 1$ U.S. dollar and let $b = 0.82$ Euros. Then the ratio can be written as either 1 to 0.82; or 1:0.82; or $\frac{1}{0.82}$.

YOUR TURN 5.28

1. On June 9, 2021, 1 U.S. dollar was worth 1.21 Canadian dollars. Write this comparison as a ratio.

EXAMPLE 5.29

Expressing the Relationship between Two Weights as a Ratio

The gravitational pull on various planetary bodies in our solar system varies. Because weight is the force of gravity acting upon a mass, the weights of objects is different on various planetary bodies than they are on Earth. For example, a person who weighs 200 pounds on Earth would weigh only 33 pounds on the moon! Write this comparison as a ratio.

Solution

Using the definition of ratio, let $a = 200$ pounds on Earth and let $b = 33$ pounds on the moon. Then the ratio can be written as either 200 to 33; or 200:33; or $\frac{200}{33}$.

YOUR TURN 5.29

1. A person who weighs 170 pounds on Earth would weigh 64 pounds on Mars. Write this comparison as a ratio.

Using and Applying Proportional Relationships to Solve Problems

Using **proportions** to solve problems is a very useful method. It is usually used when you know three parts of the proportion, and one part is unknown. Proportions are often solved by setting up like ratios. If $\frac{a}{b}$ and $\frac{c}{d}$ are two ratios such that $\frac{a}{b} = \frac{c}{d}$, then the fractions are said to be **proportional**. Also, two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are proportional $\left(\frac{a}{b} = \frac{c}{d}\right)$ if and only if $a \times d = b \times c$.

EXAMPLE 5.30

Solving a Proportion Involving Two Currencies

You are going to take a trip to France. You have $520 U.S. dollars that you wish to convert to Euros. You know that 1 U.S. dollar is worth 0.82 Euros. How much money in Euros can you get in exchange for $520?

Solution

Step 1: Set up the two ratios into a proportion; let x be the variable that represents the unknown. Notice that U.S. dollar amounts are in both numerators and Euro amounts are in both denominators.

$$\frac{1}{0.82} = \frac{520}{x}$$

Step 2: Cross multiply, since the ratios $\frac{a}{b}$ and $\frac{c}{d}$ are proportional, then $a \times d = b \times c$.

$$520\,(0.82) = 1\,(x)$$
$$426.4 = x$$

You should receive 426.4 Euros (426.4€).

YOUR TURN 5.30

1. After your trip to France, you have 180 Euros remaining. You wish to convert them back into U.S. dollars. Assuming the exchange rate is the same ($\$1 = 0.82$ €), how many dollars should you receive? Round to the nearest cent if necessary.

EXAMPLE 5.31

Solving a Proportion Involving Weights on Different Planets

A person who weighs 170 pounds on Earth would weigh 64 pounds on Mars. How much would a typical racehorse (1,000 pounds) weigh on Mars? Round your answer to the nearest tenth.

Solution

Step 1: Set up the two ratios into a proportion. Notice the Earth weights are both in the numerator and the Mars weights are both in the denominator.

$$\frac{170}{64} = \frac{1{,}000}{x}$$

Step 2: Cross multiply, and then divide to solve.

$$\begin{aligned} 170x &= 1{,}000\,(64) \\ 170x &= 64{,}000 \\ \frac{170x}{170} &= \frac{64{,}000}{170} \\ x &= 376.5 \end{aligned}$$

So the 1,000-pound horse would weigh about 376.5 pounds on Mars.

YOUR TURN 5.31

1. A person who weighs 200 pounds on Earth would weigh only 33 pounds on the moon. A 2021 Toyota Prius weighs 3,040 pounds on Earth; how much would it weigh on the moon? Round to the nearest tenth if necessary.

EXAMPLE 5.32

Solving a Proportion Involving Baking

A cookie recipe needs $2\frac{1}{4}$ cups of flour to make 60 cookies. Jackie is baking cookies for a large fundraiser; she is told she needs to bake 1,020 cookies! How many cups of flour will she need?

Solution

Step 1: Set up the two ratios into a proportion. Notice that the cups of flour are both in the numerator and the amounts of cookies are both in the denominator. To make the calculations more efficient, the cups of flour $\left(2\frac{1}{4}\right)$ is converted to a decimal number (2.25).

$$\frac{2.25}{60} = \frac{x}{1020}$$

Step 2: Cross multiply, and then simplify to solve.

$$\begin{aligned} 2.25\,(1{,}020) &= 60x \\ 2{,}295 &= 60x \\ 38.25 &= x \end{aligned}$$

Jackie will need 38.25, or $38\frac{1}{4}$, cups of flour to bake 1,020 cookies.

YOUR TURN 5.32

1. You are going to bake cookies, using the same recipe as above. You find out that you have 27 cups of flour in your pantry. Assuming you have all the other ingredients necessary, how many cookies can you make with 27

cups of flour?

Part of the definition of proportion states that two fractions $\frac{a}{b}$ *and* $\frac{c}{d}$ *are proportional if* $a \times d = b \times c$. *This is the "cross multiplication" rule that students often use (and unfortunately, often use incorrectly). The only time cross multiplication can be used is if you have two ratios (and only two ratios) set up in a proportion. For example, you cannot use cross multiplication to solve for* x *in an equation such as* $\frac{2}{5} = \frac{x}{8} + 3x$ *because you do not have just the two ratios. Of course, you could use the rules of algebra to change it to be just two ratios and then you could use cross multiplication, but in its present form, cross multiplication cannot be used.*

PEOPLE IN MATHEMATICS

Eudoxus was born around 408 BCE in Cnidus (now known as Knidos) in modern-day Turkey. As a young man, he traveled to Italy to study under Archytas, one of the followers of Pythagoras. He also traveled to Athens to hear lectures by Plato and to Egypt to study astronomy. He eventually founded a school and had many students.

Eudoxus made many contributions to the field of mathematics. In mathematics, he is probably best known for his work with the idea of proportions. He created a definition of proportions that allowed for the comparison of any numbers, even irrational ones. His definition concerning the equality of ratios was similar to the idea of cross multiplying that is used today. From his work on proportions, he devised what could be described as a method of integration, roughly 2000 years before calculus (which includes integration) would be fully developed by Isaac Newton and Gottfried Leibniz. Through this technique, Eudoxus became the first person to rigorously prove various theorems involving the volumes of certain objects. He also developed a planetary theory, made a sundial still usable today, and wrote a seven volume book on geography called *Tour of the Earth*, in which he wrote about all the civilizations on the Earth, and their political systems, that were known at the time. While this book has been lost to history, over 100 references to it by different ancient writers attest to its usefulness and popularity.

Determining and Applying a Constant of Proportionality

In the last example, we were given that $2\frac{1}{4}$ cups of flour could make 60 cookies; we then calculated that $38\frac{1}{4}$ cups of flour would make 1,020 cookies, and 720 cookies could be made from 27 cups of flour. Each of those three ratios is written as a fraction below (with the fractions converted to decimals). What happens if you divide the numerator by the denominator in each?

$$\frac{2.25}{60} = 0.0375 \qquad \frac{38.25}{1{,}020} = 0.0375 \qquad \frac{27}{720} = 0.0375.$$

The quotients in each are exactly the same! This number, determined from the ratio of cups of flour to cookies, is called the **constant of proportionality**. If the values a and b are related by the equality $\frac{a}{b} = k$, then k is the constant of proportionality between a and b. Note since $\frac{a}{b} = k$, then $b = \frac{a}{k}$. and $b = \frac{a}{k}$.

One piece of information that we can derive from the constant of proportionality is a unit rate. In our example (cups of flour divided by cookies), the constant of proportionality is telling us that it takes 0.0375 cups of flour to make one cookie. What if we had performed the calculation the other way (cookies divided by cups of flour)?

$$\frac{60}{2.25} = 26.66666... \qquad \frac{1{,}020}{38.25} = 26.66666... \qquad \frac{720}{27} = 26.66666...$$

In this case, the constant of proportionality $(26.66666\ldots = 26\frac{2}{3})$ is telling us that $26\frac{2}{3}$ cookies can be made with one cup of flour. Notice in both cases, the "one" unit is associated with the denominator. The constant of proportionality is also useful in calculations if you only know one part of the ratio and wish to find the other.

EXAMPLE 5.33

Finding a Constant of Proportionality

Isabelle has a part-time job. She kept track of her pay and the number of hours she worked on four different days, and recorded it in the table below. What is the constant of proportionality, or pay divided by hours? What does the constant of proportionality tell you in this situation?

Pay	$87.50	$50.00	$37.50	$100.00
Hours	7	4	3	8

Solution

To find the constant of proportionality, divide the pay by hours using the information from any of the four columns. For example, $\frac{87.5}{7} = 12.5$. The constant of proportionality is 12.5, or $12.50. This tells you Isabelle's hourly pay: For every hour she works, she gets paid $12.50.

YOUR TURN 5.33

1. The following table contains the lengths of four objects in both inches and centimeters. What is the constant of proportionality (centimeters divided by inches)? What does the constant of proportionality tell you in this situation?

Object	floor tile	book	table	pencil
Length (in.)	24	13	60	7.5
Length (cm)	60.96	33.02	152.4	19.05

EXAMPLE 5.34

Applying a Constant of Proportionality: Running mph

Zac runs at a constant speed: 4 miles per hour (mph). One day, Zac left his house at exactly noon (12:00 PM) to begin running; when he returned, his clock said 4:30 PM. How many miles did he run?

Solution

The constant of proportionality in this problem is 4 miles per hour (or 4 miles in 1 hour). Since $\frac{a}{b} = k$, where k is the constant of proportionality, we have

$\frac{a \text{ miles}}{b \text{ hours}} = k$

$\frac{a}{4.5} = 4$ (30 minutes is ½, or 0.5, hours)

$a = 4\,(4.5)$, since from the definition we know $a = kb$

$a = 18$

Zac ran 18 miles.

YOUR TURN 5.34

1. One week, Zac ran a total of 122 miles. How much time did he spend running in that week?

EXAMPLE 5.35

Applying a Constant of Proportionality: Filling Buckets

Joe had a job where every time he filled a bucket with dirt, he was paid \$2.50. One day Joe was paid \$337.50. How many buckets did he fill that day?

Solution

The constant of proportionality in this situation is \$2.50 per bucket (or \$2.50 for 1 bucket). Since $\frac{a}{b} = k$, where k is the constant of proportionality, we have

$$\begin{aligned} \frac{a \text{ dollars}}{b \text{ buckets}} &= k \\ \frac{337.50}{b} &= 2.50 \end{aligned}$$

Since we are solving for b, and we know from the definition that $b = \dfrac{a}{k}$:

$$\begin{aligned} b &= \frac{337.50}{2.50} \\ b &= 135 \end{aligned}$$

Joe filled 135 buckets.

YOUR TURN 5.35

1. Suppose one day Joe filled 83 buckets; how much money would he make on that day?

EXAMPLE 5.36

Applying a Constant of Proportionality: Miles vs. Kilometers

While driving in Canada, Mabel quickly noticed the distances on the road signs were in kilometers, not miles. She knew the constant of proportionality for converting kilometers to miles was about 0.62—that is, there are about 0.62 miles in 1 kilometer. If the last road sign she saw stated that Montreal is 104 kilometers away, about how many more miles does Mabel have to drive? Round your answer to the nearest tenth.

Solution

The constant of proportionality in this situation is 0.62 miles per 1 kilometer. Since $\frac{a}{b} = k$, where k is the constant of proportionality, we have

$$\begin{aligned} \frac{a \text{ miles}}{b \text{ kilometers}} &= k \\ \frac{a}{104} &= 0.62 \\ a &= 0.62(104) \\ a &= 64.48 \end{aligned}$$

Rounding the answer to the nearest tenth, Mabel has to drive 64.5 miles.

YOUR TURN 5.36

1. Later in her trip, Mabel decides to drive to the capital of Canada, Ottawa. As she left Montreal, she saw a road sign that read that Ottawa is 203 kilometers away. About how many miles is that? Round your answer to the nearest tenth.

Using Proportions to Solve Scaling Problems

Figure 5.16 A map of the northeastern United States

Ratio and proportions are used to solve problems involving **scale**. One common place you see a scale is on a map (as represented in Figure 5.16). In this image, 1 inch is equal to 200 miles. This is the scale. This means that 1 inch on the map corresponds to 200 miles on the surface of Earth. Another place where scales are used is with models: model cars, trucks, airplanes, trains, and so on. A common ratio given for model cars is 1:24—that means that 1 inch in length on the model car is equal to 24 inches (2 feet) on an actual automobile. Although these are two common places that scale is used, it is used in a variety of other ways as well.

EXAMPLE 5.37

Solving a Scaling Problem Involving Maps

Figure 5.17 is an outline map of the state of Colorado and its counties. If the distance of the southern border is 380 miles, determine the scale (i.e., 1 inch = how many miles). Then use that scale to determine the approximate lengths of the other borders of the state of Colorado.

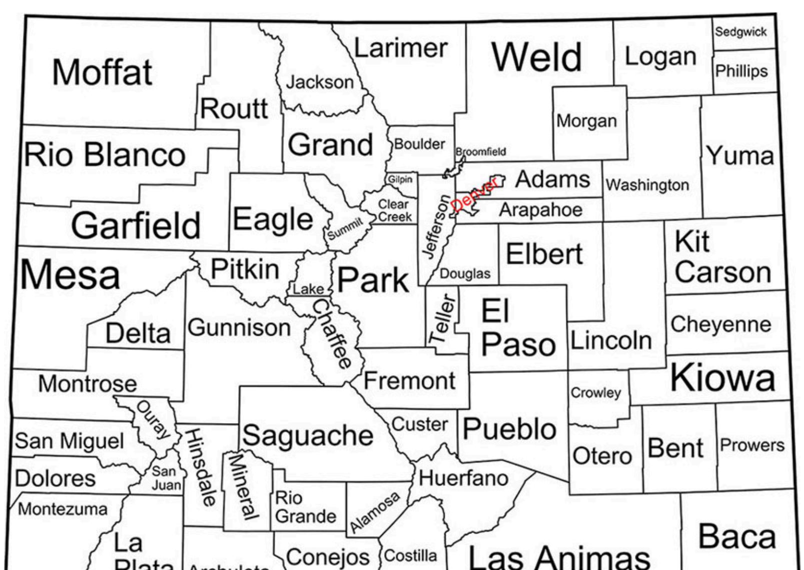


Figure 5.17 Outline Map of Colorado (credit: "Map of Colorado Counties" by David Benbennick/Wikimedia Commons,

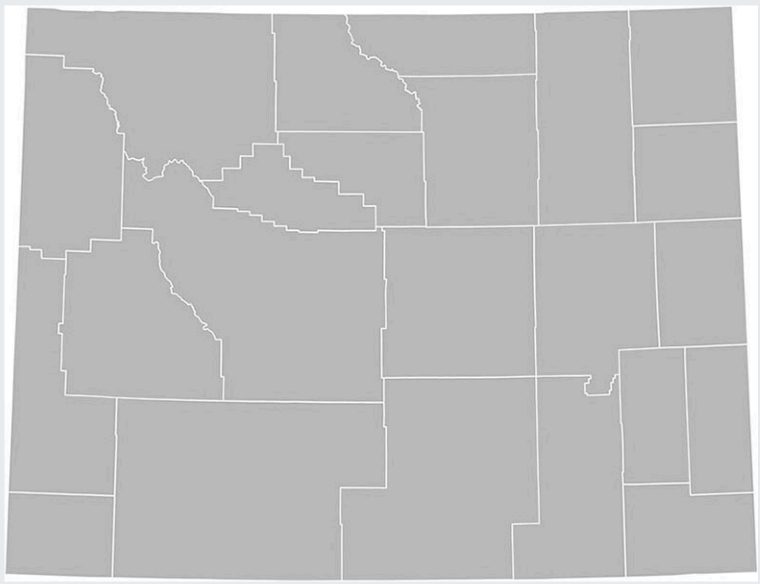
Public Domain)

Solution

When the southern border is measured with a ruler, the length is 4 inches. Since the length of the border in real life is 380 miles, our scale is 1 inch $= 95$ miles.

The eastern and western borders both measure 3 inches, so their lengths are about 285 miles. The northern border measures the same as the southern border, so it has a length of 380 miles.

YOUR TURN 5.37

1. 

Outline Map of Wyoming (credit: "Blank map of Wyoming showing counties" by David Benbennick/Wikimedia Commons, Public Domain)

Consider the outline map of the state of Wyoming and its counties. If the distance of the southern border is 365 miles, determine the scale (i.e., $1 \text{ inch} =$ how many miles). Then use that scale to determine the approximate lengths of the other borders of the state of Wyoming.

EXAMPLE 5.38

Solving a Scaling Problem Involving Model Cars

Die-cast NASCAR model cars are said to be built on a scale of 1:24 when compared to the actual car. If a model car is 9 inches long, how long is a real NASCAR automobile? Write your answer in feet.

Solution

The scale tells us that 1 inch of the model car is equal to 24 inches (2 feet) on the real automobile. So set up the two ratios into a proportion. Notice that the model lengths are both in the numerator and the NASCAR automobile lengths are both in the denominator.

$$\begin{aligned} \frac{1}{24} &= \frac{9}{x} \\ 24(9) &= x \\ 216 &= x \end{aligned}$$

This amount (216) is in inches. To convert to feet, divide by 12, because there are 12 inches in a foot (this conversion from inches to feet is really another proportion!). The final answer is:

$$\frac{216}{12} = 18$$

The NASCAR automobile is 18 feet long.

YOUR TURN 5.38

1. A toy Jeep is built on a $1:16$ scale. The website for the toy Jeep says the toy is 11.5 inches long. Based on this, how long is the real Jeep?

Check Your Understanding

32. If a:$b = c$:d, then b:$a = d$:c for all non-zero whole numbers a, b, c, and d.
  a. True
  b. False

33. If the ratio of wolves to rabbits in a national park is 3:5, then the ratio of rabbits to (wolves and rabbits) is 5:8.
  a. True
  b. False

34. All fractions are ratios but not all ratios are fractions.
  a. True
  b. False

35. In the following equation, $\frac{x}{46} = \frac{20}{90} + 4$, cross multiplication can be used as the first step towards solving for x.
  a. True
  b. False

36. All fractions are ratios but not all ratios are fractions.
  a. True
  b. False

37. There are 16 math majors and 12 non-math majors in Ms. Kraft's class. What is **not** a correct way to express the ratio of math majors to non-math majors?

$16:12$	$12:16$	$4:3$

38. There are 16 math majors and 12 non-math majors in Ms. Kraft's class. What shows the ratio of math majors to all the students in Ms. Kraft's class?

$16:12$	$12:16$	$16:28$	$28:12$	None of these

39. One U.S. dollar is worth 0.72 British pounds. Damon is traveling to Great Britain and wishes to exchange $450 U.S. dollars for British pounds. How many British pounds should Damon get in return?

625	6,250	3,456	345.6	None of these

40. The HO scale for model trains is the most common size of model trains. This scale is $1:87$. If a real locomotive is 73 feet long, how long should the model locomotive be (in inches)? Round your answer to the nearest inch.

41. Albert's Honda Civic gets 37 miles per gallon of gasoline. The gas tank on the Civic can hold 13.5 gallons of gas. Albert is driving from Tucson, Arizona to Los Angeles, California, a distance of 485 miles. Albert thinks he can make it on one full tank of gasoline. Can he? Explain.

42. The average price of a gallon of regular gasoline in the California on July 1, 2021 was $\$4.28$ per gallon. Albert stops at a gas station in California and puts 9.5 gallons of gasoline into his Civic. How much did he pay for the gas?

SECTION 5.4 EXERCISES

For the following exercises, use this scenario: Kelly opened a bag of colored chocolate coated candies and counted the number of each color of candy. She found she had 9 green, 4 yellow, 13 black, 11 orange, 8 blue, and 7 red. What is the ratio of the following candy colors?

1. Red candies to green candies
2. Green candies to black candies
3. Yellow candies to black candies
4. Black candies to blue candies
5. Orange candies to non-orange candies
6. Yellow candies to non-yellow candies
7. Red candies to all candies
8. Pink candies to all candies
9. Candies with the letter 'r' in their name to all candies
10. Candies with the letter 'r' in their name to candies without the letter 'r' in their name

For the following exercises, solve each proportion for the unknown variable.

11. $\frac{x}{46} = \frac{4}{8}$
12. $\frac{27}{x} = \frac{3.5}{14}$
13. $\frac{16}{7} = \frac{x}{14}$
14. $\frac{1}{0.8} = \frac{x}{514}$
15. $\frac{203}{g} = \frac{10}{22}$
16. $\frac{1}{29.5} = \frac{16}{m}$
17. $\frac{14}{95} = \frac{a}{19}$
18. $\frac{13}{140} = \frac{s}{2961}$
19. $\frac{1}{k} = \frac{69}{111}$ (Round answer to the nearest hundredth.)
20. $\frac{p}{1} = \frac{22}{7}$ (Round answer to the nearest hundredth.)

21. Pet Paradise has 20 cats and 16 dogs. Animal Acres has 15 cats. How many dogs must be at Animal Acres so that Pet Paradise and Animal Acres have the same ratio of cats to dogs?

22. Pet Paradise has 20 cats and 16 dogs. Critter Corral has 28 dogs. How many cats must be at Critter Corral so that Pet Paradise and Critter Corral have the same ratio of cats to dogs?

23. A high school has 960 students. The ratio of students to high school teachers is $16 : 1$. How many high school teachers are at the school?

24. A high school has 960 students. The ratio of students to high school teachers is $16 : 1$. How many more teachers are needed to have a $12 : 1$ ratio at the high school of students to teachers?

25. One U.S. dollar is worth $1.23 Canadian dollars. Bernice is traveling to Canada and wants to convert $550 U.S. to Canadian money. How much in Canadian money should she receive?

