instead of using one of the two "magic multipliers" ($\frac{b}{a}$ or $\frac{c}{a}$) immediately, we use the two known corresponding values (one of each kind)—here £100 and $150—to calculate:

- first that

  £1 (the *unit*) corresponds to $\$\left(\frac{150}{100}\right) = \$1.50$

- then to multiply the answer ($1.50) by 768 to get the number of $

  $£768 = 768 \times \mathbf{£1}$ corresponds to $768 \times \$1.50 = \ldots .$

Thus

$$\begin{array}{lll} \text{if} & \mathbf{£100} & \xrightarrow{\hspace{3cm}} \mathbf{\$150} \\ \text{then} & £1 & \xrightarrow{\hspace{3cm}} \$\left(\frac{150}{100}\right) = \$1.50 \\ \text{so} & £768 = 768 \times £1 & \xrightarrow{\hspace{3cm}} 768 \times \$1.50 = \$\ldots . \end{array}$$

2. Algebra

2.1. Structure

We noted in Part II Section 2.1.1 that *elementary algebra* has its roots in *structural arithmetic*—the art of numerical calculation which exploits *structure* rather than brute force.

- At its simplest, this appeal to "structure" may go no further than to use *place value*—as in

  $$73 + 48 + 27 = (73 + 27) + 48,$$

  or

  $$17.18 + 7460 + 22.82 = (17.18 + 22.82) + 7460.$$

- This grows into an awareness of the algebraic *structure* lurking beneath the surface of many numerical or symbolical expressions—as in

  $$3 \times 17 + 7 \times 17 = (3 + 7) \times 17,$$

  or

  $$\frac{6 \times 15}{10} = \frac{3 \times (2 \times 5) \times 3}{10} = \ldots,$$

  or

  $$16 \times 17 - 3 \times 34 = (16 - 6) \times 17.$$

- Eventually this instinct for "tidying up" applies the underlying *algebraic* rules in a way that allows us to simplify all manner of algebraic expressions—starting with the simplest examples, such as

  $$6(a - b) + 3(2b - a) = \ldots .$$

But before this third stage, pupils must first internalise these *algebraic* rules by applying them to simplify *numerical* expressions, and then learn to see symbols as "placeholders for numbers" and to calculate with symbols in this spirit.

2.2. Technique

It is not easy to illustrate what we in England need to do differently with algebra at age 11–14. The complaints of those who teach the top 20% at the start of A level mathematics at age 16 are clear and consistent: these students—who are the most successful products of Key Stage 3 and Key Stage 4—**struggle with fractions, and with the simple algebra** they need for beginning A level. Other pupils are even more ill-served by the current approach to algebra up to age 16, where this key topic is either ignored, or treated far too superficially. We need to lay much stronger foundations in algebra for **all pupils** (even if some will inevitably go further than others).

The more focused demands of the new Key Stage 4 programme of study could provide a useful indication of what needs to be done—provided these demands are understood, and are taught with a view to mastery (rather than as mindless rules). Another useful pointer as to what needs attention may be found in the released items[20] from TIMSS 2011, which compared achievement in around 50 countries. English Year 9 pupils do tolerably well as long as they only need to use their common sense (e.g. "pattern spotting"). But once their performance depends on *technique* (i.e. something that has to be **taught**), the results are less encouraging. We saw in Part II (examples 1.4A–1.4K) how the bulk of Year 9 pupils in England struggle with simple problems involving *fractions* and *decimals*. In Sections 2.4, 2.5, and 2.7 below we include a selection of items intended to support the assertion that schools also need to re-consider how they approach *algebra* at Key Stage 3. We have again avoided making comparisons with countries from the Far East, and instead compare the results of pupils in England with those from Russia, from Hungary, from the USA, and from Australia.

2.3. [Subject content: *Algebra* p. 6]

- **use and interpret algebraic notation, including**

  ab in place of $a \times b$

  $3y$ in place of $y + y + y$ and $3 \times y$

  a^2 in place of $a \times a$, a^3 in place of $a \times a \times a$, a^2b in place of $a \times a \times b$

  $\frac{a}{b}$ in place of $a \div b$

  coefficients written as fractions rather than as decimals

  brackets

It would be hard to overstate the extent to which the algebraic *notation* summarised in these six bullet points makes elementary mathematics accessible to ordinary people.

The importance of *notation* should be clear if one considers the impact of our notation for writing integers and **decimals** *in base* 10. This was

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[20] http://timss.bc.edu/timss2011/international-released-items.html

proposed in 1585 by the Dutchman *Simon Stevin* in his little book *Die Thiende*. *Stevin*'s original notation was slightly unwieldy, but it soon evolved into the astonishingly compressed form that we use today, where a single succession of digits (and a decimal point) captures everything about a number, and does so in a way that allows routine calculation in a form that everyone can master. This notation was later extended by adopting our way of writing fractions or quotients (see the fourth bullet point above), and surds.

Our **algebraic** notation then emerged in almost modern form in Descartes' book on *Geometry* in 1637. This had an even greater impact. Before that time, even the best mathematicians struggled to express general calculations using symbols. Yet within 40 years, Descartes' new symbolism had revolutionised mathematics, allowing Newton and Leibniz to invent what we now call "the calculus". And within another 100 years, the language of algebra had been streamlined further by Euler into a form that made its potential power available to everyone.

But for ordinary mortals to access this power, the conventions summarised at the start of 2.3 have to be learned and respected. It seems not to be generally understood why these conventions make such a difference; but there is nothing difficult here, and beginners need to be absolutely clear that the conventions are not optional.

The whole purpose of algebraic notation as summarised above is

- to write algebraic expressions in a *compact* form, that can be apprehended at a glance, and
- to do so in a way that reflects the rules for *priority of operations*.

The genius of the *Descartes-Euler* conventions lies in the way they ensure that:

- Multiplication and division hold together tightly, allowing products (such as "$3ab$" or "$5a^2b$") to be spatially compressed, so that the eye and the brain perceive them as a single "term".
- In the same spirit, fractions—whether as coefficients, or as terms within the overall expression—must be written with a horizontal bar (as $\frac{1}{2}a$,

never in the misleading form "1/2a"); and decimals are best avoided whenever possible, since they undermine the goal of holding terms together compactly.

- Addition and subtraction link separate multiplicative "terms", but do so more loosely, so that the visual impression at a glance reflects the priority of operations.

The result is that the eye and brain can learn to read an algebraic expression at a glance in much the same way as place value allows one to grasp the meaning of numbers. But first one has to learn to routinely and reliably *translate* mildly complicated combinations into this new algebraic script. Thus one would like almost all pupils to be able to grasp the meaning of the simplest expression, such as "$xy + 1$"—especially if all they had to do was to choose between four mostly dodgy options.

2.3A What does $xy + 1$ mean?

A add 1 to y, then multiply by x
B multiply x and y by 1
C add x to y, then add 1
D multiply x by y, then add 1

2.3A Russia 89%, USA 80%, Hungary 73%, England 72%, Australia 71%

Given the importance of algebra in elementary mathematics, we really do need to think how to get understanding at this most basic level up around 90%. So in Section 2.5 we stray from our usual "higher viewpoint" and risk a few specific suggestions to encourage schools to consider what we might be currently omitting.

One important point is obscured by the simple examples used in the requirements listed at the start of the current Section 2.3—namely that the ingredient constructions and conventions are often *combined*. Hence in a typical "sum" the terms being added may themselves already be **compound** *expressions* (as in "$(3x - 6y + 4) + (5y - 2x - 3)$"), and in a typical "square" the expression being squared may be a **compound** *expression* (as in "$(3x - 6y + 4)^2$"). Too often we stop short of adding this extra layer of complexity. In the short term, nothing may be lost; but such "layered complexity", or variation, is an integral part of the new algebraic language, which pupils need to get used to.

2.4. [Subject content: *Algebra* p. 6]

- **substitute numerical values into formulae and expressions, including scientific formulae**
- **understand and use standard mathematical formulae; rearrange formulae to change the subject**
- **understand and use the concepts and vocabulary of expressions, equations, inequalities, terms and factors**
- **simplify and manipulate algebraic expressions to maintain equivalence by:**
  - **collecting like terms**
  - **multiplying out a single term over a bracket**
  - **taking out common factors**
  - **expanding products of two or more binomials**

Section 2.4.1 addresses the content of these four requirements by providing some initial food for thought from TIMSS 2011, a study which compared the performance of Year 9 pupils in around 50 countries.

Section 2.4.3 is long and expands on our earlier remark that "Elementary algebra *copies* the structure of arithmetic (that is, the four rules, together with the commutative laws, the associative laws, and the distributive law), and *applies it to a new mixed universe of symbols* (or letters) and numbers."

Section 2.4.2 is relatively short, and refers loosely to some of the ideas from 2.4.3, so should perhaps follow it. But that would risk the basic message of 2.4.2 being obscured by the preceding detail. Since this message is simple and important, we present it *before* the details in Section 2.4.3.

2.4.1 We begin with six tasks taken from TIMSS 2011. The first three are simple exercises involving substitution, and so are directly relevant to the first listed requirement in 2.4. The last three—two of which are again simple exercises—are relevant to the second, third and fourth requirements (and especially the fourth).

2.4A $y = a + \frac{b}{c}$. $a = 8$, $b = 6$, and $c = 2$. What is the value of y?

A 7 B 10 C 11 D 14

2.4B $k = 7$ and $m = 10$. What is the value of P when $P = \frac{3}{5}km$?

2.4C Use the formula

$$y = 100 - \frac{100}{1+t}$$

to find the value of y when $t = 9$.

2.4D Which of the following is equal to $3p^2 + 2p + 2p^2 + p$?

A $8p$ B $8p^2$ C $5p^2 + 3p$ D $7p^2 + p$

2.4E Which expression is equal to $4(3 + x)$?

A $12 + x$ B $7 + x$ C $12 + 4x$ D $12x$

2.4F Simplify the expression

$$\frac{3x}{8} + \frac{x}{4} + \frac{x}{2}.$$

Show your work.

Once the conventions of elementary algebra are understood, substituting values should be entirely routine. Hence 2.4A, 2.4B, and 2.4C should be exercises in simple arithmetic—where we should expect a high level of success.

2.4D and 2.4E go beyond mere arithmetic, but remain the very simplest kind of algebraic exercises; so one should again expect success rates to be high.

The actual results for 2.4A, 2.4B, and 2.4C (see below) suggest either: that Year 9 arithmetic is weak; or that the conventions of elementary algebra are often not understood at this level. The results for 2.4D and 2.4E (multiple-choice questions with just four rather crude options) suggest that pupils' grasp of the basic algebraic conventions remains painfully weak.

2.4F is more searching. It is the simplest imaginable example of genuine algebraic simplification involving fractions (as opposed to an introductory textbook exercise); but it requires pupils to have understood that adding fractions requires one to reduce to a common denominator. This idea has to be applied in a mildly algebraic context—but it is hard to imagine what other standard principle might be elicited by the instruction to "simplify" such an expression. The results suggest that schools need to reflect on their current approach to the arithmetic of fractions and to elementary algebra.

> **2.4A** Russia 91%, USA 81%, Hungary 81%, England 73%, Australia 71%
>
> **2.4B** Russia 83%, USA 70%, Australia 46%, Hungary 46%, England 40%
>
> **2.4C** Russia 80%, USA 55%, Hungary 51%, Australia 47%, England 45%
>
> **2.4D** Russia 81%, Hungary 63%, USA 58%, Australia 56%, England 47%
>
> **2.4E** Russia 81%, Hungary 57%, USA 53%, England 41%, Australia 40%
>
> **2.4F** Russia 35%, Hungary 34%, USA 19%, Australia 14%, England 9%

2.4.2 "Substituting values into formulae and expressions" to highlight letters as placeholders for numbers The requirement to "substitute numerical values into formulae and expressions" draws attention to a basic characteristic of algebra which deserves more attention.

- In an **equation** the letters are constrained, so can only take **particular** (as yet unknown) values. So we are not free to "substitute arbitrary numerical values" for the unknown.

- A **formula** is essentially no different from an *equation* with two or more variables, in that it expresses the way one variable *depends on*, and *is determined by*, others. So we are only free to "substitute numerical values" for *certain* variables—and this then determines the value of some other quantity which depends on them.
- In contrast, the letters and numbers in an algebraic **expression** are only required to satisfy the rules of arithmetic (or of algebra), *so the letters can be replaced by **any numbers whatsoever**, provided all occurrences of the same letter are given the same value.*

Many pupils never grasp these facts, and blindly move letters around without ever realising that they are little more than "placeholders for numbers". The examples 2.4A, 2.4B, and 2.4C reinforce the impression that pupils need more varied, carefully designed experiences of "substituting given numerical values" for the letters in "formulae and expressions", so that they internalise the idea that each letter in an expression can be given any value.

The act of substituting and evaluating also provides opportunities

- to exercise mental arithmetic, and
- to check *in a numerical context* (and if necessary to correct) the way algebraic notation is understood—including brackets, the correct reading and evaluation of expressions involving exponents, the priority of operations, juxtaposition as multiplication, and the fraction bar as division.

Moreover, evaluating expressions in this way can begin to convey the idea that

- each choice of *inputs* gives rise to a single determined *output* value for the expression.

That is, that such expressions provide the simplest examples of what we will later call a *function* (of is component variables).

2.4.3 What is elementary algebra?

We saw in 2.3 how the *Descartes-Euler* notation for elementary algebra helps us to make sense of compound expressions as being made up from their atomic parts, which we call "terms". There is no strict definition of what counts as a "term", but it tends to refer to one of the products, or brackets, which are combined to make up the whole expression. For example, where an integer such as 35 can be written as a product of two integers ($35 = 5 \times 7$), the 5 and 7 are called *factors* of 35. Similarly, when a compound expression (such as $x^2 + 5x + 6$) can be written as a product of two or more brackets ($x^2 + 5x + 6 = (x + 2)(x + 3)$), each of the brackets on the RHS is a *factor* of the original expression. In the expanded form, "x^2", "$5x$", and "6" would be seen as separate terms; but in the factorised form, the separate brackets "$(x + 2)$" and "$(x + 3)$" might be referred to as constituent *terms*.

This new domain of elementary "algebra" has several distinct sub-domains, each of which sheds a slightly different light on the subject. Some of these subdomains are more natural starting points for beginners than others. The four most obvious subdomains—in approximate order of sophistication—are *formulae, equations* (and *inequalities*), *expressions*, and *identities*.

- *Formulae*. Here letters are used in place of familiar entities (e.g. $A = l \times b$ for the area A of a rectangle of length l and breadth b; or $C = 2\pi r$ for the circumference C of a circle of radius r). In each such formula, the letters can take different values. The simplest formulae are a bit like the simplest calculations that we meet at Key Stages 1 and 2, in that they tell us how the value of one entity can be calculated once we know the values of certain others. For a rectangle, $A = l \times b$ tells us that the area of a rectangle (measured in square units) is given by multiplying the length and the breadth: the entities l and b can take any value $\geqslant 0$, and the value of A is then determined.

  At Key Stage 3 it is important to stress that, even though symbols (like l and b) are often chosen so that the letters reminds us of what they represent, the symbols are not a shorthand for the *concepts* "length" and "breadth", but stand for *numbers*. Hence b stands for "the number of units in the breadth" rather than for the breadth itself, and the r in $2\pi r$ stands for "the length in units of the radius" (see Sections 2.4.1 above and Part II, Section 2.1.3).

[There may be a clash here with the way variables are used in science. In mathematics letters stand for pure numbers. But science teachers sometimes use letters to stand for *quantities*—including their units: so a letter may be used to stand for a length "3cm", rather than just for the number of centimetres—namely "3".]

- *Equations* (and *inequalities*). The first equations one meets involve a single letter (often denoted by "x"). This letter is usually referred to as the "unknown"—because an equation can be interpreted as an arithmetical constraint which some "unknown number x" has to satisfy. An equation can then be transformed using the rules of algebra to try to unmask this previously "unknown number". For example, the problem:

  > "I'm thinking of a number.
  > When I double it and add 3 the result is 15.
  > What is my number?"

  can be formulated by saying:

  > Let the unknown number be "x".
  > Then x must satisfy the equation $2x + 3 = 15$.

  Once the equation has been set up, the secret is to forget where it came from and to transform the equation according to the laws of arithmetic (or the laws of algebra) in order to recover what information we can about "x": for example,

  > adding "-3" to both sides we get $2x = 12$;
  > then dividing both sides by 2 we get $x = 6$.

  These "transformations of an equation" set the scene for the way the "=" sign will be routinely handled when pupils work with *expressions* and *identities*.

Later we meet equations involving the square or the cube of the "unknown", or equations involving more than one "unknown". For example, suppose we are asked:

> "How can I transfer exactly 76 litres from a pond into an empty tank by using two buckets—one holding exactly 8 litres and the other holding exactly 7 litres?"

We can imagine filling and pouring the 7 litre bucket "x" times and the 8 litre bucket "y" times to get 76 litres, so that $7x + 8y = 76$. Notice that in the problem as described, the two unknowns x and y are both integers $\geqslant 0$. (We ignore for the moment the fact that one could also imagine pouring 12 full 7 litre buckets into the empty tank and then removing one full 8 litre bucket, or pouring 13 full 8 litre buckets into the empty tank and then removing 4 full 7 litre buckets—which correspond to solutions in which one of x and y may be negative.)

The third requirement listed at the start of Section 2.4 refers to *inequalities*. One should probably not try to go too far in exploring inequalities at Key Stage 3. However, we already saw at the end of Section 2.2.2 in Part II:

- that understanding and solving inequalities are important in applications of elementary mathematics,
- how solving inequalities relates to solving *equations*, and
- how badly neglected the topic has been in English schools.

The third bullet point here suggests that considerable thought needs to go into how to address this requirement in the course of Key Stage 3 and Key Stage 4. Work at Key Stage 3 needs to prepare for what will be needed at Key Stage 4, so one should hesitate to offer a general way of solving inequalities at this stage, and should focus instead on lots of examples. These examples should be given in different forms, and in different contexts, with both positive and negative coefficients, and with the unknown appearing on either, and on both sides of the inequality. The solutions should be expressed in words, marked on the number line (and eventually, for the bold, written using "set notation"—as is required in the GCSE *Subject criteria*).

To cut a long story short, every linear inequality in one variable can be reduced either

(a) to the form "$ax + b < 0$", or "$ax + b \leqslant 0$" (where a and b are constants, with $a > 0$), or

(b) to the form "$ax + b > 0$", or "$ax + b \geqslant 0$" (where a and b are constants, with $a > 0$).

To consider the first case only: we can add "$-b$" to both sides of the inequality, and then multiply both sides by the positive constant $\frac{1}{a}$, to conclude that the solutions in the two cases consist of

> "all values of x satisfying $x < -\frac{b}{a}$", or "all values of x satisfying $x \leqslant -\frac{b}{a}$".

These can be shown on the x-axis, or number line, by *shading*

> "all points x to the left of $-\frac{b}{a}$", or "all points x to the left of $-\frac{b}{a}$ **together with** $x = -\frac{b}{a}$".

For the more ambitious, the solutions can later be written in the form

$$\left\{x : x < -\frac{b}{a}\right\},$$

or

$$\left\{x : x \leqslant -\frac{b}{a}\right\}.$$

Quadratic inequalities in one variable, and linear inequalities in two variables, are more interesting, but probably belong at Key Stage 4.

- *Expressions*. Given a formula, such as $C = 2\pi r$, we very soon want to move the letters around. For example, suppose we use string to measure the circumference C of a tall cylindrical lamp post and want to calculate the radius r of the lamp post—a length which we cannot measure directly. We then need to re-write the formula as

$$r = \frac{C}{2\pi}$$

so that we can calculate r as soon as we know the circumference C. We therefore need to learn how to "calculate" with expressions consisting of letters and numbers, and to move the letters around "as if they were numbers" (since this is exactly what the letters represent).

As part of this process of collecting terms, adding, subtracting, multiplying and dividing, multiplying out brackets, factorising, cancelling common factors, etc. we have to learn to forget *temporarily* the meaning of the symbols and simply to respect the laws of arithmetic (or of algebra), and the meaning of "equality"—as we did

- with the equation "$2x + 3 = 15$" to get first "$2x = 12$" and then "$x = 6$"

and as we did

- when dividing both sides of the equation "$C = 2\pi r$" by "2π" to get "$r = \frac{C}{2\pi}$".

We can work in a similar way to discover how to obtain "exactly 76 litres"—but this time we have to exploit the fact that the unknowns have to be *positive integers*.

- We can start with $7x + 8y = 76$ and add "$-8y$" to both sides to get $7x = 76 - 8y$.
- We can then take out a common factor of 4 to get $7x = 4(19 - 2y)$,

which tells us that

- the LHS "$7x$" must be a multiple of 4, and hence x must be a multiple of 4.
- But we also know that, if x and y are integers $\geqslant 0$, then $7x$ is at most 76, so $x < 12$.
- So we only need to consider $x = 4$ (which yields a solution) and $x = 0$, or $x = 8$ (which do not).

It is this art of "calculating with expressions" that allows us to transform formulae and equations in a flexible way—and to derive information that may be far from obvious. And the art of calculating with expressions requires lots of carefully graduated practise if pupils are to become fluent in simplifying the kind of complicated-looking expressions that will arise naturally later.

The fourth subdomain of elementary algebra—namely *identities*—is in some ways the most important subdomain of the four. "Identities" are not mentioned in the third requirement at the start of 2.4—but they are implicit in other requirements, so cannot be entirely avoided at Key Stage 3 (even if they feature more strongly at Key Stage 4 and beyond).

- *Identities*: In primary arithmetic the = sign is at first used to connect some required calculation such as "$13 + 29$" (on the left hand side) with the answer "42" (on the right hand side):

  $$13 + 29 = 42.$$

  But the = sign then broadens its meaning and is later used to connect any two numerically equivalent expressions—such as

  "$13 + 29 = 6 \times 7$", or "$6^2 - 1 = 5 \times 7$", or "$\frac{28}{42} = \frac{10}{15}$".

  Something similar arises in the algebra of expressions, where pupils first learn that, given a jumble of symbols on the left hand side, one is expected to simplify it in some way and set it "equal" to something a bit like an "answer" (on the right hand side). For example one might be given an expression such as

  $$\left(\frac{x}{x-1} - \frac{x+1}{x}\right)^{-1}$$

  and rewrite it as

  $$= x^2 - x.$$

  However one later broadens this use of the equals sign so that "$=$" simply links two expressions that are "algebraically equivalent"—that is, where one side can be transformed into the other side via the rules of algebra.

Any such equation that links two expressions that are algebraically equivalent is called an *identity*.

2.5. [Subject content: *Algebra* p. 6]

> – **model situations or procedures by translating them into algebraic expressions or formulae**

This requirement summarises what an idealist would like all pupils to be able to do *eventually*. However, at present very few pupils ever reach this level of fluency—even at Key Stage 4 (let alone at Key Stage 3: see examples 2.5B and 2.5C below). Hence the requirement needs to be interpreted with care.

One reason for our current limited success is that we fail to separate two stages which have here been combined in the same requirement:

- first learn to translate a numerical procedure, or a sequence of operations, into algebraic form *as an expression*;
- then learn to equate the results of two such procedures, or to take on board an additional constraint, to **derive an equation** (or a "formula").

That is, we pay too little attention to the more modest *prerequisite* requirement of getting pupils

– **to interpret descriptions, or situations, given *orally* in words and to write down the answers as expressions.**

The extent to which we need to rethink current practice is partly illustrated by the following five Year 9 tasks from TIMSS 2011. The first four tasks (2.5A–2.5D) are basic exercises. These are **not** sophisticated modelling tasks; but they indicate the kind of exercises that we may need to take more seriously, and engage with more systematically, if we are eventually to address the full-blooded requirement at the start of Section 2.5. The fifth example requires one to set up a very simple linear equation and to interpret its solution—and although the comparison countries show that this task is more demanding, the English performance on this problem is in some ways even more telling.

> **2.5A** There were m boys and n girls in a parade. Each person carried 2 balloons. Which of these expressions represents the total number of balloons carried in the parade?
>
> A $2(m+n)$ B $2+(m+n)$ C $2m+n$ D $m+2n$
>
> **2.5B** A taxi company has a basic charge of 25 zeds and a charge of 0.2 zeds for each kilometre the taxi is driven. Which of these represents the cost in zeds to hire a taxi for a trip of n kilometres?
>
> A $25+0.2n$ B $25\times 0.2n$ C $0.2\times(25+n)$
> D $0.2\times 25+n$
>
> **2.5C** What is the area of the rectangle shown? [A rectangle with *length* $x+2$ and *width* x is shown.]
>
> A x^2+2 B x^2+2x C $2x+2$ D $4x+4$
>
> **2.5D** What is the sum of three consecutive whole numbers with $2n$ as the middle number?
>
> A $6n+3$ B $6n$ C $6n-1$ D $6n-3$
>
> **2.5E** A piece of wood was 40cm long. It was cut into 3 pieces. The lengths in cm are: $2x-5$, $x+7$, $x+6$. What is the length of the longest piece?

The success rates among Year 9 pupils in our four comparison countries were as follows:

> **2.5A** Russia 90%, USA 88%, Hungary 80%, England 74%, Australia 73%
>
> **2.5B** Russia 70%, USA 61%, Hungary 50%, Australia 47%, England 45%

2.5C Russia 72%, USA 37%, England 35%, Hungary 30%, Australia 26%

2.5D Hungary 56%, Russia 53%, England 46%, Australia 45%, USA 37%

2.5E Hungary 23%, Russia 22%, Australia 7%, USA 7%, England 3%

Some observers might be satisfied with a 74% success rate for 2.5A. But the Russian, USA, and Hungarian scores should challenge such complacency. (This is a multiple choice question, and the 26% who chose options B, C, or D suggest that a significant number of pupils were simply guessing—so some of the 74% correct will have chosen option A by accident.)

The responses to 2.5B reinforce the impression that most Year 9 pupils in England are very rarely expected to formulate such simple expressions algebraically from a situation given in words. (Note that if 30% of pupils were fairly sure of option A, and the other 70% of pupils were reduced to guessing, then an additional 17.5% of the cohort would select option A by accident—so more than 45% would then have chosen the "correct" option.)

Example 2.5C would seem to be even simpler—provided that pupils can read the simplest diagram and know that "area = length × breadth".

Setting up an "equation", or a "formula", is like writing a sentence. So pupils first need to learn how to "read", then how to "spell" the ingredient words and how to build up expressions (in a way that respects the conventions of elementary algebra—see Section 2.3). They then need to learn the basic art of naming a variable, and applying a sequence of arithmetical or algebraic transformations to it in a reliable way. We infer that this is either not done, or done in a way that does not allow these key skills to take root.

We have largely resisted the temptation to offer "solutions". However, as a contribution to the challenge for schools to develop the necessary extended sequence of stages that leads to algebraic fluency, we draw attention to three ingredients that seem to be relatively neglected. These stages relate the need to learn how to "match verbal descriptions with algebraic expressions" (see TIMMS example 2.3A above).

On the simplest level pupils need exercises of the following kind (see example 2.3A above to see why).

> *Match up each expression on the left with the corresponding English description on the right.*

$4+2x$	Six less than four times x
$x-5$	Three times one more than x
$2x-4$	Two less than one quarter of x
$\frac{x+2}{4}$	Three times one less than x
$3(x+1)$	One quarter of two less than x
$4x-6$	One quarter of two more than x
$\frac{x}{4}-2$	Four less than twice x
$\frac{x-2}{4}$	Six more than half of x
$x+6$	One more than three times x
$3x+1$	Five less than x
$\frac{x}{2}+6$	Six more than x
$3(x-1)$	Four more than twice x.

Pupils then need to take the step from "matching up" verbal descriptions and *given* expressions to reading, or listening to verbal descriptions and reliably translating these into written expressions for themselves. So they need variations on the following activity to cultivate the art of *listening*, *thinking*, and *interpreting*. (We give two contexts for purposes of illustration—but many others can be imagined.) These are intended to be **oral** challenges, read slowly and clearly, leaving sufficient pauses between successive tasks—with pupils expected to **listen** and write down "answers" (preferably without the instructions being repeated).

(a) "I'm thinking of a number, which I multiply by 3. Write an expression for my final number."

"I'm thinking of a number, which I multiply by 3, and then add 2. Write an expression for my final number."

"I'm thinking of a number, to which I add 2 and then multiply the result by 3. Write an expression for my final number."

"I'm thinking of a number, to which I add 2, then multiply the result by 3, and then square the answer. Write an expression for my final number."

"I'm thinking of a number, to which I add 2, square the result, and subtract 4 times one more than the number I first thought of. Write a fully simplified expression for my final number."

(b) "A square has sides of length a. Write an expression for its perimeter. Write another expression for its area."

"A rectangle has sides of length a and b. Write an expression for its perimeter. Write another expression for its area."

"A rectangle has sides whose lengths differ by 1. Write an expression for its perimeter. Write another expression for its area."

"A rectangle has one side twice as long as the other. Write an expression for its perimeter. Write another expression for its area."

"A rectangle has sides in the ratio $3 : 2$. Write an expression for its perimeter. Write another expression for its area."

Once pupils understand how *expressions* are constructed they may be in a better position to use this skill to translate a problem, or a result, given in words into an **equation** or *formula*, as with such variations on the above **oral** tasks as the following.

(a) "I'm thinking of a number, which I multiply by 3, and the result is 27. **Express this as an equation.**"

"I'm thinking of a number, which I multiply by 3, and then add 2. The result is 41. Express this as an equation."

"I'm thinking of a number, to which I add 2 and then multiply the result by 3. The result is 39. Express this as an equation."

"I'm thinking of a number, to which I add 2, then multiply the result by 3, and then square the answer. The result is 36. Express this as an equation."

"I'm thinking of a number, to which I add 2, square the result, and subtract 4 times one more than the number I first thought of. The result is 144. Express this as an equation."

(b) "A square has sides of length a. It perimeter is 108. **Express this as an equation.**"

"A square has sides of length $2a$. Its area is 144. Express this as an equation."

"A rectangle has sides of length a and b. Its perimeter is 108. Express this as an equation."

"A rectangle has sides of length a and b. Its perimeter is 10 and its area is 6. Express these facts as two equations in a and b."

"A rectangle has sides whose lengths differ by 1. Its perimeter is 62. Express this as an equation."

"A rectangle has sides whose lengths differ by 1. Its area is 56. Express this as an equation."

"A rectangle has one side twice as long as the other. Its area is 50. Express this as an equation."

> "A rectangle has sides in the ratio 3 : 2. It perimeter is 130. Express this as an equation."

In the above examples, the numbers have been chosen so that the solutions may be accessed without requiring any special technique. This should allow pupils to check whether the evident *numerical* solution is consistent with their *algebraic* formulation. But pupils later need to progress to exercises where the solutions cannot be so easily discerned. The successful solution of any resulting equations will then depend on preparatory algebraic and arithmetical work done elsewhere (especially work with transforming algebraic expressions and with fractions).

In the official Key Stage 3 programme of study, the full version of the requirement given at the start of Section 2.5 is even more ambitious, in that it states that pupils should be taught to:

- "model situations or procedures by translating them into algebraic expressions or formulae **and by using graphs**." [emphasis added]

We have already suggested that increased success may depend on separating the art of "formulating a procedure as an algebraic expression" from, and treating it earlier than, "formulating equations". So the immediate juxtaposition of the two separate stages "expressions and formulae" could be misleading. The final four words above (in bold) would seem to constitute an even more unfortunate juxtaposition, in that two entirely separate requirements *that cannot be handled simultaneously* have been compressed into a single statement.

