TECH CHECK

Using Desmos to Find the GCD

To find the GCD of a set of numbers in Desmos, type gcd(first_number,second_number...) and Desmos will display the GCD of the numbers, as the numbers are typed!

VIDEO

[Using Desmos to Find the GCD (https://openstax.org/r/Using_Desmos_to_Find_the_GCD)](https://openstax.org/r/Using_Desmos_to_Find_the_GCD)

Applications of the Greatest Common Divisor

The greatest common divisor has uses that are related to other mathematics (reducing fractions) but also in everyday applications. We'll look at two such applications, which have very similar underlying structures. In each case, something must divide the groups or measurements equally.

EXAMPLE 3.18

Calculating Floor Tile Size

Suppose you have a rectangular room that is 570 cm wide and 450 cm long. You wish to cover the floor of the room with square tiles. What's the largest size square tile that can be used to cover this floor?

Solution

Using squares means that the length and width of the tiles are equal. To ensure we are using full tiles, the side length of the square tiles must divide the length of the room and the width of the room. Since we want the largest square tiles, we need the GCD of the width and length of the room or the GCD of 570 and 450.

Step 1: Find the prime factorizations of the numbers.

The prime factorization of 570 is $2 \times 3 \times 5 \times 19$.

The prime factorization of 450 is $2 \times 3^2 \times 5^2$.

Step 2: Identify the prime factors that appear in every number's prime factorization.

The common prime factors are 2, 3, and 5.

Step 3: Identify the smallest exponent of each prime identified in Step 2 in the prime factorizations.

The exponents of 2 in the prime factorizations of 570 and 450 are 1 and 1. So the smallest exponent for 2 is 1.

The exponents of 3 in the prime factorizations of 570 and 450 are 1 and 2, so the smallest exponent for 3 is 1.

The exponents of 5 in the prime factorizations of 570 and 450 are 1 and 2, so the smallest exponent for 5 is 1.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

This gives $2^1 \times 3^1 \times 5^1 = 30$. The GCD of 450 and 570 is 30, so the largest size square tile that can be used to cover the floor is 30 cm by 30 cm.

YOUR TURN 3.18

1. You are designing a brick patio made of square bricks 5 cm thick, but you need to determine the width and length of those bricks. The patio will be 400 cm by 540 cm. What are the largest size square bricks that can be used so that you do not need to cut any bricks?

EXAMPLE 3.19

Organizing Books Per Shelf

Suppose you want to organize books onto shelves, and you want the shelves to hold the same number of books. Each shelf will only contain one genre of book. You have 24 sci-fi, 42 fantasy, and 30 horror books. How many books can go on each shelf?

Solution

Since we want shelves that hold an equal number of books, and a shelf can only hold one genre of book, we need a number that will equally divide 24, 42, and 30. So, we need the GCD of the number of books of each genre or the GCD of 24, 42, and 30.

Step 1: Find the prime factorizations of the numbers.

The prime factorization of 24 is $2^3 \times 3$.

The prime factorization of 42 is $2 \times 3 \times 7$.

The prime factorization of 30 is $2 \times 3 \times 5$.

Step 2: Identify the prime factors that appear in every number's prime factorization.

The common prime factors are 2 and 3.

Step 3: Identify the smallest exponent of each prime identified in Step 2 in the prime factorizations.

The smallest exponent of 2 and 3 in the factorizations is 1.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

This gives $2^1 \times 3^1 = 6$. The GCD of 24, 42, and 30 is 6, so the largest number of books that can go on a shelf is 6.

YOUR TURN 3.19

1. There are three gym classes. The number of students in the classes is 21, 35, and 28. What is the largest team size that can be formed if teams from every class must have the same number of students?

VIDEO

[Applying the GCD (https://openstax.org/r/Applying_the_GCD)](https://openstax.org/r/Applying_the_GCD)

Finding the Least Common Multiple

The flip side to a divisor of a number is a **multiple** of a number. For example, 5 is a divisor of 45 and so 45 is a multiple of 5. More generally, if the number a divides the number b, then b is a multiple of a.

This drives the idea of the least common multiple of a set of numbers. A common multiple of a set of numbers is a multiple of each of those numbers. For instance, 45 is a common multiple of 9 and 5, because 45 is a multiple of 9 (9 divides 45) and 45 is also a multiple of 5 (5 divides 45). The **least common multiple** (LCM) of a set of number is the smallest positive common multiple of that set of numbers.

There are (at least) three ways to find the LCM of a set of numbers, and we will explore two of them. One way is to create a list of multiples of each number in the set, and then identify the smallest multiple that appears in those lists.

EXAMPLE 3.20

Finding the Least Common Multiple Using Lists

Find the LCM of 24 and 90 by listing multiples and choosing the smallest common multiple.

Solution

Create a list of the multiples of each number.

Step 1: The first 15 multiples for 24:

24, 48, 72, 96, 120, 144, 168, 192, 216, 240, 264, 288, 312, 336, 360.

Step 2: The first 15 multiples for 90:

90, 180, 270, 360, 450, 540, 630, 720, 810, 900, 990, 1,080, 1,170, 1,260, 1,350.

There is one multiple common to these lists, which is 360. So, 360 is the LCM of 24 and 90.

YOUR TURN 3.20

1. Use lists to find the LCM of 12 and 15.

The second method we can use is to find the prime factorizations of the number in the set to build the LCM of the numbers based on the prime divisors of the numbers. The LCM can be built from the prime factorization of the numbers in the set in a similar way as when finding the greatest common divisor. Here are the steps for using the prime factorization process for finding the LCM.

Step 1: Find the prime factorization of each number.

Step 2: Identify each prime that is present in any of the prime factorizations.

Step 3: Identify the largest exponent of each prime identified in Step 2 in the prime factorizations.

Step 4:. Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

EXAMPLE 3.21

Finding the Least Common Multiple Using Prime Factorization

Use the prime factorizations of 24 and 90 to identify their LCM.

Solution

Step 1: Find the prime factorization of each number.

$24 = 2^3 \times 3$

$90 = 2 \times 3^2 \times 5$

Step 2: Identify each prime that is present in any of the prime factorizations.

The prime numbers present in the prime factorizations are 2, 3, and 5.

Step 3: Identify the largest exponent of each prime identified in Step 2 in the prime factorizations.

Prime	2	3	5
Largest exponent	3	2	1

Step 4: Compute the LCM by multiplying the prime factors identified in Step 2 raised to the powers identified in Step 3.

The LCM for 24 and 90 is $2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5^1 = 360$.

YOUR TURN 3.21

1. Use prime factorization to find the LCM of 20 and 28.

EXAMPLE 3.22

Finding the Least Common Multiple Using Prime Factorization

Use the prime factorizations of 36, 66, and 250 to identify the LCM.

Solution

Step 1: Find the prime factorization of each number.

$36 = 2^2 \times 3^2$

$66 = 2 \times 3 \times 11$

$250 = 2 \times 5^3$

Step 2: Identify each prime that is present in any of the prime factorizations.

The prime numbers present in the prime factorizations are 2, 3, 5, and 11.

Step 3: Identify the largest exponent of each prime identified in Step 2 in the prime factorizations.

Prime	2	3	5	11
Largest exponent	2	2	3	1

Step 4: Compute the LCM by multiplying the prime factors identified in Step 2 raised to the powers identified in Step 3.

The LCM for 36, 66, and 250 is $2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 5 \times 11 = 2^2 \times 3^2 \times 5^3 \times 11^1 = 49{,}500$.

YOUR TURN 3.22

1. Use the prime factorization method to find the LCM of 150, 240, and 462.

Using lists for three or more numbers, particularly larger numbers, could take quite a bit of time. Frequently, as in this example, the prime factorization process is much quicker. In practice, you can use either listing or prime factorization to find the LCM.

EXAMPLE 3.23

Finding the Least Common Multiple Using Both Methods

Find the LCM of 20, 36, and 45 using lists and prime factorization.