26. One U.S. dollar is worth $1.23 Canadian dollars. Rene is traveling from Canada to the United States and wants to convert $550 of Canadian money to U.S. money. How much in U.S. money should he receive? Round your answer to the nearest cent.

27. One U.S. dollar is worth $1.23 Canadian dollars. What is one Canadian dollar worth in U.S. funds? Round your answer to the nearest cent.

28. A salad recipe needs one cup of crushed almonds. It will serve eight people. Rashida needs to make a salad to serve 20 people. How many cups of crushed almonds does she need?

29. A salad recipe needs one cup of crushed almonds. It will serve eight people. Elmer has 4.75 cups of crushed almonds. If he uses all of the crushed almonds he has to make this salad, how many people will it serve?

30. Jorge is 6 feet tall and casts a 7-foot shadow. At the same time, a nearby tree has a shadow of 56 feet. How tall is the tree?

31. Tony can run 4 kilometers in 30 minutes. At that rate, how far could he run in 1 hour, 45 minutes?

32. Kara's parent owns a restaurant. When she came in one day, they asked her to figure out how much they were spending per ounce on steak they were buying from a vendor. They had their last four receipts, but unfortunately they spilled liquid on them and some parts were unreadable. Find out how much Kara's parent is spending per ounce on steak; then use that information to fill in the unreadable parts of the receipts (labeled a, b, and c below).

Receipt	1	2	3	4
Ounces	128	460	b	541
Cost	$163.84	a	$277.76	c

33. The scale for a map reads "$1 \text{ inch } = 250 \text{ miles}$." You measure the distance on the map from Fargo, North Dakota to Winnipeg, Manitoba and get 1.44 inches. How far is it from Fargo to Winnipeg?

34. Hot Wheels toy cars are said to be built on a scale of $1{:}64$ when compared to the actual car. If a real car is 18 feet long, how long should the Hot Wheels toy car be (in inches)?

35. The Eiffel Tower in Paris, France, is 1,067 feet tall. The replica Eiffel Tower in Las Vegas, Nevada, is built on the scale of $1.976{:}1$. How tall is the replica Eiffel Tower in Las Vegas? Round your answer to the nearest foot.

5.5 Graphing Linear Equations and Inequalities

Figure 5.18 How much would it cost to fill up your gas tank? (credit: "Gas Under 4 Bucks" by Mark Turnauckas, Flickr/CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Graph linear equations and inequalities in two variables.
2. Solve applications of linear equations and inequalities.

In this section, we will learn how to graph linear equations and inequalities. There are several real-world scenarios that can be represented by graphs of linear inequalities. Think of filling your car up with gasoline. If gasoline is $3.99 per gallon and you put 10 gallons in your car, you will pay $39.90. Your friend buys 15 gallons of gasoline and pays $59.85. You can plot these points on a coordinate system and connect the points with a line to create the graph of a line. You'll learn to do both in this section.

Plotting Points on a Rectangular Coordinate System

Just like maps use a grid system to identify locations, a grid system is used in algebra to show a relationship between two variables in a rectangular coordinate system. The rectangular coordinate system is also called the xy-plane or the “coordinate plane.”

The rectangular coordinate system is formed by two intersecting number lines, one horizontal and one vertical. The horizontal number line is called the x-axis. The vertical number line is called the y-axis. These axes divide a plane into four regions, called quadrants. The quadrants are identified by Roman numerals, beginning on the upper right and proceeding counterclockwise. See Figure 5.19.

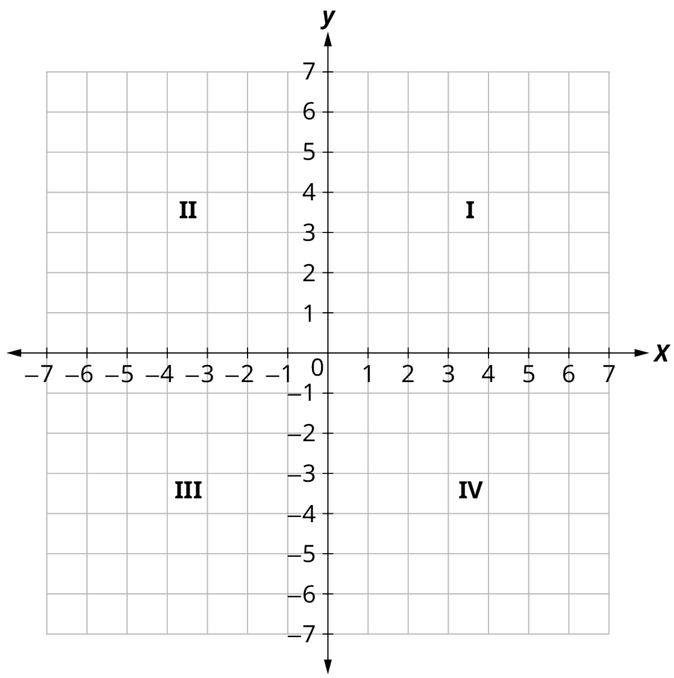


Figure 5.19 Quadrants on the Coordinate Plane

In the rectangular coordinate system, every point is represented by an **ordered pair** (Figure 5.20). The first number in the ordered pair is the x-coordinate of the point, and the second number is the y-coordinate of the point. The phrase "ordered pair" means that the order is important. At the point where the axes cross and where both coordinates are zero, the ordered pair is $(0, 0)$. The point $(0, 0)$ has a special name. It is called the **origin**.

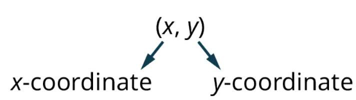


Figure 5.20 Ordered Pair

We use the coordinates to locate a point on the xy-plane. Let's plot the point $(1, 3)$ as an example. First, locate 1 on the x-axis and lightly sketch a vertical line through $x = 1$. Then, locate 3 on the y-axis and sketch a horizontal line through $y = 3$. Now, find the point where these two lines meet—that is the point with coordinates $(1, 3)$. See Figure 5.21.

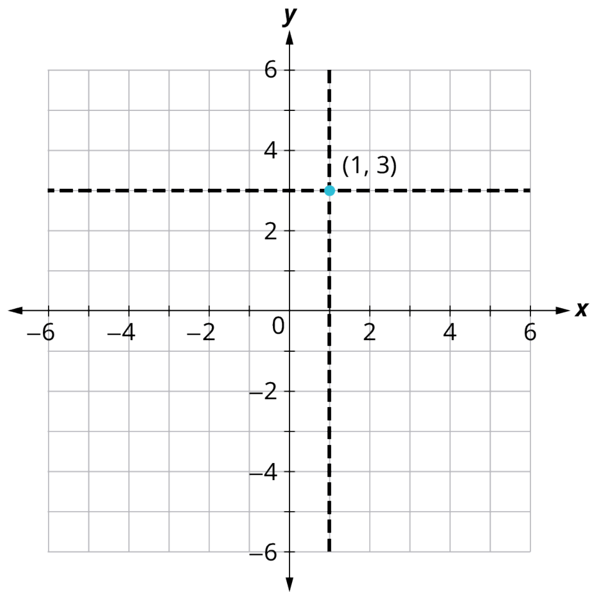


Figure 5.21 Point $(1, 3)$ Plotted on the Coordinate Plane

Notice that the vertical line through $x = 1$ and the horizontal line through $y = 3$ are not part of the graph. The dotted lines are just used to help us locate the point $(1, 3)$. When one of the coordinates is zero, the point lies on one of the axes. In Figure 5.22, the point $(0, 4)$ is on the y-axis and the point (−2, 0) is on the x-axis.

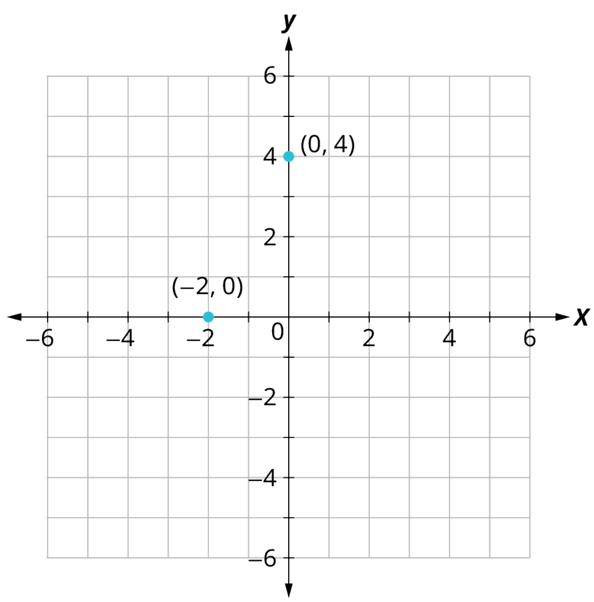


Figure 5.22 Points $(-2, 0)$ and $(0, 4)$ Plotted on the Coordinate Plane

EXAMPLE 5.39

Plotting Points on a Coordinate System

Plot the following points in the rectangular coordinate system and identify the quadrant in which the point is located:

1. $(-5, 4)$
2. $(-3, -4)$
3. $(2, -3)$
4. $(0, -1)$
5. $\left(3, \frac{5}{2}\right)$

Solution

The first number of the coordinate pair is the x-coordinate, and the second number is the y-coordinate. To plot each point, sketch a vertical line through the x-coordinate and a horizontal line through the y-coordinate (Figure 5.23). Their intersection is the point.

1. Since $x = -5$, the point is to the left of the y-axis. Also, since $y = 4$, the point is above the x-axis. The point $(-5, 4)$ is in quadrant II.
2. Since $x = -3$, the point is to the left of the y-axis. Also, since $y = -4$, the point is below the x-axis. The point $(-3, -4)$ is in quadrant III.
3. Since $x = 2$, the point is to the right of the y-axis. Since $y = -3$, the point is below the x-axis. The point $(2, -3)$ is in quadrant IV.
4. Since $x = 0$, the point whose coordinates are $(0, -1)$ is on the y-axis.
5. Since $x = 3$, the point is to the right of the y-axis. Since $y = \frac{5}{2}$, which is equal to 2.5, the point is above the x-axis. The point $\left(3, \frac{5}{2}\right)$ is in quadrant I.

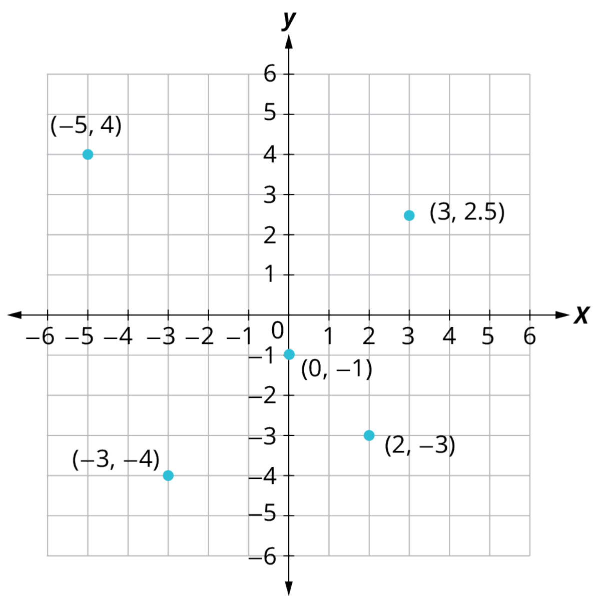


Figure 5.23

YOUR TURN 5.39

1. Plot the following points in the rectangular coordinate system and identify the quadrant in which the point is located:

a. $(-4, 2)$
b. $(-1, -2)$
c. $(3, -5)$
d. $(-3, 0)$
e. $(\frac{5}{3}, 2)$

Graphing Linear Equations in Two Variables

Up to now, all the equations you have solved were equations with just one variable. In almost every case, when you solved the equation, you got exactly one solution. But equations can have more than one variable. Equations with two variables may be of the form $Ax + By = C$. An equation of this form, where A and B are both not zero, is called a **linear equation in two variables**. Here is an example of a linear equation in two variables, x and y.

$$\begin{aligned}&\boldsymbol{Ax + By = C}\\&\boldsymbol{A + 4y = 8}\\&\boldsymbol{A = 1, B = 4, C = 8}\end{aligned}$$

The equation $y = -3x + 5$ is also a linear equation. But it does not appear to be in the form $Ax + By = C$. We can use the addition property of equality and rewrite it in $Ax + By = C$ form.

Step 1: Add $3x$ to both sides. $y + 3x = -3x + 5 + 3x$

Step 2: Simplify. $y + 3x = 5$

Step 3: Put it in $Ax + By = C$ form. $3x + y = 5$

By rewriting $y = -3x + 5$ as $3x + y = 5$, we can easily see that it is a linear equation in two variables because it is of the form $Ax + By = C$. When an equation is in the form $Ax + By = C$, we say it is in **standard form of a linear equation**. Most people prefer to have A, B, and C be integers and $A \geq 0$ when writing a linear equation in standard form, although it is not strictly necessary.

Linear equations have infinitely many solutions. For every number that is substituted for x there is a corresponding y value. This pair of values is a **solution** to the linear equation and is represented by the ordered pair (x, y). When we substitute these values of x and y into the equation, the result is a true statement, because the value on the left side is equal to the value on the right side.

We can plot these solutions in the rectangular coordinate system. The points will line up perfectly in a straight line. We connect the points with a straight line to get the graph of the linear equation. We put arrows on the ends of each side of the line to indicate that the line continues in both directions.

A graph is a visual representation of all the solutions of a linear equation. The line shows you *all* the solutions to that linear equation. Every point on the line is a solution of that linear equation. And every solution of the linear equation is on this line. This line is called the graph of the equation. Points *not* on the line are not solutions! The graph of a linear equation $Ax + By = C$ is a straight line.

- Every point on the line is a solution of the equation.
- Every solution of this equation is a point on this line.

EXAMPLE 5.40

Determining Points on a Line

Figure 5.24 is the graph of $y = 2x - 3$.

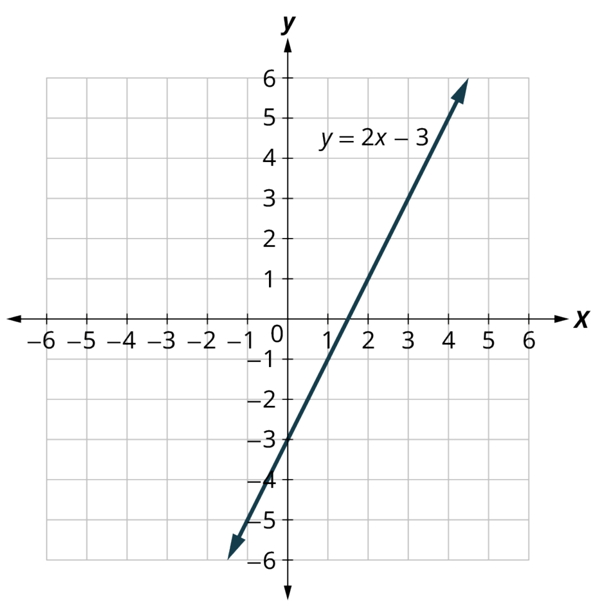


Figure 5.24 Graph of $y = 2x – 3$

For each ordered pair, decide:

I. Is the ordered pair a solution to the equation?
II. Is the point on the line?

A: $(0,-3)$ B: $(3,3)$ C: $(2,-3)$ D: $(-1,-5)$

Solution

Substitute the x- and y-values into the equation to check if the ordered pair is a solution to the equation.

I.

$$\begin{array}{cccc} \text{A}: (0,-3) & \text{B}: (3,3) & \text{C}: (2,-3) & \text{D}: (-1,-5) \\ y = 2x-3 & y = 2x-3 & y = 2x-3 & y = 2x-3 \\ -3 \stackrel{?}{=} 2(0)-3 & 3 \stackrel{?}{=} 2(3)-3 & -3 \stackrel{?}{=} 2(2)-3 & -5 \stackrel{?}{=} 2(-1)-3 \\ -3 = -3 ✓ & 3 = 3 ✓ & -3 \neq 1 & -5 = -5 ✓ \\ (0,-3) \text{ is a solution.} & (3,3) \text{ is a solution.} & (2,-3) \text{ is not a solution.} & (-1,-5) \text{ is a solution.} \end{array}$$

II. Plot the points $(0,-3)$, $(3,3)$, $(2,-3)$, and $(-1,-5)$.

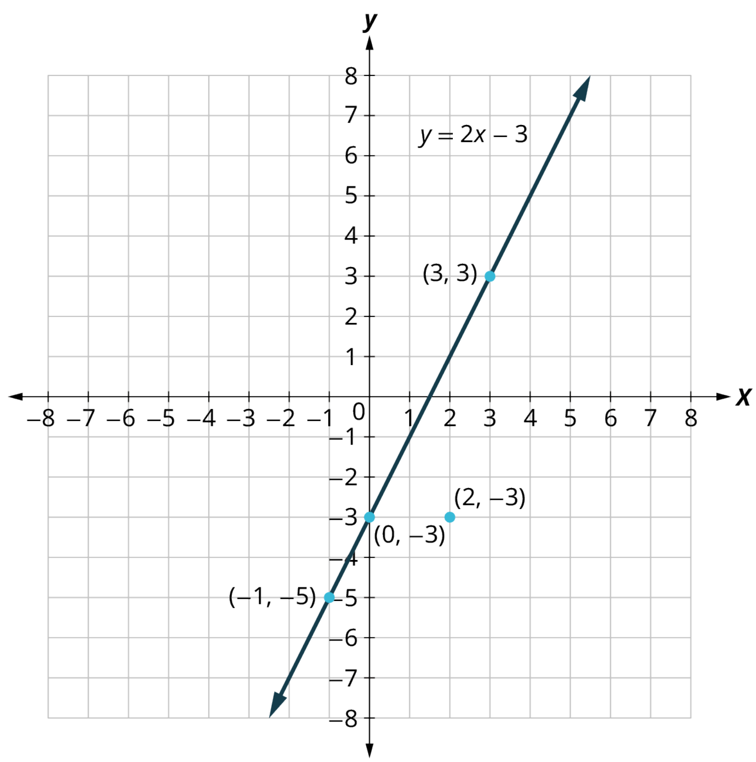
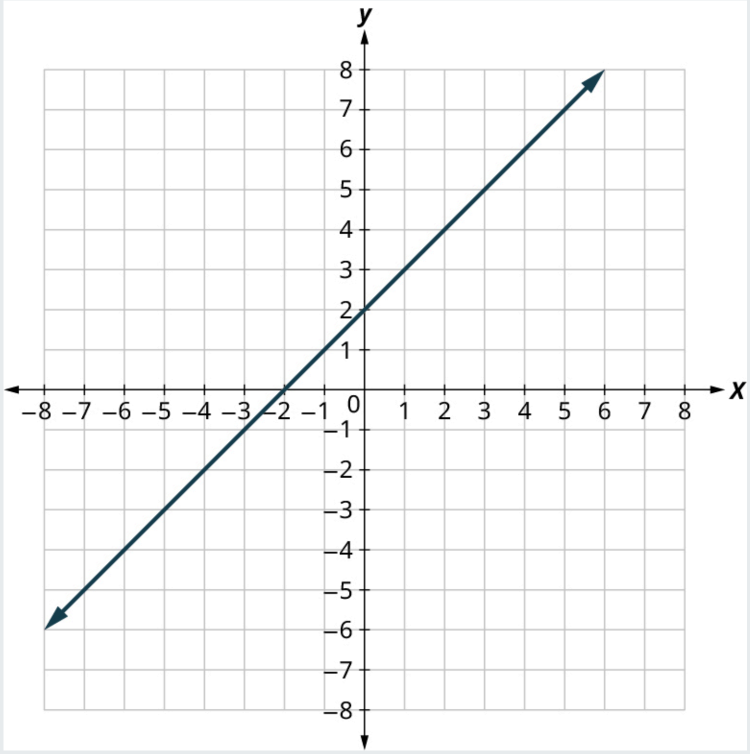


Figure 5.25

In Figure 5.25, the points $(0,3)$, $(3,-3)$, and $(-1,-5)$ are on the line $y = 2x - 3$, and the point $(2,-3)$ is not on the line. The points that are solutions to $y = 2x - 3$ are on the line, but the point that is not a solution is not on the line.

YOUR TURN 5.40

The given figure is the graph of $y = x + 2$.



Graph of $y = x + 2$

For each ordered pair below, decide:

I. Is the ordered pair a solution to the equation?

II. Is the point on the line?

1. $(0, 2)$
2. $(1, 2)$
3. $(-1, 1)$
4. $(-3, -1)$

The steps to take when graphing a linear equation by plotting points are:

Step 1: Find three points whose coordinates are solutions to the equation. Organize them in a table.

Step 2: Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.

Step 3: Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

It is true that it only takes two points to determine a line, but it is a good habit to use three points. If you only plot two points and one of them is incorrect, you can still draw a line, but it will not represent the solutions to the equation. It will be the wrong line. If you use three points, and one is incorrect, the points will not line up. This tells you something is wrong, and you need to check your work.

EXAMPLE 5.41

Graphing a Line by Plotting Points

Graph the equation: $y = \frac{1}{2}x + 3$.

Solution

Find three points that are solutions to the equation. Since this equation has the fraction $\frac{1}{2}$ as a coefficient of x, we will choose values of x carefully. We will use zero as one choice and multiples of 2 for the other choices. Why are multiples of two a good choice for values of x? By choosing multiples of 2, the multiplication by $\frac{1}{2}$ simplifies to a whole number.

$x = 0$	$x = 2$	$x = 4$
$y = \frac{1}{2}x + 3$	$y = \frac{1}{2}x + 3$	$y = \frac{1}{2}x + 3$
$y = \frac{1}{2}(0) + 3$	$y = \frac{1}{2}(2) + 3$	$y = \frac{1}{2}(4) + 3$
$y = 0 + 3$	$y = 1 + 3$	$y = 2 + 3$
$y = 3$	$y = 4$	$y = 5$

x	y	(x, y)
0	3	$(0, 3)$
2	4	$(2, 4)$
4	5	$(4, 5)$

Plot the points, check that they line up, and draw the line (Figure 5.26).