The reference to "graphs" suggests that the situation being analysed involves at least **two variables**. This in turn suggests that this requirement only becomes relevant much later. Long before one can think about "using a graph", one needs to be able to formulate the relevant algebraic equation in two variables entirely reliably—and this seems likely to take more time and effort than we have realised (see 2.5A–2.5C). Hence **schools must be prepared to use their judgement** as to when such apparently juxtaposed requirements in the official programme of study have to be separated in time, with the missing stages, or "stepping-stones", provided internally.

In this instance, the connection with graphs is likely to feature much later. Once pupils have learned to work with linear graphs, one could revisit

- "I'm thinking of a number, which I multiply by 3, and the result is 27."

and relate the algebraic solution:

> Let the unknown number be x.
>
> $\therefore 3x = 27$.

to the point of intersection $(9, 27)$ of the line $y = 3x$ and the ordinate $y = 27$.

Later in Key Stage 3, or in early Key Stage 4 (see the third, fourth and fifth requirements in section 2.7 below), pupils might relate the problem

> "I'm thinking of a number, to which I add 2, then multiply the result by 3, and then square the answer. The result is 36. Express this as an equation."

to the intersection point $(2, 36)$ of the graph of $y = 9(x+2)^2$ and the ordinate $y = 36$.

In the same spirit, the problem:

> "A rectangle has sides whose lengths differ by 1. Its area is 56. Express this as an equation."

could be linked to the *positive* intersection point $(7, 0)$ (after rejecting $(-8, 0)$) of the graph of $y = x^2 + x - 56$ with the x-axis $y = 0$.

And having learned to solve quadratic equations at Key Stage 4, pupils might relate

> "A rectangle has sides of length a and b. Its perimeter is 10 and its area is 6. Express these facts as two equations in a and b."

to the equation $x^2 - 5x + 6 = 0$ (whose roots are a and b). Some pupils could then explore the general question of whether knowing the sum $a + b$ and the product ab of two unknowns is always sufficient to determine a and b.

Teachers should know that this latter idea (namely that a rectangle is determined by its area and its semi-perimeter) is more important than one might think—both historically and at higher levels. The ancient

Babylonians and Greeks both tackled quadratics in this way (among others).

***Babylonian*: around 1700 BC** The Babylonian approach was eclectic, and essentially *algebraic*, but without symbols. The problems were expressed in words, and the solution methods were given as recipes applied to the particular numbers in the problem; but the recipes were so formulated that they would still work if the particular numbers were changed. They addressed a remarkable variety of problems which correspond to what we would call "quadratic equations". Otto Neugebauer, the leading historian of such matters in the first half of the 20th century, catalogued hundreds of examples in what he called "normal form", where two numbers were to be found if their *product* and their *sum* (or difference) were known. Neugebauer also found countless exercises designed to train young scribes how to reduce other sorts of quadratic problems to this "normal form".

***Greek*: around 300 BC** The Greek approach is harder to explain briefly, because it was expressed purely geometrically (for they had no way of writing algebraically). If we cheat a little and describe the steps in their method partly algebraically, their "normal form" for a quadratic problem was to imagine a line segment broken in to unequal lengths (so of length $a+b$, with $a>b$):

- to construct the midpoint and then construct the square on half of the complete segment
- to construct the "a by b" rectangle with the two unequal segments as sides
- to subtract the rectangle from the square
- to construct the square which was equal to this difference (whose side was therefore "$\frac{1}{2}(a-b)$")
- to combine the segments of length $\frac{1}{2}(a+b)$ and of length $\frac{1}{2}(a-b)$ to find a; then find b.

All of this was done strictly geometrically—though we would write the process algebraically as

(i) first find $\frac{1}{2}(a+b)$, and then $\left[\frac{1}{2}(a+b)\right]^2$;

(ii) find ab;

(iii) subtract $\left[\frac{1}{2}(a+b)\right]^2 - ab$ to get $\left[\frac{1}{2}(a-b)\right]^2$;

(iv) find $\frac{1}{2}(a-b)$;

(v) add to get

$$a = \frac{1}{2}(a+b) + \frac{1}{2}(a-b),$$

and subtract to get

$$b = \frac{1}{2}(a+b) - \frac{1}{2}(a-b).$$

When pupils proceed beyond GCSE, they will need to know that:

- If a quadratic $x^2 + dx + e$ has roots a and b, it can be factorised as

  $$x^2 + dx + e = (x-a)(x-b);$$

  and multiplying out the RHS shows that

  $$d = -(a+b) \text{ and } e = ab.$$

  Hence once we know the quadratic, we already know the sum and product of the roots, and "solving the equation" is a way of going from knowing "the sum and product of the roots" to finding the roots themselves.

- Though the Babylonians and Greeks did not know it, they had hit upon something important. For if a cubic $x^3 + dx^2 + ex + f$ has roots a, b and c, then it can be factorised as

  $$x^3 + dx^2 + ex + f = (x-a)(x-b)(x-c);$$

  and multiplying out the RHS shows that

  $$d = -(a+b+c), \quad e = ab + bc + ca, \text{ and } f = -abc.$$

  Hence once we know the cubic, we already know the sum of the roots ($-d$), the product of the roots ($-f$), and the sum of the products in pairs (*e*). So "solving a cubic equation" requires us to find a way of going from knowing "the sum of all three roots, the product of all three roots, and the sum of the products in pairs" to finding the three roots themselves.

2.6. [Subject content: *Algebra* p. 6]

– use algebraic methods to solve linear equations in one variable (including all forms that require rearrangement)

Pupils will no doubt already be familiar with the way general results such as $C = 2\pi r$ can be expressed using letters; but in such a formula, the letters stand for familiar entities (the radius r of the circle, and its circumference C). In contrast, solving linear equations in one unknown may well be pupils' first encounter with symbols being used to encode information about completely unknown entities. So this is likely to be the setting in which key ideas about algebra are internalised—and where misconceptions may well take root.

We have seen (Part II, Section 2.2.2.2) that "to solve equations" means to solve **exactly**—by algebraic methods. We start out with an equation which has an unknown set of solutions, or possible values for the unknown "x". "Solving the equation algebraically" is a process which pins down the unknown "x" by exploiting two kinds of "moves".

- The first kind of move allows us to replace any constituent expression on either side of the equation by another expression which is *algebraically equivalent* to it (for example, we can "collect up" separate multiples of the unknown "x" into a single term). Because this kind of move is *reversible*, we know that exactly **the same values** of the unknown "x" satisfy the new equation as satisfied the old equation.

- The second kind of move is to subject both sides of the equation to the same operation (for example, we can add the same quantity to both sides, or multiply both sides by the same quantity). As long as this operation is

> *reversible* (as it always is if we add or subtract the same quantity to both sides, or if we multiply or divide both sides by a given expression *that is never equal to zero*), then we can again be sure that **exactly the same values** of the unknown "x" satisfy the new equation as satisfied the old equation.

Pupils need to learn not only to transform equations according to these rules of algebra, but also to recognise any pair of equations which are algebraically equivalent. Thus they should be faced with such tasks as:

> *Match up each equation on the left with the equation(s) on the right to which it is equivalent.*

$$\begin{array}{lr}
x+6=11 & 7x=4 \\
2x-3=5 & 2x+9=23 \\
10=6+7x & 2=-2x \\
\dfrac{x}{3}=4 & 14=2+x \\
3=2x-11 & 10+x=15 \\
\dfrac{x}{3}-5=11 & 2x+2=10 \\
5-2x=7 & 2x=14 \\
13=x+6 & 16=\dfrac{x}{3}.
\end{array}$$

Pupils also need lots of equations to solve, and standard contexts in which they learn to set up and solve equations which reveal things that were not previously clear and that are vaguely interesting. We offer a sequence of problems based on one idea—but there are dozens of other possible settings.

(a) *I start with the fraction $\frac{1}{6}$. I wish to add the same amount to the numerator and to the denominator so that the result is equal to $\frac{1}{5}$. What amount should I add?*

I then start with the fraction $\frac{1}{5}$. I wish to add the same amount to the numerator and to the denominator so that the result is equal to $\frac{1}{4}$. What amount should I add?

I then start with the fraction $\frac{1}{4}$*. I wish to add the same amount to the numerator and to the denominator so that the result is equal to* $\frac{1}{3}$*. What amount should I add?*

I then start with the fraction $\frac{1}{3}$*. I wish to add the same amount to the numerator and to the denominator so that the result is equal to* $\frac{1}{2}$*. What amount should I add?*

(b) *I start again with the fraction* $\frac{1}{6}$*. I want to* ***add*** *some amount* a *to the numerator and* ***subtract*** *the same amount from the denominator to make the result equal to* $\frac{1}{5}$*. Find* a*.*

I then start with the fraction $\frac{1}{5}$*. I want to add some amount* b *to the numerator and subtract the same amount from the denominator to make the result equal to* $\frac{1}{4}$*. Find* b*.*

I then start with the fraction $\frac{1}{4}$*. I want to add some amount* c *to the numerator and subtract the same amount from the denominator to make the result equal to* $\frac{1}{3}$*. Find* c*.*

I then start with the fraction $\frac{1}{3}$*. I want to add some amount* d *to the numerator and subtract the same amount from the denominator to make the result equal to* $\frac{1}{2}$*. Find* d*.*

I then start with the fraction $\frac{1}{2}$*. I want to add some amount* e *to the numerator and subtract the same amount from the denominator to make the result equal to 1. Find* e*.*

2.7. [Subject content: *Algebra* pp. 6-7]

- **work with coordinates in all four quadrants**
- **reduce a given linear equation in two variables to the standard form $y = mx + c$; calculate and interpret gradients and intercepts of graphs of such linear equations numerically, graphically and algebraically**

- **recognise, sketch and produce graphs of linear and quadratic functions of one variable with appropriate scaling, using equations in x and y in the Cartesian plane**
- **use linear and quadratic graphs to estimate values of y for given values of x and vice versa and to find approximate solutions of simultaneous linear equations**
- **model situations or procedures by translating them into algebraic expressions or formulae and by using graphs**
- **interpret mathematical relationships both algebraically and graphically**
- **find approximate solutions to contextual problems from given graphs of a variety of functions, including piece-wise linear, exponential and reciprocal graphs**

[Ratio, proportion and rates of change p. 7]

- **solve problems involving direct and inverse proportion, including graphical and algebraic representations**

2.7.1 This collection of requirements linked to graphs needs to be treated with extreme care. The first requirement makes perfect sense. The *first half* of the second requirement is equally standard; but the second half is already far from clear. And as one reads on, the meanings become more opaque and the stated goals appear progressively more optimistic, or overblown at this level. For example, we have already seen that direct proportion is hard, and that its "graphical and algebraic representation" may be more appropriate at Key Stage 4 (for more confident pupils); so the inclusion of this requirement for "inverse proportion" may need to be taken with a pinch of salt.

In short, we would urge schools **to sift out what clearly belongs to Key Stage 3 and to teach it well**. Where material seems out of place at Key

Stage 3, and where the listed material in standard type in the Key Stage 4 programme of study either repeats it verbatim or does not take it much further, work at Key Stage 3 should perhaps be limited to "preparatory" experience that can then be built on in Years 10 and 11.

2.7.2 The first requirement could be interpreted as being limited to work with individual points. However, one of the characteristic features of coordinate geometry and equations is that they are ways of working with *groups* of points or lines.

- An equation represents the set of *all* points (x, y) that satisfy the equation.
- To find the equation of a straight line we use the known coordinates of two given points and the unknown coordinates of a third variable point (x, y) which lies on the line.
- And we think about the solutions of simultaneous equations as the point, or points, where two or more lines or curves intersect.

So pupils need to learn to work with several points at once. The gulf between understanding ideas or methods *in isolation* (one-piece jigsaws) and being able to handle two or more simple ideas at once is indicated by the following item for Year 9 pupils in TIMSS 2011.

> **2.7.2A** $(0, -1)$, $(1, 3)$ Which equation is satisfied by **both** of these pairs of numbers (x, y)?
>
> A $x + y = -1$ B $2x + y = 5$ C $3x - y = 0$ D $4x - y = 1$
>
> **2.7.2A** Russia 53%, USA 38%, Hungary 29%, England 24%, Australia 22%

So before pupils begin to work with equations, basic work with coordinates should include learning to think about the "relative position" of *groups of points*. For example:

- One may give the coordinates of three vertices of a square, and require them to be located, and the coordinates of the fourth (unspecified) vertex to be found and the vertex marked.

- One may specify the coordinates of two *neighbouring* vertices of a square (for example, $(-4,2)$ and $(-3,-3)$), and ask for the possible coordinates of the other two vertices.
- One may specify the coordinates of two *opposite* vertices of a square and require that the other two vertices be marked and their coordinates found.

That is, pupils need lots of work which not only establishes the underlying conventions, but which teaches them to "see", and to think about, *groups* of points (and lines) that are related to each other in some way.

2.7.3 After sufficient experience imagining, and locating individual points and groups of points in all four quadrants, pupils will be well-placed to think about what links a given set of points (preferably given as a list, rather than as a table) such as:

$$(-6,-3),(-4,-2),(-2,-1),(0,0),(2,1),(4,2),(6,3),(8,4).$$

Plotting points should convey the idea that they appear to lie on a line, and that

> each time the x-coordinate increases by 2, the y-coordinate increases by 1 ("along 2, up 1").

One can then ask for the coordinates of *intermediate* points that lie on the same line, both to establish the possibility of fractional values (such as $\left(1,\frac{1}{2}\right)$, or $(3,\frac{3}{2})$, or $\left(-1,-\frac{1}{2}\right)$), and to extract

> the **unit** step: "along 1, up $\frac{1}{2}$".

Pupils can go further and then find $\left(\frac{1}{2},\frac{1}{4}\right)$ ("along $\frac{1}{2}$, up $\frac{1}{4}$"), and $\left(\frac{1}{3},\frac{1}{6}\right)$ ("along $\frac{1}{3}$, up $\frac{1}{6}$"), etc.. They should also be challenged to identify points on the same line with much more distant coordinates (such as $(100,50)$, or $(-200,-100)$).

Once these ideas (that the line extends indefinitely, and that it includes points that are as close together was we choose) have been firmly established, one can

- look for ways of relating x- and y-coordinates of points which lie on the line,
- obtain the usual *equation* "$y = \frac{1}{2}x$", and
- check that every point on the line satisfies this equation, and that every point whose coordinates satisfy the equation must lie on the line.

It may be necessary to repeatedly reinforce the idea that

- the collection of points on the line, and
- the collection of points whose coordinates (x, y) satisfy the equation $y = \frac{1}{2}x$

are the same (i.e. that points lie on the line *precisely when* their coordinates satisfy the equation): that is, that the equation provides an *algebraic* way of reasoning about, and calculating with, the *geometrical* line.

This whole sequence can then be repeated for a new set of points

$$(-6, -2), (-4, -1), (-2, 0), (0, 1), (2, 2), (4, 3), (6, 4), (8, 5).$$

Again, plotting points will indicate that the points lie on a line, that whenever the x-coordinate increases by 2, the y-coordinate increases by 1 ("along 2, up 1"), and that this line can never meet the first line (since the first line goes through $(-6, -3)$ and follows the rule "along 2, up 1", whereas the second line follows the *same* rule "along 2, up 1", but goes through a point $(-6, -2)$ *which does not lie on the first line*). Again one can ask pupils to find the coordinates of *intermediate* points on the line, and for points on the line with much more distant coordinates (such as $(100, 51)$, or $(-200, -99)$), and can then obtain the usual *equation* $y = \frac{1}{2}x + 1$. The significance of the parameters $m = \frac{1}{2}$, and of $c = 1$ can be established. And everything can be reinforced by considering the new set of points

$$(-6, -5), (-4, -4), (-2, -3), (0, -2), (2, -1), (4, 0), (6, 1), (8, 2).$$

In making sense of the linear equation $y = mx + c$, pupils need to internalise the significance

- of c (as the y-coordinate of the point where the line crosses the y-axis).

They also need sufficient experience to establish a clear mental image of how the parameter m affects the visual impression of the represented line (assuming that equal scales are used on both axes), so that they distinguish:

- lines in which $m = 1$ (rising to the right at 45°),
- lines with $0 < m < 1$ (rising to the right less steeply than $m = 1$)
- lines with $m > 1$ (rising to the right more steeply than $m = 1$), and
- lines with $m < 0$ (falling as one moves to the right).

Schools will need to decide for themselves how much of what follows is best handled at Key Stage 3 and how much fits more naturally within Key Stage 4. But at some point, once the basic ideas have been grasped, pupils need to do lots of work in the opposite direction:

- starting with linear equations given in a variety of forms (including with terms in "x" and in "y" on both sides of the equation, and where the y-terms may have any positive or negative coefficient),
- reducing the given equation to the "standard form $y = mx + c$" (or "$x = a$")
- and then sketching the line.

Pupils eventually need to be able to find the equation of a line which satisfies certain conditions, such as:

(a) passing through a given point with a given gradient m,

(b) with a given y-intercept $(0, c)$ and passing through a given point,

(c) with a given gradient m and a given x-intercept $(a, 0)$,

(d) with a given x-intercept and passing through a given point,

(e) passing through two given points.

2.7.4 Once the basic language of straight line graphs and linear equations has been established, pupils are ready to explore the wealth of problems whose natural representation is in terms of linear equations and straight line graphs.

We have already seen how this arises whenever two quantities are related in such a way as to be "in proportion", so that doubling the first quantity (such as the number of hours worked) leads to a doubling of the second quantity (the pay that is earned: see Section 1.9.2 above and Part II, Section 2.2.1). This is clearly relevant to the last requirement listed at the start of Section 2.7. If two different quantities vary "in proportion", and we know two corresponding numerical values—one of the first kind (a), and one of the second kind (c),

$$\boldsymbol{a} \longrightarrow \boldsymbol{c}$$

then any two corresponding *unknown* amounts x and y (one of the first kind and the other of the second kind) provide the third and fourth vertices of our "rectangular template"

$$x \longrightarrow y$$

and the proportion

$$x : a = y : c$$

translates into an equality of ratios, or fractions

$$\frac{x}{a} = \frac{y}{c}$$

which in turn gives rise to the linear equation

$$y = \left(\frac{c}{a}\right) x$$

with "constant of proportionality", or gradient, $\frac{c}{a}$.

Such examples also arise whenever one changes units. If the units belong to the same system, then the constant of proportionality will be exact, and relatively simple. For example, when changing from centimetres to metres, M metres becomes $C = 100M$ centimetres. But if the units come from *different* systems, then we usually simplify by using a convenient approximation to the "constant of proportionality". For example, when changing from miles into kilometres, M miles is generally taken to be

$K = \frac{8}{5}M$km, where we use 1.6 as the approximate scale factor in place of the messy actual value of "1.609344 to 6 decimal places".

However, just as most straight line graphs $y = mx + c$ do not have $c = 0$, so we must expect most linear relations to occur with a built-in "offset" $c \neq 0$. This offset can be interpreted as saying that the two scales we are comparing need to be "re-aligned". For example, the equation which relates the temperature F in Fahrenheit with the temperature C in Centigrade, or Celsius, is

$$F = \frac{9}{5}C + 32.$$

Here the "+32" arises because there is no obvious "zero" for measuring temperature; the Celsius scale uses the freezing point of water as $0°C$, whereas the Fahrenheit scale locates this at $32°F$. (In this instance, although the units arise from different systems, the scale factor "$\frac{9}{5}$" is *exact*, because the Celsius scale from $0°C$ to $100°C$ matches up uniformly with the Fahrenheit scale from $32°F$ to $212°F$, so that each $1°C$ corresponds to *exactly* $1.8°F$.)

A straight line graph tells us that there is a "linear relation" between x and y even if the line does not go through the origin. Most instances where there is some hidden proportion occur with an "offset" (that is, with $c \neq 0$). A good example is the graph which underlies example 2.5B:

> **2.5B** A taxi company has a basic charge of 25 zeds and a charge of 0.2 zeds for each kilometre the taxi is driven. Which of these represents the cost in zeds to hire a taxi for a trip of n kilometres?
>
> A $25 + 0.2n$ B $25 \times 0.2n$ C $0.2 \times (25 + n)$ D $0.2 \times 25 + n$

Here the cost of a journey is directly proportional to the distance travelled—except for the addition of a "basic charge of 25 zeds"; hence the charge "y zeds" for a journey of length x km (priced in "zeds"—the universal currency in TIMSS problems) is given by

$$y = 0.2x + 25$$

which is better written without decimals as

$$y = \frac{1}{5}x + 25.$$

Mathematics teachers need to remember that scientists, engineers and others will go to almost any lengths to reduce more complicated relationships to ones that give rise to straight line graphs—because empirical laws are easiest to discern, or to confirm, if the approximate data can be plotted to *look as though it fits on a straight line*. For example, if a scientist believes the data should satisfy an equation of the form $y = kx^2$ for some positive constant value k, then rather than plotting x against y and having to identify a parabola, they might well

- plot values of x^2 against values of y and hope to see a straight line with gradient k, or
- plot $\log(x)$ against $\log(y)$ and expect to see a straight line with gradient 2 and with y-intercept $c = \log(k)$.

2.7.5 Such connections and applications should be part of any treatment of linear equations at Key Stage 3 and Key Stage 4, and this presumably covers at least part of what is meant by the fifth requirement listed at the start of Section 2.7 ("model situations …"), and also the sixth requirement ("interpret mathematical relationships …").

2.7.6 The third and fourth requirements at the start of Section 2.7 mention *quadratic functions* and *quadratic graphs*. These references need to be interpreted with care.

The new GCSE specification (and hence the programmes of study for Key Stage 3 and 4) deliberately downplay premature reference to abstract "functions", and to function notation—such as $f(x)$. Instead, the programmes of study would appear to be designed to emphasise the *use* of such ideas *in concrete form* before abstractions such as $f(x)$ are introduced in Year 12 (though there is nothing to prevent a school from doing both prior to GCSE).

So when the word "function" appears in the context of linear and quadratic functions, *it is being used informally*, indicating that the curriculum should prepare the ground for a more abstract formulation in Year 12. In particular, work at Key Stage 3 should take account of the fact that GCSE will **no longer** expect pupils to use the abstract notation $f(x)$. Nor will pupils be expected to make sense of *general* transformations of coordinates

- moving the y-axis by rewriting the given expression for the function in the form $f(x \pm a)$, or
- moving the x-axis by rewriting the given expression for the function in the form $f(x) \pm a$, or
- moving both axes at once by rewriting the given expression for the function in the form $f(x \pm a) \pm b$.

Instead, by the end of Key Stage 4, pupils who expect to take Higher tier GCSE need to be able to implement such transformations **in the contexts of specific linear, or quadratic, or trig functions**. Pupils will therefore work with particular functions f and with particular numerical values of the parameters a and b. But for convenience we summarise these specific numerical examples by giving them in general symbolic form.

- The general *linear* equation $y = mx + c$ can be seen to be essentially the same as $Y = mX$ in two obvious ways:
  - by moving the origin to $(0, c)$, and setting $Y = y - c$, and $X = x$, and also
  - by moving the origin to $\left(-\frac{c}{m}, 0\right)$, and setting $Y = y$, and $X = x + \frac{c}{m}$.
- Pupils need exercises that lead them to recognise that any given quadratic equation behaves essentially just like $y = x^2$ or $y = -x^2$.
  - The first step is to understand the prototype of all quadratics, namely $y = x^2$,
    - ∗ to appreciate its symmetry about the y-axis (giving the same y value for $\pm x$)
    - ∗ to recognise how it "sits on" (or is tangent to) the x-axis
    - ∗ how this relates to the fact that squaring values of x between -1 and 1 produces a *smaller* output x^2 , while squaring values of x which are greater than 1 or less than -1 gives rise to *larger and larger* outputs x^2.
  - This analysis can then be extended to graphs whose equation has the form $y = x^2 + c$, where c may be either positive or negative, and where

moving the origin to $(0, c)$ corresponds to a change of coordinates: $Y = y - c$, $X = x$, so that the original equation $y = x^2 + c$ becomes $Y = X^2$.

- The same idea extends to equations of the form $y = (x - a)^2$, and to those of the form $y = (x - a)^2 + c$.
- And one can show (via particular numerical examples) how any given quadratic equation $y = x^2 + bx + c$ can be rewritten in such a form by "completing the square"

$$y = \left(x + \frac{b}{2}\right)^2 + \left(c - \left[\frac{b}{2}\right]^2\right),$$

so that the original equation becomes $Y = X^2$, where

$$X = x + \frac{b}{2},$$

and

$$Y = y - C = y - \left(c - \left[\frac{b}{2}\right]^2\right).$$

- Later (perhaps in Year 12) those who enjoy algebra can discover how the general quadratic $y = ax^2 + bx + c$ can be rewritten as

$$ay = (ax)^2 + b(ax) + ac = a^2\left(x + \frac{b}{2a}\right)^2 + \frac{1}{4}\left[4ac - b^2\right]$$

which turns into $Y = X^2$ after shifting the origin and dividing both x and y by "a". Hence, although some quadratics appear tall and skinny, while others appear short and fat, all parabolas are in fact *similar*, just as all circles, or all squares are similar.

2.7.7 The requirement to work with "given graphs of a variety of functions, including piece-wise linear, exponential and reciprocal graphs" needs to be interpreted carefully, in the spirit of 2.7.6. Given the apparent ruling about "functions" in general at GCSE, this stated requirement would seem to have limited relevance at Key Stage 3. Even at Key Stage 4 it may mean little more than that pupils

- should ideally be familiar with the graph of $y = \frac{1}{x}$ and its obvious variants (such as $y = \frac{k}{x}$, or possibly $y = \frac{1}{x-a}$);
- should have some experience of such graphs as $y = (1.05)^x$, that arise when exploring how an investment, or a debt of £1 would grow in x years at 5% per annum; and
- should be prepared to make sense of natural problems where the given data happen to give rise in some way to a graph that is (for example) piece-wise linear.

2.7.8 The fourth requirement listed at the start of Section 2.7 states that

- "pupils should be taught to use linear and quadratic graphs to estimate values of y for given values of x and vice versa, and to find approximate solutions of simultaneous linear equations."

The seventh listed requirement at the start of Section 2.7 also mentions finding "approximate solutions".

These two requirements appear to confuse two quite different things—each of which is valuable, but whose combination here is potentially confusing.

It is important for pupils to learn to "read a graph". By this we mean:

- that pupils be confronted with a graph whose equation is unknown,
- that they be given a value a of x, and have to trace the corresponding *abscissa* $x = a$ to see where it hits the graph, and then to trace the corresponding *ordinate* from that point to the y-axis to estimate the value of y corresponding to $x = a$ (using their eyes, or their fingers, or a carefully positioned—preferably transparent—ruler), and
- that they be given a value b of y, and have to trace the corresponding *ordinate* $y = b$ to find all the points where it hits the graph, and then to trace the corresponding *abscissas* from these points on the graph to the x-axis to estimate all the values of x corresponding to $y = b$.

An entirely separate (and equally important) requirement is, given a formula or equation relating x and y, to substitute values for x, so that expressions involving x become numerical expressions, and so to use arithmetic to discover what this says about the corresponding value of y (see Section 2.4.2 above). Note however, that in this process the calculations are **exact**, not estimates.

The listed requirement appears to confound these two very different, and entirely admirable, activities, by suggesting that pupils should engage in such "estimation" with "linear and quadratic graphs". But if we know that we are working with a "linear graph" or a "quadratic graph", then we must know its equation—so substitution becomes an exact calculation, rather than a matter of *estimation*—**with one exception**.

- If the equation is linear, then given a value of x, pupils should calculate an **exact** value of y; and given a value of y, one can equally demand that they calculate an **exact** value of x.
- If the equation is quadratic, then given a value of x, pupils should calculate an **exact** value of y.
- Hence, the only obvious scope for "estimating values" would seem to arise in asking, for a given quadratic graph or equation,
  - "Which possible values of x give rise to a given value of y?"

  This is an excellent requirement (namely to draw the relevant *ordinate* parallel to the x-axis, to estimate where it cuts the graph, and to infer the approximate values (if any) of x—which one can then check by substituting the estimated values in the known equation).

The final part of the fourth requirement

> "to find **approximate** solutions of **simultaneous linear equations**"

might be fine if it was stated at a point where pupils could see how it links up with

> finding the **exact** solution (by eliminating a variable).

But there is no mention of this requirement at Key Stage 3, so the requirement to address the significant challenge of working with *simultaneous* equations purely in order to find approximate solutions seems seriously premature.

At Key Stage 4, Higher tier candidates are expected to find the intersection points of a line and a circle, so it makes sense to consider how to prepare the ground for such pupils at Key Stage 3. In general one would eventually like all pupils to understand that

- the solutions of simultaneous equations

correspond to

- the coordinates of points where two lines or curves meet.

This is an important idea, provided it is not misrepresented as an alternative to "solving the equations algebraically". So at the point where simultaneous linear equations are to be solved exactly, pupils need to understand

- that the two linear equations correspond to two straight lines in the plane,

and

- that the output from the solving process is precisely the coordinates of the point where the two lines cross.

So as and when a class is ready to handle "elimination of a variable" in order to find the exact solution, it makes sense for them

- to draw the two lines,
- to recognise that the solution (x, y) that they seek corresponds to the coordinates of the point where the two lines cross, and
- to *estimate* the solution that is being sought (as a guide for what they should expect to emerge from the subsequent algebraic exact calculation).

They would then be in a good position to confront the algebraic challenge of "how to eliminate a variable", and to use this new-found skill to tackle lots of lovely problems. But there is something wrong with a programme that requires pupils to find "approximate" solutions while not revealing the fact that one can find the **exact** solution.

One would also like pupils to tackle problems where this geometrical interpretation is an essential part of the problem (for example, where they are given the coordinates of three vertices of a triangle, and are required to find the coordinates of the point where two medians meet). However, such problems are rather hard precisely because they require pupils to coordinate several steps (find the coordinates of the midpoints of the sides; find the equations of the two medians; solve these two simultaneous equations; extract the coordinates of the point where they cross).

2.8. [Subject content: *Algebra* p. 7]

- **generate terms of a sequence from either a term-to-term rule or a position-to-term rule**
- **recognise arithmetic sequences and find the n^{th} term**
- **recognise geometric sequences and appreciate other sequences that arise.**

Work with sequences provides valuable opportunities:

- to revise and to strengthen arithmetic
- to cultivate the ability to notice basic patterns (constant, linear, powers, exponentials)
- to discover how geometrical and combinatorial sequences often give rise to familiar integer sequences
- to express numerical patterns algebraically
- to link discrete sequences with work on functions and graphs.

For example,

- if a *formula* is given for the n^{th} term, then finding the succession of terms is an exercise in substituting easy numerical (integer) values into an expression;
- if the first few terms of a sequence are given (whether 2, 4, 6, 8, …, or 4, 7, 10, 13, …, or 0, 3, 8, 15, …, or 1, 3, 7, 15, …, or 2, 5, 13, 35, …), then it is an excellent exercise to think of the *simplest* algebraic expression that could generate the given sequence.

A **sequence**

$$x_1, x_2, x_3, x_4, x_5, \ldots$$

is a way of presenting an endless amount of information in a single list. There are two quite different ways of specifying the terms of such a *sequence*.