Solution

Step 1: Use listing. List the multiples:

20	20, 40, 60, 80, 100, 120, 140, 160, 180, 200, 220
36	36, 72, 108, 144, 180, 216, 252, 288, 324, 360
45	45, 90, 135, 180, 225, 270, 315, 360, 405, 450, 495, 540

The smallest value that appears on all the lists is 180, so 180 is the LCM of 20, 36, and 45.

Step 2: Find the prime factorization of each number.

$20 = 2^2 \times 5$

$36 = 2^2 \times 3^2$

$45 = 3^2 \times 5$

Step 3: Identify each prime that is present in any of the prime factorizations.

The prime numbers present in the prime factorizations are 2, 3, 5.

Step 4: Identify the largest exponent of each prime identified in Step 3 in the prime factorizations.

Prime	2	3	5
Largest exponent	2	2	1

Step 5: Compute the LCM by multiplying the prime factors identified in Step 3 raised to the powers identified in Step 4.

The LCM for 20, 36, and 45 is $2^2 \times 3^2 \times 5^1 = 180$.

Both listing and prime factorization produced the same result: the LCM is 180.

YOUR TURN 3.23

1. Find the LCM of 18, 24, and 40 using lists and prime factorization.

VIDEO

Finding the LCM (https://openstax.org/r/Finding_the_LCM)

TECH CHECK

Using Desmos to Find the LCM

To find the LCM of a set of numbers in Desmos, type lcm(first_number,second_number...) and Desmos will display the LCM of the numbers, as the numbers are typed!

VIDEO

Using Desmos to find the LCM (https://openstax.org/r/Using_Desmos_to_find_the_LCM)

Applications of the Least Common Multiple

Some applications of LCM involve events that repeat at fixed intervals, such as visits to a location. Other applications involve getting things to be of equal magnitude when using parts that are different sizes (see Example 3.25, for instance). In each case, we may be looking to determine when two processes "line up."

EXAMPLE 3.24

Determining Scheduling Overlap Using the Least Common Multiple

Two students, João, and Amelia, meet one day at an assisted living facility where they volunteer. João volunteers every 6 days. Amelia volunteers every 10 days. How many days will it be until they are both volunteering on the same day again?

Solution

If we list the days that each student will volunteer, it becomes clear that we could solve this problem using the LCM of 6 and 10.

João will be at the assisted living facility 6, 12, 18, 24, 30, 36, 42, and 48 days later.

Amelia will be at the assisted living facility 10, 20, 30, 40, 50, and 60 days later.

The smallest number appearing on both lists is 30. João and Amelia will once more be volunteering together 30 days later.

YOUR TURN 3.24

1. The sun, Venus, and Jupiter all line up on a given day. Venus orbits the sun once every 255 days. Jupiter orbits the sun every 4,330 days (we'll ignore the decimal values of days for each orbit). How many days will it be until they line up again?

EXAMPLE 3.25

Determining the Minimum Height Using the LCM

A team-building exercise has teams build a house of cards as high as possible. However, the cards for different teams are of different sizes. Team 1 uses 10 cm × 10 cm cards, while Team 2 uses 8 cm × 8 cm cards. What is the minimum height when the two teams will be tied? Ignore the width of the cards.

Solution

This is an example where we want to put together objects with different sizes. We want to know the minimum height when they are tied, or when the houses of cards line up the first time. The heights of the towers built using the 10 cm × 10 cm cards will be 10, 20, 30, 40, 50, and 60 cm tall. When the 8 cm × 8 cm cards are used, the tower will be 8, 16, 24, 32, 40, 48, and 56 cm tall. The smallest number appearing on both lists is 40. The first time they are tied is when the two towers are 40 cm tall.

YOUR TURN 3.25

1. In an Internet giveaway, every 130th person who submits a survey receives $250, and every 900th person receives a free cell phone. How many submissions must be received for the first person to receive both prizes?

VIDEO

Application of LCM (https://openstax.org/r/Application_of_LCM)

WORK IT OUT

Prime Number Life Cycles—Cicadas

Cicadas are known to have life cycles of 13 or 17 years, which are prime numbers. Why would a prime number life cycle be an evolutionary advantage? To figure this out, we have to explore how common multiples work with prime numbers.

Make a conjecture regarding the LCM of a prime number and another number. Test this conjecture with a few examples of your own making.

Check Your Understanding

1. Identify which of the following numbers are prime and which are composite.
   31, 56, 213, 48, 701
2. Find the prime factorization of 4,570.
3. Find the greatest common divisor of 410 and 144.
4. Find the least common multiple of 45 and 70.
5. You want to fill gift bags for children in the after-school program where you volunteer. You have 30 crayons, 20

sticker sheets, and 70 bite-sized candies. If each gift bag must contain the same number of crayons, sticker sheets, and bite-sized candies, what is the maximum number of bags that can be filled?

SECTION 3.1 EXERCISES

For the following exercises, use divisibility rules to determine if each of the following is divisible by 2.

1. 24
2. 37
3. 1,345,321

For the following exercises, use divisibility rules to determine if each of the following is divisible by 3.

4. 48
5. 210
6. 5,345,324

For the following exercises, use divisibility rules to determine if each of the following is divisible by 5.

7. 130
8. 237
9. 1,345,321

For the following exercises, use divisibility rules to determine if each of the following is divisible by 9.

10. 48
11. 210
12. 5,345,325

For the following exercises, use divisibility rules to determine if each of the following is divisible by 12.

13. 48
14. 210
15. 5,355,324

16. Determine which of the following numbers are prime: {3, 27, 77, 131, 457}

17. Determine which of the following numbers are prime: {31, 97, 188, 389}

For the following exercises, find the prime factorization of the given number.

18. 12
19. 53
20. 72
21. 345
22. 938
23. 36,068
24. 8,211,679

For the following exercises, find the greatest common divisor of the given set of numbers.

25. {45, 245}
26. {11, 24}
27. {56, 44}
28. {150, 600}
29. {1,746, 28,324}
30. {30, 40, 75}
31. {19, 45, 70}
32. {293, 7,298, 19,229}
33. {3,927,473, 82,709, 1,210,121}
34. Make a list of the common divisors of 12 and 18. What is the GCD of 12 and 18? Which of the other common divisors of 12 and 18 divide the GCD?
35. Make a list of the common divisors of 20 and 84. What is the GCD? Which of the other common divisors of 20 and 84 also divide the GCD?
36. Make a list of the common divisors of 120 and 88. What is the GCD? Which of the other common divisors of 120 and 88 also divide the GCD?
37. Based on the answers to 34, 35, and 36, make a conjecture about the GCD of two numbers, and the other common divisors of those numbers.
38. Rebecca wants to cut two lengths of board into equal length pieces, with no leftover piece. The two boards are

230 cm long and 370 cm long. What is the longest length that Rebecca can cut from these boards so that all the cut boards are the same length?

39. Yasmin is playing with her younger brother, Cameron. They are grouping Skittles by color. They have 14 green, 10 yellow, and 8 purple Skittles. Each group must have the same number of green, the same number of yellow, and the same number of purple Skittles. What's the maximum number of piles that Sophia can build with Cameron?
40. Gathii is creating a tile backsplash for his kitchen. He wants to use square tiles to cover a 330 cm × 12 cm area. What is the largest size square tile he can use to create this backsplash?
41. Deiji is designing a contest where teams will be given the same number of toothpicks, 5 oz. paper cups, and 2 cm length pieces of string. She has 420 pieces of string, 300 paper cups, and 1,610 toothpicks. What is the maximum number of teams she can have so that every team gets an equal number of pieces of string, paper cups, and toothpicks?

For the following exercises, find the least common multiple of the given set of numbers.

