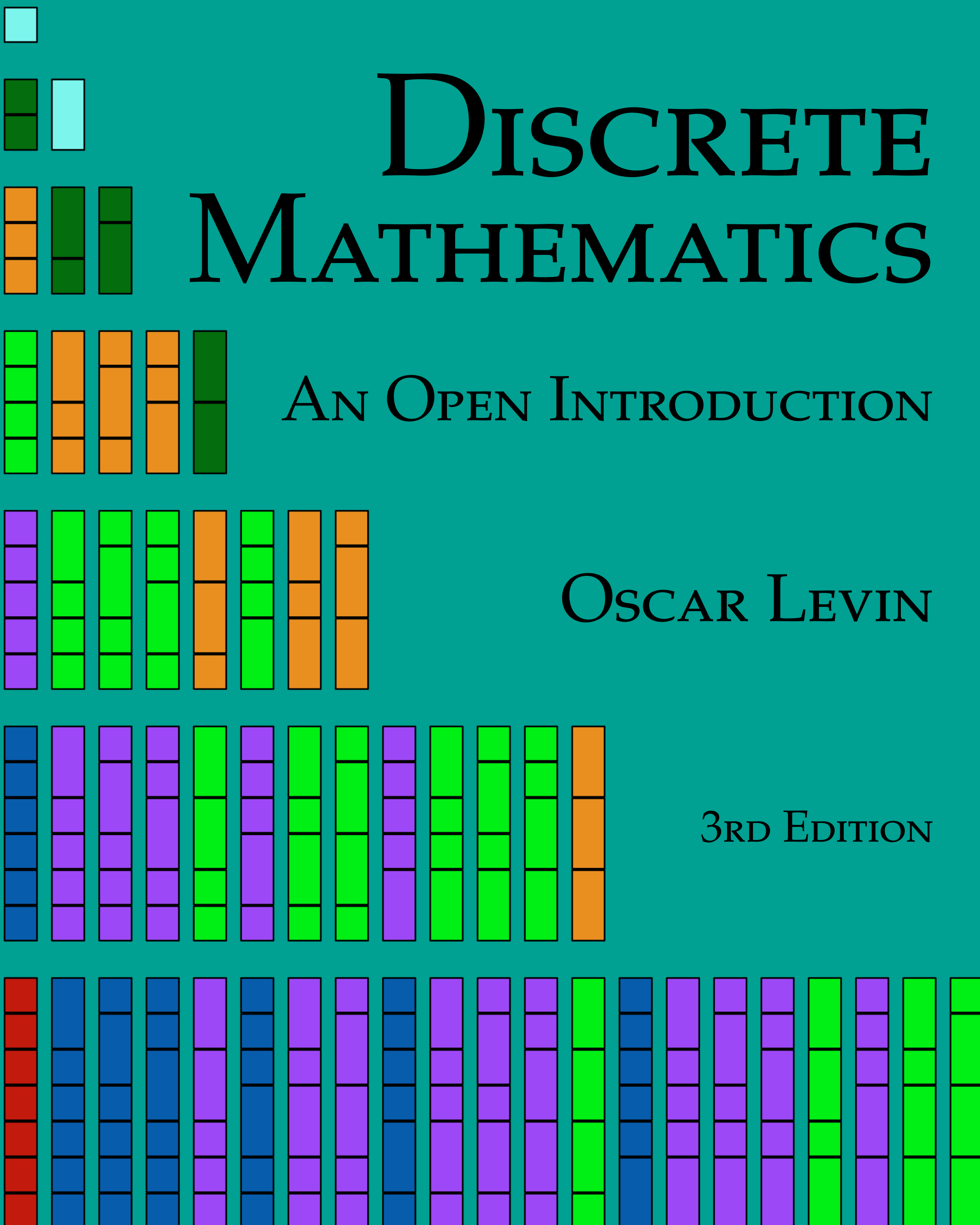
Discrete Mathematics

An Open Introduction

Oscar Levin

3rd Edition



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Cover image: *Tiling with Fibonacci and Pascal*.

For Madeline and Teagan

Acknowledgements

This book would not exist if not for "Discrete and Combinatorial Mathematics" by Richard Grassl and Tabitha Mingus. It is the book I learned discrete math out of, and taught out of the semester before I began writing this text. I wanted to maintain the inquiry based feel of their book but update, expand and rearrange some of the material. Some of the best exposition and exercises here were graciously donated from this source.

Thanks to Alees Seehausen who co-taught the Discrete Mathematics course with me in 2015 and helped develop many of the *Investigate!* activities and other problems currently used in the text. She also offered many suggestions for improvement of the expository text, for which I am quite grateful. Thanks also to Katie Morrison, Nate Eldredge and Richard Grassl (again) for their suggestions after using parts of this text in their classes.

While odds are that there are still errors and typos in the current book, there are many fewer thanks to the work of Michelle Morgan over the summer of 2016.

The book is now available in an interactive online format, and this is entirely thanks to the work of Rob Beezer, David Farmer, and Alex Jordan along with the rest of the participants of the pretext-support group.

Finally, a thank you to the numerous students who have pointed out typos and made suggestions over the years and a thanks in advance to those who will do so in the future.

Preface

This text aims to give an introduction to select topics in discrete mathematics at a level appropriate for first or second year undergraduate math majors, especially those who intend to teach middle and high school mathematics. The book began as a set of notes for the Discrete Mathematics course at the University of Northern Colorado. This course serves both as a survey of the topics in discrete math and as the "bridge" course for math majors, as UNC does not offer a separate "introduction to proofs" course. Most students who take the course plan to teach, although there are a handful of students who will go on to graduate school or study applied math or computer science. For these students the current text hopefully is still of interest, but the intent is not to provide a solid mathematical foundation for computer science, unlike the majority of textbooks on the subject.

Another difference between this text and most other discrete math books is that this book is intended to be used in a class taught using problem oriented or inquiry based methods. When I teach the class, I will assign sections for reading *after* first introducing them in class by using a mix of group work and class discussion on a few interesting problems. The text is meant to consolidate what we *discover* in class and serve as a reference for students as they master the concepts and techniques covered in the unit. None-the-less, every attempt has been made to make the text sufficient for self study as well, in a way that hopefully mimics an inquiry based classroom.

The topics covered in this text were chosen to match the needs of the students I teach at UNC. The main areas of study are combinatorics, sequences, logic and proofs, and graph theory, in that order. Induction is covered at the end of the chapter on sequences. Most discrete books put logic first as a preliminary, which certainly has its advantages. However, I wanted to discuss logic and proofs together, and found that doing both of these before anything else was overwhelming for my students given that they didn't yet have context of other problems in the subject. Also, after spending a couple weeks on proofs, we would hardly use that at all when covering combinatorics, so much of the progress we made was quickly lost. Instead, there is a short introduction section on mathematical statements, which should provide enough common language to discuss the logical content of combinatorics and sequences.

Depending on the speed of the class, it might be possible to include additional material. In past semesters I have included generating functions (after sequences) and some basic number theory (either after the logic and proofs chapter or at the very end of the course). These additional topics are covered in the last chapter.

While I (currently) believe this selection and order of topics is optimal, you should feel free to skip around to what interests you. There are occasionally examples and exercises that rely on earlier material, but I have tried to keep these to a minimum and usually can either be skipped or understood without too much additional study. If you are an instructor, feel free to edit the LaTeX or PreTeXt source to fit your needs.

Improvements to the 3rd Edition.

In addition to lots of minor corrections, both to typographical and mathematical errors, this third edition includes a few major improvements, including:

- More than 100 new exercises, bringing the total to 473. The selection of which exercises have solutions has also been improved, which should make the text more useful for instructors who want to assign homework from the book.
- A new section in on trees in the graph theory chapter.
- Substantial improvement to the exposition in chapter 0, especially the section on functions.
- The interactive online version of the book has added interactivity. Currently, many of the exercises are displayed as WeBWorK problems, allowing readers to enter answers to verify they are correct.

The previous editions (2nd edition, released in August 2016, and the Fall 2015 edition) will still be available for instructors who wish to use those versions due to familiarity.

My hope is to continue improving the book, releasing a new edition each spring in time for fall adoptions. These new editions will incorporate additions and corrections suggested by instructors and students who use the text the previous semesters. Thus I encourage you to send along any suggestions and comments as you have them.

Oscar Levin, Ph.D.
University of Northern Colorado, 2019

How to use this book

In addition to expository text, this book has a few features designed to encourage you to interact with the mathematics.

Investigate! activities.

Sprinkled throughout the sections (usually at the very beginning of a topic) you will find activities designed to get you acquainted with the topic soon to be discussed. These are similar (sometimes identical) to group activities I give students to introduce material. You really should spend some time thinking about, or even working through, these problems before reading the section. By priming yourself to the types of issues involved in the material you are about to read, you will better understand what is to come. There are no solutions provided for these problems, but don't worry if you can't solve them or are not confident in your answers. My hope is that you will take this frustration with you while you read the proceeding section. By the time you are done with the section, things should be much clearer.

Examples.

I have tried to include the "correct" number of examples. For those examples which include *problems*, full solutions are included. Before reading the solution, try to at least have an understanding of what the problem is asking. Unlike some textbooks, the examples are not meant to be all inclusive for problems you will see in the exercises. They should not be used as a blueprint for solving other problems. Instead, use the examples to deepen our understanding of the concepts and techniques discussed in each section. Then use this understanding to solve the exercises at the end of each section.

Exercises.

You get good at math through practice. Each section concludes with a small number of exercises meant to solidify concepts and basic skills presented in that section. At the end of each chapter, a larger collection of similar exercises is included (as a sort of "chapter review") which might bridge material of different sections in that chapter. Many exercise have a hint or solution (which in the PDF version of the text can be found by clicking on the exercise number—clicking on the solution number will bring you back to the exercise). Readers are encouraged to try these exercises before looking at the help.

Both hints and solutions are intended as a way to check your work, but often what would "count" as a correct solution in a math class would be quite a bit more. When I teach with this book, I assign exercises that have solutions as practice and then use them, or similar problems, on quizzes and exams. There are also problems without solutions to challenge yourself (or to be assigned as homework).

Interactive Online Version.

For those of you reading this in a PDF or in print, I encourage you to also check out the interactive online version, which makes navigating the book a little easier. Additionally, some of the exercises are implemented as WeBWorK problems, which allow you to check your work without seeing the correct answer immediately. Additional interactivity is planned, including instructional videos for examples and additional exercises at the end of sections. These "bonus" features will be added on a rolling basis, so keep an eye out!

You can view the interactive version for free at http://discrete.openmathbooks.org/ or by scanning the QR code below with your smart phone.

Contents

Chapter 0

Introduction and Preliminaries

Welcome to Discrete Mathematics. If this is your first time encountering the subject, you will probably find discrete mathematics quite different from other math subjects. You might not even know what discrete math is! Hopefully this short introduction will shed some light on what the subject is about and what you can expect as you move forward in your studies.

0.1 What is Discrete Mathematics?

dis·crete / dis'krët.

Adjective: Individually separate and distinct.

Synonyms: separate - detached - distinct - abstract.

Defining *discrete mathematics* is hard because defining *mathematics* is hard. What is mathematics? The study of numbers? In part, but you also study functions and lines and triangles and parallelepipeds and vectors and .... Or perhaps you want to say that mathematics is a collection of tools that allow you to solve problems. What sort of problems? Okay, those that involve numbers, functions, lines, triangles, ... Whatever your conception of what mathematics is, try applying the concept of "discrete" to it, as defined above. Some math fundamentally deals with *stuff* that is individually separate and distinct.

In an algebra or calculus class, you might have found a particular set of numbers (maybe the set of numbers in the range of a function). You would represent this set as an interval: $[0, \infty)$ is the range of $f(x) = x^2$ since the set of outputs of the function are all real numbers 0 and greater. This set of numbers is NOT discrete. The numbers in the set are not separated by much at all. In fact, take any two numbers in the set and there are infinitely many more between them which are also in the set.

Discrete math could still ask about the range of a function, but the set would not be an interval. Consider the function which gives the number of children of each person reading this. What is the range? I'm guessing it is something like $\{0, 1, 2, 3\}$. Maybe 4 is in there too. But certainly there is nobody reading this that has 1.32419 children. This output set *is* discrete because the elements are separate. The inputs to the function also form a discrete set because each input is an individual person.

One way to get a feel for the subject is to consider the types of problems you solve in discrete math. Here are a few simple examples:

Investigate!

Note: Throughout the text you will see Investigate! *activities like this one. Answer the questions in these as best you can to give yourself a feel for what is coming next.*

1. The most popular mathematician in the world is throwing a party for all of his friends. As a way to kick things off, they decide that everyone should shake hands. Assuming all 10 people at the party each shake hands with every other person (but not themselves, obviously) exactly once, how many handshakes take place?

2. At the warm-up event for Oscar's All Star Hot Dog Eating Contest, Al ate one hot dog. Bob then showed him up by eating three hot dogs. Not to be outdone, Carl ate five. This continued with each contestant eating two more hot dogs than the previous contestant. How many hot dogs did Zeno (the 26th and final contestant) eat? How many hot dogs were eaten all together?

3. After excavating for weeks, you finally arrive at the burial chamber. The room is empty except for two large chests. On each is carved a message (strangely in English):

If this chest is empty, then the other chest's message is true.	This chest is filled with treasure or the other chest contains deadly scorpions.

   You know exactly one of these messages is true. What should you do?

4. Back in the days of yore, five small towns decided they wanted to build roads directly connecting each pair of towns. While the towns had plenty of money to build roads as long and as winding as they wished, it was very important that the roads not intersect with each other (as stop signs had not yet been invented). Also, tunnels and bridges were not allowed. Is it possible for each of these towns to build a road to each of the four other towns without creating any intersections?

Attempt the above activity before proceeding

One reason it is difficult to define discrete math is that it is a very broad description which encapsulates a large number of subjects. In this course we will study four main topics: **combinatorics** (the theory of ways things *combine*; in particular, how to count these ways), **sequences**, **symbolic logic**, and **graph theory**. However, there are other topics that belong under the discrete umbrella, including computer science, abstract algebra, number theory, game theory, probability, and geometry (some of these, particularly the last two, have both discrete and non-discrete variants).

Ultimately the best way to learn what discrete math is about is to *do* it. Let's get started! Before we can begin answering more complicated (and fun) problems, we must lay down some foundation. We start by reviewing mathematical statements, sets, and functions in the framework of discrete mathematics.

0.2 Mathematical Statements

Investigate!

While walking through a fictional forest, you encounter three trolls guarding a bridge. Each is either a *knight*, who always tells the truth, or a *knave*, who always lies. The trolls will not let you pass until you correctly identify each as either a knight or a knave. Each troll makes a single statement:

> Troll 1: If I am a knave, then there are exactly two knights here.
>
> Troll 2: Troll 1 is lying.
>
> Troll 3: Either we are all knaves or at least one of us is a knight.

Which troll is which?

Attempt the above activity before proceeding

In order to *do* mathematics, we must be able to *talk* and *write* about mathematics. Perhaps your experience with mathematics so far has mostly involved finding answers to problems. As we embark towards more advanced and abstract mathematics, writing will play a more prominent role in the mathematical process.

Communication in mathematics requires more precision than many other subjects, and thus we should take a few pages here to consider the basic building blocks: *mathematical statements*.

Atomic and Molecular Statements

A **statement** is any declarative sentence which is either true or false. A statement is **atomic** if it cannot be divided into smaller statements, otherwise it is called **molecular**.

Example 0.2.1

These are statements (in fact, *atomic* statements):

- Telephone numbers in the USA have 10 digits.
- The moon is made of cheese.
- 42 is a perfect square.
- Every even number greater than 2 can be expressed as the sum of two primes.

- $3 + 7 = 12$

And these are not statements:

- Would you like some cake?
- The sum of two squares.
- $1 + 3 + 5 + 7 + \cdots + 2n + 1$.
- Go to your room!
- $3 + x = 12$

The reason the sentence "$3 + x = 12$" is not a statement is that it contains a variable. Depending on what x is, the sentence is either true or false, but right now it is neither. One way to make the *sentence* into a *statement* is to specify the value of the variable in some way. This could be done by specifying a specific substitution, for example, "$3 + x = 12$ where $x = 9$," which is a true statement. Or you could *capture* the free variable by *quantifying* over it, as in, "for all values of x, $3 + x = 12$," which is false. We will discuss quantifiers in more detail at the end of this section.

You can build more complicated (molecular) statements out of simpler (atomic or molecular) ones using **logical connectives**. For example, this is a molecular statement:

> Telephone numbers in the USA have 10 digits and 42 is a perfect square.

Note that we can break this down into two smaller statements. The two shorter statements are *connected* by an "and." We will consider 5 connectives: "and" (Sam is a man and Chris is a woman), "or" (Sam is a man or Chris is a woman), "if. . . , then. . . " (if Sam is a man, then Chris is a woman), "if and only if" (Sam is a man if and only if Chris is a woman), and "not" (Sam is not a man). The first four are called **binary connectives** (because they connect two statements) while "not" is an example of a **unary connective** (since it applies to a single statement).

These molecular statements are of course still statements, so they must be either true or false. The absolutely key observation here is that which **truth value** the molecular statement achieves is completely determined by the type of connective and the truth values of the parts. We do not need to know what the parts actually say, only whether those parts are true or false. So to analyze logical connectives, it is enough to consider **propositional variables** (sometimes called *sentential* variables), usually capital letters in the middle of the alphabet: $P, Q, R, S, \ldots$. We think of these as standing in for (usually atomic) statements, but there are only two *values* the variables can achieve: true or false.[1] We also have symbols for the logical connectives: $\wedge$, $\vee$, $\to$, $\leftrightarrow$, $\neg$.

Logical Connectives.

- $P \wedge Q$ is read "P and Q," and called a **conjunction**.
- $P \vee Q$ is read "P or Q," and called a **disjunction**.
- $P \to Q$ is read "if P then Q," and called an **implication** or **conditional**.
- $P \leftrightarrow Q$ is read "P if and only if Q," and called a **biconditional**.
- $\neg P$ is read "not P," and called a **negation**.

The **truth value** of a statement is determined by the truth value(s) of its part(s), depending on the connectives:

Truth Conditions for Connectives.

- $P \wedge Q$ is true when both P and Q are true.
- $P \vee Q$ is true when P or Q or both are true.
- $P \to Q$ is true when P is false or Q is true or both.
- $P \leftrightarrow Q$ is true when P and Q are both true, or both false.
- $\neg P$ is true when P is false.

Note that for us, *or* is the **inclusive or** (and not the sometimes used *exclusive or*) meaning that $P \vee Q$ is in fact true when both P and Q are true. As for the other connectives, "and" behaves as you would expect, as does negation. The biconditional (if and only if) might seem a little strange, but you should think of this as saying the two parts of the statements are *equivalent* in that they have the same truth value. This leaves only the conditional $P \to Q$ which has a slightly different meaning in mathematics than it does in ordinary usage. However, implications are so common and useful in mathematics, that we must develop fluency with their use, and as such, they deserve their own subsection.

[1]In computer programming, we should call such variables **Boolean variables**.

Implications

Implications.

An **implication** or **conditional** is a molecular statement of the form

$$P \to Q$$

where P and Q are statements. We say that

- P is the **hypothesis** (or **antecedent**).
- Q is the **conclusion** (or **consequent**).

An implication is *true* provided P is false or Q is true (or both), and *false* otherwise. In particular, the only way for $P \to Q$ to be false is for P to be true *and* Q to be false.

Easily the most common type of statement in mathematics is the implication. Even statements that do not at first look like they have this form conceal an implication at their heart. Consider the *Pythagorean Theorem*. Many a college freshman would quote this theorem as "$a^2 + b^2 = c^2$." This is absolutely not correct. For one thing, that is not a statement since it has three variables in it. Perhaps they imply that this should be true for any values of the variables? So $1^2 + 5^2 = 2^2$??? How can we fix this? Well, the equation is true as long as a and b are the legs of a right triangle and c is the hypotenuse. In other words:

> *If* a and b are the legs of a right triangle with hypotenuse c, *then* $a^2 + b^2 = c^2$.

This is a reasonable way to think about implications: our claim is that the conclusion ("then" part) is true, but on the assumption that the hypothesis ("if" part) is true. We make no claim about the conclusion in situations when the hypothesis is false.[2]

Still, it is important to remember that an implication is a statement, and therefore is either true or false. The truth value of the implication is determined by the truth values of its two parts. To agree with the usage above, we say that an implication is true either when the hypothesis is false, or when the conclusion is true. This leaves only one way for an implication to be false: when the hypothesis is true and the conclusion is false.

[2]However, note that in the case of the Pythagorean Theorem, it is also the case that *if* $a^2 + b^2 = c^2$, *then* a and b are the legs of a right triangle with hypotenuse c. So we could have also expressed this theorem as a biconditional: "a and b are the legs of a right triangle with hypotenuse c *if and only if* $a^2 + b^2 = c^2$."

Example 0.2.2

Consider the statement:

If Bob gets a 90 on the final, then Bob will pass the class.

This is definitely an implication: P is the statement "Bob gets a 90 on the final," and Q is the statement "Bob will pass the class."

Suppose I made that statement to Bob. In what circumstances would it be fair to call me a liar? What if Bob really did get a 90 on the final, and he did pass the class? Then I have not lied; my statement is true. However, if Bob did get a 90 on the final and did not pass the class, then I lied, making the statement false. The tricky case is this: what if Bob did not get a 90 on the final? Maybe he passes the class, maybe he doesn't. Did I lie in either case? I think not. In these last two cases, P was false, and the statement $P \to Q$ was true. In the first case, Q was true, and so was $P \to Q$. So $P \to Q$ is true when either P is false or Q is true.

Just to be clear, although we sometimes read $P \to Q$ as "P *implies* Q", we are not insisting that there is some causal relationship between the statements P and Q. In particular, if you claim that $P \to Q$ is *false*, you are not saying that P does not imply Q, but rather that P is true and Q is false.

Example 0.2.3

Decide which of the following statements are true and which are false. Briefly explain.

1. If $1 = 1$, then most horses have 4 legs.
2. If $0 = 1$, then $1 = 1$.
3. If 8 is a prime number, then the 7624th digit of π is an 8.
4. If the 7624th digit of π is an 8, then $2 + 2 = 4$.

Solution. All four of the statements are true. Remember, the only way for an implication to be false is for the *if* part to be true and the *then* part to be false.

1. Here both the hypothesis and the conclusion are true, so the implication is true. It does not matter that there is no meaningful connection between the true mathematical fact and the fact about horses.
2. Here the hypothesis is false and the conclusion is true, so the implication is true.

3. I have no idea what the 7624th digit of π is, but this does not matter. Since the hypothesis is false, the implication is automatically true.
4. Similarly here, regardless of the truth value of the hypothesis, the conclusion is true, making the implication true.

It is important to understand the conditions under which an implication is true not only to decide whether a mathematical statement is true, but in order to *prove* that it is. Proofs might seem scary (especially if you have had a bad high school geometry experience) but all we are really doing is explaining (very carefully) why a statement is true. If you understand the truth conditions for an implication, you already have the outline for a proof.

Direct Proofs of Implications.

To prove an implication $P \to Q$, it is enough to assume P, and from it, deduce Q.

Perhaps a better way to say this is that to prove a statement of the form $P \to Q$ directly, you must explain why Q is true, but you *get to* assume P is true first. After all, you only care about whether Q is true in the case that P is as well.

There are other techniques to prove statements (implications and others) that we will encounter throughout our studies, and new proof techniques are discovered all the time. Direct proof is the easiest and most elegant style of proof and has the advantage that such a proof often does a great job of explaining *why* the statement is true.

Example 0.2.4

Prove: If two numbers a and b are even, then their sum $a + b$ is even.

Solution.

Proof. Suppose the numbers a and b are even. This means that $a = 2k$ and $b = 2j$ for some integers k and j. The sum is then $a + b = 2k + 2j = 2(k + j)$. Since $k + j$ is an integer, this means that $a + b$ is even. ■

Notice that since we get to assume the hypothesis of the implication, we immediately have a place to start. The proof proceeds essentially by repeatedly asking and answering, "what does that mean?" Eventually, we conclude that it means the conclusion.

This sort of argument shows up outside of math as well. If you ever found yourself starting an argument with "hypothetically, let's assume ...," then you have attempted a direct proof of your desired conclusion.

An implication is a way of expressing a relationship between two statements. It is often interesting to ask whether there are other relationships between the statements. Here we introduce some common language to address this question.

Converse and Contrapositive.

- The **converse** of an implication $P \to Q$ is the implication $Q \to P$. The converse is NOT logically equivalent to the original implication. That is, whether the converse of an implication is true is independent of the truth of the implication.
- The **contrapositive** of an implication $P \to Q$ is the statement $\neg Q \to \neg P$. An implication and its contrapositive are logically equivalent (they are either both true or both false).

Mathematics is overflowing with examples of true implications which have a false converse. If a number greater than 2 is prime, then that number is odd. However, just because a number is odd does not mean it is prime. If a shape is a square, then it is a rectangle. But it is false that if a shape is a rectangle, then it is a square.

However, sometimes the converse of a true statement is also true. For example, the Pythagorean theorem has a true converse: if $a^2 + b^2 = c^2$, then the triangle with sides a, b, and c is a *right* triangle. Whenever you encounter an implication in mathematics, it is always reasonable to ask whether the converse is true.

The contrapositive, on the other hand, always has the same truth value as its original implication. This can be very helpful in deciding whether an implication is true: often it is easier to analyze the contrapositive.

Example 0.2.5

True or false: If you draw any nine playing cards from a regular deck, then you will have at least three cards all of the same suit. Is the converse true?

Solution. True. The original implication is a little hard to analyze because there are so many different combinations of nine cards. But consider the contrapositive: If you *don't* have at least three cards all of the same suit, then you don't have nine cards. It is easy to see why this is true: you can at most have two cards of each of the four suits, for a total of eight cards (or fewer).

The converse: If you have at least three cards all of the same suit, then you have nine cards. This is false. You could have three spades and nothing else. Note that to demonstrate that the converse (an implication) is false, we provided an example where the hypothesis is true (you do have three cards of the same suit), but where the conclusion is false (you do not have nine cards).

Understanding converses and contrapositives can help understand implications and their truth values:

Example 0.2.6

Suppose I tell Sue that if she gets a 93% on her final, then she will get an A in the class. Assuming that what I said is true, what can you conclude in the following cases:

1. Sue gets a 93% on her final.
2. Sue gets an A in the class.
3. Sue does not get a 93% on her final.
4. Sue does not get an A in the class.

Solution. Note first that whenever $P \rightarrow Q$ and P are both true statements, Q must be true as well. For this problem, take P to mean "Sue gets a 93% on her final" and Q to mean "Sue will get an A in the class."

1. We have $P \rightarrow Q$ and P, so Q follows. Sue gets an A.
2. You cannot conclude anything. Sue could have gotten the A because she did extra credit for example. Notice that we do not know that if Sue gets an A, then she gets a 93% on her final. That is the converse of the original implication, so it might or might not be true.
3. The contrapositive of the converse of $P \rightarrow Q$ is $\neg P \rightarrow \neg Q$, which states that if Sue does not get a 93% on the final, then she will not get an A in the class. But this does not follow from the original implication. Again, we can conclude nothing. Sue could have done extra credit.
4. What would happen if Sue does not get an A but *did* get a 93% on the final? Then P would be true and Q would be false. This makes the implication $P \rightarrow Q$ false! It must be that

Sue did not get a 93% on the final. Notice now we have the implication $\neg Q \rightarrow \neg P$ which is the contrapositive of $P \rightarrow Q$. Since $P \rightarrow Q$ is assumed to be true, we know $\neg Q \rightarrow \neg P$ is true as well.

As we said above, an implication is not logically equivalent to its converse, but it is possible that both the implication and its converse are true. In this case, when both $P \rightarrow Q$ and $Q \rightarrow P$ are true, we say that P and Q are equivalent and write $P \leftrightarrow Q$. This is the biconditional we mentioned earlier.

If and only if.

$P \leftrightarrow Q$ is logically equivalent to $(P \rightarrow Q) \wedge (Q \rightarrow P)$.

Example: Given an integer n, it is true that n is even if and only if n^2 is even. That is, if n is even, then n^2 is even, as well as the converse: if n^2 is even, then n is even.

You can think of "if and only if" statements as having two parts: an implication and its converse. We might say one is the "if" part, and the other is the "only if" part. We also sometimes say that "if and only if" statements have two directions: a forward direction ($P \rightarrow Q$) and a backwards direction ($P \leftarrow Q$, which is really just sloppy notation for $Q \rightarrow P$).

Let's think a little about which part is which. Is $P \rightarrow Q$ the "if" part or the "only if" part? Consider an example.

Example 0.2.7

Suppose it is true that I sing if and only if I'm in the shower. We know this means both that if I sing, then I'm in the shower, and also the converse, that if I'm in the shower, then I sing. Let P be the statement, "I sing," and Q be, "I'm in the shower." So $P \rightarrow Q$ is the statement "if I sing, then I'm in the shower." Which part of the if and only if statement is this?

What we are really asking for is the meaning of "I sing *if* I'm in the shower" and "I sing *only if* I'm in the shower." When is the first one (the "if" part) *false*? When I am in the shower but not singing. That is the same condition on being false as the statement "if I'm in the shower, then I sing." So the "if" part is $Q \rightarrow P$. On the other hand, to say, "I sing only if I'm in the shower" is equivalent to saying "if I sing, then I'm in the shower," so the "only if" part is $P \rightarrow Q$.

It is not terribly important to know which part is the "if" or "only if" part, but this does illustrate something very, very important: *there are many ways to state an implication!*

Example 0.2.8

Rephrase the implication, "if I dream, then I am asleep" in as many different ways as possible. Then do the same for the converse.

Solution. The following are all equivalent to the original implication:

1. I am asleep if I dream.
2. I dream only if I am asleep.
3. In order to dream, I must be asleep.
4. To dream, it is necessary that I am asleep.
5. To be asleep, it is sufficient to dream.
6. I am not dreaming unless I am asleep.

The following are equivalent to the converse (if I am asleep, then I dream):

1. I dream if I am asleep.
2. I am asleep only if I dream.
3. It is necessary that I dream in order to be asleep.
4. It is sufficient that I be asleep in order to dream.
5. If I don't dream, then I'm not asleep.

Hopefully you agree with the above example. We include the "necessary and sufficient" versions because those are common when discussing mathematics. In fact, let's agree once and for all what they mean.

Necessary and Sufficient.

- "P is necessary for Q" means $Q \to P$.
- "P is sufficient for Q" means $P \to Q$.
- If P is necessary and sufficient for Q, then $P \leftrightarrow Q$.

To be honest, I have trouble with these if I'm not very careful. I find it helps to keep a standard example for reference.

Example 0.2.9

Recall from calculus, if a function is differentiable at a point c, then it is continuous at c, but that the converse of this statement is not true (for example, $f(x) = |x|$ at the point 0). Restate this fact using "necessary and sufficient" language.

Solution. It is true that in order for a function to be differentiable at a point c, it is necessary for the function to be continuous at c. However, it is not necessary that a function be differentiable at c for it to be continuous at c.

It is true that to be continuous at a point c, it is sufficient that the function be differentiable at c. However, it is not the case that being continuous at c is sufficient for a function to be differentiable at c.

Thinking about the necessity and sufficiency of conditions can also help when writing proofs and justifying conclusions. If you want to establish some mathematical fact, it is helpful to think what other facts would *be enough* (be sufficient) to prove your fact. If you have an assumption, think about what must also be necessary if that hypothesis is true.

Predicates and Quantifiers

Investigate!

Consider the statements below. Decide whether any are equivalent to each other, or whether any imply any others.

1. You can fool some people all of the time.
2. You can fool everyone some of the time.
3. You can always fool some people.
4. Sometimes you can fool everyone.

Attempt the above activity before proceeding

It would be nice to use variables in our mathematical sentences. For example, suppose we wanted to claim that if n is prime, then $n + 7$ is not prime. This looks like an implication. I would like to write something like

$$P(n) \to \neg P(n+7)$$

where $P(n)$ means "n is prime." But this is not quite right. For one thing, because this sentence has a **free variable** (that is, a variable that we have not specified anything about), it is not a statement. A sentence that contains variables is called a **predicate**.

Now, if we plug in a specific value for n, we do get a statement. In fact, it turns out that no matter what value we plug in for n, we get a true implication in this case. What we really want to say is that *for all* values of n, if n is prime, then $n + 7$ is not. We need to *quantify* the variable.

Although there are many types of *quantifiers* in English (e.g., many, few, most, etc.) in mathematics we, for the most part, stick to two: existential and universal.

Universal and Existential Quantifiers.

The existential quantifier is $\exists$ and is read "there exists" or "there is." For example,

$$\exists x(x < 0)$$

asserts that there is a number less than 0.

The universal quantifier is $\forall$ and is read "for all" or "every." For example,

$$\forall x(x \geq 0)$$

asserts that every number is greater than or equal to 0.

As with all mathematical statements, we would like to decide whether quantified statements are true or false. Consider the statement

$$\forall x \exists y (y < x).$$

You would read this, "for every x there is some y such that y is less than x." Is this true? The answer depends on what our *domain of discourse* is: when we say "for all" x, do we mean all positive integers or all real numbers or all elements of some other set? Usually this information is implied. In discrete mathematics, we almost always quantify over the *natural numbers*, $0, 1, 2, \ldots$, so let's take that for our domain of discourse here.

For the statement to be true, we need it to be the case that no matter what natural number we select, there is always some natural number that is strictly smaller. Perhaps we could let y be $x - 1$? But here is the problem: what if $x = 0$? Then $y = -1$ and that is *not a number!* (in our domain of discourse). Thus we see that the statement is false because there is a number which is less than or equal to all other numbers. In symbols,

$$\exists x \forall y (y \geq x).$$

To show that the original statement is false, we proved that the *negation* was true. Notice how the negation and original statement compare. This is typical.

Quantifiers and Negation.

$\neg \forall x P(x)$ is equivalent to $\exists x \neg P(x)$.

$\neg \exists x P(x)$ is equivalent to $\forall x \neg P(x)$.

Essentially, we can pass the negation symbol over a quantifier, but that causes the quantifier to switch type. This should not be surprising: if not everything has a property, then something doesn't have that property. And if there is not something with a property, then everything doesn't have that property.

IMPLICIT QUANTIFIERS.

It is always a good idea to be precise in mathematics. Sometimes though, we can relax a little bit, as long as we all agree on a convention. An example of such a convention is to assume that sentences containing predicates with free variables are intended as statements, where the variables are universally quantified.

For example, do you believe that if a shape is a square, then it is a rectangle? But how can that be true if it is not a statement? To be a little more precise, we have two predicates: $S(x)$ standing for "x is a square" and $R(x)$ standing for "x is a rectangle". The *sentence* we are looking at is,

$$S(x) \to R(x).$$

This is neither true nor false, as it is not a statement. But come on! We all know that we meant to consider the statement,

$$\forall x(S(x) \to R(x)),$$

and this is what our convention tells us to consider.

Similarly, we will often be a bit sloppy about the distinction between a predicate and a statement. For example, we might write, *let $P(n)$ be the* statement, *"n is prime,"* which is technically incorrect. It is implicit that we mean that we are defining $P(n)$ to be a predicate, which for each n becomes the statement, n is prime.

Exercises

1. For each sentence below, decide whether it is an atomic statement, a molecular statement, or not a statement at all.

(a) Customers must wear shoes.

(b) The customers wore shoes.

(c) The customers wore shoes and they wore socks.

2. Classify each of the sentences below as an atomic statement, a molecular statement, or not a statement at all. If the statement is molecular, say what kind it is (conjunction, disjunction, conditional, biconditional, negation).

(a) The sum of the first 100 odd positive integers.

(b) Everybody needs somebody sometime.

(c) The Broncos will win the Super Bowl or I'll eat my hat.

(d) We can have donuts for dinner, but only if it rains.

(e) Every natural number greater than 1 is either prime or composite.

(f) This sentence is false.

3. Suppose P and Q are the statements: P: Jack passed math. Q: Jill passed math.

(a) Translate "Jack and Jill both passed math" into symbols.

(b) Translate "If Jack passed math, then Jill did not" into symbols.

(c) Translate "$P \vee Q$" into English.

(d) Translate "$\neg(P \wedge Q) \to Q$" into English.

(e) Suppose you know that if Jack passed math, then so did Jill. What can you conclude if you know that:

i. Jill passed math?

ii. Jill did not pass math?

4. Determine whether each molecular statement below is true or false, or whether it is impossible to determine. Assume you do not know what my favorite number is (but you do know that 13 is prime).

(a) If 13 is prime, then 13 is my favorite number.

(b) If 13 is my favorite number, then 13 is prime.

(c) If 13 is not prime, then 13 is my favorite number.

(d) 13 is my favorite number or 13 is prime.

(e) 13 is my favorite number and 13 is prime.

(f) 7 is my favorite number and 13 is not prime.

(g) 13 is my favorite number or 13 is not my favorite number.

5. In my safe is a sheet of paper with two shapes drawn on it in colored crayon. One is a square, and the other is a triangle. Each shape is drawn in a single color. Suppose you believe me when I tell you that *if the square is blue, then the triangle is green*. What do you therefore know about the truth value of the following statements?

(a) The square and the triangle are both blue.

(b) The square and the triangle are both green.

(c) If the triangle is not green, then the square is not blue.

(d) If the triangle is green, then the square is blue.

(e) The square is not blue or the triangle is green.

6. Again, suppose the statement "if the square is blue, then the triangle is green" is true. This time however, assume the converse is false. Classify each statement below as true or false (if possible).

(a) The square is blue if and only if the triangle is green.

(b) The square is blue if and only if the triangle is not green.

(c) The square is blue.

(d) The triangle is green.

7. Consider the statement, "If you will give me a cow, then I will give you magic beans." Decide whether each statement below is the converse, the contrapositive, or neither.

   (a) If you will give me a cow, then I will not give you magic beans.

   (b) If I will not give you magic beans, then you will not give me a cow.

   (c) If I will give you magic beans, then you will give me a cow.

   (d) If you will not give me a cow, then I will not give you magic beans.

   (e) You will give me a cow and I will not give you magic beans.

   (f) If I will give you magic beans, then you will not give me a cow.

8. Consider the statement "If Oscar eats Chinese food, then he drinks milk."

   (a) Write the converse of the statement.

   (b) Write the contrapositive of the statement.

   (c) Is it possible for the contrapositive to be false? If it was, what would that tell you?

   (d) Suppose the original statement is true, and that Oscar drinks milk. Can you conclude anything (about his eating Chinese food)? Explain.

   (e) Suppose the original statement is true, and that Oscar does not drink milk. Can you conclude anything (about his eating Chinese food)? Explain.

9. You have discovered an old paper on graph theory that discusses the *viscosity* of a graph (which for all you know, is something completely made up by the author). A theorem in the paper claims that "if a graph satisfies *condition (V)*, then the graph is *viscous*." Which of the following are equivalent ways of stating this claim? Which are equivalent to the *converse* of the claim?

   (a) A graph is viscous only if it satisfies condition (V).

   (b) A graph is viscous if it satisfies condition (V).

   (c) For a graph to be viscous, it is necessary that it satisfies condition (V).

   (d) For a graph to be viscous, it is sufficient for it to satisfy condition (V).

(e) Satisfying condition (V) is a sufficient condition for a graph to be viscous.

(f) Satisfying condition (V) is a necessary condition for a graph to be viscous.

(g) Every viscous graph satisfies condition (V).

(h) Only viscous graphs satisfy condition (V).

10. Write each of the following statements in the form, "if . . . , then . . . ." Careful, some of the statements might be false (which is alright for the purposes of this question).

(a) To lose weight, you must exercise.

(b) To lose weight, all you need to do is exercise.

(c) Every American is patriotic.

(d) You are patriotic only if you are American.

(e) The set of rational numbers is a subset of the real numbers.

(f) A number is prime if it is not even.

(g) Either the Broncos will win the Super Bowl, or they won't play in the Super Bowl.

11. Which of the following statements are equivalent to the implication, "if you win the lottery, then you will be rich," and which are equivalent to the converse of the implication?

(a) Either you win the lottery or else you are not rich.

(b) Either you don't win the lottery or else you are rich.

(c) You will win the lottery and be rich.

(d) You will be rich if you win the lottery.

(e) You will win the lottery if you are rich.

(f) It is necessary for you to win the lottery to be rich.

(g) It is sufficient to win the lottery to be rich.

(h) You will be rich only if you win the lottery.

(i) Unless you win the lottery, you won't be rich.

(j) If you are rich, you must have won the lottery.

(k) If you are not rich, then you did not win the lottery.

(l) You will win the lottery if and only if you are rich.

12. Let $P(x)$ be the predicate, "$3x + 1$ is even."

(a) Is $P(5)$ true or false?

(b) What, if anything, can you conclude about $\exists x P(x)$ from the truth value of $P(5)$?

(c) What, if anything, can you conclude about $\forall x P(x)$ from the truth value of $P(5)$?

13. Let $P(x)$ be the predicate, "$4x + 1$ is even."

(a) Is $P(5)$ true or false?

(b) What, if anything, can you conclude about $\exists x P(x)$ from the truth value of $P(5)$?

(c) What, if anything, can you conclude about $\forall x P(x)$ from the truth value of $P(5)$?