The first, and most primitive, way is to give the first few terms and then to specify a *term-to-term* rule (or "recurrence relation") that tells you how to work out the next term from the ones you already know. For example,

- $x_1 = 3$, $x_{n+1} = 2x_n$ defines the sequence 3, 6, 12, 24, 48, …;
- $x_1 = x_2 = 1$, $x_{n+1} = x_n + x_{n-1}$ defines the sequence 1, 1, 2, 3, 5, 8, 13, …
- $x_1 = 2$, $x_{n+1} = 3x_n - 2^n$ defines the sequence 2, 4, 8, 16, 32, ….

This first approach allows you to continue the sequence as far as you like, and determines the 10th, the 100th, and the 1000th terms uniquely. However, in order to find the 1000th term we first have to calculate the 1st, the 2nd, the 3rd, …, **and** the 999th terms. In other words, we can generate terms of the sequence, but it may not be easy to obtain a proven closed formula giving the n^{th} term of the sequence as a formula in terms of n. We may think we can guess how the sequence is behaving, but we are unlikely to be able to prove anything about *the sequence as a whole*.

- In the first of our three examples above, we can see that:

  the first term $x_1 = 3$ is doubled $n - 1$ times to get the nth term,
  so $x_n = 3 \times 2^{n-1}$.

- In the second example, it is easy to generate more and more terms, but it is quite unclear how to write the n^{th} term as a *closed formula* in terms of n.

- In the third example, it is easy to guess that the *closed formula* for the nth term looks as though it "has to be" $x_n = 2^n$, but it is not at all clear how to prove that this is correct.

In short, a term-to-term rule is easy to use, but it is inefficient; and it gives us no way of reasoning in general about the *n^{th} term*.

The second (and generally more powerful) way to specify a sequence is by a *position-to-term* rule, which tells you how the n^{th} term can be calculated directly in terms of n. That is, the sequence of terms

$$x_1, x_2, x_3, x_4, x_5, \ldots, x_n, \ldots$$

is simply a listing of the outputs for a single rule, or *function* f, by listing

$$f(1), f(2), f(3), f(4), f(5), \ldots, f(n), \ldots.$$

A position-to-term rule may be given explicitly by a *formula*—as with

- the sequence of squares, where $x_n = n^2$, or
- the sequence of powers of 2, where $x_n = 2^n$.

But a position-to-term rule may also define a sequence **intrinsically**, with the n^{th} term being defined to be a number which can be calculated from some algebraic procedure, or from some geometrical configuration. For example:

- Let the nth term t_n of a sequence be defined to be equal to the sum of the first n positive integers. Then

$$\begin{aligned}
t_1 &= 1, \\
t_2 &= 1 + 2 = 3, \\
t_3 &= 1 + 2 + 3 = 6, \\
&\vdots \\
t_n &= 1 + 2 + 3 + \cdots + n.
\end{aligned}$$

- Let the n^{th} term c_n of a sequence be defined to be the number of chords that can be created by joining pairs of points chosen from n points marked on a circle. Then

  1 point on a circle gives rise to $c_1 = 0$ chords;
  2 points on a circle give rise to exactly $c_2 = 1$ chord;
  3 points on a circle give rise to $c_3 = 3$ chords; etc..

- Let the n^{th} term f_n of a sequence be defined to be equal to the number of positive factors of n. Then

  $f_1 = 1$, $f_2 = 2$ (factors 1 and 2); $f_3 = 2$ (factors 1 and 3), $f_4 = 3$ (factors 1, 2, and 4); ....

In these three examples, the position-to-term rule tells us exactly how to find each term; but the underlying function, or rule, is given as a *process* or a *recipe*, rather than as a formula. This makes it possible to generate

- the first sequence, whose n^{th} term is sum of the first n positive integers, by simply working out any term we need:

  1, 1+2 = 3, 1+2+3 = 6, 1+2+3+4 = 10, 1+2+3+4+5 = 15,
  1+2+3+4+5+6 = 21, ...

- the second example, whose n^{th} term is equal to the number of chords created by n points on a circle:

$$0, 1, 3, 6, 10, 15, \ldots$$

- the third sequence, whose n^{th} term is equal to the number of positive factors of n:

$$1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2, \ldots.$$

Each sequence is well-defined, but we are not given either a term-to-term rule or a *closed formula* for any of the sequences. So any claims we might wish to make about how each sequence behaves must be deduced from the given algebraic or geometrical definition.

An *arithmetic sequence*

$$c, c+m, c+2m, c+3m, c+4m, \ldots$$

is one that goes up in constant steps: that is, where the term-to-term rule for the sequence is simply

> "add m" for some fixed constant m.

The n^{th} term is determined by the first term c and the $n-1$ steps of size m that take us from the 1st term to the n^{th} term:

> $\therefore$ the n^{th} term is equal to "$c+(n-1)m$".

The prototype of every *arithmetic sequence* is the familiar counting sequence

$$0,1,2,3,4,5,6,7,8,9,10,\ldots.$$

The *general arithmetic sequence* arises from the counting sequence

- by first multiplying the whole sequence by m (to get

  "$0, m, 2m, 3m, 4m, \ldots, (n-1)m, \ldots$")
- then adding c to every term (to get

  "$c, c+m, c+2m, c+3m, c+4m, \ldots, c+(n--1)m, \ldots$")

If we think of the n^{th} term "$c+(n-1)m$" as a function of $x=n-1$, then we see that the sequence lists the values of "$y=mx+c$ " for integer values of x. So the sequence corresponds to the sequence of points for $x=0$, $x=1$, $x=2$, etc. on the straight line $y=mx+c$; hence another name for an "*arithmetic* sequence" is a "**linear** sequence".

In general, once we have a closed formula for the n^{th} term of a sequence, we can treat $n=x$ as the dependent variable and "plot the graph of the sequence" as a "point graph", with one graph point for each positive integer value of $x=n$. The "common difference" m is then the "gradient" of this point graph (for every unit step to the right in the positive $n=x$ direction, the point graph jumps up distance m in the y direction), and the initial value c is the point at which the point graph hits the y-axis. The tradition of using "a" for the first term in place of "c", and using "d" for the common difference in place of "m" makes it much less likely that pupils will appreciate this important connection.

A *geometric sequence*

$$c, cr, cr^2, cr^3, cr^4, cr^5, \ldots$$

is one for which the term-to-term rule for the sequence is simply

> "multiply by r" for some fixed constant r.

Hence the n^{th} term is completely determined:

- by the first term c, and
- by the $n-1$ steps "multiply by r" that take us from the 1^{st} term c to the n^{th} term.
- $\therefore$ the n^{th} term is equal to "cr^{n-1}".

If we think of the n^{th} term "cr^{n-1}" as a function of $x = n-1$, then we see that the sequence is specified by the two constants c and r, and lists the values of $y = c \cdot r^x$ for integer values of x (starting at $x = 0$). Because the term number "n" appears as an exponent, a *geometric* sequence is also called an **exponential** sequence.

In the third requirement listed at the start of Section 2.9 it is unclear what exactly is meant by

> "and appreciate other sequences that arise".

However, these "other sequences" should certainly include:

- *linear* sequences (or arithmetic sequences)
- *quadratic* sequences, like the sequence of squares:

$$1^2,\ 2^2,\ 3^2,\ 4^2,\ 5^2, \ldots n^2, \ldots$$

- the sequence of *triangular* numbers

$$0,\ 1,\ 3,\ 6,\ 10,\ 15,\ 21\ , \ldots$$

- the sequence of *cubes*:

$$1,\ 8,\ 27,\ 64,\ 125,\ 216,\ 343,\ \ldots$$

- the *Fibonacci* sequence:

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ \ldots.$$

School mathematics often gives the impression that all sequences are *polynomial* sequences—that is, sequences where the n^{th} term is a polynomial function of n, as with

- *linear* sequences (or arithmetic sequences)
- *quadratic* sequences, like the sequence of squares

$$1^2,\ 2^2,\ 3^2,\ldots,n^2,\ldots$$

  or the sequence of *triangular* numbers

$$1,\ 1+2,\ 1+2+3,\ \ldots,1+2+3+\cdots+n=\frac{n(n+1)}{2},\ldots$$

- the sequence of *cubes*

$$1^3,\ 2^3,\ 3^3,\ \ldots,n^3,\ \ldots.$$

However, nature (and mathematics) often prefers geometric or exponential sequences, such as

the powers of 2:

$$2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ \ldots,$$

or the *Fibonacci* sequence:

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ \ldots.$$

One key distinction between the two types of sequences becomes apparent if we compare what happens when we look at the sequence of "term-to-term differences" for each type of sequence.

(a) The sequence of "term-to-term differences" for a *linear* sequence, such as

$$2,\ 4,\ 6,\ 8,\ 10,\ 12,\ \ldots,$$

gives rise to a *constant* sequence of differences

$$2,\ 2,\ 2,\ 2,\ 2,\ \ldots.$$

The sequence of "term-to-term differences" for a *quadratic* sequence, such as

$$1, 4, 9, 16, 25, 36, \ldots$$

gives rise to a *linear* sequence of differences

$$3, 5, 7, 9, 11, \ldots,$$

whose own sequence of differences (or "second differences") is then *constant*

$$2, 2, 2, 2, \ldots.$$

The sequence of "term-to-term differences" for the quadratic sequence of triangular numbers

$$0, 1, 3, 6, 10, 15, 21, \ldots$$

gives rise to a *linear* sequence of differences

$$1, 2, 3, 4, 5, 6, \ldots,$$

whose own sequence of differences is then *constant*

$$1, 1, 1, 1, 1, \ldots.$$

(b) The sequence of "term-to-term differences" for a geometric sequence behaves quite differently. If we consider the geometric sequence of powers of 2:

$$2, 4, 8, 16, 32, 64, 128, \ldots$$

then the sequence of "term-to-term differences" gives rise to

$$2, 4, 8, 16, 32, 64. \ldots$$

which is the same as the original sequence, so taking second and third differences will never lead to anything simpler.

If we consider the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

Then taking differences gives rise to

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

which is essentially the same sequence again. Hence taking second and third differences will never lead to anything simpler.

In short, taking differences repeatedly for a polynomial sequence seems to lead eventually to a **constant** sequence, whereas the sequence of differences for a geometric (or exponential) sequence leads only to something closely related to the original sequence (from which it never escapes).

3. Geometry and measures

3.1. Background

Geometry should be one of the highlights of mathematics teaching in lower secondary school.

- The subject matter is intuitively appealing and practical.
- It offers extensive scope for drawing intriguing figures, for implementing unexpected constructions, and for making pleasing—even beautiful—models.
- The tools and principles which allow us to analyse this wonderful world *exactly* are surprisingly simple and accessible.
- All pupils can calculate some surprising things, can solve some interesting problems, and can **prove** some strikingly useful results; and more confident pupils can **prove** a wide range of remarkable and unexpected facts.

- Applications to the world around us are immediate, convincing, and impressive.
- The material of school geometry captures the spirit of mathematics better than almost any other part of elementary mathematics.

One would be hard-pressed to discern these strengths in the published programme of study. In particular, there is little emphasis on drawing and making, no clear indication of the intended deductive structure for geometry, and there is no mention of applications to the world around us. But the Good News is that the official programme is compatible with an approach based on the above bullet points—provided schools do not simply mimic the printed sequence of official requirements.

There is considerable confusion over geometry in many apparently authoritative pronouncements—including the requirements listed in the official Key Stage 3 programme of study. To understand why, teachers need to know how we got where we are. So we begin with a thumbnail sketch of some of the relevant historical and pedagogical roots of the current approach to school geometry in England.

School work with number and algebra tends to be relatively "one dimensional". Typical problems look fairly familiar, and can usually be solved by implementing some well-rehearsed "linear" procedure.

- One is told what is to be calculated (the goal).
- It is usually fairly clear where to begin.
- And with sufficient practice, one can more-or-less follow one's nose to get from the start to what is required.

Real mathematics is not like this, and is more like what secondary school geometry ought to be.

- We are given some information about a two- or three-dimensional configuration or shape.
- We are asked to calculate something, or to prove some fact.
- We have to draw and edit a diagram as a guide.

- Then we are left to find for ourselves
  - (i) a suitable feature of the figure that might serve as a starting point, and
  - (ii) a sequence of steps from this (elusive) start to what is required.
- Because figures and diagrams are in **two** dimensions, there is often no clear starting point, and no obvious route from start to finish.

For pupils (and teachers) who have come to see school mathematics as a collection of predictable, one-dimensional procedures, this experience is unsettling. The given figure may appear elementary; and one may understand what is wanted. But one often has **no idea where to begin, or how to proceed**. As we noted in Section 2.3 of Part II (*Solving problems*), in such a setting it does not take much for a routine *exercise* to become a frustratingly elusive *problem*. Geometry reveals this distinction more strongly than most other parts of elementary mathematics.

Most mathematics educators in England are aware of "a difficulty" with geometry; but there has been very little attempt to analyse it in detail, or to explore effective ways of overcoming it. Rather than attempt some easy explanation, our concern here is to draw attention to this neglect, in the hope that once it is recognised, teachers will be more willing to question the conventional wisdoms about school geometry which often take the place of serious analysis. For example, our ambivalence towards geometry has often been mixed up with attitudes towards "proof"— because historically geometry came to be seen as the main vehicle for conveying ideas of proof in school mathematics. This has led to a view that serious geometry and proof are "only for the few". Yet, as we have tried to illustrate, "proof" (whether used to derive new methods and results on the basis of what we already know, or to make sense of standard procedures) should be an integral part of school mathematics from the earliest years, and geometry should enrich everyone's experience of school mathematics.

Proposals for major change in secondary school "geometry for all" arose in the early 1900s, with John Perry's moves to advocate measurement, drawing, trigonometry, the solution of triangles, calculation of areas and volumes, coordinate geometry, and "technical drawing". Perry's ideas met with some success—possibly more so in the USA. In England the need for change was recognised, but the traditional influence of Oxford and Cambridge on secondary school curricula resulted in a very English compromise, which lasted in some sense until the 1950s.

The reorganisation of schooling after the Second World War re-opened the question of "geometry for all". However this liberal concern was overtaken by the "modernising" reforms which gained momentum in the late 1950s and early 1960s in the wake of the Soviet Union's launch of *Sputnik*. The official shock among western governments at being "left behind in the space race" strengthened the hand of those who wanted to "sweep away the old" and replace it by something more "up to date". In the USA the preferred "modern" approach was "axiomatic synthetic geometry". In the UK we tried to replace the uneasy compromise of classical and coordinate geometry (and technical drawing) by "transformations and matrices". In France there was a strong lobby in favour of linear algebra and affine geometry. All these approaches had some advantages for the very best pupils. But all approaches proved too ambitious—even wholly inappropriate—for most pupils, and failed. The British approach through transformations had some attractive features, and some strong advocates, which seems to have made it more difficult for us to admit that it had failed, and to engage in a serious review of what was actually needed. As a result, certain themes (e.g. nets of polyhedra, and selected transformations) continue to feature, even though they no longer deliver any significant *mathematical* pay-off. In place of a considered, if overambitious, progression from naïve symmetry, through transformation geometry, to matrices and affine transformations, we are left with a residual rump of bits and pieces.

In short, since the early 1980s, geometry teaching in England has increasingly served up a mish-mash. We abandoned the grand vision of the reforms of the 1960s and 70s, while retaining some of its language and content. And in the 1990s we revived a half-hearted version of traditional Euclidean geometry without ever really sorting out what was needed. (The new curriculum illustrates our current plight. There we are exhorted to "derive and illustrate properties of triangles, quadrilaterals, [...] and other plane figures", without any recognition of the central position of *isosceles triangles*—which are never mentioned; and without any hint that the most important "quadrilaterals" of all are *parallelograms*—which are only mentioned once, in the context of "formulae to calculate area" (p. 8).)

During the same period, university mathematics departments recognised that their students lacked the geometrical background that was assumed in many courses. But universities neither got involved at school level, nor did they develop an effective university level "introduction to geometry". Hence **most of those who now teach school mathematics have never experienced a systematic study of elementary geometry**—either in school, or at university. We have therefore erred on the side of including here more than is needed for most pupils, in order to provide teachers with a brief exposition of what they have been missing. In particular, we have included many details that belong more naturally in Key Stage 4—and then sometimes only for appropriate groups of pupils. We hope this will encourage schools to consider what is genuinely accessible at this level, to experiment, and to decide for themselves what to teach, to whom, and at which level.

To cut a long story short, it is our contention (though rarely admitted explicitly):

- that secondary school *geometry* is potentially attractive, but inevitably "hard" (e.g. because it cannot be reduced to a series of well-rehearsed, one-dimensional routines);
- that no one is well-served by the present confused mish-mash;
- that, although *translations* relate to work on vectors, and although there may be unstated aesthetic reasons for introducing the language of symmetry, patterns, rotations, reflections, translations, and enlargements (and the missing isometries, the *glide reflections*), these ideas **can never** constitute an effective mathematical way of analysing geometrical figures at this level;
- that all groups would benefit from a coherent initial approach to secondary geometry in Key Stages 1–3—even if not all follow through to the same endpoint at Key Stage 4;
- that the three basic principles (*congruence, parallels, similarity*) can be appreciated by everyone, and can be used on different levels in drawing,

constructing, and analysing interesting configurations in 2D and 3D; and hence

- that we need to develop an effective approach to secondary geometry, which would be potentially accessible and appealing to most pupils, which is founded upon the *congruence criterion*, the *criterion for parallels*, and the *similarity criterion*, and which combines
    (a) drawing, measuring, and calculating (lengths, areas, volumes, angles, trigonometry),
    (b) analysing figures and configurations in terms of *points, lines, line segments, angles, triangles, parallelograms, circles*, etc.,
    (c) using a mix of deduction of key results with lots of lovely problems, and
    (d) linking with algebra and a suitable dose of coordinate geometry at Key Stage 4.

To create an internal scheme of work that reflects this, schools must be willing

(i) to interpret the official requirements intelligently,

(ii) to discriminate between what is important for their pupils' mathematical development and what is not,

(iii) "to join up the (sometimes invisible) dots" into a coherent scheme of work, and then

(iv) to review and refine the details in the light of experience.

We provide an initial supporting map by grouping most of the official requirements under three main headings:

3.2 *Drawing, measuring, and terminology*

3.3 *Perimeter, area, and volume*

3.4 *Constructions, conventions, and derivations*

Although it is left unsaid, we assume that under each heading, pupils will be expected to tackle a rich variety of suitable problems.

The remaining official requirements are then discussed in Section 3.5.

3.2. Drawing, measuring, and terminology

- **draw and measure line segments and angles in geometric figures [. . .]**
- **describe, sketch and draw using conventional terms and notations: points, lines, parallel lines, perpendicular lines, right angles, regular polygons, and other polygons that are reflectively [and/or] rotationally symmetric**
- **[. . .] illustrate properties of triangles, quadrilaterals, circles and other plane figures [for example, equal lengths and angles] using appropriate language and technologies**
- **identify properties of, and describe the results of, translations, rotations and reflections applied to given figures**
- **draw and measure line segments and angles in geometric figures, including interpreting scale drawings**
- **identify and construct congruent triangles, and similar shapes by enlargement, with and without coordinate grids**

Despite the emphasis here on "doing", the language remains vague. Teachers will need to be creative, and to identify those themes that deserve to be included but are here passed over in silence. In particular, there is no obvious mention of "applications": angles are to be drawn and measured, and scale drawings (presumably including maps) are specifically included, but there is no hint that one should include practical activities involving "bearings", or "angles of elevation"—so that these ideas will have some meaning when they arise in later paper exercises. So there is much to be "filled in".

However, if we leave aside the many ingredients which are omitted, one way to think about these six requirements is that:

- the **first two** involve basic opportunities to draw, to measure, and to describe;
- the **next two** involve more reflective preliminary analysis ("illustrating" and "identifying"—and one hopes *talking about*, and familiarising pupils with— "properties", as opposed to *deriving* them as some pupils should do later);
- in the **last two** requirements, pupils begin to grapple with the three basic principles of Euclidean 2D geometry: *congruence* and *similarity* are mentioned explicitly, while the characteristic property of *parallel lines* is implicit in the whole idea of "enlargement" and scale drawings.

Thus this first group of six requirements serves as a bridge—launching out from the familiar territory of "geometry as experience" at Key Stage 2 towards the pre-formal, more analytical world of constructions and deductions at secondary school (see Section 3.4).

3.2.1 Drawing, measuring, and describing One would like to see initial "measuring and drawing" tasks

(a) that check on, and strengthen skills from Key Stage 2;

(b) that develop pupils' facility and precision in working with ruler, protractor, and compasses;

(c) that use and establish the correct notation for line segments and for angles in *labelled* diagrams, and

(d) that give rise to slightly unexpected results, which can then be talked through in class.

The neglect (not just in England) of

(i) basic work on drawing and measuring, and

(ii) the cultivation of spatial common sense through learning to think through one's hands, fingers, and eyes,

is indicated by the following very basic Year 9 items from TIMSS 2011.

> **3.2.1A** Points A, B, and C lie in a line and B is between A and C. If $AB = 10\text{cm}$ and $BC = 5.2\text{cm}$, what is the distance between the midpoints of AB and BC?
>
> A 2.4cm B 2.6cm C 5.0cm D 7.6cm

> **3.2.1A** Russia 60%, Hungary 41%, Australia 40%, England 38%, USA 29%

> **3.2.1B** [An 8×8 square grid is shown] The length of side of each of the small squares represents 1cm. Draw an isosceles triangle with a base of 4cm and a height of 5cm.

> **3.2.1B** Russia 75%, Hungary 68%, Australia 41%, England 40%, USA 27%

The responses clearly suggest that pupils are never expected to construct the simplest diagrams for themselves. So we must be prepared to begin Year 7 with lots of drawing exercises that might once have been assumed from Key Stage 2, but which have fallen out of favour—perhaps because they cannot easily be assessed. This seems to hold for even the simplest traditional primary school activities, such as using compasses:

> "Draw a circle with centre O and with radius OA;
>
> then draw the circle with centre A passing through O, and meeting the original circle again at B and F;
>
> then draw the circles with centres at B and F and passing through O, to meet the original circle again at C and E;
>
> finally draw the circles with centres at C and E and passing through O, and notice that these circles meet the original circle at **the same point *D***."

And then colour the resulting hexagonal pattern of flower petals!

To illustrate the kind of additional tasks that one might use we offer the following examples.

- Given a drawn rectangle $ABCD$ measuring 3cm by 4cm, require that the two diagonals AC, BD be measured, along with the angles $\angle BAC$ and $\angle DCA$.
- Given a square $ABCD$ with sides of length 10cm, require that the two diagonals AC, BD be measured, along with the four angles $\angle BAC$, $\angle BCA$, $\angle DCA$, $\angle DAC$.
- Given a regular hexagon $ABCDEF$, measure the edge length AB and the length of a "long diagonal" FC, and the angles $\angle BAC$, $\angle CAD$, $\angle DAE$, $\angle EAF$.
- Given a regular pentagon $ABCDE$ with sides of length 10cm, measure the length of the diagonals AD, BD, and the angles $\angle EAD$, $\angle ADB$, $\angle BDC$, $\angle DBA$, $\angle DAB$.

Such drawing and measuring exercises are intended to feed into subsequent class discussion, for which the initial practical activity serves as the directly relevant prior experience. The above tasks provide opportunities to consider:

- Whether the two diagonals AC, BD of the rectangle $ABCD$ really are equal?
- Whether the four angles $\angle BAC$, $\angle BCA$, $\angle DCA$, $\angle DAC$ in the square $ABCD$ really are equal, whether they are all equal to 45°, and whether something else seems to be true about the two diagonals AC and BD?
- Whether the diagonal FC in the regular hexagon $ABCDEF$ really is twice as long as the side AB, whether anything else seems to be true of the lines AB, FC, ED, and whether the angles $\angle BAC$, $\angle CAD$, $\angle DAE$, $\angle EAF$ really are all equal to 30°?
- Whether in the regular pentagon $ABCDE$ there is anything else that seems to be true about the side EA and the diagonal DB, or about the diagonal AD and the side BC, whether the angles $\angle EAD$, $\angle ADB$, $\angle BDC$ really are all equal (to 36°), whether the angles $\angle DBA$ and $\angle DAB$ are equal to each other and twice the size of the previous group?

Some of the equalities and relationships that emerge from such an exercise can be justified at this level. But others should be treated as genuine "surprises", which demand explanation *later*. In particular, teachers should hesitate before giving the impression that plausible-sounding catch-all "reasons" (e.g. in terms of the presumed "symmetry" of a regular n-gon) are acceptable as explanations of what is observed.

In primary school the approach to geometry is largely rooted in experience, with properties being observed and used. But in secondary school the approach should be more analytical, and should distinguish between the (minimal) definition of an object, and any derived properties. In particular, the definition of a regular n-gon says nothing about its symmetry. A *regular n-gon* is **defined** very simply to be

> a polygon in which the n sides are all equal and the n angles are all equal.

It is not at all obvious—though not difficult to **prove** later—that a regular polygon always has a "centre", can be inscribed in a circle with that centre, and has n-fold rotational symmetry and n lines of reflection symmetry. But at secondary level it is **wrong** to convey the impression that these additional properties are part of what we "know" *a priori* about a regular polygon. Hence the reference in the second listed requirement to

> "regular polygons, and other polygons that are reflectively [and/or] rotationally symmetric"

is thoroughly misleading. At the very least *the word "other" should be deleted*.

3.2.2 Establishing a basic repertoire: "illustrating, identifying and describing" The next two requirements in the list at the start of Section 3.2 ("to illustrate properties ..." and "to identify properties ... and describe ...") are best not taken too literally, but should be interpreted as an

> "invitation to revise and to extend pupils' repertoire of language and observed facts in geometry".

In particular, schools will need to clarify for themselves

- how to interpret the indiscriminate word "quadrilaterals", by sorting out **which** quadrilaterals are most important (namely **parallelograms** including rectangles);
- what to make of the reference to "other plane figures" (which as far as one can tell should probably mean (a) properly defined "regular polygons", and (b) composite figures made from rectangles and arcs of circles which will be needed in Section 3.3);
- what is meant by the curious bracket

  > "[for example, equal lengths and angles]",

  which we take to be an unedited cryptic allusion to

  - **isosceles triangles** (which receive no mention of any kind elsewhere), and to
  - the two basic results
    (i) if $AB = AC$ (i.e. triangle ABC is isosceles with base BC), then $\angle ABC = \angle ACB$;
    (ii) if $\angle ABC = \angle ACB$, then $AB = AC$ (so the triangle ABC is isosceles with base BC);
- whether the single isolated reference to "appropriate technologies" was included for a good reason in the only appropriate place, or whether this comment should be taken as a prompt to consider carefully the potential advantages, and dangers, of technology throughout the teaching of *Geometry and measures* at this level.

On the latter point, we merely note that active drawing exercises clearly help to cultivate pupils' geometrical thinking, whereas the passive enjoyment of ready-made enhanced graphics seems to convey no similar mental benefit.

3.2.3 Towards congruence (and similarity) The last two listed requirements ("draw and measure …" and "identify and construct …") are no longer merely elaborating what pupils bring with them from Key Stage 2.

- Making and "interpreting scale drawings" is a valuable common sense exercise, which can later be related to enlargement, proportion and similarity. But at this stage, the focus should be on interpreting lengths, distances, and perhaps areas.

  (Understanding that *angles are preserved* in such scale drawings should be appreciated informally at this stage. The proof may be best left until similarity is addressed later—at which point one can explain that:

  - If two lines AB and AC meet at A, and if the points A, B, C are represented by the points A', B', C' on a scale drawing, then

    $$AB : A'B' = AC : A'C' = BC : B'C'.$$

  - Hence by the *similarity criterion* (Part II, Section 2.2.2.3 and Section 3.4.7 below), it follows that

    $$\angle BAC = \angle B'A'C' :$$

    that is, the angle between the two original lines AB and AC is the same as the angle between the lines $A'B'$ and $A'C'$ in the scale drawing. **QED**)

- "Identify and construct congruent triangles" is best tackled separately from—and long before—"similar shapes by enlargement". The goal here should be to convey

  - the central importance of triangles;
  - the idea that a triangle ABC is an *ordered* triple;
  - that such a triangle ABC gives rise to **six** pieces of data: the three sides AB, BC, CA and the three angles $\angle CAB$, $\angle ABC$, $\angle BCA$;
  - that two (*ordered*) triangles ABC and DEF are congruent precisely when their vertices match up in order

    $$A \longleftrightarrow D, \qquad B \longleftrightarrow E, \qquad C \longleftrightarrow F$$

so that the three pairs of sides in the two triangles match up exactly, with $AB = DE$, $BC = EF$, $CA = FD$, and the three pairs of angles also match up exactly, with $\angle CAB = \angle FDE$, $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$;

- but that in reality we can be sure that two triangles are congruent **without** having to check that all **six pairs** (i.e. three sides and three angles) match up: for to determine a triangle uniquely (up to congruence) we only need to know certain triples of information—namely:

  SSS: AB, BC, and CA; or
  SAS: AB, $\angle ABC$, and BC; or
  ASA: $\angle ABC$, BC, and $\angle BCA$.

To achieve this understanding, pupils need to be given specified lengths and angles and then be required to use ruler and protractor, or ruler and compasses, to construct the triangle, and hence to internalise a sense of how this limited information determines the final triangle. They should also be given lots of examples where the triangle is not determined by the given information, such as:

AAA: given $\angle ABC$, $\angle BCA$, and $\angle CAB$ only; or

ASS: given $\angle ABC$, BC, and CA only (e.g. $\angle ABC = 30^\circ$, $BC = \sqrt{3}$, and $CA = 1$).

As explained in Part II, Section 2.2.2.3, RHS congruence is a *consequence* of SSS and *Pythagoras' Theorem*, so RHS is not part of the basic *congruence criterion*. Hence it should be introduced, proved, and used somewhat later.

The reference to "similar shapes" here is clearly informal (the full notion of similarity is more subtle, and may be best postponed until later in Key Stage 3—see Part II, the end of Section 2.2.2.3, and Section 3.4.7 below). The emphasis should at first be practical: constructing "enlargements" initially in the spirit of the exercises in Section T8 of *Extension mathematics, Book Beta* by Tony Gardiner (Oxford University Press 2007), and later more formally in the spirit of section C26 in the same book. Work with "scale drawings" should be similarly practical—interpreting scale drawings and maps and using the scale factor to estimate actual distances, areas and angles, then constructing scale drawings of familiar locations. (Note that "scale factors" also feature in the requirements addressed in Section 1.9.)

3.3. Perimeter, area, and volume

- **derive and apply formulae to calculate and solve problems involving: perimeter and area of triangles, parallelograms, trapezia, volume of cuboids (including cubes) and other prisms (including cylinders)**
- **calculate and solve problems involving: perimeters of 2D shapes (including circles), areas of circles and composite shapes**

At first sight these two requirements may seem relatively straightforward. However, there is more here than may be apparent at first sight.

3.3.1 Trapezia: an example The mention of "trapezia" illustrates a general danger. Mathematical methods are too often taught by training pupils to use formulae which they do not understand, rather than by first helping them to achieve a basic understanding, and encouraging them to use their common sense. Once pupils achieve a clear understanding, that understanding may be suitably summarised in terms of a formula—provided this is never used as a substitute for thinking what they are doing.