42. {30, 40}
43. {11, 24}
44. {14, 45}
45. {200, 450}
46. {38,077, 9,088,687}
47. {36, 42, 70}
48. {7, 13, 36}
49. {4,450,864, 339,889, 157,339}
50. Benjamin and Mia both work at the Grease Fire diner, a local eatery. Benjamin has every 4th day off, and Mia has every 6th day off. How many days pass until they have another day off together?
51. A lunar month is 30 days (rounding off). A new lunar month begins on a Saturday. How many days is it until a lunar month begins on a Saturday again?
52. Isabella is creating a collage for a project and wants a horizontal cut in the collage. The cut will be made by using purple strips of cloth that are 28 mm long, and yellow strips of paper that are 36 mm long. What is the minimum length of the cut can she make using strips with those lengths?
53. Asteroids are objects that orbit the sun. The smallest distance that an asteroid gets to the sun during its orbit is called the perihelion. Asteroids also have orbital periods, or the time it takes to go around the sun exactly one time. The asteroid Ceres has an orbital period (number of days to circle the sun) of 1,680 days. The asteroid Hygiea has an orbital period of 2,031 days. Suppose they are at their perihelion on the same day. How many days will it be before Ceres and Hygiea are at their perihelion on the same day again?

3.2 The Integers

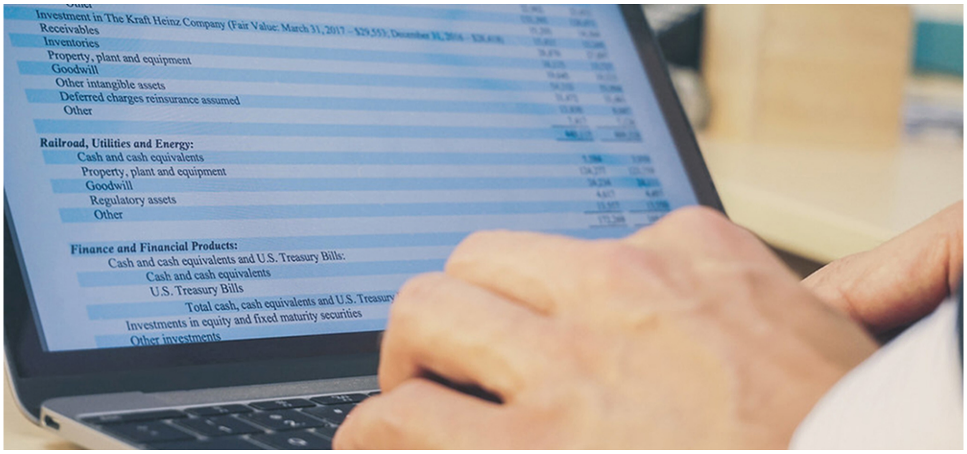


Figure 3.11 A ledger comparing assets to debts, resulting in net wealth. (credit: modification of work "Reviewing Financial Statements" by Mary Cullen/Flickr, CC BY 2.0)

After completing this section, you should be able to:
1. Define and identify numbers that are integers.
2. Graph integers on a number line.
3. Compare integers.
4. Compute the absolute value of an integer.
5. Add and subtract integers.
6. Multiply and divide integers.

Positive net wealth is when the total value of a person's assets, such as their home, their 401(k), their car, and savings account balance, exceed that of their debts, such as car loans, mortgages, or credit card debt. However, when the total value of debt exceeds the total value of assets, then the person has negative net wealth. Expressing the negative net wealth as a negative number allows people to work with the positive net values and negative net values with the same mathematical processes, and in the same applications. This section introduces the integers and operations with integers.

Defining and Identifying Integers

Extending the counting numbers to include negative numbers and zero forms the **integers**. Any other number that cannot be written as $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is not an integer.

EXAMPLE 3.26

Identifying Integers

Which of the following are integers and which are not?

−3	Is an integer, as it is the negative of a counting number
$\sqrt{24}$	This is not written as an integer. Entering the square root of 24 in a calculator, such as desmos, the result is 4.899 (rounded off). Since this is not an integer, then $\sqrt{24}$ is not an integer.
36/4	Since 36 divided by 4 is 9, and 9 is an integer, then 36/4 is an integer.
45	Is an integer, as it is a counting number
63.9	Is not an integer, because it is not a counting number and not the negative of a counting number.

2/7	Dividing 2 by 7 results in a number less than 1, but greater than 0, so is between two consecutive integers. So, 2/7 is not an integer.
−16.0	Is an integer, since the decimal part is 0

YOUR TURN 3.26

Which of the following are integers?

1. −214
2. 38/11
3. $\sqrt{90}$
4. $\sqrt{121}$
5. 420/35

Graphing Integers on a Number Line

Integers are often imagined as steps along a path. You start at 0, and going to the left is going backward, or in the negative direction, while going to the right is going forward, or in the positive direction. A number line (Figure 3.12) helps envision the integers. This also means that an integer gives magnitude (size) and direction (positive is to the right, negative is to the left). Graphing an integer on the number line means placing a solid dot at the integer on the number line.

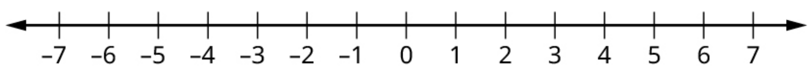


Figure 3.12

VIDEO

Graphing Integers on the Number Line (https://openstax.org/r/Graphing_Integers)

EXAMPLE 3.27

Graphing Numbers on the Number Line

Graph the following on the number line:

1. 1
2. −4
3. 3

Solution

1.

Figure 3.13

2.

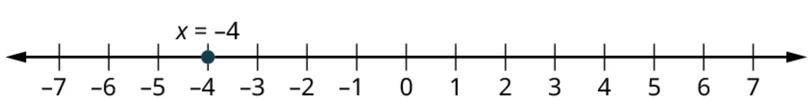


Figure 3.14

3.

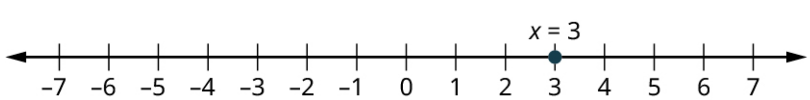


Figure 3.15

> YOUR TURN 3.27

Graph the following numbers on the number line:

1. −10
2. 4
3. 0

Comparing Integers

When determining if one quantity or size is larger than another, we know it means there is more of whatever is being discussed. In terms of positive integers, we can envision that larger integers are further to the right on the number line. This idea applies to negative integers also. This means that ***a* is greater than *b*** when a is to the right of b on the number line. We write $a > b$. When a is greater than b, we can also say that ***b* is less than *a***. On the number line, b would be to the left of a. We write $b < a$.

We need to recognize that $a > b$ means the same thing as $b < a$. This can be seen on the number line in Figure 3.16. On this number line, a is to the right of b, so $a > b$. But this means b is to the left of a, so $b < a$.

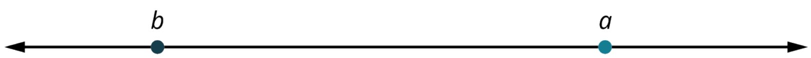


Figure 3.16

▶ VIDEO

Comparing Integers Using the Number Line (https://openstax.org/r/the_Number_Line)

EXAMPLE 3.28

Comparing Integers Using a Number Line

Determine which of −6 and 4 is larger using a number line, and express that using both the greater than and the less than notations.

Solution

To illustrate this, we use a number line (Figure 3.17).

Figure 3.17

Since −6 is to the left of 4, then −6 is less than 4. We can write this as −6 < 4. Another way of expressing this is that 4 is greater than −6. So we can also write $4 > -6$.

> YOUR TURN 3.28

1. Determine which of −38 and 27 is larger using a number line, and express that using both the greater than and the less than notations.

EXAMPLE 3.29

Comparing Negative Integers

Determine which of −6 and −2 is larger, and express that using both the greater than and the less than notations.

Solution

To illustrate this, we use a number line (Figure 3.18).

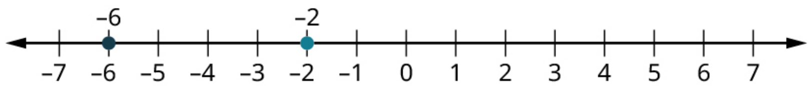


Figure 3.18

Since −6 is to the left of −2, then −6 is less than −2. We can write this as $-6 < -2$.

Another way of expressing this is that −2 is greater than −6. So we can also write $-2 > -6$.