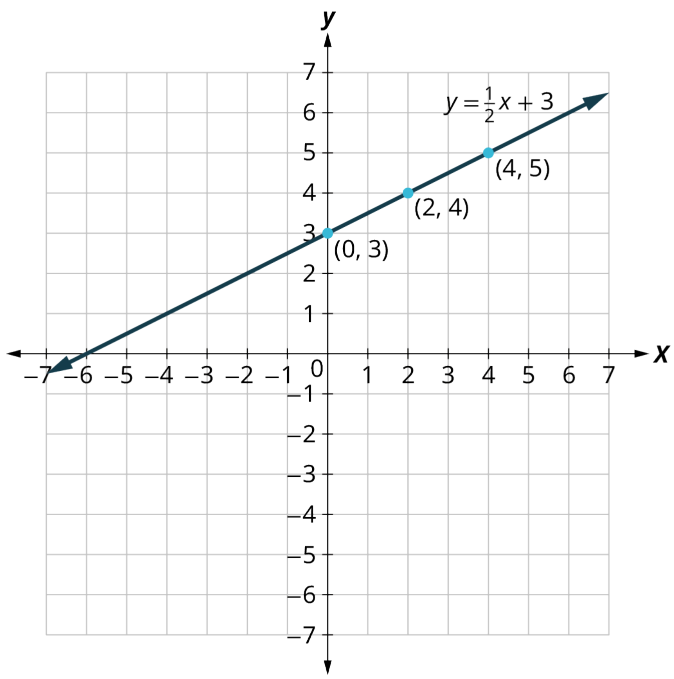


Figure 5.26

> **YOUR TURN 5.41**
>
> 1. Graph the equation $y = 3x - 1$ by plotting points.

Solving Applications Using Linear Equations in Two Variables

Many fields use linear equalities to model a problem. While our examples may be about simple situations, they give us an opportunity to build our skills and to get a feel for how they might be used.

EXAMPLE 5.42

Pumping Gas

Gasoline costs $3.53 per gallon. You put 10 gallons of gasoline in your car, and pay $35.30. Your friend puts 15 gallons of gasoline in their car and pays $52.95. Your neighbor needs 5 gallons of gasoline, how much will they pay?

Solution

Let $x =$ the number of gallons of gasoline and let $y =$ the total cost. If gas is $3.53 per gallon, then $y = 3.53x$. The two points given are (10, 35.30) and (15, 52.95). Plot the points, check that they line up, and draw the line (Figure 5.27).

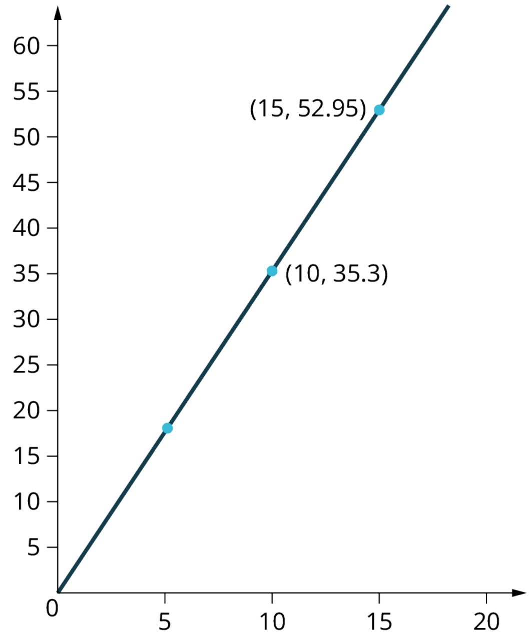


Figure 5.27

We can see the point at $x = 5$. The y-value is found by multiplying 5 by $3.53 to get $17.65. Your neighbor will pay $17.65.

YOUR TURN 5.42

1. If a stamp costs $0.55 and you buy a book of 20 stamps, then you pay $11. If you want to mail 100 letters, you can buy a roll of stamps for $55. Your friend only needs 3 stamps, how much will they pay?

PEOPLE IN MATHEMATICS

René Descartes

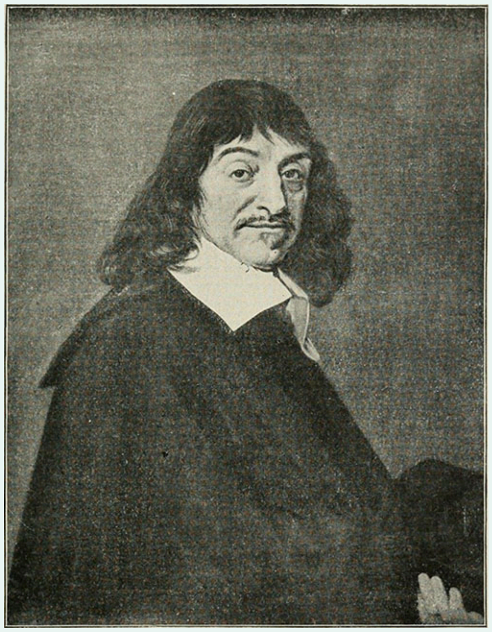


Figure 5.28 René Descartes (credit: Flickr, Public Domain)

René Descartes was born in 1596 in La Haye, France. He was sickly as a child, so much so that he was allowed to stay in bed until 11:00 AM rather than get up at 5:00 AM like the other school children. He kept this habit of rising late for most of the rest of his life.

After his primary schooling, Descartes attended the University of Poitiers, receiving a law degree in 1616. He then embarked on a myriad of journeys, joining two different militaries (one in the Netherlands, the other in Bavaria) and generally travelling around Europe until 1628, when he settled in the Netherlands. It was here that he began to delve deeply into his ideas of science, mathematics, and philosophy.

In 1637, at the urging of his friends, Descartes published *Discourse on the Method for Conducting One's Reason Well and Seeking the Truth in the Sciences*. The book had three appendices: *La Dioptrique*, a work on optics; *Les Météores*, which pertained to meteorology; and *La Géométrie*, a work on mathematics. It was in this appendix that he proposed a geometric way of representing many different algebraic expressions and equations. It is this system of representation that almost all mathematical textbooks use today.

These publications (along with several others) brought much fame to Descartes. So renowned was his reputation that late in 1649, Queen Christina of Sweden asked Descartes to come to Sweden to tutor her. However, she wished to do her studies at 5:00 in the morning; Descartes had to break his lifelong habit of sleeping in late. A few months later, in February 1650, Descartes died of pneumonia.

Graphing Linear Inequalities

Previously we learned to solve inequalities with only one variable. We will now learn about inequalities containing two variables that can be written in one of the following forms: $Ax + By \geq C$, $Ax + By > C$, $Ax + By \leq C$, and $Ax + By < C$ where A and B are not both zero. We will look at **linear inequalities in two variables,** which are very similar to linear equations in two variables.

Like linear equations, linear inequalities in two variables have many solutions. Any ordered pair (x, y) that makes an inequality true when we substitute in the values is a **solution to a linear inequality**.

EXAMPLE 5.43

Determining Solutions to an Inequality

Determine whether each ordered pair is a solution to the inequality $y > x + 4$:

1. $(0, 0)$
2. $(1, 6)$
3. $(2, 6)$
4. $(-5, -15)$
5. $(-8, 12)$

Solution

1.

$(0, 0)$	$y > x + 4$
Substitute 0 for x and 0 for y	$0 \overset{?}{>} 0 + 4$
Simplify.	$0 \not> 4$
	$(0, 0)$ is not a solution to $y > x + 4$.

2.

$(1, 6)$	$y > x + 4$
Substitute 1 for x and 6 for y	$6 \overset{?}{>} 1 + 4$
Simplify.	$6 \not> 5$
	$(1, 6)$ is not a solution to $y > x + 4$.

3.

$(2, 6)$	$y > x + 4$
Substitute 2 for x and 6 for y	$6 \overset{?}{>} 2 + 4$
Simplify.	$6 \not> 6$
	$(2, 6)$ is not a solution to $y > x + 4$.

4.

$(-5, -15)$	$y > x + 4$
Substitute -5 for x and -15 for y	$-15 \overset{?}{>} -5 + 4$
Simplify.	$-15 \not> -1$
	So, $(-5, -15)$ is not a solution to $y > x + 4$.

5.

$(-8, 12)$	$y > x + 4$
Substitute -8 for x and12 for y	$12 \overset{?}{>} -8 + 4$
Simplify.	$12 > -4$
	So, $(-8, 12)$ is a solution to $y > x + 4$.

YOUR TURN 5.43

Determine whether each ordered pair is a solution to the inequality $y > x - 1$:

1. $(0, 1)$
2. $(-4, -1)$
3. $(4, 2)$
4. $(3, 0)$
5. $(-2, -3)$

Let us think about $x > 3$. The point $x = 3$ separated that number line into two parts. On one side of 3 are all the numbers less than 3. On the other side of 3 all the numbers are greater than 3. See Figure 5.29.

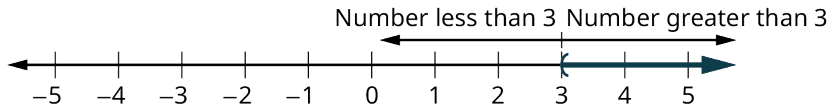


Figure 5.29 Solution to $x > 3$ on a Number Line

Similarly, the line $y = x + 4$ separates the plane into two regions. On one side of the line are points with $y < x + 4$. On the other side of the line are the points with $y > x + 4$. We call the line $y = x + 4$ a **boundary line**.

For an inequality in one variable, the endpoint is shown with a parenthesis (Figure 5.30) or a bracket (Figure 5.31) depending on whether or not a is included in the solution:

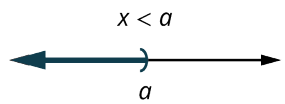


Figure 5.30 Endpoint with Parenthesis

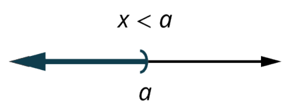


Figure 5.31 Endpoint with Bracket

Similarly, for an inequality in two variables, the boundary line is shown with a solid or dashed line to show whether or not it the line is included in the solution.

$Ax + By < C$	$Ax + By \leq C$
$Ax + By > C$	$Ax + By \geq C$
Boundary line is $Ax + By = C$	Boundary line is $Ax + By = C$
Boundary line is not included in solution.	Boundary line is included in solution.
Boundary line is dashed.	Boundary line is solid.

Now, let us take a look at what we found in Example 5.43. We will start by graphing the line $y = x + 4$, and then we will plot the five points we tested, as graphed in Figure 5.32. We found that some of the points were solutions to the inequality $y > x + 4$ and some were not. Which of the points we plotted are solutions to the inequality $y > x + 4$? The points $(1, 6)$ and $(-8, 12)$ are solutions to the inequality $y > x + 4$. Notice that they are both on the same side of the boundary line $y = x + 4$. The two points $(0, 0)$ and $(-5, -15)$ are on the other side of the boundary line $y = x + 4$, and they are not solutions to the inequality $y > x + 4$. For those two points, $y < x + 4$. What about the point $(2, 6)$? Because $6 = 2 + 4$, the point is a solution to the equation $y = x + 4$, but not a solution to the inequality $y > x + 4$. So, the point $(2, 6)$ is on the boundary line.

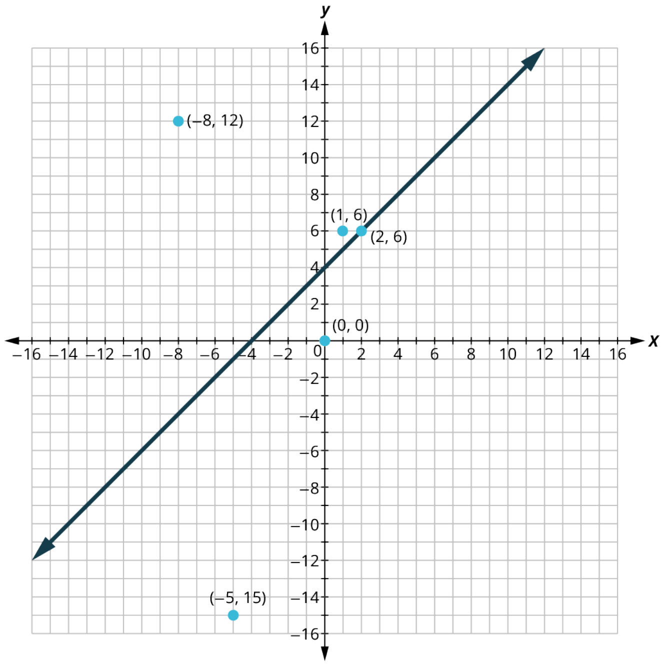


Figure 5.32 Graph of $y = x + 4$

Let us take another point above the boundary line and test whether or not it is a solution to the inequality $y > x + 4$. The point $(0, 10)$ clearly looks to be above the boundary line, doesn't it? Is it a solution to the inequality?

$$y > x + 4$$
$$10 \overset{?}{>} 0 + 4$$
$$10 > 4 ✓$$

Yes, $(0, 10)$ is a solution to $y > x + 4$. Any point you choose above the boundary line is a solution to the inequality $y > x + 4$. All points above the boundary line are solutions. Similarly, all points below the boundary line, the side with $(0, 0)$ and $(-5, -15)$, are not solutions to $y > x + 4$, as shown in Figure 5.33.

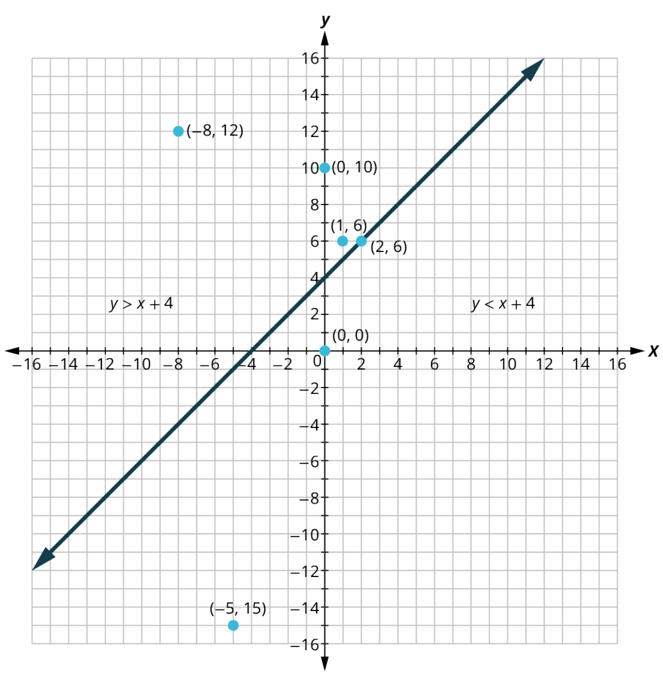


Figure 5.33 Graph of $y = x + 4$, with $y > x + 4$ Above the Boundary Line and $y < x + 4$ Below the Boundary Line

The graph of the inequality $y > x + 4$ is shown in Figure 5.34. The line $y = x + 4$ divides the plane into two regions. The shaded side shows the solutions to the inequality $y > x + 4$. The points on the boundary line, those where $y = x + 4$, are not solutions to the inequality $y > x + 4$, so the line itself is not part of the solution. We show that by making the boundary line dashed, not solid.

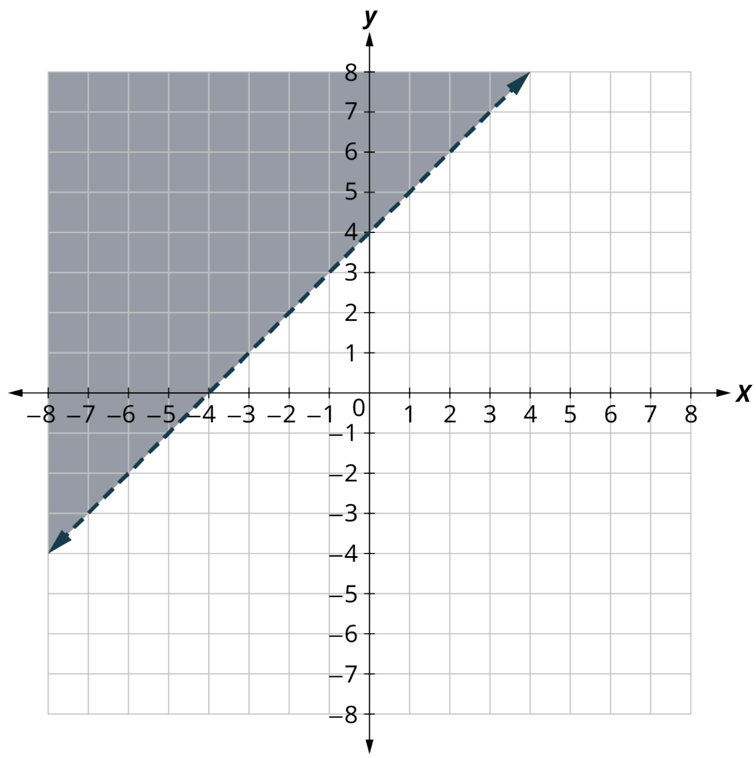


Figure 5.34 Graph of $y > x + 4$

EXAMPLE 5.44

Writing a Linear Inequality Shown by a Graph

The boundary line shown in this graph is $y = 2x - 1$. Write the inequality shown in Figure 5.35.

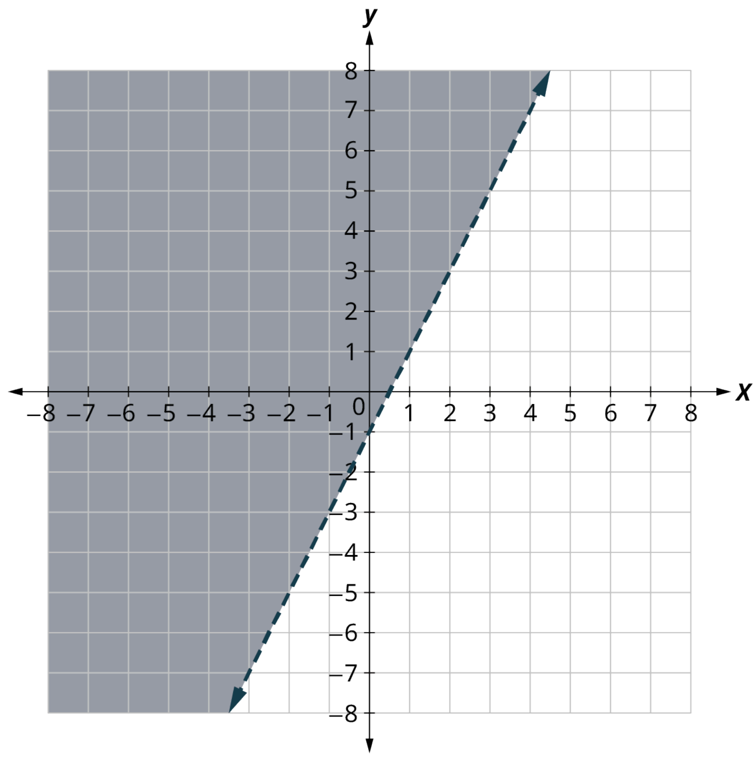


Figure 5.35

Solution

The line $y = 2x - 1$ is the boundary line. On one side of the line are the points with $y > 2x - 1$ and on the other side of the line are the points with $y < 2x - 1$. Let us test the point $(0, 0)$ and see which inequality describes its position relative to the boundary line. At $(0, 0)$, which inequality is true: $y > 2x - 1$ or $y < 2x - 1$?

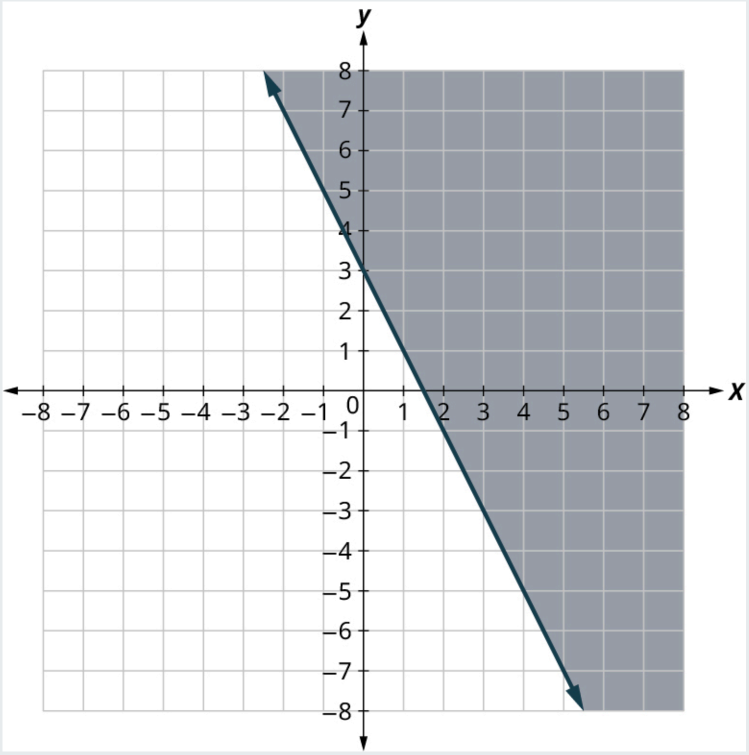
$0 > 2(0) - 1$	$0 < 2(0) - 1$
$0 > 0 - 1$	$0 < 0 - 1$
$0 > -1$	$0 < -1$
True	False

Since $y > 2x - 1$ is true, the side of the line with $(0, 0)$, is the solution. The shaded region shows the solution of the inequality $y > 2x - 1$. Since the boundary line is graphed with a dashed line, the inequality does not include the equal sign. The graph shows the inequality $y > 2x - 1$.

We could use an y point as a test point, provided it is not on the line. Why did we choose $(0, 0)$? Because it is the easiest to evaluate. You may want to pick a point on the other side of the boundary line and check that $y < 2x - 1$.