14. For a given predicate $P(x)$, you might believe that the statements $\forall x P(x)$ or $\exists x P(x)$ are either true or false. How would you decide if you were correct in each case? You have four choices: you could give an example of an element n in the domain for which $P(n)$ is true or for which $P(n)$ if false, or you could argue that no matter what n is, $P(n)$ is true or is false.

(a) What would you need to do to prove $\forall x P(x)$ is true?

(b) What would you need to do to prove $\forall x P(x)$ is false?

(c) What would you need to do to prove $\exists x P(x)$ is true?

(d) What would you need to do to prove $\exists x P(x)$ is false?

15. Suppose $P(x, y)$ is some binary predicate defined on a very small domain of discourse: just the integers 1, 2, 3, and 4. For each of the 16 pairs of these numbers, $P(x, y)$ is either true or false, according to the following table (x values are rows, y values are columns).

	1	2	3	4
1	T	F	F	F
2	F	T	T	F
3	T	T	T	T
4	F	F	F	F

For example, $P(1, 3)$ is false, as indicated by the F in the first row, third column.

Use the table to decide whether the following statements are true or false.

(a) $\forall x \exists y P(x, y)$.

(b) $\forall y \exists x P(x, y)$.

(c) $\exists x \forall y P(x, y)$.

(d) $\exists y \forall x P(x, y)$.

16. Translate into symbols. Use $E(x)$ for "x is even" and $O(x)$ for "x is odd."

(a) No number is both even and odd.

(b) One more than any even number is an odd number.

(c) There is prime number that is even.

(d) Between any two numbers there is a third number.

(e) There is no number between a number and one more than that number.

17. Translate into English:

(a) $\forall x(E(x) \to E(x+2))$.

(b) $\forall x \exists y(\sin(x) = y)$.

(c) $\forall y \exists x(\sin(x) = y)$.

(d) $\forall x \forall y(x^3 = y^3 \to x = y)$.

18. Suppose $P(x)$ is some predicate for which the statement $\forall x P(x)$ is true. Is it also the case that $\exists x P(x)$ is true? In other words, is the statement $\forall x P(x) \to \exists x P(x)$ always true? Is the converse always true? Assume the domain of discourse is non-empty.

19. For each of the statements below, give a domain of discourse for which the statement is true, and a domain for which the statement is false.

(a) $\forall x \exists y(y^2 = x)$.

(b) $\forall x \forall y(x < y \to \exists z(x < z < y))$.

(c) $\exists x \forall y \forall z(y < z \to y \leq x \leq z)$.

20. Consider the statement, "For all natural numbers n, if n is prime, then n is solitary." You do not need to know what *solitary* means for this problem, just that it is a property that some numbers have and others do not.

(a) Write the converse and the contrapositive of the statement, saying which is which. Note: the original statement claims that an implication is true for all n, and it is that implication that we are taking the converse and contrapositive of.

(b) Write the negation of the original statement. What would you need to show to prove that the statement is false?

(c) Even though you don't know whether 10 is solitary (in fact, nobody knows this), is the statement "if 10 is prime, then 10 is solitary" true or false? Explain.

(d) It turns out that 8 is solitary. Does this tell you anything about the truth or falsity of the original statement, its converse or its contrapositive? Explain.

(e) Assuming that the original statement is true, what can you say about the relationship between the *set* P of prime numbers and the *set* S of solitary numbers. Explain.

0.3 Sets

The most fundamental objects we will use in our studies (and really in all of math) are *sets*. Much of what follows might be review, but it is very important that you are fluent in the language of set theory. Most of the notation we use below is standard, although some might be a little different than what you have seen before.

For us, a **set** will simply be an unordered collection of objects. Two examples: we could consider the set of all actors who have played *The Doctor* on *Doctor Who*, or the set of natural numbers between 1 and 10 inclusive. In the first case, Tom Baker is an element (or member) of the set, while Idris Elba, among many others, is not an element of the set. Also, the two examples are of different sets. Two sets are equal exactly if they contain the exact same elements. For example, the set containing all of the vowels in the declaration of independence is precisely the same set as the set of vowels in the word "questionably" (namely, all of them); we do not care about order or repetitions, just whether the element is in the set or not.

Notation

We need some notation to make talking about sets easier. Consider,

$$A = \{1, 2, 3\}.$$

This is read, "A is the set containing the elements 1, 2 and 3." We use curly braces "$\{, \ \}$" to enclose elements of a set. Some more notation:

$$a \in \{a, b, c\}.$$

The symbol "$\in$" is read "is in" or "is an element of." Thus the above means that a is an element of the set containing the letters a, b, and c. Note that this is a true statement. It would also be true to say that d is not in that set:

$$d \notin \{a, b, c\}.$$

Be warned: we write "$x \in A$" when we wish to express that one of the elements of the set A is x. For example, consider the set,

$$A = \{1, b, \{x, y, z\}, \emptyset\}.$$

This is a strange set, to be sure. It contains four elements: the number 1, the letter b, the set $\{x, y, z\}$, and the empty set $\emptyset = \{\}$, the set containing no elements. Is x in A? The answer is no. None of the four elements in A are the letter x, so we must conclude that $x \notin A$. Similarly, consider the set $B = \{1, b\}$. Even though the elements of B are elements of A, we cannot say that the *set* B is one of the elements of A. Therefore $B \notin A$. (Soon we will see that B is a *subset* of A, but this is different from being an *element* of A.)

We have described the sets above by listing their elements. Sometimes this is hard to do, especially when there are a lot of elements in the set (perhaps infinitely many). For instance, if we want A to be the set of all even natural numbers, would could write,

$$A = \{0, 2, 4, 6, \ldots\},$$

but this is a little imprecise. A better way would be

$$A = \{x \in \mathbb{N} : \exists n \in \mathbb{N}(x = 2n)\}.$$

Let's look at this carefully. First, there are some new symbols to digest: "$\mathbb{N}$" is the symbol usually used to denote that **natural numbers**, which we will take to be the set $\{0, 1, 2, 3, \ldots\}$. Next, the colon, ":", is read *such that*; it separates the elements that are in the set from the condition that the elements in the set must satisfy. So putting this all together, we would read the set as, "the set of all x in the natural numbers, such that there exists some n in the natural numbers for which x is twice n." In other words, the set of all natural numbers, that are even. Here is another way to write the same set.

$$A = \{x \in \mathbb{N} : x \text{ is even}\}.$$

Note: Sometimes mathematicians use $|$ or $\ni$ for the "such that" symbol instead of the colon. Also, there is a fairly even split between mathematicians about whether 0 is an element of the natural numbers, so be careful there.

This notation is usually called **set builder notation**. It tells us how to *build* a set by telling us precisely the condition elements must meet to gain access (the condition is the logical statement after the ":" symbol). Reading and comprehending sets written in this way takes practice. Here are some more examples:

Example 0.3.1

Describe each of the following sets both in words and by listing out enough elements to see the pattern.

1. $\{x : x + 3 \in \mathbb{N}\}$.
2. $\{x \in \mathbb{N} : x + 3 \in \mathbb{N}\}$.
3. $\{x : x \in \mathbb{N} \vee -x \in \mathbb{N}\}$.
4. $\{x : x \in \mathbb{N} \wedge -x \in \mathbb{N}\}$.

Solution.

1. This is the set of all numbers which are 3 less than a natural number (i.e., that if you add 3 to them, you get a natural number). The set could also be written as $\{-3, -2, -1, 0, 1, 2, \ldots\}$ (note that 0 is a natural number, so -3 is in this set because $-3 + 3 = 0$).

2. This is the set of all natural numbers which are 3 less than a natural number. So here we just have $\{0, 1, 2, 3 \ldots\}$.

3. This is the set of all integers (positive and negative whole numbers, written $\mathbb{Z}$). In other words, $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

4. Here we want all numbers x such that x and $-x$ are natural numbers. There is only one: 0. So we have the set $\{0\}$.

There is also a subtle variation on set builder notation. While the condition is generally given after the "such that", sometimes it is hidden in the first part. Here is an example.

Example 0.3.2

List a few elements in the sets below and describe them in words. The set $\mathbb{Z}$ is the set of **integers**; positive and negative whole numbers.

1. $A = \{x \in \mathbb{Z} : x^2 \in \mathbb{N}\}$

2. $B = \{x^2 : x \in \mathbb{N}\}$

Solution.

1. The set of integers that pass the condition that their square is a natural number. Well, every integer, when you square it, gives you a non-negative integer, so a natural number. Thus $A = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$.

2. Here we are looking for the set of all x^2s where x is a natural number. So this set is simply the set of perfect squares. $B = \{0, 1, 4, 9, 16, \ldots\}$.

   Another way we could have written this set, using more strict set builder notation, would be as $B = \{x \in \mathbb{N} : x = n^2 \text{ for some } n \in \mathbb{N}\}$.

We already have a lot of notation, and there is more yet. Below is a handy chart of symbols. Some of these will be discussed in greater detail as we move forward.

Special sets.

$\emptyset$	The **empty set** is the set which contains no elements.
$\mathcal{U}$	The **universe set** is the set of all elements.
$\mathbb{N}$	The set of natural numbers. That is, $\mathbb{N} = \{0, 1, 2, 3 \ldots\}$.
$\mathbb{Z}$	The set of integers. That is, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$.
$\mathbb{Q}$	The set of rational numbers.
$\mathbb{R}$	The set of real numbers.
$\mathcal{P}(A)$	The **power set** of any set A is the set of all subsets of A.

Set Theory Notation.

$\{,\}$	We use these **braces** to enclose the elements of a set. So $\{1, 2, 3\}$ is the set containing 1, 2, and 3.
$:$	$\{x : x > 2\}$ is the set of all x **such that** x is greater than 2.
$\in$	$2 \in \{1, 2, 3\}$ asserts that 2 is **an element of** the set $\{1, 2, 3\}$.
$\notin$	$4 \notin \{1, 2, 3\}$ because 4 **is not an element of** the set $\{1, 2, 3\}$.
$\subseteq$	$A \subseteq B$ asserts that A **is a subset of** B: every element of A is also an element of B.
$\subset$	$A \subset B$ asserts that A **is a proper subset of** B: every element of A is also an element of B, but $A \neq B$.
$\cap$	$A \cap B$ is the **intersection of** A **and** B: the set containing all elements which are elements of both A and B.
$\cup$	$A \cup B$ is the **union of** A **and** B: is the set containing all elements which are elements of A or B or both.
$\times$	$A \times B$ is the **Cartesian product of** A **and** B: the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$.
$\setminus$	$A \setminus B$ is **set difference between** A **and** B: the set containing all elements of A which are not elements of B.
$\overline{A}$	The **complement of** A is the set of everything which is not an element of A.
$\lvert A\rvert$	The **cardinality (or size) of** A is the number of elements in A.

Investigate!

1. Find the cardinality of each set below.

   (a) $A = \{3, 4, \ldots, 15\}$.

   (b) $B = \{n \in \mathbb{N} : 2 < n \leq 200\}$.

   (c) $C = \{n \leq 100 : n \in \mathbb{N} \wedge \exists m \in \mathbb{N}(n = 2m + 1)\}$.

2. Find two sets A and B for which $|A| = 5$, $|B| = 6$, and $|A \cup B| = 9$. What is $|A \cap B|$?

3. Find sets A and B with $|A| = |B|$ such that $|A \cup B| = 7$ and $|A \cap B| = 3$. What is $|A|$?

4. Let $A = \{1, 2, \ldots, 10\}$. Define $\mathcal{B}_2 = \{B \subseteq A : |B| = 2\}$. Find $|\mathcal{B}_2|$.

5. For any sets A and B, define $AB = \{ab : a \in A \wedge b \in B\}$. If $A = \{1, 2\}$ and $B = \{2, 3, 4\}$, what is $|AB|$? What is $|A \times B|$?

STOP **Attempt the above activity before proceeding** STOP

Relationships Between Sets

We have already said what it means for two sets to be equal: they have exactly the same elements. Thus, for example,

$$\{1, 2, 3\} = \{2, 1, 3\}.$$

(Remember, the order the elements are written down in does not matter.) Also,

$$\{1, 2, 3\} = \{1, 1 + 1, 1 + 1 + 1\} = \{I, II, III\} = \{1, 2, 3, 1 + 2\}$$

since these are all ways to write the set containing the first three positive integers (how we write them doesn't matter, just what they are).

What about the sets $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$? Clearly $A \neq B$, but notice that every element of A is also an element of B. Because of this we say that A is a *subset* of B, or in symbols $A \subset B$ or $A \subseteq B$. Both symbols are read "is a subset of." The difference is that sometimes we want to say that A is either equal to or is a subset of B, in which case we use $\subseteq$. This is analogous to the difference between $<$ and $\leq$.

Example 0.3.3

Let $A = \{1,2,3,4,5,6\}$, $B = \{2,4,6\}$, $C = \{1,2,3\}$ and $D = \{7,8,9\}$. Determine which of the following are true, false, or meaningless.

1. $A \subset B$.
2. $B \subset A$.
3. $B \in C$.
4. $\emptyset \in A$.
5. $\emptyset \subset A$.
6. $A < D$.
7. $3 \in C$.
8. $3 \subset C$.
9. $\{3\} \subset C$.

Solution.

1. False. For example, $1 \in A$ but $1 \notin B$.
2. True. Every element in B is an element in A.
3. False. The elements in C are 1, 2, and 3. The *set* B is not equal to 1, 2, or 3.
4. False. A has exactly 6 elements, and none of them are the empty set.
5. True. Everything in the empty set (nothing) is also an element of A. Notice that the empty set is a subset of every set.
6. Meaningless. A set cannot be less than another set.
7. True. 3 is one of the elements of the set C.
8. Meaningless. 3 is not a set, so it cannot be a subset of another set.
9. True. 3 is the only element of the set $\{3\}$, and is an element of C, so every element in $\{3\}$ is an element of C.

In the example above, B is a subset of A. You might wonder what other sets are subsets of A. If you collect all these subsets of A into a new set, we get a set of sets. We call the set of all subsets of A the **power set** of A, and write it $\mathcal{P}(A)$.

Example 0.3.4

Let $A = \{1,2,3\}$. Find $\mathcal{P}(A)$.

Solution. $\mathcal{P}(A)$ is a set of sets, all of which are subsets of A. So

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Notice that while $2 \in A$, it is wrong to write $2 \in \mathcal{P}(A)$ since none of the elements in $\mathcal{P}(A)$ are numbers! On the other hand, we do have $\{2\} \in \mathcal{P}(A)$ because $\{2\} \subseteq A$.

What does a subset of $\mathcal{P}(A)$ look like? Notice that $\{2\} \nsubseteq \mathcal{P}(A)$ because not everything in $\{2\}$ is in $\mathcal{P}(A)$. But we do have $\{\{2\}\} \subseteq \mathcal{P}(A)$. The only element of $\{\{2\}\}$ is the set $\{2\}$ which is also an element of $\mathcal{P}(A)$. We could take the collection of all subsets of $\mathcal{P}(A)$ and call that $\mathcal{P}(\mathcal{P}(A))$. Or even the power set of that set of sets of sets.

Another way to compare sets is by their *size*. Notice that in the example above, A has 6 elements and B, C, and D all have 3 elements. The size of a set is called the set's **cardinality** . We would write $|A| = 6$, $|B| = 3$, and so on. For sets that have a finite number of elements, the cardinality of the set is simply the number of elements in the set. Note that the cardinality of $\{1, 2, 3, 2, 1\}$ is 3. We do not count repeats (in fact, $\{1, 2, 3, 2, 1\}$ is exactly the same set as $\{1, 2, 3\}$). There are sets with infinite cardinality, such as $\mathbb{N}$, the set of rational numbers (written $\mathbb{Q}$), the set of even natural numbers, and the set of real numbers ($\mathbb{R}$). It is possible to distinguish between different infinite cardinalities, but that is beyond the scope of this text. For us, a set will either be infinite, or finite; if it is finite, the we can determine its cardinality by counting elements.

Example 0.3.5

1. Find the cardinality of $A = \{23, 24, \ldots, 37, 38\}$.
2. Find the cardinality of $B = \{1, \{2, 3, 4\}, \emptyset\}$.
3. If $C = \{1, 2, 3\}$, what is the cardinality of $\mathcal{P}(C)$?

Solution.

1. Since $38 - 23 = 15$, we can conclude that the cardinality of the set is $|A| = 16$ (you need to add one since 23 is included).
2. Here $|B| = 3$. The three elements are the number 1, the set $\{2, 3, 4\}$, and the empty set.
3. We wrote out the elements of the power set $\mathcal{P}(C)$ above, and there are 8 elements (each of which is a set). So $|\mathcal{P}(C)| = 8$. (You might wonder if there is a relationship between $|A|$ and $|\mathcal{P}(A)|$ for all sets A. This is a good question which we will return to in Chapter 1.)

Operations On Sets

Is it possible to add two sets? Not really, however there is something similar. If we want to combine two sets to get the collection of objects that are in either set, then we can take the **union** of the two sets. Symbolically,

$$C = A \cup B,$$

read, "C is the union of A and B," means that the elements of C are exactly the elements which are either an element of A or an element of B (or an element of both). For example, if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, then $A \cup B = \{1, 2, 3, 4\}$.

The other common operation on sets is **intersection**. We write,

$$C = A \cap B$$

and say, "C is the intersection of A and B," when the elements in C are precisely those both in A and in B. So if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, then $A \cap B = \{2, 3\}$.

Often when dealing with sets, we will have some understanding as to what "everything" is. Perhaps we are only concerned with natural numbers. In this case we would say that our **universe** is $\mathbb{N}$. Sometimes we denote this universe by $\mathcal{U}$. Given this context, we might wish to speak of all the elements which are *not* in a particular set. We say B is the **complement** of A, and write,

$$B = \overline{A}$$

when B contains every element not contained in A. So, if our universe is $\{1, 2, \ldots, 9, 10\}$, and $A = \{2, 3, 5, 7\}$, then $\overline{A} = \{1, 4, 6, 8, 9, 10\}$.

Of course we can perform more than one operation at a time. For example, consider

$$A \cap \overline{B}.$$

This is the set of all elements which are both elements of A and not elements of B. What have we done? We've started with A and removed all of the elements which were in B. Another way to write this is the **set difference**:

$$A \cap \overline{B} = A \setminus B.$$

It is important to remember that these operations (union, intersection, complement, and difference) on sets produce other sets. Don't confuse these with the symbols from the previous section (element of and subset of). $A \cap B$ is a set, while $A \subseteq B$ is true or false. This is the same difference as between $3 + 2$ (which is a number) and $3 \leq 2$ (which is false).

Example 0.3.6

Let $A = \{1,2,3,4,5,6\}$, $B = \{2,4,6\}$, $C = \{1,2,3\}$ and $D = \{7,8,9\}$. If the universe is $\mathcal{U} = \{1,2,\ldots,10\}$, find:

1. $A \cup B$.
2. $A \cap B$.
3. $B \cap C$.
4. $A \cap D$.
5. $\overline{B \cup C}$.
6. $A \setminus B$.
7. $(D \cap \overline{C}) \cup \overline{A \cap B}$.
8. $\emptyset \cup C$.
9. $\emptyset \cap C$.

Solution.

1. $A \cup B = \{1,2,3,4,5,6\} = A$ since everything in B is already in A.
2. $A \cap B = \{2,4,6\} = B$ since everything in B is in A.
3. $B \cap C = \{2\}$ as the only element of both B and C is 2.
4. $A \cap D = \emptyset$ since A and D have no common elements.
5. $\overline{B \cup C} = \{5,7,8,9,10\}$. First we find that $B \cup C = \{1,2,3,4,6\}$, then we take everything not in that set.
6. $A \setminus B = \{1,3,5\}$ since the elements 1, 3, and 5 are in A but not in B. This is the same as $A \cap \overline{B}$.
7. $(D \cap \overline{C}) \cup \overline{A \cap B} = \{1,3,5,7,8,9,10\}$. The set contains all elements that are either in D but not in C (i.e., $\{7,8,9\}$), or not in both A and B (i.e., $\{1,3,5,7,8,9,10\}$).
8. $\emptyset \cup C = C$ since nothing is added by the empty set.
9. $\emptyset \cap C = \emptyset$ since nothing can be both in a set and in the empty set.

Having notation like this is useful. We will often want to add or remove elements from sets, and our notation allows us to do so precisely.

Example 0.3.7

If $A = \{1,2,3\}$, then we can describe the set we get by adding the number 4 as $A \cup \{4\}$. If we want to express the set we get by removing the number 2 from A we can do so by writing $A \setminus \{2\}$.

Careful though. If you add an element to the set, you get a new set! So you would have $B = A \cup \{4\}$ and then correctly say that B contains 4, but A does not.

You might notice that the symbols for union and intersection slightly resemble the logic symbols for "or" and "and." This is no accident. What does it mean for x to be an element of $A \cup B$? It means that x is an element of A *or* x is an element of B (or both). That is,

$$x \in A \cup B \qquad \Leftrightarrow \qquad x \in A \vee x \in B.$$

Similarly,

$$x \in A \cap B \qquad \Leftrightarrow \qquad x \in A \wedge x \in B.$$

Also,

$$x \in \overline{A} \qquad \Leftrightarrow \qquad \neg(x \in A).$$

which says x is an element of the complement of A if x is not an element of A.

There is one more way to combine sets which will be useful for us: the **Cartesian product**, $A \times B$. This sounds fancy but is nothing you haven't seen before. When you graph a function in calculus, you graph it in the Cartesian plane. This is the set of all ordered pairs of real numbers (x, y). We can do this for *any* pair of sets, not just the real numbers with themselves.

Put another way, $A \times B = \{(a, b) : a \in A \wedge b \in B\}$. The first coordinate comes from the first set and the second coordinate comes from the second set. Sometimes we will want to take the Cartesian product of a set with itself, and this is fine: $A \times A = \{(a, b) : a, b \in A\}$ (we might also write A^2 for this set). Notice that in $A \times A$, we still want *all* ordered pairs, not just the ones where the first and second coordinate are the same. We can also take products of 3 or more sets, getting ordered triples, or quadruples, and so on.

Example 0.3.8

Let $A = \{1, 2\}$ and $B = \{3, 4, 5\}$. Find $A \times B$ and $A \times A$. How many elements do you expect to be in $B \times B$?

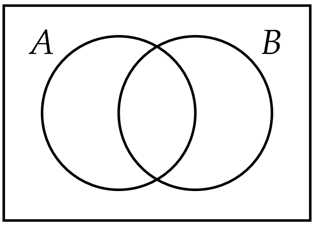
Solution. $A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)\}$.

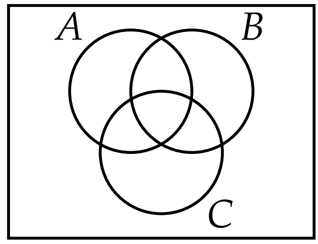
$A \times A = A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

$|B \times B| = 9$. There will be 3 pairs with first coordinate 3, three more with first coordinate 4, and a final three with first coordinate 5.

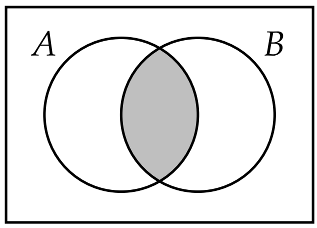
Venn Diagrams

There is a very nice visual tool we can use to represent operations on sets. A **Venn diagram** displays sets as intersecting circles. We can shade the region we are talking about when we carry out an operation. We can also represent cardinality of a particular set by putting the number in the corresponding region.

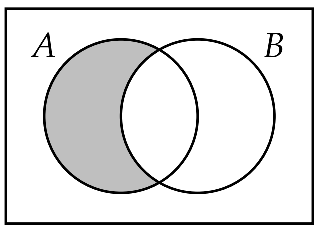




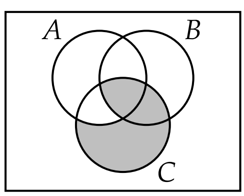
Each circle represents a set. The rectangle containing the circles represents the universe. To represent combinations of these sets, we shade the corresponding region. For example, we could draw $A \cap B$ as:



Here is a representation of $A \cap \overline{B}$, or equivalently $A \setminus B$:



A more complicated example is $(B \cap C) \cup (C \cap \overline{A})$, as seen below.



Notice that the shaded regions above could also be arrived at in another way. We could have started with all of C, then excluded the region where C and A overlap outside of B. That region is $(A \cap C) \cap \overline{B}$. So the above Venn diagram also represents $C \cap \overline{\left((A \cap C) \cap \overline{B}\right)}$. So using just the picture, we have determined that

$$(B \cap C) \cup (C \cap \overline{A}) = C \cap \overline{\left((A \cap C) \cap \overline{B}\right)}.$$

Exercises

1. Let $A = \{1, 4, 9\}$ and $B = \{1, 3, 6, 10\}$. Find each of the following sets.
   (a) $A \cup B$.
   (b) $A \cap B$.
   (c) $A \setminus B$.
   (d) $B \setminus A$.
2. Find the least element of each of the following sets, if there is one.
   (a) $\{n \in \mathbb{N} : n^2 - 3 \geq 2\}$.
   (b) $\{n \in \mathbb{N} : n^2 - 5 \in \mathbb{N}\}$.
   (c) $\{n^2 + 1 : n \in \mathbb{N}\}$.
   (d) $\{n \in \mathbb{N} : n = k^2 + 1 \text{ for some } k \in \mathbb{N}\}$.
3. Find the following cardinalities:
   (a) $|A|$ when $A = \{4, 5, 6, \ldots, 37\}$.
   (b) $|A|$ when $A = \{x \in \mathbb{Z} : -2 \leq x \leq 100\}$.
   (c) $|A \cap B|$ when $A = \{x \in \mathbb{N} : x \leq 20\}$ and $B = \{x \in \mathbb{N} : x \text{ is prime}\}$.
4. Find a set of largest possible size that is a subset of both $\{1, 2, 3, 4, 5\}$ and $\{2, 4, 6, 8, 10\}$.
5. Find a set of smallest possible size that has both $\{1, 2, 3, 4, 5\}$ and $\{2, 4, 6, 8, 10\}$ as subsets.
6. Let $A = \{n \in \mathbb{N} : 20 \leq n < 50\}$ and $B = \{n \in \mathbb{N} : 10 < n \leq 30\}$. Suppose C is a set such that $C \subseteq A$ and $C \subseteq B$. What is the largest possible cardinality of C?
7. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 3, 4\}$. How many sets C have the property that $C \subseteq A$ and $B \subseteq C$.
8. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$, and $C = \{2, 3, 5\}$.
   (a) Find $A \cap B$.
   (b) Find $A \cup B$.
   (c) Find $A \setminus B$.
   (d) Find $A \cap \overline{(B \cup C)}$.
9. Let $A = \{x \in \mathbb{N} : 4 \leq x < 12\}$ and $B = \{x \in \mathbb{N} : x \text{ is even}\}$.
   (a) Find $A \cap B$.
   (b) Find $A \setminus B$.

10. Let $A = \{x \in \mathbb{N} : 3 \leq x \leq 13\}$, $B = \{x \in \mathbb{N} : x \text{ is even}\}$, and $C = \{x \in \mathbb{N} : x \text{ is odd}\}$.

(a) Find $A \cap B$.

(b) Find $A \cup B$.

(c) Find $B \cap C$.

(d) Find $B \cup C$.

11. Find an example of sets A and B such that $A \cap B = \{3, 5\}$ and $A \cup B = \{2, 3, 5, 7, 8\}$.

12. Find an example of sets A and B such that $A \subseteq B$ and $A \in B$.

13. Recall $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (the integers). Let $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ be the positive integers. Let $2\mathbb{Z}$ be the even integers, $3\mathbb{Z}$ be the multiples of 3, and so on.

(a) Is $\mathbb{Z}^+ \subseteq 2\mathbb{Z}$? Explain.

(b) Is $2\mathbb{Z} \subseteq \mathbb{Z}^+$? Explain.

(c) Find $2\mathbb{Z} \cap 3\mathbb{Z}$. Describe the set in words, and using set notation.

(d) Express $\{x \in \mathbb{Z} : \exists y \in \mathbb{Z}(x = 2y \vee x = 3y)\}$ as a union or intersection of two sets already described in this problem.

14. Let A_2 be the set of all multiples of 2 except for 2. Let A_3 be the set of all multiples of 3 except for 3. And so on, so that A_n is the set of all multiples of n except for n, for any $n \geq 2$. Describe (in words) the set $\overline{A_2 \cup A_3 \cup A_4 \cup \cdots}$.

15. Draw a Venn diagram to represent each of the following:

(a) $A \cup \overline{B}$

(b) $\overline{(A \cup B)}$

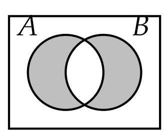
(c) $A \cap (B \cup C)$

(d) $(A \cap B) \cup C$

(e) $\overline{A} \cap B \cap \overline{C}$

(f) $(A \cup B) \setminus C$

16. Describe a set in terms of A and B (using set notation) which has the following Venn diagram:



17. Let $A = \{a, b, c, d\}$. Find $\mathcal{P}(A)$.

18. Let $A = \{1, 2, \ldots, 10\}$. How many subsets of A contain exactly one element (i.e., how many singleton subsets are there)? singleton set

How many doubleton subsets (containing exactly two elements) are there? doubleton set

19. Let $A = \{1, 2, 3, 4, 5, 6\}$. Find all sets $B \in \mathcal{P}(A)$ which have the property $\{2, 3, 5\} \subseteq B$.

20. Find an example of sets A and B such that $|A| = 4$, $|B| = 5$, and $|A \cup B| = 9$.

21. Find an example of sets A and B such that $|A| = 3$, $|B| = 4$, and $|A \cup B| = 5$.

22. Are there sets A and B such that $|A| = |B|$, $|A \cup B| = 10$, and $|A \cap B| = 5$? Explain.

23. Let $A = \{2, 4, 6, 8\}$. Suppose B is a set with $|B| = 5$.

(a) What are the smallest and largest possible values of $|A \cup B|$? Explain.

(b) What are the smallest and largest possible values of $|A \cap B|$? Explain.

(c) What are the smallest and largest possible values of $|A \times B|$? Explain.

24. Let $X = \{n \in \mathbb{N} : 10 \leq n < 20\}$. Find examples of sets with the properties below and very briefly explain why your examples work.

(a) A set $A \subseteq \mathbb{N}$ with $|A| = 10$ such that $X \setminus A = \{10, 12, 14\}$.

(b) A set $B \in \mathcal{P}(X)$ with $|B| = 5$.

(c) A set $C \subseteq \mathcal{P}(X)$ with $|C| = 5$.

(d) A set $D \subseteq X \times X$ with $|D| = 5$

(e) A set $E \subseteq X$ such that $|E| \in E$.

25. Let A, B and C be sets.

(a) Suppose that $A \subseteq B$ and $B \subseteq C$. Does this mean that $A \subseteq C$? Prove your answer. Hint: to prove that $A \subseteq C$ you must prove the implication, "for all x, if $x \in A$ then $x \in C$."

(b) Suppose that $A \in B$ and $B \in C$. Does this mean that $A \in C$? Give an example to prove that this does NOT always happen (and explain why your example works). You should be able to give an example where $|A| = |B| = |C| = 2$.

26. In a regular deck of playing cards there are 26 red cards and 12 face cards. Explain, using sets and what you have learned about cardinalities, why there are only 32 cards which are either red or a face card.

27. Find an example of a set A with $|A| = 3$ which contains only other sets and has the following property: for all sets $B \in A$, we also have $B \subseteq A$. Explain why your example works. (FYI: sets that have this property are called **transitive**.)

28. Consider the sets A and B, where $A = \{3, |B|\}$ and $B = \{1, |A|, |B|\}$. What are the sets?

29. Explain why there is no set A which satisfies $A = \{2, |A|\}$.

30. Find all sets A, B, and C which satisfy the following.

$$A = \{1, |B|, |C|\}$$
$$B = \{2, |A|, |C|\}$$
$$C = \{1, 2, |A|, |B|\}.$$

0.4 Functions

A **function** is a rule that assigns each input exactly one output. We call the output the **image** of the input. The set of all inputs for a function is called the **domain**. The set of all allowable outputs is called the **codomain**. We would write $f : X \to Y$ to describe a function with name f, domain X and codomain Y. This does not tell us *which* function f is though. To define the function, we must describe the rule. This is often done by giving a formula to compute the output for any input (although this is certainly not the only way to describe the rule).

For example, consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2 + 3$. Here the domain and codomain are the same set (the natural numbers). The rule is: take your input, multiply it by itself and add 3. This works because we can apply this rule to every natural number (every element of the domain) and the result is always a natural number (an element of the codomain). Notice though that not every natural number is actually an output (there is no way to get 0, 1, 2, 5, etc.). The set of natural numbers that *are* outputs is called the **range** of the function (in this case, the range is $\{3, 4, 7, 12, 19, 28, \ldots\}$, all the natural numbers that are 3 more than a perfect square).

The key thing that makes a rule a *function* is that there is *exactly one* output for each input. That is, it is important that the rule be a good rule. What output do we assign to the input 7? There can only be one answer for any particular function.

Example 0.4.1

The following are all examples of functions:

1. $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 3n$. The domain and codomain are both the set of integers. However, the range is only the set of integer multiples of 3.
2. $g : \{1, 2, 3\} \to \{a, b, c\}$ defined by $g(1) = c$, $g(2) = a$ and $g(3) = a$. The domain is the set $\{1, 2, 3\}$, the codomain is the set $\{a, b, c\}$ and the range is the set $\{a, c\}$. Note that $g(2)$ and $g(3)$ are the same element of the codomain. This is okay since each element in the domain still has only one output.
3. $h : \{1, 2, 3, 4\} \to \mathbb{N}$ defined by the table:

x	1	2	3	4
$h(x)$	3	6	9	12

Here the domain is the finite set $\{1, 2, 3, 4\}$ and to codomain is the set of natural numbers, $\mathbb{N}$. At first you might think this function is the same as f defined above. It is absolutely not. Even though the rule is the same, the domain and codomain are different, so these are two different functions.

Example 0.4.2

Just because you can describe a rule in the same way you would write a function, does not mean that the rule is a function. The following are NOT functions.

1. $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = \frac{n}{2}$. The reason this is not a function is because not every input has an output. Where does f send 3? The rule says that $f(3) = \frac{3}{2}$, but $\frac{3}{2}$ is not an element of the codomain.

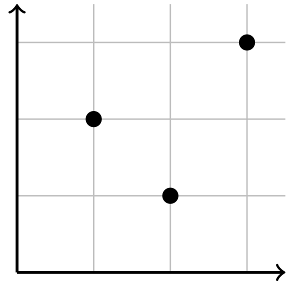
2. Consider the rule that matches each person to their phone number. If you think of the set of people as the domain and the set of phone numbers as the codomain, then this is not a function, since some people have two phone numbers. Switching the domain and codomain sets doesn't help either, since some phone numbers belong to multiple people (assuming some households still have landlines when you are reading this).

Describing Functions

It is worth making a distinction between a function and its description. The function is the abstract mathematical object that in some way exists whether or not anyone ever talks about it. But when we *do* want to talk about the function, we need a way to describe it. A particular function can be described in multiple ways.

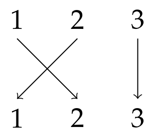
Some calculus textbooks talk about the *Rule of Four*, that every function can be described in four ways: algebraically (a formula), numerically (a table), graphically, or in words. In discrete math, we can still use any of these to describe functions, but we can also be more specific since we are primarily concerned with functions that have $\mathbb{N}$ or a finite subset of $\mathbb{N}$ as their domain.

Describing a function graphically usually means drawing the graph of the function: plotting the points on the plane. We can do this, and might get a graph like the following for a function $f : \{1, 2, 3\} \to \{1, 2, 3\}$.



It would be absolutely WRONG to connect the dots or try to fit them to some curve. There are only three elements in the domain. A curve would mean that the domain contains an entire interval of real numbers.

Here is another way to represent that same function:



This shows that the function f sends 1 to 2, 2 to 1 and 3 to 3: just follow the arrows.

The arrow diagram used to define the function above can be very helpful in visualizing functions. We will often be working with functions with *finite* domains, so this kind of picture is often more useful than a traditional graph of a function.

Note that for finite domains, finding an algebraic formula that gives the output for any input is often impossible. Of course we could use a piecewise defined function, like

$$f(x) = \begin{cases} x+1 & \text{if } x = 1 \\ x-1 & \text{if } x = 2 \\ x & \text{if } x = 3 \end{cases}.$$

This describes exactly the same function as above, but we can all agree is a ridiculous way of doing so.

Since we will so often use functions with small domains and codomains, let's adopt some notation to describe them. All we need is some clear way of denoting the image of each element in the domain. In fact, writing a table of values would work perfectly:

x	0	1	2	3	4
$f(x)$	3	3	2	4	1

We simplify this further by writing this as a "matrix" with each input directly over its output:

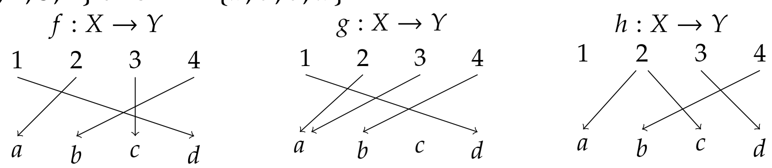
$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 3 & 2 & 4 & 1 \end{pmatrix}.$$

Note this is just notation and not the same sort of matrix you would find in a linear algebra class (it does not make sense to do operations with these matrices, or row reduce them, for example).

One advantage of the two-line notation over the arrow diagrams is that it is harder to accidentally define a rule that is not a function using two-line notation.

Example 0.4.3

Which of the following diagrams represent a function? Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$.



Solution. f is a function. So is g. There is no problem with an element of the codomain not being the image of any input, and there is no problem with a from the codomain being the image of both 2 and 3 from the domain. We could use our two-line notation to write these as

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ d & a & c & b \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ d & a & a & b \end{pmatrix}.$$

However, h is NOT a function. In fact, it fails for two reasons. First, the element 1 from the domain has not been mapped to any element from the codomain. Second, the element 2 from the domain has been mapped to more than one element from the codomain (a and c). Note that either one of these problems is enough to make a rule not a function. In general, neither of the following mappings are functions:

It might also be helpful to think about how you would write the two-line notation for h. We would have something like:

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & a, c? & d & b \end{pmatrix}.$$

There is nothing under 1 (bad) and we needed to put more than one thing under 2 (very bad). With a rule that is actually a function, the two-line notation will always "work".

We will also be interested in functions with domain $\mathbb{N}$. Here two-line notation is no good, but describing the function algebraically is often possible. Even tables are a little awkward, since they do not describe the function completely. For example, consider the function $f : \mathbb{N} \to \mathbb{N}$ given by the table below.

x	0	1	2	3	4	5	...
$f(x)$	0	1	4	9	16	25	...

Have I given you enough entries for you to be able to determine $f(6)$? You might guess that $f(6) = 36$, but there is no way for you to *know* this for sure. Maybe I am being a jerk and intended $f(6) = 42$. In fact, for every natural number n, there is a function that agrees with the table above, but for which $f(6) = n$.

Okay, suppose I really did mean for $f(6) = 36$, and in fact, for the rule that you think is governing the function to actually be the rule. Then I should say what that rule is. $f(n) = n^2$. Now there is no confusion possible.

Giving an explicit formula that calculates the image of any element in the domain is a great way to describe a function. We will say that these explicit rules are **closed formulas** for the function.

There is another very useful way to describe functions whose domain is $\mathbb{N}$, that rely specifically on the structure of the natural numbers. We can define a function *recursively*!

Example 0.4.4

Consider the function $f : \mathbb{N} \to \mathbb{N}$ given by $f(0) = 0$ and $f(n+1) = f(n) + 2n + 1$. Find $f(6)$.

Solution. The rule says that $f(6) = f(5) + 11$ (we are using $6 = n + 1$ so $n = 5$). We don't know what $f(5)$ is though. Well, we know that $f(5) = f(4) + 9$. So we need to compute $f(4)$, which will require knowing $f(3)$, which will require $f(2)$,. . . will it ever end?

Yes! In fact, this process will always end because we have $\mathbb{N}$ as our domain, so there is a least element. And we gave the value of $f(0)$ explicitly, so we are good. In fact, we might decide to work up to $f(6)$ instead of working down from $f(6)$:

$$\begin{aligned}
f(1) = & f(0) + 1 = & 0 + 1 = 1 \\
f(2) = & f(1) + 3 = & 1 + 3 = 4 \\
f(3) = & f(2) + 5 = & 4 + 5 = 9 \\
f(4) = & f(3) + 7 = & 9 + 7 = 16 \\
f(5) = & f(4) + 9 = & 16 + 9 = 25
\end{aligned}$$

$$f(6) = f(5) + 11 = \qquad 25 + 11 = 36$$

It looks that this recursively defined function is the same as the explicitly defined function $f(n) = n^2$. Is it? Later we will prove that it is.

Recursively defined functions are often easier to create from a "real world" problem, because they describe how the values of the functions are changing. However, this comes with a price. It is harder to calculate the image of a single input, since you need to know the images of other (previous) elements in the domain.

Recursively Defined Functions.