Warning: People often ignore the negative signs, and think than since 6 is greater than 2, −6 is greater than −2. To avoid that error, remember that the greater number is to the right on the number line.

YOUR TURN 3.29

1. Determine which of −63 and −213 is larger, and express that using both the greater than and the less than notations.

EXAMPLE 3.30

Comparing Integers by Quantity

Determine which of 27 and 410 is larger, and express that using both the greater than and the less than notations.

Solution

When thinking about quantity, 410 is more than 27. So, 410 is greater than 27 and 27 is less than 410. We can write this as $410 > 27$ or as $27 < 410$.

YOUR TURN 3.30

1. Determine which of 101 and 98 is larger, and express that using both the greater than and the less than notations.

The Absolute Value of an Integer

When talking about graphing integers on the number line, one interpretation suggests it is like walking along a path. Negative is going to the left of 0, and positive going to the right. If you take 30 steps to the right, you are 30 steps away from 0. On the other hand, when you take 30 steps to the left, you are still 30 steps away from 0. So, in a way, even though one is negative and the other positive, these two numbers, 30 and −30, are equal since both are 30 steps away from 0. The **absolute value of an integer** n is the distance from n to 0, regardless of the direction. The notation for absolute value of the integer n is $|n|$.

If we think of an integer as both direction and magnitude (size), absolute value is the magnitude part.

Calculating the absolute value of an integer is very straightforward. If the integer is positive, then the absolute value of the integer is just the integer itself. If the integer is negative, then to compute the absolute value of the integer, simply remove the negative sign. Keeping in mind the number line as a path, when you've gone 10 steps to the left of 0, you have still taken 10 steps, and the direction does not matter.

VIDEO

[Evaluating the Absolute Value of an Integer (https://openstax.org/r/Absolute_Value_of_an_Integer)](https://openstax.org/r/Absolute_Value_of_an_Integer)

EXAMPLE 3.31

Calculating the Absolute Value of a Positive Integer

Calculate $|19|$.

Solution

Since the number inside the absolute value symbol is positive, the absolute value is just the number itself. So $|19| = 19$.

YOUR TURN 3.31

1. Calculate $|38|$.

EXAMPLE 3.32

Calculating the Absolute Value of a Negative Integer

Calculate $|-435|$.

Solution

Since the number inside the absolute value is negative, the absolute value removes the negative sign. So $|-435| = 435$.

YOUR TURN 3.32

1. Calculate $|-81|$.

Adding and Subtracting Integers

You may recall having approached adding and subtracting integers using the number line from earlier in your academic life. Adding a positive integer results in moving to the right on the number line. Adding a negative integer results in moving to the left. Subtracting a positive integer results in a move to the left on the number line. But subtracting a negative integer results in a move to the right.

This leads to a few adding and subtracting rules, such as:

Rule 1: Subtracting a negative is the same as adding a positive.

Rule 2: Adding two negative integers always results in a negative integer.

Rule 3: Adding two positive integers always results in a positive integer.

Rule 4: The sign when adding integers with opposite signs is the same as the integer with the larger absolute value.

These rules are good to keep in the back of your mind, as they can serve as a quick error check when you use a calculator.

EXAMPLE 3.33

Adding Integers

Use your calculator to calculate $4 + (-7)$. Explain how the answer agrees with what was expected.

Solution

Using a calculator, we find that $4 + (-7) = -3$. Since we are adding integers with opposite signs, the sign of the answer matches the sign of the integer with the larger absolute value which $|-7|=7$.

YOUR TURN 3.33

1. Use your calculator to calculate (−18) + 11. Explain how the answer agrees with what was expected.

EXAMPLE 3.34

Subtracting Positive Integers
Use your calculator to calculate 18 − 9. Explain how the answer agrees with what was expected.

Solution
Using a calculator, we find that 18 − 9 = 9. Since 18 was larger than 9, we expected the difference to be positive.

YOUR TURN 3.34

1. Use your calculator to calculate 38 − 100. Explain how the answer agrees with what was expected.

EXAMPLE 3.35

Subtracting with Negative Integers
Use your calculator to calculate 27 − (−13). Explain how the answer agrees with what was expected.

Solution
Using a calculator, we find that 27 – (−13) = 40. Since we're subtracting a negative number, it is the same as adding a positive, so this is the same as 27 + 13 = 40.

YOUR TURN 3.35

1. Use your calculator to calculate 45 − (−26). Explain how the answer agrees with what was expected.

EXAMPLE 3.36

Adding Integers with Opposite Signs
Use your calculator to calculate (−13) + 90. Explain how the answer agrees with what was expected.

Solution
Using a calculator, we find that (−13) + 90 = 77. Since we are adding integers with opposite signs, the sign of the answer matches the sign of the integer with the larger absolute value, which is positive since 90 is positive.

YOUR TURN 3.36

1. Use your calculator to calculate 19 + (−36). Explain how the answer agrees with what was expected.

One use of negative numbers is determining **net worth**, which is all the weath someone owns less all that someone owes. Sometimes net worth is positive (which is good), and sometimes net worth is negative (which can be stressful).

EXAMPLE 3.37

Calculating Net Worth
Jennifer is owed $50 from her friend Janice, but owes her friend Pat $87. What is Jennifer's net worth?

Solution

Net worth is the amount that one is owed minus the amount one owes. Jennifer is owed \$50 but owes \$87. So, her net worth is \$50 – \$87 = −\$37. The negative indicates that Jennifer owes more than she is owed.

YOUR TURN 3.37

1. Christian is owed \$180 from his friend Chanel, but owes his friend Jeff \$91. What is Christian's net worth?

Multiplying and Dividing Integers

Similar to addition and subtraction, the signs of the integers impact the results when multiplying and dividing integers. The rules are fairly straightforward, but again rely on the direction on the number line. There are only two rules.

Rule 1: When multiplying or dividing two integers with the same sign, the result is positive.

Rule 2: When multiplying or dividing two integers with opposite signs, the result is negative.

Just as before, these rules can serve as a quick error check when using a calculator.

EXAMPLE 3.38

Multiplying Positive Integers

Use your calculator to calculate 4 × 8. Explain how the answer agrees with what was expected.

Solution

Entering 4 × 8 into your calculator, the result is 32. This agrees with our expectation. The numbers have the same signs, so the result is positive.

YOUR TURN 3.38

1. Use your calculator to calculate 81 × 26. Explain how the answer agrees with what was expected.

EXAMPLE 3.39

Multiplying Integers with Different Signs

Use your calculator to calculate 9 × (−10). Explain how the answer agrees with what was expected.

Solution

Entering 9 × (−10) into your calculator, the result is −90. This agrees with our expectation. The numbers have opposite signs, so the result is negative.

YOUR TURN 3.39

1. Use your calculator to calculate (−18) × 13. Explain how the answer agrees with what was expected.

EXAMPLE 3.40

Dividing Integers with Different Signs

Use your calculator to calculate 400/(−25). Explain how the answer agrees with what was expected.

Solution

Entering 400/(−25) into your calculator, the result is −16. This agrees with our expectation. The numbers have opposite signs, so the result is negative.

YOUR TURN 3.40

1. Use your calculator to calculate (−116)/4. Explain how the answer agrees with what was expected.

EXAMPLE 3.41

Dividing Negative Integers

Use your calculator to calculate −750/(−3). Explain how the answer agrees with what was expected.

Solution

Entering −750/(−3) into your calculator, the result is 250. This agrees with our expectation. The numbers have the same signs, so the result is positive.

YOUR TURN 3.41

1. Use your calculator to calculate (−77)/(−11). Explain how the answer agrees with what was expected.

At the end of a season, a team may wish to buy their coach an end-of season gift. It makes sense to share the cost equally among the members. To do so, the team would need to find the average (or mean) cost per member. The **average (or mean) of a set of** numbers is the sum of the numbers divided by the number values that are being averaged.