The first listed requirement in Section 3.3 tries to compress too many ideas into one bullet point. Whenever the official programme of study tries to compress distinct topics into a single requirement in this way, the result is to distort the message—especially at the two ends of the spectrum of difficulty.

- It is unfortunate that the first requirement in Section 3.3 seems to suggest that a formula be used to calculate the "**perimeter of triangles**". There is no "formula". Common sense is all that is needed.

- At the other end of the difficulty scale, the apparent requirement that all pupils should

  > "derive and apply formulae to calculate [...the] area of [...] trapezia"

  cannot mean what it appears to say. For in the GCSE *Subject criteria* we are told (p. 15) that the formula for the area of a trapezium "is not specified in the content". So knowing and using the formula **cannot be intended for everyone** as part of the Key Stage 3 programme of study.

It is clearly more appropriate at Key Stage 3 for pupils to know

- that a quadrilateral $PQRS$ with two parallel sides PQ and SR is called a *trapezium*, and
- that, if the parallel sides have lengths $PQ = a$ and $SR = b$, and the perpendicular height is h, then the area of the shape can be found by dropping two perpendiculars, from P meeting SR at X and from Q meeting SR at Y, to produce a rectangle $PQYX$ and two right angled triangles PXS and QYR, whose areas can be combined (using addition or subtraction— depending on the shape of the trapezium) to find the area of $PQRS$.

This primitive method can later lead to a proof of the well-known formula—at least in the simplest cases.

Claim Suppose X and Y are "internal" to the line segment SR. Then

$$\text{area}(PQRS) = \frac{1}{2}(a+b) \times h.$$

Proof Since $PQYX$ is a rectangle, we know that: $XY = PQ = a$, and $PX = QY = h$, and that $SX + YR = SR - XY = b - a$.

$$\begin{aligned} \therefore \quad \text{area}(PQRS) &= \text{area}(PQYX) + \text{area}(\triangle PXS) + \text{area}(\triangle QYR) \\ &= a \times h + \frac{1}{2}(SX \times h) + \frac{1}{2}(YR \times h) \\ &= a \times h + \frac{1}{2}((b-a) \times h) \\ &= \frac{1}{2}(a+b) \times h. \quad \textbf{QED} \end{aligned}$$

3.3.2 "Composite shapes" The simple-minded approach to trapezia (by reducing the problem of finding the area of the trapezium to that of a rectangle and two right angled triangles) illustrates the reference to "composite shapes" in the second requirement. The kinds of combinations that are relevant here are very restricted, but they lie at the heart of all work with length, area, and volume.

- We calculate more complicated **lengths** (such as the perimeter of a polygon) by breaking them up into, or approximating them in terms of, combinations of *line segments* (or "one dimensional rectangles").
- We calculate the **area** of more complicated shapes in 2D by breaking them up into, or approximating them in terms of, combinations of *rectangles*.
- We calculate the **volume** of more complicated shapes in 3D by breaking them into, or approximating them in terms of, combinations of *cuboids* (or "three-dimensional rectangles").

In one dimension one may fudge the idea of "length" for the circumference of a circle by imagining a string wrapped round the circle, which is then "straightened out and measured". This is fine—both as a way of conveying what we mean by the "circumference", and to obtain a physical approximation. But it is not a *mathematical* method: the string is a *physical* object; the result is *approximate*—with no control over the error; and there is no way to be sure that the string does not change its length as one "straightens it out". However, the most serious objection is that **the idea does not extend to 2D and 3D**. For example, one cannot take a curved 2D shape like a circular disc, cut it up and rearrange the pieces exactly to find its exact area; and one cannot take a curved surface, like the surface of an orange and "straighten it out, or lay it flat" to find its surface area. The idea that can be made to work in all dimensions is to concentrate on **approximating more complicated shapes by "rectangles"** (line segments, 2D rectangles, cuboids, etc.).

It is true that in two dimensions we often dissect polygons and other shapes into triangles rather than rectangles. But this trick has to be interpreted carefully. When we move from 2D to 3D, there is no way to extend the idea of a "triangle" as a way of making sense of "calculating volumes": for there is no elementary way of finding the volume of a pyramid or tetrahedron. So we are free to use triangles in 2D, but we should think of each triangle as "**half a rectangle**" (on the same base, and with the same height), since the idea that works in all dimensions is to approximate shapes in terms of "n-dimensional rectangles". That is,

- the basic building blocks for length are *line segments* (one dimensional rectangles);
- the basic building blocks for area are *rectangles* (two dimensional rectangles);
- the basic building blocks for volume are *cuboids* (three dimensional rectangles).

We also use composite shapes to *approximate* more awkward figures.

- The circumference of a circle is approximated by the perimeter of an inscribed or a circumscribed regular n-gon.
- The area of a circle is approximated from below by counting the number of unit squares inside it, and from above by counting the number of unit squares needed to just cover it.

3.3.3 Understanding first, formulae second We repeat: pupils should be discouraged from using formulae *ab initio*. Instead they should be encouraged to imagine each "perimeter" as a sequence of separate line segments, and each "area" as being made up from, or approximated by, rectangles, or triangles (or a combination of both). In particular, they should use their common sense in working from the very beginning with composite shapes made from line segments, or from rectangles (or triangles), or from cuboids (or "wedges" as "half cuboids"). This helps to strengthen pupils' basic understanding, since such composite shapes admit no general formula.

The extent to which pupils do not at present use their common sense in such matters is indicated by the following Year 9 items from TIMSS 2011.

> **3.3.3A** The perimeter of a square is 36cm. What is the area of the square?

A 81cm^2 B 36cm^2 C 24cm^2 D 18cm^2

3.3.3B The area of a square is 144cm^2. What is the perimeter of the square?

A 12cm B 48cm C 288cm D 576cm

These are multiple choice items—so pupils were **not** required to calculate the answers. The false options here are either hard to obtain, or reflect severe mental sloppiness. So the results should provide serious food for thought (and not only in England).

3.3.3A Russia 62%, Hungary 55%, Australia 54%, USA 53%, England 51%

3.3.3B Russia 62%, Hungary 49%, Australia 48%, England 47%, USA 46%

3.3.4 Length There is more to Section 3.3.2 than may appear: in simple language, it incorporates a *definition* of what we mean by "length", of what we mean by "area", and of what we mean by "volume".

Pupils should understand the "perimeter of a *rectangle*" not via a formula, but using the common sense fact that it is made up of *four line segments*, whose lengths add up to give the perimeter (see examples 3.3.3A and 3.3.3B). The same idea applies to any polygon—where the perimeter is made up of a finite number of line segments, whose lengths can be added to give the perimeter of the polygon.

However, at first it is completely unclear how to extend this idea to measure the lengths of *curves*—such as the circumference of a circle of radius r. The *physical* idea of "the circumference of a circular, or cylindrical, lamp post" may be adequately captured by a piece of string that can be wound round the post and then straightened out and measured. But mathematics cannot depend on string. To capture the "length of a circular arc" *mathematically* we need

- to approximate it by a succession of line segments (such as the perimeter of a regular n-gon inscribed in, or circumscribed around, the circle), and

- then to realise that, as the number n of sides increases, the perimeter of the polygon gets closer and closer to the circle itself.

The cases which can be calculated easily, exactly, and instructively, without using trigonometry are:

$n = 3$: *Pythagoras' Theorem* gives

$$\text{“perimeter of inscribed regular 3-gon”} = 3r\sqrt{3}.$$

$n = 4$: *Pythagoras' Theorem* gives

$$\text{“perimeter of inscribed regular 4-gon”} = 4r\sqrt{2}.$$

$n = 6$: simple geometry gives

$$\text{“perimeter of inscribed regular 6-gon”} = 6r.$$

$n = 8$: *Pythagoras' Theorem* gives

$$\text{“perimeter of inscribed regular 8-gon”} = 8r\sqrt{2-\sqrt{2}}.$$

An inscribed regular octagon is still a long way from the circle itself, but we can see that the circumference of a circle of radius r is approximated ever more closely from below by the sequence

$$r \cdot 3\sqrt{3} < r \cdot 4\sqrt{2} < r \cdot 6 < r \cdot [8\sqrt{2-\sqrt{2}}] < \dots$$
$$\cdots < \text{“circumference } C \text{ of circle of radius } r\text{”}.$$

The required circumference C of a circle would seem to be “some multiple of the radius r”, and the mysterious multiplier “$\frac{C}{r}$” satisfies

$$5.1961\cdots < 5.6568\cdots < 6 < 6.1229\cdots < \cdots < \frac{C}{r}.$$

The multiplier “$\frac{C}{r}$” can also be bounded from above by considering circumscribed regular n-gons:

$n = 3$: *Pythagoras' Theorem* gives

$$\text{“perimeter of circumscribed regular 3-gon”} = 6r\sqrt{3}.$$

$n = 4$: *Pythagoras' Theorem* gives

"perimeter of circumscribed regular 4-gon" $= 8r$.

$n = 6$: simple geometry gives

"perimeter of circumscribed regular 6-gon" $= 4r\sqrt{3}$.

$n = 8$: *Pythagoras' Theorem* gives

"perimeter of circumscribed regular 8-gon" $= 16r(\sqrt{2} - 1)$.

Hence

$$5.1961\cdots < 5.6568\cdots < 6 < 6.1229\cdots < \cdots < \tfrac{C}{r} < \ldots$$

$$\cdots < 6.6274\cdots < 6.9282\cdots < 8 < 10.3922\ .$$

The mysterious multiplier "$\frac{C}{r}$" is clearly somewhere around 6.3. Once we give it a name "2π", and declare its actual value, we have the formula "$C = 2\pi r$" for the full circumference of a circle of radius r. We can then extend this to find the length of a semi-circle of radius r (πr), or of a quadrant ($\frac{\pi r}{2}$), or of the length of a circular arc of radius r with angle θ at the centre.

One can then pose lots of moderately challenging problems to find the perimeters of composite shapes which are made entirely of rectangles (such as staircase-shaped figures), or which combine rectangles and circular arcs (such as a "running track", or shapes made of rectangles and quadrants—both protruding and indented).

3.3.5 Area We make sense of "area" in much the same way.

- If we take the area of a unit square as "1", an m by n rectangle is made up of $m \times n$ unit squares, and so has area mn (square units)
- We can break up the sides of the unit square into unit fractions, and conclude that mn copies of a $\frac{1}{m}$ by $\frac{1}{n}$ rectangle have total area 1, so that each has area
$$\frac{1}{m} \times \frac{1}{n} = \frac{1}{mn}.$$

- A $\frac{p}{m}$ by $\frac{q}{n}$ rectangle can then be split into $p \times q$ rectangles each of which is $\frac{1}{m}$ by $\frac{1}{n}$, and so has area

$$pq \times \frac{1}{mn} = \frac{pq}{mn} = \frac{p}{m} \times \frac{q}{n}.$$

In short, the area of a rectangle with sides of lengths a units and b units can be shown to be equal to $a \times b$ square units for all possible values of a and b.

When we later come to consider "scaling" and "similarity", the two facts:

- that the area of any rectangle is equal to "length × breadth", and
- that the area of any more general shape is defined in terms of approximating the shape by combinations of rectangles

have an important hidden consequence. Whatever the area of a given shape may be, if we enlarge it (or "en-small" it) by multiplying all lengths by the same scale factor "r", then the area of each and every approximating rectangle is multiplied by r^2, so **the area of the shape** being approximated **is multiplied by r^2**. So if one square has sides that are three times as long as another, then its area is nine times as large; and a circle of radius 4 has area 16 times as large as a circle of radius 1.

Long before we attempt a formal treatment of enlargement, or similarity, we need to build up the repertoire of basic results involving measures (as listed in the official requirements at the start of Section 3.3) using congruence. In particular, we need to connect the area of other plane figures to our fundamental shape—namely rectangles. And the most important of these "other figures" are parallelograms and triangles.

Suppose a parallelogram $ABCD$ has longest diagonal AC. Let the perpendicular from A meet the line CD (extended) at X, and the perpendicular from C meet the line AB (extended) at Y. Then the parallelogram $ABCD$ is completely enclosed in the rectangle $AXCY$, and the excess is formed by the two right angled triangles AXD and CYB—which fit together to make a smaller rectangle. Hence $ABCD$ is equal to the *difference* between the large rectangle (with base XC and height XA) and the excess rectangle (with base XD and height XA)—whence:

Claim Area(parallelogram $ABCD$) = "base DC × height XA".

Given any triangle ABC with "base AB", we can draw the line through C parallel to the base AB, and the line through A parallel to the side BC. If these two lines meet at D, then $ABCD$ is a parallelogram.

Claim $\triangle ABC$ is congruent to $\triangle CDA$

Proof $\angle BAC = \angle DCA$ (alternate angles—see Part II, section 2.2.2.3)

$AC = CA$ (same side)

$\angle BCA = \angle DAC$ (alternate angles)

$\therefore$ $\triangle ABC$ is congruent to $\triangle CDA$ by the ASA congruence criterion. **QED**

Corollary
$$\begin{aligned}\text{Area}(\triangle ABC) &= \tfrac{1}{2}\times \text{area(parallelogram } ABCD)\\ &= \tfrac{1}{2}(\text{base } AB\times \text{height}).\end{aligned}$$

Pupils may think they already "know" the Corollary. What is new at Key Stage 3 is the idea that one can organise the vast lit of "known facts" in a way that identifies which are the "most basic" (namely *congruence* and the area of a rectangle), and how everything else can be derived from these basic results. Hence, one would like as many pupils as possible to appreciate

- that the result for the area of a triangle follows from
  - (i) congruence and
  - (ii) the result for the area of a parallelogram, and
- that the result for the area of a parallelogram follows from that for a rectangle.

In other words, we first highlight the *congruence criterion*, and then use it to reduce every question about the areas of other shapes (first parallelograms, then triangles, polygons, circles, etc.) to the basic question about the area of a rectangle. This is in some sense what we find in Book I of Euclid's *Elements* (c. 300BC), where he goes on to show (in Proposition 47) the remarkable fact that this is all that is needed to prove *Pythagoras' Theorem*.

Claim Let $\triangle ABC$ be a right angled triangle with a right angle at C. Then the square $ABPQ$ on the hypotenuse AB is equal to the sum of the squares $CARS$ on side AC and $BCTU$ on side BC.

Proof Let the perpendicular from C to AB meet AB at X and QP at Y.

It suffices to show that (half of) the square $CARS$ is equal to (half of) the rectangle $AXYQ$.

AR is parallel to BS.

$\therefore \triangle ARC$ and $\triangle ARB$ have the same base AR and the same height RS, so have the same area.

Also $\triangle ARB \equiv \triangle ACQ$ (by SAS), so $\triangle ARC$ and $\triangle ACQ$ have the same area.

AQ is parallel to XY.

$\therefore \triangle ACQ$ and $\triangle AXQ$ have the same base AQ and the same height AX, so have the same area.

Hence $\triangle ARC$ and $\triangle AXQ$ have equal area. **QED**

The proof needs to be acted out and expanded, but it has several advantages over most other proofs:

- It is very basic, in that it only uses congruence and the area of a triangle.
- It explains *why* the "square on the hypotenuse AB" is equal to a **sum** in a way that most proofs do not.
- The construction used is entirely natural: indeed, given a right angled triangle ABC with a right angle at C, the line CXY is the only way of splitting the "square on AB" into two parts that could possibly produce one part equal to the square on CA and the other equal to the square on CB.

In John Aubrey's *Brief lives* (1694) we read of the philosopher Thomas Hobbes, that:

> He was 40 years old before he looked on Geometry; which happened accidentally. Being in a Gentleman's Library, Euclid's **Elements** lay open, and 'twas the *[Proposition] 47 [Book I]*. He read the Proposition. By God, sayd he (he would now and then swear an emphaticall Oath by way of emphasis) this is impossible! So he reads the Demonstration of it, which

> referred him back to such a Proposition; which proposition he read. That referred him back to another, which he also read. *[Continuing in this way, checking one thing after another]* at last he was demonstratively convinced of that trueth. This made him in love with Geometry.

It is worth pondering on Hobbes' scepticism and astonishment. *Pythagoras' Theorem* is a completely unexpected result—and yet one that heralds much that lies ahead (from the Cosine rule, to scalar products, vector analysis, linear algebra, quadratic forms, and much, much more). One would like all pupils to recognise something of Hobbes' surprise: Who would think of squaring lengths before adding?

Meantime, once we know how to calculate the area of a triangle, we can use this as required to calculate the area of any polygon by breaking it up into triangles and rectangles. For example, we saw in Section 3.3.1 that:

- if a trapezium $ABCD$ has AB parallel to DC, then we can drop perpendiculars to break up the problem of finding the area of $ABCD$ into that of finding the area of a rectangle and two right angled triangles;
- by cutting a regular n-gon into n congruent isosceles triangles we show later in the section that

  $$\textbf{area(regular } n\textbf{-gon with incircle of radius } r\textbf{)} = \tfrac{1}{2}\textbf{(perimeter} \times \textbf{radius } r\textbf{)}$$

The area enclosed by any shape (including *curved* shapes such as a circular disc), is a measure of the "size" of the enclosed region. For a circle of radius r the exact value may prove elusive, but it can be approximated internally and externally to provide lower and upper bounds. For example, if we draw a circle of radius 5 centred at the origin $(0,0)$ on a square grid, the circle passes through the twelve grid points $(\pm 5,0)$, $(0,\pm 5)$, $(\pm 3,\pm 4)$, $(\pm 4,\pm 3)$. Counting unit squares inside the circle and those which just surround it then leads to the crude estimate

$$60 < \text{area of circle of radius } 5 < 88.$$

If we divide each unit in two and use squares of side $\frac{1}{2}$, the circle with centre $(0,0)$ passes through $(\pm 5,0)$, $(0,\pm 5)$, $(\pm 3,\pm 4)$, $(\pm 4,\pm 3)$, with the points

$(\pm\frac{7}{2}, 0)$, $(0, \pm\frac{7}{2})$ just inside the circle. Counting squares of size $\frac{1}{2} \times \frac{1}{2}$ leads to the slightly better estimate

$$69 < \text{area of circle of radius } 5 < 86.$$

However, merely counting smaller and smaller squares does not in itself suggest the crucial fact that the desired area is equal to **a constant multiple of r^2**. For that we need something more systematic. There are two natural approaches to this: one is highly suggestive, but mathematically less precise; one is more precise and initially less suggestive (though ultimately suggestive in a different way).

The less precise (but more intuitive) approach is to cut the circular disc into $2n$ equal *sectors*, or "cake slices", and arrange the pieces alternately pointing up and down, to form an "almost rectangle" with slightly sloping ends (each of length r—the radius) and slightly bumpy top and bottom edges (each of length equal to exactly half the perimeter of the circle—which we now know to be πr from Section 3.3.4). For larger and larger values of n—that is, for sectors with smaller and smaller angle $\left(\frac{180}{n}\right)^\circ$ at the centre—the rearranged shape is more and more like a rectangle. This suggests that the total area of the circular disc is very close to that of an r by πr rectangle—namely $r \times \pi r = \pi r^2$.

The more precise approach is to consider regular n-gons inscribed in, and circumscribed around, a circle of radius r. One should start by carrying out the exact calculations for $n = 4$ and $n = 6$ as a concrete preliminary to the beautiful, and highly suggestive, general argument for regular n-gons that follows:

$$\begin{aligned} \text{if } n = 4\text{:}\quad & \text{area(inscribed square)} = 2r^2 < \text{area(circle radius } r) \\ & < 4r^2 = \text{area(circumscribed square)}; \end{aligned}$$

$$\begin{aligned} \text{if } n = 6\text{:}\quad & \text{area(inscribed hexagon)} = \tfrac{3\sqrt{3}}{2} \cdot r^2 < \text{area(circle)} \\ & < 2\sqrt{3} \cdot r^2 = \text{area(circumscribed hexagon)}. \end{aligned}$$

These calculations suggest that the area A of a circle of radius r is some "constant" multiple of r^2, and that the mysterious *constant* satisfies

$$2 < \tfrac{3\sqrt{3}}{2} = 2.598\cdots < \textit{constant} < 3.464\cdots = 2\sqrt{3} < 4.$$

In general, if a regular polygon $ABCDEFG\ldots$ has an inscribed circle with centre O and radius r, then joining all vertices to the centre breaks up the polygon into n congruent isosceles triangles ABO, BCO, CDO, $\ldots$. We know that

$$AB = BC = CD = \ldots,$$

that the area of each triangle such as ABO is equal to

$$\frac{1}{2}(\text{base } AB \times \text{height } r),$$

and that the regular n-gon is equal to the sum of exactly n such triangles. Hence

$$\begin{aligned}\text{area}(ABCD\ldots) &= \frac{1}{2}(\text{base } AB \times \text{radius } r) + \frac{1}{2}(\text{base } BC \times \text{radius } r)\\ &\quad + \frac{1}{2}(\text{base } CD \times \text{radius } r) + \ldots\\ &= \frac{1}{2}([AB + BC + CD + \ldots] \times \text{radius } r)\\ &= \frac{1}{2}(\text{perimeter of regular } n\text{-gon } ABCD\cdots \times \text{radius } r).\end{aligned}$$

As the number n of sides increases, the regular n-gon approximates the circle more and more accurately and its area approaches the area of the circular disc.

$$\begin{aligned}\therefore \text{area}(\text{circle of radius } r) &= \frac{1}{2}(\text{circumference of circle} \times \text{radius } r)\\ &= \pi r \times r = \pi r^2.\end{aligned}$$

This links what we know about the **circumference** of a circle of radius r with the **area** of a circular disc of radius r, and shows that the mysterious "constant multiplier" is *exactly* π (that is, "half of the constant multiplier" 2π for the circumference of a circle).

Once the area of a circle of radius r is determined, one can pin down the area of a semicircle of radius r, of a quarter of a circle, and the area of a circular sector of radius r with angle θ at the centre. Pupils can then be asked to find the areas of all sorts of lovely composite shapes made from rectangles, triangles, and, circular sectors (both protruding and indented).

3.3.6 Volume In one dimension there is really only one possible "shape", namely a *line segment*. And the basic unit for "area" in 2D (namely the *rectangle*) is obtained by moving this 1D shape "perpendicular to itself in 2D". Hence in 2D there is only one possible shape that results from moving a 1D figure (a line segment) perpendicular to itself—namely a rectangle. Our whole approach to area started by assuming that we know how to find the area of a rectangle. And the step from 1D to 2D was so short and sweet that we hardly noticed it.

But in 2D there are many different shapes, each of which can be moved "perpendicular to itself in 3D" to obtain a right prism with the given shape as base.

- Our basic unit of volume, the cuboid, is obtained by moving a rectangle perpendicular to itself in 3D to create a right prism with a rectangular base.
- We could just as easily start with a *triangular* base and move that perpendicular to itself.
- Or we could start with a *regular polygon* as base.
- Or we could start with a *circle* as base.

So there is much more initial work to be done before we begin to worry about how to find the volume of curved shapes— such as cones and spheres.

The first move is to establish the formula for the volume of a general cuboid. A cuboid with integer length sides can be built up by taking an integer number of unit cubes in each of the three directions. The formula can be extended to cuboids with fractional length sides in the same way that we extended the formula for the area of a rectangle. It follows that the volume of a general cuboid with sides of lengths a units, b units, and c units can be shown to be equal to $a \times b \times c$ (cubic units) for all possible values of a, b and c giving the familiar formula:

$$\text{volume(cuboid)} = \text{length} \times \text{breadth} \times \text{height}.$$

Once this has result been established, the following sequence of steps allows us to calculate the volume of many other 3D shapes.

- First consider the cuboid as a *right prism*, (that is, as a three dimensional shape formed by moving the base rectangle at "right" angles to its plane) and interpret the formula for its volume as being

  volume(right rectangular prism) = (area of base rectangle) × height .

- Then cut the base rectangle into two congruent right angled triangles, and so extend this formula to give the volume of a right prism with a right angled triangle as base (a "right triangular wedge") as

  volume(right prism with right triangular base)
  = (area of base) × height.

- Then extend this formula to give the volume of any right prism with a *parallelogram* as base (surround the parallelogram by a rectangle, and hence surround the prism by a cuboid; then, just as in two dimensions, obtain the volume of the right prism by subtracting two "right triangular wedges" from the surrounding cuboid) to get:

  volume(right prism with parallelogram base)
  = (area of base parallelogram) × height.

- Then use the fact that any right triangular prism is half of a right prism with a parallelogram as base to show that its volume is given once more by:

  volume(right prism with triangle as base)
  = (area of base triangle) × height.

- We can then extend this same formula to any right prism with a polygon as base (by cutting up the base into triangles, and then adding up the volumes of the right triangular prisms):

  volume of any right prism = (area of base figure) × height.

- Finally we extend this formula once more to a *right circular cylinder* (by approximating the base circle by regular polygons).

All these formulae can be explained and understood—and can then be used to find the volumes of an interesting variety of compound shapes. The formulae for the volumes of more complicated shapes (such as pyramids, cones, spheres) are more subtle, and are best delayed until Key Stage 4.

When we come to consider "scaling" and "similarity", the two facts:

- that the volume of any cuboid is equal to

  "(area of base) × height",

  or

  "length × breadth × height",

and

- that the volume of any more general shape in 3D is defined in terms of approximating them by combinations of cuboids (including "half cuboids", or triangular wedges)

have a hidden consequence. Whatever the volume of a given shape may be, if we enlarge it (or "en-small" it) by multiplying all lengths by the same scale factor "r", then the volume of each and every approximating cuboid **is also multiplied by r^3**, so the volume of the shape being approximated is multiplied by r^3. If one cube has sides that are three times as long as another, then its volume is 27 times as large; and a sphere of radius 4 has volume 64 times as large as a sphere of radius 1.

3.4. Constructions, conventions, and derivations

> – **use the standard conventions for labelling the sides and angles of triangle *ABC*, and know and use the criteria for congruence of triangles**
>
> – **derive and use the standard ruler and compass constructions (perpendicular bisector of a line segment, constructing a perpendicular from/at a given point,**

bisecting an angle); recognise and use the perpendicular distance from a point to a line as the shortest distance to the line

- **apply the properties of angles at a point, angles at a point on a straight line, vertically opposite angles**
- **apply [...] triangle congruence [...] to derive results about angles and sides [...], and use known results to obtain simple proofs**
- **derive and illustrate properties of triangles, quadrilaterals, circles and other plane figures [for example, equal lengths and angles] using appropriate language and technologies**
- **understand and use the relationship between parallel lines and alternate and corresponding angles**
- **derive and use the sum of angles in a triangle and use it to deduce the angle sum in any polygon, and to derive properties of regular polygons**
- **apply angle facts, triangle congruence, similarity and properties of quadrilaterals to derive results about angles and sides, including *Pythagoras' Theorem*, and use known results to obtain simple proofs**

Congruence has already been introduced and used; and parallels have also featured (e.g. in parallelograms and trapezia). So this group of requirements, taken together, amounts to a relatively systematic "Euclidean" reorganisation of pupils' geometrical knowledge and methods. But this "reorganisation" is not an end in itself. Once the three basic principles (congruence, parallels, similarity) have been clarified, once the backbone sequence of basic results has been established, and once the idea of only using previously proved results has been grasped, pupils gain access to what should be the main educational content of secondary school geometry—namely the **wonderful world of accessible, yet elusive**

problems. To keep things relatively short, the exposition here focuses mainly on the underlying framework of basic results and methods which is needed to support this pupil activity. However, it is essential for teachers not only to grasp the underlying framework, but also to engage with the kinds of problems this framework opens up for pupils (and teachers) to enjoy. For a systematic development of deductive problems for mid-late Key Stage 3, we recommend the book *Crossing the Bridge* by G. Leversha (UKMT Publications 2008). Dedicated sets of problems can also be found in the series *Extension Mathematics* by Tony Gardiner (Oxford University Press 2007):

- *Book Alpha*: T5 (perimeters); T9, E2 (angles); T11, C7 (drawing conclusions); C17 (triangles); C19, E14 (areas and volumes)
- *Book Beta*: T11, T15 (drawing conclusions); C4, C7, C15 (congruence); T17, C11, E4 (angles); T20 (triangles); T26, C18 (areas and perimeters); C2 (parallel lines); C5 (ruler and compass constructions); C27 (volumes)
- *Book Gamma*: T10 (parallel lines); T17, C35 (Pythagoras' Theorem); T24 (loci); T8, C8 (circles); C10 (angles in regular polygons); C15 (volumes and prisms); C3, C39 (miscellaneous problems).

After a brief general introduction (Section 3.4.1) we address the very first listed requirement in two parts (Sections 3.4.2 and 3.4.3). We then discuss the role of the standard "ruler and compass constructions" in Section 3.4.4, before focusing on angles, and deriving the simplest consequences of the congruence criteria (relating to isosceles triangles and regular polygons) in Section 3.4.5. Section 3.4.6 examines the consequences of the parallel criterion—in particular the sum of angles in a triangle and results relating to parallelograms. Finally Section 3.4.7 comments briefly on the requirements relating to similarity. (The two remaining official requirements under the heading of *Geometry and measures* are discussed briefly in Section 3.5.)

3.4.1 Towards formal geometry As with all aspects of elementary mathematics, there is no "royal road" to success in geometry. The approaches adopted in England since the 1960s introduced all manner of delights, which one may hesitate to discard. *But they have singularly failed to produce school leavers able to analyse configurations in two- and three-dimensions.*

During this period a number of teachers and authors have continued to insist, and to demonstrate, that the most effective framework within which ordinary students can apprehend and 'calculate exactly' with geometrical information is that which analyses more complicated figures in terms of *triangles*. This is the thrust of the **Euclidean** framework illustrated by the sequence of official requirements listed at the start of Section 3.4.

Informal work at Key Stage 1 and Key Stage 2 to make sense of shapes and patterns in 2D and 3D prepares the ground for the 'more formal' treatment later in Key Stage 3 and at Key Stage 4. We have already stressed the need for structured work at Key Stage 2 and in early Key Stage 3 to include drawing and measuring (Section 3.2.1), calculating angles (described briefly in Part II, Section 2.3.5), and work with lengths, areas and volumes (Section 3.3.1). Such work develops the ideas and language that are needed when we begin to reorganise our approach to **Euclidean geometry** during Key Stage 3 (in terms of congruence, parallels, and similarity). The sequence of requirements listed at the start of Section 3.4 should be seen as ushering in this semi-formal phase.

The full thrust of formal Euclidean geometry only takes root late in Key Stage 3. And though the foundations are laid in Years 7 and 8, it is not surprising that most of the released Year 9 items from TIMSS 2011 focus on *calculation* and *construction*, rather than on *deduction*. However, one item is perhaps relevant.

> **3.4.1A** [A convex pentagon labelled $ABCDE$ is shown, including diagonals AC and AD.] What is the sum of all the interior angles of pentagon $ABCDE$? Show your work.