For a function $f : \mathbb{N} \to \mathbb{N}$, a **recursive definition** consists of an **initial condition** together with a **recurrence relation**. The initial condition is the explicitly given value of $f(0)$. The recurrence relation is a formula for $f(n+1)$ in terms for $f(n)$ (and possibly n itself).

Example 0.4.5

Give recursive definitions for the functions described below.

1. $f : \mathbb{N} \to \mathbb{N}$ gives the number of snails in your terrarium n years after you built it, assuming you started with 3 snails and the number of snails doubles each year.

2. $g : \mathbb{N} \to \mathbb{N}$ gives the number of push-ups you do n days after you started your push-ups challenge, assuming you could do 7 push-ups on day 0 and you can do 2 more push-ups each day.

3. $h : \mathbb{N} \to \mathbb{N}$ defined by $h(n) = n!$. Recall that $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$ is the product of all numbers from 1 through n. We also define $0! = 1$.

Solution.

1. The initial condition is $f(0) = 3$. To get $f(n+1)$ we would double the number of snails in the terrarium the previous year, which is given by $f(n)$. Thus $f(n+1) = 2f(n)$. The full recursive definition contains both of these, and would be written,
$$f(0) = 3; \ f(n+1) = 2f(n).$$

2. We are told that on day 0 you can do 7 push-ups, so $g(0) = 7$. The number of push-ups you can do on day $n + 1$ is 2 more than the number you can do on day n, which is given by $g(n)$. Thus
$$g(0) = 7;\ g(n+1) = g(n) + 2.$$

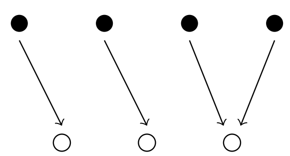
3. Here $h(0) = 1$. To get the recurrence relation, think about how you can get $h(n+1) = (n+1)!$ from $h(n) = n!$. If you write out both of these as products, you see that $(n+1)!$ is just like $n!$ except you have one more term in the product, an extra $n + 1$. So we have,
$$h(0) = 1;\ h(n+1) = (n+1) \cdot h(n).$$

Surjections, Injections, and Bijections

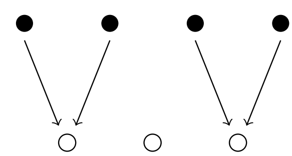
We now turn to investigating special properties functions might or might not possess.

In the examples above, you may have noticed that sometimes there are elements of the codomain which are not in the range. When this sort of the thing *does not* happen, (that is, when everything in the codomain is in the range) we say the function is **onto** or that the function maps the domain *onto* the codomain. This terminology should make sense: the function puts the domain (entirely) on top of the codomain. The fancy math term for an onto function is a **surjection**, and we say that an onto function is a **surjective** function.

In pictures:



Surjective



Not surjective

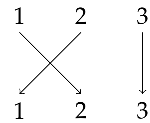
Example 0.4.6

Which functions are surjective (i.e., onto)?

1. $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 3n$.

2. $g : \{1, 2, 3\} \to \{a, b, c\}$ defined by $g = \begin{pmatrix} 1 & 2 & 3 \\ c & a & a \end{pmatrix}$.

3. $h : \{1, 2, 3\} \to \{1, 2, 3\}$ defined as follows:



Solution.

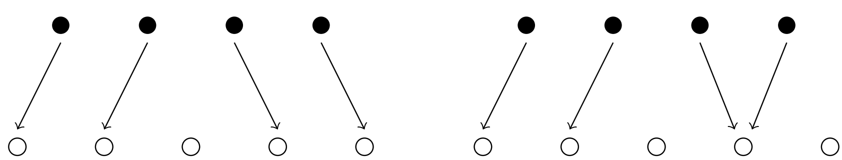
1. f is not surjective. There are elements in the codomain which are not in the range. For example, no $n \in \mathbb{Z}$ gets mapped to the number 1 (the rule would say that $\frac{1}{3}$ would be sent to 1, but $\frac{1}{3}$ is not in the domain). In fact, the range of the function is $3\mathbb{Z}$ (the integer multiples of 3), which is not equal to $\mathbb{Z}$.

2. g is not surjective. There is no $x \in \{1, 2, 3\}$ (the domain) for which $g(x) = b$, so b, which is in the codomain, is not in the range. Notice that there is an element from the codomain "missing" from the bottom row of the matrix.

3. h is surjective. Every element of the codomain is also in the range. Nothing in the codomain is missed.

To be a function, a rule cannot assign a single element of the domain to two or more different elements of the codomain. However, we have seen that the reverse *is* permissible: a function might assign the same element of the codomain to two or more different elements of the domain. When this *does not* occur (that is, when each element of the codomain is the image of at most one element of the domain) then we say the function is **one-to-one**. Again, this terminology makes sense: we are sending at most one element from the domain to one element from the codomain. One input to one output. The fancy math term for a one-to-one function is an **injection**. We call one-to-one functions **injective** functions.

In pictures:



Injective　　　　Not injective

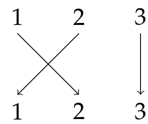
Example 0.4.7

Which functions are injective (i.e., one-to-one)?

1. $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 3n$.

2. $g : \{1, 2, 3\} \to \{a, b, c\}$ defined by $g = \begin{pmatrix} 1 & 2 & 3 \\ c & a & a \end{pmatrix}$.

3. $h : \{1, 2, 3\} \to \{1, 2, 3\}$ defined as follows:



Solution.

1. f is injective. Each element in the codomain is assigned to at *most* one element from the domain. If x is a multiple of three, then only $x/3$ is mapped to x. If x is not a multiple of 3, then there is no input corresponding to the output x.

2. g is not injective. Both inputs 2 and 3 are assigned the output a. Notice that there is an element from the codomain that appears more than once on the bottom row of the matrix.

3. h is injective. Each output is only an output once.

Be careful: "surjective" and "injective" are NOT opposites. You can see in the two examples above that there are functions which are surjective but not injective, injective but not surjective, both, or neither. In the case when a function is both one-to-one and onto (an injection and surjection), we say the function is a **bijection**, or that the function is a **bijective** function.

To illustrate the contrast between these two properties, consider a more formal definition of each, side by side.

Injective vs Surjective.

A function is **injective** provided every element of the codomain is the image of *at most* one element from the domain.

A function is **surjective** provided every element of the codomain is the image of *at least* one element from the domain.

Notice both properties are determined by what happens to elements of the codomain: they could be repeated as images or they could be "missed" (not be images). Injective functions do not have repeats but might or might not miss elements. Surjective functions do not miss elements, but might or might not have repeats. The bijective functions are those that do not have repeats and do not miss elements.

Image and Inverse Image

When discussing functions, we have notation for talking about an element of the domain (say x) and its corresponding element in the codomain (we write $f(x)$, which *is* the image of x). Sometimes we will want to talk about all the elements that are images of some subset of the domain. It would also be nice to start with some element of the codomain (say y) and talk about which element or elements (if any) from the domain it is the image of. We could write "those x in the domain such that $f(x) = y$," but this is a lot of writing. Here is some notation to make our lives easier.

To address the first situation, what we are after is a way to describe the *set* of images of elements in some subset of the domain. Suppose $f : X \to Y$ is a function and that $A \subseteq X$ is some subset of the domain (possibly all of it). We will use the notation $f(A)$ to denote the **image of** A **under** f, namely the set of elements in Y that are the image of elements from A. That is, $f(A) = \{f(a) \in Y : a \in A\}$.

We can do this in the other direction as well. We might ask which elements of the domain get mapped to a particular set in the codomain. Let $f : X \to Y$ be a function and suppose $B \subseteq Y$ is a subset of the codomain. Then we will write $f^{-1}(B)$ for the **inverse image of** B **under** f, namely the set of elements in X whose image are elements in B. In other words, $f^{-1}(B) = \{x \in X : f(x) \in B\}$.

Often we are interested in the element(s) whose image is a particular element y of in the codomain. The notation above works: $f^{-1}(\{y\})$ is the set of all elements in the domain that f sends to y. It makes sense to think of this as a set: there might not be anything sent to y (if y is not in the range), in which case $f^{-1}(\{y\}) = \emptyset$. Or f might send multiple elements to y (if f is not injective). As a notational convenience, we usually drop the set braces around the y and write $f^{-1}(y)$ instead for this set.

WARNING: $f^{-1}(y)$ is not an inverse function! Inverse functions only exist for bijections, but $f^{-1}(y)$ is defined for any function f. The point: $f^{-1}(y)$ is a *set*, not an *element* of the domain. This is just sloppy notation for $f^{-1}(\{y\})$. To help make this distinction, we would call $f^{-1}(y)$ the **complete inverse image of** y **under** f. It is not the image of y under f^{-1} (since the function f^{-1} might not exist).

Example 0.4.8

Consider the function $f : \{1,2,3,4,5,6\} \to \{a,b,c,d\}$ given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & a & b & b & b & c \end{pmatrix}.$$

Find $f(\{1,2,3\})$, $f^{-1}(\{a,b\})$, and $f^{-1}(d)$.

Solution. $f(\{1,2,3\}) = \{a,b\}$ since a and b are the elements in the codomain to which f sends 1 and 2.

$f^{-1}(\{a,b\}) = \{1,2,3,4,5\}$ since these are exactly the elements that f sends to a and b.

$f^{-1}(d) = \emptyset$ since d is not in the range of f.

Example 0.4.9

Consider the function $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(n) = n^2 + 1$. Find $g(1)$ and $g(\{1\})$. Then find $g^{-1}(1)$, $g^{-1}(2)$, and $g^{-1}(3)$.

Solution. Note that $g(1) \neq g(\{1\})$. The first is an element: $g(1) = 2$. The second is a set: $g(\{1\}) = \{2\}$.

To find $g^{-1}(1)$, we need to find all integers n such that $n^2 + 1 = 1$. Clearly only 0 works, so $g^{-1}(1) = \{0\}$ (note that even though there is only one element, we still write it as a set with one element in it).

To find $g^{-1}(2)$, we need to find all n such that $n^2 + 1 = 2$. We see $g^{-1}(2) = \{-1,1\}$.

Finally, if $n^2 + 1 = 3$, then we are looking for an n such that $n^2 = 2$. There are no such integers so $g^{-1}(3) = \emptyset$.

Since $f^{-1}(y)$ is a set, it makes sense to ask for $\left|f^{-1}(y)\right|$, the number of elements in the domain which map to y.

Example 0.4.10

Find a function $f : \{1,2,3,4,5\} \to \mathbb{N}$ such that $\left|f^{-1}(7)\right| = 5$.

Solution. There is only one such function. We need five elements of the domain to map to the number $7 \in \mathbb{N}$. Since there are only five elements in the domain, all of them must map to 7. So

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 7 & 7 & 7 & 7 & 7 \end{pmatrix}.$$

Function Definitions.

Here is a summary of all the main concepts and definitions we use when working with functions.

- A **function** is a rule that assigns each element of a set, called the **domain**, to exactly one element of a second set, called the **codomain**.
- Notation: $f : X \to Y$ is our way of saying that the function is called f, the domain is the set X, and the codomain is the set Y.
- To specify the rule for a function with small domain, use **two-line notation** by writing a matrix with each output directly below its corresponding input, as in:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 1 \end{pmatrix}.$$

- $f(x) = y$ means the element x of the domain (input) is assigned to the element y of the codomain. We say y is an output. Alternatively, we call y the **image of** x **under** f.
- The **range** is a subset of the codomain. It is the set of all elements which are assigned to at least one element of the domain by the function. That is, the range is the set of all outputs.
- A function is **injective** (an **injection** or **one-to-one**) if every element of the codomain is the image of **at most** one element from the domain.
- A function is **surjective** (a **surjection** or **onto**) if every element of the codomain is the image of **at least** one element from the domain.
- A **bijection** is a function which is both an injection and surjection. In other words, if every element of the codomain is the image of **exactly one** element from the domain.
- The **image** of an element x in the domain is the element y in the codomain that x is mapped to. That is, the image of x under f is $f(x)$.
- The **complete inverse image** of an element y in the codomain, written $f^{-1}(y)$, is the *set* of all elements in the domain which are assigned to y by the function.
- The **image** of a subset A of the domain is the set $f(A) = \{f(a) \in Y : a \in A\}$.
- The **inverse image** of a subset B of the codomain is the set $f^{-1}(B) = \{x \in X : f(x) \in B\}$.

Exercises

1. Consider the function $f : \{1,2,3,4\} \to \{1,2,3,4\}$ given by

$$f(n) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 4 \end{pmatrix}.$$

(a) Find $f(1)$.

(b) Find an element n in the domain such that $f(n) = 1$.

(c) Find an element n of the domain such that $f(n) = n$.

(d) Find an element of the codomain that is not in the range.

2. The following functions all have $\{1,2,3,4,5\}$ as both their domain and codomain. For each, determine whether it is (only) injective, (only) surjective, bijective, or neither injective nor surjective.

(a) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}$.

(b) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$.

(c) $f(x) = 6 - x$.

(d) $f(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ (x+1)/2 & \text{if } x \text{ is odd} \end{cases}$.

3. The following functions all have domain $\{1,2,3,4,5\}$ and codomain $\{1,2,3\}$. For each, determine whether it is (only) injective, (only) surjective, bijective, or neither injective nor surjective.

(a) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 1 & 2 & 1 \end{pmatrix}$.

(b) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 1 & 2 \end{pmatrix}$.

(c) $f(x) = \begin{cases} x & \text{if } x \leq 3 \\ x-3 & \text{if } x > 3 \end{cases}$.

4. The following functions all have domain $\{1,2,3,4\}$ and codomain $\{1,2,3,4,5\}$. For each, determine whether it is (only) injective, (only) surjective, bijective, or neither injective nor surjective.

(a) $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 5 & 4 \end{pmatrix}$.

(b) $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 2 \end{pmatrix}$.

(c) $f(x)$ gives the number of letters in the English word for the number x. For example, $f(1) = 3$ since "one" contains three letters.

5. Write out all functions $f : \{1, 2, 3\} \to \{a, b\}$ (using two-line notation).
How many functions are there?
How many are injective?
How many are surjective?
How many are bijective?

6. Write out all functions $f : \{1, 2\} \to \{a, b, c\}$ (in two-line notation).
How many functions are there?
How many are injective?
How many are surjective?
How many are bijective?

7. Consider the function $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4\}$ given by the table below:

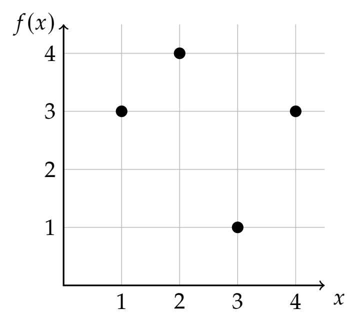
x	1	2	3	4	5
$f(x)$	3	2	4	1	2

(a) Is f injective? Explain.

(b) Is f surjective? Explain.

(c) Write the function using two-line notation.

8. Consider the function $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ given by the graph below.



(a) Is f injective? Explain.

(b) Is f surjective? Explain.

(c) Write the function using two-line notation.

9. Consider the function $f : \mathbb{N} \to \mathbb{N}$ given *recursively* by $f(0) = 1$ and $f(n+1) = 2 \cdot f(n)$. Find $f(10)$.

10. Suppose $f : \mathbb{N} \to \mathbb{N}$ satisfies the recurrence $f(n+1) = f(n) + 3$. Note that this is not enough information to define the function, since we don't have an initial condition. For each of the initial conditions below, find the value of $f(5)$.

(a) $f(0) = 0$.

(b) $f(0) = 1$.

(c) $f(0) = 2$.

(d) $f(0) = 100$.

11. Suppose $f : \mathbb{N} \to \mathbb{N}$ satisfies the recurrence relation

$$f(n+1) = \begin{cases} \frac{f(n)}{2} & \text{if } f(n) \text{ is even} \\ 3f(n)+1 & \text{if } f(n) \text{ is odd} \end{cases}.$$

Note that with the initial condition $f(0) = 1$, the values of the function are: $f(1) = 4$, $f(2) = 2$, $f(3) = 1$, $f(4) = 4$, and so on, the images cycling through those three numbers. Thus f is NOT injective (and also certainly not surjective). Might it be under other initial conditions?[3]

(a) If f satisfies the initial condition $f(0) = 5$, is f injective? Explain why or give a specific example of two elements from the domain with the same image.

(b) If f satisfies the initial condition $f(0) = 3$, is f injective? Explain why or give a specific example of two elements from the domain with the same image.

(c) If f satisfies the initial condition $f(0) = 27$, then it turns out that $f(105) = 10$ and no two numbers less than 105 have the same image. Could f be injective? Explain.

(d) Prove that no matter what initial condition you choose, the function cannot be surjective.

12. For each function given below, determine whether or not the function is injective and whether or not the function is surjective.

(a) $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n + 4$.

(b) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = n + 4$.

(c) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 5n - 8$.

[3]It turns out this is a *really* hard question to answer in general. The *Collatz conjecture* is that no matter what the initial condition is, the function will eventually produce 1 as an output. This is an open problem in mathematics: nobody knows the answer.

(d) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$

13. Let $A = \{1, 2, 3, \ldots, 10\}$. Consider the function $f : \mathcal{P}(A) \to \mathbb{N}$ given by $f(B) = |B|$. That is, f takes a subset of A as an input and outputs the cardinality of that set.

(a) Is f injective? Prove your answer.

(b) Is f surjective? Prove your answer.

(c) Find $f^{-1}(1)$.

(d) Find $f^{-1}(0)$.

(e) Find $f^{-1}(12)$.

14. Let $X = \{n \in \mathbb{N} : 0 \le n \le 999\}$ be the set of all numbers with three or fewer digits. Define the function $f : X \to \mathbb{N}$ by $f(abc) = a + b + c$, where a, b, and c are the digits of the number in X (write numbers less than 100 with leading 0's to make them three digits). For example, $f(253) = 2 + 5 + 3 = 10$.

(a) Let $A = \{n \in X : 113 \le n \le 122\}$. Find $f(A)$.

(b) Find $f^{-1}(\{1, 2\})$

(c) Find $f^{-1}(3)$.

(d) Find $f^{-1}(28)$.

(e) Is f injective? Explain.

(f) Is f surjective? Explain.

15. Consider the set $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, the set of all ordered pairs (a, b) where a and b are natural numbers. Consider a function $f : \mathbb{N}^2 \to \mathbb{N}$ given by $f((a, b)) = a + b$.

(a) Let $A = \{(a, b) \in \mathbb{N}^2 : a, b \le 10\}$. Find $f(A)$.

(b) Find $f^{-1}(3)$ and $f^{-1}(\{0, 1, 2, 3\})$.

(c) Give geometric descriptions of $f^{-1}(n)$ and $f^{-1}(\{0, 1, \ldots, n\})$ for any $n \ge 1$.

(d) Find $\left|f^{-1}(8)\right|$ and $\left|f^{-1}(\{0, 1, \ldots, 8\})\right|$.

16. Let $f : X \to Y$ be some function. Suppose $3 \in Y$. What can you say about $f^{-1}(3)$ if you know,

(a) f is injective? Explain.

(b) f is surjective? Explain.

(c) f is bijective? Explain.

17. Find a set X and a function $f : X \to \mathbb{N}$ so that $f^{-1}(0) \cup f^{-1}(1) = X$.

18. What can you deduce about the sets X and Y if you know,

(a) there is an injective function $f : X \to Y$? Explain.

(b) there is a surjective function $f : X \to Y$? Explain.

(c) there is a bijective function $f : X \to Y$? Explain.

19. Suppose $f : X \to Y$ is a function. Which of the following are possible? Explain.

(a) f is injective but not surjective.

(b) f is surjective but not injective.

(c) $|X| = |Y|$ and f is injective but not surjective.

(d) $|X| = |Y|$ and f is surjective but not injective.

(e) $|X| = |Y|$, X and Y are finite, and f is injective but not surjective.

(f) $|X| = |Y|$, X and Y are finite, and f is surjective but not injective.

20. Let $f : X \to Y$ and $g : Y \to Z$ be functions. We can define the **composition** of f and g to be the function $g \circ f : X \to Z$ for which the image of each $x \in X$ is $g(f(x))$. That is, plug x into f, then plug the result into g (just like composition in algebra and calculus).

(a) If f and g are both injective, must $g \circ f$ be injective? Explain.

(b) If f and g are both surjective, must $g \circ f$ be surjective? Explain.

(c) Suppose $g \circ f$ is injective. What, if anything, can you say about f and g? Explain.

(d) Suppose $g \circ f$ is surjective. What, if anything, can you say about f and g? Explain.

21. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-3 & \text{if } n \text{ is odd.} \end{cases}$

(a) Is f injective? Prove your answer.

(b) Is f surjective? Prove your answer.

22. At the end of the semester a teacher assigns letter grades to each of her students. Is this a function? If so, what sets make up the domain and codomain, and is the function injective, surjective, bijective, or neither?

23. In the game of *Hearts*, four players are each dealt 13 cards from a deck of 52. Is this a function? If so, what sets make up the domain and codomain, and is the function injective, surjective, bijective, or neither?

24. Seven players are playing 5-card stud. Each player initially receives 5 cards from a deck of 52. Is this a function? If so, what sets make up the domain and codomain, and is the function injective, surjective, bijective, or neither?

25. Consider the function $f : \mathbb{N} \to \mathbb{N}$ that gives the number of handshakes that take place in a room of n people assuming everyone shakes hands with everyone else. Give a recursive definition for this function.

26. Let $f : X \to Y$ be a function and $A \subseteq X$ be a finite subset of the domain. What can you say about the relationship between $|A|$ and $\left|f(A)\right|$? Consider both the general case and what happens when you know f is injective, surjective, or bijective.

27. Let $f : X \to Y$ be a function and $B \subseteq Y$ be a finite subset of the codomain. What can you say about the relationship between $|B|$ and $\left|f^{-1}(B)\right|$? Consider both the general case and what happens when you know f is injective, surjective, or bijective.

28. Let $f : X \to Y$ be a function, $A \subseteq X$ and $B \subseteq Y$.

(a) Is $f^{-1}\left(f(A)\right) = A$? Always, sometimes, never? Explain.

(b) Is $f\left(f^{-1}(B)\right) = B$? Always, sometimes, never? Explain.

(c) If one or both of the above do not always hold, is there something else you can say? Will equality always hold for particular types of functions? Is there some other relationship other than equality that would always hold? Explore.

29. Let $f : X \to Y$ be a function and $A, B \subseteq X$ be subsets of the domain.

(a) Is $f(A \cup B) = f(A) \cup f(B)$? Always, sometimes, or never? Explain.

(b) Is $f(A \cap B) = f(A) \cap f(B)$? Always, sometimes, or never? Explain.

30. Let $f : X \to Y$ be a function and $A, B \subseteq Y$ be subsets of the codomain.

(a) Is $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$? Always, sometimes, or never? Explain.

(b) Is $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$? Always, sometimes, or never? Explain.

Chapter 1

Counting

One of the first things you learn in mathematics is how to count. Now we want to count large collections of things quickly and precisely. For example:

- In a group of 10 people, if everyone shakes hands with everyone else exactly once, how many handshakes took place?
- How many ways can you distribute 10 girl scout cookies to 7 boy scouts?
- How many anagrams are there of "anagram"?

Before tackling questions like these, let's look at the basics of counting.

1.1 Additive and Multiplicative Principles

Investigate!

1. A restaurant offers 8 appetizers and 14 entrées. How many choices do you have if:
   (a) you will eat one dish, either an appetizer or an entrée?
   (b) you are extra hungry and want to eat both an appetizer and an entrée?
2. Think about the methods you used to solve question 1. Write down the rules for these methods.
3. Do your rules work? A standard deck of playing cards has 26 red cards and 12 face cards.
   (a) How many ways can you select a card which is either red or a face card?
   (b) How many ways can you select a card which is both red and a face card?
   (c) How many ways can you select two cards so that the first one is red and the second one is a face card?

STOP **Attempt the above activity before proceeding** STOP

Consider this rather simple counting problem: at Red Dogs and Donuts, there are 14 varieties of donuts, and 16 types of hot dogs. If you want either a donut or a dog, how many options do you have? This isn't too hard, just add 14 and 16. Will that always work? What is important here?

Additive Principle.

The **additive principle** states that if event A can occur in m ways, and event B can occur in n *disjoint* ways, then the event "A or B" can occur in $m + n$ ways.

It is important that the events be **disjoint**: i.e., that there is no way for A and B to both happen at the same time. For example, a standard deck of 52 cards contains 26 red cards and 12 face cards. However, the number of ways to select a card which is either red or a face card is not $26 + 12 = 38$. This is because there are 6 cards which are both red and face cards.

Example 1.1.1

How many two letter "words" start with either A or B? (A **word** is just a string of letters; it doesn't have to be English, or even pronounceable.)

Solution. First, how many two letter words start with A? We just need to select the second letter, which can be accomplished in 26 ways. So there are 26 words starting with A. There are also 26 words that start with B. To select a word which starts with either A or B, we can pick the word from the first 26 or the second 26, for a total of 52 words.

The additive principle also works with more than two events. Say, in addition to your 14 choices for donuts and 16 for dogs, you would also consider eating one of 15 waffles? How many choices do you have now? You would have $14 + 16 + 15 = 45$ options.

Example 1.1.2

How many two letter words start with one of the 5 vowels?

Solution. There are 26 two letter words starting with A, another 26 starting with E, and so on. We will have 5 groups of 26. So we add 26 to itself 5 times. Of course it would be easier to just multiply $5 \cdot 26$. We are really using the additive principle again, just using multiplication as a shortcut.

Example 1.1.3

Suppose you are going for some fro-yo. You can pick one of 6 yogurt choices, and one of 4 toppings. How many choices do you have?

Solution. Break your choices up into disjoint events: A are the choices with the first topping, B the choices featuring the second topping, and so on. There are four events; each can occur in 6 ways (one for each yogurt flavor). The events are disjoint, so the total number of choices is $6 + 6 + 6 + 6 = 24$.

Note that in both of the previous examples, when using the additive principle on a bunch of events all the same size, it is quicker to multiply. This really is the same, and not just because $6 + 6 + 6 + 6 = 4 \cdot 6$. We can first select the topping in 4 ways (that is, we first select which of the disjoint events we will take). For each of those first 4 choices, we now have 6 choices of yogurt. We have:

Multiplicative Principle.

The **multiplicative principle** states that if event A can occur in m ways, and each possibility for A allows for exactly n ways for event B, then the event "A and B" can occur in $m \cdot n$ ways.

The multiplicative principle generalizes to more than two events as well.

Example 1.1.4

How many license plates can you make out of three letters followed by three numerical digits?

Solution. Here we have six events: the first letter, the second letter, the third letter, the first digit, the second digit, and the third digit. The first three events can each happen in 26 ways; the last three can each happen in 10 ways. So the total number of license plates will be $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10$, using the multiplicative principle.

Does this make sense? Think about how we would pick a license plate. How many choices we would have? First, we need to pick the first letter. There are 26 choices. Now for each of those, there are 26 choices for the second letter: 26 second letters with first letter A, 26 second letters with first letter B, and so on. We add 26 to itself 26 times. Or quicker: there are $26 \cdot 26$ choices for the first two letters.

Now for each choice of the first two letters, we have 26 choices for the third letter. That is, 26 third letters for the first two letters

AA, 26 choices for the third letter after starting AB, and so on. There are $26 \cdot 26$ of these 26 third letter choices, for a total of $(26 \cdot 26) \cdot 26$ choices for the first three letters. And for each of these $26 \cdot 26 \cdot 26$ choices of letters, we have a bunch of choices for the remaining digits.

In fact, there are going to be exactly 1000 choices for the numbers. We can see this because there are 1000 three-digit numbers (000 through 999). This is 10 choices for the first digit, 10 for the second, and 10 for the third. The multiplicative principle says we multiply: $10 \cdot 10 \cdot 10 = 1000$.

All together, there were 26^3 choices for the three letters, and 10^3 choices for the numbers, so we have a total of $26^3 \cdot 10^3$ choices of license plates.

Careful: "and" doesn't mean "times." For example, how many playing cards are both red and a face card? Not $26 \cdot 12$. The answer is 6, and we needed to know something about cards to answer that question.

Another caution: how many ways can you select two cards, so that the first one is a red card and the second one is a face card? This looks more like the multiplicative principle (you are counting two separate events) but the answer is not $26 \cdot 12$ here either. The problem is that while there are 26 ways for the first card to be selected, it is not the case that *for each* of those there are 12 ways to select the second card. If the first card was both red and a face card, then there would be only 11 choices for the second card.[1]

Example 1.1.5 Counting functions.

How many functions $f : \{1, 2, 3, 4, 5\} \to \{a, b, c, d\}$ are there?

Solution. Remember that a function sends each element of the domain to exactly one element of the codomain. To determine a function, we just need to specify the image of each element in the domain. Where can we send 1? There are 4 choices. Where can we send 2? Again, 4 choices. What we have here is 5 "events" (picking the image of an element in the domain) each of which can happen in 4 ways (the choices for that image). Thus there are $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^5$ functions.

This is more than just an example of how we can use the multiplicative principle in a particular counting question. What

[1]To solve this problem, you could break it into two cases. First, count how many ways there are to select the two cards when the first card is a red non-face card. Second, count how many ways when the first card is a red face card. Doing so makes the events in each separate case independent, so the multiplicative principle can be applied.

we have here is a general interpretation of certain applications of the multiplicative principle using rigorously defined mathematical objects: functions. Whenever we have a counting question that asks for the number of outcomes of a repeated event, we can interpret that as asking for the number of functions from $\{1, 2, \ldots, n\}$ (where n is the number of times the event is repeated) to $\{1, 2, \ldots, k\}$ (where k is the number of ways that event can occur).

Counting With Sets

Do you believe the additive and multiplicative principles? How would you convince someone they are correct? This is surprisingly difficult. They seem so simple, so obvious. But why do they work?

To make things clearer, and more mathematically rigorous, we will use sets. Do not skip this section! It might seem like we are just trying to give a proof of these principles, but we are doing a lot more. If we understand the additive and multiplicative principles rigorously, we will be better at applying them, and knowing when and when not to apply them at all.

We will look at the additive and multiplicative principles in a slightly different way. Instead of thinking about event A and event B, we want to think of a set A and a set B. The sets will contain all the different ways the event can happen. (It will be helpful to be able to switch back and forth between these two models when checking that we have counted correctly.) Here's what we mean:

Example 1.1.6

Suppose you own 9 shirts and 5 pairs of pants.

1. How many outfits can you make?
2. If today is half-naked-day, and you will wear only a shirt or only a pair of pants, how many choices do you have?

Solution. By now you should agree that the answer to the first question is $9 \cdot 5 = 45$ and the answer to the second question is $9 + 5 = 14$. These are the multiplicative and additive principles. There are two events: picking a shirt and picking a pair of pants. The first event can happen in 9 ways and the second event can happen in 5 ways. To get both a shirt and a pair of pants, you multiply. To get just one article of clothing, you add.

Now look at this using sets. There are two sets, call them S and P. The set S contains all 9 shirts so $|S| = 9$ while $|P| = 5$, since there are 5 elements in the set P (namely your 5 pairs of pants). What are we asking in terms of these sets? Well in question 2, we really want $|S \cup P|$, the number of elements in the union of shirts and pants. This is just $|S| + |P|$ (since there is no overlap; $|S \cap P| = 0$). Question 1 is slightly more complicated. Your first guess might be to find $|S \cap P|$, but this is not right (there is nothing in the intersection). We are not asking for how many clothing items are both a shirt and a pair of pants. Instead, we want one of each. We could think of this as asking how many pairs (x, y) there are, where x is a shirt and y is a pair of pants. As we will soon verify, this number is $|S| \cdot |P|$.

From this example we can see right away how to rephrase our additive principle in terms of sets:

Additive Principle (with sets).

Given two sets A and B, if $A \cap B = \emptyset$ (that is, if there is no element in common to both A and B), then

$$|A \cup B| = |A| + |B|.$$

This hardly needs a proof. To find $A \cup B$, you take everything in A and throw in everything in B. Since there is no element in both sets already, you will have $|A|$ things and add $|B|$ new things to it. This is what adding does! Of course, we can easily extend this to any number of disjoint sets.

From the example above, we see that in order to investigate the multiplicative principle carefully, we need to consider ordered pairs. We should define this carefully:

Cartesian Product.

Given sets A and B, we can form the *set*

$$A \times B = \{(x, y) : x \in A \wedge y \in B\}$$

to be the set of all ordered pairs (x, y) where x is an element of A and y is an element of B. We call $A \times B$ the **Cartesian product** of A and B.

Example 1.1.7

Let $A = \{1, 2\}$ and $B = \{3, 4, 5\}$. Find $A \times B$.

Solution. We want to find ordered pairs (a, b) where a can be either 1 or 2 and b can be either 3, 4, or 5. $A \times B$ is the set of all of these pairs:

$$A \times B = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}.$$

The question is, what is $|A \times B|$? To figure this out, write out $A \times B$. Let $A = \{a_1, a_2, a_3, \ldots, a_m\}$ and $B = \{b_1, b_2, b_3, \ldots, b_n\}$ (so $|A| = m$ and $|B| = n$). The set $A \times B$ contains all pairs with the first half of the pair being some $a_i \in A$ and the second being one of the $b_j \in B$. In other words:

$$\begin{aligned} A \times B = \{&(a_1, b_1), (a_1, b_2), (a_1, b_3), \ldots (a_1, b_n), \\ &(a_2, b_1), (a_2, b_2), (a_2, b_3), \ldots, (a_2, b_n), \\ &(a_3, b_1), (a_3, b_2), (a_3, b_3), \ldots, (a_3, b_n), \\ &\vdots \\ &(a_m, b_1), (a_m, b_2), (a_m, b_3), \ldots, (a_m, b_n)\}. \end{aligned}$$

Notice what we have done here: we made m rows of n pairs, for a total of $m \cdot n$ pairs.

Each row above is really $\{a_i\} \times B$ for some $a_i \in A$. That is, we fixed the A-element. Broken up this way, we have

$$A \times B = (\{a_1\} \times B) \cup (\{a_2\} \times B) \cup (\{a_3\} \times B) \cup \cdots \cup (\{a_m\} \times B).$$

So $A \times B$ is really the union of m disjoint sets. Each of those sets has n elements in them. The total (using the additive principle) is $n + n + n + \cdots + n = m \cdot n$.

To summarize:

Multiplicative Principle (with sets).

Given two sets A and B, we have $|A \times B| = |A| \cdot |B|$.

Again, we can easily extend this to any number of sets.

Principle of Inclusion/Exclusion

Investigate!

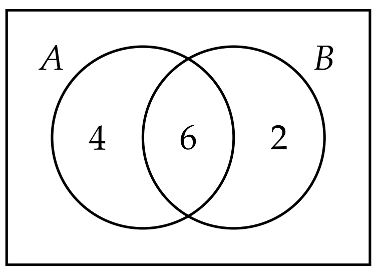
A recent buzz marketing campaign for *Village Inn* surveyed patrons on their pie preferences. People were asked whether they enjoyed (A) Apple, (B) Blueberry or (C) Cherry pie (respondents answered yes or no to each type of pie, and could say yes to more than one type). The following table shows the results of the survey.

Pies enjoyed:	A	B	C	AB	AC	BC	ABC
Number of people:	20	13	26	9	15	7	5

How many of those asked enjoy at least one of the kinds of pie? Also, explain why the answer is not 95.

Attempt the above activity before proceeding

While we are thinking about sets, consider what happens to the additive principle when the sets are NOT disjoint. Suppose we want to find $|A \cup B|$ and know that $|A| = 10$ and $|B| = 8$. This is not enough information though. We do not know how many of the 8 elements in B are also elements of A. However, if we also know that $|A \cap B| = 6$, then we can say exactly how many elements are in A, and, of those, how many are in B and how many are not (6 of the 10 elements are in B, so 4 are in A but not in B). We could fill in a Venn diagram as follows:



This says there are 6 elements in $A \cap B$, 4 elements in $A \setminus B$ and 2 elements in $B \setminus A$. Now *these* three sets *are* disjoint, so we can use the additive principle to find the number of elements in $A \cup B$. It is $6+4+2 = 12$.

This will always work, but drawing a Venn diagram is more than we need to do. In fact, it would be nice to relate this problem to the case where A and B are disjoint. Is there one rule we can make that works in either case?

Here is another way to get the answer to the problem above. Start by just adding $|A| + |B|$. This is $10 + 8 = 18$, which would be the answer if $|A \cap B| = 0$. We see that we are off by exactly 6, which just so happens to be $|A \cap B|$. So perhaps we guess,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This works for this one example. Will it always work? Think about what we are doing here. We want to know how many things are either in A or B (or both). We can throw in everything in A, and everything in B. This would give $|A| + |B|$ many elements. But of course when you actually take the union, you do not repeat elements that are in both. So far we have counted every element in $A \cap B$ exactly twice: once when we put in the elements from A and once when we included the elements from B. We correct by subtracting out the number of elements we have counted twice. So we added them in twice, subtracted once, leaving them counted only one time.

In other words, we have:

Cardinality of a union (2 sets).

For any finite sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

We can do something similar with three sets.

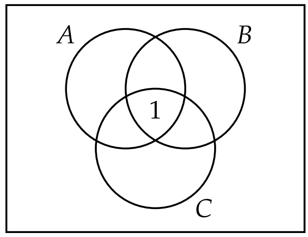
Example 1.1.8

An examination in three subjects, Algebra, Biology, and Chemistry, was taken by 41 students. The following table shows how many students failed in each single subject and in their various combinations:

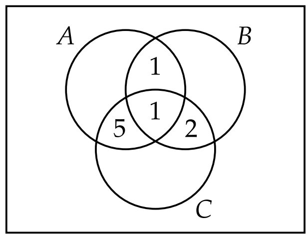
Subject:	A	B	C	AB	AC	BC	ABC
Failed:	12	5	8	2	6	3	1

How many students failed at least one subject?

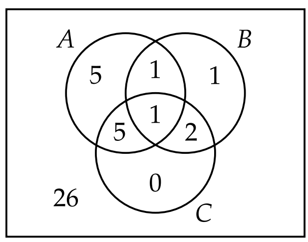
Solution. The answer is not 37, even though the sum of the numbers above is 37. For example, while 12 students failed Algebra, 2 of those students also failed Biology, 6 also failed Chemistry, and 1 of those failed all three subjects. In fact, that 1 student who failed all three subjects is counted a total of 7 times in the total 37. To clarify things, let us think of the students who failed Algebra as the elements of the set A, and similarly for sets B and C. The one student who failed all three subjects is the lone element of the set $A \cap B \cap C$. Thus, in Venn diagrams:



Now let's fill in the other intersections. We know $A \cap B$ contains 2 elements, but 1 element has already been counted. So we should put a 1 in the region where A and B intersect (but C does not). Similarly, we calculate the cardinality of $(A \cap C) \setminus B$, and $(B \cap C) \setminus A$:



Next, we determine the numbers which should go in the remaining regions, including outside of all three circles. This last number is the number of students who did not fail any subject:



We found 5 goes in the "A only" region because the entire circle for A needed to have a total of 12, and 7 were already accounted for. Similarly, we calculate the "B only" region to contain only 1 student and the "C only" region to contain no students.

Thus the number of students who failed at least one class is 15 (the sum of the numbers in each of the eight disjoint regions). The number of students who passed all three classes is 26: the total number of students, 41, less the 15 who failed at least one class.

Note that we can also answer other questions. For example, how many students failed just Chemistry? None. How many passed Algebra but failed both Biology and Chemistry? This corresponds to the region inside both B and C but outside of A, containing 2 students.

Could we have solved the problem above in an algebraic way? While the additive principle generalizes to any number of sets, when we add a third set here, we must be careful. With two sets, we needed to know the cardinalities of A, B, and $A \cap B$ in order to find the cardinality of $A \cup B$. With three sets we need more information. There are more ways the sets can combine. Not surprisingly then, the formula for cardinality of the union of three non-disjoint sets is more complicated:

Cardinality of a union (3 sets).

For any finite sets A, B, and C,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

To determine how many elements are in at least one of A, B, or C we add up all the elements in each of those sets. However, when we do that, any element in both A and B is counted twice. Also, each element in both A and C is counted twice, as are elements in B and C, so we take each of those out of our sum once. But now what about the elements which are in $A \cap B \cap C$ (in all three sets)? We added them in three times, but also removed them three times. They have not yet been counted. Thus we add those elements back in at the end.

Returning to our example above, we have $|A| = 12$, $|B| = 5$, $|C| = 8$. We also have $|A \cap B| = 2$, $|A \cap C| = 6$, $|B \cap C| = 3$, and $|A \cap B \cap C| = 1$. Therefore:

$$|A \cup B \cup C| = 12 + 5 + 8 - 2 - 6 - 3 + 1 = 15.$$

This is what we got when we solved the problem using Venn diagrams.

This process of adding in, then taking out, then adding back in, and so on is called the *Principle of Inclusion/Exclusion*, or simply PIE. We will return to this counting technique later to solve for more complicated problems (involving more than 3 sets).

Exercises

1. Your wardrobe consists of 5 shirts, 3 pairs of pants, and 17 bow ties. bow ties How many different outfits can you make?

2. For your college interview, you must wear a tie. You own 3 regular (boring) ties and 5 (cool) bow ties.

   (a) How many choices do you have for your neck-wear?

   (b) You realize that the interview is for clown college, so you should probably wear both a regular tie and a bow tie. How many choices do you have now?