EXAMPLE 3.42

Finding the Average of a Set of Numbers

The daily low temperatures in Barrie, Ontario, for the week of February 14, 2021, were −20°, −12°, −15°, −23°, −17°, −13°, and −19° degrees Celsius. What was the average daily temperature for the week of February 14, 2021, in Barrie?

Solution

Step 1: To find the average daily temperature, we first need to add the temperatures.

(−20) + (−12) + (−15) + (−23) + (−17) + (−13) + (−19) = −119

Step 2: That sum will then be divided by 7 since we are averaging over seven days, giving −119/7 = −17. So, the average daily temperature in Barrie, Ontario the week of February 14, 2021, was −17° Celsius.

YOUR TURN 3.42

1. Banks and credit cards often base their interest on the average daily balance of an account, which is the average of the balance from each day of the period. The account balance of Jada's checking account on each day of the week of December 13, 2020, was $1,250, $673, −$1,500, $1,000, $785, $785, and $710. What was Jada's average daily balance for the week of December 13, 2020? Assume Jada pays no fees for a negative balance.

Check Your Understanding

6. Identify which of the following numbers are integers:
   −4, 15.2, $\sqrt{2}$, $\frac{3}{20}$, 430

7. Graph the following integers on the number line: 4, −2, 7.

8. Place these integers in increasing order: 4, −2, −7, 10, −13.

9. Calculate $|-7|$.

10. Calculate 4 – (−9).

11. Calculate $(-3) \times (-12)$.

SECTION 3.2 EXERCISES

1. Identify all the integers in the following list: 4, −17, 8, 0.5, $\sqrt{7}$, $\frac{1}{9}$, −300.

2. Identify all integers in the following list: −9.2, 13, −1, $\sqrt{47}$, $\frac{\sqrt{3}}{2}$, 567, −300.

For the following exercises, plot the integers on the same number line.

3. 4, −2, 10, 0
4. −6, −3, 10, 1, 4
5. −3, −10, 7, 2
6. 2, 4, 8, −2, −5

For the following exercises, determine if the comparison is true or false.

7. $-3 < -10$
8. $-10 > -3$
9. $7 > -6$
10. $-6 < 7$
11. $18 < 20$
12. $20 < 18$
13. What are two numbers with an absolute value of 76?
14. What are two numbers with an absolute value 87?
15. Determine $|-67|$.
16. Determine $|98|$.
17. Determine $|61|$.
18. Determine $|-903|$.
19. What are two numbers that are 23 away from 0?
20. What are two numbers 13 away from 0?

For the following exercises, complete the indicated calculation.

21. 47 + 200
22. 67 + (−86)
23. (−86) + 104
24. 13 – (−54)
25. (−45) – (−26)
26. (−13) + (−102)
27. 38×12
28. $(-12) \times (-10)$
29. $14 \times (-5)$
30. $421 \times (-13)$
31. $(-412) \times 504$
32. $(-8,919) \times (-3,401)$
33. $(-130) \div (-65)$
34. $450 \div 9$
35. $2000 \div (-40)$
36. $(-910) \div 14$
37. The daily low temperatures, in degrees centigrade, in Fargo for the week of January 17, 2021, were −9, −17, −18, −14, −17, −19, and −11. What was the average low temperature in Fargo that week?
38. Riley collects checks for a fundraiser supporting the homeless in town. Through Venmo, they collect the following amounts: \$20, \$20, \$50, \$75, \$250, \$10, \$15, \$65, \$30, \$15. What was the average donation that Riley collected?
39. Heath has \$495 in an account. They will collect two paychecks this week, one for \$150 and the other for \$250. Heath also will pay three bills, one for \$50, one for \$110, and one for \$300. After all those transactions, how much will Heath have in their account?
40. Five diners decide to split the check evenly. The total bill comes to \$475. How much does each diner owe?

41. There are many people who are single-issue voters, which means that they will vote for (or against!) a candidate based on one issue and one issue only. Suppose a politician wants to earn votes based on single issues: Issue 1, Issue 2, and Issue 3. By publicly supporting issue 1, the politician gains 127 voter but loses 154. By publicly supporting issue 2, the politician gains 350 voters but loses 83. By publicly denouncing Issue 3, the politician gains 306 voters but loses 158. By publicly taking those stances, what is the politician's net gain or loss in number of voters?

3.3 Order of Operations

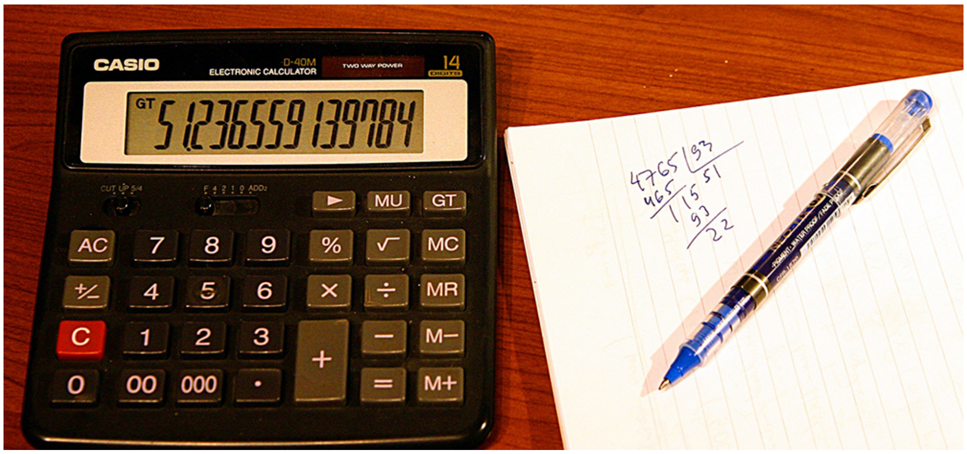


Figure 3.19 Calculators may automatically apply order of operations to calculations. (credit: "Precision" by Leonid Mamchenkov/Flickr, CC BY 2.0)

After completing this module, you should be able to:
1. Simplify expressions using order of operations.
2. Simplify expressions using order of operations involving grouping symbols.

Calculates else sure someone be rules expect explicit we what that needs to need make, we to that them what be calculate to calculated first.

You probably read that sentence and couldn't make heads or tails of it. Seems like it might concern calculations, but maybe it concerns needs? You may even be attempting to unscramble the sentence as you read it, placing words in the order you might expect them to appear in. The reason that the sentence makes no sense is that the words don't follow the order you expect them to follow. Unscrambled, the sentence was intended to be "To be sure that someone else calculates when we expect them to calculate, we need rules that make explicit what needs to be calculated first."

Similarly, when working with math expressions and equations, if we don't follow the rules for **order of operations**, arithmetic expressions make no sense. Just a simple expression would be problematic if we didn't have some rules to tell us what to calculate first. For instance, $4 \times 2^2 + 3 + 5^2$ can be calculated in many ways. You could get 5,184. Or, you could get 80. Or, 96. The issue is that without following a set of rules for calculation, the same expression will give various results. In case you are curious, using the appropriate order of operations, we find $4 \times 2^2 + 3 + 5^2 = 44$.

Simplify Expressions Using Order of Operations

The order in which mathematical operations is performed is a convention that makes it easier for anyone to correctly calculate. They follow the acronym EMDAS:

E	Exponents
M/D	Multiplication and division
A/S	Addition and Subtraction

So, what does EMDAS tell us to do? In an equation, moving left to right, we begin by calculating all the exponents first. Once the exponents have been calculated, we again move left to right, calculating the multiplications and divisions, one at a time. Multiplication and division hold the same position in the ordering, so when you encounter one or the other at this step, do it. Once the multiplications and divisions have been calculated, we again move left to right, calculating the additions and subtractions, one at a time. Additions and subtractions hold the same position in the ordering, so when you encounter one or the other at this step, do it. (You may have previously learned the order of operations as PEMDAS, with parentheses first; we will add that aspect later on.) We'll explore this as we work an example.

EXAMPLE 3.43

Using Two Order of Operations

Calculate $21 - 4 \times 13$.

Solution

There are no exponents in this expression, so the next operations to check are multiplication and division.