The dissection of the pentagon in the accompanying diagram into three triangles ABC, CAD, and DEA invites (but does not force) pupils to use the "known fact" that the angles in any triangle add to 180°. Since most Year 9 pupils have known this "fact" for several years, it seems reasonable to hope that significant numbers might manage to produce the answer of 540°, with an acceptable justification—even if expressed rather crudely as:

$$\triangle ABC + \triangle ACD + \triangle ADE = 180 + 180 + 180 = 540,$$

or

$$\square ABCD + \triangle ADE = 360 + 180 = 540.$$

The reported results therefore underline the challenge of trying to get pupils to "reason geometrically". The mark scheme awarded 2 points for an acceptable solution (*including* a justification), with 1 point for the numerically correct answer, but with an incorrect reason (and maybe for an acceptable reason, with an incorrect answer). We give the percentage of pupils scoring 2 points (with the percentage scoring at least 1 point in brackets):

> **3.4.1A** Hungary 22% (29); Russia 19% (35);
> England 17% (20); Australia 13% (19); USA 12% (16)

It is probably worth noting three additional results from the Far East. The Japanese scores of 72% (and 81%) show that it is possible to do considerably better than we do at present. At the same time the Singapore scores of 55% (and 60%), and the Hong Kong scores of 38% (and 51%), suggest that it would be rash to expect too much, too soon, from too many pupils.

3.4.2 Conventions The details relating to the first half of the first listed requirement were explored at length in Part II, Section 2.2.2.3, namely for pupils to learn

- **to use the standard conventions for labelling the sides and angles of triangle ABC.**

These conventions establish the language and grammar of all "geometrical calculation".

Mathematics in general succeeds by translating sense impressions, and language or sounds, into **symbols** which allow *exact calculation*. For example, we replace "words for numbers" by *numerals* and *place value*, which then makes it possible to develop exact methods for "numerical calculation". Similarly, it is only when general relations are expressed as *algebraic expressions* that we have a chance of making deductions we might otherwise overlook. For example, as long as the problem

> "Find a prime number that is one less than a square"

is presented in non-mathematical language, its analysis remains elusive. But as soon as it we translate this into the appropriate mathematical language:

> "When is $n^2 - 1$ prime?"

we immediately have the chance of seeing how to proceed by engaging in "algebraic calculation", since "$n^2 - 1$" should trigger the well-known factorisation

$$n^2 - 1 = (n - 1)(n + 1),$$

so $n^2 - 1$ can only be prime if $n - 1 = 1$.

In the same spirit, the English words "triangle" or "quadrilateral" conjure up a visual impression, or imagined shape. But one cannot calculate with such a visual impression. If we wish to refer to, and to calculate with, a particular triangle or quadrilateral, we need to **give it a name** in accordance with certain conventions.

The labelling conventions are chosen to communicate reliably between individuals, and to reflect the geometric structure of the object being labelled. Points are routinely denoted by capital letters (preferably *italic*). Two points A, B determine a **line** AB. But in England we use *the same notation* for the line **segment** which starts at A, runs to B, and then stops. And we use *the same notation* again for the **length** of the line segment! In other countries, these three different ideas are given different notations. It is unclear who has the power to change this confusion. But it is completely clear that, as long as we continue to use "AB" to denote all three ideas, it is essential for teachers to make sure that the *associated language* used in the classroom and in pupils' written solutions always makes it clear which meaning is intended.

A polygon is a "broken" (or bent) **sequence** of line segments. Hence, when labelling a polygon, the **sequence** in which the vertices are named matters. A quadrilateral $ABCD$ has to be labelled **in cyclic order**, where the edges are the successive line segments, or edges, that make up the quadrilateral: with edges AB and BC (meeting at the vertex B), BC and CD (meeting at vertex C), CD and DA (meeting at vertex D), and DA and AB (meeting at vertex A).

The whole of geometry in 2D and in 3D rests on the discovery that **triangles** hold the key to the construction and analysis of more complicated shapes. When we label the vertices of a triangle $\triangle ABC$, the cyclic order is not a problem: because there are only three vertices, the only choice is to list the vertices in *clockwise*, or in *anticlockwise* order. Each of the three vertices gives rise to an (internal) *angle*:

> $\angle ABC$ (often abbreviated as "$\angle B$", or just "B"), $\angle BCA$ (abbreviated as "$\angle C$", or "C"), and $\angle CAB$ (abbreviated as "$\angle A$", or "A").

And the *length of each side* of the triangle is conventionally labelled with the lower case version of the opposite vertex:

> side AB (opposite vertex C) has length c, side BC has length a, side CA has length b.

More awkward is the fact that whenever push comes to shove, a 'triangle' is not just a three-cornered shape: it is a *labelled*, or *ordered, triple* ABC, where **the order matter**s. If one only knows the three vertices, but not the order, then this corresponds to several *different* triangles: the triangles $\triangle ABC, \triangle BCA, \triangle CAB, \triangle BAC, \ldots$ are in some sense *different* (as becomes clear when aligning triangles to demonstrate congruence—see Section 3.4.3). Even if we choose not to insist on such precision all the time, whenever we come to do some kind of calculation with a triangle, or a quadrilateral, we find that the **order** matters.

In a similar spirit, Key Stage 3 should witness a marked shift in how geometric objects are **defined**.

- In primary school, an object is pinned down (or "apprehended") by accumulating an ever-increasing list of "properties" (so that a "rectangle" is understood through *all* its properties: opposite pairs of sides equal and parallel, four right angles, equal diagonals which bisect each other, and so on).
- In Key Stage 3 this "encyclopedic" approach to the question "What is a rectangle?" should be replaced by the idea of a *definition* as a **minimal specification**. Hence

- a "rectangle" is defined to be "a parallelogram with one right angle";
- a "parallelogram" is defined to be "a quadrilateral with opposite pairs of sides parallel"; and
- a "right angle" is defined to be "half a straight angle".

This not only makes it clear what exactly we mean by a "rectangle", it also makes it much easier to check that a given quadrilateral is in fact a rectangle (since we only have to check (a) that it is a parallelogram, and (b) that it has at least one right angle). Once we have done this, we know that every other property of a rectangle comes for free—without the need to check.

3.4.3 Congruence The second half of the first listed requirement, namely

- **to know and use the criteria for congruence of triangles**

was explored in Section 3.2.3 above and in Part II, Section 2.2.2.3. Further consequences arise in Section 3.4.4, 3.4.5, and 3.4.6 below.

Two (ordered) triangles $\triangle ABC$ and $\triangle DEF$ are *congruent* if the (ordered) correspondence

$$A \longleftrightarrow D, \qquad B \longleftrightarrow E, \qquad C \longleftrightarrow F$$

matches up each of the six ingredients of triangle $\triangle ABC$ with those of triangle $\triangle DEF$ in such a way that

- all three *corresponding* pairs of line segments are equal:

$$AB = DE,\ BC = EF,\ CA = FD,$$

and

- all three *corresponding* pairs of angles are equal:

$$\angle A = \angle D,\ \angle B = \angle E,\ \angle C = \angle F.$$

We write this as: $\triangle ABC \equiv \triangle DEF$ (which we read as "triangle ABC is *congruent* to triangle DEF").

"Congruence of triangles" only makes sense between **ordered** triangles. And it can help pupils to see more clearly which vertex of the first triangle corresponds to which in the second triangle, and which side of the first triangle corresponds to which in the second triangle if pupils initially write

$$\begin{aligned} &\triangle ABC \\ \equiv\ &\triangle DEF \end{aligned}$$

lining up corresponding vertices and edges vertically over each other (as with column arithmetic):

- with vertex A directly above vertex D, B directly above E, C directly above F, and
- with edge AB directly above edge DE, BC directly above EF, CA directly above FD.

The three basic *congruence criteria* (SSS, SAS, and ASA) arise naturally from drawing and construction exercises, and the SSS-congruence criterion plays a significant role in the next Section 3.4.4 to show that the standard ruler and compass constructions do what they claim:

> triangles $\triangle ABC$ and $\triangle DEF$ are congruent (by SSS) if $AB = DE$, $BC = EF$, and $CA = FD$;
>
> triangles $\triangle ABC$ and $\triangle DEF$ are congruent (by SAS) if $AB = DE$, $\angle BAC = \angle EDF$, and $AC = DF$;
>
> triangles $\triangle ABC$ and $\triangle DEF$ are congruent (by ASA) if $\angle BAC = \angle EDF$, $AB = DE$, and $\angle ABC = \angle DEF$.

The RHS congruence criterion is not part of this basic congruence criterion, so does not really belong at this stage. It arises as the degenerate instance of the failed ASS criterion (where the angle "A" in "ASS" is a right angle, and so is neither acute nor obtuse). The fact that RHS guarantees congruence depends on *Pythagoras' Theorem*, since knowing two sides and a right angle then determines the third side. So RHS is a special case of SSS.

3.4.4 Congruence and ruler and compass constructions "Construction" at Key Stage 3 takes on a slightly different meaning, moving

- **from measuring work with rulers and protractors at Key Stage 2 and early Key Stage 3**
- **to a simple, hands-on, geometrical framework using "ruler and compasses",** ***which avoids measuring altogether,*** **in which the familiar "measuring ruler" becomes a** ***straightedge*** **(that is, a mere straight-line-drawer), and the focus switches from measuring lengths to "equality" of line segments (e.g. as radii of a given circle, created by a pair of compasses).**

We stick to the tradition of referring to these latter constructions as **ruler and compass constructions**—even though the ruler is being used as an "ideal" *mental straightedge* (and its crude, approximate markings play no role).

- Given two points A and B, the "ruler" is simply a way of physically capturing the idea that one can imagine the *line or line segment "AB"* determined by these two points; and
- given a point O (as centre) and another point P, the "compasses" are a way of physically capturing the "ideal" construction of the circle with centre O and passing through P.

That is, the two instruments are in some sense not being used to perform actual constructions, but to illustrate *imagined ideal constructions* (performed with 'heavenly' straightedge and compasses).

Ruler and compass constructions offer a natural psycho-kinetic embodiment of the simplest parts of *formal* geometry (for example, allowing pupils to experience SSS-congruence directly). The constructions themselves are experienced directly; and the *proofs* that the basic constructions do what they claim constitute an introduction to the subsequent transition from physical to *formal* geometry. Hence ruler and compass constructions embody four rather different aspects of secondary mathematics.

- The first is the clean simplicity of the basic moves:
  - to construct the line AB through two known points A and B,
  - to construct the circle with known centre A passing through a known point B, and

  - to obtain "new known points" as the points of intersection of two constructed lines, or of a constructed line and a circle, or of two constructed circles.

- The second aspect is the act of drawing itself (which may at first be ungainly, but which benefits hugely from practice, which exploits the links between hand, eye, and brain, gives physical substance to geometrical ideas, and leads ultimately to quiet satisfaction after a well-implemented construction).

- The third aspect is to *imagine* the act of drawing without first carrying out each construction, so that one can begin to combine standard constructions as basic moves in a chain that achieves some more complicated goal: (for example, we can imagine how one might use ruler and compasses to construct an equilateral triangle—or a square, or a regular pentagon, or a regular hexagon, or a regular octagon—inscribed in a circle with centre at O and passing through the point A).

- The fourth aspect is the simple *deductive structure*, based mainly on the SSS-congruence criterion, that shows how "equal lengths" (which is all one can create using compasses, where two radii of the same circle are necessarily equal) leads to congruence, and hence forces certain *angles* to be equal.

The idea that mathematical objects need to be "constructed", rather than "postulated" or plucked out of thin air, lay at the heart of ancient Greek mathematics. The assumptions which underlie ruler and compass constructions were declared as the first three of their five "axioms" or principles:

- to construct a line segment AB joining two known points A, B;
- to extend this line segment as far as one wishes in either direction;
- to construct the circle with known centre O and passing through a known point A.

Many of the results they proved were presented as *constructions*. For example, the very first Proposition in Book I of *Euclid*'s *Elements*:

> "On a given finite straight line [segment AB] to construct an equilateral triangle [ABC]."
>
> **Construction:** Draw the circle with centre A passing through B, and the circle with centre B passing through A. Let these two circles meet at C and at D.
>
> $\therefore AB = AC$ (radii of the same circle) and $BA = BC$ (radii of the same circle).
>
> $\therefore \triangle ABC$ is equilateral. **QEF**

Genuine *proofs* ended with a statement (in Greek)

> "which is that which was to be *proved*".

This is rendered in Latin as "*Quod Erat Demonstrandum*" and abbreviated as "QED". In contrast, *constructions* like the one above ended with the statement

> "being what it was required to *do*",

which is rendered in Latin as "*Quod Erat Faciendum*" and abbreviated as "QEF".

This may all seem to have little to do with school mathematics. But it is worth reflecting on the links between this "constructive" approach to mathematical concepts and the *psychology of the learner*. As mathematics became more abstruse in the eighteenth, nineteenth, and early twentieth centuries, its ideas and methods were increasingly postulated abstractly. This approach proved exceedingly powerful; but it also made the subject less accessible, and led to philosophical difficulties. The advent of computers has reminded us afresh of the need to be able to construct the ideas about which we reason in mathematics: knowing that a curve crosses the x-axis and so has a "root" is one thing; but we also need effective methods for **finding** that root. Something analogous applies to learners, where a constructive approach often allows a more meaningful kind of engagement than a purely logical analysis. (This observation also seems to have been behind John Perry's proposed reforms in the early 1900s.)

The official requirement that pupils should

- **derive and use the standard ruler and compass constructions (perpendicular bisector of a line segment, constructing a perpendicular from/at a given point, bisecting an angle); recognise and use the perpendicular distance from a point to a line as the shortest distance to the line**

encourages this kind of healthy, constructive engagement, and does so using "ruler and compasses" in a way that is consistent with the Euclidean reworking of geometry. It is therefore to be welcomed (though, as we shall see, the final reference to "shortest distance to the line" is slightly out of place).

The first "standard construction" is implicit in *Euclid*'s Proposition 1.

> **To construct** the perpendicular bisector of a given line segment AB.
>
> **Construction**: Draw the circle with centre A passing through B, and the circle with centre B passing through A. Let these two circles meet at C and at D.
>
> We may not yet know how to *construct* the midpoint of the line segment AB; but the midpoint certainly exists, so let us "imagine" it (somewhere between A and B) and give it a name, M.
>
> We claim that $\triangle CMA \equiv \triangle CMB$ (by SSS).
>
> $\therefore \angle CMA = \angle CMB$, so each angle is half a straight angle, and CM is perpendicular to AB.
>
> Similarly, DM is perpendicular to AB.
>
> $\therefore$ CMD is a straight line, so the line CD crosses AB at its midpoint M.
>
> $\therefore$ CD is the required perpendicular bisector. **QEF**

The second standard construction uses the same idea.

Given a line segment AB and a point P, to construct the perpendicular from P to AB.

Construction:

(i) Suppose first that P lies on the line AB.

Clearly P cannot be the same point as both A and B. So we may suppose that $P \neq B$.

Draw the circle with centre P passing through B, and let it meet the line AB (= BP) again at C.

Then P is the midpoint of BC.

Use the first standard construction to find the perpendicular bisector of BC, and this will be the perpendicular to AB at the point P.

(ii) Suppose next that P does not lie on the line AB.

By drawing the circles with centre P passing through A and through B we can choose the point *furthest* from P—which we may suppose is B.

Draw the circle with centre P passing through B, and let it meet the **line** AB again at C. (The point C lies on the line AB, but is not internal to the line segment AB.)

Use the first standard construction to find the midpoint M of BC.

$\therefore \triangle PMB \equiv \triangle PMC$ (by SSS)

$\therefore \angle PMB = \angle PMC$, so each is half a straight angle.

$\therefore PM$ is the perpendicular from P to AB. **QEF**

The third standard construction is slightly different. Because ruler and compasses can only make *lengths* equal, it again uses SSS-congruence—this time to conclude that two angles are equal.

Given two lines BA and BC meeting at the point B, to construct the bisector of $\angle ABC$.

> **Construction:** Draw the circle with centre B passing through C and let this circle meet the segment BA (extended if necessary beyond the point A) at D.
>
> $\therefore BC = BD$ (radii of same circle)
>
> Draw the circle with centre C passing through B, and the circle with centre D passing through B, and let these two circles meet at B and again at E.
>
> $\therefore CB = CE$ (radii of same circle) and $DB = DE$ (radii of same circle).
>
> $\therefore \triangle CBE \equiv \triangle DBE$ (by SSS)
>
> $\therefore \angle CBE = \angle DBE$, so the line BE bisects $\angle ABC$. **QEF**

There are lots of lovely problems which exploit these three basic constructions. Once we are in a position to use "equal alternate (or corresponding) angles" as a criterion for two lines to be parallel, we can extend the second standard construction to obtain the line through P parallel to AB.

> **Given** a line AB and a point P not on AB, to construct the line through P parallel to AB.
>
> **Construction:** Construct the perpendicular from P to AB, meeting AB at the point X.
>
> Then construct the perpendicular PY to PX at the point P.
>
> The fact that $\angle AXP$ and $\angle XPY$ are right angles, then implies that PY is parallel to AB. **QEF**

We can also explore the question of constructing regular polygons. The flower petal construction described in Section 3.2.1 shows how to construct a **regular hexagon** $ABCDEF$ inscribed in a given circle with centre O and passing through A. By taking every second vertex, we obtain a way of constructing an **equilateral triangle** ACE inscribed in the circle with centre O.

The question as to which other regular polygons can be constructed in this way was addressed (and answered completely) by *Carl Friedrich Gauss* in his late teens in the mid-late 1790s, and published in his famous book *Disquisitiones arithmeticae*, 1801 (at the time, Latin was still the main international language for communicating scientific results). To construct a **square**, let AO meet the circle again at C, construct the perpendicular bisector of the line segment AC, and let this meet the circle at B and at D; then one can prove that $ABCD$ is a square. One can also find relatively simple ways of constructing a **regular pentagon** in the circle with centre O (though proving that they really work may have to wait until Key Stage 4). And once we know how to construct a regular 4-gon $ACEG$ in the circle with centre O passing through A, we can use the first standard construction to construct the perpendicular bisector of each side and so find the points B, D, F, H where these perpendicular bisectors cut the circle—thus constructing a **regular 8-gon** $ABCDEFGH$. Similarly, once we know how to construct a regular 5-gon, we can construct a **regular 10-gon**. But it is **impossible** to construct a regular 7-gon, or a regular 9-gon, or a regular 11-gon with ruler and compasses.

The final requirement for pupils to:

> **recognise and use the perpendicular distance from a point to a line as the shortest distance to the line**

is slightly out of place here. We saw how to construct the perpendicular from a point P to meet AB at the point X. But it is not obvious that PX is the *shortest* distance from P to AB. (The easiest way to see this is to consider any other point Y on the line AB and then to apply *Pythagoras' Theorem* to the right angled triangle PXY to see that PY is greater than PX.)

3.4.5 The basic consequences of congruence The simplest geometrical result of all is that "vertically opposite angles are always equal" as required by

> - **apply the properties of angles at a point, angles at a point on a straight line, vertically opposite angles.**

Claim Whenever two lines cross at a point P, any pair of vertically opposite angles A and A' at P are necessarily equal.

Proof: Let B be the angle "between" the two vertically opposite angles A and A' at P.
Then $A + B$ is the straight angle on one line, and $B + A'$ is the straight angle on the other line.
$\therefore A + B = B + A'$, so $A = A'$. **QED**

In general the size of an angle is defined in terms of fractions of a "straight angle" (the "angle" at a point P on a straight line). For example, if we *bisect* a straight angle, then each half is a *right angle*. Thanks to the ancient Babylonians, we still measure angles in *degrees*, with each straight angle equal to 180°, so each right angle is equal to 90°. We are not sure why they chose 360° for a full turn. However, it may be related in some way

(a) to their use of the sexagesimal numeral system (base 60), and

(b) to their use of angles in astronomy, and the connection between the apparent movement of the observed stars and what they took to be the number of days in a year.

The rest of this section focuses on the SSS, SAS, and ASA congruence criteria. These are in many ways more fundamental than the criterion for two lines to be parallel (which we address in Section 3.4.6), in that they apply to geometries where the parallel criterion fails—allowing us to show that certain angles, or line segments, are equal (as in Section 3.4.4, where we dropped perpendiculars, and where we bisected any given angle). The miracle of Euclidean geometry is how much more one can prove by *combining* these two principles.

We start by developing the "backbone" of results that depend only on congruence. This obliges us to interpret the two slightly confused official requirements:

- **apply [...] triangle congruence [...] to derive results about angles and sides [...], and use known results to obtain simple proofs**

> – **derive and illustrate properties of triangles, quadrilaterals, circles and other plane figures [for example, equal lengths and angles] using appropriate language and technologies**

The problem here is that the wording in the full requirement (including the parts which have here been omitted) confuses

- the experiential Key Stage 2 approach to geometry (where one "collects and uses facts" without any definitions or proofs) and
- the Key Stage 3 approach, which begins to organise geometrical knowledge through *minimal definitions*, respecting conventions, emphasising the three basic *principles* (the congruence criterion, the parallel criterion, and the similarity criterion), and **deriving** those "facts", or "properties", which are most useful.

If we disentangle this confusion, and focus on what should be the distinctive Key Stage 3 approach, then the first move has to be to prove the basic facts about isosceles triangles. A triangle ABC in which $AB = AC$ is called *isosceles*, with *base* BC and with *apex* A. ("Iso" means "same" in Greek; and "sceles" means "legs", or sides.)

Claim if $AB = AC$, then $\angle ABC = \angle ACB$ ("the base angles of any isosceles triangle are equal"). Moreover, the line AM joining the apex A to the midpoint M of the base BC (the "median") is also the perpendicular bisector of the base BC, and the bisector of the apex angle $\angle BAC$.

Proof Construct the midpoint M of the base BC.

Then $\triangle ABM \equiv \triangle ACM$ (by SSS).

$\therefore \angle ABM = \angle ACM$, so the two base angles $\angle ABC$ and $\angle ACB$ are equal.

Also $\angle AMB = \angle AMC$, so each is equal to half a straight angle—that is, a right angle.

And $\angle BAM = \angle CAM$, so AM bisects the angle $\angle BAC$. **QED**

Claim If $\angle ABC = \angle ACB$, then $AB = AC$ ("if the base angles are equal, the triangle is isosceles").

Proof $\triangle ABC \equiv \triangle ACB$ (by ASA).

$\therefore AB = AC$. **QED**

Claim (a) If M is the midpoint of BC, and MX is perpendicular to BC, then $XB = XC$. That is, each point on the perpendicular bisector of BC is equidistant from B and from C.

(b) Conversely, if Y is equidistant from B and from C, then Y lies on the perpendicular bisector of BC.

Hence the perpendicular bisector of a line segment BC is precisely the *locus* of all points that are equidistant from B and from C.

Proof (a) $\triangle XMB \equiv \triangle XMC$ (by SAS, since $XM = XM$, $\angle XMB = \angle XMC$, $MB = MC$).

$\therefore XB = XC$.

(b) Join YM. Then $\triangle YMB \equiv \triangle YMC$ (by SSS).

$\therefore \angle YMB = \angle YMC$, so each is half of a straight angle. **QED**

Isosceles triangles arise naturally when working with circles: if A and B lie on the circle with centre O, then $OA = OB$, so $\triangle OAB$ is isosceles. Hence the perpendicular from O to AB bisects the base AB and also bisects the angle $\angle AOB$. Isosceles triangles also feature in the following useful result.

Claim A regular n-gon $ABCDEF\ldots$ has a centre O, and is inscribed in a circle with centre O.

Proof Let the perpendicular bisector of AB meet the perpendicular bisector of BC at O. By a previous result, $OA = OB$, and $OB = OC$. Hence $OA = OC$ and the circle with centre O passing through A, also passes through B and C. We show that this circle necessarily passes through the vertex D, and hence through all vertices of the regular polygon.

$\therefore \triangle OAB$ is isosceles, so $\angle OAB = \angle OBA$, and $\triangle OBC$ is isosceles, so $\angle OBC = \angle OCB$.

Moreover $\triangle OAB \equiv \triangle OBC$ (by SSS: since $OA = OB$, $OB = OC$, and $AB = BC$).

$\therefore \angle OAB = \angle OBC$, so

$$\angle OAB = \angle OBA = \angle OBC = \angle OCB = \frac{1}{2}(\angle ABC).$$

$\therefore \angle OCD = \angle BCD - \angle OCB = \frac{1}{2}(\angle ABC) = \angle OBC.$

$\therefore \triangle OBC \equiv \triangle OCD$ (by SAS: $OB = OC$, $\angle OBC = \angle OCD$, $BC = CD$).

$\therefore OC$ (in $\triangle OBC$) $= OD$ (in $\triangle OCD$) so the circle with centre O passing through A also passes through D.

Continuing in this way shows that the circle passes through every vertex of the regular polygon. **QED**

3.4.6 The parallel criterion and angles in a triangle To prove more interesting results—such as to

> – **derive and use the sum of angles in a triangle and use it to deduce the angle sum in any polygon**

we need more than just the congruence criterion. In particular, we need to

> – **understand and use the relationship between parallel lines and alternate and corresponding angles.**

This is the second organising principle in geometry—namely the criterion for two lines in the plane to be *parallel*. Given any two lines in the plane, a *transversal* is a third line that cuts both of the two given lines. The *parallel criterion* declares that:

- two lines are *parallel* precisely when the *alternate angles* (or the *corresponding angles*) created by a transversal are equal.

This is a rather subtle criterion, but one which can be made thoroughly plausible.

The formal proof that the three angles in any triangle ABC add to a straight angle echoes the primary school activity of tearing off the three corners and fitting the pieces together crudely against a ruler. But here we use "God's ruler" (namely the line through C parallel to AB), and we fit the three angles together *perfectly* and in a very particular order ($\angle A + \angle C + \angle B$).

> **Claim** The three angles in any triangle $\triangle ABC$ add to a straight angle.
>
> **Proof** Construct the line XCY through C parallel to AB (with X on the same side of CB as A).
>
> Then $\angle XCA = \angle BAC = \angle A$ (alternate angles)
>
> and $\angle YCB = \angle ABC = \angle B$ (alternate angles).
>
> $\therefore \angle A + \angle C + \angle B = \angle XCA + \angle ACB + \angle YCB$. **QED**

A quadrilateral $ABCD$ can be split into **two** triangles (by drawing one of the diagonals AC, BD), so the sum of the four angles in any quadrilateral is "$2 \times 180°$". The same idea shows that the angles in any polygon with n sides have sum $(n-2) \times 180°$. These simple observations open the door to hundreds of wonderful (non-obvious, multi-step) problems involving *angle chasing* (see, for example, *Extension mathematics*, Tony Gardiner: *Book Alpha*, Sections T9, E2; and *Book Beta*, Sections T17, C11, E4).

The last seven words of the requirement

> – **derive and use the sum of angles in a triangle and use it to deduce the angle sum in any polygon, and to derive properties of regular polygons**

are slightly out of place here. A *regular polygon* is defined to be a polygon whose sides are all equal and whose angles are all equal. It should be a major focus of secondary geometry to explore the geometry of regular polygons—at least including regular 3-gons, regular 4-gons, regular 5-gons, regular 6-gons, and regular 8-gons. And whilst it follows from the above that each angle in a regular n-gon is equal to

$$\left(1 - \frac{2}{n}\right) \times 180°,$$

almost anything else one might prove about regular polygons depends on the congruence criteria (in particular, properties of "isosceles triangles").

This observation even applies to proving that certain diagonals and sides are parallel.

Claim Let $ABCDE$ be a regular pentagon. Then each diagonal is parallel to the opposite side.

Proof We show that AC is parallel to ED.

$\angle ABC = \left(1 - \frac{2}{5}\right) \times 180^\circ = 108^\circ$.

$BA = BC$, so $\triangle BAC$ is isosceles. Hence $\angle BAC = \angle BCA = 36^\circ$.

$\therefore \angle CAE = 72^\circ$, so $\angle CAE + \angle DEA = 180^\circ$ whence AC is parallel to ED. **QED**

The most important application of the basic property of parallel lines is to derive results about parallelograms. A *parallelogram* is a quadrilateral $ABCD$ in which opposite pairs of sides AB, DC and BC, AD are parallel. Most results relating to parallelograms depend on the congruence criteria. But two results depend only on the basic property of parallel lines.

Claim If $ABCD$ is a parallelogram, then opposite angles are equal: $\angle A = \angle C$, $\angle B = \angle D$.

Conversely, if $ABCD$ is a quadrilateral with $\angle A = \angle C$, $\angle B = \angle D$, then $ABCD$ is a parallelogram.

Proof Suppose $ABCD$ is a parallelogram. Then AB is parallel to DC, so $\angle A + \angle D = 180^\circ$.

And AD is parallel to BC, so $\angle D + \angle C = 180^\circ$.

$\therefore \angle A = \angle C$, and $\angle B = 180^\circ - \angle A = 180^\circ - \angle C = \angle D$.

Conversely, suppose $ABCD$ is any quadrilateral in which opposite angles are equal in pairs: $\angle A = \angle C$, $\angle B = \angle D$. Since

$$\angle A + \angle B + \angle C + \angle D = 360^\circ,$$

it follows that $\angle A + \angle B = 180^\circ$, so AD is parallel to BC. Similarly $\angle B + \angle C = 180^\circ$, so AB is parallel to DC. **QED**

A *rectangle* is defined to be "a parallelogram with (at least one) right angle". If the rectangle is $ABCD$, and if the right angle is at A, then from the above result it follows that $\angle C$ is also a right angle. Hence $\angle B + \angle D = 180^\circ$; since $\angle B = \angle D$, it follows that $\angle B$ and $\angle D$ are also right angles. However, there is no simple way to conclude that "opposite sides of a rectangle are equal" other than by proving the result for parallelograms in general (using ASA-congruence).

Claim If $ABCD$ is a parallelogram, then opposite sides are equal in pairs: $AB = DC$ and $BC = AD$.

Proof Draw the diagonal AC.

$\therefore \angle BAC = \angle DCA$ (alternate angles)

$AC = CA$

$\angle BCA = \angle DAC$ (alternate angles)

$\therefore \triangle BAC \equiv \triangle DCA$ (by ASA)

$\therefore BA = DC$ and $BC = DA$. **QED**

Claim If $ABCD$ is a rectangle, then $AC = BD$.

Proof We claim that $\triangle ABC \equiv \triangle BAD$ (by SAS: since $AB = BA$, $\angle ABC = \angle BAD = 90^\circ$, and $BC = AD$ (opposite sides of parallelogram)).

$\therefore AC = BD$. **QED**

Each result one can prove for parallelograms has a converse which (if true) should also be proved, since it allows us to identify a parallelogram on the basis of other characteristic properties.

Claim If $ABCD$ is a quadrilateral in which $AB = DC$ and $BC = AD$, then $ABCD$ is a parallelogram.

Proof Draw the diagonal AC.

$\therefore \triangle BAC \equiv \triangle DCA$ (by SSS).

$\therefore \angle BAC = \angle DCA$, so AB is parallel to DC (alternate angles equal), and $\angle BCA = \angle DAC$, so BC is parallel to AD (alternate angles equal). **QED**

Claim If $ABCD$ is a parallelogram, then the diagonals AC and BD bisect each other.

Conversely, any quadrilateral $ABCD$ whose diagonals bisect each other is a parallelogram.

Proof Let the two diagonals meet at X.

$\therefore \angle ADX = \angle CBX$ (alternate angles)

$DA = BC$ (opposite sides of a parallelogram)

$\angle DAX = \angle BCX$ (alternate angles).

$\therefore \triangle ADX \equiv \triangle CBX$ (by ASA)

$\therefore DX = BX$ and $AX = CX$, so the diagonals bisect each other.

Now let $ABCD$ be any quadrilateral whose diagonals AC, BD bisect each other at X.

Then $AX = CX$ and $DX = BX$, and $\angle AXD = \angle CXB$ (vertically opposite angles).