(c) For the rest of your outfit, you have 5 shirts, 4 skirts, 3 pants, and 7 dresses. You want to select either a shirt to wear with a skirt or pants, or just a dress. How many outfits do you have to choose from?

3. Your Blu-ray collection consists of 9 comedies and 7 horror movies. Give an example of a question for which the answer is:

(a) 16.

(b) 63.

4. hexadecimal We usually write numbers in decimal form (or base 10), meaning numbers are composed using 10 different "digits" $\{0, 1, \ldots, 9\}$. Sometimes though it is useful to write numbers hexadecimal or base 16. Now there are 16 distinct digits that can be used to form numbers: $\{0, 1, \ldots, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}\}$. So for example, a 3 digit hexadecimal number might be 2B8.

(a) How many 2-digit hexadecimals are there in which the first digit is E or F? Explain your answer in terms of the additive principle (using either events or sets).

(b) Explain why your answer to the previous part is correct in terms of the multiplicative principle (using either events or sets). Why do both the additive and multiplicative principles give you the same answer?

(c) How many 3-digit hexadecimals start with a letter (A-F) and end with a numeral (0-9)? Explain.

(d) How many 3-digit hexadecimals start with a letter (A-F) or end with a numeral (0-9) (or both)? Explain.

5. Suppose you have sets A and B with $|A| = 10$ and $|B| = 15$.

(a) What is the largest possible value for $|A \cap B|$?

(b) What is the smallest possible value for $|A \cap B|$?

(c) What are the possible values for $|A \cup B|$?

6. If $|A| = 8$ and $|B| = 5$, what is $|A \cup B| + |A \cap B|$?

7. A group of college students were asked about their TV watching habits. Of those surveyed, 28 students watch *The Walking Dead*, 19 watch *The Blacklist*, and 24 watch *Game of Thrones*. Additionally, 16 watch *The Walking Dead* and *The Blacklist*, 14 watch *The Walking Dead* and *Game of Thrones*, and 10 watch *The Blacklist* and *Game of Thrones*. There are 8 students who watch all three shows. How many students surveyed watched at least one of the shows?

8. In a recent survey, 30 students reported whether they liked their potatoes Mashed, French-fried, or Twice-baked. 15 liked them mashed, 20 liked French fries, and 9 liked twice baked potatoes. Additionally, 12 students liked both mashed and fried potatoes, 5 liked French fries and twice baked potatoes, 6 liked mashed and baked, and 3 liked all three styles. How many students *hate* potatoes? Explain why your answer is correct.

9. For how many $n \in \{1, 2, \ldots, 500\}$ is n a multiple of one or more of 5, 6, or 7?

10. How many positive integers less than 1000 are multiples of 3, 5, or 7? Explain your answer using the Principle of Inclusion/Exclusion.

11. Let A, B, and C be sets.

(a) Find $|(A \cup C) \setminus B|$ provided $|A| = 50$, $|B| = 45$, $|C| = 40$, $|A \cap B| = 20$, $|A \cap C| = 15$, $|B \cap C| = 23$, and $|A \cap B \cap C| = 12$.

(b) Describe a set in terms of A, B, and C with cardinality 26.

12. Consider all 5 letter "words" made from the letters a through h. (Recall, words are just strings of letters, not necessarily actual English words.)

(a) How many of these words are there total?

(b) How many of these words contain no repeated letters?

(c) How many of these words start with the sub-word "aha"?

(d) How many of these words either start with "aha" or end with "bah" or both?

(e) How many of the words containing no repeats also do not contain the sub-word "bad"?

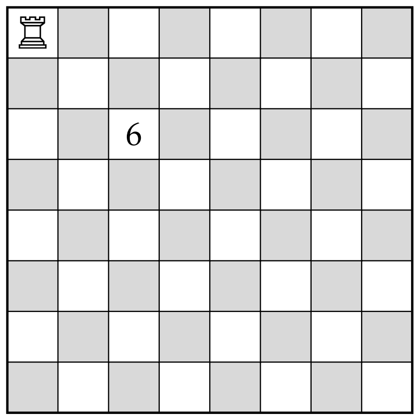
13. For how many three digit numbers (100 to 999) is the *sum of the digits* even? (For example, 343 has an even sum of digits: $3 + 4 + 3 = 10$ which is even.) Find the answer and explain why it is correct in at least two *different* ways.

14. The number 735000 factors as $2^3 \cdot 3 \cdot 5^4 \cdot 7^2$. How many divisors does it have? Explain your answer using the multiplicative principle.

1.2 Binomial Coefficients

Investigate!

In chess, a rook can move only in straight lines (not diagonally). Fill in each square of the chess board below with the number of different shortest paths the rook, in the upper left corner, can take to get to that square. For example, one square is already filled in. There are six different paths from the rook to the square: DDRR (down down right right), DRDR, DRRD, RDDR, RDRD and RRDD.



Attempt the above activity before proceeding

Here are some apparently different discrete objects we can count: subsets, bit strings, lattice paths, and binomial coefficients. We will give an example of each type of counting problem (and say what these things even are). As we will see, these counting problems are surprisingly similar.

Subsets

Subsets should be familiar, otherwise read over Section 0.3 again. Suppose we look at the set $A = \{1, 2, 3, 4, 5\}$. How many subsets of A contain exactly 3 elements?

First, a simpler question: How many subsets of A are there total? In other words, what is $|\mathcal{P}(A)|$ (the cardinality of the power set of A)? Think about how we would build a subset. We need to decide, for each of the elements of A, whether or not to include the element in our subset. So we need to decide "yes" or "no" for the element 1. And for each choice we make, we need to decide "yes" or "no" for the element 2. And so on. For each of the 5 elements, we have 2 choices. Therefore the number of subsets is simply $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$ (by the multiplicative principle).

Of those 32 subsets, how many have 3 elements? This is not obvious. Note that we cannot just use the multiplicative principle. Maybe we want to say we have 2 choices (yes/no) for the first element, 2 choices for the second, 2 choices for the third, and then only 1 choice for the other two. But what if we said "no" to one of the first three elements? Then we would have two choices for the 4th element. What a mess!

Another (bad) idea: we need to pick three elements to be in our subset. There are 5 elements to choose from. So there are 5 choices for the first element, and for each of those 4 choices for the second, and then 3 for the third (last) element. The multiplicative principle would say then that there are a total of $5 \cdot 4 \cdot 3 = 60$ ways to select the 3-element subset. But this cannot be correct ($60 > 32$ for one thing). One of the outcomes we would get from these choices would be the set $\{3, 2, 5\}$, by choosing the element 3 first, then the element 2, then the element 5. Another outcome would be $\{5, 2, 3\}$ by choosing the element 5 first, then the element 2, then the element 3. But these are the same set! We can correct this by dividing: for each set of three elements, there are 6 outcomes counted among our 60 (since there are 3 choices for which element we list first, 2 for which we list second, and 1 for which we list last). So we expect there to be 10 3-element subsets of A.

Is this right? Well, we could list out all 10 of them, being very systematic in doing so, to make sure we don't miss any or list any twice. Or we could try to count how many subsets of *A don't* have 3 elements in them. How many have no elements? Just 1 (the empty set). How many have 5? Again, just 1. These are the cases in which we say "no" to all elements, or "yes" to all elements. Okay, what about the subsets which contain a single element? There are 5 of these. We must say "yes" to exactly one element, and there are 5 to choose from. This is also the number of subsets containing 4 elements. Those are the ones for which we must say "no" to exactly one element.

So far we have counted 12 of the 32 subsets. We have not yet counted the subsets with cardinality 2 and with cardinality 3. There are a total of 20 subsets left to split up between these two groups. But the number of each must be the same! If we say "yes" to exactly two elements, that can be accomplished in exactly the same number of ways as the number of ways we can say "no" to exactly two elements. So the number of 2-element subsets is equal to the number of 3-element subsets. Together there are 20 of these subsets, so 10 each.

Number of elements:	0	1	2	3	4	5
Number of subsets:	1	5	10	10	5	1

BIT STRINGS

"Bit" is short for "binary digit," so a **bit string** is a string of binary digits. The **binary digits** are simply the numbers 0 and 1. All of the following are bit strings:

$$1001 \quad 0 \quad 1111 \quad 1010101010.$$

The number of bits (0's or 1's) in the string is the **length** of the string; the strings above have lengths 4, 1, 4, and 10 respectively. We also can ask how many of the bits are 1's. The number of 1's in a bit string is the **weight** of the string; the weights of the above strings are 2, 0, 4, and 5 respectively.

Bit Strings.

- An n**-bit string** is a bit string of length n. That is, it is a string containing n symbols, each of which is a bit, either 0 or 1.
- The **weight** of a bit string is the number of 1's in it.
- $\mathbf{B}^n$ is the *set* of all n-bit strings.
- $\mathbf{B}^n_k$ is the set of all n-bit strings of weight k.

For example, the elements of the set $\mathbf{B}^3_2$ are the bit strings 011, 101, and 110. Those are the only strings containing three bits exactly two of which are 1's.

The counting questions: How many bit strings have length 5? How many of those have weight 3? In other words, we are asking for the cardinalities $|\mathbf{B}^5|$ and $|\mathbf{B}^5_3|$.

To find the number of 5-bit strings is straight forward. We have 5 bits, and each can either be a 0 or a 1. So there are 2 choices for the first bit, 2 choices for the second, and so on. By the multiplicative principle, there are $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$ such strings.

Finding the number of 5-bit strings of weight 3 is harder. Think about how such a string could start. The first bit must be either a 0 or a 1. In the first case (the string starts with a 0), we must then decide on four more bits. To have a total of three 1's, among those four remaining bits there must be three 1's. To count all of these strings, we must include all 4-bit strings of weight 3. In the second case (the string starts with a 1), we still have four bits to choose, but now only two of them can be 1's, so we should look at all the 4-bit strings of weight 2. So the strings in $\mathbf{B}^5_3$ all have the form $1\mathbf{B}^4_2$ (that is, a 1 followed by a string from $\mathbf{B}^4_2$) or $0\mathbf{B}^4_3$. These two sets are disjoint, so we can use the additive principle:

$$|\mathbf{B}^5_3| = |\mathbf{B}^4_2| + |\mathbf{B}^4_3|.$$

This is an example of a **recurrence relation**. We represented one instance of our counting problem in terms of two simpler instances of the problem. If only we knew the cardinalities of $\mathbf{B}^4_2$ and $\mathbf{B}^4_3$. Repeating the same reasoning,

$$|\mathbf{B}^4_2| = |\mathbf{B}^3_1| + |\mathbf{B}^3_2| \quad \text{and} \quad |\mathbf{B}^4_3| = |\mathbf{B}^3_2| + |\mathbf{B}^3_3|.$$

We can keep going down, but this should be good enough. Both $\mathbf{B}^3_1$ and $\mathbf{B}^3_2$ contain 3 bit strings: we must pick one of the three bits to be a 1 (three ways to do that) or one of the three bits to be a 0 (three ways to do that). Also, $\mathbf{B}^3_3$ contains just one string: 111. Thus $|\mathbf{B}^4_2| = 6$ and $|\mathbf{B}^4_3| = 4$, which puts $\mathbf{B}^5_3$ at a total of 10 strings.

But wait —32 and 10 were the answers to the counting questions about subsets. Coincidence? Not at all. Each bit string can be thought of as a *code* for a subset. To represent the subsets of $A = \{1, 2, 3, 4, 5\}$, we can use 5-bit strings, one bit for each element of A. Each bit in the string is a 0 if its corresponding element of A is not in the subset, and a 1 if the element of A is in the subset. Remember, deciding the subset amounted to a sequence of five yes/no votes for the elements of A. Instead of yes, we put a 1; instead of no, we put a 0.

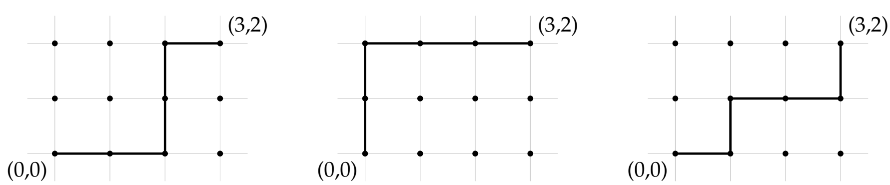
For example, the bit string 11001 represents the subset $\{1, 2, 5\}$ since the first, second and fifth bits are 1's. The subset $\{3, 5\}$ would be coded by the string 00101. What we really have here is a bijection from $\mathcal{P}(A)$ to $\mathbf{B}^5$.

Now for a subset to contain exactly three elements, the corresponding bit string must contain exactly three 1's. In other words, the weight must be 3. Thus counting the number of 3-element subsets of A is the same as counting the number 5-bit strings of weight 3.

Lattice Paths

The **integer lattice** is the set of all points in the Cartesian plane for which both the x and y coordinates are integers. If you like to draw graphs on graph paper, the lattice is the set of all the intersections of the grid lines.

A **lattice path** is one of the shortest possible paths connecting two points on the lattice, moving only horizontally and vertically. For example, here are three possible lattice paths from the point $(0, 0)$ to $(3, 2)$:

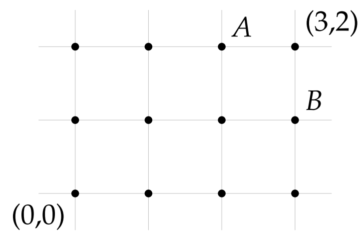


Notice to ensure the path is the *shortest* possible, each move must be either to the right or up. Additionally, in this case, note that no matter what path we take, we must make three steps right and two steps up. No matter what order we make these steps, there will always be 5 steps. Thus each path has *length* 5.

The counting question: how many lattice paths are there between $(0,0)$ and $(3,2)$? We could try to draw all of these, or instead of drawing them, maybe just list which direction we travel on each of the 5 steps. One path might be RRUUR, or maybe UURRR, or perhaps RURRU (those correspond to the three paths drawn above). So how many such strings of R's and U's are there?

Notice that each of these strings must contain 5 symbols. Exactly 3 of them must be R's (since our destination is 3 units to the right). This seems awfully familiar. In fact, what if we used 1's instead of R's and 0's instead of U's? Then we would just have 5-bit strings of weight 3. There are 10 of those, so there are 10 lattice paths from (0,0) to (3,2).

The correspondence between bit strings and lattice paths does not stop there. Here is another way to count lattice paths. Consider the lattice shown below:



Any lattice path from (0,0) to (3,2) must pass through exactly one of A and B. The point A is 4 steps away from (0,0) and two of them are towards the right. The number of lattice paths to A is the same as the number of 4-bit strings of weight 2, namely 6. The point B is 4 steps away from (0,0), but now 3 of them are towards the right. So the number of paths to point B is the same as the number of 4-bit strings of weight 3, namely 4. So the total number of paths to (3,2) is just $6 + 4$. This is the same way we calculated the number of 5-bit strings of weight 3. The point: the exact same recurrence relation exists for bit strings and for lattice paths.

Binomial Coefficients

Binomial coefficients are the coefficients in the expanded version of a binomial, such as $(x + y)^5$. What happens when we multiply such a binomial out? We will expand $(x + y)^n$ for various values of n. Each of these are done by multiplying everything out (i.e., FOIL-ing) and then collecting like terms.

$$(x + y)^1 = x + y$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

In fact, there is a quicker way to expand the above binomials. For example, consider the next one, $(x+y)^5$. What we are really doing is multiplying out,

$$(x+y)(x+y)(x+y)(x+y)(x+y).$$

If that looks daunting, go back to the case of $(x+y)^3 = (x+y)(x+y)(x+y)$. Why do we only have one x^3 and y^3 but three x^2y and xy^2 terms? Every time we distribute over an $(x+y)$ we create two copies of what is left, one multiplied by x, the other multiplied by y. To get x^3, we need to pick the "multiplied by x" side every time (we don't have any y's in the term). This will only happen once. On the other hand, to get x^2y we need to select the x side twice and the y side once. In other words, we need to pick one of the three $(x+y)$ terms to "contribute" their y.

Similarly, in the expansion of $(x+y)^5$, there will be only one x^5 term and one y^5 term. This is because to get an x^5, we need to use the x term in each of the copies of the binomial $(x+y)$, and similarly for y^5. What about x^4y? To get terms like this, we need to use four x's and one y, so we need exactly one of the five binomials to contribute a y. There are 5 choices for this, so there are 5 ways to get x^4y, so the coefficient of x^4y is 5. This is also the coefficient for xy^4 for the same (but opposite) reason: there are 5 ways to pick which of the 5 binomials contribute the single x. So far we have

$$(x+y)^5 = x^5 + 5x^4y + \underline{\ ?\ }\, x^3y^2 + \underline{\ ?\ }\, x^2y^3 + 5xy^4 + y^5.$$

We still need the coefficients of x^3y^2 and x^2y^3. In both cases, we need to pick exactly 3 of the 5 binomials to contribute one variable, the other two to contribute the other. Wait. This sounds familiar. We have 5 things, each can be one of two things, and we need a total of 3 of one of them. That's just like taking 5 bits and making sure exactly 3 of them are 1's. So the coefficient of x^3y^2 (and also x^2y^3) will be exactly the same as the number of bit strings of length 5 and weight 3, which we found earlier to be 10. So we have:

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

These numbers we keep seeing over and over again. They are the number of subsets of a particular size, the number of bit strings of a particular weight, the number of lattice paths, and the coefficients of these binomial products. We will call them **binomial coefficients**. We even have a special symbol for them: $\binom{n}{k}$.

Binomial Coefficients.

For each integer $n \geq 0$ and integer k with $0 \leq k \leq n$ there is a number

$$\binom{n}{k},$$

read "n choose k." We have:

- $\binom{n}{k} = |\mathbf{B}^n_k|$, the number of n-bit strings of weight k.
- $\binom{n}{k}$ is the number of subsets of a set of size n each with cardinality k.
- $\binom{n}{k}$ is the number of lattice paths of length n containing k steps to the right.
- $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$.
- $\binom{n}{k}$ is the number of ways to select k objects from a total of n objects.

The last bullet point is usually taken as the definition of $\binom{n}{k}$. Out of n objects we must choose k of them, so there are n choose k ways of doing this. Each of our counting problems above can be viewed in this way:

- How many subsets of $\{1,2,3,4,5\}$ contain exactly 3 elements? We must choose 3 of the 5 elements to be in our subset. There are $\binom{5}{3}$ ways to do this, so there are $\binom{5}{3}$ such subsets.
- How many bit strings have length 5 and weight 3? We must choose 3 of the 5 bits to be 1's. There are $\binom{5}{3}$ ways to do this, so there are $\binom{5}{3}$ such bit strings.
- How many lattice paths are there from (0,0) to (3,2)? We must choose 3 of the 5 steps to be towards the right. There are $\binom{5}{3}$ ways to do this, so there are $\binom{5}{3}$ such lattice paths.
- What is the coefficient of x^3y^2 in the expansion of $(x+y)^5$? We must choose 3 of the 5 copies of the binomial to contribute an x. There are $\binom{5}{3}$ ways to do this, so the coefficient is $\binom{5}{3}$.

It should be clear that in each case above, we have the right answer. All we had to do is phrase the question correctly and it became obvious that $\binom{5}{3}$ is correct. However, this does not tell us that the answer is in fact 10 in each case. We will eventually find a formula for $\binom{n}{k}$, but for now, look back at how we arrived at the answer 10 in our counting problems above. It all came down to bit strings, and we have a recurrence relation for bit strings:

$$|\mathbf{B}^n_k| = |\mathbf{B}^{n-1}_{k-1}| + |\mathbf{B}^{n-1}_k|.$$

Remember, this is because we can start the bit string with either a 1 or a 0. In both cases, we have $n-1$ more bits to pick. The strings starting with 1 must contain $k-1$ more 1's, while the strings starting with 0 still need k more 1's.

Since $|\mathbf{B}_k^n| = \binom{n}{k}$, the same recurrence relation holds for binomial coefficients:

Recurrence relation for $\binom{n}{k}$.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Pascal's Triangle

Let's arrange the binomial coefficients $\binom{n}{k}$ into a triangle like follows:

$$\begin{array}{ccccccccc} & & & & \binom{0}{0} & & & & \\ & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \end{array}$$

This can continue as far down as we like. The recurrence relation for $\binom{n}{k}$ tells us that each entry in the triangle is the sum of the two entries above it. The entries on the sides of the triangle are always 1. This is because $\binom{n}{0} = 1$ for all n since there is only one way to pick 0 of n objects and $\binom{n}{n} = 1$ since there is one way to select all n out of n objects. Using the recurrence relation, and the fact that the sides of the triangle are 1's, we can easily replace all the entries above with the correct values of $\binom{n}{k}$. Doing so gives us **Pascal's triangle**.

We can use Pascal's triangle to calculate binomial coefficients. For example, using the triangle below, we can find $\binom{12}{6} = 924$.

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1
1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1
1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 560 120 16 1

EXERCISES

1. Let $S = \{1, 2, 3, 4, 5, 6\}$

(a) How many subsets are there total?

(b) How many subsets have $\{2, 3, 5\}$ as a subset?

(c) How many subsets contain at least one odd number?

(d) How many subsets contain exactly one even number?

2. Let $S = \{1, 2, 3, 4, 5, 6\}$

(a) How many subsets are there of cardinality 4?

(b) How many subsets of cardinality 4 have $\{2, 3, 5\}$ as a subset?

(c) How many subsets of cardinality 4 contain at least one odd number?

(d) How many subsets of cardinality 4 contain exactly one even number?

3. Let $A = \{1, 2, 3, \ldots, 9\}$.

(a) How many subsets of A are there? That is, find $|\mathcal{P}(A)|$. Explain.

(b) How many subsets of A contain exactly 5 elements? Explain.

(c) How many subsets of A contain only even numbers? Explain.

(d) How many subsets of A contain an even number of elements? Explain.

4. How many 9-bit strings (that is, bit strings of length 9) are there which:

(a) Start with the sub-string 101? Explain.

(b) Have weight 5 (i.e., contain exactly five 1's) and start with the sub-string 101? Explain.

(c) Either start with 101 or end with 11 (or both)? Explain.

(d) Have weight 5 and either start with 101 or end with 11 (or both)? Explain.

5. You break your piggy-bank to discover lots of pennies and nickels. You start arranging these in rows of 6 coins.

(a) You find yourself making rows containing an equal number of pennies and nickels. For fun, you decide to lay out every possible such row. How many coins will you need?

(b) How many coins would you need to make all possible rows of 6 coins (not necessarily with equal number of pennies and nickels)?

6. How many 10-bit strings contain 6 or more 1's?

7. How many subsets of $\{0, 1, \ldots, 9\}$ have cardinality 6 or more?

8. What is the coefficient of x^{12} in $(x + 2)^{15}$?

9. What is the coefficient of x^9 in the expansion of $(x + 1)^{14} + x^3(x + 2)^{15}$?

10. How many lattice paths start at (3,3) and

(a) end at (10,10)?

(b) end at (10,10) and pass through (5,7)?

(c) end at (10,10) and avoid (5,7)?

11. Gridtown USA, besides having excellent donut shops, is known for its precisely laid out grid of streets and avenues. Streets run east-west, and avenues north-south, for the entire stretch of the town, never curving and never interrupted by parks or schools or the like.

Suppose you live on the corner of 3rd and 3rd and work on the corner of 12th and 12th. Thus you must travel 18 blocks to get to work as quickly as possible.

(a) How many different routes can you take to work, assuming you want to get there as quickly as possible? Explain.

(b) Now suppose you want to stop and get a donut on the way to work, from your favorite donut shop on the corner of 10th ave and 8th st. How many routes to work, stopping at the donut shop, can you take (again, ensuring the shortest possible route)? Explain.

(c) Disaster Strikes Gridtown: there is a pothole on 4th ave between 5th st and 6th st. How many routes to work can you take avoiding that unsightly (and dangerous) stretch of road? Explain.

(d) The pothole has been repaired (phew) and a new donut shop has opened on the corner of 4th ave and 5th st. How many routes to work drive by one or the other (or both) donut shops? Hint: the donut shops serve PIE.

12. Suppose you are ordering a large pizza from *D.P. Dough*. You want 3 distinct toppings, chosen from their list of 11 vegetarian toppings.

(a) How many choices do you have for your pizza?

(b) How many choices do you have for your pizza if you refuse to have pineapple as one of your toppings?

(c) How many choices do you have for your pizza if you *insist* on having pineapple as one of your toppings?

(d) How do the three questions above relate to each other? Explain.

13. Explain why the coefficient of x^5y^3 the same as the coefficient of x^3y^5 in the expansion of $(x+y)^8$?

1.3 Combinations and Permutations

Investigate!

You have a bunch of chips which come in five different colors: red, blue, green, purple and yellow.

1. How many different two-chip stacks can you make if the bottom chip must be red or blue? Explain your answer using both the additive and multiplicative principles.
2. How many different three-chip stacks can you make if the bottom chip must be red or blue and the top chip must be green, purple or yellow? How does this problem relate to the previous one?
3. How many different three-chip stacks are there in which no color is repeated? What about four-chip stacks?
4. Suppose you wanted to take three different colored chips and put them in your pocket. How many different choices do you have? What if you wanted four different colored chips? How do these problems relate to the previous one?

Attempt the above activity before proceeding

A **permutation** is a (possible) rearrangement of objects. For example, there are 6 permutations of the letters a, b, c:

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba.$$

We know that we have them all listed above —there are 3 choices for which letter we put first, then 2 choices for which letter comes next, which leaves only 1 choice for the last letter. The multiplicative principle says we multiply $3 \cdot 2 \cdot 1$.

Example 1.3.1

How many permutations are there of the letters a, b, c, d, e, f?

Solution. We do NOT want to try to list all of these out. However, if we did, we would need to pick a letter to write down first. There are 6 choices for that letter. For each choice of first letter, there are 5 choices for the second letter (we cannot repeat the first letter; we are rearranging letters and only have one of each), and for each of those, there are 4 choices for the third, 3 choices for the fourth, 2 choices for the fifth and finally only 1 choice for the last letter. So there are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ permutations of the 6 letters.

A piece of notation is helpful here: $n!$, read "n factorial", is the product of all positive integers less than or equal to n (for reasons of convenience, we also define $0!$ to be 1). So the number of permutation of 6 letters, as seen in the previous example is $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. This generalizes:

Permutations of n elements.

There are $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$ permutations of n (distinct) elements.

Example 1.3.2 Counting Bijective Functions.

How many functions $f : \{1, 2, \ldots, 8\} \to \{1, 2, \ldots, 8\}$ are *bijective*?

Solution. Remember what it means for a function to be bijective: each element in the codomain must be the image of exactly one element of the domain. Using two-line notation, we could write one of these bijections as

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 5 & 8 & 7 & 6 & 2 & 4 \end{pmatrix}.$$

What we are really doing is just rearranging the elements of the codomain, so we are creating a permutation of 8 elements. In fact, "permutation" is another term used to describe bijective functions from a finite set to itself.

If you believe this, then you see the answer must be $8! = 8 \cdot 7 \cdot \cdots \cdot 1 = 40320$. You can see this directly as well: for each element of the domain, we must pick a distinct element of the codomain to map to. There are 8 choices for where to send 1, then 7 choices for where to send 2, and so on. We multiply using the multiplicative principle.

Sometimes we do not want to permute all of the letters/numbers/elements we are given.

Example 1.3.3

How many 4 letter "words" can you make from the letters *a* through *f*, with no repeated letters?

Solution. This is just like the problem of permuting 4 letters, only now we have more choices for each letter. For the first letter, there are 6 choices. For each of those, there are 5 choices for the second letter. Then there are 4 choices for the third letter, and 3 choices for the last letter. The total number of words is $6 \cdot 5 \cdot 4 \cdot 3 = 360$. This is not 6! because we never multiplied by 2 and 1. We could start with 6! and then cancel the 2 and 1, and thus write $\frac{6!}{2!}$.

In general, we can ask how many permutations exist of k objects choosing those objects from a larger collection of n objects. (In the example above, $k = 4$, and $n = 6$.) We write this number $P(n, k)$ and sometimes call it a **k-permutation of n elements**. From the example above, we see that to compute $P(n, k)$ we must apply the multiplicative principle to k numbers, starting with n and counting backwards. For example

$$P(10, 4) = 10 \cdot 9 \cdot 8 \cdot 7.$$

Notice again that $P(10, 4)$ starts out looking like 10!, but we stop after 7. We can formally account for this "stopping" by dividing away the part of the factorial we do not want:

$$P(10, 4) = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{10!}{6!}.$$

Careful: The factorial in the denominator is not 4! but rather $(10 - 4)!$.

k-permutations of n elements.

$P(n, k)$ is the number of **k-permutations of n elements**, the number of ways to *arrange* k objects chosen from n distinct objects.

$$P(n, k) = \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-(k-1)).$$

Note that when $n = k$, we have $P(n, n) = \frac{n!}{(n-n)!} = n!$ (since we defined 0! to be 1). This makes sense —we already know $n!$ gives the number of permutations of all n objects.

Example 1.3.4 Counting injective functions.

How many functions $f : \{1, 2, 3\} \to \{1, 2, 3, 4, 5, 6, 7, 8\}$ are *injective*?

Solution. Note that it doesn't make sense to ask for the number of *bijections* here, as there are none (because the codomain is larger than the domain, there are no surjections). But for a function to be injective, we just can't use an element of the codomain more than once.

We need to pick an element from the codomain to be the image of 1. There are 8 choices. Then we need to pick one of the remaining 7 elements to be the image of 2. Finally, one of the remaining 6 elements must be the image of 3. So the total number of functions is $8 \cdot 7 \cdot 6 = P(8, 3)$.

What this demonstrates in general is that the number of injections $f : A \to B$, where $|A| = k$ and $|B| = n$, is $P(n, k)$.

Here is another way to find the number of k-permutations of n elements: first select which k elements will be in the permutation, then count how many ways there are to arrange them. Once you have selected the k objects, we know there are $k!$ ways to arrange (permute) them. But how do you select k objects from the n? You have n objects, and you need to *choose* k of them. You can do that in $\binom{n}{k}$ ways. Then for each choice of those k elements, we can permute *them* in $k!$ ways. Using the multiplicative principle, we get another formula for $P(n, k)$:

$$P(n, k) = \binom{n}{k} \cdot k!.$$

Now since we have a closed formula for $P(n, k)$ already, we can substitute that in:

$$\frac{n!}{(n-k)!} = \binom{n}{k} \cdot k!.$$

If we divide both sides by $k!$ we get a closed formula for $\binom{n}{k}$.

Closed formula for $\binom{n}{k}$.

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)(n-2)\cdots(n-(k-1))}{k(k-1)(k-2)\cdots 1}.$$

We say $P(n, k)$ counts *permutations*, and $\binom{n}{k}$ counts *combinations*. The formulas for each are very similar, there is just an extra $k!$ in the denominator of $\binom{n}{k}$. That extra $k!$ accounts for the fact that $\binom{n}{k}$ does not distinguish between the different orders that the k objects can appear in. We are just selecting (or choosing) the k objects, not arranging them. Perhaps "combination" is a misleading label. We don't mean it like a combination lock (where the order would definitely matter). Perhaps a better metaphor is a combination of flavors — you just need to decide which flavors to combine, not the order in which to combine them.

To further illustrate the connection between combinations and permutations, we close with an example.

Example 1.3.5

You decide to have a dinner party. Even though you are incredibly popular and have 14 different friends, you only have enough chairs to invite 6 of them.

1. How many choices do you have for which 6 friends to invite?
2. What if you need to decide not only which friends to invite but also where to seat them along your long table? How many choices do you have then?

Solution.

1. You must simply choose 6 friends from a group of 14. This can be done in $\binom{14}{6}$ ways. We can find this number either by using Pascal's triangle or the closed formula: $\frac{14!}{8!\cdot 6!} = 3003$.
2. Here you must count all the ways you can permute 6 friends chosen from a group of 14. So the answer is $P(14,6)$, which can be calculated as $\frac{14!}{8!} = 2162160$.

   Notice that we can think of this counting problem as a question about counting functions: how many injective functions are there from your set of 6 chairs to your set of 14 friends (the functions are injective because you can't have a single chair go to two of your friends).

How are these numbers related? Notice that $P(14,6)$ is *much* larger than $\binom{14}{6}$. This makes sense. $\binom{14}{6}$ picks 6 friends, but $P(14,6)$ arranges the 6 friends as well as picks them. In fact, we can say exactly how much larger $P(14,6)$ is. In both counting problems we choose 6 out of 14 friends. For the first one, we stop there, at 3003 ways. But for the second counting problem, each of those 3003 choices of 6 friends can be arranged in exactly 6! ways. So now we have $3003 \cdot 6!$ choices and that is exactly 2162160.

Alternatively, look at the first problem another way. We want to select 6 out of 14 friends, but we do not care about the order they are selected in. To select 6 out of 14 friends, we might try this:

$$14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9.$$

This is a reasonable guess, since we have 14 choices for the first guest, then 13 for the second, and so on. But the guess is wrong (in fact, that product is exactly $2162160 = P(14,6)$). It distinguishes between the different orders in which we could invite the guests.

To correct for this, we could divide by the number of different arrangements of the 6 guests (so that all of these would count as just one outcome). There are precisely 6! ways to arrange 6 guests, so the correct answer to the first question is

$$\frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{6!}.$$

Note that another way to write this is

$$\frac{14!}{8! \cdot 6!}.$$

which is what we had originally.

Exercises

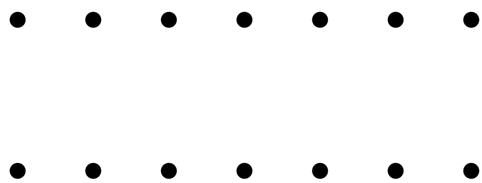
1. A pizza parlor offers 10 toppings.
   (a) How many 3-topping pizzas could they put on their menu? Assume double toppings are not allowed.
   (b) How many total pizzas are possible, with between zero and ten toppings (but not double toppings) allowed?
   (c) The pizza parlor will list the 10 toppings in two equal-sized columns on their menu. How many ways can they arrange the toppings in the left column?
2. A combination lock consists of a dial with 40 numbers on it. To open the lock, you turn the dial to the right until you reach a first number, then to the left until you get to second number, then to the right again to the third number. The numbers must be distinct. How many different combinations are possible?
3. Using the digits 2 through 8, find the number of different 5-digit numbers such that:
   (a) Digits can be used more than once.
   (b) Digits cannot be repeated, but can come in any order.
   (c) Digits cannot be repeated and must be written in increasing order.
   (d) Which of the above counting questions is a combination and which is a permutation? Explain why this makes sense.

4. In an attempt to clean up your room, you have purchased a new floating shelf to put some of your 17 books you have stacked in a corner. These books are all by different authors. The new book shelf is large enough to hold 10 of the books.

(a) How many ways can you select and arrange 10 of the 17 books on the shelf? Notice that here we will allow the books to end up in any order. Explain.

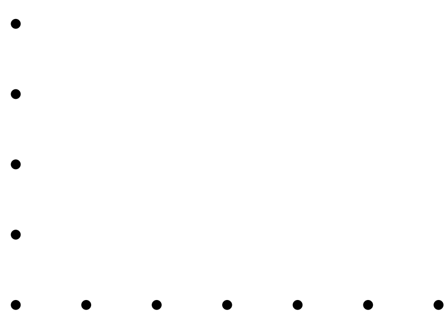
(b) How many ways can you arrange 10 of the 17 books on the shelf if you insist they must be arranged alphabetically by author? Explain.

5. Suppose you wanted to draw a quadrilateral using the dots below as vertices (corners). The dots are spaced one unit apart horizontally and two units apart vertically.



How many quadrilaterals are possible?
How many are squares?
How many are rectangles?
How many are parallelograms?
How many are trapezoids? (Here, as in calculus, a trapezoid is defined as a quadrilateral with *at least* one pair of parallel sides. In particular, parallelograms are trapezoids.)
How many are trapezoids that are not parallelograms?

6. How many triangles are there with vertices from the points shown below? Note, we are not allowing degenerate triangles - ones with all three vertices on the same line, but we do allow non-right triangles. Explain why your answer is correct.



7. An *anagram* of a word is just a rearrangement of its letters. How many different anagrams of "uncopyrightable" are there? (This happens to be the longest common English word without any repeated letters.)

8. How many anagrams are there of the word "assesses" that start with the letter "a"?

9. How many anagrams are there of "anagram"?

10. On a business retreat, your company of 20 executives go golfing.

(a) You need to divide up into foursomes (groups of 4 people): a first foursome, a second foursome, and so on. How many ways can you do this?

(b) After all your hard work, you realize that in fact, you want each foursome to include one of the five Board members. How many ways can you do this?

11. How many different seating arrangements are possible for King Arthur and his 9 knights around their round table?

12. Consider sets A and B with $|A| = 10$ and $|B| = 17$.

(a) How many functions $f : A \to B$ are there?

(b) How many functions $f : A \to B$ are injective?

13. Consider functions $f : \{1,2,3,4\} \to \{1,2,3,4,5,6\}$.

(a) How many functions are there total?

(b) How many functions are injective?

(c) How many of the injective functions are *increasing*? To be increasing means that if $a < b$ then $f(a) < f(b)$, or in other words, the outputs get larger as the inputs get larger.

14. We have seen that the formula for $P(n,k)$ is $\dfrac{n!}{(n-k)!}$. Your task here is to explain *why* this is the right formula.

(a) Suppose you have 12 chips, each a different color. How many different stacks of 5 chips can you make? Explain your answer and why it is the same as using the formula for $P(12,5)$.

(b) Using the scenario of the 12 chips again, what does 12! count? What does 7! count? Explain.

(c) Explain why it makes sense to divide 12! by 7! when computing $P(12,5)$ (in terms of the chips).

(d) Does your explanation work for numbers other than 12 and 5? Explain the formula $P(n,k) = \frac{n!}{(n-k)!}$ using the variables n and k.

1.4 Combinatorial Proofs

Investigate!

1. The Stanley Cup is decided in a best of 7 tournament between two teams. In how many ways can your team win? Let's answer this question two ways:

   (a) How many of the 7 games does your team need to win? How many ways can this happen?

   (b) What if the tournament goes all 7 games? So you win the last game. How many ways can the first 6 games go down?

   (c) What if the tournament goes just 6 games? How many ways can this happen? What about 5 games? 4 games?

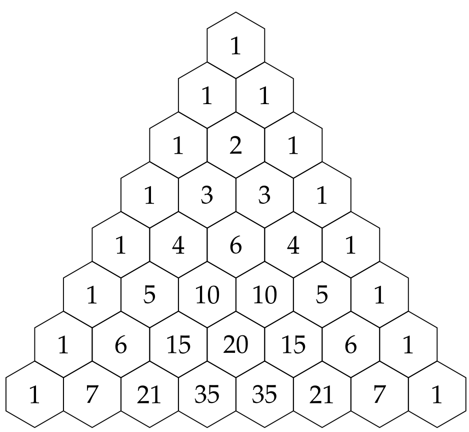
   (d) What are the two different ways to compute the number of ways your team can win? Write down an equation involving binomial coefficients (that is, $\binom{n}{k}$'s). What pattern in Pascal's triangle is this an example of?

2. Generalize. What if the rules changed and you played a best of 9 tournament (5 wins required)? What if you played an n game tournament with k wins required to be named champion?

Attempt the above activity before proceeding

Patterns in Pascal's Triangle

Have a look again at Pascal's triangle. Forget for a moment where it comes from. Just look at it as a mathematical object. What do you notice?



There are lots of patterns hidden away in the triangle, enough to fill a reasonably sized book. Here are just a few of the most obvious ones:

1. The entries on the border of the triangle are all 1.
2. Any entry not on the border is the sum of the two entries above it.
3. The triangle is symmetric. In any row, entries on the left side are mirrored on the right side.
4. The sum of all entries on a given row is a power of 2. (You should check this!)

We would like to state these observations in a more precise way, and then prove that they are correct. Now each entry in Pascal's triangle is in fact a binomial coefficient. The 1 on the very top of the triangle is $\binom{0}{0}$. The next row (which we will call row 1, even though it is not the top-most row) consists of $\binom{1}{0}$ and $\binom{1}{1}$. Row 4 (the row 1, 4, 6, 4, 1) consists of the binomial coefficients

$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}.$$

Given this description of the elements in Pascal's triangle, we can rewrite the above observations as follows:

1. $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$.
2. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
3. $\binom{n}{k} = \binom{n}{n-k}$.
4. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

Each of these is an example of a **binomial identity** : an identity (i.e., equation) involving binomial coefficients.

Our goal is to establish these identities. We wish to prove that they hold for all values of n and k. These proofs can be done in many ways. One option would be to give algebraic proofs, using the formula for $\binom{n}{k}$:

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}.$$

Here's how you might do that for the second identity above.

Example 1.4.1

Give an algebraic proof for the binomial identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Solution.