Step 1: Moving left to right, the first multiplication encountered is 4 multiplied by 13. We perform that operation first.

$$21 - \mathbf{4 \times 13}$$
$$= 21 \mathbf{- 52}$$

Step 2: The only operation remaining is the subtraction.

$21 - 52 = -31.$

So, $21 - 4 \times 13 = -31.$

YOUR TURN 3.43

1. Calculate $43 + 18 \times 15$.

EXAMPLE 3.44

Using Two Order of Operations

Calculate 4×8^3.

Solution

Step 1: Moving left to right, we see there is an exponent. We calculate the exponent first.

$$4 \times \mathbf{8^3} = 4 \times \mathbf{(8 \times 8 \times 8)} = 4 \times \mathbf{512}$$

Step 2: The only operation remaining is the multiplication.

$4 \times 512 = 2{,}048$

So, $4 \times 8^3 = 2{,}048.$

YOUR TURN 3.44

1. Calculate $60^2/50$.

EXAMPLE 3.45

Using Three Order of Operations

Calculate $2 + 3^2 \times 4$.

Solution

Step 1: To calculate this, move left to right, and compute all the exponents first. The only exponent we see is the squaring of the 3, so that is calculated first.

$$2 + 3^2 \times 4 = 2 + \mathbf{(3 \times 3)} \times 4 = 2 + \mathbf{9} \times 4$$

Step 2: Since the exponents are all calculated, now calculate all the multiplications and divisions moving left to right. The only multiplication or division present is 9 times 4.

$2 + 9 \times 4 = 2 + \mathbf{36}$

Step 3: Moving left to right, perform the additions and subtractions. There is only one such operation, 2 plus 36.

$2 + 36 = 38$

So, $2 + 3^2 \times 4 = 38$.

YOUR TURN 3.45

1. Calculate $3 \times 6^3 - 18$.

Even if the expression being calculated gets more complicated, we perform the operations in the order: EMDAS.

VIDEO

[Order of Operations 1 (https://openstax.org/r/Order_of_Operations_1)](https://openstax.org/r/Order_of_Operations_1)

EXAMPLE 3.46

Using Eight Order of Operations

Correctly apply the order of operations to compute the following:

$4 - 25 \times 6/10 \times 3^2 + 7 \times 2^3$.

Solution

Step 1: To do so, calculate the exponents first, moving left to right. There are two occurrences of exponents in the expression, 3 squared and 2 cubed.

$4 - 25 \times 6/10 \times \mathbf{3^2} + 7 \times \mathbf{2^3} = 4 - 25 \times 6/10 \times \mathbf{9} + 7 \times \mathbf{8}$

Step 2: Now that the exponents are calculated, perform the multiplication and division, moving left to right. The first is the product of 25 and 6.

$$\begin{aligned} & 4 - \mathbf{25 \times 6}/10 \times 9 + 7 \times 8 \\ & = 4 - \mathbf{150}/10 \times 9 + 7 \times 8 \end{aligned}$$

Step 3: Next is the 150 divided by 10.

$$\begin{aligned} & 4 - \mathbf{150/10} \times 9 + 7 \times 8 \\ & = 4 - \mathbf{15} \times 9 + 7 \times 8 \end{aligned}$$

Step 4: Next is 15 multiplied by 9.

$$\begin{aligned} & 4 - \mathbf{15 \times 9} + 7 \times 8 \\ & = 4 - \mathbf{135} + 7 \times 8 \end{aligned}$$

Step 5: Finally, multiply the 7 and 8.

$$\begin{aligned} & 4 - 135 + \mathbf{7 \times 8} \\ & = 4 - 135 + \mathbf{56} \end{aligned}$$

As all the multiplications and divisions have been calculated, the additions and subtractions are performed, moving left to right.

$$\begin{aligned} & \mathbf{4 - 135} + 56 \\ & = \mathbf{-131} + 56 \\ & = \mathbf{-131 + 56} \\ & = \mathbf{-75} \end{aligned}$$

The computed value is −75.

YOUR TURN 3.46

1. Correctly apply the order of operations to compute the following:

$13 + 3^4 \times 8/6 \times 5^2 - 3 \times 4.$

VIDEO

[Order of Operations 2 (https://openstax.org/r/Order_of_Operations_2)](https://openstax.org/r/Order_of_Operations_2)

EXAMPLE 3.47

Using Six Order of Operations

Correctly apply the rules for the order of operations to accurately compute the following:

$10 - 3 \times 5^3/15 + 56/4.$

Solution

Step 1: Calculate exponents first, moving left to right:

$$10 - 3 \times \mathbf{5^3}/15 + 56/4$$
$$= 10 - 3 \times \mathbf{125}/15 + 56/4$$

Step 2: Multiply and divide, moving left to right:

$$10 - \mathbf{3 \times 125}/15 + 56/4$$
$$= 10 - \mathbf{375/15} + 56/4$$
$$= 10 - 25 + \mathbf{56/4}$$
$$= 10 - 25 + \mathbf{14}$$

Step 3: Add and subtract, moving left to right:

$$\mathbf{10 - 25} + 14$$
$$= \mathbf{-15 + 14}$$
$$= -1$$

YOUR TURN 3.47

1. Correctly apply the rules for the order of operations to accurately compute the following:

$12/(-4) + 8 \times 9/12 \times 2^3 - 24 \times 25/10.$

EXAMPLE 3.48

Using Order of Operations

Correctly apply the rules for the order of operations to accurately compute the following:
$(-8)/2 \times 3 - 9 \times 2^4/12 + 9 \times (-4)^2/2^3.$

Solution

Step 1: Calculate the exponents first, moving left to right:

$$(-8)/2 \times 3 - 9 \times \mathbf{2^4}/12 + 9 \times (-4)^2/2^3$$
$$= (-8)/2 \times 3 - 9 \times 16/12 + 9 \times \mathbf{12^2}/2^3$$
$$= (-8)/2 \times 3 - 9 \times 16/12 + 9 \times 144/\mathbf{2^3}$$
$$= (-8)/2 \times 3 - 9 \times 16/12 + 9 \times 144/8$$

Step 2: Multiply and divide, moving left to right:

$$= \mathbf{(-8)/2} \times 3 - 9 \times 16/12 + 9 \times 144/8$$
$$= \mathbf{(-4) \times 3} - 9 \times 16/12 + 9 \times 144/8$$
$$= (-12) - \mathbf{9 \times 16}/12 + 9 \times 144/8$$
$$= (-12) - \mathbf{144/12} + 9 \times 144/8$$
$$= (-12) - 12 + \mathbf{9 \times 144}/8$$
$$= (-12) - 12 + \mathbf{1{,}296/8}$$
$$= (-12) - 12 + 162$$

Step 3: Add and subtract, moving left to right:

$$= \mathbf{(-12) - 12} + 162 = \mathbf{(-24) + 162} = 138$$

YOUR TURN 3.48

1. Correctly apply the rules for the order of operations to accurately compute the following:

   $3 \times 4^3 \times 7 + 24/6 \times 7^2 - 9/3 \times 8.$

Using the Order of Operations Involving Grouping Symbols

We have examined how to use the order of operations, denoted by EMDAS, to correctly calculate expressions. However, there may be expressions where a multiplication should happen before an exponent, or a subtraction before a division. To indicate an operation should be performed out of order, the operation is placed inside parentheses. When parentheses are present, the operations inside the parentheses are performed first. Adding the parentheses to our list, we now have PEMDAS, as shown below.

P	Parentheses	
E	Exponents	
M/D	Multiplication and division	(division is just the multiplication by the reciprocal)
A/S	Addition and subtraction	(subtraction is just the addition of the negative)

As said previously, parentheses indicate that some operation or operations will be performed outside the standard order of operation rules. For instance, perhaps you want to multiply 4 and 7 before squaring. To indicate that the multiplication happens before the exponent, the multiplication is placed inside parentheses: $(4 \times 7)^2$.

This means operations inside the parentheses take precedence, or happen before other operations. Now, the first step in calculating arithmetic expressions using the order of operations is to perform operations inside parentheses first. Inside the parentheses, you follow the order of operation rules EMDAS.