YOUR TURN 5.44

1. Write the inequality shown by the graph with the boundary line $y = -2x + 3$.

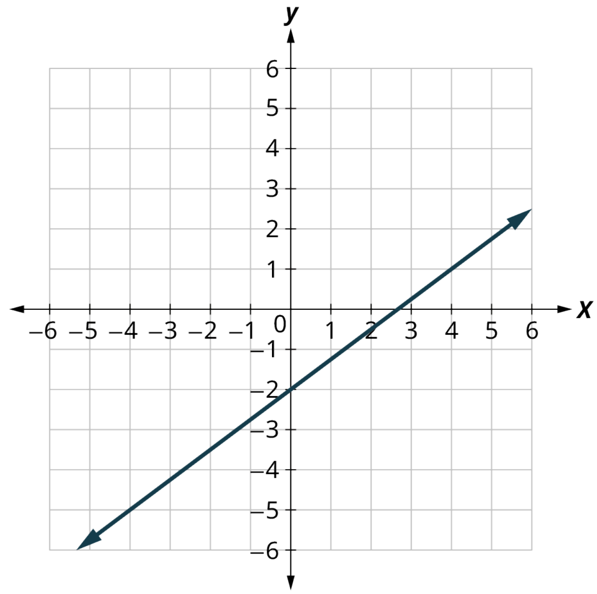
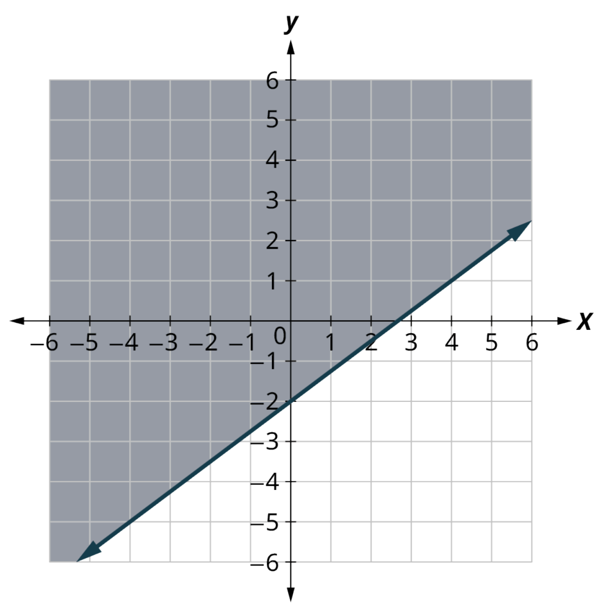


EXAMPLE 5.45

Graphing a Linear Inequality

Graph the linear inequality $y \geq \frac{3}{4}x - 2$.

Solution

Step 1. Identify and graph the boundary line (Figure 5.36). • If the inequality is ≤ or ≥, the boundary line is solid. • If the inequality is < or >, the boundary line is dashed.	Replace the inequality sign with an equal sign to find the boundary line. Graph the boundary line $y = \frac{3}{4}x - 2$. The inequality sign is ≥, so we draw a solid line.	 **Figure 5.36**
Step 2. Test a point that is not on the boundary line. Is it a solution of the inequality?	We'll test $(0, 0)$. Is it a solution of the inequality?	At $(0, 0)$, is $y \geq \frac{3}{4}x - 2$? $0 \overset{?}{\geq} \frac{3}{4}(0) - 2$ $0 \geq -2$ So, $(0, 0)$ is a solution.
Step 3. Shade in one side of the boundary line (Figure 5.37). • If the test point is a solution, shade in the side that includes the point. • If the test point is not a solution, shade in the opposite side.	The test point $(0, 0)$ is a solution to $y \geq \frac{3}{4}x - 2$. So we shade in that side.	 **Figure 5.37** All points in the shaded region and on the boundary line represent the solution to $y \geq \frac{3}{4}x - 2$.

YOUR TURN 5.45

1. Graph the linear inequality: $y > \frac{2x}{3} - 1$.

VIDEO

[Graphing Linear Inequalities in Two Variables (https://openstax.org/r/Graphing_linear)](https://openstax.org/r/Graphing_linear)

Solving Applications Using Linear Inequalities in Two Variables

Many fields use linear inequalities to model a problem. While our examples may be about simple situations, they give us an opportunity to build our skills and to get a feel for how they might be used.

EXAMPLE 5.46

Working Multiple Jobs

Hilaria works two part time jobs to earn enough money to meet her obligations of at least $240 a week. Her job in food service pays $10 an hour and her tutoring job on campus pays $15 an hour. How many hours does Hilaria need to work at each job to earn at least $240?

1. Let x be the number of hours she works at the job in food service and let y be the number of hours she works tutoring. Write an inequality that would model this situation.
2. Graph the inequality.
3. Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Hilaria.

Solution

1. Let x be the number of hours she works at the job in food service and let y be the number of hours she works tutoring. She earns $10 per hour at the job in food service and $15 an hour tutoring. At each job, the number of hours multiplied by the hourly wage will give the amount earned at that job.

$$\underbrace{\text{Amount earned at the food service job}}_{10x}\ \underset{+}{\text{plus}}\ \underbrace{\text{the amount earned tutoring}}_{15y}\ \underbrace{\text{is at least}}_{\geq 240}$$

2. Graph the inequality:
Step 1: Graph the boundary line $10x + 15y = 240$

Create a table of values

x	y
0	$10(0) + 15y = 240 -> y = 16$
6	$10(6) + 15y = 240 -> y = 12$
12	$10(12) + 15y = 240 -> y = 8$

Step 2: Pick a test point. Let us pick $(0, 0)$ again:

$10(0) + 15(0) \geq 240?$

$0 \geq 240$ is false and not a solution so the shading happens on the other side of the boundary line (Figure 5.38).

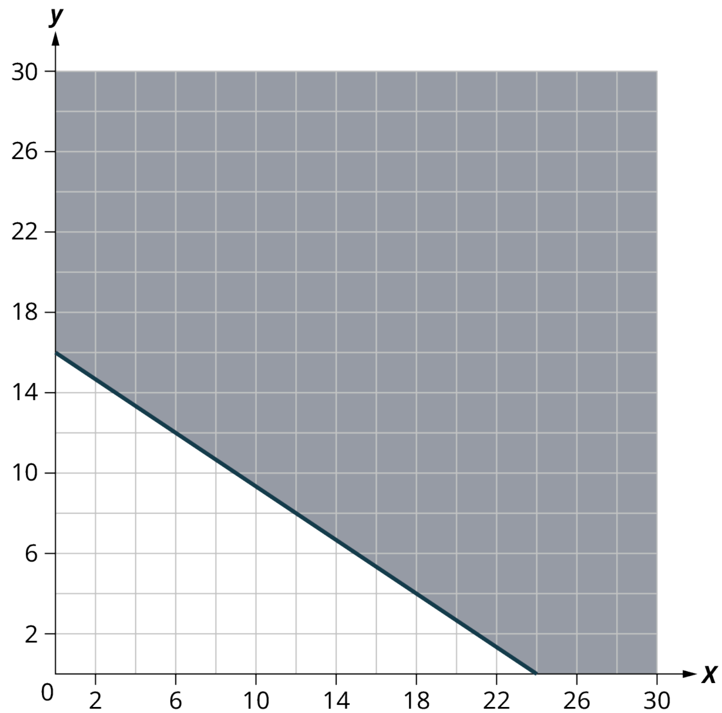


Figure 5.38

3. From the graph, we see that the ordered pairs $(15, 10)$, $(0, 16)$, $(24, 0)$ represent three of infinitely many solutions. Check the values in the inequality.

$$\begin{array}{ccc} (15, 10) & (0, 16) & (24, 0) \\ 10x + 15y \geq 240 & 10x + 15y \geq 240 & 10x + 15y \geq 240 \\ 10(15) + 15(10) \overset{?}{\geq} 240 & 10(0) + 15(16) \overset{?}{\geq} 240 & 10(24) + 15(0) \overset{?}{\geq} 240 \\ 300 \geq 240 \text{ True} & 240 \geq 240 \text{ True} & 240 \geq 240 \text{ True} \end{array}$$

For Hilaria, it means that to earn at least \$240, she can work 15 hours tutoring and 10 hours at her food service job, earn all her money tutoring for 16 hours, or earn all her money while working 24 hours at the job in food service.

YOUR TURN 5.46

Harrison works two part time jobs. One at a gas station that pays \$11 an hour and the other is as an IT consultant for \$16.50 an hour. Between the two jobs, Harrison wants to earn at least \$330 a week. How many hours does Harrison need to work at each job to earn at least \$330?

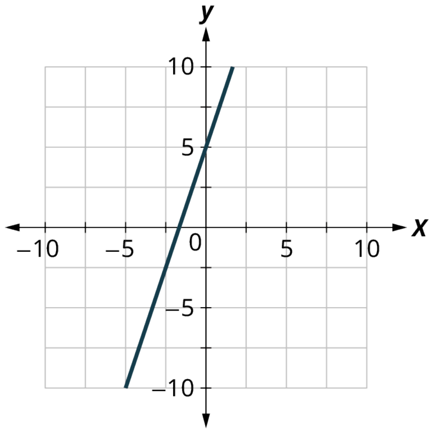
1. Let x be the number of hours he works at the gas station and let y be the number of hours he works as an IT consultant. Write an inequality that would model this situation.
2. Graph the inequality.
3. Find three ordered pairs (x, y) that would be solutions to the inequality. Then, explain what that means for Harrison.

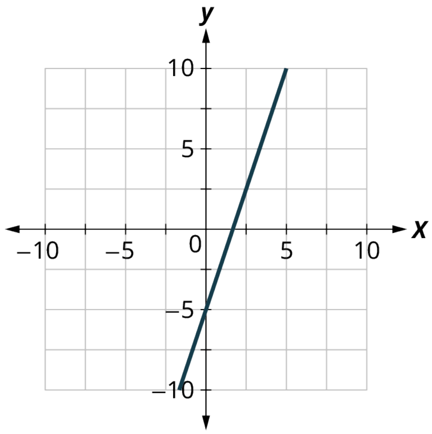
Check Your Understanding

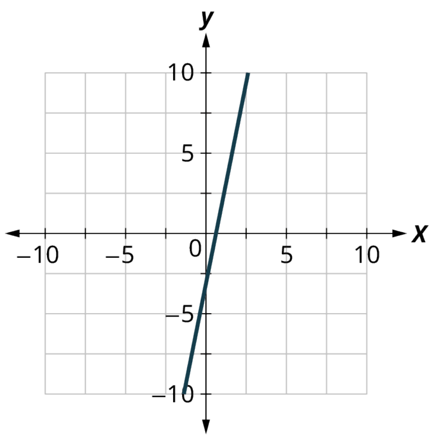
43. Choose the correct solution to the equation $6y + 10 = 12y$.

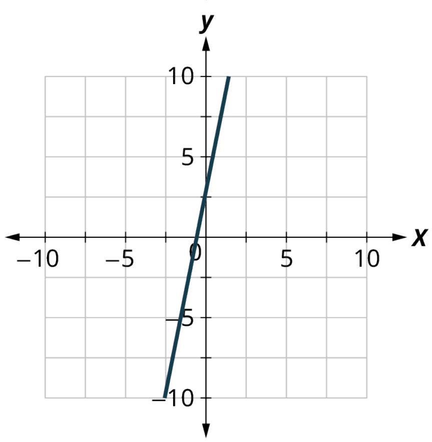
a. $y = 5$
b. $y = -1$
c. $y = \frac{1}{2}$
d. $y = \frac{5}{3}$

44. Choose the correct graph for $y = 3x + 5$.

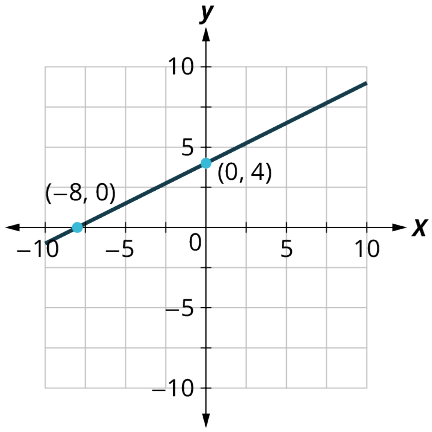






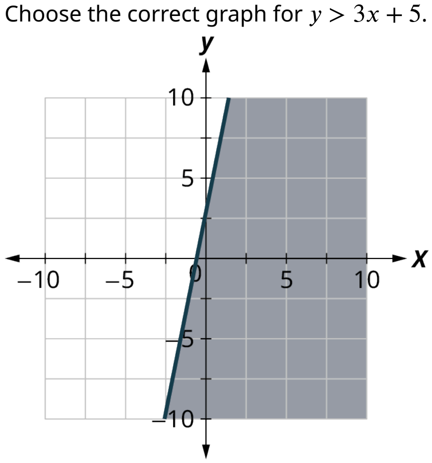


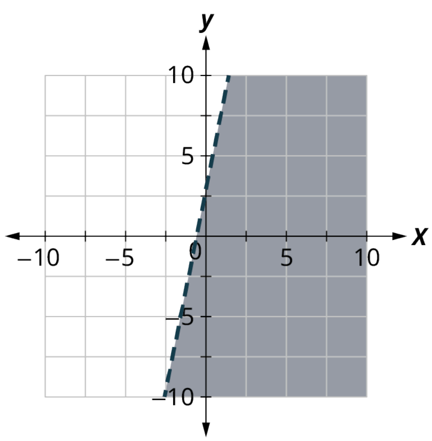
45. Choose the correct equation for the graph shown:

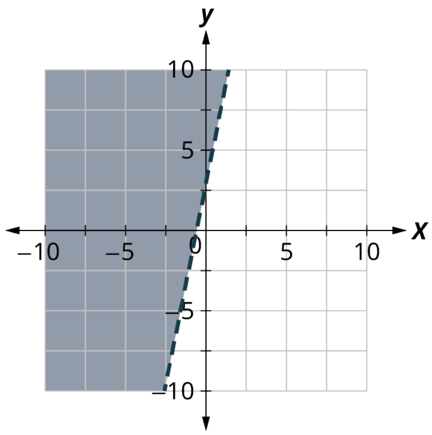


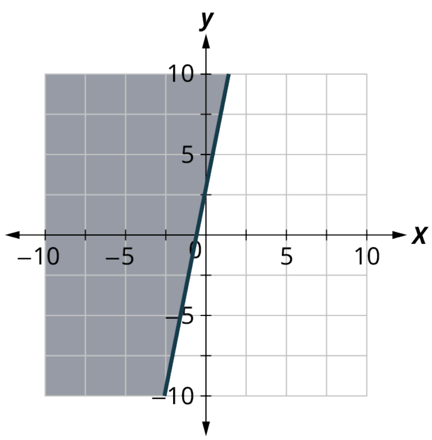
a. $y = 2x + 4$
b. $y = \frac{1}{2}x + 4$
c. $y = -2x + 4$
d. $y = \frac{1}{2}x + 4$

46. Choose the correct graph for $y > 3x + 5$.

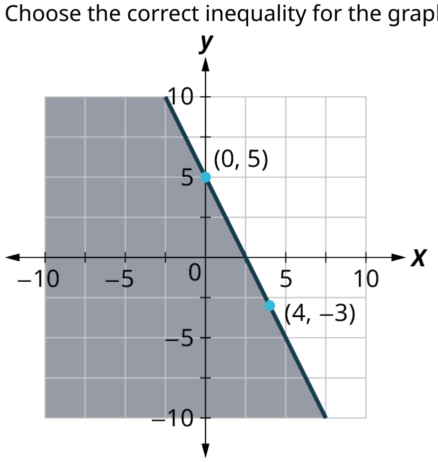








47. Choose the correct inequality for the graph shown.



a. $y = -2x + 5$
b. $y \leq -2x + 5$
c. $y \geq -2x + 5$
d. $y < -2x + 5$

SECTION 5.5 EXERCISES

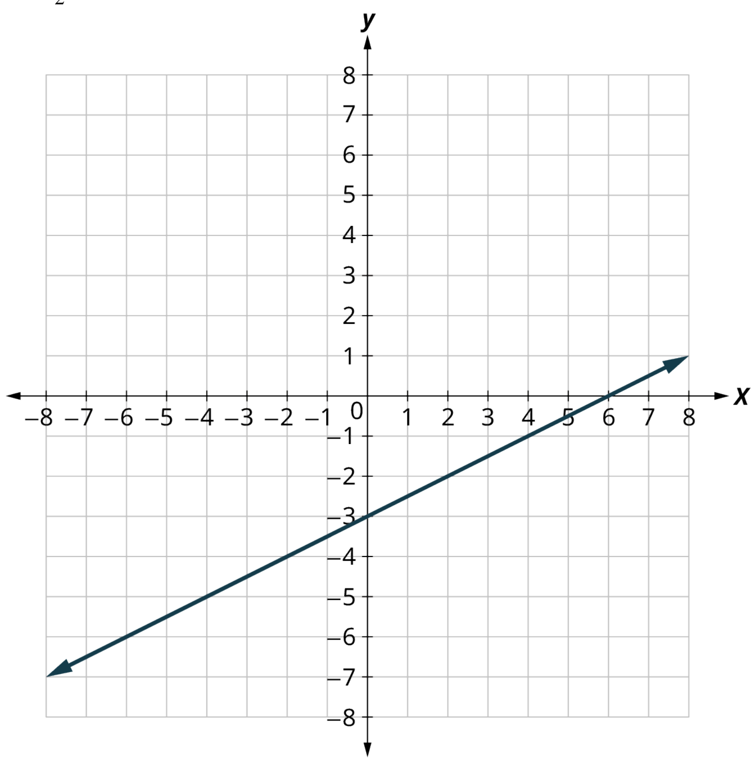
1. Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located.
a. $(3, -1)$
b. $(-3, 1)$
c. $(-2, 0)$
d. $(-4, -3)$
e. $(1, \frac{14}{5})$

For each ordered pair below, decide:

I. Is the ordered pair a solution to the equation?

II. Is the point on the line in the given graph?

$y = \frac{1}{2}x - 3$



2. $(0, -3)$

3. $(2, -2)$

4. $(-2, -4)$

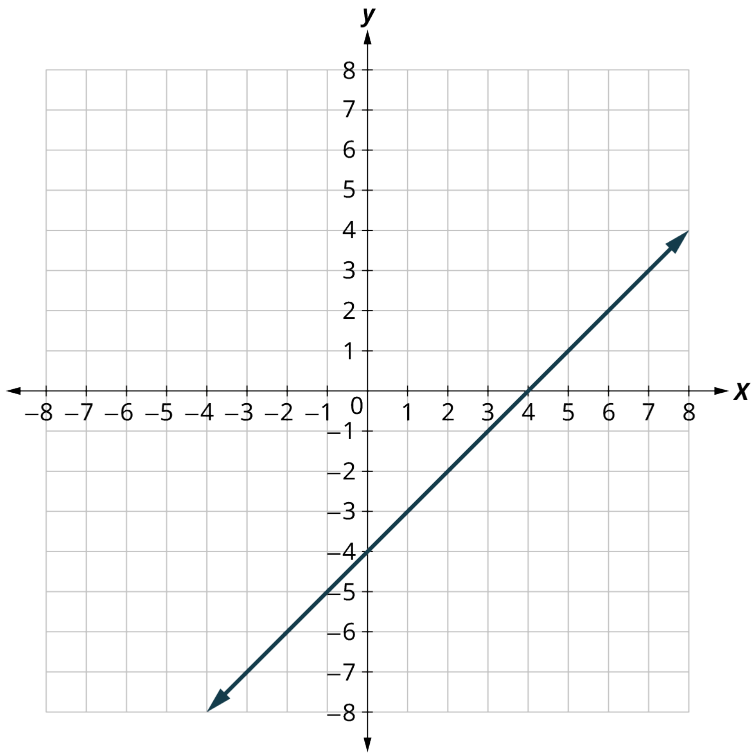
5. $(4, 1)$

For each ordered pair below, decide:

I. Is the ordered pair a solution to the equation?

II. Is the point on the line in the given graph?

$y = x - 4$

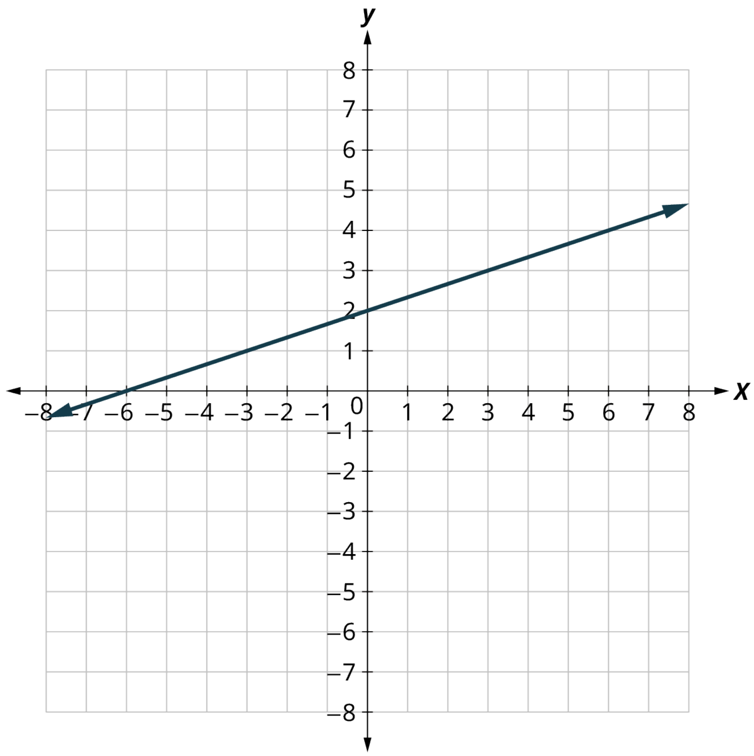


6. $(0, -4)$
7. $(3, -1)$
8. $(2, 2)$
9. $(1, -5)$

For each ordered pair below, decide:

I. Is the ordered pair a solution to the equation?
II. Is the point on the line in the given graph?

$y = \frac{1}{3}x + 2$



10. $(0, 2)$
11. $(3, 3)$
12. $(-3, 2)$
13. $(-6, 0)$

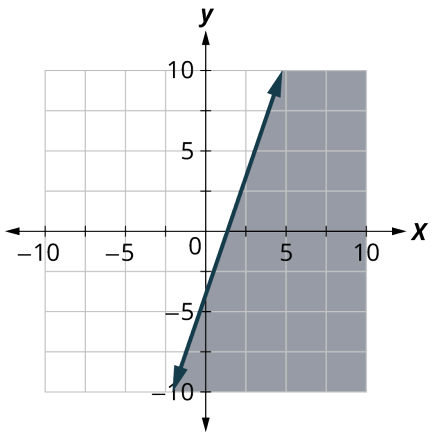
For the following exercises, graph by plotting points.