Proof. By the definition of $\binom{n}{k}$, we have

$$\binom{n-1}{k-1} = \frac{(n-1)!}{(n-1-(k-1))!(k-1)!} = \frac{(n-1)!}{(n-k)!(k-1)!}$$

and

$$\binom{n-1}{k} = \frac{(n-1)!}{(n-1-k)!k!}.$$

Thus, starting with the right-hand side of the equation:

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(n-k)!(k-1)!} + \frac{(n-1)!}{(n-1-k)!\,k!} \\ &= \frac{(n-1)!k}{(n-k)!\,k!} + \frac{(n-1)!(n-k)}{(n-k)!\,k!} \\ &= \frac{(n-1)!(k+n-k)}{(n-k)!\,k!} \\ &= \frac{n!}{(n-k)!\,k!} \\ &= \binom{n}{k}.\end{aligned}$$

The second line (where the common denominator is found) works because $k(k-1)! = k!$ and $(n-k)(n-k-1)! = (n-k)!$. ■

This is certainly a valid proof, but also is entirely useless. Even if you understand the proof perfectly, it does not tell you *why* the identity is true. A better approach would be to explain what $\binom{n}{k}$ *means* and then say why that is also what $\binom{n-1}{k-1} + \binom{n-1}{k}$ means. Let's see how this works for the four identities we observed above.

Example 1.4.2

Explain why $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$.

Solution. What do these binomial coefficients tell us? Well, $\binom{n}{0}$ gives the number of ways to select 0 objects from a collection of n objects. There is only one way to do this, namely to not select any of the objects. Thus $\binom{n}{0} = 1$. Similarly, $\binom{n}{n}$ gives the number of ways to select n objects from a collection of n objects. There is only one way to do this: select all n objects. Thus $\binom{n}{n} = 1$.

Alternatively, we know that $\binom{n}{0}$ is the number of n-bit strings with weight 0. There is only one such string, the string of all 0's. So

$\binom{n}{0} = 1$. Similarly $\binom{n}{n}$ is the number of n-bit strings with weight n. There is only one string with this property, the string of all 1's.

Another way: $\binom{n}{0}$ gives the number of subsets of a set of size n containing 0 elements. There is only one such subset, the empty set. $\binom{n}{n}$ gives the number of subsets containing n elements. The only such subset is the original set (of all elements).

Example 1.4.3

Explain why $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Solution. The easiest way to see this is to consider bit strings. $\binom{n}{k}$ is the number of bit strings of length n containing k 1's. Of all of these strings, some start with a 1 and the rest start with a 0. First consider all the bit strings which start with a 1. After the 1, there must be $n-1$ more bits (to get the total length up to n) and exactly $k-1$ of them must be 1's (as we already have one, and we need k total). How many strings are there like that? There are exactly $\binom{n-1}{k-1}$ such bit strings, so of all the length n bit strings containing k 1's, $\binom{n-1}{k-1}$ of them start with a 1. Similarly, there are $\binom{n-1}{k}$ which start with a 0 (we still need $n-1$ bits and now k of them must be 1's). Since there are $\binom{n-1}{k}$ bit strings containing $n-1$ bits with k 1's, that is the number of length n bit strings with k 1's which start with a 0. Therefore $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Another way: consider the question, how many ways can you select k pizza toppings from a menu containing n choices? One way to do this is just $\binom{n}{k}$. Another way to answer the same question is to first decide whether or not you want anchovies. If you do want anchovies, you still need to pick $k-1$ toppings, now from just $n-1$ choices. That can be done in $\binom{n-1}{k-1}$ ways. If you do not want anchovies, then you still need to select k toppings from $n-1$ choices (the anchovies are out). You can do that in $\binom{n-1}{k}$ ways. Since the choices with anchovies are disjoint from the choices without anchovies, the total choices are $\binom{n-1}{k-1} + \binom{n-1}{k}$. But wait. We answered the same question in two different ways, so the two answers must be the same. Thus $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

You can also explain (prove) this identity by counting subsets, or even lattice paths.

Example 1.4.4

Prove the binomial identity $\binom{n}{k} = \binom{n}{n-k}$.

Solution. Why is this true? $\binom{n}{k}$ counts the number of ways to select k things from n choices. On the other hand, $\binom{n}{n-k}$ counts the number of ways to select $n-k$ things from n choices. Are these really the same? Well, what if instead of selecting the $n-k$ things you choose to exclude them. How many ways are there to choose $n-k$ things to exclude from n choices. Clearly this is $\binom{n}{n-k}$ as well (it doesn't matter whether you include or exclude the things once you have chosen them). And if you exclude $n-k$ things, then you are including the other k things. So the set of outcomes should be the same.

Let's try the pizza counting example like we did above. How many ways are there to pick k toppings from a list of n choices? On the one hand, the answer is simply $\binom{n}{k}$. Alternatively, you could make a list of all the toppings you don't want. To end up with a pizza containing exactly k toppings, you need to pick $n-k$ toppings to not put on the pizza. You have $\binom{n}{n-k}$ choices for the toppings you don't want. Both of these ways give you a pizza with k toppings, in fact all the ways to get a pizza with k toppings. Thus these two answers must be the same: $\binom{n}{k} = \binom{n}{n-k}$.

You can also prove (explain) this identity using bit strings, subsets, or lattice paths. The bit string argument is nice: $\binom{n}{k}$ counts the number of bit strings of length n with k 1's. This is also the number of bit string of length n with k 0's (just replace each 1 with a 0 and each 0 with a 1). But if a string of length n has k 0's, it must have $n-k$ 1's. And there are exactly $\binom{n}{n-k}$ strings of length n with $n-k$ 1's.

Example 1.4.5

Prove the binomial identity $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

Solution. Let's do a "pizza proof" again. We need to find a question about pizza toppings which has 2^n as the answer. How about this: If a pizza joint offers n toppings, how many pizzas can you build using any number of toppings from no toppings to all toppings, using each topping at most once?

On one hand, the answer is 2^n. For each topping you can say "yes" or "no," so you have two choices for each topping.

On the other hand, divide the possible pizzas into disjoint groups: the pizzas with no toppings, the pizzas with one topping, the pizzas with two toppings, etc. If we want no toppings, there is only one pizza like that (the empty pizza, if you will) but it would be better to think of that number as $\binom{n}{0}$ since we choose 0 of the n toppings. How many pizzas have 1 topping? We need to choose 1 of the n toppings, so $\binom{n}{1}$. We have:

- Pizzas with 0 toppings: $\binom{n}{0}$
- Pizzas with 1 topping: $\binom{n}{1}$
- Pizzas with 2 toppings: $\binom{n}{2}$
- $\vdots$
- Pizzas with n toppings: $\binom{n}{n}$.

The total number of possible pizzas will be the sum of these, which is exactly the left-hand side of the identity we are trying to prove.

Again, we could have proved the identity using subsets, bit strings, or lattice paths (although the lattice path argument is a little tricky).

Hopefully this gives some idea of how explanatory proofs of binomial identities can go. It is worth pointing out that more traditional proofs can also be beautiful.[2] For example, consider the following rather slick proof of the last identity.

Expand the binomial $(x+y)^n$:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x \cdot y^n + \binom{n}{n}y^n.$$

Let $x = 1$ and $y = 1$. We get:

$$(1+1)^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n-1}1 \cdot 1^n + \binom{n}{n}1^n.$$

Of course this simplifies to:

$$(2)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}.$$

Something fun to try: Let $x = 1$ and $y = 2$. Neat huh?

[2]Most every binomial identity can be proved using mathematical induction, using the recursive definition for $\binom{n}{k}$. We will discuss induction in Section 2.5.

More Proofs

The explanatory proofs given in the above examples are typically called **combinatorial proofs**. In general, to give a combinatorial proof for a binomial identity, say $A = B$ you do the following:

1. Find a counting problem you will be able to answer in two ways.
2. Explain why one answer to the counting problem is A.
3. Explain why the other answer to the counting problem is B.

Since both A and B are the answers to the same question, we must have $A = B$.

The tricky thing is coming up with the question. This is not always obvious, but it gets easier the more counting problems you solve. You will start to recognize types of answers as the answers to types of questions. More often what will happen is you will be solving a counting problem and happen to think up two different ways of finding the answer. Now you have a binomial identity and the proof is right there. The proof *is* the problem you just solved together with your two solutions.

For example, consider this counting question:

> How many 10-letter words use exactly four A's, three B's, two C's and one D?

Let's try to solve this problem. We have 10 spots for letters to go. Four of those need to be A's. We can pick the four A-spots in $\binom{10}{4}$ ways. Now where can we put the B's? Well there are only 6 spots left, we need to pick 3 of them. This can be done in $\binom{6}{3}$ ways. The two C's need to go in two of the 3 remaining spots, so we have $\binom{3}{2}$ ways of doing that. That leaves just one spot of the D, but we could write that 1 choice as $\binom{1}{1}$. Thus the answer is:

$$\binom{10}{4}\binom{6}{3}\binom{3}{2}\binom{1}{1}.$$

But why stop there? We can find the answer another way too. First let's decide where to put the one D: we have 10 spots, we need to choose 1 of them, so this can be done in $\binom{10}{1}$ ways. Next, choose one of the $\binom{9}{2}$ ways to place the two C's. We now have 7 spots left, and three of them need to be filled with B's. There are $\binom{7}{3}$ ways to do this. Finally the A's can be placed in $\binom{4}{4}$ (that is, only one) ways. So another answer to the question is

$$\binom{10}{1}\binom{9}{2}\binom{7}{3}\binom{4}{4}.$$

Interesting. This gives us the binomial identity:

$$\binom{10}{4}\binom{6}{3}\binom{3}{2}\binom{1}{1} = \binom{10}{1}\binom{9}{2}\binom{7}{3}\binom{4}{4}.$$

Here are a couple more binomial identities with combinatorial proofs.

Example 1.4.6

Prove the identity

$$1n + 2(n-1) + 3(n-2) + \cdots + (n-1)2 + n1 = \binom{n+2}{3}.$$

Solution. To give a combinatorial proof we need to think up a question we can answer in two ways: one way needs to give the left-hand-side of the identity, the other way needs to be the right-hand-side of the identity. Our clue to what question to ask comes from the right-hand side: $\binom{n+2}{3}$ counts the number of ways to select 3 things from a group of $n + 2$ things. Let's name those things $1, 2, 3, \ldots, n + 2$. In other words, we want to find 3-element subsets of those numbers (since order should not matter, subsets are exactly the right thing to think about). We will have to be a bit clever to explain why the left-hand-side also gives the number of these subsets. Here's the proof.

Proof. Consider the question "How many 3-element subsets are there of the set $\{1, 2, 3, \ldots, n + 2\}$?" We answer this in two ways:

Answer 1: We must select 3 elements from the collection of $n+2$ elements. This can be done in $\binom{n+2}{3}$ ways.

Answer 2: Break this problem up into cases by what the middle number in the subset is. Say each subset is $\{a, b, c\}$ written in increasing order. We count the number of subsets for each distinct value of b. The smallest possible value of b is 2, and the largest is $n + 1$.

When $b = 2$, there are $1 \cdot n$ subsets: 1 choice for a and n choices (3 through $n + 2$) for c.

When $b = 3$, there are $2 \cdot (n-1)$ subsets: 2 choices for a and $n-1$ choices for c.

When $b = 4$, there are $3 \cdot (n-2)$ subsets: 3 choices for a and $n-2$ choices for c.

And so on. When $b = n + 1$, there are n choices for a and only 1 choice for c, so $n \cdot 1$ subsets.

Therefore the total number of subsets is

$$1n + 2(n-1) + 3(n-2) + \cdots + (n-1)2 + n1.$$

Since Answer 1 and Answer 2 are answers to the same question, they must be equal. Therefore

$$1n + 2(n-1) + 3(n-2) + \cdots + (n-1)2 + n1 = \binom{n+2}{3}.$$

■

Example 1.4.7

Prove the binomial identity

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Solution. We will give two different proofs of this fact. The first will be very similar to the previous example (counting subsets). The second proof is a little slicker, using lattice paths.

Proof. Consider the question: "How many pizzas can you make using n toppings when there are $2n$ toppings to choose from?"

Answer 1: There are $2n$ toppings, from which you must choose n. This can be done in $\binom{2n}{n}$ ways.

Answer 2: Divide the toppings into two groups of n toppings (perhaps n meats and n veggies). Any choice of n toppings must include some number from the first group and some number from the second group. Consider each possible number of meat toppings separately:

0 meats: $\binom{n}{0}\binom{n}{n}$, since you need to choose 0 of the n meats and n of the n veggies.

1 meat: $\binom{n}{1}\binom{n}{n-1}$, since you need 1 of n meats so $n-1$ of n veggies.

2 meats: $\binom{n}{2}\binom{n}{n-2}$. Choose 2 meats and the remaining $n-2$ toppings from the n veggies.

And so on. The last case is n meats, which can be done in $\binom{n}{n}\binom{n}{0}$ ways.

Thus the total number of pizzas possible is

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}.$$

This is not quite the left-hand side ... yet. Notice that $\binom{n}{n} = \binom{n}{0}$ and $\binom{n}{n-1} = \binom{n}{1}$ and so on, by the identity in Example 1.4.4. Thus we do indeed get

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2.$$

Since these two answers are answers to the same question, they must be equal, and thus

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

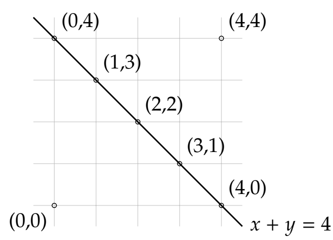
■

For an alternative proof, we use lattice paths. This is reasonable to consider because the right-hand side of the identity reminds us of the number of paths from $(0,0)$ to (n,n).

Proof. Consider the question: How many lattice paths are there from $(0,0)$ to (n,n)?

Answer 1: We must travel $2n$ steps, and n of them must be in the up direction. Thus there are $\binom{2n}{n}$ paths.

Answer 2: Note that any path from $(0,0)$ to (n,n) must cross the line $x+y=n$. That is, any path must pass through exactly one of the points: $(0,n), (1,n-1), (2,n-2), \ldots, (n,0)$. For example, this is what happens in the case $n=4$:



How many paths pass through $(0,n)$? To get to that point, you must travel n units, and 0 of them are to the right, so there are $\binom{n}{0}$ ways to get to $(0,n)$. From $(0,n)$ to (n,n) takes n steps, and 0 of them are up. So there are $\binom{n}{0}$ ways to get from $(0,n)$ to (n,n). Therefore there are $\binom{n}{0}\binom{n}{0}$ paths from $(0,0)$ to (n,n) through the point $(0,n)$.

What about through $(1,n-1)$. There are $\binom{n}{1}$ paths to get there (n steps, 1 to the right) and $\binom{n}{1}$ paths to complete the journey to (n,n) (n steps, 1 up). So there are $\binom{n}{1}\binom{n}{1}$ paths from $(0,0)$ to (n,n) through $(1,n-1)$.

In general, to get to (n,n) through the point $(k,n-k)$ we have $\binom{n}{k}$ paths to the midpoint and then $\binom{n}{k}$ paths from the midpoint to (n, n). So there are $\binom{n}{k}\binom{n}{k}$ paths from $(0, 0)$ to (n, n) through $(k, n-k)$.

All together then the total paths from $(0, 0)$ to (n, n) passing through exactly one of these midpoints is

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2.$$

Since these two answers are answers to the same question, they must be equal, and thus

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

■

Exercises

1. Give a combinatorial proof of the identity $2 + 2 + 2 = 3 \cdot 2$.

2. Suppose you own x fezzes and y bow ties. Of course, x and y are both greater than 1.

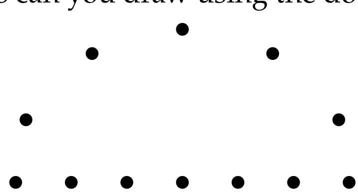
(a) How many combinations of fez and bow tie can you make? You can wear only one fez and one bow tie at a time. Explain.

(b) Explain why the answer is *also* $\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2}$. (If this is what you claimed the answer was in part (a), try it again.)

(c) Use your answers to parts (a) and (b) to give a combinatorial proof of the identity

$$\binom{x+y}{2} - \binom{x}{2} - \binom{y}{2} = xy..$$

3. How many triangles can you draw using the dots below as vertices?

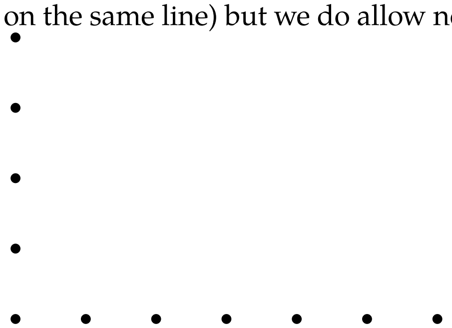


(a) Find an expression for the answer which is the sum of three terms involving binomial coefficients.

(b) Find an expression for the answer which is the difference of two binomial coefficients.

(c) Generalize the above to state and prove a binomial identity using a combinatorial proof. Say you have x points on the horizontal axis and y points in the semi-circle.

4. Consider all the triangles you can create using the points shown below as vertices. Note, we are not allowing degenerate triangles (ones with all three vertices on the same line) but we do allow non-right triangles.



(a) Find the number of triangles, and explain why your answer is correct.

(b) Find the number of triangles again, using a different method. Explain why your new method works.

(c) State a binomial identity that your two answers above establish (that is, give the binomial identity that your two answers a proof for). Then generalize this using m's and n's.

5. A woman is getting married. She has 15 best friends but can only select 6 of them to be her bridesmaids, one of which needs to be her maid of honor. How many ways can she do this?

(a) What if she first selects the 6 bridesmaids, and then selects one of them to be the maid of honor?

(b) What if she first selects her maid of honor, and then 5 other bridesmaids?

(c) Explain why $6\binom{15}{6} = 15\binom{14}{5}$.

6. Consider the identity:

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

(a) Is this true? Try it for a few values of n and k.

(b) Use the formula for $\binom{n}{k}$ to give an algebraic proof of the identity.

(c) Give a combinatorial proof of the identity.

7. Give a combinatorial proof of the identity $\binom{n}{2}\binom{n-2}{k-2} = \binom{n}{k}\binom{k}{2}$.

8. Consider the binomial identity

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

(a) Give a combinatorial proof of this identity. Hint: What if some number of a group of n people wanted to go to an escape room, and among those going, one needed to be the team captain?

(b) Give an alternate proof by multiplying out $(1+x)^n$ and taking derivatives of both sides.

9. Give a combinatorial proof for the identity $1 + 2 + 3 + \cdots + n = \binom{n+1}{2}$.

10. Consider the bit strings in $\mathbf{B}_2^6$ (bit strings of length 6 and weight 2).

(a) How many of those bit strings start with 1?

(b) How many of those bit strings start with 01?

(c) How many of those bit strings start with 001?

(d) Are there any other strings we have not counted yet? Which ones, and how many are there?

(e) How many bit strings are there total in $\mathbf{B}_2^6$?

(f) What binomial identity have you just given a combinatorial proof for?

11. Let's count **ternary** digit strings, that is, strings in which each digit can be 0, 1, or 2.

(a) How many ternary digit strings contain exactly n digits?

(b) How many ternary digit strings contain exactly n digits and n 2's.

(c) How many ternary digit strings contain exactly n digits and $n-1$ 2's. (Hint: where can you put the non-2 digit, and then what could it be?)

(d) How many ternary digit strings contain exactly n digits and $n-2$ 2's. (Hint: see previous hint)

(e) How many ternary digit strings contain exactly n digits and $n-k$ 2's.

(f) How many ternary digit strings contain exactly n digits and no 2's. (Hint: what kind of a string is this?)

(g) Use the above parts to give a combinatorial proof for the identity

$$\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + 2^3\binom{n}{3} + \cdots + 2^n\binom{n}{n} = 3^n.$$

12. How many ways are there to rearrange the letters in the word "rearrange"? Answer this question in at least two different ways to establish a binomial identity.

13. Establish the identity below using a combinatorial proof.

$$\binom{2}{2}\binom{n}{2} + \binom{3}{2}\binom{n-1}{2} + \binom{4}{2}\binom{n-2}{2} + \cdots + \binom{n}{2}\binom{2}{2} = \binom{n+3}{5}.$$

14. In Example 1.4.5 we established that the sum of any row in Pascal's triangle is a power of two. Specifically,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

The argument given there used the counting question, "how many pizzas can you build using any number of n different toppings?" To practice, give new proofs of this identity using different questions.

(a) Use a question about counting subsets.

(b) Use a question about counting bit strings.

(c) Use a question about counting lattice paths.

1.5 Stars and Bars

Investigate!

Suppose you have some number of identical Rubik's cubes to distribute to your friends. Imagine you start with a single row of the cubes.

1. Find the number of different ways you can distribute the cubes provided:

   (a) You have 3 cubes to give to 2 people.

   (b) You have 4 cubes to give to 2 people.

   (c) You have 5 cubes to give to 2 people.

   (d) You have 3 cubes to give to 3 people.

   (e) You have 4 cubes to give to 3 people.

   (f) You have 5 cubes to give to 3 people.

2. Make a conjecture about how many different ways you could distribute 7 cubes to 4 people. Explain.

3. What if each person were required to get *at least one* cube? How would your answers change?

Attempt the above activity before proceeding

Consider the following counting problem:

> You have 7 cookies to give to 4 kids. How many ways can you do this?

Take a moment to think about how you might solve this problem. You may assume that it is acceptable to give a kid no cookies. Also, the cookies are all identical and the order in which you give out the cookies does not matter.

Before solving the problem, here is a wrong answer: You might guess that the answer should be 4^7 because for each of the 7 cookies, there are 4 choices of kids to which you can give the cookie. This is reasonable, but wrong. To see why, consider a few possible outcomes: we could assign the first six cookies to kid A, and the seventh cookie to kid B. Another outcome would assign the first cookie to kid B and the six remaining cookies to kid A. Both outcomes are included in the 4^7 answer. But for our counting problem, both outcomes are really the same – kid A gets six cookies and kid B gets one cookie.

What do outcomes actually look like? How can we represent them? One approach would be to write an outcome as a string of four numbers like this:

$$3112,$$

which represent the outcome in which the first kid gets 3 cookies, the second and third kid each get 1 cookie, and the fourth kid gets 2 cookies. Represented this way, the order in which the numbers occur matters. 1312 is a different outcome, because the first kid gets a one cookie instead of 3. Each number in the string can be any integer between 0 and 7. But the answer is not 7^4. We need the *sum* of the numbers to be 7.

Another way we might represent outcomes is to write a string of seven letters:

ABAADCD,

which represents that the first cookie goes to kid A, the second cookie goes to kid B, the third and fourth cookies go to kid A, and so on. In fact, this outcome is identical to the previous one—A gets 3 cookies, B and C get 1 each and D gets 2. Each of the seven letters in the string can be any of the 4 possible letters (one for each kid), but the number of such strings is not 4^7, because here order does *not* matter. In fact, another way to write the same outcome is

AAABCDD.

This will be the preferred representation of the outcome. Since we can write the letters in any order, we might as well write them in *alphabetical* order for the purposes of counting. So we will write all the A's first, then all the B's, and so on.

Now think about how you could specify such an outcome. All we really need to do is say when to switch from one letter to the next. In terms of cookies, we need to say after how many cookies do we stop giving cookies to the first kid and start giving cookies to the second kid. And then after how many do we switch to the third kid? And after how many do we switch to the fourth? So yet another way to represent an outcome is like this:

$$***|*|*|**.$$

Three cookies go to the first kid, then we switch and give one cookie to the second kid, then switch, one to the third kid, switch, two to the fourth kid. Notice that we need 7 stars and 3 bars – one star for each cookie, and one bar for each switch between kids, so one fewer bars than there are kids (we don't need to switch after the last kid – we are done).

Why have we done all of this? Simple: to count the number of ways to distribute 7 cookies to 4 kids, all we need to do is count how many *stars and bars* charts there are. But a **stars and bars chart** is just a string of symbols, some stars and some bars. If instead of stars and bars we would use 0's and 1's, it would just be a bit string. We know how to count those.

Before we get too excited, we should make sure that really *any* string of (in our case) 7 stars and 3 bars corresponds to a different way to distribute cookies to kids. In particular consider a string like this:

$$| * * * || * * * *.$$

Does that correspond to a cookie distribution? Yes. It represents the distribution in which kid A gets 0 cookies (because we switch to kid B before any stars), kid B gets three cookies (three stars before the next bar), kid C gets 0 cookies (no stars before the next bar) and kid D gets the remaining 4 cookies. No matter how the stars and bars are arranged, we can distribute cookies in that way. Also, given any way to distribute cookies, we can represent that with a stars and bars chart. For example, the distribution in which kid A gets 6 cookies and kid B gets 1 cookie has the following chart:

$$* * * * * * | * ||.$$

After all that work we are finally ready to count. Each way to distribute cookies corresponds to a stars and bars chart with 7 stars and 3 bars. So there are 10 symbols, and we must choose 3 of them to be bars. Thus:

$$\text{There are } \binom{10}{3} \text{ ways to distribute 7 cookies to 4 kids.}$$

While we are at it, we can also answer a related question: how many ways are there to distribute 7 cookies to 4 kids so that each kid gets at least one cookie? What can you say about the corresponding stars and bars charts? The charts must start and end with at least one star (so that kids A and D) get cookies, and also no two bars can be adjacent (so that kids B and C are not skipped). One way to assure this is to place bars only in the spaces *between* the stars. With 7 stars, there are 6 spots between the stars, so we must choose 3 of those 6 spots to fill with bars. Thus there are $\binom{6}{3}$ ways to distribute 7 cookies to 4 kids giving at least one cookie to each kid.

Another (and more general) way to approach this modified problem is to first give each kid one cookie. Now the remaining 3 cookies can be distributed to the 4 kids without restrictions. So we have 3 stars and 3 bars for a total of 6 symbols, 3 of which must be bars. So again we see that there are $\binom{6}{3}$ ways to distribute the cookies.

Stars and bars can be used in counting problems other than kids and cookies. Here are a few examples:

Example 1.5.1

Your favorite mathematical ice-cream parlor offers 10 flavors. How many milkshakes could you create using exactly 6, not necessarily distinct scoops? The order you add the flavors does not matter (they will be blended up anyway) but you are allowed repeats. So one possible shake is triple chocolate, double cherry, and mint chocolate chip.

Solution. We get six scoops, each of which could be one of ten possible flavors. Represent each scoop as a star. Think of going down the counter one flavor at a time: you see vanilla first, and skip to the next, chocolate. You say yes to chocolate three times (use three stars), then switch to the next flavor. You keep skipping until you get to cherry, which you say yes to twice. Another switch and you are at mint chocolate chip. You say yes once. Then you keep switching until you get past the last flavor, never saying yes again (since you already have said yes six times). There are ten flavors to choose from, so we must switch from considering one flavor to the next nine times. These are the nine bars.

Now that we are confident that we have the right number of stars and bars, we answer the question simply: there are 6 stars and 9 bars, so 15 symbols. We need to pick 9 of them to be bars, so the number of milkshakes possible is $\binom{15}{9}$.

Example 1.5.2

How many 7 digit phone numbers are there in which the digits are non-increasing? That is, every digit is less than or equal to the previous one.

Solution. We need to decide on 7 digits so we will use 7 stars. The bars will represent a switch from each possible single digit number down to the next smaller one. So the phone number 866-5221 is represented by the stars and bars chart

$$| * || * * | * ||| * * | * |.$$

There are 10 choices for each digit (0-9) so we must switch between choices 9 times. We have 7 stars and 9 bars, so the total number of phone numbers is $\binom{16}{9}$.

Example 1.5.3

How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13.$$

(An **integer solution** to an equation is a solution in which the unknown must have an integer value.)

1. where $x_i \geq 0$ for each x_i?
2. where $x_i > 0$ for each x_i?
3. where $x_i \geq 2$ for each x_i?

Solution. This problem is just like giving 13 cookies to 5 kids. We need to say how many of the 13 units go to each of the 5 variables. In other words, we have 13 stars and 4 bars (the bars are like the "+" signs in the equation).

1. If x_i can be 0 or greater, we are in the standard case with no restrictions. So 13 stars and 4 bars can be arranged in $\binom{17}{4}$ ways.
2. Now each variable must be at least 1. So give one unit to each variable to satisfy that restriction. Now there are 8 stars left, and still 4 bars, so the number of solutions is $\binom{12}{4}$.
3. Now each variable must be 2 or greater. So before any counting, give each variable 2 units. We now have 3 remaining stars and 4 bars, so there are $\binom{7}{4}$ solutions.

Counting with Functions.

Many of the counting problems in this section might at first appear to be examples of counting *functions*. After all, when we try to count the number of ways to distribute cookies to kids, we are assigning each cookie to a kid, just like you assign elements of the domain of a function to elements in the codomain. However, the number of ways to assign 7 cookies to 4 kids is $\binom{10}{7} = 120$, while the number of functions $f : \{1, 2, 3, 4, 5, 6, 7\} \to \{a, b, c, d\}$ is $4^7 = 16384$. What is going on here?

When we count functions, we consider the following two functions, for example, to be different:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ a & b & c & c & c & c & c \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ b & a & c & c & c & c & c \end{pmatrix}.$$

But these two functions would correspond to the *same* cookie distribution: kids a and b each get one cookie, kid c gets the rest (and none for kid d).

The point: elements of the domain are distinguished, cookies are indistinguishable. This is analogous to the distinction between permutations (like counting functions) and combinations (not).

EXERCISES

1. A multiset is a collection of objects, just like a set, but can contain an object more than once (the order of the elements still doesn't matter). For example, $\{1, 1, 2, 5, 5, 7\}$ is a multiset of size 6.

(a) How many *sets* of size 5 can be made using the 10 numeric digits 0 through 9?

(b) How many *multi*sets of size 5 can be made using the 10 numeric digits 0 through 9?

2. Using the digits 2 through 8, find the number of different 5-digit numbers such that:

(a) Digits cannot be repeated and must be written in increasing order. For example, 23678 is okay, but 32678 is not.

(b) Digits *can* be repeated and must be written in *non-decreasing* order. For example, 24448 is okay, but 24484 is not.

3. Each of the counting problems below can be solved with stars and bars. For each, say what outcome the diagram

$$* * *| * || * *|$$

represents, if there are the correct number of stars and bars for the problem. Otherwise, say why the diagram does not represent any outcome, and what a correct diagram would look like.

(a) How many ways are there to select a handful of 6 jellybeans from a jar that contains 5 different flavors?

(b) How many ways can you distribute 5 identical lollipops to 6 kids?

(c) How many 6-letter words can you make using the 5 vowels in alphabetical order?

(d) How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 6$.

4. After gym class you are tasked with putting the 14 identical dodgeballs away into 5 bins.

(a) How many ways can you do this if there are no restrictions?

(b) How many ways can you do this if each bin must contain at least one dodgeball?

5. How many integer solutions are there to the equation $x + y + z = 8$ for which

(a) x, y, and z are all positive?

(b) x, y, and z are all non-negative?

(c) x, y, and z are all greater than or equal to -3.

6. When playing Yahtzee, you roll five regular 6-sided dice. How many different outcomes are possible from a single roll? The order of the dice does not matter.

7. Your friend tells you she has 7 coins in her hand (just pennies, nickels, dimes and quarters). If you guess how many of each kind of coin she has, she will give them to you. If you guess randomly, what is the probability that you will be correct?

8. How many integer solutions to $x_1 + x_2 + x_3 + x_4 = 25$ are there for which $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$ and $x_4 \geq 4$?

9. Solve the three counting problems below. Then say why it makes sense that they all have the same answer. That is, say how you can interpret them as each other.

(a) How many ways are there to distribute 8 cookies to 3 kids?

(b) How many solutions in non-negative integers are there to $x + y + z = 8$?

(c) How many different packs of 8 crayons can you make using crayons that come in red, blue and yellow?

10. Consider functions $f : \{1, 2, 3, 4, 5\} \to \{0, 1, 2, \ldots, 9\}$.

(a) How many of these functions are strictly increasing? Explain. (A function is strictly increasing provided if $a < b$, then $f(a) < f(b)$.)

(b) How many of the functions are non-decreasing? Explain. (A function is non-decreasing provided if $a < b$, then $f(a) \leq f(b)$.)

11. *Conic*, your favorite math themed fast food drive-in offers 20 flavors which can be added to your soda. You have enough money to buy a large soda with 4 added flavors. How many different soda concoctions can you order if:

(a) You refuse to use any of the flavors more than once?

(b) You refuse repeats but care about the order the flavors are added?

(c) You allow yourself multiple shots of the same flavor?

(d) You allow yourself multiple shots, and care about the order the flavors are added?

1.6 ADVANCED COUNTING USING PIE

Investigate!

You have 11 identical mini key-lime pies to give to 4 children. However, you don't want any kid to get more than 3 pies. How many ways can you distribute the pies?

1. How many ways are there to distribute the pies without any restriction?
2. Let's get rid of the ways that one or more kid gets too many pies. How many ways are there to distribute the pies if Al gets too many pies? What if Bruce gets too many? Or Cat? Or Dent?
3. What if two kids get too many pies? How many ways can this happen? Does it matter which two kids you pick to overfeed?
4. Is it possible that three kids get too many pies? If so, how many ways can this happen?
5. How should you combine all the numbers you found above to answer the original question?

Suppose now you have 13 pies and 7 children. No child can have more than 2 pies. How many ways can you distribute the pies?

STOP **Attempt the above activity before proceeding** STOP

Stars and bars allows us to count the number of ways to distribute 10 cookies to 3 kids and natural number solutions to $x + y + z = 11$, for example. A relatively easy modification allows us to put a *lower bound* restriction on these problems: perhaps each kid must get at least two cookies or $x, y, z \geq 2$. This was done by first assigning each kid (or variable) 2 cookies (or units) and then distributing the rest using stars and bars.

What if we wanted an *upper bound* restriction? For example, we might insist that no kid gets more than 4 cookies or that $x, y, z \leq 4$. It turns out this is considerably harder, but still possible. The idea is to count all the distributions and then remove those that violate the condition. In other words, we must count the number of ways to distribute 11 cookies to 3 kids in which *one or more* of the kids gets more than 4 cookies. For any particular kid, this is not a problem; we do this using stars and bars. But how to combine the number of ways for kid A, or B or C? We must use the PIE.

The Principle of Inclusion/Exclusion (PIE) gives a method for finding the cardinality of the union of not necessarily disjoint sets. We saw in Section 1.1 how this works with three sets. To find how many things are in *one or more* of the sets A, B, and C, we should just add up the number of things in each of these sets. However, if there is any overlap among the sets, those elements are counted multiple times. So we subtract the things in each intersection of a pair of sets. But doing this removes elements which are in all three sets once too often, so we need to add it back in. In terms of cardinality of sets, we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Example 1.6.1

Three kids, Alberto, Bernadette, and Carlos, decide to share 11 cookies. They wonder how many ways they could split the cookies up provided that none of them receive more than 4 cookies (someone receiving no cookies is for some reason acceptable to these kids).

Solution. Without the "no more than 4" restriction, the answer would be $\binom{13}{2}$, using 11 stars and 2 bars (separating the three kids). Now count the number of ways that one or more of the kids violates the condition, i.e., gets at least 4 cookies.

Let A be the set of outcomes in which Alberto gets more than 4 cookies. Let B be the set of outcomes in which Bernadette gets more than 4 cookies. Let C be the set of outcomes in which Carlos gets more than 4 cookies. We then are looking (for the sake of subtraction) for the size of the set $A \cup B \cup C$. Using PIE, we must find the sizes of $|A|$, $|B|$, $|C|$, $|A \cap B|$ and so on. Here is what we find.

- $|A| = \binom{8}{2}$. First give Alberto 5 cookies, then distribute the remaining 6 to the three kids without restrictions, using 6 stars and 2 bars.
- $|B| = \binom{8}{2}$. Just like above, only now Bernadette gets 5 cookies at the start.
- $|C| = \binom{8}{2}$. Carlos gets 5 cookies first.
- $|A \cap B| = \binom{3}{2}$. Give Alberto and Bernadette 5 cookies each, leaving 1 (star) to distribute to the three kids (2 bars).
- $|A \cap C| = \binom{3}{2}$. Alberto and Carlos get 5 cookies first.
- $|B \cap C| = \binom{3}{2}$. Bernadette and Carlos get 5 cookies first.

- $|A \cap B \cap C| = 0$. It is not possible for all three kids to get 4 or more cookies.

Combining all of these we see

$$|A \cup B \cup C| = \binom{8}{2} + \binom{8}{2} + \binom{8}{2} - \binom{3}{2} - \binom{3}{2} - \binom{3}{2} + 0 = 75.$$

Thus the answer to the original question is $\binom{13}{2} - 75 = 78 - 75 = 3$. This makes sense now that we see it. The only way to ensure that no kid gets more than 4 cookies is to give two kids 4 cookies and one kid 3; there are three choices for which kid that should be. We could have found the answer much quicker through this observation, but the point of the example is to illustrate that PIE works!

For four or more sets, we do not write down a formula for PIE. Instead, we just think of the principle: add up all the elements in single sets, then subtract out things you counted twice (elements in the intersection of a *pair* of sets), then add back in elements you removed too often (elements in the intersection of groups of three sets), then take back out elements you added back in too often (elements in the intersection of groups of four sets), then add back in, take back out, add back in, etc. This would be very difficult if it wasn't for the fact that in these problems, all the cardinalities of the single sets are equal, as are all the cardinalities of the intersections of two sets, and that of three sets, and so on. Thus we can group all of these together and multiply by how many different combinations of 1, 2, 3, ... sets there are.

Example 1.6.2

How many ways can you distribute 10 cookies to 4 kids so that no kid gets more than 2 cookies?

Solution. There are $\binom{13}{3}$ ways to distribute 10 cookies to 4 kids (using 10 stars and 3 bars). We will subtract all the outcomes in which a kid gets 3 or more cookies. How many outcomes are there like that? We can force kid A to eat 3 or more cookies by giving him 3 cookies before we start. Doing so reduces the problem to one in which we have 7 cookies to give to 4 kids without any restrictions. In that case, we have 7 stars (the 7 remaining cookies) and 3 bars (one less than the number of kids) so we can distribute the cookies in $\binom{10}{3}$ ways. Of course we could choose any one of the 4 kids to give too many cookies, so it would appear that there are $\binom{4}{1}\binom{10}{3}$ ways to distribute the cookies giving too many to one kid. But in fact, we have over counted.

We must get rid of the outcomes in which two kids have too many cookies. There are $\binom{4}{2}$ ways to select 2 kids to give extra cookies. It takes 6 cookies to do this, leaving only 4 cookies. So we have 4 stars and still 3 bars. The remaining 4 cookies can thus be distributed in $\binom{7}{3}$ ways (for each of the $\binom{4}{2}$ choices of which 2 kids to over-feed).

But now we have removed too much. We must add back in all the ways to give too many cookies to three kids. This uses 9 cookies, leaving only 1 to distribute to the 4 kids using stars and bars, which can be done in $\binom{4}{3}$ ways. We must consider this outcome for every possible choice of which three kids we over-feed, and there are $\binom{4}{3}$ ways of selecting that set of 3 kids.

Next we would subtract all the ways to give four kids too many cookies, but in this case, that number is 0.

All together we get that the number of ways to distribute 10 cookies to 4 kids without giving any kid more than 2 cookies is:

$$\binom{13}{3} - \left[\binom{4}{1}\binom{10}{3} - \binom{4}{2}\binom{7}{3} + \binom{4}{3}\binom{4}{3}\right]$$

which is

$$286 - [480 - 210 + 16] = 0.$$

This makes sense: there is NO way to distribute 10 cookies to 4 kids and make sure that nobody gets more than 2. It is slightly surprising that

$$\binom{13}{3} = \left[\binom{4}{1}\binom{10}{3} - \binom{4}{2}\binom{7}{3} + \binom{4}{3}\binom{4}{3}\right],$$

but since PIE works, this equality must hold.

Just so you don't think that these problems always have easier solutions, consider the following example.

Example 1.6.3

Earlier (Example 1.5.3) we counted the number of solutions to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13,$$

where $x_i \geq 0$ for each x_i.

How many of those solutions have $0 \le x_i \le 3$ for each x_i?

Solution. We must subtract off the number of solutions in which one or more of the variables has a value greater than 3. We will need to use PIE because counting the number of solutions for which each of the five variables separately are greater than 3 counts solutions multiple times. Here is what we get:

- Total solutions: $\binom{17}{4}$.
- Solutions where $x_1 > 3$: $\binom{13}{4}$. Give x_1 4 units first, then distribute the remaining 9 units to the 5 variables.
- Solutions where $x_1 > 3$ and $x_2 > 3$: $\binom{9}{4}$. After you give 4 units to x_1 and another 4 to x_2, you only have 5 units left to distribute.
- Solutions where $x_1 > 3$, $x_2 > 3$ and $x_3 > 3$: $\binom{5}{4}$.
- Solutions where $x_1 > 3$, $x_2 > 3$, $x_3 > 3$, and $x_4 > 3$: 0.

We also need to account for the fact that we could choose any of the five variables in the place of x_1 above (so there will be $\binom{5}{1}$ outcomes like this), any pair of variables in the place of x_1 and x_2 ($\binom{5}{2}$ outcomes) and so on. It is because of this that the double counting occurs, so we need to use PIE. All together we have that the number of solutions with $0 \le x_i \le 3$ is

$$\binom{17}{4} - \left[\binom{5}{1}\binom{13}{4} - \binom{5}{2}\binom{9}{4} + \binom{5}{3}\binom{5}{4}\right] = 15.$$

Counting Derangements

Investigate!