EXAMPLE 3.49

Prioritizing Parentheses in the Order of Operations

Correctly apply the rules for the order of operations to accurately compute the following:

$(10 - 3) \times 5^3$.

Solution

Step 1: Perform all calculations within the parentheses before all other operations.

$(\mathbf{10 - 3}) \times 5^3 = \mathbf{7} \times 5^3$

Step 2: Since all parentheses have been cleared, move left to right, and compute all the exponents next.

$7 \times \mathbf{5^3} = 7 \times \mathbf{125}$

Step 3: Perform all multiplications and divisions moving left to right.

$\mathbf{7 \times 125 = 875}$

YOUR TURN 3.49

1. Correctly apply the rules for the order of operations to accurately compute the following:

   $8 - (25 - 2^2)/7$.

Be aware that there can be more than one set of parentheses, and parentheses within parentheses. When one set of parentheses is inside another set, do the innermost set first, and then work outward.

VIDEO

[Order of Operations 3 (https://openstax.org/r/Order_of_Operations_3)](https://openstax.org/r/Order_of_Operations_3)

EXAMPLE 3.50

Working Innermost Parentheses in the Order of Operations

Correctly apply the rules for order of operations to accurately compute the following:

$4 + 2 \times (3^2 - (2 + 5)^2 \times 4)/(3 + 8)$.

Solution

Step 1: Perform all calculations within the parentheses before other operations. Evaluate the innermost parentheses first. We can work separate parentheses expressions at the same time. The innermost set of parentheses has the 2 + 5 inside. The 3 + 8 is in a separate set of parentheses, so that addition can occur at the same time as the 2 + 5.

$$4 + 2 \times (3^2 - (\mathbf{2 + 5})^2 \times 4)/(\mathbf{3 + 8})$$
$$= 4 + 2 \times (3^2 - (\mathbf{7})^2 \times 4)/(\mathbf{11})$$

Step 2: Now that those parentheses have been handled, move on to the next set of parentheses. Applying the order of operation rules inside that set of parentheses, the exponent is evaluated first, then the multiplication, and then the addition.

$$4 + 2 \times (\mathbf{3^2 - 7^2 \times 4})/11$$
$$= 4 + 2 \times (\mathbf{3^2 - 49 \times 4})/11$$
$$= 4 + 2 \times (\mathbf{-187})/11$$

Step 3: Since all parentheses have been cleared, apply the EMDAS rules to finish the calculation.

$4 + \mathbf{2 \times (-187)}/11 = 4 - \mathbf{374/11} = \mathbf{4 - 34} = -30$

YOUR TURN 3.50

1. Correctly apply the rules for the order of operations to accurately compute the following:

$(8-6)^2 \times 100 - ((48/6-3)^2 - 4 \times 7)$.

VIDEO

[Order of Operations 4 (https://openstax.org/r/Order_of_Operations_4)](https://openstax.org/r/Order_of_Operations_4)

Check Your Understanding

12. Which operation has highest precedence?

13. Which is performed first, exponents or addition?

14. Calculate $2 \times 3^2 - 5 \times 8$.

15. What is used to indicate operations that should be performed out of order?

16. Calculate $(4-3)^2 + 27 \times 8^2 \div 6^2$.

SECTION 3.3 EXERCISES

1. Which operations have the lowest precedence in order of operations?

2. If many operations have the same precedence in an expression, in what order should the operations be performed?

3. Which operations have the same precedence in order of operations?

4. After all operations in parentheses have been performed, which operations should be performed next?

For the following exercises, perform the indicated calculation.

5. $4 - 5 \times 6$
6. $34 - 10 \times 6$
7. $18 + 8 \times 13$
8. $40 + 12 \times 17$
9. $50 \div 2 + 8$
10. $72 \div 6 + 18$
11. $13 + 4 \times 3^2$
12. $45 - 6 \times 7^3$
13. $6^2 \times 5 - 13 \times 9^2$
14. $14^2 \times 8 - 5 \times 2^4$
15. $450 \div 3^2 - 56 \div 2^2$
16. $1{,}000 \div 5^3 - 7 \times 8^4$
17. $38 \times 6 - 4 + 5 \times 18 \div 10$
18. $15 \times 7 + 23 - 6 \times 40 \div 24$
19. $600 \div 12 \times 5 + 40 - 6^2$
20. $2 \times 12^3 \div 8 \times 3 - 4^5$
21. $10 \times 6^3 \div 2 \times 5 + 3^2 - 240 \div 8 \times 9$
22. $45 \div 15 \times 6 + 7^2 - 4 \times 15 \times 18 \div 12 \times 3$
23. $(4+3) \times 2$
24. $6 \times (12+8)$
25. $(14-25) \times 2$
26. $(86-61) \div 5$
27. $(3+2)^3$
28. $(17-12)^4$
29. $(45-60)^2$
30. $(90-101)^3$
31. $(3 \times 4)^3 \div 8 + 5$
32. $(6-9)^2 \times 5 - 4$
33. $4 - (6+3) \times 5 + 11$
34. $12 + (13-6) \times 8 - 130$

35. $15 \times (4 - 1)^2 + 5 \times (17 + 2 \times 3) + 4$
36. $18 \times (12 - 6)^4 - 8 \times (30 - 14 \times 16) - 90$
37. $4 \times (5 + 2 \times (6 + 8)) + 10 \times 3^2$
38. $9 \times (13 - 12 \times (41 - 32)) - 8 \times 45$
39. $21 - 3 \times (5 \times (2 + 6) - 3 \times (18 - 11) + 25) \div 2$
40. $48 - 6 \times (10 \times (18 + 12) - 25 \times (16 - 9) + 19) \div 4$

3.4 Rational Numbers

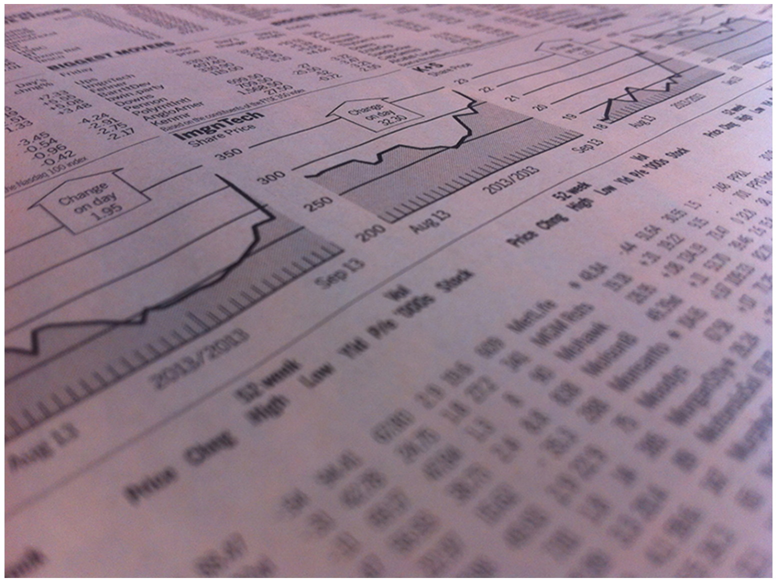


Figure 3.20 Stock gains and losses are often represented as percentages.(credit: "stock market quotes in newspaper" by Andreas Poike/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Define and identify numbers that are rational.
2. Simplify rational numbers and express in lowest terms.
3. Add and subtract rational numbers.
4. Convert between improper fractions and mixed numbers.
5. Convert rational numbers between decimal and fraction form.
6. Multiply and divide rational numbers.
7. Apply the order of operations to rational numbers to simplify expressions.
8. Apply density property of rational numbers.
9. Solve problems involving rational numbers.
10. Use fractions to convert between units.
11. Define and apply percent.
12. Solve problems using percent.