14. $y = 3x - 1$
15. $y = -x - 3$
16. $y = 2x$
17. $y = \frac{1}{2}x + 2$
18. $y = \frac{4}{3}x - 5$
19. $y = \frac{2}{5}x + 1$
20. $y = \frac{3}{2}x + 2$

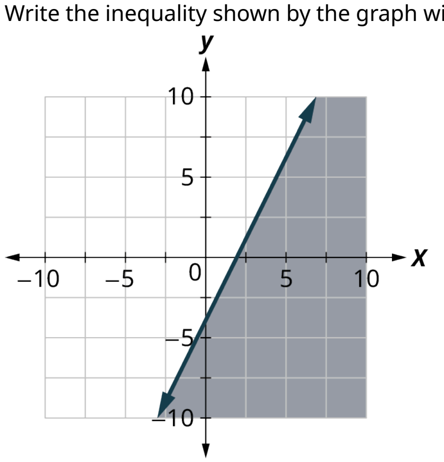
For the following exercises, determine whether each ordered pair is a solution to the inequality.

21. $y > x - 3$
A: (0, 0) B: (2, 1) C: (-1, -5) D: (-6, -3) E: (1, 0)
22. $y < 3x + 2$
A: (0, 3) B: (-3, -2) C: (-2, 0) D: (0, 0) E: (-1, 4)
23. $y < -2x + 5$
A: (-3, 0) B: (1, 6) C: (-6, -2) D: (0, 1) E: (5, -4)
24. $3x - 4y > 4$
A: (5, 1) B: (-2, 6) C: (3, 2) D: (10, -5) E: (0, 0)
25. $2x + 3y > 2$
A: (1, 1) B: (4, -3) C: (0, 0) D: (-8, 12) E: (3, 0)

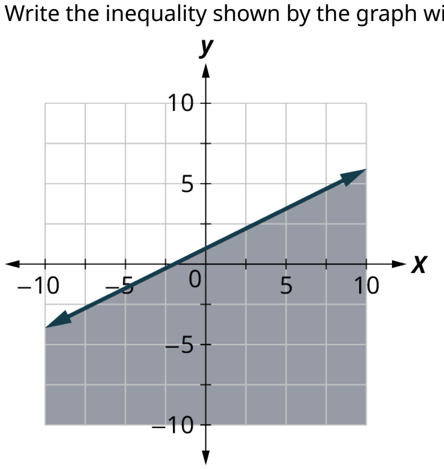
26. Write the inequality shown by the graph with the boundary line $y = 3x - 4$.



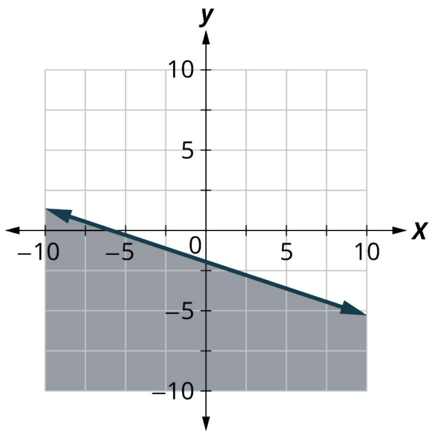
27. Write the inequality shown by the graph with the boundary line $y = 2x - 4$.



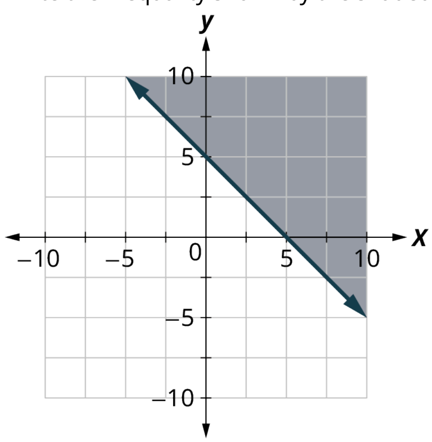
28. Write the inequality shown by the graph with the boundary line $y = \frac{1}{2}x + 1$.



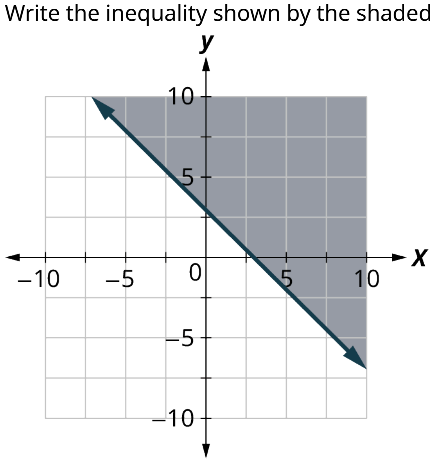
29. Write the inequality shown by the graph with the boundary line $y = -\frac{1}{3}x - 2$.



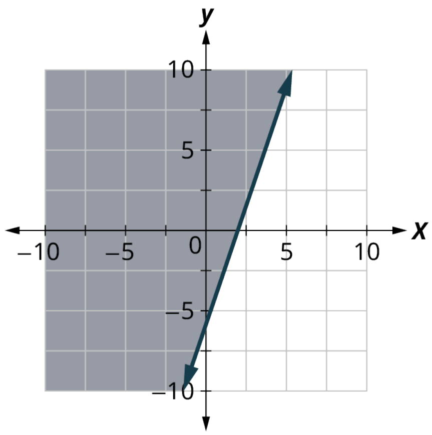
30. Write the inequality shown by the shaded region in the graph with the boundary line $x + y = 5$.



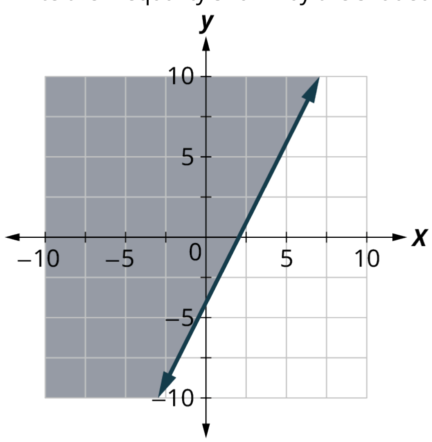
31. Write the inequality shown by the shaded region in the graph with the boundary line $x + y = 3$.



32. Write the inequality shown by the shaded region in the graph with the boundary line $3x - y = 6$.



33. Write the inequality shown by the shaded region in the graph with the boundary line $2x - y = 4$.



For the following exercises, graph the linear inequality.

34. $y < \frac{3}{5}x + 2$
35. $y \leq -\frac{1}{2}x + 4$
36. $y \geq -\frac{1}{3}x - 2$
37. $x - y \leq 3$
38. $x - y \geq -2$
39. $4x + y > -4$
40. $x + 5y < -5$
41. $3x + 2y \geq -6$
42. $4x + 2y \geq -8$
43. $y > 4x$
44. $y \leq -3x$

5.6 Quadratic Equations with Two Variables with Applications

Figure 5.39 The Gateway Arch in St. Louis, Missouri (credit: modification of work "Gateway Arch - St. Louis - Missouri" by Sam valadi/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:
1. Multiply binomials.
2. Factor trinomials.
3. Solve quadratic equations by graphing.
4. Solve quadratic equations by factoring.
5. Solve quadratic equations using square root method.
6. Solve quadratic equations using the quadratic formula.
7. Solve real world applications modeled by quadratic equations.

In this section, we will discuss quadratic equations. There are several real-world scenarios that can be represented by the graph of a quadratic equation. Think of the Gateway Arch in St. Louis, Missouri. Both ends of the arch are 630 feet apart and the arch is 630 feet tall. You can plot these points on a coordinate system and create a parabola to graph the quadratic equation.

Identify Polynomials, Monomials, Binomials and Trinomials

You have learned that a term is a constant, or the product of a constant and one or more variables. When it is of the form ax^m, where a is a constant and x^m is a positive whole number, it is called a **monomial**. Some examples of monomial are 8, $-2x^2$, $4y^3$, and $11z$.

A monomial or two or more monomials combined by addition or subtraction is a **polynomial**. Some examples include: $b + 11$, $4y^2 - 7y + 2$, and $4x^4 + x^3 + 8x^2 - 9x + 1$. Some polynomials have special names, based on the number of terms. A monomial is a polynomial with exactly one term (examples: 14, $8y^2$, $-9x^3y^5$, and -13). A **binomial** has exactly two terms (examples: $a + 7$, $4b - 5$, $y^2 - 16$, and $3x^3 - 9x^2$), and a trinomial has exactly three terms (examples: $x^2 - 7x + 12$, $9y^2 + 2y - 8$, $6m^4 - m^3 + 8m$, and $x^4 + 3x^2 - 1$).

Notice that every monomial, binomial, and trinomial is also a polynomial. They are just special members of the “family” of polynomials and so they have special names. We use the words monomial, binomial, and trinomial when referring to these special polynomials and just call all the rest polynomials.

Multiply Binomials

Recall multiplying algebraic expressions from Algebraic Expressions. In this section, we will continue that work and multiply binomials as well. We can use an area model to do multiplication.

EXAMPLE 5.47

Multiply Binomials

Multiply $(x + 2)(x + 3)$.

Solution

Step 1: Use the distributive property:

$$(x)(x) + (x)(3) + (2)(x) + (2)(3)$$

In the area model (Figure 5.40) multiply each term on the side by each term on the top (think of it as a multiplication table).

	x	+	2
x	x^2		$2x$
+			
3	$3x$		6

Figure 5.40

Step 2: After we multiply, we get the following equation:

$$x^2 + 3x + 2x + 6$$

Step 3: Combine the like terms to arrive at:

$$x^2 + 5x + 6$$

YOUR TURN 5.47

1. Multiply $(x + 3)(x + 1)$.

EXAMPLE 5.48

Multiplying More Complex Binomials

Multiply $(2x + 7)(3x - 5)$.

Solution

Step 1: Use the Distributive Property:

$$(2x)(3x) - (2x)(5) + (7)(3x) - (7)(5)$$

	$2x$	$+7$
$3x$	$6x^2$	$21x$
-5	$-10x$	-35

Figure 5.41

Step 2: After multiplying, get the following equation:

$$6x^2 - 10x + 21x - 35$$

Step 3: Combine the like terms to arrive at:

$$6x^2 + 11x - 35$$

YOUR TURN 5.48

1. Multiply $(x-3)(2x+1)$.

WHO KNEW?

They Are Teaching Multiplication of Binomials in Elementary School

Manipulatives are often used in elementary school for students to experience a hands-on way to experience the mathematics they are learning. Base Ten Blocks, or Dienes Blocks, are often used to introduce place value and the operation of whole numbers. When multiplying two-digit numbers, students can make an array to visualize the Distributive Property. Figure 5.42 shows the value of each Base Ten Block and Figure 5.43 shows how to multiply 17 and 23 using an area model and Base Ten Blocks. You can see how this helps students visualize the multiplication using the Distributive Property. Consider how $(10+7)(20+3)$ can easily extend to $(10+x)(20+x)$ in algebra!

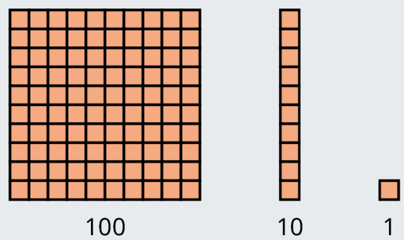


Figure 5.42 The Value of Each Base Ten Block

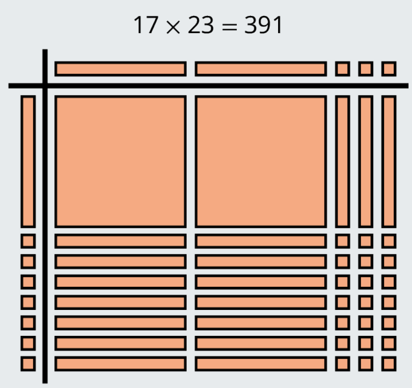


Figure 5.43 How to Multiply 17 and 23 Using an Area Model and Base Ten Blocks

Factoring Trinomials

We've just covered how to multiply binomials. Now you will need to "undo" this multiplication—to start with the product and end up with the factors. Let us review an example of multiplying binomials to refresh your memory.

$$(x+2)(x+3)=x^2+5x+6$$

To factor the trinomial means to start with the product, x^2+5x+6, and end with the factors, $(x+2)(x+3)$. You need to think about where each of the terms in the trinomial came from. The first term came from multiplying the first term in each binomial. So, to get x^2 in the product, each binomial must start with an x.

$$x^2+5x+6$$
$$\left(x\quad\right)\left(x\quad\right)$$

The last term in the trinomial came from multiplying the last term in each binomial. So, the last terms must multiply to 6. What two numbers multiply to 6? The factors of 6 could be 1 and 6, or 2 and 3. How do you know which pair to use? Consider the middle term. It came from adding the outer and inner terms. So the numbers that must have a product of 6 will need a sum of 5.

We'll test both possibilities and summarize the results in the following table, which will be very helpful when you work with numbers that can be factored in many different ways.

Factors of 6	Sum of Factors
1, 6	$1 + 6 = 7$
2, 3	$2 + 3 = 5$

We see that 2 and 3 are the numbers that multiply to 6 and add to 5. We have the factors of $x^2 + 5x + 6$. They are $(x + 2)(x + 3)$.

$$\begin{array}{ll} x^2 + 5x + 6 & \text{product} \\ (x + 2)(x + 3) & \text{factors} \end{array}$$

You can check if the factors are correct by multiplying. Looking back, we started with $x^2 + 5x + 6$, which is of the form $x^2 + bx + c$, where $b = 5$ and $c = 6$. We factored it into two binomials of the form $(x + m)$ and $(x + n)$.

$$\begin{array}{ll} x^2 + 5x + 6 & x^2 + bx + c \\ (x + 2)(x + 3) & (x + m)(x + n) \end{array}$$

To get the correct factors, we found two number m and n whose product is c and sum is b. With the area model (Figure 5.44), start with an empty box and then put in the x^2 term and c.

x^2	
	6

Figure 5.44

Continue by putting in two terms that add up to $5x$: $2x$ and $3x$ (Figure 5.45):

x^2	$2x$
$3x$	6

Figure 5.45

Then you find the terms of the binomials on the top and side (Figure 5.46):

	x	+ 2
x	x^2	$2x$
+ 3	$3x$	6

Figure 5.46

EXAMPLE 5.49

Factoring Trinomials

Factor $x^2 + 7x + 12$.

Solution

The numbers that must have a product of 12 will need a sum of 7. We will summarize the results in a table below.

Factors of 12	Sum of Factors
1, 12	$1 + 12 = 13$
2, 6	$2 + 6 = 8$
3, 4	$3 + 4 = 7$

We see that 3 and 4 are the numbers that multiply to 12 and add to 7. The factors of $x^2 + 7x + 12$ are $(x + 3)(x + 4)$.

YOUR TURN 5.49

1. Factor $x^2 + 6x + 8$.

EXAMPLE 5.50

Factoring More Complex Trinomials

Factor $x^2 - 11x + 28$.

Solution

The numbers that must have a product of 28 will need a sum of -11. We will summarize the results in a table.

Factors of 28	Sum of Factors
$1, 28$	$1 + 28 = 29$
$2, 14$	$2 + 14 = 16$
$4, 7$	$4 + 7 = 11$

We see that 4 and 7 are the numbers that multiply to 28 and add to 11. But we needed -11, so we will need to use -4 and -7 because $(-4)(-7) = 28$ and $(-4) + (-7) = -11$. The factors of $x^2 - 11x + 28$ are $(x - 4)(x - 7)$.

YOUR TURN 5.50

1. Factor: $x^2 - 16x + 63$.

VIDEO

[Factoring with the Box Method (Area Model) (https://openstax.org/r/Factoring_with_the_Box)](https://openstax.org/r/Factoring_with_the_Box)

Solving Quadratic Equations by Graphing

We have already solved and graphed linear equations in Graphing Linear Equations and Inequalities, equations of the form $Ax + By = C$. In linear equations, the variables have no exponents. **Quadratic equations** are equations in which the variable is squared. The following are some examples of quadratic equations:

$$x^2 + 5x + 6 = 0 \quad 3y^2 + 4y = 106 \quad 4u^2 - 81 = 0 \quad n(n + 1) = 42$$

The last equation does not appear to have the variable squared, but when we simplify the expression on the left, we will get $n^2 + n$. The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a, b, and c are real numbers, with $a \neq 0$.

Remember that a solution of an equation is a value of a variable that makes a true statement when substituted into the equation. The solutions of quadratic equations are the values of the variables that make the quadratic equation $ax^2 + bx + c = 0$ true.

To solve quadratic equations, we need methods different than the ones we used in solving linear equations. We will start by solving a quadratic equation from its graph. Just like we started graphing linear equations by plotting points, we will do the same for quadratic equations. Let us look first at graphing the quadratic equation $y = x^2$. We will choose integer values of x between -2 and 2 and find their y values, as shown in the table below.

$y = x^2$	
x	y
0	0
1	1
-1	1
2	4
-2	4

Notice when we let $x = 1$ and $x = -1$, we got the same value for y.

$$\begin{array}{lll} y = x^2 & y = 1^2 & y = 1 \\ y = x^2 & y = (-1)^2 & y = 1 \end{array}$$

The same thing happened when we let $x = 2$ and $x = -2$. Now, we will plot the points to show the graph of $y = x^2$. See Figure 5.47.

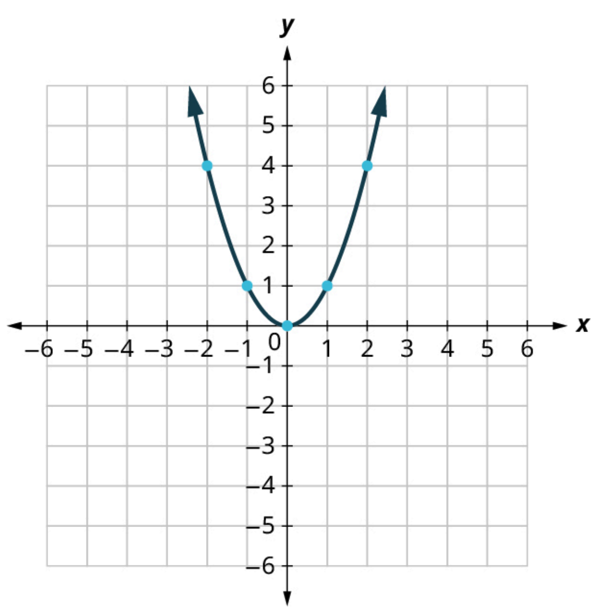


Figure 5.47

The graph is not a line. This figure is called a parabola. Every quadratic equation has a graph that looks like this. When $y = 0$ the solution to the quadratic $y = x^2$ is 0 because $x^2 = 0$ at $x = 0$.

EXAMPLE 5.51

Graphing a Quadratic Equation

Graph $y = x^2 - 1$ and list the solutions to the quadratic equation.

Solution

We will graph the equation by plotting points.

Step 1: Choose integer values for x, substitute them into the equation, and solve for y.

Step 2: Record the values of the ordered pairs in the chart.

$y = x^2 - 1$	
x	y
0	-1
1	0
-1	0
2	3
-2	3

Step 3: Plot the points and then connect them with a smooth curve. The result will be the graph of the equation $y = x^2 - 1$ (Figure 5.48). The solutions are $x = 1$ and $x = -1$.

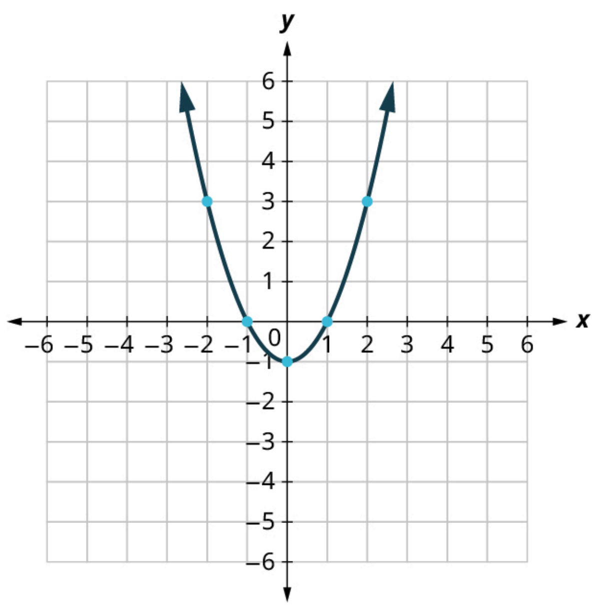


Figure 5.48

YOUR TURN 5.51

1. Graph $y = -x^2$.

EXAMPLE 5.52

Solving a Quadratic Equation From Its Graph

Find the solutions of $y = x^2 + 5x + 4$ from its graph (Figure 5.49).