For your senior prank, you decide to switch the nameplates on your favorite 5 professors' doors. So that none of them feel left out, you want to make sure that all of the nameplates end up on the wrong door. How many ways can this be accomplished?

Attempt the above activity before proceeding

The advanced use of PIE has applications beyond stars and bars. A **derangement** of n elements $\{1, 2, 3, \ldots, n\}$ is a permutation in which no element is fixed. For example, there are 6 permutations of the three elements $\{1, 2, 3\}$:

$$123 \;\; 132 \;\; 213 \;\; 231 \;\; 312 \;\; 321.$$

but most of these have one or more elements fixed: 123 has all three elements fixed since all three elements are in their original positions, 132 has the first element fixed (1 is in its original first position), and so on. In fact, the only derangements of three elements are

231 and 312.

If we go up to 4 elements, there are 24 permutations (because we have 4 choices for the first element, 3 choices for the second, 2 choices for the third leaving only 1 choice for the last). How many of these are derangements? If you list out all 24 permutations and eliminate those which are not derangements, you will be left with just 9 derangements. Let's see how we can get that number using PIE.

Example 1.6.4

How many derangements are there of 4 elements?

Solution. We count all permutations, and subtract those which are not derangements. There are $4! = 24$ permutations of 4 elements. Now for a permutation to *not* be a derangement, at least one of the 4 elements must be fixed. There are $\binom{4}{1}$ choices for which single element we fix. Once fixed, we need to find a permutation of the other three elements. There are 3! permutations on 3 elements.

But now we have counted too many non-derangements, so we must subtract those permutations which fix two elements. There are $\binom{4}{2}$ choices for which two elements we fix, and then for each pair, 2! permutations of the remaining elements. But this subtracts too many, so add back in permutations which fix 3 elements, all $\binom{4}{3}1!$ of them. Finally subtract the $\binom{4}{4}0!$ permutations (recall $0! = 1$) which fix all four elements. All together we get that the number of derangements of 4 elements is:

$$4! - \left[\binom{4}{1}3! - \binom{4}{2}2! + \binom{4}{3}1! - \binom{4}{4}0!\right] = 24 - 15 = 9.$$

Of course we can use a similar formula to count the derangements of any number of elements. However, the more elements we have, the longer the formula gets. Here is another example:

Example 1.6.5

Five gentlemen attend a party, leaving their hats at the door. At the end of the party, they hastily grab hats on their way out. How many different ways could this happen so that none of the gentlemen leave with his own hat?

Solution. We are counting derangements on 5 elements. There are 5! ways for the gentlemen to grab hats in any order—but many of these permutations will result in someone getting their own hat. So we subtract all the ways in which one or more of the men get their own hat. In other words, we subtract the non-derangements. Doing so requires PIE. Thus the answer is:

$$5! - \left[\binom{5}{1}4! - \binom{5}{2}3! + \binom{5}{3}2! - \binom{5}{4}1! + \binom{5}{5}0!\right].$$

Counting Functions

Investigate!

1. Consider all functions $f : \{1,2,3,4,5\} \to \{1,2,3,4,5\}$. How many functions are there all together? How many of those are injective? Remember, a function is an injection if every input goes to a different output.
2. Consider all functions $f : \{1,2,3,4,5\} \to \{1,2,3,4,5\}$. How many of the *injections* have the property that $f(x) \neq x$ for any $x \in \{1,2,3,4,5\}$?

   Your friend claims that the answer is:

   $$5! - \left[\binom{5}{1}4! - \binom{5}{2}3! + \binom{5}{3}2! - \binom{5}{4}1! + \binom{5}{5}0!\right].$$

   Explain why this is correct.
3. Recall that a *surjection* is a function for which every element of the codomain is in the range. How many of the functions $f : \{1,2,3,4,5\} \to \{1,2,3,4,5\}$ are surjective? Use PIE!

Attempt the above activity before proceeding

We have seen throughout this chapter that many counting questions can be rephrased as questions about counting functions with certain properties. This is reasonable since many counting questions can be thought of as counting the number of ways to assign elements from one set to elements of another.

Example 1.6.6

You decide to give away your video game collection so as to better spend your time studying advanced mathematics. How many ways can you do this, provided:

1. You want to distribute your 3 different PS4 games among 5 friends, so that no friend gets more than one game?
2. You want to distribute your 8 different 3DS games among 5 friends?
3. You want to distribute your 8 different SNES games among 5 friends, so that each friend gets at least one game?

In each case, model the counting question as a function counting question.

Solution.

1. We must use the three games (call them 1, 2, 3) as the domain and the 5 friends (a,b,c,d,e) as the codomain (otherwise the function would not be defined for the whole domain when a friend didn't get any game). So how many functions are there with domain $\{1, 2, 3\}$ and codomain $\{a, b, c, d, e\}$? The answer to this is $5^3 = 125$, since we can assign any of 5 elements to be the image of 1, any of 5 elements to be the image of 2 and any of 5 elements to be the image of 3.

   But this is not the correct answer to our counting problem, because one of these functions is $f = \begin{pmatrix} 1 & 2 & 3 \\ a & a & a \end{pmatrix}$; one friend can get more than one game. What we really need to do is count *injective* functions. This gives $P(5, 3) = 60$ functions, which is the answer to our counting question.

2. Again, we need to use the 8 games as the domain and the 5 friends as the codomain. We are counting all functions, so the number of ways to distribute the games is 5^8.
3. This question is harder. Use the games as the domain and friends as the codomain (the reverse would not give a function). To ensure that every friend gets at least one game means that every element of the codomain is in the range. In other words, we are looking for *surjective* functions. How do you count those??

In Example 1.1.5 we saw how to count all functions (using the multiplicative principle) and in Example 1.3.4 we learned how to count injective functions (using permutations). Surjective functions are not as easily counted (unless the size of the domain is smaller than the codomain, in which case there are none).

The idea is to count the functions which are *not* surjective, and then subtract that from the total number of functions. This works very well when the codomain has two elements in it:

Example 1.6.7

How many functions $f : \{1, 2, 3, 4, 5\} \to \{a, b\}$ are surjective?

Solution. There are 2^5 functions all together, two choices for where to send each of the 5 elements of the domain. Now of these, the functions which are *not* surjective must exclude one or more elements of the codomain from the range. So first, consider functions for which a is not in the range. This can only happen one way: everything gets sent to b. Alternatively, we could exclude b from the range. Then everything gets sent to a, so there is only one function like this. These are the only ways in which a function could not be surjective (no function excludes both a and b from the range) so there are exactly $2^5 - 2$ surjective functions.

When there are three elements in the codomain, there are now three choices for a single element to exclude from the range. Additionally, we could pick pairs of two elements to exclude from the range, and we must make sure we don't over count these. It's PIE time!

Example 1.6.8

How many functions $f : \{1, 2, 3, 4, 5\} \to \{a, b, c\}$ are surjective?

Solution. Again start with the total number of functions: 3^5 (as each of the five elements of the domain can go to any of three elements of the codomain). Now we count the functions which are *not* surjective.

Start by excluding a from the range. Then we have two choices (b or c) for where to send each of the five elements of the domain. Thus there are 2^5 functions which exclude a from the range. Similarly, there are 2^5 functions which exclude b, and another 2^5 which exclude c. Now have we counted all functions which are not surjective? Yes, but in fact, we have counted some multiple times. For example, the function which sends everything to c was one of the 2^5 functions we counted when we excluded a from the range, and also one of the 2^5 functions we counted when we excluded b from the range. We must subtract out all the functions which specifically exclude two elements from the range. There is 1 function when we exclude a and b (everything goes to c), one function when we exclude a and c, and one function when we exclude b and c.

We are using PIE: to count the functions which are not surjective, we added up the functions which exclude a, b, and c separately, then subtracted the functions which exclude pairs of elements. We would then add back in the functions which exclude groups of three elements, except that there are no such functions. We find that the number of functions which are *not* surjective is

$$2^5 + 2^5 + 2^5 - 1 - 1 - 1 + 0.$$

Perhaps a more descriptive way to write this is

$$\binom{3}{1}2^5 - \binom{3}{2}1^5 + \binom{3}{3}0^5.$$

since each of the 2^5's was the result of choosing 1 of the 3 elements of the codomain to exclude from the range, each of the three 1^5's was the result of choosing 2 of the 3 elements of the codomain to exclude. Writing 1^5 instead of 1 makes sense too: we have 1 choice of where to send each of the 5 elements of the domain.

Now we can finally count the number of surjective functions:

$$3^5 - \left[\binom{3}{1}2^5 - \binom{3}{2}1^5\right] = 150.$$

You might worry that to count surjective functions when the codomain is larger than 3 elements would be too tedious. We need to use PIE but with more than 3 sets the formula for PIE is very long. However, we have lucked out. As we saw in the example above, the number of functions which exclude a single element from the range is the same no matter which single element is excluded. Similarly, the number of functions which exclude a pair of elements will be the same for every pair. With larger codomains, we will see the same behavior with groups of 3, 4, and more elements excluded. So instead of adding/subtracting each of these, we can simply add or subtract all of them at once, if you know how many there are. This works just like it did in for the other types of counting questions in this section, only now the size of the various combinations of sets is a number raised to a power, as opposed to a binomial coefficient or factorial. Here's what happens with 4 and 5 elements in the codomain.

Example 1.6.9

1. How many functions $f : \{1, 2, 3, 4, 5\} \to \{a, b, c, d\}$ are surjective?
2. How many functions $f : \{1, 2, 3, 4, 5\} \to \{a, b, c, d, e\}$ are surjective?

Solution.

1. There are 4^5 functions all together; we will subtract the functions which are not surjective. We could exclude any one of the four elements of the codomain, and doing so will leave us with 3^5 functions for each excluded element. This counts too many so we subtract the functions which exclude two of the four elements of the codomain, each pair giving 2^5 functions. But this excludes too many, so we add back in the functions which exclude three of the four elements of the codomain, each triple giving 1^5 function. There are $\binom{4}{1}$ groups of functions excluding a single element, $\binom{4}{2}$ groups of functions excluding a pair of elements, and $\binom{4}{3}$ groups of functions excluding a triple of elements. This means that the number of functions which are *not* surjective is:

$$\binom{4}{1}3^5 - \binom{4}{2}2^5 + \binom{4}{3}1^5.$$

We can now say that the number of functions which are surjective is:

$$4^5 - \left[\binom{4}{1}3^5 - \binom{4}{2}2^5 + \binom{4}{3}1^5\right].$$

2. The number of surjective functions is:

$$5^5 - \left[\binom{5}{1}4^5 - \binom{5}{2}3^5 + \binom{5}{3}2^5 - \binom{5}{4}1^5\right].$$

We took the total number of functions 5^5 and subtracted all that were not surjective. There were $\binom{5}{1}$ ways to select a single element from the codomain to exclude from the range, and for each there were 4^5 functions. But this double counts, so we use PIE and subtract functions excluding two elements from the range: there are $\binom{5}{2}$ choices for the two elements to exclude, and for each pair, 3^5 functions. This takes out too many functions, so we add back in functions which exclude 3 elements from the range: $\binom{5}{3}$ choices for which three to exclude, and then 2^5 functions for each choice of elements. Finally we take back out the 1 function which excludes 4 elements for each of the $\binom{5}{4}$ choices of 4 elements.

If you happen to calculate this number precisely, you will get 120 surjections. That happens to also be the value of 5!. This might seem like an amazing coincidence until you realize that every surjective function $f : X \to Y$ with $|X| = |Y|$ finite must necessarily be a bijection. The number of bijections is always $|X|!$ in this case. What we have here is a *combinatorial proof* of the following identity:

$$n^n - \left[\binom{n}{1}(n-1)^n - \binom{n}{2}(n-2)^n + \cdots + \binom{n}{n-1}1^n\right] = n!.$$

We have seen that counting surjective functions is another nice example of the advanced use of the Principle of Inclusion/Exclusion. Also, counting injective functions turns out to be equivalent to permutations, and counting all functions has a solution akin to those counting problems where order matters but repeats are allowed (like counting the number of words you can make from a given set of letters).

These are not just a few more examples of the techniques we have developed in this chapter. Quite the opposite: everything we have learned in this chapter are examples of *counting functions*!

Example 1.6.10

How many 5-letter words can you make using the eight letters a through h? How many contain no repeated letters?

Solution. By now it should be no surprise that there are 8^5 words, and $P(8,5)$ words without repeated letters. The new piece here is that we are actually counting functions. For the first problem, we are counting all functions from $\{1,2,\ldots,5\}$ to $\{a,b,\ldots,h\}$. The numbers in the domain represent the *position* of the letter in the word, the codomain represents the letter that could be assigned to that position. If we ask for no repeated letters, we are asking for injective functions.

If A and B are *any* sets with $|A| = 5$ and $|B| = 8$, then the number of functions $f : A \to B$ is 8^5 and the number of injections is $P(8, 5)$. So if you can represent your counting problem as a function counting problem, most of the work is done.

Example 1.6.11

How many subsets are there of $\{1, 2, \ldots, 9\}$? How many 9-bit strings are there (of any weight)?

Solution. We saw in Section 1.2 that the answer to both these questions is 2^9, as we can say yes or no (or 0 or 1) to each of the 9 elements in the set (positions in the bit-string). But 2^9 also looks like the answer you get from counting functions. In fact, if you count all functions $f : A \to B$ with $|A| = 9$ and $|B| = 2$, this is exactly what you get.

This makes sense! Let $A = \{1, 2, \ldots, 9\}$ and $B = \{y, n\}$. We are assigning each element of the set either a yes or a no. Or in the language of bit-strings, we would take the 9 positions in the bit string as our domain and the set $\{0, 1\}$ as the codomain.

So far we have not used a function as a model for binomial coefficients (combinations). Think for a moment about the relationship between combinations and permutations, say specifically $\binom{9}{3}$ and $P(9, 3)$. We *do* have a function model for $P(9, 3)$. This is the number of *injective* functions from a set of size 3 (say $\{1, 2, 3\}$ to a set of size 9 (say $\{1, 2, \ldots, 9\}$) since there are 9 choices for where to send the first element of the domain, then only 8 choices for the second, and 7 choices for the third. For example, the function might look like this:

$$f(1) = 5 \qquad f(2) = 8 \qquad f(3) = 4.$$

This is a different function from:

$$f(1) = 4 \qquad f(2) = 5 \qquad f(3) = 8.$$

Now $P(9, 3)$ counts these as different outcomes correctly, but $\binom{9}{3}$ will count these (among others) as just one outcome. In fact, in terms of functions $\binom{9}{3}$ just counts the number of different ranges possible of injective functions. This should not be a surprise since binomial coefficients counts subsets, and the range is a possible subset of the codomain.[3]

[3]A more mathematically sophisticated interpretation of combinations is that we are defining two injective functions to be *equivalent* if they have the same range, and then counting the number of equivalence classes under this notion of equivalence.

While it is possible to interpret combinations as functions, perhaps the better advice is to instead use combinations (or stars and bars) when functions are not quite the right way to interpret the counting question.

EXERCISES

1. The dollar menu at your favorite tax-free fast food restaurant has 7 items. You have \$10 to spend. How many different meals can you buy if you spend all your money and:

(a) Purchase at least one of each item.

(b) Possibly skip some items.

(c) Don't get more than 2 of any particular item.

2. After a late night of math studying, you and your friends decide to go to your favorite tax-free fast food Mexican restaurant, *Burrito Chime*. You decide to order off of the dollar menu, which has 7 items. Your group has \$16 to spend (and will spend all of it).

(a) How many different orders are possible? Explain. (The *order* in which the order is placed does not matter - just which and how many of each item that is ordered.)

(b) How many different orders are possible if you want to get at least one of each item? Explain.

(c) How many different orders are possible if you don't get more than 4 of any one item? Explain.

3. After another gym class you are tasked with putting the 14 identical dodgeballs away into 5 bins. This time, no bin can hold more than 6 balls. How many ways can you clean up?

4. Consider the equation $x_1 + x_2 + x_3 + x_4 = 15$. How many solutions are there with $2 \leq x_i \leq 5$ for all $i \in \{1, 2, 3, 4\}$?

5. Suppose you planned on giving 7 gold stars to some of the 13 star students in your class. Each student can receive at most one star. How many ways can you do this?

Use PIE. Then, find the numeric answer in Pascal's triangle and explain why that makes sense.

6. Based on the previous question, give a combinatorial proof for the identity:

$$\binom{n}{k} = \binom{n+k-1}{k} - \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} \binom{n+k-(2j+1)}{k-2j}.$$

7. Illustrate how the counting of derangements works by writing all permutations of $\{1, 2, 3, 4\}$ and the crossing out those which are not derangements. Keep track of the permutations you cross out more than once, using PIE.

8. How many permutations of $\{1, 2, 3, 4, 5\}$ leave exactly 1 element fixed?

9. Ten ladies of a certain age drop off their red hats at the hat check of a museum. As they are leaving, the hat check attendant gives the hats back randomly. In how many ways can exactly six of the ladies receive their own hat (and the other four not)? Explain.

10. The Grinch sneaks into a room with 6 Christmas presents to 6 different people. He proceeds to switch the name-labels on the presents. How many ways could he do this if:

    (a) No present is allowed to end up with its original label? Explain what each term in your answer represents.

    (b) Exactly 2 presents keep their original labels? Explain.

    (c) Exactly 5 presents keep their original labels? Explain.

11. Consider functions $f : \{1, 2, 3, 4\} \to \{a, b, c, d, e, f\}$. How many functions have the property that $f(1) \neq a$ or $f(2) \neq b$, or both?

12. Consider sets A and B with $|A| = 10$ and $|B| = 5$. How many functions $f : A \to B$ are surjective?

13. Let $A = \{1, 2, 3, 4, 5\}$. How many injective functions $f : A \to A$ have the property that for each $x \in A$, $f(x) \neq x$?

14. Let d_n be the number of derangements of n objects. For example, using the techniques of this section, we find

$$d_3 = 3! - \left(\binom{3}{1}2! - \binom{3}{2}1! + \binom{3}{3}0!\right).$$

We can use the formula for $\binom{n}{k}$ to write this all in terms of factorials. After simplifying, for d_3 we would get

$$d_3 = 3!\left(1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{6}\right).$$

Generalize this to find a nicer formula for d_n. Bonus: For large n, approximately what fraction of all permutations are derangements? Use your knowledge of Taylor series from calculus.

1.7 Chapter Summary

Investigate!

Suppose you have a huge box of animal crackers containing plenty of each of 10 different animals. For the counting questions below, carefully examine their similarities and differences, and then give an answer. The answers are all one of the following:

$$P(10,6) \qquad \binom{10}{6} \qquad 10^6 \qquad \binom{15}{9}.$$

1. How many animal parades containing 6 crackers can you line up?
2. How many animal parades of 6 crackers can you line up so that the animals appear in alphabetical order?
3. How many ways could you line up 6 different animals in alphabetical order?
4. How many ways could you line up 6 different animals if they can come in any order?
5. How many ways could you give 6 children one animal cracker each?
6. How many ways could you give 6 children one animal cracker each so that no two kids get the same animal?
7. How many ways could you give out 6 giraffes to 10 kids?
8. Write a question about giving animal crackers to kids that has the answer $\binom{10}{6}$.

STOP **Attempt the above activity before proceeding** STOP

With all the different counting techniques we have mastered in this last chapter, it might be difficult to know when to apply which technique. Indeed, it is very easy to get mixed up and use the wrong counting method for a given problem. You get better with practice. As you practice you start to notice some trends that can help you distinguish between types of counting problems. Here are some suggestions that you might find helpful when deciding how to tackle a counting problem and checking whether your solution is correct.

- Remember that you are counting the number of items in some *list of outcomes*. Write down part of this list. Write down an element in the middle of the list – how are you deciding whether your element really is in the list. Could you get this element more than once using your proposed answer?

- If generating an element on the list involves selecting something (for example, picking a letter or picking a position to put a letter, etc), can the things you select be repeated? Remember, permutations and combinations select objects from a set *without* repeats.

- Does order matter? Be careful here and be sure you know what your answer really means. We usually say that order matters when you get different outcomes when the same objects are selected in different orders. Combinations and "Stars & Bars" are used when order *does not* matter.

- There are four possibilities when it comes to order and repeats. If order matters and repeats are allowed, the answer will look like n^k. If order matters and repeats are not allowed, we have $P(n,k)$. If order doesn't matter and repeats are allowed, use stars and bars. If order doesn't matter and repeats are not allowed, use $\binom{n}{k}$. But be careful: this only applies when you are selecting things, and you should make sure you know exactly what you are selecting before determining which case you are in.

- Think about how you would represent your counting problem in terms of sets or functions. We know how to count different sorts of sets and different types of functions.

- As we saw with combinatorial proofs, you can often solve a counting problem in more than one way. Do that, and compare your numerical answers. If they don't match, something is amiss.

While we have covered many counting techniques, we have really only scratched the surface of the large subject of *enumerative combinatorics*. There are mathematicians doing original research in this area even as you read this. Counting can be really hard.

In the next chapter, we will approach counting questions from a very different direction, and in doing so, answer infinitely many counting questions at the same time. We will create *sequences* of answers to related questions.

Chapter Review

1. You have 9 presents to give to your 4 kids. How many ways can this be done if:

(a) The presents are identical, and each kid gets at least one present?

(b) The presents are identical, and some kids might get no presents?

(c) The presents are unique, and some kids might get no presents?

(d) The presents are unique and each kid gets at least one present?

2. For each of the following counting problems, say whether the answer is $\binom{10}{4}$, $P(10,4)$, or neither. If you answer is "neither," say what the answer should be instead.

(a) How many shortest lattice paths are there from $(0,0)$ to $(10,4)$?

(b) If you have 10 bow ties, and you want to select 4 of them for next week, how many choices do you have?

(c) Suppose you have 10 bow ties and you will wear a different one on each of the next 4 days. How many choices do you have?

(d) If you want to wear 4 of your 10 bow ties next week (Monday through Sunday), how many ways can this be accomplished?

(e) Out of a group of 10 classmates, how many ways can you rank your top 4 friends?

(f) If 10 students come to their professor's office but only 4 can fit at a time, how different combinations of 4 students can see the prof first?

(g) How many 4 letter words can be made from the first 10 letters of the alphabet?

(h) How many ways can you make the word "cake" from the first 10 letters of the alphabet?

(i) How many ways are there to distribute 10 identical apples among 4 children?

(j) If you have 10 kids (and live in a shoe) and 4 types of cereal, how many ways can your kids eat breakfast?

(k) How many ways can you arrange exactly 4 ones in a string of 10 binary digits?

(l) You want to select 4 distinct, single-digit numbers as your lotto picks. How many choices do you have?

(m) 10 kids want ice-cream. You have 4 varieties. How many ways are there to give the kids as much ice-cream as they want?

(n) How many 1-1 functions are there from $\{1, 2, \ldots, 10\}$ to $\{a, b, c, d\}$?

(o) How many surjective functions are there from $\{1, 2, \ldots, 10\}$ to $\{a, b, c, d\}$?

(p) Each of your 10 bow ties match 4 pairs of suspenders. How many outfits can you make?

(q) After the party, the 10 kids each choose one of 4 party-favors. How many outcomes?

(r) How many 6-elements subsets are there of the set $\{1, 2, \ldots, 10\}$

(s) How many ways can you split up 11 kids into 5 named teams?

(t) How many solutions are there to $x_1 + x_2 + \cdots + x_5 = 6$ where each x_i is a non-negative integer?

(u) Your band goes on tour. There are 10 cities within driving distance, but only enough time to play 4 of them. How many choices do you have for the cities on your tour?

(v) In how many different ways can you play the 4 cities you choose?

(w) Out of the 10 breakfast cereals available, you want to have 4 bowls. How many ways can you do this?

(x) There are 10 types of cookies available. You want to make a 4 cookie stack. How many different stacks can you make?

(y) From your home at (0,0) you want to go to either the donut shop at (5,4) or the one at (3,6). How many paths could you take?

(z) How many 10-digit numbers do not contain a sub-string of 4 repeated digits?

3. bow tiesRecall, you own 3 regular ties and 5 bow ties. You realize that it would be okay to wear more than two ties to your clown college interview.

(a) You must select some of your ties to wear. Everything is okay, from no ties up to all ties. How many choices do you have?

(b) If you want to wear at least one regular tie and one bow tie, but are willing to wear up to all your ties, how many choices do you have for which ties to wear?

(c) How many choices of which ties to wear do you have if you wear exactly 2 of the 3 regular ties and 3 of the 5 bow ties?

(d) Once you have selected 2 regular and 3 bow ties, in how many orders could you put the ties on, assuming you must have one of the three bow ties on top?

4. Give a counting question where the answer is $8 \cdot 3 \cdot 3 \cdot 5$. Give another question where the answer is $8 + 3 + 3 + 5$.

5. Consider five digit numbers $\alpha = a_1a_2a_3a_4a_5$, with each digit from the set $\{1, 2, 3, 4\}$.

(a) How many such numbers are there?

(b) How many such numbers are there for which the *sum* of the digits is even?

(c) How many such numbers contain more even digits than odd digits?

6. In a recent small survey of airline passengers, 25 said they had flown American in the last year, 30 had flown Jet Blue, and 20 had flown Continental. Of those, 10 reported they had flown on American and Jet Blue, 12 had flown on Jet Blue and Continental, and 7 had flown on American and Continental. 5 passengers had flown on all three airlines.

How many passengers were surveyed? (Assume the results above make up the entire survey.)

7. Recall, by 8-bit strings, we mean strings of binary digits, of length 8.

(a) How many 8-bit strings are there total?

(b) How many 8-bit strings have weight 5?

(c) How many subsets of the set $\{a, b, c, d, e, f, g, h\}$ contain exactly 5 elements?

(d) Explain why your answers to parts (b) and (c) are the same. Why are these questions equivalent?

8. What is the coefficient of x^{10} in the expansion of $(x+1)^{13} + x^2(x+1)^{17}$?

9. How many 8-letter words contain exactly 5 vowels? (One such word is "aaioobtt"; don't consider "y" a vowel for this exercise.)

What if repeated letters were not allowed?

10. For each of the following, find the number of shortest lattice paths from $(0,0)$ to $(8,8)$ which:

(a) pass through the point $(2,3)$.

(b) avoid (do not pass through) the point $(7,5)$.

(c) either pass through $(2,3)$ or $(5,7)$ (or both).

11. You live in Grid-Town on the corner of 2nd and 3rd, and work in a building on the corner of 10th and 13th. How many routes are there which take you from home to work and then back home, but by a different route?

12. How many 10-bit strings start with 111 or end with 101 or both?

13. How many 10-bit strings of weight 6 start with 111 or end with 101 or both?

14. How many 6 letter words made from the letters a, b, c, d, e, f without repeats do not contain the sub-word "bad" in consecutive letters?

How many don't to contain the subword "bad" in not-necessarily consecutive letters (but in order)?

15. Explain using lattice paths why $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

16. Suppose you have 20 one-dollar bills to give out as prizes to your top 5 discrete math students. How many ways can you do this if:

(a) Each of the 5 students gets at least 1 dollar?

(b) Some students might get nothing?

(c) Each student gets at least 1 dollar but no more than 7 dollars?

17. How many functions $f : \{1, 2, 3, 4, 5\} \to \{a, b, c, d, e\}$ are there satisfying:

(a) $f(1) = a$ or $f(2) = b$ (or both)?

(b) $f(1) \neq a$ or $f(2) \neq b$ (or both)?

(c) $f(1) \neq a$ *and* $f(2) \neq b$, and f is injective?

(d) f is surjective, but $f(1) \neq a$, $f(2) \neq b$, $f(3) \neq c$, $f(4) \neq d$ and $f(5) \neq e$?

18. How many functions map $\{1, 2, 3, 4, 5, 6\}$ *onto* $\{a, b, c, d\}$ (i.e., how many *surjections* are there)?

19. To thank your math professor for doing such an amazing job all semester, you decide to bake Oscar cookies. You know how to make 10 different types of cookies.

(a) If you want to give your professor 4 different types of cookies, how many different combinations of cookie type can you select? Explain your answer.

(b) To keep things interesting, you decide to make a different number of each type of cookie. If again you want to select 4 cookie types, how many ways can you select the cookie types and decide for which there will be the most, second most, etc. Explain your answer.

(c) You change your mind again. This time you decide you will make a total of 12 cookies. Each cookie could be any one of the 10 types of cookies you know how to bake (and it's okay if you leave some types out). How many choices do you have? Explain.

(d) You realize that the previous plan did not account for presentation. This time, you once again want to make 12 cookies, each one could be any one of the 10 types of cookies. However, now you plan to shape the cookies into the numerals 1, 2, ..., 12 (and probably arrange them to make a giant clock, but you haven't decided on that yet). How many choices do you have for which types of cookies to bake into which numerals? Explain.

(e) The only flaw with the last plan is that your professor might not get to sample all 10 different varieties of cookies. How many choices do you have for which types of cookies to make into which numerals, given that each type of cookie should be present at least once? Explain.

20. For which of the parts of the previous problem (Exercise 1.7.19) does it make sense to interpret the counting question as counting some number of functions? Say what the domain and codomain should be, and whether you are counting all functions, injections, surjections, or something else.

Chapter 2

Sequences

Investigate!

There is a monastery in Hanoi, as the legend goes, with a great hall containing three tall pillars. Resting on the first pillar are 64 giant disks (or washers), all different sizes, stacked from largest to smallest. The monks are charged with the following task: they must move the entire stack of disks to the third pillar. However, due to the size of the disks, the monks cannot move more than one at a time. Each disk must be placed on one of the pillars before the next disk is moved. And because the disks are so heavy and fragile, the monks may never place a larger disk on top of a smaller disk. When the monks finally complete their task, the world shall come to an end. Your task: figure out how long before we need to start worrying about the end of the world.

1. First, let's find the minimum number of moves required for a smaller number of disks. Collect some data. Make a table.
2. Conjecture a formula for the minimum number of moves required to move n disks. Test your conjecture. How do you know your formula is correct?
3. If the monks were able to move one disk every second without ever stopping, how long before the world ends?

Attempt the above activity before proceeding

This puzzle is called the *Tower of Hanoi*. You are tasked with finding the minimum number of moves to complete the puzzle. This certainly sounds like a counting problem. Perhaps you have an answer? If not, what else could we try?

The answer depends on the number of disks you need to move. In fact, we could answer the puzzle first for 1 disk, then 2, then 3 and so on. If we list out all of the answers for each number of disks, we will get a **sequence** of numbers. The nth term in the sequence is the answer to the question, "what is the smallest number of moves required to complete the Tower of Hanoi puzzle with n disks?" You might wonder why we would create such a sequence instead of just answering the question. By looking at how the sequence of numbers grows, we gain insight into the problem. It is easy to count the number of moves required for a small number of disks. We can then look for a pattern among the first few terms of the sequence. Hopefully this will suggest a method for finding the nth term of the sequence, which is the answer to our question. Of course we will also need to verify that our suspected pattern is correct, and that this correct pattern really does give us the nth term we think it does, but it is impossible to prove that your formula is correct without having a formula to start with.

Sequences are also interesting mathematical objects to study in their own right. Let's see why.

2.1 Describing Sequences

Investigate!

You have a large collection of 1×1 squares and 1×2 dominoes. You want to arrange these to make a 1×15 strip. How many ways can you do this?

1. Start by collecting data. How many length 1×1 strips can you make? How many 1×2 strips? How many 1×3 strips? And so on.
2. How are the 1×3 and 1×4 strips related to the 1×5 strips?
3. How many 1×15 strips can you make?
4. What if I asked you to find the number of 1×1000 strips? Would the method you used to calculate the number fo 1×15 strips be helpful?

Attempt the above activity before proceeding

A **sequence** is simply an ordered list of numbers. For example, here is a sequence: $0, 1, 2, 3, 4, 5, \ldots$. This is different from the set $\mathbb{N}$ because, while the sequence is a complete list of every element in the set of natural numbers, in the sequence we very much care what order the numbers come in. For this reason, when we use variables to represent terms in a sequence they will look like this:

$$a_0, a_1, a_2, a_3, \ldots.$$

To refer to the *entire* sequence at once, we will write $(a_n)_{n \in \mathbb{N}}$ or $(a_n)_{n \ge 0}$, or sometimes if we are being sloppy, just (a_n) (in which case we assume we start the sequence with a_0).

We might replace the a with another letter, and sometimes we omit a_0, starting with a_1, in which case we would use $(a_n)_{n \ge 1}$ to refer to the sequence as a whole. The numbers in the subscripts are called **indices** (the plural of **index**).

While we often just think of a sequence as an ordered list of numbers, it is really a type of function. Specifically, the sequence $(a_n)_{n\geq 0}$ is a function with domain $\mathbb{N}$ where a_n is the image of the natural number n. Later we will manipulate sequences in much the same way you have manipulated functions in algebra or calculus. We can shift a sequence up or down, add two sequences, or ask for the rate of change of a sequence. These are done exactly as you would for functions.

That said, while keeping the rigorous mathematical definition in mind is helpful, we often describe sequences by writing out the first few terms.

Example 2.1.1

Can you find the next term in the following sequences?

1. $7, 7, 7, 7, 7, \ldots$
2. $3, -3, 3, -3, 3, \ldots$
3. $1, 5, 2, 10, 3, 15, \ldots$
4. $1, 2, 4, 8, 16, 32, \ldots$
5. $1, 4, 9, 16, 25, 36, \ldots$
6. $1, 2, 3, 5, 8, 13, 21, \ldots$
7. $1, 3, 6, 10, 15, 21, \ldots$
8. $2, 3, 5, 7, 11, 13, \ldots$
9. $3, 2, 1, 0, -1, \ldots$
10. $1, 1, 2, 6, \ldots$

Solution. No you cannot. You might guess that the next terms are:

1. 7
2. −3
3. 4
4. 64
5. 49
6. 34
7. 28
8. 17
9. −2
10. 24

In fact, those are the next terms of the sequences I had in mind when I made up the example, but there is no way to be sure they are correct.

Still, we will often do this. Given the first few terms of a sequence, we can ask what the pattern in the sequence suggests the next terms are.

Given that no number of initial terms in a sequence is enough to say for certain which sequence we are dealing with, we need to find another way to specify a sequence. We consider two ways to do this:

Closed formula.

A **closed formula** for a sequence $(a_n)_{n\in\mathbb{N}}$ is a formula for a_n using a fixed finite number of operations on n. This is what you normally think of as a formula in n, just as if you were defining a function in terms of n (because that is exactly what you are doing).

Recursive definition.

A **recursive definition** (sometimes called an **inductive definition**) for a sequence $(a_n)_{n\in\mathbb{N}}$ consists of a **recurrence relation** : an equation relating a term of the sequence to previous terms (terms with smaller index) and an **initial condition**: a list of a few terms of the sequence (one less than the number of terms in the recurrence relation).

It is easier to understand what is going on here with an example:

Example 2.1.2

Here are a few closed formulas for sequences:

- $a_n = n^2$.
- $a_n = \dfrac{n(n+1)}{2}$.
- $a_n = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^{-n}}{\sqrt{5}}$.

Note in each formula, if you are given n, you can calculate a_n directly: just plug in n. For example, to find a_3 in the second sequence, just compute $a_3 = \frac{3(3+1)}{2} = 6$.

Here are a few recursive definitions for sequences:

- $a_n = 2a_{n-1}$ with $a_0 = 1$.
- $a_n = 2a_{n-1}$ with $a_0 = 27$.
- $a_n = a_{n-1} + a_{n-2}$ with $a_0 = 0$ and $a_1 = 1$.

In these formulas, if you are given n, you cannot calculate a_n directly, you first need to find a_{n-1} (or a_{n-1} and a_{n-2}). In the second sequence, to find a_3 you would take $2a_2$, but to find $a_2 = 2a_1$ we would need to know $a_1 = 2a_0$. We do know this, so we could trace back through these equations to find $a_1 = 54$, $a_2 = 108$ and finally $a_3 = 216$.

You might wonder why we would bother with recursive definitions for sequences. After all, it is harder to find a_n with a recursive definition than with a closed formula. This is true, but it is also harder to find a closed formula for a sequence than it is to find a recursive definition. So to find a useful closed formula, we might first find the recursive definition, then use that to find the closed formula.

This is not to say that recursive definitions aren't useful in finding a_n. You can always calculate a_n given a recursive definition, it might just take a while.

Example 2.1.3

Find a_6 in the sequence defined by $a_n = 2a_{n-1} - a_{n-2}$ with $a_0 = 3$ and $a_1 = 4$.

Solution. We know that $a_6 = 2a_5 - a_4$. So to find a_6 we need to find a_5 and a_4. Well

$$a_5 = 2a_4 - a_3 \qquad \text{and} \qquad a_4 = 2a_3 - a_2,$$

so if we can only find a_3 and a_2 we would be set. Of course

$$a_3 = 2a_2 - a_1 \qquad \text{and} \qquad a_2 = 2a_1 - a_0,$$

so we only need to find a_1 and a_0. But we are given these. Thus

$$\begin{aligned}
a_0 &= 3 \\
a_1 &= 4 \\
a_2 &= 2 \cdot 4 - 3 = 5 \\
a_3 &= 2 \cdot 5 - 4 = 6 \\
a_4 &= 2 \cdot 6 - 5 = 7 \\
a_5 &= 2 \cdot 7 - 6 = 8 \\
a_6 &= 2 \cdot 8 - 7 = 9.
\end{aligned}$$

Note that now we can guess a closed formula for the nth term of the sequence: $a_n = n + 3$. To be sure this will always work, we could plug in this formula into the recurrence relation:

$$\begin{aligned}
2a_{n-1} - a_{n-2} &= 2((n-1)+3) - ((n-2)+3) \\
&= 2n + 4 - n - 1 \\
&= n + 3 = a_n.
\end{aligned}$$

That is not quite enough though, since there can be multiple closed formulas that satisfy the same recurrence relation; we must also check that our closed formula agrees on the initial terms of the sequence. Since $a_0 = 0 + 3 = 3$ and $a_1 = 1 + 3 = 4$ are the correct initial conditions, we can now conclude we have the correct closed formula.

Finding closed formulas, or even recursive definitions, for sequences is not trivial. There is no one method for doing this. Just as in evaluating integrals or solving differential equations, it is useful to have a bag of tricks you can apply, but sometimes there is no easy answer.

One useful method is to relate a given sequence to another sequence for which we already know the closed formula. To do this, we need a few "known sequences" to compare mystery sequences to. Here are a few that are good to know. We will verify the formulas for these in the coming sections.

Common Sequences.

$1, 4, 9, 16, 25, \ldots$
The **square numbers**. The sequence $(s_n)_{n\geq 1}$ has closed formula $s_n = n^2$

$1, 3, 6, 10, 15, 21, \ldots$
The **triangular numbers**. The sequence $(T_n)_{n\geq 1}$ has closed formula $T_n = \frac{n(n+1)}{2}$.

$1, 2, 4, 8, 16, 32, \ldots$
The **powers of 2**. The sequence $(a_n)_{n\geq 0}$ with closed formula $a_n = 2^n$.

$1, 1, 2, 3, 5, 8, 13, \ldots$
The **Fibonacci numbers** (or Fibonacci sequence), defined recursively by $F_n = F_{n-1} + F_{n-2}$ with $F_1 = F_2 = 1$.

Example 2.1.4

Use the formulas $T_n = \frac{n(n+1)}{2}$ and $a_n = 2^n$ to find closed formulas that agree with the following sequences. Assume each first term corresponds to $n = 0$.

1. (b_n): $1, 2, 4, 7, 11, 16, 22, \ldots$.
2. (c_n): $3, 5, 9, 17, 33, \ldots$.
3. (d_n): $0, 2, 6, 12, 20, 30, 42, \ldots$.
4. (e_n): $3, 6, 10, 15, 21, 28, \ldots$.

5. (f_n): $0, 1, 3, 7, 15, 31, \ldots$.
6. (g_n) $3, 6, 12, 24, 48, \ldots$.
7. (h_n): $6, 10, 18, 34, 66, \ldots$.
8. (j_n): $15, 33, 57, 87, 123, \ldots$.

Solution. We wish to compare these sequences to the triangular numbers $(0, 1, 3, 6, 10, 15, 21, \ldots)$, when we start with $n = 0$, and the powers of 2: $(1, 2, 4, 8, 16, \ldots)$.

1. $(1, 2, 4, 7, 11, 16, 22, \ldots)$. Note that if subtract 1 from each term, we get the sequence (T_n). So we have $b_n = T_n + 1$. Therefore a closed formula is $b_n = \frac{n(n+1)}{2} + 1$. A quick check of the first few n confirms we have it right.
2. $(3, 5, 9, 17, 33, \ldots)$. Each term in this sequence is one more than a power of 2, so we might guess the closed formula is $c_n = a_n + 1 = 2^n + 1$. If we try this though, we get $c_0 = 2^0 + 1 = 2$ and $c_1 = 2^1 + 1 = 3$. We are off because the indices are shifted. What we really want is $c_n = a_{n+1} + 1$ giving $c_n = 2^{n+1} + 1$.
3. $(0, 2, 6, 12, 20, 30, 42, \ldots)$. Notice that all these terms are even. What happens if we factor out a 2? We get (T_n)! More precisely, we find that $d_n/2 = T_n$, so this sequence has closed formula $d_n = n(n + 1)$.
4. $(3, 6, 10, 15, 21, 28, \ldots)$. These are all triangular numbers. However, we are starting with 3 as our initial term instead of as our third term. So if we could plug in 2 instead of 0 into the formula for T_n, we would be set. Therefore the closed formula is $e_n = \frac{(n+2)(n+3)}{2}$ (where $n + 3$ came from $(n + 2) + 1$). Thinking about sequences as functions, we are doing a horizontal shift by 2: $e_n = T_{n+2}$ which would cause the graph to shift 2 units to the left.
5. $(0, 1, 3, 7, 15, 31, \ldots)$. Try adding 1 to each term and we get powers of 2. You might guess this because each term is a little more than twice the previous term (the powers of 2 are *exactly* twice the previous term). Closed formula: $f_n = 2^n - 1$.
6. $(3, 6, 12, 24, 48, \ldots)$. These numbers are also doubling each time, but are also all multiples of 3. Dividing each by 3 gives $1, 2, 4, 8, \ldots$. Aha. We get the closed formula $g_n = 3 \cdot 2^n$.