$\therefore \triangle ADX \equiv \triangle CBX$ (by SAS).

$\therefore \angle DAX = \angle BCX$, so DA is parallel to CB (alternate angles equal), and

Similarly we can show that $\triangle ABX \equiv \triangle CDX$ (by SAS).

Hence $\angle BAX = \angle DCX$, so AB is parallel to DC (alternate angles equal). **QED**

A *rhombus* is a parallelogram $ABCD$ with adjacent sides equal; $AB = AD$. And a *square* is a rhombus which is also a rectangle.

Claim The two diagonals of a rhombus $ABCD$ are perpendicular.

Proof Let the two diagonals meet at X. Then $DX = BX$ so $\triangle AXD \equiv \triangle AXB$ (by SSS).

$\therefore \angle AXD = \angle AXB$. **QED**

In a rhombus $ABCD$, each diagonal splits the rhombus into two *isosceles triangles*. Hence other properties of a rhombus (and their converses) tend to exploit the basic property of isosceles triangles (and its converse).

3.4.7 Similarity (from 3.2)

- **identify and construct congruent triangles, and similar shapes by enlargement, with and without coordinate grids**
- **apply angle facts, triangle congruence, similarity and properties of quadrilaterals to derive results about angles and sides, including *Pythagoras' Theorem*, and use known results to obtain simple proofs**
- **use *Pythagoras' Theorem* and trigonometric ratios in similar triangles to solve problems involving right angled triangles**

The first two requirements have both been addressed elsewhere (in Section 3.2 and 3.2.3, and in Section 3.4.5 respectively). They are linked here because both mention "similar shapes" or "similarity", and this idea has to be addressed to prepare the way for simple trigonometry (as in the third listed requirement).

We noted in Section 3.2 that the reference to "similar shapes" in the first of the above requirements is largely "informal", and that the initial emphasis here should be practical. The formal notion of *similarity* should emerge from pupils' own experience. For example, they should construct "enlargements" in the spirit of the exercises in sections T8 and C26 of *Extension mathematics, Book Beta* by Tony Gardiner (Oxford University Press 2007). And their understanding and interpretation of "scale drawings", and the effect of scale factors on lengths, areas and volumes, should also be rooted in practical work and calculation (see, for example, sections T21, C41 in *Extension mathematics Book Gamma*).

However, pupils need more than this in preparation for simple trigonometry (see Section 3.5). So once sufficient foundations-in-experience have been laid (as indicated below), it is certainly worth explaining clearly what it means for two figures to be similar: namely that two polygons $ABCD\ldots$ and $A'B'C'D'\ldots$ are **similar** if

- corresponding *angles* are *equal*:

$$\angle A = \angle A', \angle B = \angle B', \angle C = \angle C', \angle D = \angle D', \ldots$$

and

- corresponding *sides* are *proportional*:

$$AB : A'B' = BC : B'C' = CD : C'D' = \ldots.$$

Pupils need to recognise that these two conditions seem to capture what we mean when we say that "two polygons have the same shape".

A square (or *regular* 4-*gon*) is defined as "a quadrilateral having all sides equal and all angles equal". Two different squares $ABCD$ and $A'B'C'D'$ have all angles equal to 90°; hence they automatically satisfy the first bullet point. And the four sides of each square are equal: if the first square has sides of length a and the second has sides of length b; then each ratio in the second bullet point is equal to $a : b$, so the second bullet point is satisfied. Hence any two squares are mathematically *similar*. They are also "physically similar-looking", in that a large square that is some distance away leaves the same image on the retina as a nearby smaller square.

To establish that **both** bullet points are needed, pupils should think of examples

- of two rectangles whose angles clearly match up in pairs, but whose sides are **definitely not** proportional (such as a 1 by 1 square and a 2 by 1 rectangle), or
- of two parallelograms whose sides are in proportion, but whose angles are not equal in pairs (such as a 1 by 1 square and a 60° rhombus with sides of length 1).

It should then be clear that our idea of "same shape" requires **both** conditions.

However, there is a remarkable difference between polygons with more than three sides (such as quadrilaterals), and polygons with exactly three sides (i.e. triangles). Any two *equilateral* triangles of different sizes are similar—and for much the same reason as any two squares are similar: (i) all angles are equal to 60°, so are certainly "equal in pairs", and (ii) all three sides of one triangle have equal length (say a), and all three sides of the other triangle have equal length (b say), so the ratios of corresponding sides are all equal to $a : b$. But, unlike the case of squares (where both conditions are needed), one of these conditions for equilateral triangles comes for free. As the name implies, for $\triangle ABC$ to be equilateral, all we need is

> "that the three sides are all equal: $AB = BC = CA$".

The fact that the three angles are all equal to 60° then comes for free—thanks to the SSS congruence criterion (since $\triangle ABC \equiv \triangle BCA$, so $\angle ABC = \angle BCA$ and $\angle BCA = \angle CAB$). Hence, to check the claim that any two equilateral triangles are similar, it is enough to observe that the second bullet point is satisfied (and the first then comes for free).

The same is true whenever we apply the idea of "similarity" to triangles in general. Officially two (ordered) triangles $\triangle ABC$ and $\triangle DEF$ are *similar* (which we write as $\triangle ABC \sim \triangle DEF$) if

- corresponding *angles* are equal: $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$,

and

- corresponding *sides* are proportional: $AB : DE = BC : EF = CA : FD$.

Yet the challenge, to think of

- two triangles whose angles match up in pairs, but whose sides are **not** proportional, or
- two triangles whose sides are proportional but whose angles are **not** equal in pairs,

leads to a surprise.

- If $\triangle ABC$ and $\triangle DEF$ have angles equal in pairs, the three pairs of corresponding sides always turn out to be proportional; and
- if $\triangle ABC$ and $\triangle DEF$ have corresponding sides proportional, then corresponding angles are automatically equal.

This fact is unlikely to make sense if simply stated in the way we have stated it here. So pupils need prior experience of drawing and measuring that makes this important statement meaningful and plausible: the *similarity criterion* states that, **for triangles**, each of the above bullet points implies the other. (See, for example, "Problem 0" in Section T13 of *Extension mathematics Book Gamma*.)

Initially pupils need the idea of similar triangles for simple trigonometry: i.e. only for *right-angled* triangles. We can even restrict to right angled triangles $\triangle OBC$, with a right angle at B, and $\triangle OB'C'$ with a right angle at B', sharing a common vertex O (which we may take to be the origin), with B and B' lying on the positive x-axis. If we fix the angle at O $\angle BOC = \theta$, and choose C' to lie on the line OC, then the angles of the two triangles $\triangle OBC$ and $\triangle OB'C'$ are equal in pairs (namely to θ, 90°, and $90^\circ - \theta$); and we can establish as a fact of experience (by drawing and measuring; or partly of deduction—first for $k = 2$ or $k = \frac{1}{2}$, then for any integer or unit fraction, and finally for any fraction) that

> if $OB' = k \cdot OB$ (so to get from O to B' we go "k times as far **along**" as we did to get to B)
>
> then $B'C' = k \cdot BC$ (so to get to B' from C' we go "k times as far **up**" as we did to get from B to C)

It follows (by *Pythagoras' Theorem*) that $OC' : OC = k$. So the first bullet point implies the second. Hence if $\angle BOC = \angle B'OC' = \theta$ (and $\angle OBC = \angle OB'C' = 90^\circ$), then corresponding sides are in proportion:

$$OB' : OB = OC' : OC = B'C' : BC.$$

Cross-multiplying shows that

$$B'C' : OB' = BC : OB,$$

so the quotient "$\frac{\text{opposite}}{\text{adjacent}}$" depends only on the angle $\angle BOC = \theta$, and not on the choice of triangle. Hence we can safely write it as "$\tan\theta$"—that is as a *function* that only depends on the angle θ. (See Section C20 in *Extension mathematics Book Gamma*.)

Similarly $B'C' : OC' = BC : OC$, so the quotient "$\frac{\text{opposite}}{\text{hypotenuse}}$" depends only on the angle $\angle BOC = \theta$ and not on the choice of triangle, so we can safely write it as "$\sin\theta$"—that is, as a function of θ. And the ratio $OB' : OC' = OB : OC$, so the quotient "$\frac{\text{adjacent}}{\text{hypotenuse}}$" depends only on the angle $\angle BOC = \theta$ and not on the choice of triangle, so we can safely write it as "$\cos\theta$". (See Section C33 in *Extension mathematics Book Gamma*.)

The congruence criterion and the parallel criterion allow one to transfer *exact* relations (such as *equality* of line segments or of angles) from one place to another. The *similarity criterion* goes beyond this world of *exact equality* to allow one to deal with ratios, scaling, and enlargement. Hence this criterion is probably best delayed until the basic consequences of congruence and parallelism have been sufficiently explored, and until pupils are sufficiently confident in working with ratio. (The *similarity criterio*n may be thought of as a substitute for the evidently false "AAA congruence criterion". The criterion can also be re-formulated as SAS-similarity: (see Section C13 of *Extension mathematics Book Gamma*).

As hinted above, special cases of the similarity criterion can actually be proved using the congruence criterion and the parallel criterion—namely where the ratio between corresponding sides in the second bullet point is a fraction. The most important example occurs when this ratio is equal to 2 (or to $\frac{1}{2}$) and is called the *Midpoint Theorem*, which says that:

> if in $\triangle ABC$, M is the midpoint of AB and N is the midpoint of AC,
>
> then MN is parallel to BC and $BC : MN = 2 : 1$.

That is $\triangle ABC \sim \triangle AMN$, with the corresponding scale factor

$$AB : AM = AC : AN = BC : MN = 2 : 1$$

(see section T13, Problem 6 in *Extension mathematics Book Gamma*).

The third requirement listed at the start of Section 3.4.7 concerns applications of these ideas. Once we know *Pythagoras' Theorem* we can use it to find lengths exactly (in surd form). An equilateral triangle of side 2 has height equal to $\sqrt{3}$. A square $ABCD$ of side 1 has diagonal AC of length $\sqrt{2}$. A regular pentagon $ABCDE$ of side length 1 has diagonal AC

of length $\frac{1+\sqrt{5}}{2}$. A regular hexagon $ABCDEF$ with sides of length 1 has two different length diagonals—a diameter AD of length exactly 2, and a shorter diagonal AC of length exactly $\sqrt{3}$. The square of side 1 allows one to write down the exact values for $\tan 45^\circ = 1$, and for

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \cos 45^\circ.$$

In the equilateral triangle of side 2, the perpendicular from the apex to the base bisects the apex angle into two angles of 30°, and meets the base at its midpoint. Hence we can write down the exact value for

$$\sin 30^\circ = \frac{1}{2} = \cos 60^\circ,$$

for

$$\sin 60^\circ = \frac{\sqrt{3}}{2} = \cos 30^\circ,$$

for $\tan 30^\circ = \frac{1}{\sqrt{3}}$, and for $\tan 60^\circ = \sqrt{3}$. One can also use *Pythagoras' Theorem* to find the distance between any two points whose coordinates are given (in 2D or in 3D).

Wherever right angled triangles appear, one can use sin, cos and tan (or similar triangles) to find missing angles or lengths. Classical applications include

- "angles of elevation (or depression)", where we might know that "from the top of a vertical cliff 40m high, we can see a buoy whose angle of depression (from our position on top of the cliff) is 35°. How far is the buoy from the base of the cliff?", or
- the traditional exercise of "calculating the height of a tree without measuring directly", where we line up our eye (at ground level), the top of a pupil's head and the top of a tree, and then measure
  (i) the pupil's height and
  (ii) the distances from our eye to the pupil's feet, and to the base of the tree.

One would also like to see other applications of angles which do not involve right angled triangles directly (e.g. angle problems involving bearings).

3.5. The remaining requirements

- **use the properties of faces, surfaces, edges and vertices of cubes, cuboids, prisms, cylinders, pyramids, cones and spheres to solve problems in 3D**
- **interpret mathematical relationships both algebraically and geometrically**

These two final requirements look very much like a collection of "remnants". Both seem to relate to rather late in Key Stage 3 or even to Key Stage 4.

Pythagoras' Theorem and similarity (or trig) feature in solving problems relating to regular polygons or familiar figures in 3D—whether calculating the lengths of ladders leaning against walls, or the height of some point above the ground or table, or surface areas and volumes. However, the examples listed seem better suited to Key Stage 4 than to Key Stage 3. Nevertheless one would love to see problems at some stage that involve finding and using the slant height of a cone, or the height of a pyramid, or the distance between two opposite corners of a cube, or the angles between lines in 3D figures, or the angle between a slanting face and the base of a pyramid.

The final requirement is admirable as a general idea. But it is also rather too vague for us to try to interpret it reliably here.

4. Probability and Statistics

The requirements under these headings leave many questions unanswered. It is not always clear how to interpret them as they stand, so we have tried to suggest "alternative readings". We have also taken the opportunity to discuss some of the background which needs to be borne in mind when devising a scheme of work.

4.1. Probability

4.1.1 Introduction

- **record, describe and analyse the frequency of outcomes of simple probability experiments involving randomness, fairness, equally and unequally likely outcomes, using appropriate language and the 0–1 probability scale**

Our understanding of how to teach probability is less well developed than our understanding of how to teach geometry. So it is difficult to know exactly where the problems lie. But there would seem to be considerable potential for confusion here between

> the language of messy "experiments" in the real world,

and

> language that belongs to a pristine mathematical universe (namely *probability*).

This confusion is especially awkward given the explicit mention of "using appropriate language".

Of course the mathematical universe often has its roots in the real world, so terms and expressions may at times inhabit both worlds. Nevertheless it may be easier to interpret the above official requirement if one imagines added quotation marks (and the extra word "eventually") roughly as follows:

- record, describe and analyse the frequency of outcomes of simple "probability experiments" involving "randomness", "fairness", "equally and unequally likely outcomes", using appropriate language and [eventually] the 0–1 probability scale.

"Record, describe and analyse the frequency of outcomes of simple [...] experiments" is an excellent requirement: pupils need such experience in order to develop their ideas of variability, and to understand how these are ultimately captured by the universal model of a sample space (S, p), where p assigns values between 0 and 1 to subsets of S according to certain rules (e.g. for a single toss of a fair coin, $S = \{H, T\}$, with $p(H) = p(T) = \frac{1}{2}$). However, this step lies some way off—though it is alluded to vaguely in the next batch of requirements, where we read (see Section 4.1.2):

> "generate theoretical sample spaces for simple and combined events with equally likely, mutually exclusive outcomes".

The idea also features in the GCSE *Subject Criteria* using curious, non-standard language

> "construct theoretical *possibility* spaces for single and combined experiments with equally likely outcomes" [emphasis added].

In contrast, there is no hint of "sample spaces" in the Key Stage 4 programme of study.

However, given the explicit mention of "theoretical sample spaces" in the next official requirement (see Section 4.1.2), we assume that the "experiments" referred to in the first Key Stage 3 requirement are intended to open up *informal* consideration of questions involving "fairness", "randomness", and the crucial idea of "equally likely". And if these informal considerations are to lead (eventually!) to the idea of a sample space (S, p), we may need a shift of focus from arbitrary real-world experiments to more carefully chosen settings (such as coin tossing, or dice rolling, or equally divided spinners), where a theoretical analysis is possible.

Hence, if pupils' understanding of probability is to progress, we may need to distinguish three separate settings:

> experiments in the *real world* of messy data;
>
> experiments and analysis in the *in-between world* of controlled data (fair coins, dice, etc.);

and

> the *mathematical* world of theoretical probability.

We may choose to start in the real world of messy data: for example, with pupils examining the apparent likelihood of being born on each day of the week. The obvious "sample", or experiment, (namely, collecting all the results for pupils in the class) leads first to the need for them to use their known birthday and age to discover the day of the week when they were born; the class can then record numbers for each day of the week; and finally one can introduce the idea of using "relative frequencies" as a better measure than the raw numbers. The resulting distribution will inevitably raise the question of "fair sampling" and "randomness" (for it is almost bound to contradict pupils' gut feeling by deviating from the expectation that each day should be "equally likely"). More representative data—if it can be procured—is just as likely to challenge this understandable assumption.

The use of "relative frequencies" introduces the idea of a 0–1 scale (though not at this stage a "*probability* scale"). And one can emphasise the fact that the relative frequency of those born on a weekday (say) is obtained by adding the five separate relative frequencies for Monday-Friday.

But relative frequencies only tell us what **was** observed—**once**; and this would seem to tell us nothing about what *will* be observed *in the future*. This is the whole point of non-deterministic data. We may know that the recorded relative frequencies add up to 1; and that the relative frequency of a combined event is equal to the sum of the ingredient relative frequencies. But this only tells us *what happened last time*. We cannot calculate with observed relative frequencies to learn anything more general—as one can to some extent with probabilities. So it should soon become clear that this is not a mathematical world, where one can answer more interesting questions using exact calculation.

Classical science is deterministic, and reported results in classical science must be replicable: if you or I repeat a deterministic experiment as it was reported, we expect to replicate the stated results. And if we fail, then we have to question either the reported result or our own attempted replication. But with stochastic processes, the situation is completely different. When we repeat a "probability experiment", the observed outcomes vary considerably. Yet within the observed variations one can discern certain clear *trends*. This new science is no longer to be judged by, or analysed through, the outcomes of a single experiment, but through patterns in the variation of the outcomes of repetitions of the experiment. Single snapshots are of little relevance; instead we try to summarise the background reality that lies behind what we observe by integrating *all possible snapshots* into a single model of "probabilistic reality".

4.1.2 Theoretical probability

- **understand that the probabilities of all possible outcomes sum to 1**
- **enumerate sets and unions/intersections of sets systematically, using tables, grids and Venn diagrams**
- **generate theoretical sample spaces for single and combined events with equally likely, mutually exclusive outcomes and use these to calculate theoretical probabilities**

So there are strong reasons to move beyond messy real-world "experiments", and to focus on a more restricted (or more artificial) *mathematical* universe—such as coin tossing or dice rolling—where everything is much more clearly defined. Here one can perform *repeated* experiments relatively easily. And one can also analyse the background situation precisely—by counting.

Even in this restricted world, there are elephant traps to be identified and avoided. For example, when tossing two coins, time is needed to clarify the expected ratios of the three possible outcomes—"two Heads", "two Tails", and "one Head and one Tail". But unlike the messy real world, it is now natural to imagine an idealised version where one thinks of a "fair coin", with Head and Tail truly "equally likely" (or a "fair die", where each of the six outcomes is "equally likely"). Experiments show that the observed "relative frequencies" of Heads and Tails (or of the six possible outcomes for rolling a die), vary significantly. But they always add to 1, and can always be combined to find the relative frequencies of compound events (such as "rolling an odd number").

More importantly, experimental results can now be compared with what one would "expect" on the basis of the idealised model. This background "expectation", based on counting within the idealised model, is quite different from "the recorded results of experiments". And whereas the results from successive experiments will vary, the "expected" results stay the same. It is as though the calculated expected results are some kind of "ideal summary", and each experiment is only an approximation to, or a flickering shadow of this ideal summary. For example, within the model we can *count exactly*: there are 2^4 possible sequences of 4 tosses of the coin, and exactly 4 of these sequences have just one "Tail"—which seems to say that, if we record 100 such sequences of 4 tosses, then we should "expect" $\frac{4}{16}$ of them to have just one "Tail". The reality will of course usually be different; but pupils may gradually come to realise that the existence of the idealised model provides a *fixed reference point* with which we can compare the results of different experiments, and provides the key to making sense of their variability (as "deviations from the expected ideal").

There are many advantages in working within these carefully chosen, controlled settings. In particular, they clarify the difference between observed "relative frequencies" from a single experiment and the expected frequencies, or "theoretical probabilities". The theoretical model also allows us to see more clearly how the cumulative results of many experiments tend to "average out", and how this long-term average tends to approximate the theoretical probability ever more closely.

An experiment, and the associated set of observed frequencies, is like a single snapshot *of a ghost*, or *a shadow* of some hidden object. This is especially true if the experiment involves messy real world data. The snapshot gives one a record of vague outlines—hints of something substantial. Yet one cannot be sure of the precise outline or shape which gave rise to this impression; that is, one may at first have no knowledge of the "background reality" that caused the impression or shadow. The observed results (of say "days of birth") may suggest a surprising pattern, but it is only a hint: the actual reality that lies behind the observations remains elusive. Subsequent snapshots of apparently the same object may vary greatly from each other—and yet between them reveal patterns that suggest that there really is some “background reality” that lurks out of sight.

This is a classic instance of Plato’s parable of *The Cave*. We can only discern shadows of some presumed “Platonic reality” (“theoretical probability” in this instance), and must somehow *infer* what we can about the hidden reality that is casting the shadow, or leaving a ghostly impression. And the test for any inferred “reality” is whether it explains the shadows that we do see, and *why we do* **not** *see the shadows that we do* **not** *see*. If the observed shadows were always the same shape (as would be the case if the object were a solid statue, and the light source remained constant), then the “Platonic reality” might be a classical numerical measurement from elementary mathematics (like “the height of Nelson’s column”).

Probability and statistics are different, in that the observed “facts” differ each time we look. Yet there is still something substantial behind the observation. A single experiment, or sample, and the associated set of “observed relative frequencies”, is but a single shadow of an elusive, moving object. And our inferred “Platonic reality” must somehow combine all *conceivable* observations into a single idea, which somehow incorporates the observed variability, and explains how each snapshot arises as a single view, or aspect of it. That is the role played here by the idea of a **sample space**: a set of atomic outcomes, with a *probability* assigned to each, so that their sum is 1.

Elementary mathematics can be largely summarised as *the art of exact calculation* with numbers, symbols, geometrical entities, etc. If we wish to find “the height of Nelson’s column”, though we do not know the answer, it is natural to *assume it has a definite value*, and then use the methods of elementary mathematics to calculate this presumed “definite value” using other known facts (e.g. properties of similar triangles). That is, the objects to which this “art of exact calculation” applies—whether represented by numerals or letters—are usually assumed to have *definite* values (possibly unknown). The associated mathematical universe may be abstract; but its objects have specific values, which remain constant throughout any subsequent calculation. Such entities are relatively tame, and static; they can be imagined relatively easily.

However, the numerical data related to probability and statistics is more elusive than this—though the fact that it is clearly still numerical (in some sense), may tempt us to overlook its more elusive character. Consider, for example,

> the number of "Heads" obtained in a sequence of 4 coin tosses.

Each particular "instance" (toss a coin 4 times and keep track of the number of "Heads") gives rise to a *single value*—namely, the number of "Heads" obtained. So this "number of Heads in 4 tosses" is superficially *like* "the height of Nelson's column". However, the object of thought is not the individual value that we obtained on this one occasion, but

> the *totality* of all possible "numbers of Heads" that could be obtained if we repeated the experiment,

together with

> the way these "numbers of Heads" are *distributed* between 0 and 4.

This object of thought is multi-layered: there is a **sample space** S (the set of integers between 0 and 4), with each member having an attached *number* (the relative frequency with which this number of "Heads" occurs). If we shift from repeated experiments and observed "relative frequencies", we can use the idealised model of a "fair coin" with

$$p(H) = p(T) = \frac{1}{2}$$

to calculate exactly the expected frequency for each number of "Heads". This "expected frequency" varies with the number (though 0 "Heads" turns out to be exactly as likely, or as unlikely, as 4 "Heads", and 1 "Head" turns out to be exactly as likely as 3 "Heads"). We therefore get a new probability P for this sample space S, where

$$P(0) = P(4) = \frac{1}{16}, \quad P(1) = P(3) = \frac{4}{16}, \quad P(2) = \frac{6}{16}.$$

The nature of variability, and the difference between

(i) deterministic systems (such as classical science, where one expects to be able to replicate an experiment and observe the same results), and

(ii) stochastic data

can be explored in the messy real world. But at this level, the analysis of stochastic data is largely restricted to discussion and qualitative statistics. So one needs to move the field of play to the in-between world of controlled data (fair coins, dice, etc.). Here one can still do experiments; but one can also analyse things in a way that offers a bridge to *theoretical probability*. One can construct a natural God-given model, and can compare its predictions with the results of experiments to see just how variable things can be. In particular, it makes didactical sense to choose in-between examples with *finitely many* atomic outcomes, and to focus on examples where symmetry guarantees that the atomic outcomes are *equally likely*. **Everything then reduces to counting.** And one can compare the relative frequencies that arise in experiments with the "God-given" relative frequencies derived from counting (which form the model for our idea of a "sample space and probability").

With luck it may now be a bit clearer why we questioned the informal mix of words "probability experiment", "randomness", "fairness", "equally and unequally likely outcomes", "0–1 probability scale" in Subsection 4.1.1 and suggested there was a danger they might blur the distinction between

(i) the world of observed real-world data, and

(ii) the hidden Platonic reality, or the theoretical model.

The requirement in Subsection 4.1.1 used language that ultimately belongs to our inferred Platonic model, and imposed it upon the world of shadows, or observed data. Pupils should definitely "record, describe, and analyse the frequency of outcomes from simple [...] experiments", whose results are "non-deterministic" in that the data vary from one experiment to the next; but these are not really *probability* experiments. Repeated coin tossing, or dice rolling, or drawing pin tossing force us to address the underlying issue of "variable outcomes" and to nurture ideas of probability. But they can only be described as "probability experiments" in retrospect—once we have the background notion of an underlying sample space S and a probability function p. The language used in this official requirement is the language that should emerge as a result of a carefully chosen sequence of such experiments and analysis: it should not really be used "up front". Hence its use is best interpreted as a summary of what is needed—used informally for ease of communication between parties who are already "in the know".

We end this rather heavy digression on a lighter note. During a test in which all the questions required only a true/false response, a pupil was observed to be repeatedly tossing a coin until he had answered every question. When asked what he was doing, he replied: "I have no idea of what is going on in this course. So I flip the coin; if it turns up Heads, I choose true and if it turns up Tails I choose false." The invigilator tried to keep a straight face, and moved on. Later, the invigilator announced: "You have five minutes remaining" and was surprised to see the pupil madly tossing a coin once more. Puzzled, he asked: "What exactly are you doing now?", only to be told: "I'm checking my answers!"

4.2. Statistics

No-one should doubt the increasing importance of statistics in the modern world. But it is less clear how this fact should influence the school curriculum—and in particular, the school *mathematics* curriculum. The world is awash with data. But the information available to decision makers in government, in business, in management, in operating public utilities, etc. only tells part of the story. They may collect "random samples" to try to eliminate bias, but the result is still an incomplete "snapshot". What can one infer about the true situation from such a snapshot? And how much confidence can one place in the resulting inference? If one takes a second sample from a different source, or at a different time, it is bound to differ from the first. But when are the differences such as to suggest that "something has really changed"? These are the kinds of questions addressed by statistics. It is one thing to suggest (rightly) that the mathematics curriculum must think carefully about how to prepare pupils so that they have a chance of making sense of the way statistics is used at Key Stage 5 and beyond; it is quite another thing to suggest that significant chunks of elementary mathematics should be sidelined, or de-emphasised, in order to make room at Key Stage 1–3 for possibly premature, low grade statistical content.

The situation we face should sound familiar. No one disputed the realisation in the 1950s and 1960s of the increasing importance of a kind of "modern mathematics" that was very different from school mathematics as then taught. However, the inference that school mathematics should be re-formed into something closer to the said "modern mathematics" proved to be thoroughly misguided—and it took us two decades before we finally admitted this fact. In much the same way, no-one doubted the claim that the 1970s and 1980s witnessed the beginning of a revolution in computational technology, that led to a marked shift in the way mathematics was being used in the outside world; yet the assertion that primary school mathematics "therefore" needed to be radically re-formed to incorporate calculators proved once again to be misguided. Claims were made at the highest level that pupils no longer needed to "learn their tables"; and it again took twenty years for us to discover that "learning one's tables" is important **not** so that we can compete with a calculator, but because it is part of the way young minds internalise an understanding of *the way numbers work*—and so is essential if we are to help pupils prepare to make use of the new technology **later**.

Hence, while welcoming the commitment and enthusiasm of those with a special interest in statistics, it is important not to repeat the same mistake of unquestioningly accepting the claims of those who may have allowed their enthusiasm to run away with them. We would all like the next generation to be well-placed to use statistics intelligently in adult life. But the experience of the last 25 years should convince us that this is not likely to be achieved by neglecting parts of elementary mathematics that are needed for the subsequent effective analysis of statistical methods in favour of low-level qualitative methods of little lasting value. In particular, the basic framework for statistics depends on having a firm grasp of theoretical probability.

Given the ubiquity of statistical data, some understanding of the associated problems deserves attention. But it remains unclear how this experience should be embedded within the wider curriculum, and how much of it, and which aspects, are best treated in the time allocated to *mathematics* (and at what stage). The dilemmas were clearly indicated in the analysis and recommendations of the Smith report *Making mathematics count* (2004)—see para 0.28, paras 4.16–4.18 and **Recommendation 4.4**.[21]

Curricula since 1988 have allocated significant amounts of classroom time to *Handling data* many years before pupils master the mathematics that is needed for statistical calculation. As a result, the content listed under *Handling data* has been largely restricted to *descriptive statistics*. Whilst there is some value in using common sense to extract simple information from statistical data in all subjects, and to use this to draw pupils' attention to misconceptions, we need to consider carefully how much of the necessary time should be taken from that allocated for *mathematics*. There is a balance to be struck between on the one hand alerting pupils to the challenge presented by statistical data, and on the other developing the mathematical tools that will subsequently allow pupils to engage in some more significant analysis of problems—including statistical problems. If the necessary tools are not mastered, pupils are likely to be reduced to applying cookbook procedures which they cannot possibly understand. Moreover, this contradicts the declared *Aims* of the curriculum, and the idea that one should insist on *meaning* and *understanding*. So we should perhaps look for ways of treating this material at a later stage when pupils can make sense of it using mathematics that they understand.

4.2.1

- **describe, interpret and compare observed distributions of a single variable through: appropriate graphical representation involving discrete, continuous and grouped data; and appropriate measures of central tendency (mean, mode, median) and spread (range, consideration of outliers)**

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[21] http://www.mathsinquiry.org.uk/report/MathsInquiryFinalReport.pdf

- **construct and interpret appropriate tables, charts, and diagrams, including frequency tables, bar charts, pie charts, and pictograms for categorical data, and vertical line (or bar) charts for ungrouped and grouped numerical data**

The listed requirements have acquired a fairly standard interpretation in current textbooks and assessments. Yet it is worth asking how well this standard interpretation *prepares* pupils to understand the more serious statistics that is used in many subjects *beyond Key Stage 4*.

Most of the elementary mathematics we have covered so far can be summarised *as the art of exact calculation* with numbers, symbols, geometrical entities, etc. Suppose we wish to find "the height of the school building", or "the height of Nelson's column". Though we do not know the answer, we assume it has a definite value. We then use the methods of elementary mathematics to calculate this value. That is, the objects to which this "art of exact calculation" applies (whether represented by numerals or by letters) can be assumed to have *definite* values, which remain constant throughout any calculation. Such entities are static, and can be imagined relatively easily.

However, stochastic, or statistical data—though still numerical—is not quite like this. Consider, for example, "the height of a UK adult male in 2014". Each particular instance of such data ("choose one adult UK male, then measure and record his height") gives rise to a *single value*—the height of that particular individual. So one might think that "the height of a UK adult male in 2014" is like "the height of Nelson's column". But the object of thought here is not the single value obtained by choosing and measuring the height of one adult male: we are interested in the *totality of individual heights*, and the way these individual heights are distributed throughout the whole population of "UK adult males in 2014". This object of thought has several 'layers':

- there is a **population** S (the set of UK adult males in 2014);
- each member has an attached *number* (his height);

- this attached number varies as one varies the choice of individual, and does so in such a way as to give rise to a *distribution* of possible values, where each "height" occurs with its own *frequency*, or **probability**.