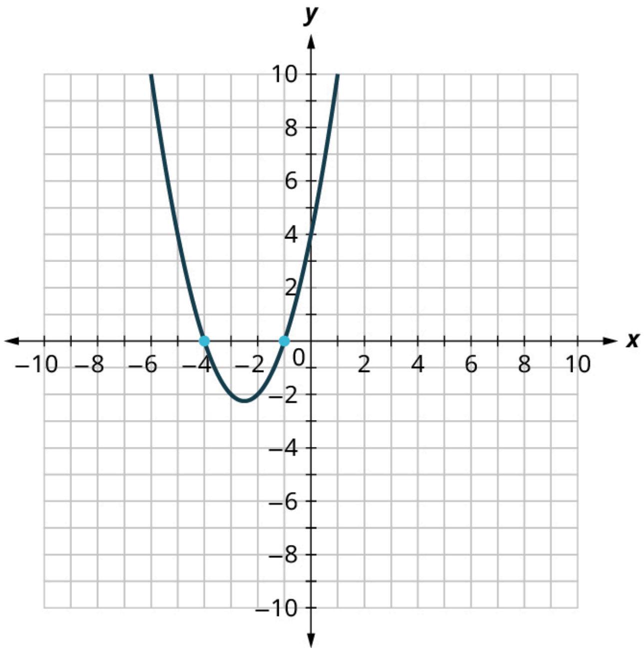


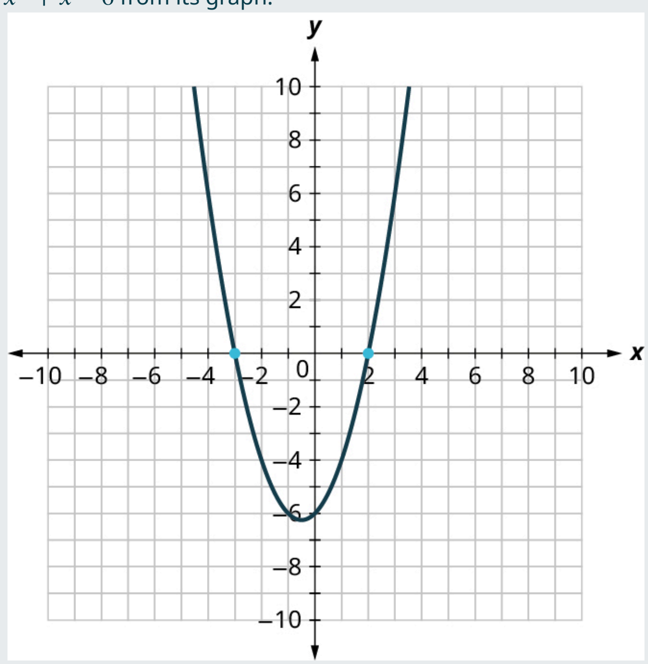
Figure 5.49

Solution

The solutions of a quadratic equations are the values of x that make the equation a true statement when set equal to zero (i.e. when $y = 0$). $x^2 + 5x + 4 = 0$ at $x = -4$ and $x = -1$.

YOUR TURN 5.52

1. Find the solutions of $y = x^2 + x - 6$ from its graph.



Solving Quadratic Equations by Factoring

Another way of solving quadratic equations is by factoring. We will use the **Zero Product Property** that says that if the product of two quantities is zero, it must be that at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

EXAMPLE 5.53

Solving a Quadratic Equation by Factoring

Solve $(x+1)(x-4)=0$.

Solution

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(x+1)(x-4)=0$ $x+1=0 \quad \text{or} \quad x-4=0$
Step 2. Solve the linear equations.	Solve each equation.	$x=-1 \quad \text{or} \quad x=4$
Step 3. Check.	Substitute each solution separately into the original equation.	$x=\mathbf{-1}$ $(x+1)(x-4)=0$ $(\mathbf{-1}+1)(\mathbf{-1}-4)\stackrel{?}{=}0$ $(0)(-5)\stackrel{?}{=}0$ $0=0$✓
		$x=\mathbf{4}$ $(x+1)(x-4)=0$ $(\mathbf{4}+1)(\mathbf{4}-4)\stackrel{?}{=}0$ $(5)(0)\stackrel{?}{=}0$ $0=0$✓

YOUR TURN 5.53

1. Solve $(x-2)(x+5)=0$.

EXAMPLE 5.54

Solve Another Quadratic Equation by Factoring

Solve $x^2+2x-8=0$.

Solution

Step 1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.	The equation is already in standard form.	$x^2+2x-8=0$
Step 2. Factor the quadratic expression.	Factor x^2+2x-8 $(x+4)(x-2)$	$(x+4)(x-2)=0$

Step 3. Use the Zero Product Property.	Set each factor equal to zero.	$x + 4 = 0$ or $x - 2 = 0$
Step 4. Solve the linear equations.	We have two linear equations.	$x = -4$ or $x = 2$
Step 5. Check.	Substitute each solution separately into the original equation.	$x^2 + 2x - 8 = 0$ $x = \mathbf{-4}$ $(\mathbf{-4})^2 - 2(\mathbf{-4}) - 8 \stackrel{?}{=} 0$ $16 + (-8) - 8 \stackrel{?}{=} 0$ $0 = 0$✓
		$x^2 + 2x - 8 = 0$ $x = \mathbf{2}$ $\mathbf{2}^2 - 2(\mathbf{2}) - 8 \stackrel{?}{=} 0$ $4 + 4 - 8 \stackrel{?}{=} 0$ $0 = 0$✓

YOUR TURN 5.54

1. Solve $x^2 + 2x - 15 = 0$.

VIDEO

Solving Quadratics with the Zero Property (https://openstax.org/r/Zero_Property)

Be careful to write the quadratic equation in standard form first. The equation must be set equal to zero in order for you to use the Zero Product Property! Often students start in Step 2 resulting in an incorrect solution. For example, $x^2 + 2x - 15 = -7$ *cannot be factored to* $(x - 3)(x + 5) = -7$ *and then solved by setting each factor equal to* -7.

The correct way to solve this quadratic equation is to set it equal to zero FIRST: $x^2 + 2x - 15 + 7 = -7 + 7$ *which becomes* $x^2 + 2x - 8 = 0$, *then continue to factor. See the table below for the correct way to apply the Zero Product Property.*

	$x^2 + 2x - 15 = -7$		$x^2 + 2x - 15 = -7$	
Step 1	Skipped		$x^2 + 2x - 15 + 7 = -7 + 7$ $x^2 + 2x - 8 = 0$	
Step 2	$(x - 3)(x + 5) = -7$		$(x - 2)(x + 4) = 0$	
Step 3	$x - 3 = -7$	$x + 5 = -7$	$x - 2 = 0$	$x + 4 = 0$

Step 4	$x = -4$	$x = -12$	$x = 2$	$x = -4$
Step 5	$(-4)^2 + 2(-4) - 15$ $= 16 - 8 - 15$ $= -7$	$(-12)^2 + 2(-12) - 15$ $= 144 - 24 - 1 = 105$ $\neq -7$	$(2)^2 + 2(2) - 8$ $= 4 + 4 - 8$ $= 0$	$(-4)^2 + 2(-4) - 18$ $= 16 - 8 - 8$ $= 0$

Solving Quadratic Equations Using the Square Root Property

We just solved some quadratic equations by factoring. Let us use factoring to solve the quadratic equation $x^2 = 9$.

Step 1: Put the equation in standard form.	$x^2 - 9 = 0$	
Step 2: Factor the left side.	$(x + 3)(x - 3) = 0$	
Step 3: Use the Zero Product Property.	$x + 3 = 0$	$x - 3 = 0$
Step 4: Solve each equation.	$x = -3$	$x = 3$
Step 5: Combine the two solutions into ±	$x = \pm 3$	

The solution is read as "x is equal to positive or negative 3."

What happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring. These equations are all of the form $x^2 = k$. We define the square root of a number in this way: If $n^2 = m$, then n is a square root of m. This leads to the **Square Root Property**.

EXAMPLE 5.55

Using the Square Root Property to Solve a Quadratic Equation

Solve using the square Root Property: $x^2 = 169$.

Solution

Step 1: Use the Square Root Property.	$x = \pm\sqrt{169}$	
Step 2: Simplify the radical.	$x = \pm 13$	
Step 3: Rewrite to show the two solutions.	$x = 13$	$x = -13$

YOUR TURN 5.55

1. Solve using the Square Root Property: $x^2 = 25$.

EXAMPLE 5.56

Using the Square Root Property to Solve Another Quadratic Equation

Solve using the Square Root Property: $144q^2 = 25$.

Solution

Step 1: Solve for q.

$q^2 = \frac{25}{144}$

Step 2: Use the Square Root Property.

$q = \pm\sqrt{\frac{25}{144}}$

Step 3: Simplify the radical.

$$q = \pm\frac{5}{12}$$

Step 4: Rewrite to show the two solutions.
$q = \frac{5}{12}, q = -\frac{5}{12}$

YOUR TURN 5.56

1. Solve using the Square Root Property: $25p^2 = 49$.

Solving Quadratic Equations Using the Quadratic Formula

This last method we will look at for solving quadratic equations is the **quadratic formula**. This method works for all quadratic equations, even the quadratic equations we could not factor! To use the quadratic formula, we substitute the values of a, b, and c into the expression on the right side of the formula. Then, we do all the math to simplify the expression. The result gives the solution(s) to the quadratic equation.

EXAMPLE 5.57

Solving a Quadratic Equation Using the Quadratic Formula
Solve using the quadratic formula: $x^2 - 6x + 5 = 0$.

Solution

$x^2 - 6x + 5 = 0$

This equation is in standard form.

$ax^2 + bx + c = 0$
$x^2 - 6x + 5 = 0$

Step 1: Identify the a, b, and c values.

$a = 1, b = -6, c = 5$

Step 2: Write the quadratic formula.

$x = \frac{-b\pm\sqrt{b^2 - 4ac}}{2a}$

Step 3: Substitute in the values of a, b, c.

$x = \frac{(-6) \pm \sqrt{(-6)^2 - 4(1)(5)}}{2.1}$

Step 4: Simplify.

$x = \frac{6\pm\sqrt{36 - 20}}{2}$

$x = \frac{6\pm\sqrt{16}}{2}$

$x = \frac{6 \pm 4}{2}$

Step 5: Rewrite to show two solutions.

$x = \frac{6+4}{2}, x = \frac{6-4}{2}$

Step 6: Simplify.

$x = \frac{10}{2}, x = \frac{2}{2}$

$x = 5, x = 1$

Step 7: Check.

$$\begin{array}{rr} x^2 - 6x + 5 = 0 & x^2 - 6x + 5 = 0 \\ 5^2 - 6 \cdot 5 + 5 \stackrel{?}{=} 0 & 1^2 - 6 \cdot 1 + 5 \stackrel{?}{=} 0 \\ 25 - 30 + 5 \stackrel{?}{=} 0 & 1 - 6 + 5 \stackrel{?}{=} 0 \\ 0 = 0 \checkmark & 0 = 0 \checkmark \end{array}$$

YOUR TURN 5.57

1. Solve using the quadratic formula: $a^2 - 2a - 15 = 0$.

EXAMPLE 5.58

Solving Another Quadratic Equation Using the Quadratic Formula

Solve using the quadratic formula: $2x^2 + 9x - 5 = 0$.

Solution

Step 1. Write the quadratic equation in standard form. Identify the a, b, c values.	This equation is in standard form.	$\boldsymbol{ax^2 + bx + c = 0}$ $2x^2 + 9x - 5 = 0$ $a = 2,\ b = 9,\ c = -5$
Step 2. Write the quadratic formula. Then substitute in the values of a, b, c.	Substitute in $\boldsymbol{a = 2}$, $\boldsymbol{b = 9}$, $\boldsymbol{c = -5}$	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{\mathbf{-9} \pm \sqrt{\mathbf{9}^2 - 4(\mathbf{2})(\mathbf{-5})}}{2 \cdot \mathbf{2}}$
Step 3. Simplify the fraction, and solve for x.		$x = \frac{-9 \pm \sqrt{81 - (-40)}}{4}$ $x = \frac{-9 \pm \sqrt{121}}{4}$ $x = \frac{-9 \pm 11}{4}$ $x = \frac{-9+11}{4}$ $x = \frac{-9-11}{4}$ $x = \frac{2}{4}$ $x = \frac{-20}{4}$ $x = \frac{1}{2}$ $x = -5$

Step 4. Check the solutions.	Put each answer in the original equation to check. Substitute $x = \frac{1}{2}$.	$2x^2 + 9x - 5 = 0$ $2\left(\frac{1}{2}\right)^2 + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $\frac{1}{2} + \frac{9}{2} - 5 \stackrel{?}{=} 0$ $\frac{10}{2} - 5 \stackrel{?}{=} 0$ $5 - 5 \stackrel{?}{=} 0$ $0 = 0$✓
	Substitute $x = -5$.	$2x^2 + 9x - 5 = 0$ $2(-5)^2 + 9(-5) - 5 \stackrel{?}{=} 0$ $2 \cdot 25 - 45 - 5 \stackrel{?}{=} 0$ $50 - 45 - 5 \stackrel{?}{=} 0$ $0 = 0$✓

YOUR TURN 5.58

1. Solve using the quadratic formula: $3y^2 - 5y + 2 = 0$.

VIDEO

Solving Quadratics with the Quadratic Formula (https://openstax.org/r/Solving_Quadratics)

Solving Real-World Applications Modeled by Quadratic Equations

There are problem solving strategies that will work well for applications that translate to quadratic equations. Here's a problem-solving strategy to solve word problems:

Step 1: Read the problem. Make sure all the words and ideas are understood.

Step 2: Identify what we are looking for.

Step 3: Name what we are looking for. Choose a variable to represent that quantity.

Step 4: Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.

Step 5: Solve the equation using good algebra techniques.

Step 6: Check the answer in the problem and make sure it makes sense.

Step 7: Answer the question with a complete sentence.

EXAMPLE 5.59

Finding Consecutive Integers

The product of two consecutive integers is 132. Find the integers.

Solution

Step 1: Read the problem.

Step 2: Identify what we are looking for.

We are looking for two consecutive integers.

Step 3: Name what we are looking for.

Let $n =$ the first integer

Let $n + 1 =$ the next consecutive integer.

Step 4: Translate into an equation. Restate the problem in a sentence.

$n(n + 1) = 132$

The product of the two consecutive integers is 132. The first integer times the next integer is 132.

Step 5: Solve the equation.

$n^2 + n = 132$

Bring all the terms to one side.

$n^2 + n - 132 = 0$

Factor the trinomial.

$(n - 11)(n + 12) = 0$

Use the zero product property.

$n - 11 = 0 \text{ or } n + 12 = 0$

Solve the equations.

$n = 11, n = -12$

There are two values for n that are solutions to this problem. So, there are two sets of consecutive integers that will work.

If the first integer is $n = 11$, then the next integer is 12. If the first integer is $n = -12$, then the next integer is -11.

Step 6: Check the answer.

The consecutive integers are 11, 12 and $-11, -12$. The product of 11 and $12 = 132$ and the product of $-11(-12) = 132$. Both pairs of consecutive integers are solutions.

Step 7: Answer the question.

The consecutive integers are 11, 12, and $-11, -12$.

YOUR TURN 5.59

1. The product of two consecutive odd integers is 240. Find the integers.

Were you surprised by the pair of negative integers that is one of the solutions? In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

EXAMPLE 5.60

Finding Length and Width of a Garden

A rectangular garden has an area 15 square feet. The length of the garden is 2 feet more than the width. Find the length and width of the garden.

Solution

Step 1: Read the problem. In problems involving geometric figures, a sketch can help you visualize the situation (Figure 5.50).

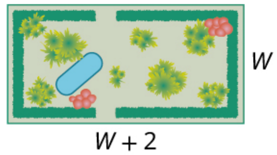


Figure 5.50

Step 2: Identify what you are looking for.

We are looking for the length and width.

Step 3: Name what you are looking for. The length is 2 feet more than width.

Let W = the width of the garden.

$W + 2 =$ the length of the garden.

Step 4: Translate into an equation.

Restate the important information in a sentence.

The area of the rectangular garden is 15 square feet.

Use the formula for the area of a rectangle.

$A = L \times W$

Substitute in the variables.

$15 = (W + 2)\,W$

Step 5: Solve the equation. Distribute first.

$15 = W^2 + 2W$

Get zero on one side.

$0 = W^2 + 2W - 15$

Factor the trinomial.

$0 = (W + 5)\,(W - 3)$

Use the Zero Product Property.

$0 = W + 5$

$0 = W - 3$

Solve each equation.

$-5 = W$

$3 = W$

Since W is the width of the garden, it does not make sense for it to be negative. We eliminate that value for W.

$W = 3$

Width is 3 feet.

Find the value of the length.

$W + 2 =$ length.

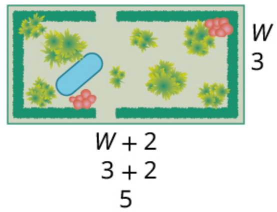
$3 + 2$

5

Length is 5 feet.

Step 6: Check the answer (Figure 5.51).

Does the answer make sense?



$A = L \cdot W$
$A = 3 \cdot 5$
$A = 15$

Figure 5.51

Yes, this makes sense.

Step 7: Answer the question.

The width of the garden is 3 feet and the length is 5 feet.

YOUR TURN 5.60

1. A rectangular sign has an area of 30 square feet. The length of the sign is 1 foot more than the width. Find the length and width of the sign.

WORK IT OUT

Completing the Square

Recall the two methods used to solve quadratic equations of the form $ax^2 + bx + c$: by factoring and by using the quadratic formula. There are, however, many different methods for solving quadratic equations that were developed throughout history. Egyptian, Mesopotamian, Chinese, Indian, and Greek mathematicians all solved various types of quadratic equations, as did Arab mathematicians of the ninth through the twelfth centuries. It is one of these Arab mathematicians' methods that we wish to investigate with this activity.

Muhammad ibn Musa al-Khwarizmi was employed as a scholar at the House of Wisdom in Baghdad, located in present day Iraq. One of the many accomplishments of Al-Khwarizmi was his book on the topic of algebra. In that book, he asks, “What must be the square which, when increased by ten of its own roots, amounts to 39?” Al-Khwarizmi, like many Arab mathematicians of his time, was well versed in Euclid's Elements. Like Euclid, he viewed algebra very geometrically, and thus had a geometric approach to solving a problem like the one above. In his approach, he used a method which today we refer to as *completing the square*.

His description of the solution method for the above problem is: halve the number of roots, which in the present instance yields 5. This you multiply by itself; the product is 25. Add this to 39; the sum is 64. Now take the root of this which is 8, and subtract from it half the number of the roots, which is 5; the remainder is 3. This is the root of the square which you sought for. Thus the square is 9.

So, what does all of this mean? Al-Khwarizmi would start with a square of unknown length of side (we will label the side length x). See Figure 5.52 So, this square has area x^2

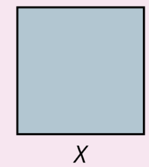


Figure 5.52

x

He would then proceed to halve the number of roots (i.e., there are 10 roots by which the square is increasing) to get 5; this he would add to the first square. See Figure 5.53 The area of the two new pieces added into the original square are both $5x$. At this point, we have $x^2 + 10x = 39$.

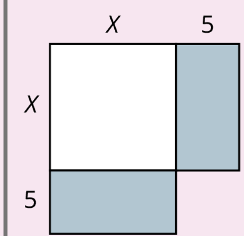


Figure 5.53

Now Al-Khwarizmi needed to "complete the square" by adding into the drawing a small square. See Figure 5.54 This square has an area of 25.

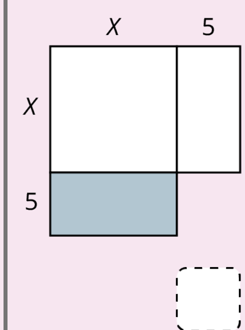


Figure 5.54

$x^2 + 10x + 25 = 39 + 25$, or $x^2 + 10x + 25 = 64$.

Notice that the completed square has side length $x + 5$, so the large square has area $(x + 5)^2$. (Notice algebraically that the left half of the equation $x^2 + 10x + 25 = 64$ factors to $(x + 5)^2 = 64$.) This means the area of large square equals 64. If $(x + 5)^2 = 64$, then $x + 5 = 8$; so x must be equal to 3 or -13 to make this true. Note that Al-Kwarimi would not have considered the possibility of a negative solution, since he approached the solution geometrically, and negative distances do not exist.