7. $(6, 10, 18, 34, 66, \ldots)$. To get from one term to the next, we almost double each term. So maybe we can relate this back to 2^n. Yes, each term is 2 more than a power of 2. So we get $h_n = 2^{n+2} + 2$ (the $n + 2$ is because the first term is 2 more than 2^2, not 2^0). Alternatively, we could have related this sequence to the second sequence in this example: starting with 3, 5, 9, 17, . . . we see that this sequence is twice the terms from that sequence. That sequence had closed formula $c_n = 2^{n+1} + 1$. Our sequence here would be twice this, so $h_n = 2(2^n + 1)$, which is the same as we got before.

8. $(15, 33, 57, 87, 123, \ldots)$. Try dividing each term by 3. That gives the sequence $5, 11, 19, 29, 41, \ldots$. Now add 1 to each term: $6, 12, 20, 30, 42, \ldots$, which is (d_n) in this example, except starting with 6 instead of 0. So let's start with the formula $d_n = n(n + 1)$. To start with the 6, we shift: $(n + 2)(n + 3)$. But this is one too many, so subtract 1: $(n + 2)(n + 3) - 1$. That gives us our sequence, but divided by 3. So we want $j_n = 3((n + 2)(n + 3) - 1)$.

Partial sums.

Some sequences naturally arise as the sum of terms of another sequence.

Example 2.1.5

Sam keeps track of how many push-ups she does each day of her "do lots of push-ups challenge." Let $(a_n)_{n\geq 1}$ be the sequence that describes the number of push-ups done on the nth day of the challenge. The sequence starts

$$3, 5, 6, 10, 9, 0, 12, \ldots.$$

Describe a sequence $(b_n)_{n\geq 1}$ that describes the total number of push-ups done by Sam after the nth day.

Solution. We can find the terms of this sequence easily enough.

$$3, 8, 14, 24, 33, 33, 45, \ldots.$$

Here b_1 is just a_1, but then

$$b_2 = 3 + 5 = a_1 + a_2,$$

$$b_3 = 3 + 5 + 6 = a_1 + a_2 + a_3,$$

and so on.

There are a few ways we might describe b_n in general. We could do so recursively as,

$$b_n = b_{n-1} + a_n,$$

since the total number of push-ups done after n days will be the number done after $n-1$ days, plus the number done on day n.

For something closer to a closed formula, we could write

$$b_n = a_1 + a_2 + a_3 + \cdots + a_n,$$

or the same thing using *summation notation*:

$$b_n = \sum_{i=1}^{n} a_i.$$

However, note that these are not really closed formulas since even if we had a formula for a_n, we would still have an increasing number of computations to do as n increases.

Given any sequence $(a_n)_{n\in\mathbb{N}}$, we can always form a new sequence $(b_n)_{n\in\mathbb{N}}$ by

$$b_n = a_0 + a_1 + a_2 + \cdots + a_n.$$

Since the terms of (b_n) are the sums of the initial part of the sequence (a_n) ways call (b_n) the **sequence of partial sums of** (a_n). Soon we will see that it is sometimes possible to find a closed formula for (b_n) from the closed formula for (a_n).

To simplify writing out these sums, we will often use notation like $\sum_{k=1}^{n} a_k$. This means add up the a_k's where k changes from 1 to n.

Example 2.1.6

Use $\sum$ notation to rewrite the sums:

1. $1 + 2 + 3 + 4 + \cdots + 100$
2. $1 + 2 + 4 + 8 + \cdots + 2^{50}$
3. $6 + 10 + 14 + \cdots + (4n - 2)$.

Solution.

1. $\sum_{k=1}^{100} k$
2. $\sum_{k=0}^{50} 2^k$
3. $\sum_{k=2}^{n} (4k - 2)$

If we want to multiply the a_k instead, we could write $\prod_{k=1}^{n} a_k$. For example, $\prod_{k=1}^{n} k = n!$.

EXERCISES

1. Find the closed formula for each of the following sequences by relating them to a well known sequence. Assume the first term given is a_1.

(a) $2, 5, 10, 17, 26, \ldots$

(b) $0, 2, 5, 9, 14, 20, \ldots$

(c) $8, 12, 17, 23, 30, \ldots$

(d) $1, 5, 23, 119, 719, \ldots$

2. For each sequence given below, find a closed formula for a_n, the nth term of the sequence (assume the first terms are a_0) by relating it to another sequence for which you already know the formula. In each case, briefly say how you got your answers.

(a) 4, 5, 7, 11, 19, 35, . . .

(b) 0, 3, 8, 15, 24, 35, . . .

(c) 6, 12, 20, 30, 42, . . .

(d) 0, 2, 7, 15, 26, 40, 57, . . . (Cryptic Hint: these might be called "house numbers")

3. Write out the first 5 terms (starting with a_0) of each of the sequences described below. Then give either a closed formula or a recursive definition for the sequence (whichever is NOT given in the problem).

(a) $a_n = \frac{1}{2}(n^2 + n)$.

(b) $a_n = 2a_{n-1} - a_{n-2}$ with $a_0 = 0$ and $a_1 = 1$.

(c) $a_n = na_{n-1}$ with $a_0 = 1$.

4. Consider the sequence $(a_n)_{n\geq1}$ that starts $1, 3, 5, 7, 9, \ldots$ (i.e., the odd numbers in order).

(a) Give a recursive definition and closed formula for the sequence.

(b) Write out the sequence $(b_n)_{n\geq2}$ of partial sums of (a_n). Write down the recursive definition for (b_n) and guess at the closed formula.

5. The Fibonacci sequence is $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ (where $F_0 = 0$).

(a) Write out the first few terms of the sequence of partial sums: 0, $0 + 1, 0 + 1 + 1, \ldots$

(b) Guess a formula for the sequence of partial sums expressed in terms of a single Fibonacci number. For example, you might say $F_0 + F_1 + \cdots + F_n = 3F_{n-1}^2 + n$, although that is definitely not correct.

6. Consider the three sequences below. For each, find a recursive definition. How are these sequences related?

(a) $2, 4, 6, 10, 16, 26, 42, \ldots$.

(b) $5, 6, 11, 17, 28, 45, 73, \ldots$.

(c) $0, 0, 0, 0, 0, 0, 0, \ldots$.

7. Write out the first few terms of the sequence given by $a_1 = 3$; $a_n = 2a_{n-1} + 4$. Then find a recursive definition for the sequence $10, 24, 52, 108, \ldots$.

8. Write out the first few terms of the sequence given by $a_n = n^2 - 3n + 1$. Then find a closed formula for the sequence (starting with a_1) $0, 2, 6, 12, 20, \ldots$.

9. Show that $a_n = 3 \cdot 2^n + 7 \cdot 5^n$ is a solution to the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$. What would the initial conditions need to be for this to be the closed formula for the sequence?

10. Show that $a_n = 2^n - 5^n$ is also a solution to the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$. What would the initial conditions need to be for this to be the closed formula for the sequence?

11. Find a closed formula for the sequence with recursive definition $a_n = 2a_{n-1} - a_{n-2}$ with $a_1 = 1$ and $a_2 = 2$.

12. Give two different recursive definitions for the sequence with closed formula $a_n = 3 + 2n$. Prove you are correct. At least one of the recursive definitions should makes use of two previous terms and no constants.

13. Use summation ($\sum$) or product ($\prod$) notation to rewrite the following.

(a) $2 + 4 + 6 + 8 + \cdots + 2n$.

(b) $1 + 5 + 9 + 13 + \cdots + 425$.

(c) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{50}$.

(d) $2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n$.

(e) $(\frac{1}{2})(\frac{2}{3})(\frac{3}{4}) \cdots (\frac{100}{101})$.

14. Expand the following sums and products. That is, write them out the long way.

(a) $\sum_{k=1}^{100}(3+4k)$.

(b) $\sum_{k=0}^{n} 2^k$.

(c) $\sum_{k=2}^{50} \frac{1}{(k^2-1)}$.

(d) $\prod_{k=2}^{100} \frac{k^2}{(k^2-1)}$.

(e) $\prod_{k=0}^{n}(2+3k)$.

15. Suppose you draw n lines in the plane so that every pair of lines cross (no lines are parallel) and no three lines cross at the same point. This will create some number of regions in the plane, including some unbounded regions. Call the number of regions R_n. Find a recursive formula for the number of regions created by n lines, and justify why your recursion is correct.

16. A **ternary** string is a sequence of 0's, 1's and 2's. Just like a bit string, but with three symbols.

Let's call a ternary string *good* provided it never contains a 2 followed immediately by a 0. Let G_n be the number of good strings of length n. For example, $G_1 = 3$, and $G_2 = 8$ (since of the 9 ternary strings of length 2, only one is not good).

Find, with justification, a recursive formula for G_n, and use it to compute G_5.

17. Consider bit strings with length l and weight k (so strings of l 0's and 1's, including k 1's). We know how to count the number of these for a fixed l and k. Now, we will count the number of strings for which the *sum* of the length and the weight is fixed. For example, let's count all the bit strings for which $l + k = 11$.

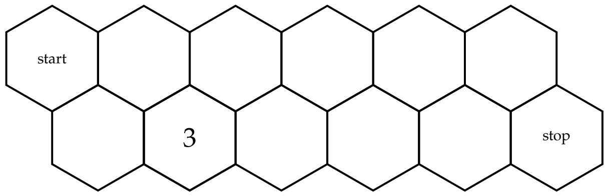
(a) Find examples of these strings of different lengths. What is the longest string possible? What is the shortest?

(b) How many strings are there of each of these lengths. Use this to count the total number of strings (with sum 11).

(c) The other approach: Let $n = l + k$ vary. How many strings have sum $n = 1$? How many have sum $n = 2$? And so on. Find and explain a recurrence relation for the sequence (a_n) which gives the number of strings with sum n.

(d) Describe what you have found above in terms of Pascal's Triangle. What pattern have you discovered?

18. When bees play chess, they use a hexagonal board like the one shown below. The queen bee can move one space at a time either directly to the right or angled up-right or down-right (but can never move leftwards). How many different paths can the queen take from the top left hexagon to the bottom right hexagon? Explain your answer, and this relates to the previous question. (As an example, there are three paths to get to the second hexagon on the bottom row.)

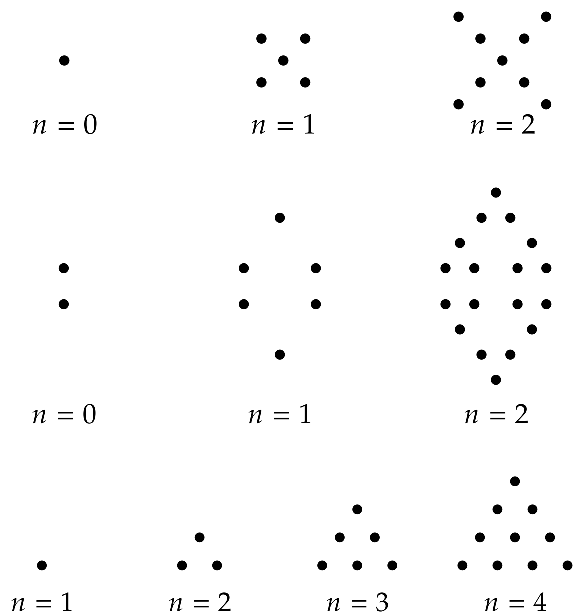


19. Let t_n denote the number of ways to tile a $2 \times n$ chessboard using 1×2 dominoes. Write out the first few terms of the sequence $(t_n)_{n \geq 1}$ and then give a recursive definition. Explain why your recursive formula is correct.

2.2 Arithmetic and Geometric Sequences

Investigate!

For the patterns of dots below, draw the next pattern in the sequence. Then give a recursive definition and a closed formula for the number of dots in the nth pattern.



Attempt the above activity before proceeding

We now turn to the question of finding closed formulas for particular types of sequences.

Arithmetic Sequences.

If the terms of a sequence differ by a constant, we say the sequence is **arithmetic**. If the initial term (a_0) of the sequence is a and the **common difference** is d, then we have,

Recursive definition: $a_n = a_{n-1} + d$ with $a_0 = a$.

Closed formula: $a_n = a + dn$.

How do we know this? For the recursive definition, we need to specify a_0. Then we need to express a_n in terms of a_{n-1}. If we call the first term a, then $a_0 = a$. For the recurrence relation, by the definition of an arithmetic sequence, the difference between successive terms is some constant, say d. So $a_n - a_{n-1} = d$, or in other words,

$$a_0 = a \qquad a_n = a_{n-1} + d.$$

To find a closed formula, first write out the sequence in general:

$$\begin{aligned} a_0 &= a \\ a_1 &= a_0 + d = a + d \\ a_2 &= a_1 + d = a + d + d = a + 2d \\ a_3 &= a_2 + d = a + 2d + d = a + 3d \\ &\vdots \end{aligned}$$

We see that to find the nth term, we need to start with a and then add d a bunch of times. In fact, add it n times. Thus $a_n = a + dn$.

Example 2.2.1

Find recursive definitions and closed formulas for the arithmetic sequences below. Assume the first term listed is a_0.

1. $2, 5, 8, 11, 14, \ldots$.
2. $50, 43, 36, 29, \ldots$.

Solution. First we should check that these sequences really are arithmetic by taking differences of successive terms. Doing so will reveal the common difference d.

1. $5 - 2 = 3$, $8 - 5 = 3$, etc. To get from each term to the next, we add three, so $d = 3$. The recursive definition is therefore $a_n = a_{n-1} + 3$ with $a_0 = 2$. The closed formula is $a_n = 2 + 3n$.
2. Here the common difference is -7, since we add -7 to 50 to get 43, and so on. Thus we have a recursive definition of $a_n = a_{n-1} - 7$ with $a_0 = 50$. The closed formula is $a_n = 50 - 7n$.

What about sequences like $2, 6, 18, 54, \ldots$? This is not arithmetic because the difference between terms is not constant. However, the *ratio* between successive terms is constant. We call such sequences **geometric**.

The recursive definition for the geometric sequence with initial term a and common ratio r is $a_n = a_{n-1} \cdot r; a_0 = a$. To get the next term we multiply the previous term by r. We can find the closed formula like we did for the arithmetic progression. Write

$$\begin{aligned} a_0 &= a \\ a_1 &= a_0 \cdot r \\ a_2 &= a_1 \cdot r = a_0 \cdot r \cdot r = a_0 \cdot r^2 \\ &\vdots \end{aligned}$$

We must multiply the first term a by r a number of times, n times to be precise. We get $a_n = a \cdot r^n$.

Geometric Sequences.

A sequence is called **geometric** if the ratio between successive terms is constant. Suppose the initial term a_0 is a and the **common ratio** is r. Then we have,

Recursive definition: $a_n = ra_{n-1}$ with $a_0 = a$.
Closed formula: $a_n = a \cdot r^n$.

Example 2.2.2

Find the recursive and closed formula for the geometric sequences below. Again, the first term listed is a_0.

1. $3, 6, 12, 24, 48, \ldots$
2. $27, 9, 3, 1, 1/3, \ldots$

Solution. Start by checking that these sequences really are geometric by dividing each term by its previous term. If this ratio really is constant, we will have found r.

1. $6/3 = 2$, $12/6 = 2$, $24/12 = 2$, etc. Yes, to get from any term to the next, we multiply by $r = 2$. So the recursive definition is $a_n = 2a_{n-1}$ with $a_0 = 3$. The closed formula is $a_n = 3 \cdot 2^n$.
2. The common ratio is $r = 1/3$. So the sequence has recursive definition $a_n = \frac{1}{3}a_{n-1}$ with $a_0 = 27$ and closed formula $a_n = 27 \cdot \frac{1}{3}^n$.

In the examples and formulas above, we assumed that the *initial* term was a_0. If your sequence starts with a_1, you can easily find the term that would have been a_0 and use that in the formula. For example, if we want a formula for the sequence $2, 5, 8, \ldots$ and insist that $2 = a_1$, then we can find $a_0 = -1$ (since the sequence is arithmetic with common difference 3, we have $a_0 + 3 = a_1$). Then the closed formula will be $a_n = -1 + 3n$.

Remark 2.2.3 If you look at other textbooks or online, you might find that their closed formulas for arithmetic and geometric sequences differ from ours. Specifically, you might find the formulas $a_n = a + (n-1)d$ (arithmetic) and $a_n = a \cdot r^{n-1}$ (geometric). Which is correct? Both! In our case, we take a to be a_0. If instead we had a_1 as our initial term, we would get the (slightly more complicated) formulas you find elsewhere.

Sums of Arithmetic and Geometric Sequences

Investigate!

Your neighborhood grocery store has a candy machine full of Skittles.

1. Suppose that the candy machine currently holds exactly 650 Skittles, and every time someone inserts a quarter, exactly 7 Skittles come out of the machine.

   (a) How many Skittles will be left in the machine after 20 quarters have been inserted?

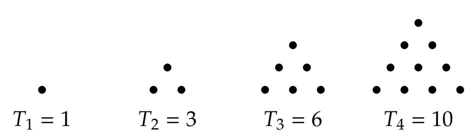
   (b) Will there ever be exactly zero Skittles left in the machine? Explain.

2. What if the candy machine gives 7 Skittles to the first customer who put in a quarter, 10 to the second, 13 to the third, 16 to the fourth, etc. How many Skittles has the machine given out after 20 quarters are put into the machine?

3. Now, what if the machine gives 4 Skittles to the first customer, 7 to the second, 12 to the third, 19 to the fourth, etc. How many Skittles has the machine given out after 20 quarters are put into the machine?

Attempt the above activity before proceeding

Look at the sequence $(T_n)_{n\geq 1}$ which starts $1, 3, 6, 10, 15, \ldots$. These are called the **triangular numbers** since they represent the number of dots in an equilateral triangle (think of how you arrange 10 bowling pins: a row of 4 plus a row of 3 plus a row of 2 and a row of 1).



Is this sequence arithmetic? No, since $3 - 1 = 2$ and $6 - 3 = 3 \neq 2$, so there is no common difference. Is the sequence geometric? No. $3/1 = 3$ but $6/3 = 2$, so there is no common ratio. What to do?

Notice that the *differences* between terms *do* form an arithmetic sequence: $2, 3, 4, 5, 6, \ldots$. This means that the nth term of the sequence (T_n) is the *sum* of the first n terms in the sequence $1, 2, 3, 4, 5, \ldots$. We say that (T_n) is the **sequence of partial sums** of the sequence $1, 2, 3, \ldots$ (*partial* sums because we are not taking the sum of all infinitely many terms).

This should become clearer if we write the triangular numbers like this:

$$
\begin{aligned}
1 &= 1 \\
3 &= 1 + 2 \\
6 &= 1 + 2 + 3 \\
10 &= 1 + 2 + 3 + 4 \\
\vdots &\quad \vdots \\
T_n &= 1 + 2 + 3 + \cdots + n.
\end{aligned}
$$

If we know how to add up the terms of an arithmetic sequence, we could find a closed formula for a sequence whose differences are the terms of that arithmetic sequence. Consider how we could find the sum of the first 100 positive integers (that is, T_{100}). Instead of adding them in order, we regroup and add $1 + 100 = 101$. The next pair to combine is $2 + 99 = 101$. Then $3 + 98 = 101$. Keep going. This gives 50 pairs which each add up to 101, so $T_{100} = 101 \cdot 50 = 5050$.[1]

In general, using this same sort of regrouping, we find that $T_n = \frac{n(n+1)}{2}$. Incidentally, this is exactly the same as $\binom{n+1}{2}$, which makes sense if you think of the triangular numbers as counting the number of handshakes that take place at a party with $n + 1$ people: the first person shakes n hands, the next shakes an additional $n - 1$ hands and so on.

The point of all of this is that some sequences, while not arithmetic or geometric, can be interpreted as the sequence of partial sums of arithmetic and geometric sequences. Luckily there are methods we can use to compute these sums quickly.

Summing Arithmetic Sequences: Reverse and Add

Here is a technique that allows us to quickly find the sum of an arithmetic sequence.

Example 2.2.4

Find the sum: $2 + 5 + 8 + 11 + 14 + \cdots + 470$.

Solution. The idea is to mimic how we found the formula for triangular numbers. If we add the first and last terms, we get 472. The second term and second-to-last term also add up to 472. To

[1]This insight is usually attributed to Carl Friedrich Gauss, one of the greatest mathematicians of all time, who discovered it as a child when his unpleasant elementary teacher thought he would keep the class busy by requiring them to compute the lengthy sum.

keep track of everything, we might express this as follows. Call the sum S. Then,

$$\begin{array}{rrcrcrcrcrcr} & S = & 2 & + & 5 & + & 8 & +\cdots+ & 467 & + & 470 \\ + & S = & 470 & + & 467 & + & 464 & +\cdots+ & 5 & + & 2 \\ \hline & 2S = & 472 & + & 472 & + & 472 & +\cdots+ & 472 & + & 472 \end{array}$$

To find $2S$ then we add 472 to itself a number of times. What number? We need to decide how many terms (**summands**) are in the sum. Since the terms form an arithmetic sequence, the nth term in the sum (counting 2 as the 0th term) can be expressed as $2 + 3n$. If $2 + 3n = 470$ then $n = 156$. So n ranges from 0 to 156, giving 157 terms in the sum. This is the number of 472's in the sum for $2S$. Thus

$$2S = 157 \cdot 472 = 74104.$$

It is now easy to find S:

$$S = 74104/2 = 37052.$$

This will work for the sum of any *arithmetic* sequence. Call the sum S. Reverse and add. This produces a single number added to itself many times. Find the number of times. Multiply. Divide by 2. Done.

Example 2.2.5

Find a closed formula for $6 + 10 + 14 + \cdots + (4n - 2)$.

Solution. Again, we have a sum of an arithmetic sequence. How many terms are in the sequence? Clearly each term in the sequence has the form $4k-2$ (as evidenced by the last term). For which values of k though? To get 6, $k = 2$. To get $4n - 2$ take $k = n$. So to find the number of terms, we must count the number of integers in the range $2, 3, \ldots, n$. This is $n - 1$. (There are n numbers from 1 to n, so one less if we start with 2.)

Now reverse and add:

$$\begin{array}{rrcrcrcrcr} & S = & 6 & + & 10 & +\cdots+ & 4n-6 & + & 4n-2 \\ + & S = & 4n-2 & + & 4n-6 & +\cdots+ & 10 & + & 6 \\ \hline & 2S = & 4n+4 & + & 4n+4 & +\cdots+ & 4n+4 & + & 4n+4 \end{array}$$

Since there are $n - 1$ terms, we get

$$2S = (n-1)(4n+4) \qquad \text{so} \qquad S = \frac{(n-1)(4n+4)}{2}.$$

Besides finding sums, we can use this technique to find closed formulas for sequences we recognize as sequences of partial sums.

Example 2.2.6

Use partial sums to find a closed formula for $(a_n)_{n\geq 0}$ which starts $2, 3, 7, 14, 24, 37, \ldots\ldots$

Solution. First, if you look at the differences between terms, you get a sequence of differences: $1, 4, 7, 10, 13, \ldots$, which is an arithmetic sequence. Written another way:

$$\begin{aligned} a_0 &= 2 \\ a_1 &= 2+1 \\ a_2 &= 2+1+4 \\ a_3 &= 2+1+4+7 \end{aligned}$$

and so on. We can write the general term of (a_n) in terms of the arithmetic sequence as follows:

$$a_n = 2+1+4+7+10+\cdots+(1+3(n-1))$$

(we use $1+3(n-1)$ instead of $1+3n$ to get the indices to line up correctly; for a_3 we add up to 7, which is $1+3(3-1)$).

We can reverse and add, but the initial 2 does not fit our pattern. This just means we need to keep the 2 out of the reverse part:

$$\begin{array}{rrcccccccc} & a_n = & 2 & + & 1 & + & 4 & +\cdots+ & 1+3(n-1) \\ + & a_n = & 2 & + & 1+3(n-1) & + & 1+3(n-2) & +\cdots+ & 1 \\ \hline & 2a_n = & 4 & + & 2+3(n-1) & + & 2+3(n-1) & +\cdots+ & 2+3(n-1) \end{array}$$

Not counting the first term (the 4) there are n summands of $2+3(n-1) = 3n-1$ so the right-hand side becomes $4+(3n-1)n$.

Finally, solving for a_n we get

$$a_n = \frac{4+(3n-1)n}{2}.$$

Just to be sure, we check $a_0 = \frac{4}{2} = 2$, $a_1 = \frac{4+2}{2} = 3$, etc. We have the correct closed formula.

Summing Geometric Sequences: Multiply, Shift and Subtract

To find the sum of a geometric sequence, we cannot just reverse and add. Do you see why? The reason we got the same term added to itself many times is because there was a constant difference. So as we added that difference in one direction, we subtracted the difference going the other way, leaving a constant total. For geometric sums, we have a different technique.

Example 2.2.7

What is $3 + 6 + 12 + 24 + \cdots + 12288$?

Solution. Multiply each term by 2, the common ratio. You get $2S = 6+12+24+\cdots+24576$. Now subtract: $2S - S = -3 + 24576 = 24573$. Since $2S - S = S$, we have our answer.

To better see what happened in the above example, try writing it this way:

$$\begin{array}{rrlll} & S = & 3+ & 6 + 12 + 24 + \cdots + 12288 & \\ - & 2S = & & 6 + 12 + 24 + \cdots + 12288 & +24576 \\ \hline & -S = & 3+ & 0 + 0 + 0 + \cdots + 0 & -24576 \end{array}$$

Then divide both sides by -1 and we have the same result for S. The idea is, by multiplying the sum by the common ratio, each term becomes the next term. We shift over the sum to get the subtraction to mostly cancel out, leaving just the first term and new last term.

Example 2.2.8

Find a closed formula for $S(n) = 2 + 10 + 50 + \cdots + 2 \cdot 5^n$.

Solution. The common ratio is 5. So we have

$$\begin{array}{rrll} & S & = 2 + 10 + 50 + \cdots + 2 \cdot 5^n & \\ - & 5S & = \quad 10 + 50 + \cdots + 2 \cdot 5^n & + 2 \cdot 5^{n+1} \\ \hline & -4S & = 2 - 2 \cdot 5^{n+1} & \end{array}$$

Thus $S = \dfrac{2 - 2 \cdot 5^{n+1}}{-4}$

Even though this might seem like a new technique, you have probably used it before.

Example 2.2.9

Express $0.464646\ldots$ as a fraction.

Solution. Let $N = 0.46464646\ldots$. Consider $0.01N$. We get:

$$\begin{array}{rrl} & N = & 0.4646464\ldots \\ - & 0.01N = & 0.00464646\ldots \\ \hline & 0.99N = & 0.46 \end{array}$$

So $N = \frac{46}{99}$. What have we done? We viewed the repeating decimal $0.464646\ldots$ as a sum of the geometric sequence $0.46, 0.0046, 0.000046, \ldots$ The common ratio is 0.01. The only real difference is that we are now computing an *infinite* geometric sum, we do not have the extra "last" term to consider. Really, this is the result of taking a limit as you would in calculus when you compute *infinite* geometric sums.

Exercises

1. Consider the sequence $5, 9, 13, 17, 21, \ldots$ with $a_1 = 5$

(a) Give a recursive definition for the sequence.

(b) Give a closed formula for the nth term of the sequence.

(c) Is 2013 a term in the sequence? Explain.

(d) How many terms does the sequence $5, 9, 13, 17, 21, \ldots, 533$ have?

(e) Find the sum: $5 + 9 + 13 + 17 + 21 + \cdots + 533$. Show your work.

(f) Use what you found above to find b_n, the n^{th} term of $1, 6, 15, 28, 45, \ldots$, where $b_0 = 1$

2. Consider the sequence $(a_n)_{n\geq 0}$ which starts $8, 14, 20, 26, \ldots$.

(a) What is the next term in the sequence?

(b) Find a formula for the nth term of this sequence.

(c) Find the sum of the first 100 terms of the sequence: $\sum_{k=0}^{99} a_k$.

3. Consider the sum $4 + 11 + 18 + 25 + \cdots + 249$.

(a) How many terms (summands) are in the sum?

(b) Compute the sum using a technique discussed in this section.

4. Consider the sequence $1, 7, 13, 19, \ldots, 6n + 7$.

(a) How many terms are there in the sequence? Your answer will be in terms of n.

(b) What is the second-to-last term?

(c) Find the sum of all the terms in the sequence, in terms of n.

5. Find $5 + 7 + 9 + 11 + \cdots + 521$ using a technique from this section.

6. Find $5 + 15 + 45 + \cdots + 5 \cdot 3^{20}$.

7. Find $1 - \frac{2}{3} + \frac{4}{9} - \cdots + \frac{2^{30}}{3^{30}}$.

8. Find x and y such that $27, x, y, 1$ is part of an arithmetic sequence.

Then find x and y so that the sequence is part of a geometric sequence.

(Warning: x and y might not be integers.)

9. Find x and y such that $5, x, y, 32$ is part of an arithmetic sequence.

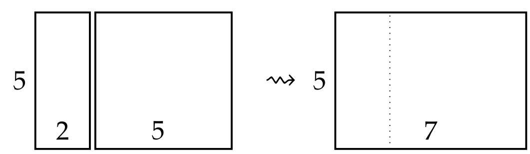
Then find x and y so that the sequence is part of a geometric sequence.

(Warning: x and y might not be integers.)

10. Is there a pair of integers (a, b) such that a, x_1, y_1, b is part of an arithmetic sequences and a, x_2, y_2, b is part of a geometric sequence with x_1, x_2, y_1, y_2 all integers?

11. Consider the sequence $2, 7, 15, 26, 40, 57, \ldots$ (with $a_0 = 2$). By looking at the differences between terms, express the sequence as a sequence of partial sums. Then find a closed formula for the sequence by computing the nth partial sum.

12. Starting with any rectangle, we can create a new, larger rectangle by attaching a square to the longer side. For example, if we start with a 2×5 rectangle, we would glue on a 5×5 square, forming a 5×7 rectangle:



The next rectangle would be formed by attaching a 7×7 square to the top or bottom of the 5×7 rectangle.

(a) Create a sequence of rectangles using this rule starting with a 1×2 rectangle. Then write out the sequence of *perimeters* for the rectangles (the first term of the sequence would be 6, since the perimeter of a 1×2 rectangle is 6 - the next term would be 10).

(b) Repeat the above part this time starting with a 1×3 rectangle.

(c) Find recursive formulas for each of the sequences of perimeters you found in parts (a) and (b). Don't forget to give the initial conditions as well.

(d) Are the sequences arithmetic? Geometric? If not, are they *close* to being either of these (i.e., are the differences or ratios *almost* constant)? Explain.

13. If you have enough toothpicks, you can make a large triangular grid. Below, are the triangular grids of size 1 and of size 2. The size 1 grid requires 3 toothpicks, the size 2 grid requires 9 toothpicks.

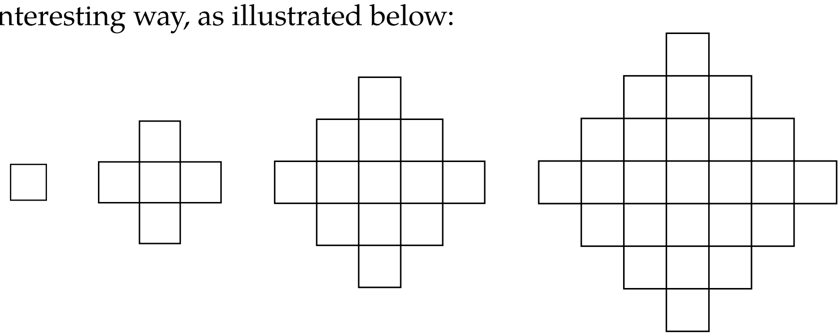
(a) Let t_n be the number of toothpicks required to make a size n triangular grid. Write out the first 5 terms of the sequence $t_1, t_2, \ldots$.

(b) Find a recursive definition for the sequence. Explain why you are correct.

(c) Is the sequence arithmetic or geometric? If not, is it the sequence of partial sums of an arithmetic or geometric sequence? Explain why your answer is correct.

(d) Use your results from part (c) to find a closed formula for the sequence. Show your work.

14. If you were to shade in a $n \times n$ square on graph paper, you could do it the boring way (with sides parallel to the edge of the paper) or the interesting way, as illustrated below:



The interesting thing here, is that a 3×3 square now has area 13. Our goal is the find a formula for the area of a $n \times n$ (diagonal) square.

(a) Write out the first few terms of the sequence of areas (assume $a_1 = 1$, $a_2 = 5$, etc). Is the sequence arithmetic or geometric? If not, is it the sequence of partial sums of an arithmetic or geometric sequence? Explain why your answer is correct, referring to the diagonal squares.

(b) Use your results from part (a) to find a closed formula for the sequence. Show your work. Note, while there are lots of ways to find a closed formula here, you should use partial sums specifically.

(c) Find the closed formula in as many other interesting ways as you can.

15. Here is a surprising use of sequences to answer a counting question: How many license plates consist of 6 symbols, using only the three numerals 1, 2, and 3 and the four letters a, b, c, and d, so that no numeral appears after any letter? For example, "31ddac", "123321", and "ababab" are each acceptable license plates, but "13ba2c" is not.

(a) First answer this question by considering different cases: how many of the license plates contain no numerals? How many contain one numeral, etc.

(b) Now use the techniques of this section to show why the answer is $4^7 - 3^7$.

2.3 Polynomial Fitting

Investigate!

A standard 8×8 chessboard contains 64 squares. Actually, this is just the number of unit squares. How many squares of all sizes are there on a chessboard? Start with smaller boards: 1×1, 2×2, 3×3, etc. Find a formula for the total number of squares in an $n \times n$ board.

Attempt the above activity before proceeding

So far we have seen methods for finding the closed formulas for arithmetic and geometric sequences. Since we know how to compute the sum of the first n terms of arithmetic and geometric sequences, we can compute the closed formulas for sequences which have an arithmetic (or geometric) sequence of differences between terms. But what if we consider a sequence which is the sum of the first n terms of a sequence which is itself the sum of an arithmetic sequence?

Before we get too carried away, let's consider an example: How many squares (of all sizes) are there on a chessboard? A chessboard consists of 64 squares, but we also want to consider squares of longer side length. Even though we are only considering an 8×8 board, there is already a lot to count. So instead, let us build a sequence: the first term will be the number of squares on a 1×1 board, the second term will be the number of squares on a 2×2 board, and so on. After a little thought, we arrive at the sequence

$$1, 5, 14, 30, 55, \ldots.$$

This sequence is not arithmetic (or geometric for that matter), but perhaps it's sequence of differences is. For differences we get

$$4, 9, 16, 25, \ldots.$$

Not a huge surprise: one way to count the number of squares in a 4×4 chessboard is to notice that there are 16 squares with side length 1, 9 with side length 2, 4 with side length 3 and 1 with side length 4. So the original sequence is just the sum of squares. Now this sequence of differences is not arithmetic since it's sequence of differences (the differences of the differences of the original sequence) is not constant. In fact, this sequence of **second differences** is

$$5, 7, 9, \ldots,$$

which *is* an arithmetic sequence (with constant difference 2). Notice that our original sequence had **third differences** (that is, differences of differences of differences of the original) constant. We will call such a sequence Δ^3-constant. The sequence $1, 4, 9, 16, \ldots$ has second differences constant, so it will be a Δ^2-constant sequence. In general, we will say a sequence is a Δ^k**-constant** sequence if the kth differences are constant.

Example 2.3.1

Which of the following sequences are Δ^k-constant for some value of k?

1. $2, 3, 7, 14, 24, 37, \ldots$.
2. $1, 8, 27, 64, 125, 216, \ldots$.
3. $1, 2, 4, 8, 16, 32, 64, \ldots$.

Solution.

1. This is the sequence from Example 2.2.6, in which we found a closed formula by recognizing the sequence as the sequence of partial sums of an arithmetic sequence. Indeed, the sequence of first differences is $1, 4, 7, 10, 13, \ldots$, which itself has differences $3, 3, 3, 3, \ldots$. Thus $2, 3, 7, 14, 24, 37, \ldots$ is a Δ^2-constant sequence.
2. These are the perfect cubes. The sequence of first differences is $7, 19, 37, 61, 91, \ldots$; the sequence of second differences is $12, 18, 24, 30, \ldots$; the sequence of third differences is constant: $6, 6, 6, \ldots$. Thus the perfect cubes are a Δ^3-constant sequence.
3. If we take first differences we get $1, 2, 4, 8, 16, \ldots$. Wait, what? That's the sequence we started with. So taking second differences will give us the same sequence again. No matter how many times we repeat this we will always have the same sequence, which in particular means no finite number of differences will be constant. Thus this sequence is not Δ^k-constant for any k.

The Δ^0-constant sequences are themselves constant, so a closed formula for them is easy to compute (it's just the constant). The Δ^1-constant sequences are arithmetic and we have a method for finding closed formulas for them as well. Every Δ^2-constant sequence is the sum of an arithmetic sequence so we can find formulas for these as well. But notice that the format of the closed formula for a Δ^2-constant sequence is always quadratic. For example, the square numbers are Δ^2-constant with closed formula $a_n = n^2$. The triangular numbers (also Δ^2-constant) have closed formula $a_n = \frac{n(n+1)}{2}$, which when multiplied out gives you an n^2 term as well. It appears that every time we increase the complexity of the sequence, that is, increase the number of differences before we get constants, we also increase the degree of the polynomial used for the closed formula. We go from constant to linear to quadratic. The sequence of differences between terms tells us something about the rate of growth of the sequence. If a sequence is growing at a constant rate, then the formula for the sequence will be linear. If the sequence is growing at a rate which itself is growing at a constant rate, then the formula is quadratic. You have seen this elsewhere: if a function has a constant second derivative (rate of change) then the function must be quadratic.

This works in general:

Finite Differences.

The closed formula for a sequence will be a degree k polynomial if and only if the sequence is Δ^k-constant (i.e., the kth sequence of differences is constant).

This tells us that the sequence of numbers of squares on a chessboard, $1, 5, 14, 30, 55, \ldots$, which we saw to be Δ^3-constant, will have a cubic (degree 3 polynomial) for its closed formula.

Now once we know what format the closed formula for a sequence will take, it is much easier to actually find the closed formula. In the case that the closed formula is a degree k polynomial, we just need $k+1$ data points to "fit" the polynomial to the data.

Example 2.3.2

Find a formula for the sequence $3, 7, 14, 24, \ldots$. Assume $a_1 = 3$.

Solution. First, check to see if the formula has constant differences at some level. The sequence of first differences is $4, 7, 10, \ldots$ which is arithmetic, so the sequence of second differences is constant. The sequence is Δ^2-constant, so the formula for a_n will be a degree 2 polynomial. That is, we know that for some constants a, b, and c,

$$a_n = an^2 + bn + c.$$

Now to find a, b, and c. First, it would be nice to know what a_0 is, since plugging in $n = 0$ simplifies the above formula greatly. In this case, $a_0 = 2$ (work backwards from the sequence of constant differences). Thus

$$a_0 = 2 = a \cdot 0^2 + b \cdot 0 + c,$$

so $c = 2$. Now plug in $n = 1$ and $n = 2$. We get

$$a_1 = 3 = a + b + 2$$

$$a_2 = 7 = a4 + b2 + 2.$$

At this point we have two (linear) equations and two unknowns, so we can solve the system for a and b (using substitution or elimination or even matrices). We find $a = \frac{3}{2}$ and $b = \frac{-1}{2}$, so $a_n = \frac{3}{2}n^2 - \frac{1}{2}n + 2$.

Example 2.3.3

Find a closed formula for the number of squares on an $n \times n$ chessboard.

Solution. We have seen that the sequence $1, 5, 14, 30, 55, \ldots$ is Δ^3-constant, so we are looking for a degree 3 polynomial. That is,

$$a_n = an^3 + bn^2 + cn + d.$$

We can find d if we know what a_0 is. Working backwards from the third differences, we find $a_0 = 0$ (unsurprisingly, since there are no squares on a 0×0 chessboard). Thus $d = 0$. Now plug in $n = 1$, $n = 2$, and $n = 3$:

$$\begin{aligned} 1 =& a + b + c \\ 5 =& 8a + 4b + 2c \\ 14 =& 27a + 9b + 3c. \end{aligned}$$

If we solve this system of equations we get $a = \frac{1}{3}$, $b = \frac{1}{2}$ and $c = \frac{1}{6}$. Therefore the number of squares on an $n \times n$ chessboard is $a_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{1}{6}n(n+1)(2n+1)$.