Later, these multi-layered objects will be formalised as *random variables*, and captured via *distributions*. No matter how they may eventually be formalised, all we need to notice here is that they are clearly more elusive than the numbers studied elsewhere in elementary mathematics.

And this is just the easy part of the story. The harder part is that we rarely know the underlying *distribution* precisely. So we try to draw inferences about the underlying distribution on the basis of some more-or-less representative *random* **sample**! (The word "random" deserves a whole mini-essay of its own; but it indicates that the sampling is done in a way that avoids giving a systematically false impression of the population being sampled.) Or we may want to decide whether the apparent differences between two different random samples can be explained by "natural variation", or whether the differences suggest that something significant has changed.

The specific (possibly unknown, but fixed) numbers of more familiar elementary mathematics have here been replaced by *distributions*, where a range of possible values can occur—each with its own frequency. The background *distribution* may be unknown—and instead all we know is information from one or more *samples*. And the goal is to decide what one can infer about the (unknown) background *distribution*, or whether the differences between two different samples are significant. This is an important art. But it is very different from (and conceptually much more demanding than) the mathematics of numbers, measures, symbols, or functions that is studied elsewhere in Key Stages 1–3.

4.2.2

> **– describe simple mathematical relationships between two variables (bivariate data) in observational and experimental contexts and illustrate using scatter graphs**

Any tabulation, or graphical representation, involves two variables! In Section 4.2.1 there was an initial imagined “single variable”, whose *frequency* of occurrence was being recorded. So one was dealing with two linked variables: the original variable, and the counting numbers. But in some sense the counting numbers did not have an independent interest. In Section 4.2.2 we are concerned with two independently existing variables which may be related (such as height and weight among adult males), and where we wish to understand the possible linkage better.

An obvious trick is to plot linked pairs (x, y), with one variable along the x-axis, and the other variable along the y-axis. The resulting collection of points in 2D is called a *scatter graph*. This is not the graph of a function, since

(a) not all possible x-values occur, and

(b) those x-values that do occur may occur more than once (with different y-values).

The idea that there might be a “connection” between the two variables then translates into the idea that the scatter graph may reveal some structure.

The simplest imaginable structure would be for the plotted points to lie along some straight line, or to reflect some other functional dependency of one variable upon the other. A non-statistical example might plot the temperature in “degrees Fahrenheit” against the temperature in “degrees Centigrade”: here because the relationship is deterministic and exact, the data sits along a perfect straight line $y = \frac{9}{5}x + 32$. But statistical data is never quite so well-behaved.

When trying to spot a hidden relationship with messy data it can help to impose an additional **constraint**. For example, we may consider whether there is some special point that should be forced to lie on any possible curve which links the two variables x and y. The data points themselves are all as reliable, or as unreliable, as each other. But examples can be used to support the idea that the point $(\mathrm{Av}(x), \mathrm{Av}(y))$, where $\mathrm{Av}(x)$ is the average of all the x-values, $\mathrm{Av}(y)$ is the average of all the y-values, serves as a kind of “representative centre” for the set of data points, and so should lie on any resulting curve. In particular, if we decide that the relationship is approximately linear, then requiring the line to pass through the point $(\mathrm{Av}(x), \mathrm{Av}(y))$ makes it much easier to choose the gradient "by eye" so that we get a line that seems to follow the data approximately, and which leaves deviant data points (x, y) in some sense "equally distributed" above and below the line. The whole thrust of this analysis is to try to see patterns in the data that might not be apparent from a mere list of numbers. However, the analysis remains at best weakly "mathematical": we are not yet sufficiently well-placed to engage in genuine calculations.

In the relatively tame world of elementary mathematics we have already highlighted the difference between *direct* calculation, where the answer can be ground out deterministically, and *inverse problems*, whose solution forces us to "work backwards" from some "output" in search of some direct calculation that might give rise to the given data (see Part II, Section 1.2.3, and Part III, Sections 1.2.2, 1.2.4). The art of analysing statistical data mathematically would seem to be an important instance—and a rather subtle instance—of such *inverse problems*. This art is therefore doubly challenging. Not only are the objects of the relevant *direct* statistical calculations more subtle than those we meet in the rest of elementary mathematics; but *handling data* is useful precisely because statistical problems are *inverse* problems—we typically know only selected information (from some presumed *random sample*), and we need to assess what we may infer from this *sampled* data about the unknown background *distribution* of the whole *population*—and what degree of confidence we may attach to such inferences.

Despite the difficulties, this material plays such an important role in modern society that it is natural for educators to try to find ways of introducing pupils to the underlying ideas. It is not easy to summarise the experience of the last 25 years; but it is probably fair to say that the rhetoric has been consistently ahead of the reality. Thus there are many outstanding issues which a programme of study, or a scheme of work, needs to weigh up and resolve. Three important questions concern

- the age, or prerequisite maturity, that is required before simple mathematical analysis of statistical material can be handled effectively;
- the technical prerequisites that pupils need to master before this analysis can make worthwhile progress;

- the time that is needed to make the engagement with statistical questions worthwhile at a given stage, the likely progress that might be made at that stage, and (crucially) *what other topics would have to be sidelined* in order to make that time available.

IV. A sample curriculum for all—written from a humane mathematical viewpoint

What follows stems from an attempt to consider what a "humane mathematician and educator" might expect to see included at each Key Stage of a National Curriculum—under the assumption that:

- one would like to see number, measures and calculation grasped (in some sense) at primary level, with
- further work on number, together with algebra, geometry, and trigonometry being mastered by age 16 to a level that would allow those who proceed to further studies, in whatever subjects, to be in a position to use the mathematics they have learned.

In particular, the sample curriculum tries to set realistic (rather than ambitious) goals for primary mathematics. It also tries to restrict the extent of abstraction at Key Stage 3, on the assumption that for a significant group of pupils this material may form the core of work at secondary level (age 11–16), along with consolidation of material from upper primary school. More abstract material has been delayed wherever possible until Key Stage 4, where it serves as a transition from elementary mathematics to higher mathematics for all pupils who intend to proceed to further academic studies beyond age 16 (in any subject). Hence teachers may choose to blur the boundary between Key Stages 3 and 4 for many pupils, taking much longer to cover the listed Key Stage 3 material for some, while others may treat Key Stage 3 material more abstractly than is suggested here in preparation for Key Stage 4.

Having produced a *Brief Version* (which was already fairly detailed—see Section 2 below), we then drafted two further versions:

- a *Fuller Version*, which unpacked in greater detail some of the cryptic references we were advised might not be immediately understood; and
- a *Very Brief Version* (see Section 1 below) which rashly tried to compress the essence of each Key Stage into a single page.

We have chosen to reproduce here (and to improve) the *Brief Version* and the *Very Brief Version*, to provide an easily accessible reference for the reader of Parts I, II, and III. Some words that appear in **bold type** have a technical meaning that will have been explained in detail in Part II or Part III. If some aspect remains unclear, we recommend that readers refer to the *Fuller Version*, which is freely available at The De Morgan Forum.[22]

1. Very Brief version

1.1 Key Stage 1

By the end of Year 2 pupils:

Counting, reading and recording number

- use the language for numbers and quantities in everyday settings
- count accurately; read, write and order numbers to at least 100; understand place value, know what each digit of any two-digit number represents, and know that the position of a digit determines its "value"

Recalling facts

- have instant recall of addition and subtraction facts for numbers to 10; have instant recall of ×2, ×5, ×10 multiplication tables, and derive corresponding division facts

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[22] http://education.lms.ac.uk/wp-content/uploads/2012/02/KS_1-4_DMJ.pdf

Calculating

- use the language for simple calculations in everyday settings
- carry out mental and informal written calculations using the four operations of addition, subtraction, multiplication and division
- recognise and use effectively the fact that subtraction is the inverse of addition, and in simple cases that division is the inverse of multiplication
- handle confidently two-digit addition and subtraction in **standard written column format**

Describing shapes and measuring

- recognise, name, and describe properties of common 2D and 3D shapes
- measure and draw straight lines accurate to the nearest centimetre; estimate lengths and other quantities; tell the time to the nearest quarter of an hour, compare durations using standard units, and order events chronologically; use measuring instruments to measure length (cm, m), weight (kg), capacity (l), reading and interpreting scales to the nearest labelled division; use money

Solving problems, reasoning, and using language and symbols

- apply their understanding of number and arithmetic to work with measures and to solve **word problems**
- use mathematical language accurately; read and interpret text, diagrams, and symbols when solving problems; record their results clearly; explain their methods and reasoning

1.2 Key Stage 2

By the end of Year 6 pupils:

Place value

- handle place value to represent and order numbers to 10 000 and beyond; extend this to negative integers and decimals; work with decimals and measures involving tenths, hundredths and thousandths; multiply and divide integers and decimals by 10, 100, 1000

- round integers, decimals and measures to the nearest “ten”, integer, or tenth

Recalling facts; correct use of language and symbols

- recall instantly addition and subtraction facts for numbers to 20; “know by heart” multiplication tables to 10 × 10 and corresponding division facts; factorise any two-digit integer; recognise primes and squares
- use mathematical language and notation correctly; understand that some statements are **exact** and can be clearly demonstrated

Structural arithmetic

- add and subtract positive and negative integers; multiply and divide positive integers; use place value and the structure of arithmetic to **simplify** calculations
- work flexibly with fractions and percentages; switch freely between equivalent fractions; add and subtract simple fractions
- understand the order of operations and the use of brackets

Calculating

- add and subtract any two two-digit integers mentally, and three-and four-digit integers using **standard written column format**
- multiply and divide mentally a two-digit integer by any one-digit integer; complete written short multiplication and division of three-digit and four-digit integers by numbers up to 12, and long multiplication of three-digit by two-digit integers

Geometry and measures

- copy simple figures; work with common 2D and 3D shapes; find unknown angles in simple figures; plot points with given coordinates
- measure and draw line segments accurate to the nearest millimetre and angles to the nearest degree; calculate reliably with standard measures; find the areas of rectangles and shapes made from rectangles, and the volumes of cuboids and shapes made from cuboids

- use and calculate with money; tell the time to the nearest minute; read scales—interpolating between marks; convert between related units

Solving problems

- tackle and solve **word problems** and simple **multi-step** problems involving numbers, measures, and shapes; make sensible estimates; make **connections** between topics; explain their reasoning

1.3 Key Stage 3

Key Stage 3 revisits important Key Stage 2 material—partly for revision, but mainly to reinterpret old material from a new "viewpoint" (extending the written algorithms to decimals, shifting the focus from bare hands mental methods to structural arithmetic and the simplification of expressions, etc.).

By the end of Year 9 pupils:

Place value

- handle **place value** to represent and order integers and decimals with up to six digits; multiply and divide by 10, 100, 1000
- work with decimals and measures involving up to four decimal places; write terminating decimals as fractions and vice versa, and know that some fractions have decimals that recur
- round numbers and measures freely and flexibly

Calculating

- use multiplication tables freely to multiply and divide mentally in context;
- compute with integers and decimals using **standard column format**
- compare and compute with fractions; work flexibly with fractions, **ratios**, percentages

Structural arithmetic

- use **place value**, factorisation, and the algebraic structure of arithmetic to **simplify** and to evaluate numerical expressions and calculations—including with fractions and negatives

- test for divisibility by 2, 3, 4, 5, 10; find HCFs and LCMs; factorise integers as a product of primes; recognise squares and cubes; find or estimate square roots
- represent numbers as powers—including simple fractional powers; work with powers of 10
- use the algebraic equivalence of expressions to **simplify** calculations

Simplification of algebraic expressions; solving linear equations

- use unknowns and variables in context (formulae); substitute values in expressions; use algebraic rules to **simplify** expressions and calculations—collect like terms, expand and factorise simple expressions; work with simple sequences
- set up and solve a single linear equation in one unknown in complete generality; use the rules of algebra to "change the subject of", or to rearrange, equations and formulae; solve two simultaneous linear equations; solve linear inequalities in one unknown

Geometry and measures

- measure and draw accurately; read scales; change units; understand and use basic formulae; find lengths, areas, and volumes for common 2D or 3D shapes—including triangles, parallelograms, circles, cuboids, and prisms; calculate reliably with standard measures
- plot points in all four quadrants; find the midpoint of a line segment, and the distance between two given points; understand and work with linear equations and straight line graphs; interpret gradient as a **ratio** or rate; use trig ratios in right-angled triangles
- use basic **ruler and compass constructions**, parallels, angles in a triangle, **angle-chasing**, congruence; establish a preliminary basis for 2D **Euclidean geometry**; prove and use *Pythagoras' Theorem*

Solving problems

- tackle and solve **word problems** and simple **multi-step** and **inverse** problems involving numbers, measures, symbols and shapes

- use the **unitary method** to solve **proportion** problems involving rates and **ratios**
- make sensible **estimates**
- make **connections** between topics; explain their reasoning

1.4 Key Stage 4

Key stage 4 revisits important Key Stage 3 material—partly as revision, but also to interpret it afresh. For some pupils, this re-working and strengthening of Key Stage 3 material (together with consolidation of Key Stage 2 material) will be their main goal at Key Stage 4; other pupils may supplement revision of Key Stage 3 material with a programme that covers selected parts of what is summarised here. Those who expect to continue to further academic studies beyond Key Stage 4 should aim to cover everything summarised here.

By the end of Year 11 pupils who complete the Key Stage 1–4 programme:

Number and measures

- handle (positive and negative) large numbers and decimals, with and without units, possibly expressed using powers or standard form
- move freely between fractions and decimals
- use rounding and **exact** arithmetic to work with **approximations**
- calculate **probabilities** in standard models; analyse sampled data

Calculating and simplifying

- compute with fractions; work flexibly with fractions, **ratios**, percentages
- solve problems involving **proportion** (including **the unitary method**)
- use algebraic structure and multiplication facts to simplify numerical expressions—including those involving fractions and powers
- calculate with surds and mixed surd expressions (without evaluating)

Algebra (expressions, formulae, equations, identities) and graphs

- use algebraic equivalence (including the index laws) to **simplify** expressions
- know, use, and rearrange standard formulae
- work in all four quadrants; work with equations of straight lines in 2D
- solve any linear equation or inequality in one unknown; solve any pair of simultaneous linear equations or inequalities in two unknowns; interpret the solutions graphically
- know and use standard quadratic identities; solve any quadratic equation or inequality; solve easy simultaneous equations—one linear and one quadratic; interpret solutions graphically
- understand linear and quadratic expressions in one variable as functions; sketch and analyse linear and quadratic graphs

Geometry

- use *Pythagoras' Theorem* to solve problems in 2D and 3D; find lengths, surface areas, and volumes for common 2D and 3D shapes—including regular polygons
- use basic trigonometry and the Sine and Cosine rules to "solve triangles"
- understand and use basic **ruler and compass constructions**
- understand how congruence, parallels, and similarity provide a basis for **Euclidean geometry**; use these to derive results and to solve problems
- understand, prove, and use the basic properties of circles
- understand how scaling affects angles, lengths, areas, and volumes analyse standard 2D and 3D figures—including prisms, pyramids, cylinders, cones, spheres

Solving problems

- solve **word problems** and simple **multi-step** and **inverse** problems

- make **connections** between topics; write well-presented **proofs**; explain their reasoning
- calculate with standard and compound measures; work with "rates"

2. Brief version

2.1 Key Stage 1

Breadth of study

1. During the Key Stage pupils should be taught the required *Knowledge, Skills, and Understanding* through:

(a) practical activity, exploration and discussion

(b) linking the language of mathematics with spoken and written English

(c) **learning key facts by heart**; learning to store tens and units temporarily in the mind (including as intermediate outputs in a longer calculation) to support the development of mental calculation strategies

(d) using mathematical ideas in practical activities, then recording these ideas using objects, pictures, diagrams, tables, words, and numbers

(e) developing rich mental calculation strategies, and **standard written procedures** for addition and subtraction

(f) drawing, measuring and estimating in a range of practical contexts

Knowledge, Skills, and Understanding

Teaching should ensure that appropriate connections are made between the section *Number and measures* and the section *Shape, space, and measures*.

Ma1 Number and measures

1. *Using and applying "Number and measures"*

Pupils should:

Solving problems

(a) explore, interpret, develop flexible approaches to, and persist with problems involving number and measures in a variety of forms

Communicating

(a) use correct language, symbols, and vocabulary associated with number and measures

(b) explain and record methods and results in spoken, pictorial, and written form

Reasoning

(a) present results in an organised way; sort and classify numbers according to given criteria

(b) understand that some statements are **exact** and can be clearly demonstrated

2. *Numbers and the number system*

Pupils should: **Counting**

(a) count reliably at first up to 20 objects, later extending counting to 100 and beyond (to 120 say), remaining secure across "tens boundaries" [e.g. from 19 to 20, or from 99 to 100]; recognise the invariance of quantity

(b) estimate a number of objects that can be checked by counting; round two digit numbers to the nearest 10

The base 10 number system

(a) understand the groupings into units and 10s (and later into 100s) that underpin **place value**; know what each digit represents (including 0 as a number, and as a placeholder), and how the "value" represented by each digit is determined by its position

(b) read and write two-digit and three-digit numbers in figures and words

(c) order two-digit numbers and position them on a number line; use =, <, and > and the associated language

Number patterns and sequences

(a) create, describe, and explore basic number patterns and sequences—including odd and even numbers, multiples of 2, multiples of 5, and multiples of 10

3. *Calculation*

Pupils should:

Number operations and the relationship between them

(a) understand addition and use related vocabulary and notation; understand subtraction (as "take away" and as "difference") and use the related vocabulary and notation; recognise that subtraction is the inverse of addition

(b) identify and use the calculations needed to solve simple **word problems** and **inverse** problems [e.g. oral "I'm thinking of a number" problems]

(c) understand simple instances of multiplication as repeated addition, and division as "grouping", and as "sharing"; use the vocabulary and notation associated with multiplication and division; find one half, or one quarter of a familiar shape, or of a small set of objects

Mental, informal, and standard written methods

(a) develop instant recall of number facts; know addition and subtraction facts with totals less than 10, and use these to derive other facts; learn addition facts with totals up to 20

(b) know ×2, ×5, and ×10 multiplication tables, and derive the corresponding division facts; know the doubles of numbers to 20 and the corresponding halves

(c) use practical and informal written methods to add and subtract two-digit numbers

(d) develop mental methods which flexibly use known facts to calculate the answer to less familiar "sums" [e.g. working out 4 × 6 by doubling 2 × 6, or by doubling 4 × 3]; add 10 to any single digit number, then add and subtract a multiple of 10 to or from a two-digit number

(e) make sense of number sentences involving all four operations

(f) lay out and complete simple two-digit additions and subtractions in **standard column format**

(g) use practical and informal written methods and related vocabulary to support multiplication and division, including calculations with remainder

4. ***Solving numerical problems***

Pupils should:

(a) choose sensible calculation methods to solve simple **word problems** involving whole numbers—including problems involving money or measures, drawing on their understanding of arithmetical operations

5. ***Processing, representing, and interpreting data***

Pupils should:

(a) solve suitable problems using simple lists, tables, and charts to sort, classify, and organise information; discuss the methods they use and explain what they find

Ma2: Shape, space, and measures

1. ***Using and applying "Shape, space, and measures"***

Pupils should:

Solving problems

(a) follow instructions to construct simple 2D and 3D objects; represent 3D objects via 2D drawings

Communicating

(a) use correct language and vocabulary for shape, space, and measures

(b) measure objects using ad hoc informal as well as standard measures; record measurements in ordered tables

Reasoning

(a) recognise simple spatial patterns and relationships; sort and classify shapes according to given criteria

2. ***Understanding properties of shapes, position, and movement***

Pupils should:

(a) describe relationships using the language "larger – smaller", "higher – lower", "longer – shorter", "above – below", "left of – right of"

(b) draw and describe properties of 2D and 3D shapes; recognise, name, and sort common 2D and 3D shapes—including triangles, rectangles (including squares), circles, cubes, cuboids, hexagons, pentagons, cylinders, pyramids, cones, and spheres

(c) recognise right angles; understand whole turns, and quarter- and half-turns (clockwise and anticlockwise)

3. ***Understanding measures***

Pupils should:

(a) use direct comparison to order objects by size, using appropriate language; put familiar events in chronological order

(b) measure and draw straight lines accurate to the nearest centimetre

(c) estimate, compare, and measure lengths, weights, and capacities; choose and use standard units (m, cm, kg, litre); compare durations (using seconds, minutes, hours, days); read and interpret numbers on scales to the nearest labelled division, interpreting the divisions between them; identify time intervals, including those that cross the hour

2.2 Key stage 2

Breadth of Study

1. During the Key Stage pupils should be taught the required *Knowledge, Skills, and Understanding* through:

(a) extending **place value** to larger integers and to simple decimals

(b) extending their understanding of the number system to include integers, fractions, and decimals

(c) **learning key facts by heart**; learning to store hundreds, tens and units temporarily in the mind (including as intermediate outputs in a longer calculation) to support the development of mental calculation strategies

(d) extending **exact** arithmetic to the **standard written algorithms** for integers and simple decimals

(e) using **structural arithmetic** to calculate efficiently and to develop (pre-)algebraic thinking

(f) drawing and measuring; using **exact** arithmetic to make good **estimates** when solving problems; recording results using words, pictures, numbers, diagrams, and tables (and symbols where appropriate)

(g) linking the language of mathematics with spoken and written English using carefully crafted problems; solving **word problems**; establishing **connections** between number work, measures, geometry, and practical tasks; distinguishing between sensible and misleading uses of mathematics

Knowledge, Skills, and Understanding

Teaching should ensure that appropriate connections are made between the section *Number and measures* and the section *Geometry and measures*.

Ma1 Number and measures

1. *Using and applying "Number and measures"*

Pupils should:

Solving problems

(a) extract numerical, geometrical, and logical information from simple problems expressed in words

(b) make **connections**; use integers, decimals, and fractions (and arithmetic) when solving problems involving measures, and in other settings

(c) solve **multi-step**, and simple **inverse** problems

(d) solve problems involving tables, lists, and information presented pictorially;

(e) use knowledge of **exact** arithmetic to make good mental **estimates**

Communicating

(a) use notation, terminology, symbols, and language correctly

(b) present results and solutions to problems clearly; explain reasoning, methods, and conclusions

(c) interpret tables, lists, and charts; construct and interpret frequency tables

Reasoning

(a) present results in an organised way; sort and classify numbers and shapes according to given criteria

(b) investigate apparent patterns; understand that some statements are **exact** and can be clearly explained

2. ***Numbers and the number system***

Pupils should:

Counting

(a) count reliably beyond 100, passing smoothly from any given set of "90s" onto the next hundred

(b) count on and back in steps of constant size, starting from any integer, extending to negative integers

The base 10 number system

(a) use **place value** in representing numbers first up to 1000, then up to 10 000 and beyond; extend to decimals with up to three decimal places

Number patterns and sequences

(a) recognise two- and three-digit multiples of 2, 5 and 10; find the factors of a given integer, and the common factors of two given integers; find the HCF and the LCM of two given integers; recognise prime numbers to 50, and square numbers to 10 × 10; find factor pairs and all the factors of any two-digit integer; double or halve any two-digit integer

Integers

(a) read, write (in figures and words), and order whole numbers to 10 000

(b) multiply, and divide, any integer by 10 or 100, and then by 1000; round integers to the nearest 10 or 100, and then 1000

(c) understand and use negative integers; order a given set of positive and negative integers

Integers and decimals

(a) use decimal notation for tenths, hundredths, and thousandths; order a set of numbers or measurements

(b) compare and order integers and decimals in different contexts; locate integers (positive and negative), fractions, and decimals on the number line; use correctly the symbols $=, \neq, <, \leqslant, >, \geqslant$

(c) multiply and divide, any integer or decimal by 10 or 100; round integers and decimals to the nearest integer, to the nearest ten, and to the nearest tenth

Fractions, percentages and ratio

(a) understand fractions; locate fractions on a number line; find fractional parts of shapes or quantities

(b) understand equivalent fractions; **simplify** by cancelling common factors

(c) order simple fractions

(d) understand percentage; use simple percentages for comparison; find fractions and percentages of whole number quantities, and express part of a given whole as a percentage; express one whole number quantity as a fraction of another

(e) divide a given quantity into two parts in a given **ratio** (both part-to-part and part-to-whole); compare quantities in a given (external) ratio; solve simple problems involving ratios

3. *Calculation*

Pupils should:

Number operations and the relationship between them

(a) develop their understanding of the four number operations—including inverses, and operations with zero

(b) find remainders after division; express a quotient as a fraction or decimal; relate $\frac{p}{q}$ to $p \div q$

(c) know and use the conventions for the order of operations; understand and use **structural arithmetic** to **simplify** calculations; write numerical expressions involving brackets; group related terms in a sum and related factors in a product to simplify, and hence evaluate, numerical expressions

Mental methods

(a) achieve instant recall of all addition and subtraction facts for integers up to 20

(b) add or subtract any pair of two-digit integers; handle suitable three-digit and four-digit additions and subtractions presented in written form

(c) add and subtract positive and negative integers mentally

(d) achieve instant recall of (i.e. **know by heart**) multiplication tables to 10×10 and use them to derive division facts

(e) multiply and divide in the range 1 to 100, then for larger numbers

(f) derive multiplication and division facts involving decimals

(g) relate fractions to multiplication and division; work with simple quotients as fractions and as decimals; switch freely between equivalent fractions; add and subtract simple fractions by reducing to a common denominator

Written methods

(a) use the **standard written method** in column format to add and subtract three-digit positive integers, then four-digit positive integers; add and subtract numbers involving decimals

(b) use the *standard written method* in column format for short multiplication (of two- and three-digit integers by a single digit multiplier), then long multiplication of two-digit and three-digit integers by two-digit multipliers; extend to simple decimal multiplication

(c) use short division of two-digit and three-digit integers by numbers up to 12

(d) use **approximations** and other strategies to check that answers are reasonable

Measures

(a) calculate reliably with standard measures, money, and time; convert measures from one unit to a related unit

(b) relate distance, time, and speed in uniform rectilinear motion; work with other simple rates and compound measures

4. *Solving numerical problems*

Pupils should:

(a) use the four number operations to solve **word problems** involving numbers, or money, or measures of length and area, mass, capacity, or time

(b) solve **multi-step** and **inverse** problems with confidence

(c) check that their results are reasonable; explain why their answers are correct

5. *Processing, representing, and interpreting data*

Pupils should:

(a) solve suitable problems using simple lists, tables, and charts to sort, classify, and organise information, discuss the methods they use, interpret their results, and explain what they find

(b) explore the notions of “centre” and “spread” for numerical data sets

Ma2: Geometry and measures

1. *Using and applying "Geometry and measures"*

Pupils should:

Solving problems

(a) recognise standard geometrical figures; use their properties to select and perform appropriate calculations; measure and draw accurately to construct 2D and 3D figures

(b) use standard units of measurement and simple compound measures; convert reliably between related units

Communicating

(a) use geometrical notation, terminology, and symbols correctly; interpret solutions to problems involving geometrical figures and measures; organise work and record findings clearly

Reasoning

(a) analyse standard 2D and 3D figures; calculate efficiently and make simple deductions with angles, lengths, areas, volumes, time, and other measures

2. *Understanding properties of shape*

Pupils should:

(a) recognise right angles, perpendicular and parallel lines; know that angles at a point total 360°, that angles at a point on a straight line total 180°, and that angles in a triangle total 180°

(b) describe relationships using the language "larger – smaller", "higher – lower", "longer – shorter", "above – below", "left of – right of", "top – bottom", "in front of – behind", "closer – further away", "between"

(c) talk clearly about common 2D and 3D shapes; visualise 3D shapes from 2D drawings

(d) make and draw shapes with increasing accuracy, and analyse their geometrical properties

3. ***Understanding properties of position and movement***

Pupils should:

(a) read and plot coordinates—eventually in all four quadrants; draw, or locate, shapes with given properties in the coordinate plane

(b) visualise, predict, and represent the position of a shape following a rotation, reflection, translation, or glide reflection

4. ***Understanding measures***

Pupils should:

(a) draw and measure lines to the nearest millimetre; combine linear measurements to measure perimeters

(b) draw and measure acute and obtuse angles of a given size to the nearest degree; estimate the size of given angles and order them; draw angles reliably as parts of compound shapes

(c) read the time to the nearest minute; calculate time intervals from clocks, from timetables, and from calendars

(d) use standard units of length, area, volume, mass, and capacity; measure and weigh items; convert between related units

(e) find areas of rectangles and of shapes composed of rectangles

(f) measure and compare capacities; understand conservation of volume; find volumes of cuboids and of simple shapes composed of cuboids

(g) read scales with increasing accuracy; record measurements using decimal notation

2.3 Key stage 3

Breadth of Study

1. During the Key Stage pupils should be taught the required *Knowledge, Skills, and Understanding* through:

(a) extending **place value** to arbitrary integers and decimals

(b) extending their understanding of numbers to include integers (positive and negative), fractions, and decimals

(c) extending **exact** arithmetic to the **standard written algorithms** for integers and decimals, and the standard procedures for calculating with fractions

(d) using **structural arithmetic** for efficient numerical calculation, and for algebraic **simplification** of numerical, fractional, and symbolic expressions

(e) representing unknowns and variables by letters; using formulae; solving linear equations; representing and interpreting straight lines and linear equations

(f) engaging in tasks that develop short chains of deductive reasoning and that bring out the centrality of **proof** in number, algebra, and geometry

(g) drawing and measuring; using **ruler and compass constructions**; calculating areas and volumes; recording results using diagrams, words, numbers, and symbols; **angle-chasing** and analysing more complex figures in terms of triangles

(h) linking the language of mathematics with spoken and written English; building simple logical expressions such as "... and ...", "... or ...", "if ..., then ...", "not only ..., but also ..."; interpreting carefully crafted "realistic" problems; solving **word problems**; distinguishing between sensible and misleading uses of mathematics

(i) routinely tackling familiar and unfamiliar problems, including **multi-step** and **inverse** problems; recognising that mathematical operations often come in 'direct-inverse' pairs, and that the inverse operation depends on robust fluency in the direct operation

(j) practical work in which they draw inferences from a mathematical analysis of data, and consider how statistics can be used to inform decisions

Knowledge, Skills, and Understanding

Teaching should ensure that appropriate connections are made between the section on *Number and algebra* and the section on *Geometry and measures*.