Check Your Understanding

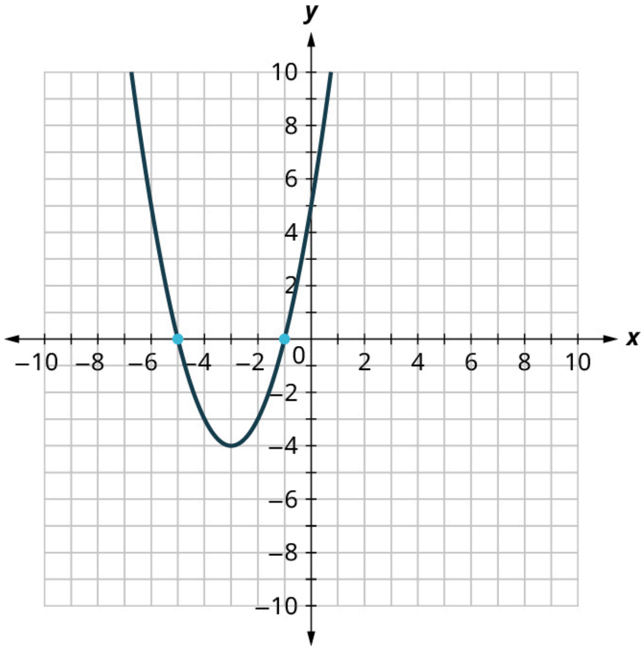
48. Which quadratic equation equals $(x - 3)(x + 5)$?

a. $x^2 - 15$
b. $x^2 - 3x + 15$
c. $x^2 - 2x - 15$
d. $x^2 + 2x - 15$

49. Which product is equal to $x^2 - 8x + 15$?

a. $(x - 3)(x + 5)$
b. $(x - 3)(x - 5)$
c. $(x + 3)(x + 5)$
d. $(x + 3)(x - 5)$

50. The graph shown is the graph of which quadratic equation?



a. $x^2 - x + 5 = 0$
b. $x^2 - 5x + 1 = 0$
c. $x^2 + 6x + 5 = 0$
d. $x^2 - 4x + 5 = 0$

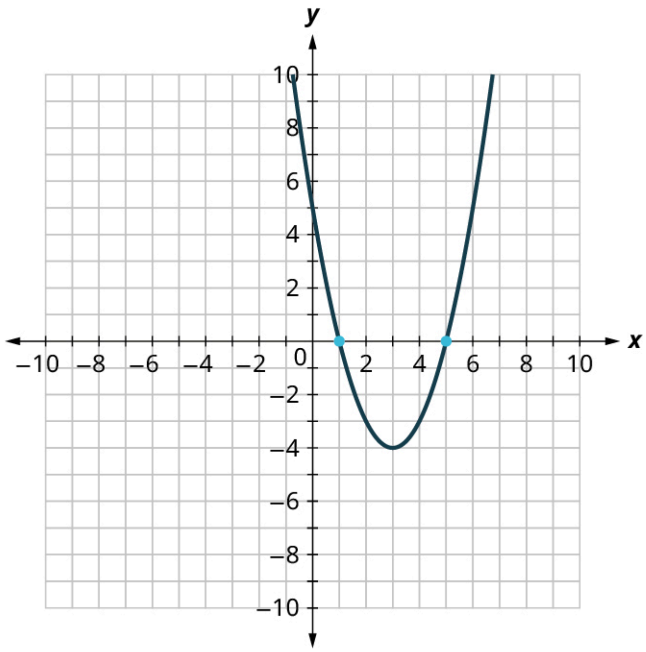
51. What is the solution to $x^2 - 49 = 0$?
a. $x = 7$
b. $x = -7$
c. $x = \pm 49$
d. $x = \pm 7$

52. $x^2 - 5x + 5 = 0$ can be factored to $(x - 1)(x - 5)$.
a. True
b. False

53. $x^2 - 5x + 5 = 0$ can be solved using the square root method.
a. True
b. False

54. $x^2 - 5x + 5 = 0$ can be solved using the quadratic formula.
a. True
b. False

55. $x^2 - 5x + 5 = 0$ can be graphed as:



a. True
b. False

56. Using the square root method, find the solutions to $x^2 - 5x + 5 = 0$.

SECTION 5.6 EXERCISES

For the following exercises, multiply the binomials.

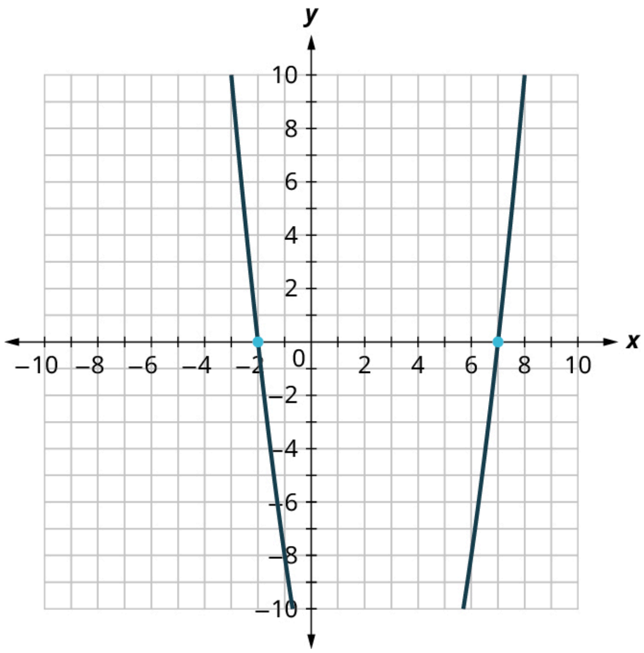
1. $(w + 5)(w + 7)$
2. $(y + 9)(y + 3)$
3. $(p + 11)(p - 4)$
4. $(q + 4)(q - 8)$
5. $(x + 8)(x + 3)$
6. $(2x - 1)(x + 6)$

For the following exercises, factor the trinomials.

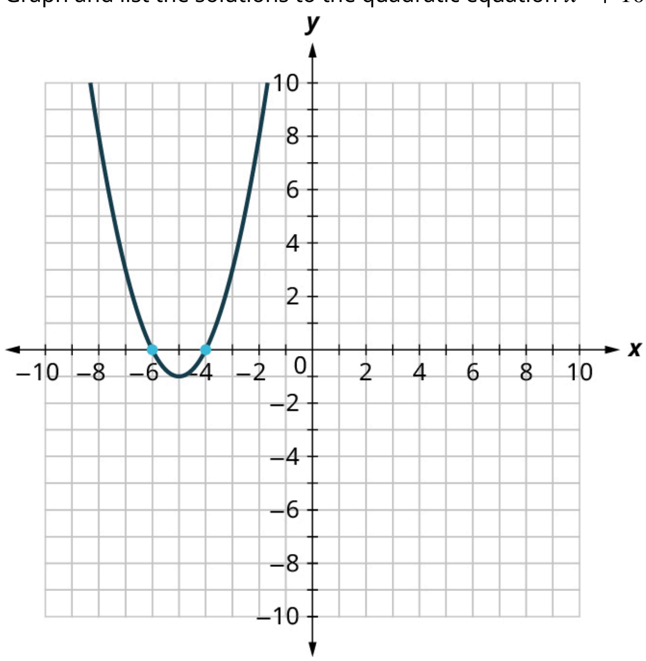
7. $a^2 - 5a - 14$
8. $n^2 + 10n + 24$
9. $u^2 - 16$
10. $x^2 + 4x - 21$
11. $y^2 - 8y + 15$
12. $x^2 - 25$

For the following exercises, solve the quadratic equations by graphing.

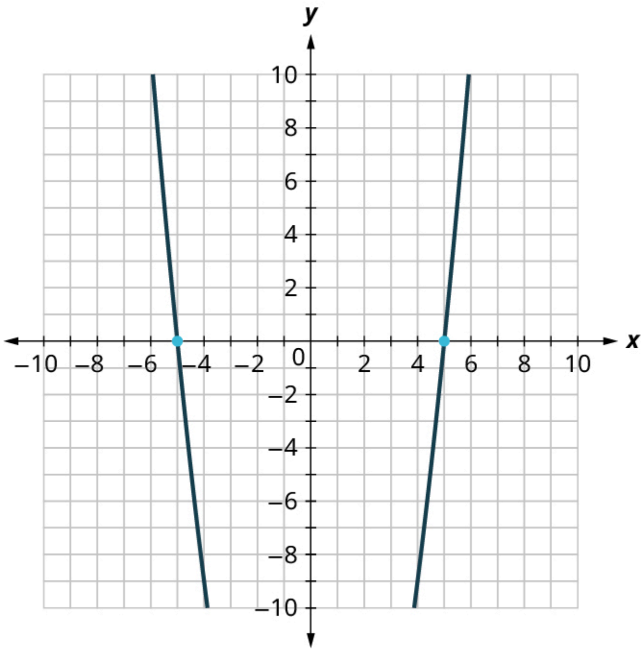
13. Graph and list the solutions to the quadratic equation $x^2 - 5x - 14 = 0$.



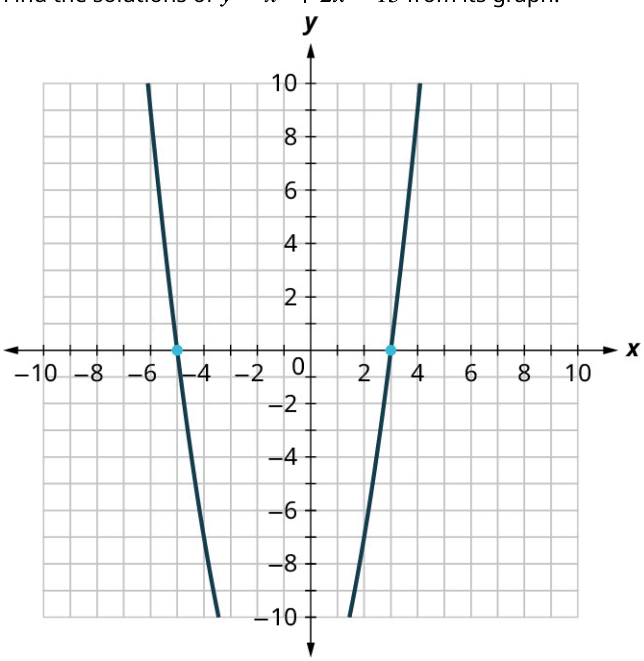
14. Graph and list the solutions to the quadratic equation $x^2 + 10x + 24 = 0$.



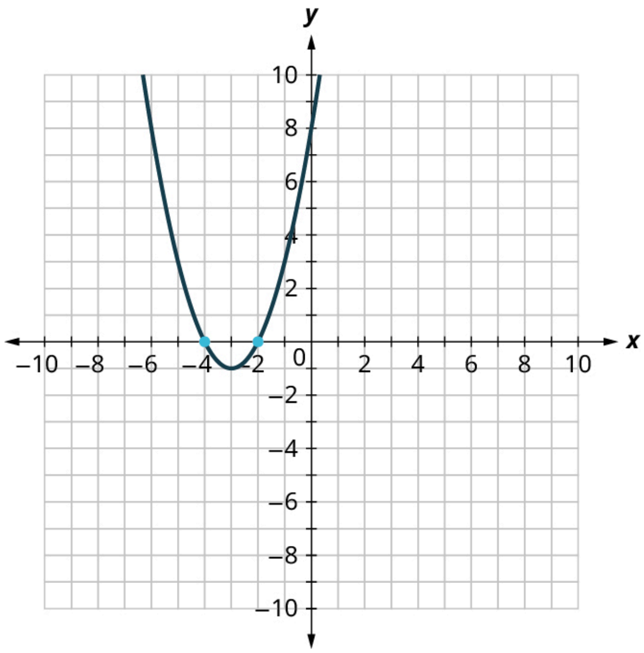
15. Graph and list the solutions to the quadratic equation $x^2 - 25 = 0$.



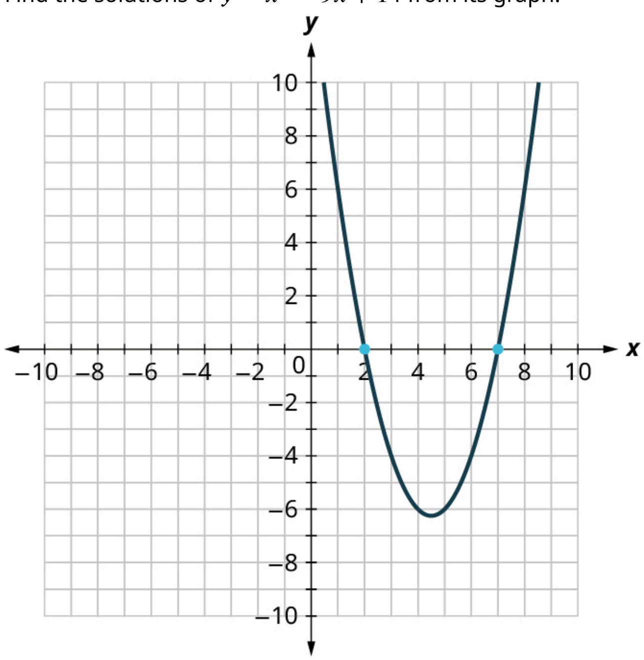
16. Find the solutions of $y = x^2 + 2x - 15$ from its graph.



17. Find the solutions of $y = x^2 + 6x + 8$ from its graph.



18. Find the solutions of $y = x^2 - 9x + 14$ from its graph.



For the following exercises, solve the quadratic equation by factoring.

19. $x^2 + 7x + 12 = 0$
20. $y^2 - 8y + 15 = 0$
21. $n^2 - 5n + 6 = 0$
22. $a^2 + 2a = a$
23. $b^2 = -b$
24. $x^2 - 9x + 14 = 0$

For the following exercises, solve the quadratic equation using the square root method.

25. $x^2 = 16$
26. $49m^2 = 144$
27. $625 = x^2$

For the following exercises, solve the quadratic equation using the quadratic formula.

28. $16p^2 = 24p - 9$

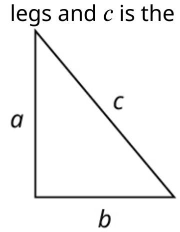
29. $m^2 - 2m = 1$
30. $4x^2 - 2x - 3 = 0$
31. $y^2 - 6y + 7 = 0$
32. $x^2 - 3x - 7 = 0$
33. $2x^2 - 4x - 11 = 0$

34. The product of two consecutive odd integers is 99. Find the integers.

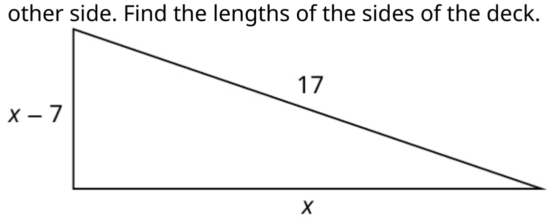
35. The product of two consecutive even integers is 168. Find the integers.

36. A rectangular patio has an area of 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the patio.

For the following exercises, use the Pythagorean Theorem: In any right triangle, where a and b are the lengths of the legs and c is the length of the hypotenuse, as shown in the given figure, $a^2 + b^2 = c^2$.



37. Justine wants to put a deck in the corner of her backyard in the shape of a right triangle, as shown in the given figure. The hypotenuse will be 17 feet long. The length of one side will be 7 feet less than the length of the other side. Find the lengths of the sides of the deck.



38. A boat's sail is a right triangle. The length of one side of the sail is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the sail.
39. The sun casts a shadow from a flagpole. The height of the flagpole is three times the length of its shadow. The distance between the end of the shadow and the top of the flagpole is 21 feet. Find the length of the shadow and the length of the flagpole. Round to the nearest tenth of a foot.
40. Rene is setting up a holiday light display. He wants to make a "tree" in the shape of two right triangles and has two 10-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be? Round to the nearest tenth of a foot.

5.7 Functions

Figure 5.55 A small group of elementary students learning from their teacher. (credit: modification of work "Our school" by Woodleywonderworks/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Use function notation.
2. Determine if a relation is a function with different representations.
3. Apply the vertical line test.
4. Determine the domain and range of a function.

In this section, we will learn about relations and functions. As we go about our daily lives, we have many data items or quantities that are paired to our names. Our social security number, student ID number, email address, phone number, and our birthday are matched to our name. There is a relationship between our name and each of those items. When your teacher gets their class roster, the names of all the students in the class are listed in one column and then the student ID number is likely to be in the next column. If we think of the correspondence as a set of ordered pairs, where the first element is a student name and the second element is that student's ID number, we call this a relation.

(Student name, Student ID #)

The set of all the names of the students in the class is called the domain of the relation and the set of all student ID numbers paired with these students is the range of the relation. In general terms, a **relation** is any set of ordered pairs, (x, y). All the x-values in the ordered pairs together make up the **domain**. All the y-values in the ordered pairs together make up the range.

There are many situations similar to the student's name and student ID # where one variable is paired or matched with another. The set of ordered pairs that records this matching is a relation. A special type of relation, called a **function**, occurs extensively in mathematics. A function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each x-value is matched with only one y-value.

Let us look at the relation between your friends and their birthdays in Figure 5.56. Every friend has a birthday, but no one has two birthdays. It is okay for two people to share a birthday. It is okay that Danny and Stephen share July 24 as their birthday and that June and Liz share August 2. Since each person has exactly one birthday, the relation is a function.

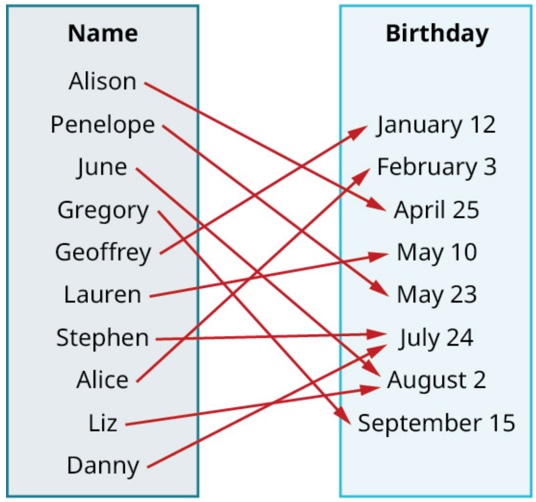


Figure 5.56 Birthday Mapping

Use Function Notation

It is very convenient to name a function; most often functions are named f, g, h, F, G, or H. In any function, for each x-value from the domain, we get a corresponding y-value in the range. In the function f, we write this range value y as $f(x)$. This notation $f(x)$ is called function notation and is read "f of x" or "the value of f at x." In this case the parentheses do not indicate multiplication.

We call x the independent variable as it can be any value in the domain. We call y the dependent variable as its value depends on x. Much like when you first encountered the variable x, function notation may be rather unsettling. But the more you use the notation, the more familiar you become with the notation, and the more comfortable you will be with it.

Let's review the equation $y = 4x - 5$. To find the value of y when $x = 2$, we know to substitute $x = 2$ into the equation and then simplify.

	$y = 4x - 5$
Let $x = 2$.	$y = 4 \cdot \mathbf{2} - 5$ $y = 3$

The value of the function at $x = 2$ is 3. We do the same thing using function notation, the equation $y = 4x - 5$ can be written as $f(x) = 4x - 5$. To find the value when $x = 2$, we write:

	$f(x) = 4x - 5$
Let $x = 2$.	$f(2) = 4 \cdot \mathbf{2} - 5$ $f(2) = 3$

The value of the function at $x = 2$ is 3. This process of finding the value of $f(x)$ for a given value of x is called **evaluating the function**.

EXAMPLE 5.61

Evaluating the Function

For the function $f(x) = 2x^2 + 3x - 1$, evaluate the function.

1. $f(3)$
2. $f(-2)$
3. $f(a)$

Solution

1. To evaluate $f(3)$, substitute 3, for x.
   Simplify.

$$f(x) = 2x^2 + 3x - 1$$
$$f(\mathbf{3}) = 2(\mathbf{3})^2 + 3 \cdot \mathbf{3} - 1$$
$$f(3) = 2 \cdot 9 + 3 \cdot 3 - 1$$
$$f(3) = 18 + 9 - 1$$
$$f(3) = 26$$

2. To evaluate $f(-2)$, substitute $\mathbf{-2}$ for x.
   Simplify.

$$f(x) = 2x^2 + 3x - 1$$
$$f(\mathbf{-2}) = 2(\mathbf{-2})^2 + 3(\mathbf{-2}) - 1$$
$$f(-2) = 2 \cdot 4 + (-6) - 1$$
$$f(-2) = 8 + (-6) - 1$$
$$f(-2) = 1$$

3. To evaluate $f(a)$, substitute a for x.
   Simplify.

$$f(x) = 2x^2 + 3x - 1$$
$$f(\boldsymbol{a}) = 2(\boldsymbol{a})^2 + 3 \cdot \boldsymbol{a} - 1$$
$$f(a) = 2a^2 + 3a - 1$$

YOUR TURN 5.61

For the function $f(x) = 3x^2 - 2x + 1$, evaluate the function.

1. $f(3)$
2. $f(-1)$
3. $f(t)$

EXAMPLE 5.62

Evaluating the Function in an Application

The number of unread emails in Sylvia's inbox is 75. This number grows by 10 unread emails a day. The function $N(t) = 75 + 10t$ represents the relation between the number of emails, N, and the time, t, measured in days. Find N(5). Explain what this result means.

Solution

Find N(5). Explain what this result means.

Substitute in $t = 5$.
Simplify.

$$N(\mathbf{5}) = 75 + 10 \cdot \mathbf{5}$$
$$N(t) = 75 + 10t$$
$$N(5) = 75 + 50$$
$$N(5) = 125$$

If 5 is the number of days, $N(5)$ is the number of unread emails after 5 days. After 5 days, there are 125 unread emails in Sylvia's inbox.

YOUR TURN 5.62

1. The number of unread emails in Bryan's account is 100. This number grows by 15 unread emails a day. The function $N(t) = 100 + 15t$ represents the relation between the number of emails, N, and the time, t, measured in days. Find $N(7)$. Explain what the result means.

Determining If a Relation Is a Function with Different Representations

We can determine whether a relation is a function by identifying the input and the output values. If each input value leads to only one output value, classify the relation as a function. If any input value leads to two or more outputs, do not classify the relation as a function.

We will review three different representations of relations and determine if they are functions: ordered pairs, mapping, and equations.

EXAMPLE 5.63

Determining If a Relation Is a Function with a Set of Ordered Pairs

Use the set of ordered pairs to determine whether the relation is a function.

1. $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$
2. $\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$

Solution

1. Each x-value is matched with only one y-value. This relation is a function.
2. The x-value 9 is matched with two y-values, both 3 and -3. This relation is not a function.

YOUR TURN 5.63

Use the set of ordered pairs to determine whether the relation is a function.

1. $\{(-3, -6), (-2, -4), (-1, -2), (0, 0), (1, 2), (2, 4), (3, 6)\}$
2. $\{ (8, -4), (4, -2), (2, -1), (0,0), (2,1), (4,2), (8,4)\}$

A **mapping** is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range. Consider the example of the relation between your friends and their birthdays used in Figure 5.57. In this particular example, the domain is the set of people's names, and the range is the set of their birthdays. This mapping was a function because everybody's name maps to exactly one birthday.

EXAMPLE 5.64

Determining If a Relation Is a Function with Mapping

Use the mapping in Figure 5.57 to determine whether the relation is a function.