Note: Since the squares-on-a-chessboard problem is really asking for the sum of squares, we now have a nice formula for $\sum_{k=1}^{n} k^2$.

Not all sequences will have polynomials as their closed formula. We can use the theory of finite differences to identify these.

Example 2.3.4

Determine whether the following sequences can be described by a polynomial, and if so, of what degree.

1. $1, 2, 4, 8, 16, \ldots$

2. $0, 7, 50, 183, 484, 1055, \ldots$

3. $1, 1, 2, 3, 5, 8, 13, \ldots$

Solution.

1. As we saw in Example 2.3.1, this sequence is not Δ^k-constant for any k. Therefore the closed formula for the sequence is not a polynomial. In fact, we know the closed formula is $a_n = 2^n$, which grows faster than any polynomial (so is not a polynomial).
2. The sequence of first differences is $7, 43, 133, 301, 571, \ldots$. The second differences are: $36, 90, 168, 270, \ldots$. Third difference: $54, 78, 102, \ldots$. Fourth differences: $24, 24, \ldots$. As far as we can tell, this sequence of differences is constant so the sequence is Δ^4-constant and as such the closed formula is a degree 4 polynomial.
3. This is the Fibonacci sequence. The sequence of first differences is $0, 1, 1, 2, 3, 5, 8, \ldots$, the second differences are $1, 0, 1, 1, 2, 3, 5 \ldots$. We notice that after the first few terms, we get the original sequence back. So there will never be constant differences, so the closed formula for the Fibonacci sequence is not a polynomial.

EXERCISES

1. Use polynomial fitting to find the formula for the nth term of the sequence $(a_n)_{n\geq 0}$ which starts,

$$0, 2, 6, 12, 20, \ldots.$$

2. Use polynomial fitting to find the formula for the nth term of the sequence $(a_n)_{n\geq 0}$ which starts,

$$1, 2, 4, 8, 15, 26 \ldots.$$

3. Use polynomial fitting to find the formula for the nth term of the sequence $(a_n)_{n\geq 0}$ which starts,

$$2, 5, 11, 21, 36, \ldots.$$

4. Use polynomial fitting to find the formula for the nth term of the sequence $(a_n)_{n\geq 0}$ which starts,

$$3, 6, 12, 22, 37, 58, \ldots.$$

5. Make up sequences that have

(a) $3, 3, 3, 3, \ldots$ as its second differences.

(b) $1, 2, 3, 4, 5, \ldots$ as its third differences.

(c) $1, 2, 4, 8, 16, \ldots$ as its 100th differences.

6. Consider the sequence $1, 3, 7, 13, 21, \ldots$. Explain how you know the closed formula for the sequence will be quadratic. Then "guess" the correct formula by comparing this sequence to the squares $1, 4, 9, 16, \ldots$ (do not use polynomial fitting).

7. Use a similar technique as in the previous exercise to find a closed formula for the sequence $2, 11, 34, 77, 146, 247, \ldots$.

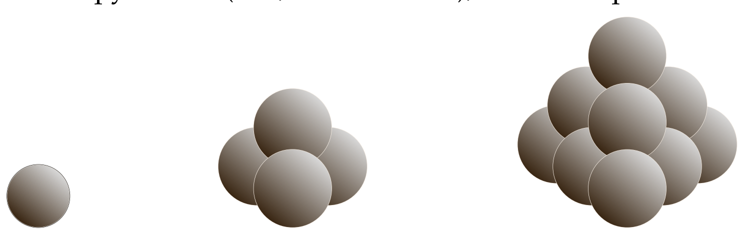
8. Suppose $a_n = n^2 + 3n + 4$. Find a closed formula for the sequence of differences by computing $a_n - a_{n-1}$.

9. Generalize Exercise 2.3.8: Find a closed formula for the sequence of differences of $a_n = an^2 + bn + c$. That is, prove that every quadratic sequence has arithmetic differences.

10. Can you use polynomial fitting to find the formula for the nth term of the sequence $4, 7, 11, 18, 29, 47, \ldots$? Explain why or why not.

11. Will the nth sequence of differences of $2, 6, 18, 54, 162, \ldots$ ever be constant? Explain.

12. In their down time, ghost pirates enjoy stacking cannonballs in triangular based pyramids (aka, tetrahedrons), like those pictured here:



Note, these are solid tetrahedrons, so there will be some cannonballs obscured from view (the picture on the right has one cannonball in the back not shown in the picture, for example)

The pirates wonder how many cannonballs would be required to build a pyramid 15 layers high (thus breaking the world cannonball stacking record). Can you help?

(a) Let $P(n)$ denote the number of cannonballs needed to create a pyramid n layers high. So $P(1) = 1$, $P(2) = 4$, and so on. Calculate $P(3)$, $P(4)$ and $P(5)$.

(b) Use polynomial fitting to find a closed formula for $P(n)$. Show your work.

(c) Answer the pirate's question: how many cannonballs do they need to make a pyramid 15 layers high?

(d) Bonus: Locate this sequence in Pascal's triangle. Why does that make sense?

2.4 Solving Recurrence Relations

Investigate!

Consider the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}.$$

1. What sequence do you get if the initial conditions are $a_0 = 1$, $a_1 = 2$? Give a closed formula for this sequence.
2. What sequence do you get if the initial conditions are $a_0 = 1$, $a_1 = 3$? Give a closed formula.
3. What if $a_0 = 2$ and $a_1 = 5$? Find a closed formula.

Attempt the above activity before proceeding

We have seen that it is often easier to find recursive definitions than closed formulas. Lucky for us, there are a few techniques for converting recursive definitions to closed formulas. Doing so is called **solving a recurrence relation**. Recall that the recurrence relation is a recursive definition without the initial conditions. For example, the recurrence relation for the Fibonacci sequence is $F_n = F_{n-1} + F_{n-2}$. (This, together with the initial conditions $F_0 = 0$ and $F_1 = 1$ give the entire recursive *definition* for the sequence.)

Example 2.4.1

Find a recurrence relation and initial conditions for $1, 5, 17, 53, 161, 485 \ldots$.

Solution. Finding the recurrence relation would be easier if we had some context for the problem (like the Tower of Hanoi, for example). Alas, we have only the sequence. Remember, the recurrence relation tells you how to get from previous terms to future terms. What is going on here? We could look at the differences between terms: $4, 12, 36, 108, \ldots$. Notice that these are growing by a factor of 3. Is the original sequence as well? $1 \cdot 3 = 3$, $5 \cdot 3 = 15$, $17 \cdot 3 = 51$ and so on. It appears that we always end up with 2 less than the next term. Aha!

So $a_n = 3a_{n-1} + 2$ is our recurrence relation and the initial condition is $a_0 = 1$.

We are going to try to *solve* these recurrence relations. By this we mean something very similar to solving differential equations: we want to find a function of n (a closed formula) which satisfies the recurrence relation, as well as the initial condition.[2] Just like for differential equations, finding a solution might be tricky, but checking that the solution is correct is easy.

Example 2.4.2

Check that $a_n = 2^n + 1$ is a solution to the recurrence relation $a_n = 2a_{n-1} - 1$ with $a_1 = 3$.

Solution. First, it is easy to check the initial condition: a_1 should be $2^1 + 1$ according to our closed formula. Indeed, $2^1 + 1 = 3$, which is what we want. To check that our proposed solution satisfies the recurrence relation, try plugging it in.

$$\begin{aligned} 2a_{n-1} - 1 &= 2(2^{n-1} + 1) - 1 \\ &= 2^n + 2 - 1 \\ &= 2^n + 1 \\ &= a_n. \end{aligned}$$

That's what our recurrence relation says! We have a solution.

Sometimes we can be clever and solve a recurrence relation by inspection. We generate the sequence using the recurrence relation and keep track of what we are doing so that we can see how to jump to finding just the a_n term. Here are two examples of how you might do that.

Telescoping refers to the phenomenon when many terms in a large sum cancel out—so the sum "telescopes." For example:

$$(2 - 1) + (3 - 2) + (4 - 3) + \cdots + (100 - 99) + (101 - 100) = -1 + 101$$

because every third term looks like: $2 + -2 = 0$, and then $3 + -3 = 0$ and so on.

We can use this behavior to solve recurrence relations. Here is an example.

Example 2.4.3

Solve the recurrence relation $a_n = a_{n-1} + n$ with initial term $a_0 = 4$.

Solution. To get a feel for the recurrence relation, write out the first few terms of the sequence: $4, 5, 7, 10, 14, 19, \ldots$. Look at the difference between terms. $a_1 - a_0 = 1$ and $a_2 - a_1 = 2$ and so on. The key thing here is that the difference between terms is n. We

[2] Recurrence relations are sometimes called difference equations since they can describe the difference between terms and this highlights the relation to differential equations further.

can write this explicitly: $a_n - a_{n-1} = n$. Of course, we could have arrived at this conclusion directly from the recurrence relation by subtracting a_{n-1} from both sides.

Now use this equation over and over again, changing n each time:

$$\begin{aligned} a_1 - a_0 &= 1 \\ a_2 - a_1 &= 2 \\ a_3 - a_2 &= 3 \\ \vdots \quad & \quad \vdots \\ a_n - a_{n-1} &= n. \end{aligned}$$

Add all these equations together. On the right-hand side, we get the sum $1 + 2 + 3 + \cdots + n$. We already know this can be simplified to $\frac{n(n+1)}{2}$. What happens on the left-hand side? We get

$$(a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}).$$

This sum telescopes. We are left with only the $-a_0$ from the first equation and the a_n from the last equation. Putting this all together we have $-a_0 + a_n = \frac{n(n+1)}{2}$ or $a_n = \frac{n(n+1)}{2} + a_0$. But we know that $a_0 = 4$. So the solution to the recurrence relation, subject to the initial condition is

$$a_n = \frac{n(n+1)}{2} + 4.$$

(Now that we know that, we should notice that the sequence is the result of adding 4 to each of the triangular numbers.)

The above example shows a way to solve recurrence relations of the form $a_n = a_{n-1} + f(n)$ where $\sum_{k=1}^{n} f(k)$ has a known closed formula. If you rewrite the recurrence relation as $a_n - a_{n-1} = f(n)$, and then add up all the different equations with n ranging between 1 and n, the left-hand side will always give you $a_n - a_0$. The right-hand side will be $\sum_{k=1}^{n} f(k)$, which is why we need to know the closed formula for that sum.

However, telescoping will not help us with a recursion such as $a_n = 3a_{n-1} + 2$ since the left-hand side will not telescope. You will have $-3a_{n-1}$'s but only one a_{n-1}. However, we can still be clever if we use **iteration**.

We have already seen an example of iteration when we found the closed formula for arithmetic and geometric sequences. The idea is, we *iterate* the process of finding the next term, starting with the known initial condition, up until we have a_n. Then we simplify. In the arithmetic sequence example, we simplified by multiplying d by the number of times we add it to a when we get to a_n, to get from $a_n = a + d + d + d + \cdots + d$ to $a_n = a + dn$.

To see how this works, let's go through the same example we used for telescoping, but this time use iteration.

Example 2.4.4

Use iteration to solve the recurrence relation $a_n = a_{n-1} + n$ with $a_0 = 4$.

Solution. Again, start by writing down the recurrence relation when $n = 1$. This time, don't subtract the a_{n-1} terms to the other side:

$$a_1 = a_0 + 1.$$

Now $a_2 = a_1 + 2$, but we know what a_1 is. By substitution, we get

$$a_2 = (a_0 + 1) + 2.$$

Now go to $a_3 = a_2 + 3$, using our known value of a_2:

$$a_3 = ((a_0 + 1) + 2) + 3.$$

We notice a pattern. Each time, we take the previous term and add the current index. So

$$a_n = ((((a_0 + 1) + 2) + 3) + \cdots + n - 1) + n.$$

Regrouping terms, we notice that a_n is just a_0 plus the sum of the integers from 1 to n. So, since $a_0 = 4$,

$$a_n = 4 + \frac{n(n+1)}{2}.$$

Of course in this case we still needed to know formula for the sum of $1, \ldots, n$. Let's try iteration with a sequence for which telescoping doesn't work.

Example 2.4.5

Solve the recurrence relation $a_n = 3a_{n-1} + 2$ subject to $a_0 = 1$.

Solution. Again, we iterate the recurrence relation, building up to the index n.

$$a_1 = 3a_0 + 2$$
$$a_2 = 3(a_1) + 2 = 3(3a_0 + 2) + 2$$

$$
\begin{aligned}
a_3 &= 3[a_2] + 2 = 3[3(3a_0 + 2) + 2] + 2\\
\vdots \quad & \quad \vdots \qquad\qquad \vdots\\
a_n &= 3(a_{n-1}) + 2 = 3(3(3(3\cdots(3a_0 + 2) + 2) + 2)\cdots + 2) + 2.
\end{aligned}
$$

It is difficult to see what is happening here because we have to distribute all those 3's. Let's try again, this time simplifying a bit as we go.

$$
\begin{aligned}
a_1 &= 3a_0 + 2\\
a_2 &= 3(a_1) + 2 = 3(3a_0 + 2) + 2 = 3^2a_0 + 2 \cdot 3 + 2\\
a_3 &= 3[a_2] + 2 = 3[3^2a_0 + 2 \cdot 3 + 2] + 2 = 3^3a_0 + 2 \cdot 3^2 + 2 \cdot 3 + 2\\
\vdots \quad & \quad \vdots \qquad\qquad\qquad\qquad \vdots\\
a_n &= 3(a_{n-1}) + 2 = 3(3^{n-1}a_0 + 2 \cdot 3^{n-2} + \cdots + 2) + 2\\
&\qquad\qquad = 3^n a_0 + 2 \cdot 3^{n-1} + 2 \cdot 3^{n-2} + \cdots + 2 \cdot 3 + 2.
\end{aligned}
$$

Now we simplify. $a_0 = 1$, so we have $3^n + \langle\text{stuff}\rangle$. Note that all the other terms have a 2 in them. In fact, we have a geometric sum with first term 2 and common ratio 3. We have seen how to simplify $2 + 2 \cdot 3 + 2 \cdot 3^2 + \cdots + 2 \cdot 3^{n-1}$. We get $\frac{2 - 2 \cdot 3^n}{-2}$ which simplifies to $3^n - 1$. Putting this together with the first 3^n term gives our closed formula:

$$
a_n = 2 \cdot 3^n - 1.
$$

Iteration can be messy, but when the recurrence relation only refers to one previous term (and maybe some function of n) it can work well. However, trying to iterate a recurrence relation such as $a_n = 2a_{n-1} + 3a_{n-2}$ will be way too complicated. We would need to keep track of two sets of previous terms, each of which were expressed by two previous terms, and so on. The length of the formula would grow exponentially (double each time, in fact). Luckily there happens to be a method for solving recurrence relations which works very well on relations like this.

The Characteristic Root Technique

Suppose we want to solve a recurrence relation expressed as a combination of the two previous terms, such as $a_n = a_{n-1} + 6a_{n-2}$. In other words, we want to find a function of n which satisfies $a_n - a_{n-1} - 6a_{n-2} = 0$. Now iteration is too complicated, but think just for a second what would happen if we *did* iterate. In each step, we would, among other things, multiply a previous iteration by 6. So our closed formula would include 6 multiplied some number of times. Thus it is reasonable to guess the solution will contain parts that look geometric. Perhaps the solution will take the form r^n for some constant r.

The nice thing is, we know how to check whether a formula is actually a solution to a recurrence relation: plug it in. What happens if we plug in r^n into the recursion above? We get

$$r^n - r^{n-1} - 6r^{n-2} = 0.$$

Now solve for r:

$$r^{n-2}(r^2 - r - 6) = 0,$$

so by factoring, $r = -2$ or $r = 3$ (or $r = 0$, although this does not help us). This tells us that $a_n = (-2)^n$ is a solution to the recurrence relation, as is $a_n = 3^n$. Which one is correct? They both are, unless we specify initial conditions. Notice we could also have $a_n = (-2)^n + 3^n$. Or $a_n = 7(-2)^n + 4 \cdot 3^n$. In fact, for any a and b, $a_n = a(-2)^n + b3^n$ is a solution (try plugging this into the recurrence relation). To find the values of a and b, use the initial conditions.

This points us in the direction of a more general technique for solving recurrence relations. Notice we will always be able to factor out the r^{n-2} as we did above. So we really only care about the other part. We call this other part the **characteristic equation** for the recurrence relation. We are interested in finding the roots of the characteristic equation, which are called (surprise) the **characteristic roots**.

Characteristic Roots.

Given a recurrence relation $a_n + \alpha a_{n-1} + \beta a_{n-2} = 0$, the **characteristic polynomial** is

$$x^2 + \alpha x + \beta$$

giving the **characteristic equation**:

$$x^2 + \alpha x + \beta = 0.$$

If r_1 and r_2 are two distinct roots of the characteristic polynomial (i.e., solutions to the characteristic equation), then the solution to the recurrence relation is

$$a_n = ar_1^n + br_2^n,$$

where a and b are constants determined by the initial conditions.

Example 2.4.6

Solve the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$ with $a_0 = 2$ and $a_1 = 3$.

Solution. Rewrite the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$. Now form the characteristic equation:

$$x^2 - 7x + 10 = 0$$

and solve for x:

$$(x-2)(x-5) = 0$$

so $x = 2$ and $x = 5$ are the characteristic roots. We therefore know that the solution to the recurrence relation will have the form

$$a_n = a2^n + b5^n.$$

To find a and b, plug in $n = 0$ and $n = 1$ to get a system of two equations with two unknowns:

$$\begin{aligned} 2 &= a2^0 + b5^0 = a + b \\ 3 &= a2^1 + b5^1 = 2a + 5b \end{aligned}$$

Solving this system gives $a = \frac{7}{3}$ and $b = -\frac{1}{3}$ so the solution to the recurrence relation is

$$a_n = \frac{7}{3}2^n - \frac{1}{3}5^n.$$

Perhaps the most famous recurrence relation is $F_n = F_{n-1} + F_{n-2}$, which together with the initial conditions $F_0 = 0$ and $F_1 = 1$ defines the Fibonacci sequence. But notice that this is precisely the type of recurrence relation on which we can use the characteristic root technique. When you do, the only thing that changes is that the characteristic equation does not factor, so you need to use the quadratic formula to find the characteristic roots. In fact, doing so gives the third most famous irrational number, φ, the **golden ratio**.

Before leaving the characteristic root technique, we should think about what might happen when you solve the characteristic equation. We have an example above in which the characteristic polynomial has two distinct roots. These roots can be integers, or perhaps irrational numbers (requiring the quadratic formula to find them). In these cases, we know what the solution to the recurrence relation looks like.

However, it is possible for the characteristic polynomial to have only one root. This can happen if the characteristic polynomial factors as $(x - r)^2$.

It is still the case that r^n would be a solution to the recurrence relation, but we won't be able to find solutions for all initial conditions using the general form $a_n = ar_1^n + br_2^n$, since we can't distinguish between r_1^n and r_2^n. We are in luck though:

Characteristic Root Technique for Repeated Roots.

Suppose the recurrence relation $a_n = \alpha a_{n-1} + \beta a_{n-2}$ has a characteristic polynomial with only one root r. Then the solution to the recurrence relation is

$$a_n = ar^n + bnr^n$$

where a and b are constants determined by the initial conditions.

Notice the extra n in bnr^n. This allows us to solve for the constants a and b from the initial conditions.

Example 2.4.7

Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 4$.

Solution. The characteristic polynomial is $x^2 - 6x + 9$. We solve the characteristic equation

$$x^2 - 6x + 9 = 0$$

by factoring:

$$(x - 3)^2 = 0$$

so $x = 3$ is the only characteristic root. Therefore we know that the solution to the recurrence relation has the form

$$a_n = a3^n + bn3^n$$

for some constants a and b. Now use the initial conditions:

$$\begin{aligned} a_0 &= 1 = a3^0 + b \cdot 0 \cdot 3^0 = a \\ a_1 &= 4 = a \cdot 3 + b \cdot 1 \cdot 3 = 3a + 3b. \end{aligned}$$

Since $a = 1$, we find that $b = \frac{1}{3}$. Therefore the solution to the recurrence relation is

$$a_n = 3^n + \frac{1}{3}n3^n.$$

Although we will not consider examples more complicated than these, this characteristic root technique can be applied to much more complicated recurrence relations. For example, $a_n = 2a_{n-1} + a_{n-2} - 3a_{n-3}$ has characteristic polynomial $x^3 - 2x^2 - x + 3$. Assuming you see how to factor such a degree 3 (or more) polynomial you can easily find the characteristic roots and as such solve the recurrence relation (the solution would look like $a_n = ar_1^n + br_2^n + cr_3^n$ if there were 3 distinct roots). It is also possible that the characteristics roots are complex numbers.

However, the characteristic root technique is only useful for solving recurrence relations in a particular form: a_n is given as a linear combination of some number of previous terms. These recurrence relations are called **linear homogeneous recurrence relations with constant coefficients**. The "homogeneous" refers to the fact that there is no additional term in the recurrence relation other than a multiple of a_j terms. For example, $a_n = 2a_{n-1} + 1$ is *non-homogeneous* because of the additional constant 1. There are general methods of solving such things, but we will not consider them here, other than through the use of telescoping or iteration described above.

Exercises

1. Find the next two terms in $(a_n)_{n\geq 0}$ beginning $3, 5, 11, 21, 43, 85\ldots$. Then give a recursive definition for the sequence. Finally, use the characteristic root technique to find a closed formula for the sequence.

2. Consider the sequences $2, 5, 12, 29, 70, 169, 408, \ldots$ (with $a_0 = 2$).
   (a) Describe the rate of growth of this sequence.
   (b) Find a recursive definition for the sequence.
   (c) Find a closed formula for the sequence.
   (d) If you look at the sequence of differences between terms, and then the sequence of second differences, the sequence of third differences, and so on, will you ever get a constant sequence? Explain how you know.

3. Solve the recurrence relation $a_n = a_{n-1} + 2^n$ with $a_0 = 5$.

4. Show that 4^n is a solution to the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$.

5. Find the solution to the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$ with initial terms $a_0 = 2$ and $a_1 = 3$.

6. Find the solution to the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$ with initial terms $a_0 = 5$ and $a_1 = 8$.

7. Solve the recurrence relation $a_n = 3a_{n-1} + 10a_{n-2}$ with initial terms $a_0 = 4$ and $a_1 = 1$.

8. Suppose that r^n and q^n are both solutions to a recurrence relation of the form $a_n = \alpha a_{n-1} + \beta a_{n-2}$. Prove that $c \cdot r^n + d \cdot q^n$ is also a solution to the recurrence relation, for any constants c, d.

9. Think back to the magical candy machine at your neighborhood grocery store. Suppose that the first time a quarter is put into the machine 1 Skittle comes out. The second time, 4 Skittles, the third time 16 Skittles, the fourth time 64 Skittles, etc.

   (a) Find both a recursive and closed formula for how many Skittles the nth customer gets.

   (b) Check your solution for the closed formula by solving the recurrence relation using the Characteristic Root technique.

10. Let a_n be the number of $1 \times n$ tile designs you can make using 1×1 squares available in 4 colors and 1×2 dominoes available in 5 colors.

    (a) First, find a recurrence relation to describe the problem. Explain why the recurrence relation is correct (in the context of the problem).

    (b) Write out the first 6 terms of the sequence $a_1, a_2, \ldots$.

    (c) Solve the recurrence relation. That is, find a closed formula.

11. You have access to 1×1 tiles which come in 2 different colors and 1×2 tiles which come in 3 different colors. We want to figure out how many different $1 \times n$ path designs we can make out of these tiles.

    (a) Find a recursive definition for the number of paths of length n.

    (b) Solve the recurrence relation using the Characteristic Root technique.

12. Solve the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$.

    (a) What is the solution if the initial terms are $a_0 = 1$ and $a_1 = 2$?

    (b) What do the initial terms need to be in order for $a_9 = 30$?

    (c) For which x are there initial terms which make $a_9 = x$?

13. Consider the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$.

    (a) Find the general solution to the recurrence relation (beware the repeated root).

    (b) Find the solution when $a_0 = 1$ and $a_1 = 2$.

    (c) Find the solution when $a_0 = 1$ and $a_1 = 8$.

2.5 Induction

Mathematical induction is a proof technique, not unlike direct proof or proof by contradiction or combinatorial proof.[3] In other words, induction is a style of argument we use to convince ourselves and others that a mathematical statement is always true. Many mathematical statements can be proved by simply explaining what they mean. Others are very difficult to prove—in fact, there are relatively simple mathematical statements which nobody yet knows how to prove. To facilitate the discovery of proofs, it is important to be familiar with some standard styles of arguments. Induction is one such style. Let's start with an example:

Stamps

Investigate!

You need to mail a package, but don't yet know how much postage you will need. You have a large supply of 8-cent stamps and 5-cent stamps. Which amounts of postage can you make exactly using these stamps? Which amounts are impossible to make?

Attempt the above activity before proceeding

Perhaps in investigating the problem above you picked some amounts of postage, and then figured out whether you could make that amount using just 8-cent and 5-cent stamps. Perhaps you did this in order: can you make 1 cent of postage? Can you make 2 cents? 3 cents? And so on. If this is what you did, you were actually answering a *sequence* of questions. We have methods for dealing with sequences. Let's see if that helps.

Actually, we will not make a sequence of questions, but rather a sequence of statements. Let $P(n)$ be the statement "you can make n cents of postage using just 8-cent and 5-cent stamps." Since for each value of n, $P(n)$ is a statement, it is either true or false. So if we form the sequence of statements

$$P(1), P(2), P(3), P(4), \ldots,$$

the sequence will consist of T's (for true) and F's (for false). In our particular case the sequence starts

$$F, F, F, F, T, F, F, T, F, T, F, F, T, \ldots$$

because $P(1), P(2), P(3), P(4)$ are all false (you cannot make 1, 2, 3, or 4 cents of postage) but $P(5)$ is true (use one 5-cent stamp), and so on.

Let's think a bit about how we could find the value of $P(n)$ for some specific n (the "value" will be either T or F). How did we find the value of

[3] You might or might not be familiar with these yet. We will consider these in Chapter 3.

the nth term of a sequence of numbers? How did we find a_n? There were two ways we could do this: either there was a closed formula for a_n, so we could plug in n into the formula and get our output value, or we had a recursive definition for the sequence, so we could use the previous terms of the sequence to compute the nth term. When dealing with sequences of statements, we could use either of these techniques as well. Maybe there is a way to use n itself to determine whether we can make n cents of postage. That would be something like a closed formula. Or instead we could use the previous terms in the sequence (of statements) to determine whether we can make n cents of postage. That is, if we know the value of $P(n-1)$, can we get from that to the value of $P(n)$? That would be something like a recursive definition for the sequence. Remember, finding recursive definitions for sequences was often easier than finding closed formulas. The same is true here.

Suppose I told you that $P(43)$ was true (it is). Can you determine from this fact the value of $P(44)$ (whether it true or false)? Yes you can. Even if we don't know how exactly we made 43 cents out of the 5-cent and 8-cent stamps, we do know that there was some way to do it. What if that way used at least three 5-cent stamps (making 15 cents)? We could replace those three 5-cent stamps with two 8-cent stamps (making 16 cents). The total postage has gone up by 1, so we have a way to make 44 cents, so $P(44)$ is true. Of course, we assumed that we had at least three 5-cent stamps. What if we didn't? Then we must have at least three 8-cent stamps (making 24 cents). If we replace those three 8-cent stamps with five 5-cent stamps (making 25 cents) then again we have bumped up our total by 1 cent so we can make 44 cents, so $P(44)$ is true.

Notice that we have not said how to make 44 cents, just that we can, on the basis that we can make 43 cents. How do we know we can make 43 cents? Perhaps because we know we can make 42 cents, which we know we can do because we know we can make 41 cents, and so on. It's a recursion! As with a recursive definition of a numerical sequence, we must specify our initial value. In this case, the initial value is "$P(1)$ is false." That's not good, since our recurrence relation just says that $P(k+1)$ is true *if* $P(k)$ is also true. We need to start the process with a true $P(k)$. So instead, we might want to use "$P(28)$ is true" as the initial condition.

Putting this all together we arrive at the following fact: it is possible to (exactly) make any amount of postage greater than 27 cents using just 5-cent and 8-cent stamps.[4] In other words, $P(k)$ is true for any $k \geq 28$. To prove this, we could do the following:

1. Demonstrate that $P(28)$ is true.

[4]This is not claiming that there are no amounts less than 27 cents which can also be made.

2. Prove that if $P(k)$ is true, then $P(k+1)$ is true (for any $k \geq 28$).

Suppose we have done this. Then we know that the 28th term of the sequence above is a T (using step 1, the initial condition or **base case**), and that every term after the 28th is T also (using step 2, the recursive part or **inductive case**). Here is what the proof would actually look like.

Proof. Let $P(n)$ be the statement "it is possible to make exactly n cents of postage using 5-cent and 8-cent stamps." We will show $P(n)$ is true for all $n \geq 28$.

First, we show that $P(28)$ is true: $28 = 4 \cdot 5 + 1 \cdot 8$, so we can make 28 cents using four 5-cent stamps and one 8-cent stamp.

Now suppose $P(k)$ is true for some arbitrary $k \geq 28$. Then it is possible to make k cents using 5-cent and 8-cent stamps. Note that since $k \geq 28$, it cannot be that we use fewer than three 5-cent stamps *and* fewer than three 8-cent stamps: using two of each would give only 26 cents. Now if we have made k cents using at least three 5-cent stamps, replace three 5-cent stamps by two 8-cent stamps. This replaces 15 cents of postage with 16 cents, moving from a total of k cents to $k+1$ cents. Thus $P(k+1)$ is true. On the other hand, if we have made k cents using at least three 8-cent stamps, then we can replace three 8-cent stamps with five 5-cent stamps, moving from 24 cents to 25 cents, giving a total of $k+1$ cents of postage. So in this case as well $P(k+1)$ is true.

Therefore, by the principle of mathematical induction, $P(n)$ is true for all $n \geq 28$. QED

Formalizing Proofs

What we did in the stamp example above works for many types of problems. Proof by induction is useful when trying to prove statements about all natural numbers, or all natural numbers greater than some fixed first case (like 28 in the example above), and in some other situations too. In particular, induction should be used when there is some way to go from one case to the next – when you can see how to always "do one more."

This is a big idea. Thinking about a problem *inductively* can give new insight into the problem. For example, to really understand the stamp problem, you should think about how any amount of postage (greater than 28 cents) can be made (this is non-inductive reasoning) and also how the ways in which postage can be made *changes* as the amount increases (inductive reasoning). When you are asked to provide a proof by induction, you are being asked to think about the problem *dynamically*; how does increasing n change the problem?

But there is another side to proofs by induction as well. In mathematics, it is not enough to understand a problem, you must also be able to communicate the problem to others. Like any discipline, mathematics has standard language and style, allowing mathematicians to share their ideas efficiently. Proofs by induction have a certain formal style, and being able to write in this style is important. It allows us to keep our ideas organized and might even help us with formulating a proof.

Here is the general structure of a proof by mathematical induction:

Induction Proof Structure.

Start by saying what the statement is that you want to prove: "Let $P(n)$ be the statement. . . " To prove that $P(n)$ is true for all $n \geq 0$, you must prove two facts:

1. Base case: Prove that $P(0)$ is true. You do this directly. This is often easy.
2. Inductive case: Prove that $P(k) \to P(k+1)$ for all $k \geq 0$. That is, prove that for any $k \geq 0$ if $P(k)$ is true, then $P(k+1)$ is true as well. This is the proof of an if . . . then . . . statement, so you can assume $P(k)$ is true ($P(k)$ is called the *inductive hypothesis*). You must then explain why $P(k+1)$ is also true, given that assumption.

Assuming you are successful on both parts above, you can conclude, "Therefore by the principle of mathematical induction, the statement $P(n)$ is true for all $n \geq 0$."

Sometimes the statement $P(n)$ will only be true for values of $n \geq 4$, for example, or some other value. In such cases, replace all the 0's above with 4's (or the other value).

The other advantage of formalizing inductive proofs is it allows us to verify that the logic behind this style of argument is valid. Why does induction work? Think of a row of dominoes set up standing on their edges. We want to argue that in a minute, all the dominoes will have fallen down. For this to happen, you will need to push the first domino. That is the base case. It will also have to be that the dominoes are close enough together that when any particular domino falls, it will cause the next domino to fall. That is the inductive case. If both of these conditions are met, you push the first domino over and each domino will cause the next to fall, then all the dominoes will fall.

Induction is powerful! Think how much easier it is to knock over dominoes when you don't have to push over each domino yourself. You just start the chain reaction, and the rely on the relative nearness of the dominoes to take care of the rest.

Think about our study of sequences. It is easier to find recursive definitions for sequences than closed formulas. Going from one case to the next is easier than going directly to a particular case. That is what is so great about induction. Instead of going directly to the (arbitrary) case for n, we just need to say how to get from one case to the next.

When you are asked to prove a statement by mathematical induction, you should first think about *why* the statement is true, using inductive reasoning. Explain why induction is the right thing to do, and roughly why the inductive case will work. Then, sit down and write out a careful, formal proof using the structure above.

Examples

Here are some examples of proof by mathematical induction.

Example 2.5.1

Prove for each natural number $n \geq 1$ that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Solution. First, let's think inductively about this equation. In fact, we know this is true for other reasons (reverse and add comes to mind). But why might induction be applicable? The left-hand side adds up the numbers from 1 to n. If we know how to do that, adding just one more term $(n + 1)$ would not be that hard. For example, if $n = 100$, suppose we know that the sum of the first 100 numbers is 5050 (so $1 + 2 + 3 + \cdots + 100 = 5050$, which is true). Now to find the sum of the first 101 numbers, it makes more sense to just add 101 to 5050, instead of computing the entire sum again. We would have $1 + 2 + 3 + \cdots + 100 + 101 = 5050 + 101 = 5151$. In fact, it would always be easy to add just one more term. This is why we should use induction.

Now the formal proof:

Proof. Let $P(n)$ be the statement $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$. We will show that $P(n)$ is true for all natural numbers $n \geq 1$.

Base case: $P(1)$ is the statement $1 = \frac{1(1+1)}{2}$ which is clearly true.

Inductive case: Let $k \geq 1$ be a natural number. Assume (for induction) that $P(k)$ is true. That means $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$. We will prove that $P(k + 1)$ is true as well. That is, we must prove that $1 + 2 + 3 + \cdots + k + (k + 1) = \frac{(k+1)(k+2)}{2}$. To prove this equation, start by adding $k + 1$ to both sides of the inductive hypothesis:

$$1 + 2 + 3 + \cdots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1).$$

Now, simplifying the right side we get:

$$\frac{k(k + 1)}{2} + k + 1 = \frac{k(k + 1)}{2} + \frac{2(k + 1)}{2}$$

$$= \frac{k(k+1)+2(k+1)}{2}$$
$$= \frac{(k+2)(k+1)}{2}.$$

Thus $P(k+1)$ is true, so by the principle of mathematical induction $P(n)$ is true for all natural numbers $n \geq 1$. ■

Note that in the part of the proof in which we proved $P(k+1)$ from $P(k)$, we used the equation $P(k)$. This was the inductive hypothesis. Seeing how to use the inductive hypotheses is usually straight forward when proving a fact about a sum like this. In other proofs, it can be less obvious where it fits in.

Example 2.5.2

Prove that for all $n \in \mathbb{N}$, $6^n - 1$ is a multiple of 5.

Solution. Again, start by understanding the dynamics of the problem. What does increasing n do? Let's try with a few examples. If $n = 1$, then yes, $6^1 - 1 = 5$ is a multiple of 5. What does incrementing n to 2 look like? We get $6^2 - 1 = 35$, which again is a multiple of 5. Next, $n = 3$: but instead of just finding $6^3 - 1$, what did the increase in n do? We will still subtract 1, but now we are multiplying by another 6 first. Viewed another way, we are multiplying a number which is one more than a multiple of 5 by 6 (because $6^2 - 1$ is a multiple of 5, so 6^2 is one more than a multiple of 5). What do numbers which are one more than a multiple of 5 look like? They must have last digit 1 or 6. What happens when you multiply such a number by 6? Depends on the number, but in any case, the last digit of the new number must be a 6. And then if you subtract 1, you get last digit 5, so a multiple of 5.

The point is, every time we multiply by just one more six, we still get a number with last digit 6, so subtracting 1 gives us a multiple of 5. Now the formal proof:

Proof. Let $P(n)$ be the statement, "$6^n - 1$ is a multiple of 5." We will prove that $P(n)$ is true for all $n \in \mathbb{N}$.

Base case: $P(0)$ is true: $6^0 - 1 = 0$ which is a multiple of 5.

Inductive case: Let k be an arbitrary natural number. Assume, for induction, that $P(k)$ is true. That is, $6^k - 1$ is a multiple of 5. Then $6^k - 1 = 5j$ for some integer j. This means that $6^k = 5j + 1$. Multiply both sides by 6:

$$6^{k+1} = 6(5j+1) = 30j + 6.$$

But we want to know about $6^{k+1} - 1$, so subtract 1 from both sides:

$$6^{k+1} - 1 = 30j + 5.$$

Of course $30j + 5 = 5(6j + 1)$, so is a multiple of 5.

Therefore $6^{k+1} - 1$ is a multiple of 5, or in other words, $P(k+1)$ is true. Thus, by the principle of mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$. ∎

We had to be a little bit clever (i.e., use some algebra) to locate the $6^k - 1$ inside of $6^{k+1} - 1$ before we could apply the inductive hypothesis. This is what can make inductive proofs challenging.

In the two examples above, we started with $n = 1$ or $n = 0$. We can start later if we need to.

Example 2.5.3

Prove that $n^2 < 2^n$ for all integers $n \geq 5$.

Solution. First, the idea of the argument. What happens when we increase n by 1? On the left-hand side, we increase the base of the square and go to the next square number. On the right-hand side, we increase the power of 2. This means we double the number. So the question is, how does doubling a number relate to increasing to the next square? Think about what the difference of two consecutive squares looks like. We have $(n+1)^2 - n^2$. This factors:

$$(n+1)^2 - n^2 = (n+1-n)(n+1+n) = 2n+1.$$

But doubling the right-hand side increases it by 2^n, since $2^{n+1} = 2^n + 2^n$. When n is large enough, $2^n > 2n + 1$.

What we are saying here is that each time n increases, the left-hand side grows by less than the right-hand side. So if the left-hand side starts smaller (as it does when $n = 5$), it will never catch up. Now the formal proof:

Proof. Let $P(n)$ be the statement $n^2 < 2^n$. We will prove $P(n)$ is true for all integers $n \geq 5$.

Base case: $P(5)$ is the statement $5^2 < 2^5$. Since $5^2 = 25$ and $2^5 = 32$, we see that $P(5)$ is indeed true.

Inductive case: Let $k \geq 5$ be an arbitrary integer. Assume, for induction, that $P(k)$ is true. That is, assume $k^2 < 2^k$. We will prove that $P(k+1)$ is true, i.e., $(k+1)^2 < 2^{k+1}$. To prove such an inequality, start with the left-hand side and work towards the right-hand side:

$$(k+1)^2 = k^2 + 2k + 1$$

$$< 2^k + 2k + 1 \qquad \ldots \text{by the inductive hypothesis.}$$
$$< 2^k + 2^k \qquad \ldots \text{ since } 2k + 1 < 2^k \text{ for } k \geq 5.$$
$$= 2^{k+1}.$$

Following the equalities and inequalities through, we get $(k + 1)^2 < 2^{k+1}$, in other words, $P(k + 1)$. Therefore by the principle of mathematical induction, $P(n)$ is true for all $n \geq 5$. ■

The previous example might remind you of the *racetrack principle* from calculus, which says that if $f(a) < g(a)$, and $f'(x) < g'(x)$ for $x > a$, then $f(x) < g(x)$ for $x > a$. Same idea: the larger function is increasing at a faster rate than the smaller function, so the larger function will stay larger. In discrete math, we don't have derivatives, so we look at differences. Thus induction is the way to go.

WARNING:.

With great power, comes great responsibility. Induction isn't magic. It seems very powerful to be able to assume $P(k)$ is true. After all, we are trying to prove $P(n)$ is true and the only difference is in the variable: k vs. n. Are we assuming that what we want to prove is true? Not really. We assume $P(k)$ is true only for the sake of proving that $P(k + 1)$ is true.

Still you might start to believe that you can prove anything with induction. Consider this incorrect "proof" that every Canadian has the same eye color: Let $P(n)$ be the statement that any n Canadians have the same eye color. $P(1)$ is true, since everyone has the same eye color as themselves. Now assume $P(k)$ is true. That is, assume that in any group of k Canadians, everyone has the same eye color. Now consider an arbitrary group of $k + 1$ Canadians. The first k of these must all have the same eye color, since $P(k)$ is true. Also, the last k of these must have the same eye color, since $P(k)$ is true. So in fact, everyone the group must have the same eye color. Thus $P(k + 1)$ is true. So by the principle of mathematical induction, $P(n)$ is true for all n.

Clearly something went wrong. The problem is that the proof that $P(k)$ implies $P(k + 1)$ assumes that $k \geq 2$. We have only shown $P(1)$ is true. In fact, $P(2)$ is false.