Ma1 Number and algebra

1. ***Using and applying "Number and algebra"***

Pupils should:

Solving problems

(a) use numerical, geometrical, and logical information in analysing data and in solving simple problems

(b) make connections; use the arithmetic of integers, decimals, and fractions when solving problems

(c) regularly solve **multi-step** problems and **inverse** problems

(d) solve problems involving measures, rates and compound measures, ratio and proportion; make and justify **estimates**

Communicating

(a) use spoken and written language, notation, diagrams, terminology, and symbols correctly

(b) recognise when information is presented in a misleading way

(c) present results and solutions to problems clearly, declare unknowns explicitly, and lay out solutions logically **line-by-line**

(d) interpret tables, lists, and information presented graphically; construct and interpret frequency tables; use precise measures of "centre" and "spread"

Reasoning

(a) understand that some statements can be clearly **proved**, and that other statements can be shown to be false

(b) use **place value** and **structural arithmetic** to **simplify** calculations and expressions; recognise and use the fact that mathematical operations often come in "direct-inverse" pairs

(c) use basic results and step-by-step deduction to draw conclusions; investigate apparent patterns and test the validity of statements—proving or disproving these statements conclusively where possible

2. *Numbers, the number system, structural arithmetic, simplification, and algebra*

Pupils should:

Counting and numbers

(a) count reliably forwards and backwards across hundreds and thousands boundaries

(b) solve problems involving **counting** [e.g. How many pages from page 171 to 263?—inclusive and exclusive; How many dots are in a 5 by 7 rectangular array? How many chords are there joining 10 points on a circle?]

(c) use **place value** in representing integers to 1 000 000, and decimals with up to four decimal digits; express position as a "power of 10"; choose the power of 10 to transform a given decimal to an integer (by multiplying)

Sequences; powers and roots

(a) recognise multiples of 2, 4, 5, 10, 20, 25, 50, 100; factorise instantly any output from multiplication tables to 10×10; recognise (or test quickly) prime numbers to 100 and test possible primes up to 500; recognise square numbers to 20×20; find all the factors of a given integer

(b) recognise powers of 2, powers of 3, and powers of 5; recognise square and cube roots of familiar squares and cubes; understand and find, or estimate, the square root of any positive number; use index notation for small positive integer powers

(c) find specified terms of a sequence given a **term-to-term** or a **position-to-term** rule; guess the simplest position-to-term rule for the nth term given the first few terms of a sequence

Integers and decimals

(a) read, write (in figures and words), and order whole numbers and decimals with up to six digits; understand, use, and calculate freely with (positive and negative) integers; use correctly the symbols $=$, $\neq$, $<$, $\leqslant$, $>$, $\geqslant$ and the associated language; order a set of positive and negative integers and decimals, or measurements

(b) use correctly the terms factor, multiple, common factor, common multiple; find and use the HCF and LCM of two given integers; test for divisibility by 2, by 3, by 4, by 5, by 9, by 10

(c) multiply, and divide, any integer or decimal by 10, 100, 1000, or 10 000; know the multiplicative complements for 10 (2×5), for 100, and for 1000, and the corresponding decimals [e.g. $\frac{1}{2} = 0.5$, $\frac{1}{5} = 0.2$, $\frac{1}{8} = 0.125$]; recognise as alternative representations the decimal and fraction forms of simple fractions

(d) express any given large number as a number less than 10 times a power of 10, and a small number as a number greater than or equal to 1 times a power of 10

(e) compare measurements (in various contexts); round integers and decimals

Fractions, percentages and ratio

(a) understand general fractions in terms of **unit fractions**; switch freely between mixed numbers (with fractional part < 1) and standard fractional form $\frac{p}{q}$

(b) find fractional parts of shapes and quantities, and recognise the fractional part represented; solve simple **ratio** problems

(c) understand equivalent fractions; express two given fractions with a common denominator; simplify a given fraction; order a list of integers and fractions

(d) understand "percentage" as a fractional operator with denominator 100; find fractions and percentages of given quantities; express one quantity as a fraction of another; use the multiplicative character of percentage as an operator in calculations involving percentage increase and decrease; distinguish between absolute and relative increase and decrease

(e) reduce a *ratio* to its simplest form, and establish the connection with "fractional parts"; divide a given quantity into two parts in a given *ratio*; solve problems involving **ratio and proportion**

3. *Calculation*

Pupils should:

Number operations and mental methods

(a) extend existing mental calculation to include negative numbers, decimals and fractions

(b) calculate effectively in solving problems

Structural arithmetic

(a) use multiplication tables freely to simplify fractional expressions; convert fractions to decimals and terminating decimals to their simplified fraction equivalents

(b) obtain the prime-power factorisation of a given integer by successive division

(c) understand and use **place value**, inverse operations [e.g. cancellation], and **structural arithmetic** to **simplify** calculations; represent numbers and roots as powers, including fractional powers; work with powers of 10

(d) understand why $(-1) \times (-1) = 1$ and why $a - (-b) = a + b$; use these to simplify and to evaluate numerical expressions

(e) use the idea of choosing a suitable (common) denominator to add, subtract, multiply and divide fractions

(f) solve **word problems** involving rates and **ratios**, including the **unitary method**

(g) give both roots of simple quadratic equations; simplify numerical expressions involving simple surds [e.g. $\sqrt{8} = 2\sqrt{2}$ because both are positive and have the same square]

Algebraic simplification

(a) substitute numerical values into **formulae** and **expressions** and evaluate; multiply out brackets, collect like terms, identify and take out common factors to simplify expressions; recognise that different-looking expressions may be identical; prove simple algebraic **identities**, and explain why two given expressions are not identical

Written methods

(a) relate decimal arithmetic to integer arithmetic; use **standard written methods** in column format for addition and subtraction, short and long multiplication, short (and long) division of integers and decimals

Inequalities

(a) solve simple linear **inequalities** in one variable and represent solutions on a number line

Measures

(a) calculate and work with perimeters, areas, volumes, durations, capacities, and simple compound measures; use standard units of length, area, volume, mass, and capacity; read scales with appropriate rounding; record and order measurements using decimal notation; convert between related units

(b) estimate the size of any given angle; draw and measure angles reliably to the nearest degree

(c) calculate reliably with measures; extract and use information from tables and charts; solve **word problems** involving money, time, length, and compound measures (speed, rates)

4. ***Algebra: equations, formulae, identities, and functions***

Pupils should:

(a) set up and solve linear equations in complete generality [e.g.

$$2 - \frac{3}{4}x = \frac{2 - 4x}{5}];$$

reduce a linear equation in two variables to standard form ($ax + by = c$, or $y = mx + c$); eliminate a variable from two simultaneous linear equations in two unknowns; solve linear inequalities in one unknown

(b) change the subject of a formula; draw the graph of a linear function, identifying its gradient, and interpreting its position; construct linear functions arising from real problems, sketch and interpret their graphs; establish the link to **ratio and proportion**

(c) use letters in general expressions; use index notation for small positive integer powers; **simplify** given expressions

(d) use algebra to find the exact solution of two simultaneous linear equations in two unknowns by eliminating a variable

(e) sketch the graphs of simple quadratic functions; solve simple quadratic equations

5. ***Solving numerical problems***

Pupils should:

(a) solve arithmetical problems, **word problems**, and geometry problems involving numbers and measures; check that their results are reasonable

(b) solve **multi-step** and **inverse** problems with confidence

(c) use the **unitary method** to solve **proportion** problems and problems involving ratios and rates

(d) use algebraic formulae; set up and solve equations

6. ***Processing, representing, and interpreting data***

Pupils should:

(a) solve problems involving lists, tables, charts, and graphs; sort, classify, and organise information; discuss the methods they use and explain what they find

(b) find the average (i.e. mean) and other measures of "centre", and measures of spread for small datasets; identify the modal class for grouped data; interpret frequency diagrams and histograms; use cumulative frequency

(c) use **counting** where each outcome is "equally likely" to calculate **probabilities**

Ma2: Geometry

1. ***Using and applying "Geometry"***

Pupils should:

Solving problems

(a) solve geometrical problems involving standard geometrical figures in 2D and 3D, and angles, length, area, and volume

(b) measure and calculate accurately to construct and analyse 2D and 3D figures; use standard units in geometry

Communicating

(a) use geometrical language, notation, terminology, and symbols correctly

(b) work in all four quadrants of the coordinate plane

(c) lay out calculations, constructions, and **proofs line-by-line**

Reasoning

(a) use basic geometrical principles to justify each step in a calculation or deduction

(b) analyse 2D and 3D configurations in terms of triangles

2. ***Constructing and analysing geometrical configurations***

Pupils should:

Know and analyse

(a) recognise right angles, perpendicular and parallel lines, and use the associated language precisely; know that angles at a point total 360°, and that angles at a point on a straight line total 180°

(b) know that two lines are parallel precisely when alternate angles are equal (or, equivalently, when corresponding angles are equal); prove and use the usual consequences (including the angle-sum in any triangle)

(c) use known angles and angle properties to find unknown angles in given configurations (i.e. **angle-chasing**)

(d) motivate the formula for the circumference of the circle and estimate π; solve related problems

(e) talk about and work with common 2D and 3D shapes (including triangles [e.g. right angled, isosceles, and equilateral], quadrilaterals [e.g. parallelograms, rhombuses, rectangles, squares, and trapezia], cuboids, and prisms); correctly copy drawings from the board; make and draw shapes with increasing accuracy, and analyse their geometrical properties

Constructions and congruence

(a) use ruler and protractor to draw triangles with given data; extract and apply the basic congruence criteria (SAS, SSS, ASA; RHS) to prove standard results (including that the base angles in any isosceles triangle are equal, and the converse)

(b) draw specified figures using "ruler" (i.e. *straightedge*) and compasses only; use the basic **ruler and compass constructions** to complete other constructions

Area and Pythagoras

(a) find the area of rectangles and shapes made from rectangles; find the area of right angled triangles and of general triangles; find the area of a general parallelogram

(b) relate the formula for the area of a circle to the formula for the circumference; use the formula to solve related problems

(c) state, **prove**, and use *Pythagoras' Theorem*

Circles

(a) understand and use the terms centre, radius, chord, diameter, circumference, tangent, arc, sector, segment

(b) **prove** the basic properties of a circle [e.g. centre and any chord form an isosceles triangle; angle in a semicircle is a right angle; tangent is perpendicular to radius; tangents form an external point are equal]; apply these results to solve problems

Volume and 3D

(a) calculate volumes of cuboids and shapes made of cuboids; calculate volumes of a "wedge" (half a cuboid), polygonal right prisms, and cylinders

(b) find lengths and angles in simple 3D figures by considering 2D cross-sections

Scaling and enlargement

(a) draw figures to scale; interpret distances, angles, and areas on maps and other scale drawings

Loci

(a) interpret a circle as a locus; interpret the perpendicular bisector of a given line segment as a locus

3. ***Coordinates and graphs***

Pupils should:

(a) read and plot coordinates in all four quadrants

(b) use *Pythagoras' Theorem* to calculate the distance between two given points (simple cases); find the coordinates of the midpoint of a line segment (simple cases)

(c) establish the link between straight lines in the coordinate plane and linear equations in x and y; understand that parallel lines have the same gradient; find the intersection of two given straight lines

(d) sketch the graphs of simple quadratic functions

(e) explore and use coordinates in 3D

2.4 Key stage 4

Breadth of Study

1. During the Key Stage pupils should be taught the required *Knowledge, Skills, and Understanding* through:

(a) activities that revisit and extend material from Key Stage 3, moving on to achieve **fluency** and **automaticity** in using a wide range of procedures

(b) using language, terminology, and logic precisely and correctly; linking the language of mathematics with spoken and written English

(c) learning basic facts and techniques by heart; using them to tackle two-step and **multi-step** exercises and **problems** in different contexts, and in solving unfamiliar problems (including **word problems**)

(d) exploiting **connections** between superficially different topics; compressing ideas and techniques

(e) recognising that operations often come in "direct-inverse" pairs, that the **inverse** operation is often the more demanding one, and that its mastery depends on robust fluency in the direct operation

(f) using **calculators** intelligently where needed, whilst avoiding inappropriate dependence

(g) extending **exact** arithmetic (without calculators) to fractions, surds, and numerical and algebraic expressions involving powers; routinely using algebraic structure to **simplify** numerical, fractional, and symbolic expressions

(h) making intelligent **estimates** and **approximations** and handling the associated calculations reliably

(i) combining **congruence** and **ruler and compass constructions, parallels**, and **similarity** to establish a formal basis for elementary **Euclidean geometry**

(j) working with tables and information presented graphically; drawing inferences from a mathematical analysis of data drawn from a population with inherent variability; considering how statistics can be used to inform decisions

Knowledge, Skills, and Understanding

Teaching should ensure that appropriate connections are made between the section on *Number and algebra* and the section on *Geometry*.

Ma1 Number and algebra

1. *Using and applying "Number and algebra"*

Pupils should:

Solving problems

(a) use numerical, algebraic, geometrical, and logical information in tackling problems in *Number and algebra*, in solving **word problems**, and in analysing data

(b) use the structure of arithmetic and the laws of algebra when working with integers, decimals, fractions, surds, and algebraic expressions in solving problems

(c) regularly solve **multi-step** problems and **inverse** problems

(d) make use of relevant **connections** between topics

(e) solve problems involving measures, rates and compound measures, ratio and **proportion**

(f) make and justify **estimates**

Communicating

(a) use spoken and written language, notation, diagrams, terminology, and symbols correctly

(b) recognise when information is presented in a misleading way

(c) present results and solutions to problems clearly, declare unknowns explicitly, and lay out solutions and **proofs** logically **line-by-line**

(d) interpret tables, lists, and charts; present information graphically

(e) construct and interpret frequency tables; use precise measures of "centre" and of spread

Reasoning

(a) investigate apparent patterns; generate, interpret, test, and prove (or disprove) simple conjectures

(b) use **place value**, index laws, and **structural arithmetic** to **simplify** calculations and expressions, and to justify the extension of known conventions (including $(-1) \times (-1) = 1$, $2^0 = 1$, $\cos 120^\circ = -\frac{1}{2}$)

(c) use known results and step-by-step deduction to draw conclusions

2. ***From numbers to algebra (including calculation)***

Pupils should:

Numbers and arithmetic

(a) use **place value** in calculating with decimals; work effectively with very large numbers

(b) multiply, and divide, any integer or decimal by any power of 10; know the multiplicative complements for powers of 10 [e.g. $1000 = 8 \times 125$], and the corresponding decimals [e.g. $\frac{1}{8} = 0.125$]; recognise the decimal forms of simple fractions

(c) understand and use **divisibility tests**

(d) understand why $(-1) \times (-1) = 1$; work with integers, decimals, fractions, and surds, **simplifying** routinely

(e) solve problems involving **counting**; understand and use the **product rule** for counting

(f) consolidate and extend short and long division

Measures

(a) compare measurements; round integers and decimals appropriately

(b) calculate and work with perimeters, areas, volumes, durations, capacities; use standard units of length, area, volume, mass, capacity, and simple compound measures (speed, density, and other "rates"); read scales with appropriate rounding; record and order measurements using decimal notation; change between related units—in numerical and algebraic contexts; solve **word problems** involving money, time, length, and compound measures

Bounds and estimation

(a) understand the limits of accuracy implied by a given measurement in decimal form and interpret the result of an arithmetical calculation

(b) establish bounds on the accuracy of an **estimate** and understand how this affects a calculation

Integer factorisation, fractions, and surds

(a) use the terms factor, multiple, common factor, common multiple; find the *HCF* and *LCM* of given integers

(b) recognise (or test quickly) prime numbers to 120; use the "square root test" to identify primes to 1000

(c) obtain the prime power factorisation of a given integer; list all factors of a given integer

(d) move freely between "mixed" fractions and fractions in standard fractional form $\frac{p}{q}$; reduce a given fraction to lowest terms; rewrite two given fractions with a common denominator; order a list of fractions

(e) use factorisation to simplify surd expressions [e.g.

$$\sqrt{12} = \sqrt{2^2 \times 3} = 2 \times \sqrt{3}]$$

Fractions

(a) understand the **unit fraction** $\frac{1}{q}$ as "that part, of which q identical copies make 1"; understand a general fraction $\frac{p}{q}$ as a multiple $p \times \frac{1}{q}$ of a unit fraction; move freely from a given fraction to a suitable equivalent fraction

(b) add and subtract fractions; multiply and divide fractions; simplify, and hence evaluate, compound expressions involving fractions

Fractions and decimals

(a) move freely between terminating decimals and decimal fractions

(b) know the equivalence of the **exact** (unevaluated) fraction notation $\frac{p}{q}$ and the result of evaluating $p \div q$; find the decimal of any given fraction; understand why the decimal form of $\frac{p}{q}$ must terminate, or recur

(c) change any terminating decimal into a fraction in its lowest terms; change any recurring decimal into a fraction

Surds

(a) recognise $\sqrt{k}$ (for $k > 0$) as the **exact** positive real number whose square is equal to k; given $k > 0$ find the exact or approximate value of $\sqrt{k}$; use the algebra of surds—including rationalising denominators [e.g.

$$\frac{1}{\sqrt{2}+1} = \sqrt{2} - 1];$$

use surds (and π) to calculate *exactly* in geometric contexts; give lengths arising from applications of *Pythagoras' Theorem* and solutions to quadratic equations in exact (mixed surd) form

(b) use the standard notation for, and calculate with, cube roots;

Powers, roots, and the index laws

(a) factorise instantly any output from multiplication tables to 10×10; recognise square numbers to 25×25; recognise cubes to 6^3.

(b) recognise powers of 2, 3, 4, 5; recognise square and cube roots of familiar squares and cubes; extend powers and roots to simple fractions and decimals; find, or estimate, the square root or cube root of any positive number

(c) know, understand, and use the index laws; use index notation to present expressions in simplified power form; calculate freely with numerical and algebraic expressions involving powers

(d) write any given number in standard form and translate a given standard form into the (approximate) number it represents; calculate with numbers given in standard form "as though they are exact"

Fractions, decimals, and percentages

(a) find and recognise fractional parts of shapes and quantities; express one quantity as a fraction of another

(b) understand percentage as a fractional operator with denominator 100; know and use the percentage equivalents of familiar fractional parts;

work freely with percentages; use the multiplicative character of percentage increase and decrease; solve problems involving percentage change (including inverse problems and compound interest)

Sequences

(a) work with standard integer sequences; generate terms given a **term-to-term** rule, or a **position-to-term** rule; guess the simplest position-to-term rule for the nth term given the first few terms of a sequence

(b) use a given term-to-term rule to find a **closed formula** for the position-to-term rule

(c) find the term-to-term rule and the position-to-term rule for sequences **defined intrinsically**

(d) understand that when $x < 1$ (or $|x| < 1$) the sequence of powers (x^n) tends rapidly to 0, and when $x > 1$ (or $|x| > 1$) the sequence of powers (x^n) grows rapidly without bound; link to compound interest, to population growth, to doubling times and to radioactive half-life

Ratio and proportion

(a) divide a given quantity into two parts in a given part-to-part, or part-to-whole **ratio**; express the division of a quantity into two parts as a ratio; work with separate quantities in a given (external) ratio; reduce a ratio to its simplest form

(b) calculate the result of a change of units; draw and use scale diagrams and maps; understand the effect of scaling and enlargement on different quantities (including angles, lengths, areas, and volumes)

(c) solve proportion problems (where three of the four variables are given, determine the fourth); use the **unitary method**, and then the general method, to solve *proportion* problems

(d) understand and use "X is inversely proportional to Y" as meaning "X is proportional to $\frac{1}{Y}$"

Algebraic expressions

(a) substitute numerical values into **formulae** and **expressions**

(b) multiply out brackets, collect like terms, and take out common factors to simplify linear, quadratic and higher order expressions; *simplify* general expressions (possibly involving powers and roots) by using additive simplification, the distributive law, and cancellation—giving answers in factorised form; work with algebraic fractions having linear and quadratic denominators

(c) rearrange formulae; solve problems using standard formulae

(d) set up linear equations; solve the general linear equation in one unknown

(e) set up linear equations in two unknowns; interpret a linear equation in two unknowns in the coordinate plane as representing a straight line; draw the graph of a linear function, identifying its gradient, and interpreting its position; find the gradient from an equation given in any form; transform a given equation into the form $y = mx + c$ (or $x = a$); construct linear functions arising from real problems, sketch and interpret their graphs; establish the link to **ratio and proportion**

(f) solve any pair of simultaneous linear equations by eliminating a variable; interpret the analytic solution as "finding the point of intersection" (if any) of the two lines

(g) factorise quadratic expressions in one variable; solve quadratic equations by factorising; interpret solutions as those points where the graph crosses the x-axis; solve fractional equations that reduce to quadratics [e.g.

$$\frac{1}{x+1} = \frac{x-1}{x}];$$

(h) factorise and use the difference of two squares; conclude that, if $k > 0$, the equation $x^2 = k$ has two solutions ($\pm\sqrt{k}$); interpret this as a statement about the graph of $y = x^2 - k$

(i) know and use the expansion of $(x + a)^2$; extend to $(x + a)^3$; use this to "complete the square" for any given quadratic; obtain the formula for the solutions of the general quadratic

$$y = ax^2 + bx + c;$$

use this formula to solve quadratic equations; deduce the symmetry of the graphs of quadratic functions

(j) solve two simultaneous equations where one is linear and the other quadratic; use completing the square to find the centre and radius, given the equation of a circle; find the points where two circles intersect

(k) understand the difference between an equation and an **identity**; decide whether two given expressions are identical or not—then **prove** they are, or show that they are not

(l) solve linear inequalities in one and two variables; interpret the solution graphically

3. *Coordinates, graphs, and functions*

Pupils should:

(a) read and plot coordinates in all four quadrants; move freely between straight lines in the coordinate plane and linear equations in x and y; derive the equation of a line through two given points, and the equation of a line through a given point with a given gradient

(b) find the coordinates of the midpoint of a line segment; calculate the distance between two points in 2D or 3D

(c) interpret straight line graphs arising in real situations

(d) know and use the general form $y = mx + c$ (or $x = a$) for a straight line; use gradient and intercept; find the point of intersection of two given straight lines

(e) know that parallel lines have the same gradient; prove and use the fact that two lines with gradients m and m' are perpendicular precisely when $m \cdot m' = -1$

(f) for particular values of m and c interpret the standard form $y = mx + c$ as '$Y = mX$' relative to an origin at $(0, c)$ or at $(-\frac{c}{m}, 0)$

(g) sketch the graph of any given quadratic function by completing the square

(h) sketch other graphs—including simple cubic functions, the reciprocal function $y = \frac{1}{x}$, the exponential function $y = k^x$ for easy (positive) values of k, the circular functions $y = \sin x$, $y = \cos x$, and $y = \tan x$

(i) use coordinates to solve simple problems in 3D

4. ***Processing, representing, and interpreting data***

Pupils should:

(a) engage in practical and theoretical work to construct and interpret tables, lists, and information presented graphically; use precise measures of "centre" and spread; sort, classify, and organise information

(b) discuss variability; distinguish between data representing a single idealised measure and informal "random variables" **sampled** from a **population** or distribution

(c) calculate the mean of a set of numbers or measures; use mode or median as appropriate to summarise the "centre"; identify the modal class for grouped data; refine measures of spread and "central tendency"

(d) introduce ideas of **probability** via standard examples of discrete **sample spaces** in which each outcome is equally likely; explore the general notion of an "event"

(e) understand why

$$prob(A \cup B) = prob(A) + prob(B)$$

for disjoint events A, B; use simple counting to calculate probabilities in discrete sample spaces; understand and use the inclusion/exclusion formula

$$prob(A \cup B) = prob(A) + prob(B) - prob(A \cap B)$$

for events which are not necessarily disjoint

Ma2: Geometry

1. ***Using and applying "Geometry"***

Pupils should:

Solving problems

(a) know and understand basic **ruler and compass constructions**; use these to devise simple constructions

(b) use standard units in geometry; solve geometrical problems in 2D and 3D involving calculation, construction, and deduction

(c) measure and calculate accurately to construct and analyse 2D and 3D figures in terms of triangles; use known results to construct simple proofs

Communicating

(a) use geometrical language, notation, terminology, and symbols correctly

(b) work in all four quadrants of the coordinate plane; interpret a given equation as the graph of a function or a circle

(c) lay out calculations, constructions, and **proofs line-by-line**

Reasoning

(a) use the basic principles of **Euclidean geometry** and results derived from them to justify each step in a calculation, construction, or deduction

(b) analyse 2D and 3D configurations [e.g. by singling out, and using known properties of triangles]

2. ***From naïve construction to Euclidean geometry***

Pupils should:

Ruler and compass constructions revisited and organised

(a) know that two points A, B determine a line AB, a line segment $\overline{AB}$, and a circle with centre A and radius $\overline{AB}$; relate this to ideal **ruler and compass constructions**

(b) know and use the conventional notation for labelling the angles and sides of $\triangle ABC$

(c) accept and use the SAS, SSS, ASA (and later RHS) congruence criteria; prove the basic properties of isosceles triangles; justify the basic ruler and compass constructions and use them to devise other constructions

(d) prove that the perpendicular bisector of a given line segment $\overline{BC}$ is the locus of points X equidistant from B and C; construct the circumcentre of any triangle

(e) recognise the "perpendicular distance" from a point X to a line as the (shortest) distance to the line; prove the angle bisector of $\angle BAC$ is the (part-) locus of points equidistant from the lines AB and AC; construct the incentre of $\triangle ABC$

(f) prove that the three altitudes of a triangle are concurrent

Parallel lines and angles in a triangle

(a) know that angles at a point total 360°, and that angles at a point on a straight line total 180°; conclude that "vertically opposite angles are equal"

(b) recognise that "two lines are parallel precisely when alternate angles (or equivalently, when corresponding angles) created by any transversal are equal"; derive the basic properties of a parallelogram and of a rhombus; where possible prove the converse results; prove and use the *Midpoint Theorem*

(c) prove that the angles in any triangle add to 180° and that the exterior angle at any vertex is equal to the sum of the two interior opposite angles; deduce that the angles in any quadrilateral add to 360°; calculate the angle-sum in an n-gon, and the angle size in a regular n-gon

(d) combine known results about angles to find unknown angles, and to show that certain pairs of lines are parallel

(e) know and use the fact that the tangent and radius at a point on a circle are perpendicular; conclude that tangents from an external point are equal; prove that the angle subtended by a chord on the major arc is half the angle subtended at the centre O; conclude that angles subtended in the same segment are equal, and that opposite angles of a cyclic quadrilateral add to 180°; prove and use the *Alternate Segment Theorem*

(f) prove that the area of a parallelogram is equal to that of a rectangle on the same base and between the same parallels and deduce the formula for the area of a triangle; use this to prove *Pythagoras' Theorem*

Similarity

(a) establish and use the AAA similarity criterion and the SAS similarity criterion for general triangles; prove basic results using similarity

(b) extend the *Midpoint Theorem* to divide a given segment into any number of equal parts; prove and use the *Intercept Theorem*

3. *Geometric calculation*

Pupils should:

Trigonometry

(a) show that the standard trig ratios for acute angles θ depend only on the angle θ; understand that $\sin\theta$, $\cos\theta$ take values between 0 and 1

(b) find the exact values for $\theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ$; plot graphs of $y = \sin\theta$, $y = \cos\theta$, and $y = \tan\theta$ for $0^\circ \leqslant \theta < 90^\circ$; understand why $\cos\theta = \sin(90^\circ - \theta)$

(c) calculate missing lengths and angles in a given triangle ABC

(d) given triangle ABC, derive and use the formula

$$\text{area}(ABC) = \frac{1}{2}ab\sin C;$$

deduce the *Sine Rule* and use it to "solve triangles"; prove that

$$\frac{a}{\sin A} = 2R,$$

where R is the circumradius of triangle ABC

(e) show that on the unit circle with centre at the origin O, the point P for which the radius OP makes an angle θ with the positive x-axis has coordinates $(\cos\theta, \sin\theta)$; apply *Pythagoras' Theorem* to derive the identity

$$\sin^2\theta + \cos^2\theta = 1;$$

use this identity to find values of $\cos\theta$ given the value of $\sin\theta$ (and *vice versa*), and the value of $\tan\theta$ given the value of $\cos\theta$

(f) prove the *Cosine Rule*, and use it to find unknown lengths and angles in triangles and other 2D and 3D figures

(g) extend the definition of $\sin\theta$ and $\cos\theta$ to $\theta > 90^\circ$; extend the graphs of $y = \sin\theta, y = \cos\theta$ to $180^\circ < \theta < 0^\circ$, and to $-180^\circ < \theta < 0^\circ$

(h) show that in the "ambiguous (ASS) case", the data may determine two possible triangles

2D and 3D figures

(a) work freely with standard 2D figures

(b) draw figures to scale; interpret maps and other scale drawings; apply similarity in analysing problems; understand how enlargement and scaling (or similarity) affects angles, lengths, areas, and volumes

(c) find lengths and angles in 3D figures by considering 2D cross-sections; calculate the angle between two planes

(d) calculate surface areas and volumes of standard figures

Circles

(a) understand and use the terms centre, radius, chord, diameter, circumference, tangent, arc, sector, segment

(b) understand and use the formula for the circumference of a circle; calculate the length of circular arcs

(c) relate the formula for the area of a circle to the formula for the circumference; calculate the area of a sector

(d) calculate the circumradius and inradius of a triangle

(e) use *Pythagoras' Theorem* to find the equation of a circle of radius r centred at the origin and at the point (c, d); complete the square to identify easy quadratic equations as circles and find their centre and radius

(f) find the equation of the tangent to a given circle at a specified point

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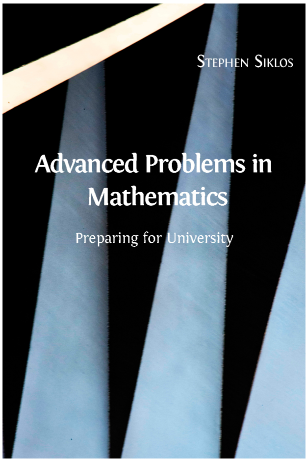

This book is intended to help candidates prepare for STEP examinations. STEP (Sixth Term Examination Paper) is an examination used by Cambridge colleges as the basis for conditional offers. They are used by Cambridge, Warwick, and many other mathematics departments recommend that their applicants practice on the past papers even if they do not take the examination. Advanced Problems in Mathematics is also recommended as preparation for any undergraduate mathematics course, even for students who do not plan to take the Sixth Term Examination Paper. The questions are all based on recent STEP questions selected to address the syllabus for Papers I and II, which is the A-level core (i.e. C1 to C4) with a few additions. Each question is followed by a comment and a full solution. The comments directs the reader's attention to key points and puts the question in its true mathematical context. The solutions point students to the methodology required to address advanced mathematical problems critically and independently. This book is a must read for any student wishing to apply to scientific subjects at university level and for anybody interested in advanced mathematics.

Teaching Mathematics at Secondary Level

Tony Gardiner

Teaching Mathematics is nothing less than a mathematical manifesto. Arising in response to a limited National Curriculum, and engaged with secondary schooling for those aged 11–14 (Key Stage 3) in particular, this handbook for teachers will help them broaden and enrich their students' mathematical education. It avoids specifying how to teach, and focuses instead on the central principles and concepts that need to be borne in mind by all teachers and textbook authors—but which are little appreciated in the UK at present.

This study is aimed at anyone who would like to think more deeply about the discipline of 'elementary mathematics', in England and Wales and anywhere else. By analysing and supplementing the current curriculum, *Teaching Mathematics* provides food for thought for all those involved in school mathematics, whether as aspiring teachers or as experienced professionals. It challenges us all to reflect upon what it is that makes secondary school mathematics educationally, culturally, and socially important.

Tony Gardiner, former Reader in Mathematics and Mathematics Education at the University of Birmingham, was responsible for the foundation of the United Kingdom Mathematics Trust in 1996, one of the UK's largest mathematics enrichment programs. In 1997 Gardiner served as President of the Mathematical Association, and in 2011 was elected Education Secretary of the London Mathematical Society.

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