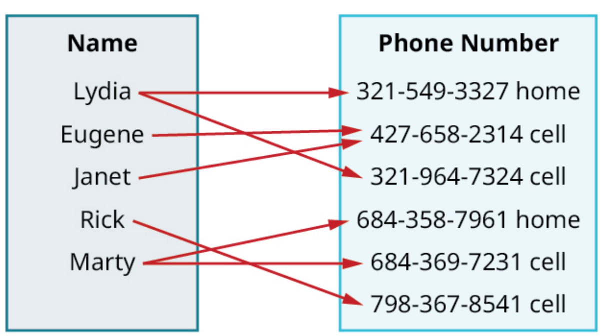


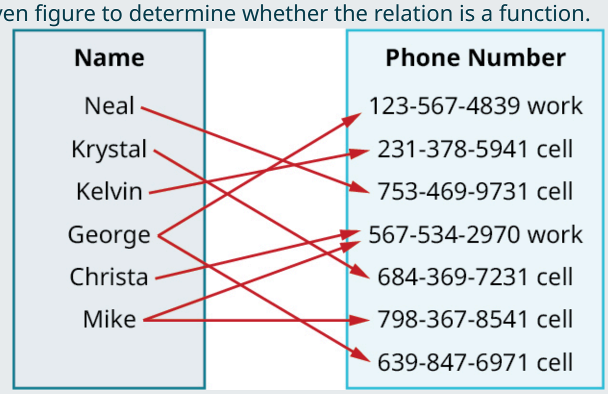
Figure 5.57

Solution

Both Lydia and Marty have two phone numbers. Each x-value is not matched with only one y-value. This relation is not a function.

YOUR TURN 5.64

1. Use the mapping in the given figure to determine whether the relation is a function.



In algebra functions will usually be represented by an equation. It is easiest to see if the equation is a function when it is solved for y. If each value of x results in only one value of y, then the equation defines a function.

EXAMPLE 5.65

Determining If a Relation Is a Function with an Equation

Determine whether each equation is a function. Assume x is the independent variable.

1. $2x + y = 7$
2. $y = x^2 + 1$
3. $x + y^2 = 3$

Solution

1. $2x + y = 7$
   For each value of x, we multiply it by -2 and then add 7 to get the y-value.

   For example, if $x = 3$:

$$y = -2x + 7$$
$$y = -2 \cdot \mathbf{3} + 7$$
$$y = 1$$

   We have that when $x = 3$, then $y = 1$. It would work similarly for any value of x. Since each value of x, corresponds to only one value of y the equation defines a function.

2. $y = x^2 + 1$
   For each value of x, we square it and then add 1 to get the y-value.

   For example, if $x = 2$

$$y = x^2 + 1$$
$$y = \mathbf{2}^2 + 1$$
$$y = 5$$

   We have that when $x = 2$, then $y = 5$. It would work similarly for any value of x. Since each value of x corresponds to only one value of y, the equation defines a function.

3. $x + y^2 = 3$

$x + y^2 = 3$

Isolate the y term.

$y^2 = -x + 3$

Let us substitute $x = 2$.

$y^2 = \mathbf{-2} + 3$

$y^2 = 1$

This gives us two values for y.

$y = 1, \ \ y = -1$

We have shown that when $x = 2$, then $y = 1$ and $y = -1$. It would work similarly for any value of x. Since each value of x does not corresponds to only one value of y the equation does not define a function.

YOUR TURN 5.65

Determine whether each equation is a function.

1. $4x + y = -3$
2. $x + y^2 = 1$
3. $y - x^2 = 2$

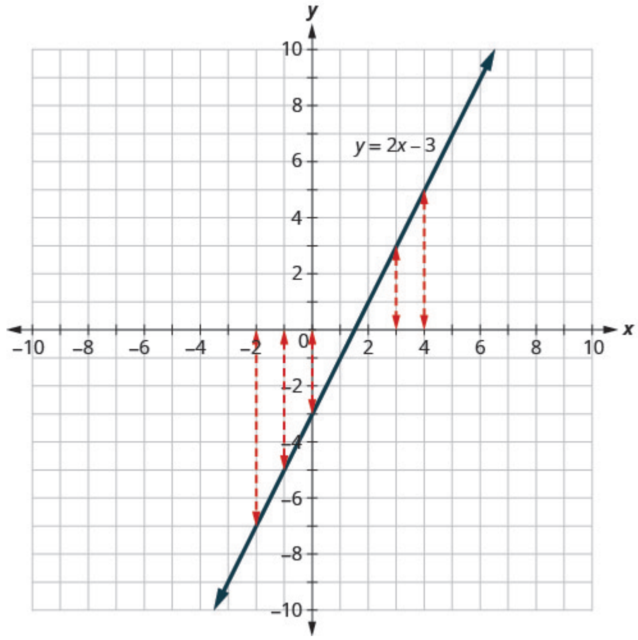
VIDEO

Relations and Functions (https://openstax.org/r/Relationsp_and_Functions)

Applying the Vertical Line Test

We reviewed how to determine if a relation is a function. The relations we looked at were expressed as a set of ordered pairs, a mapping, or an equation. We will now cover how to tell if a graph is that of a function.

An ordered pair (x, y) is a solution of a linear equation, if the equation is a true statement when the x-values and y-values of the ordered pair are substituted into the equation. The graph of a linear equation is a straight line where every point on the line is a solution of the equation, and every solution of this equation is a point on this line. Figure 5.58 we can see that in the graph of the equation $y = 2x - 3$, for every x-value there is only one y-value, as shown in the accompanying table.



y = 2x – 3		
x	y	(x, y)
–2	–7	(–2, –7)
–1	–5	(–1, –5)
0	–3	(0, –3)
3	3	(3, 3)
4	5	(4, 5)

Figure 5.58 Graph of the Equation $y = 2x - 3$

A relation is a function if every element of the domain has exactly one value in the range. The relation defined by the equation $y = 2x - 3$ is a function. If we look at the graph, each vertical dashed line only intersects the solid line at one point. This makes sense as in a function, for every x-value there is only one y-value. If the vertical line hit the graph twice, the x-value would be mapped to two y-values, and so the graph would not represent a function. This leads us a graphical method of determining functions called the **vertical line test**, which states that a set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. If any vertical line intersects the graph in more than one point, the graph does not represent a function.

EXAMPLE 5.66

Applying the Vertical Line Test

Determine whether the graph (Figure 5.59) is the graph of a function applying the vertical line test.

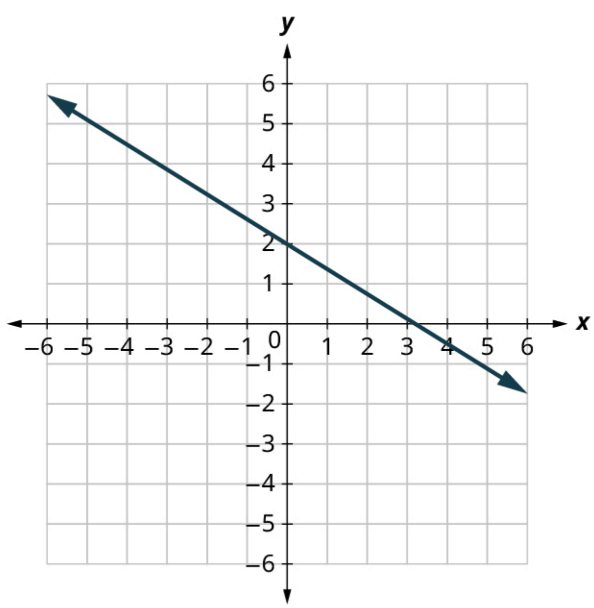


Figure 5.59

Solution

On the graph (Figure 5.60), only three vertical dashed lines are drawn. However, it can be determined that any vertical dashed line that is drawn will intersect the solid line at exactly one point. It is the graph of a function.

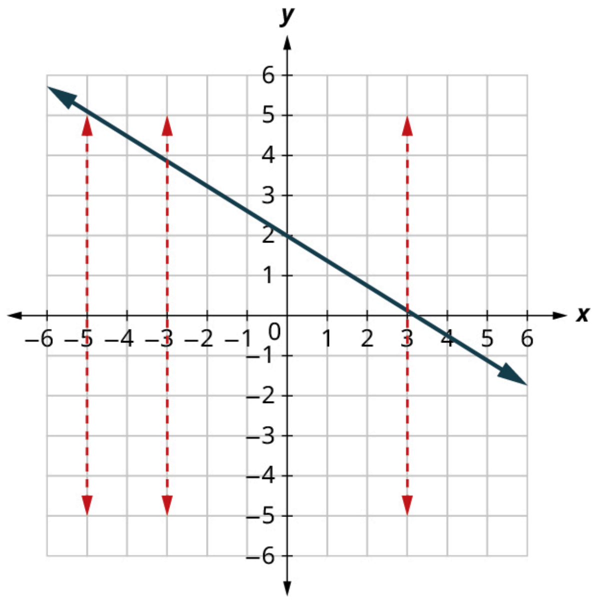
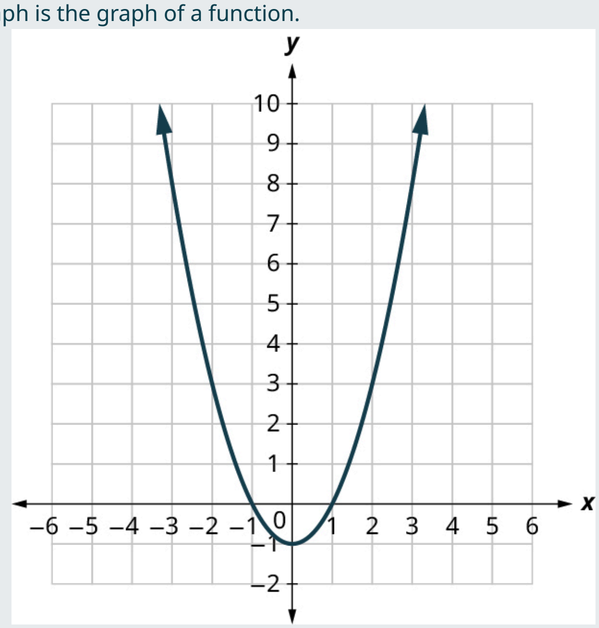


Figure 5.60

YOUR TURN 5.66

1. Determine whether the graph is the graph of a function.



EXAMPLE 5.67

Applying the Vertical Line Test to a Parabola

Determine whether the graph is the graph of a function (Figure 5.61).

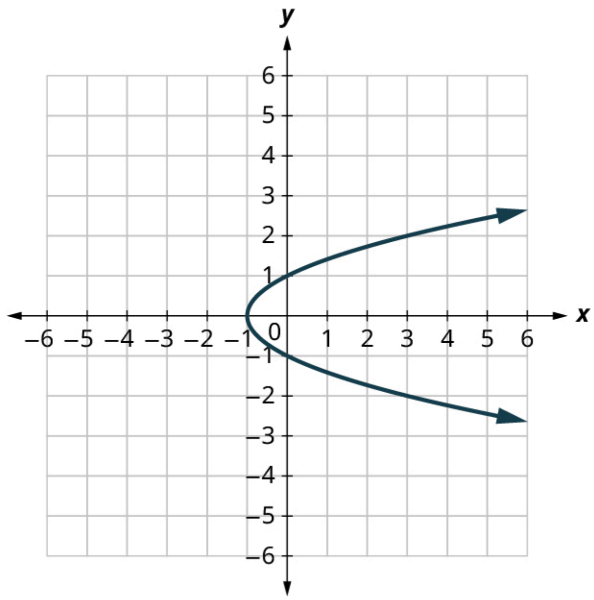


Figure 5.61

Solution

Figure 5.62 does not represent a function since the vertical dashed lines shown on the graph below intersect the solid line at two points.

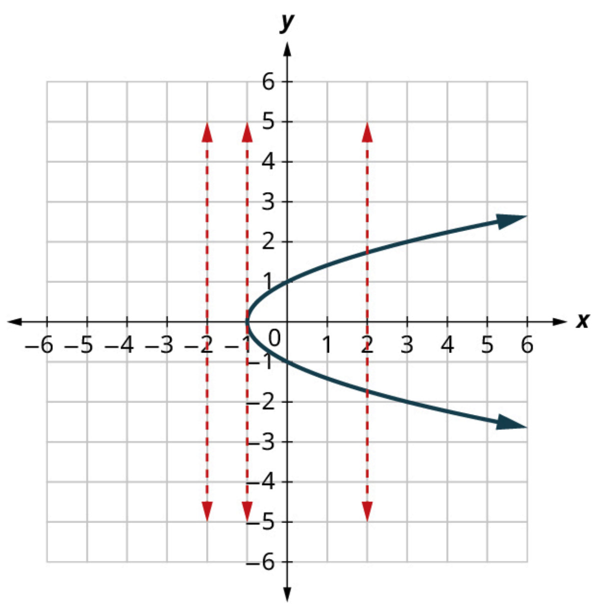
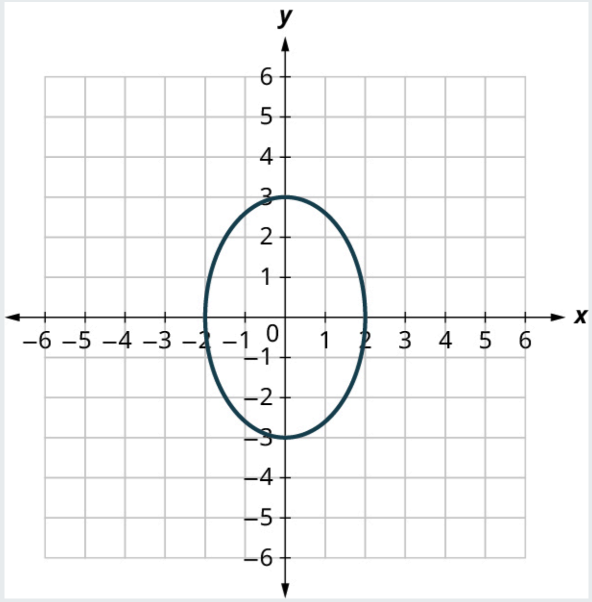


Figure 5.62

YOUR TURN 5.67

1. Determine whether the graph is the graph of a function.



Determining the Domain and Range of a Function

For the function $y = f(x)$, x is the independent variable as it can be any value in the domain, and y is the dependent variable since its value depends on x. For the function $y = f(x)$, the values of x make up the domain and the values of y make up the range.

EXAMPLE 5.68

Finding the Domain and Range of Ordered Pairs

For $\{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$:

1. Find the domain of the relation.
2. Find the range of the relation.

Solution

1. The domain is the set of all x-values of the relation: $\{1, 2, 3, 4, 5\}$
2. The range is the set of all y-values of the relation: $\{1, 4, 9, 16, 25\}$

YOUR TURN 5.68

For the relation $\{(1, 1), (2, 8), (3, 27), (4, 64), (5, 125)\}$:

1. Find the domain of the relation.
2. Find the range of the relation.

EXAMPLE 5.69

Finding the Domain and Range on a Graph

Use Figure 5.63 to:

1. List the ordered pairs of the relation.
2. Find the domain of the relation.
3. Find the range of the relation.

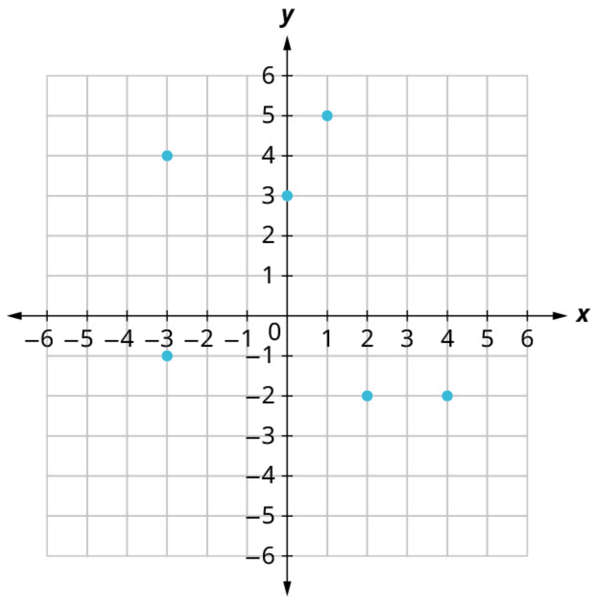


Figure 5.63

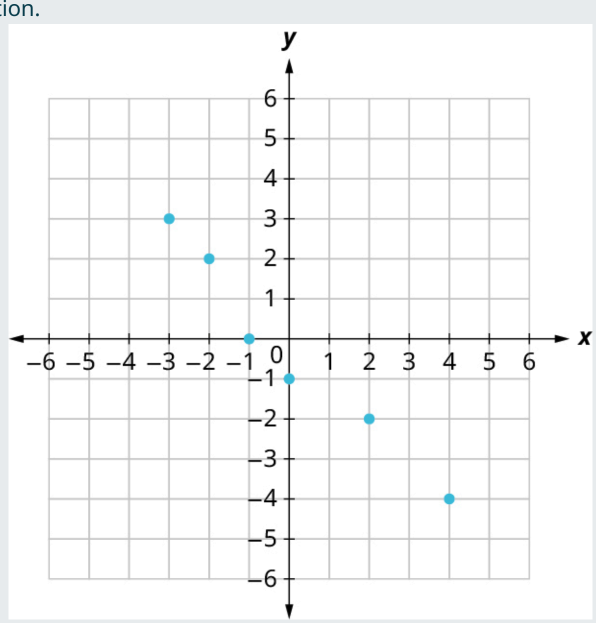
Solution

1. The ordered pairs of the relation are: $\{(1,5), (-3,-1), (4,-2), (0,3), (2,-2), (-3,4)\}$.
2. The domain is the set of all x-values of the relation: $\{-3, 0, 1, 2, 4\}$. Notice that while -3 repeats, it is only listed once.
3. The range is the set of all y-values of the relation: $\{-2, -1, 3, 4, 5\}$. Notice that while -2 repeats, it is only listed once.

YOUR TURN 5.69

Use the given figure to:

1. List the ordered pairs of the relation.
2. Find the domain of the relation.
3. Find the range of the relation.



VIDEO

[Domain and Range on Graphs (https://openstax.org/r/Domain)](https://openstax.org/r/Domain)

WHO KNEW?

Function and Function Notation

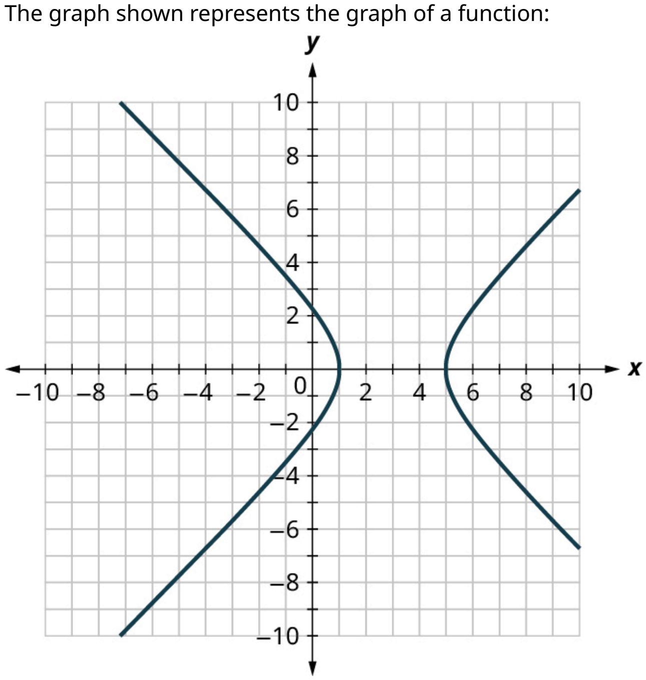
In 1673, Gottfried Leibniz, the German mathematician who co-invented calculus, seems to be the first person to use the word *function* in a mathematical sense, although his use of it does not exactly fit with the modern use and definition. The person who is credited with the modern definition of function is Swiss mathematician Johann Bernoulli, who wrote about it in a letter to Leibniz in 1698. Supposedly, Leibniz wrote Bernoulli back, approving of this use of the word. In 1734, the use of the notation $f(x)$ for a function was first used by Swiss mathematician Leonhard Euler (pronounced "Oiler"). Euler had a knack for inventing notation. He also introduced the notation e for the base of natural logs (1727), i for the square root of -1 (1777), $\sum$ for summation (1755), and many others. Euler also introduced many other ideas associated with functions. Euler defined exponential functions and defined logarithmic functions as their inverse; he also introduced the beta and gamma functions, and was the first person to consider the trigonometric identities (sine, cosine, etc.) as functions.

Check Your Understanding

57. If $f(x) = 2x - 8$ then $f(3) = -2$.
 a. True
 b. False

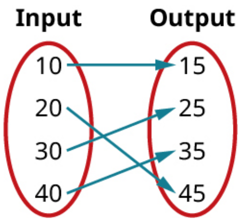
58. $\{(1,2), (2,3), (3,4), (2,1), (3,2), (4,3)\}$ represent the ordered pairs of a function.
 a. True
 b. False

59. The graph shown represents the graph of a function:



 a. True
 b. False

60. The figure shown represents the mapping of a function.



a. True
b. False

61. The domain of the mapping in the figure is $\{15, 25, 35, 45\}$.
a. True
b. False

SECTION 5.7 EXERCISES

For the following exercises, evaluate the functions at the values $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$.

1. $f(x) = 4 - 2x$
2. $f(x) = 8 - 3x$
3. $f(x) = 8x^2 - 7x + 3$
4. $f(x) = 3 + \sqrt{x + 3}$
5. $f(x) = \frac{x-2}{x+3}$
6. $f(x) = x^3$

For the following exercises, determine whether the ordered pairs represent a function.

7. $\{(-1, -1), (-2, -2), (-3, -3)\}$
8. $\{(3, 4), (4, 5), (5, 6)\}$
9. $\{(2, 5), (7, 11), (15, 8), (7, 9)\}$
10. $\{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$
11. $\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$

For the following exercises, determine whether the mapping represented a function.

12.