We are often presented with percentages or fractions to explain how much of a population has a certain feature. For example, the 6-year graduation rate of college students at public institutions is 57.6%, or 72/125. That fraction may be unsettling. But without the context, the percentage is hard to judge. So how does that compare to private institutions? There, the 6-year graduation rate is 65.4%, or 327/500. Comparing the percentages is straightforward, but the fractions are harder to interpret due to different denominators. For more context, historical data could be found. One study reported that the 6-year graduation rate in 1995 was 56.4%. Comparing that historical number to the recent 6-year graduation rate at public institutions of 57.6% shows that there hasn't been much change in that rate.

Defining and Identifying Numbers That Are Rational

A **rational number** (called rational since it is a ratio) is just a fraction where the numerator is an integer and the denominator is a non-zero integer. As simple as that is, they can be represented in many ways. It should be noted here that any integer is a rational number. An integer, n, written as a fraction of two integers is $\frac{n}{1}$.

In its most basic representation, a rational number is an integer divided by a non-zero integer, such as $\frac{3}{12}$. Fractions may be used to represent parts of a whole. The denominator is the total number of parts to the object, and the numerator is how many of those parts are being used or selected. So, if a pizza is cut into 8 equal pieces, each piece is $\frac{1}{8}$ of the pizza. If you take three slices, you have $\frac{3}{8}$ of the pizza (Figure 3.21). Similarly, if in a group of 20 people, 5 are wearing hats, then $\frac{5}{20}$ of the people are wearing hats (Figure 3.22).

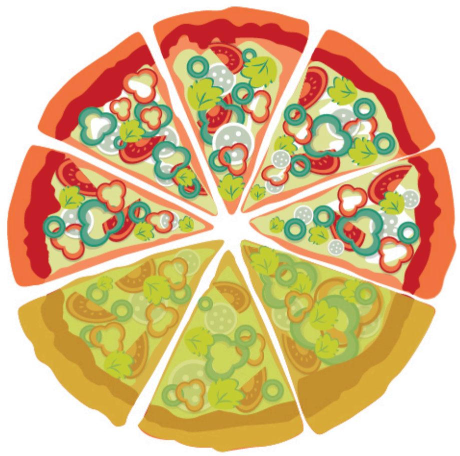


Figure 3.21 Pizza cut in 8 slices, with 3 slices highlighted

Figure 3.22 Group of 20 people, with 5 people wearing hats

Another representation of rational numbers is as a mixed number, such as $2\frac{5}{8}$ (Figure 3.23). This represents a whole number (2 in this case), plus a fraction (the $\frac{5}{8}$).

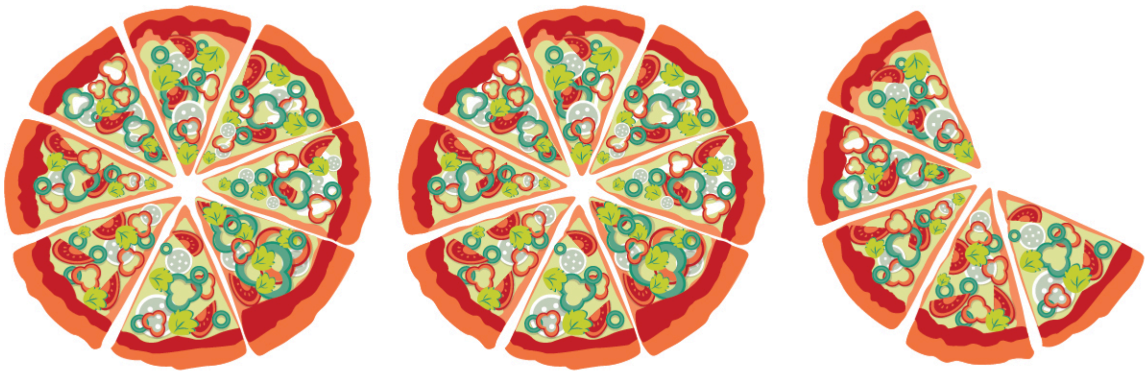


Figure 3.23 Two whole pizzas and one partial pizza

Rational numbers may also be expressed in decimal form; for instance, as 1.34. When 1.34 is written, the decimal part, 0.34, represents the fraction $\frac{34}{100}$, and the number 1.34 is equal to $1\frac{34}{100}$. However, not all decimal representations are rational numbers.

A number written in decimal form where there is a last decimal digit (after a given decimal digit, all following decimal digits are 0) is a **terminating decimal**, as in 1.34 above. Alternately, any decimal numeral that, after a finite number of decimal digits, has digits equal to 0 for all digits following the last non-zero digit.

All numbers that can be expressed as a terminating decimal are rational. This comes from what the decimal represents. The decimal part is the fraction of the decimal part divided by the appropriate power of 10. That power of 10 is the number of decimal digits present, as for 0.34, with two decimal digits, being equal to $\frac{34}{100}$.

Another form that is a rational number is a decimal that repeats a pattern, such as 67.1313... When a rational number is expressed in decimal form and the decimal is a repeated pattern, we use special notation to designate the part that repeats. For example, if we have the repeating decimal 4.3636..., we write this as $4.\overline{36}$. The bar over the 36 indicates that the 36 repeats forever.

If the decimal representation of a number does not terminate or form a repeating decimal, that number is not a rational number.

One class of numbers that is not rational is the **square roots** of integers or rational numbers that are not **perfect squares**, such as $\sqrt{10}$ and $\sqrt{\frac{25}{6}}$. More generally, the number b is the square root of the number a if $a = b^2$. The notation for this is $b = \sqrt{a}$, where the symbol $\sqrt{\ }$ is the square root sign. An integer perfect square is any integer that can be written as the square of another integer. A rational perfect square is any rational number that can be written as a fraction of two integers that are perfect squares.

Sometimes you may be able to identify a perfect square from memory. Another process that may be used is to factor the number into the product of an integer with itself. Or a calculator (such as Desmos) may be used to find the square root of the number. If the calculator yields an integer, the original number was a perfect square.

TECH CHECK

Using Desmos to Find the Square Root of a Number

When Desmos is used, there is a tab at the bottom of the screen that opens the keyboard for Desmos. The keyboard is shown below. On the keyboard (Figure 3.24) is the square root symbol $\left(\sqrt{\ }\right)$. To find the square root of a number, click the square root key, and then type the number. Desmos will automatically display the value of the square root as you enter the number.

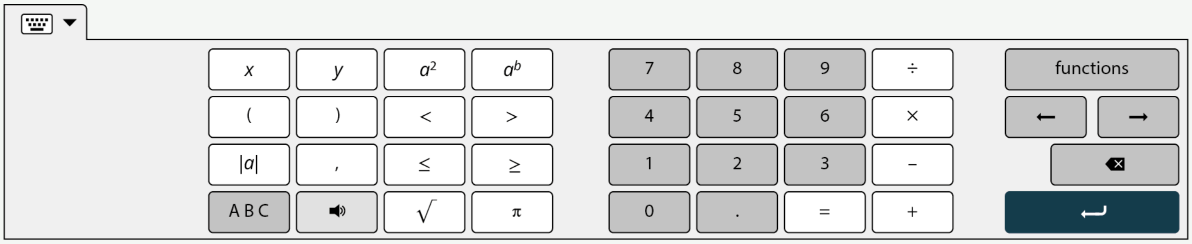


Figure 3.24 Desmos keyboard with square root key

EXAMPLE 3.51

Identifying Perfect Squares

Which of the following are perfect squares?

1. 45
2. 144

Solution

1. We could attempt to find the perfect square by factoring. Writing all the factor pairs of 45 results in $1 \times 45, 3 \times 15$, and 5×9. None of the pairs is a square, so 45 is not a perfect square. Using a calculator to find the square root of 45, we obtain 6.708 (rounded to three decimal places). Since this was not an integer, the original number was not a perfect square.
2. We could attempt to find the perfect square by factoring. Writing all the factor pairs of 144 results in $1 \times 144, 2 \times 72, 3 \times 48, 6 \times 24, 8 \times 18$, and 12×12. Since the last pair is an integer multiplied by itself, 144 is a perfect square. Alternately, using Desmos to find the square root of 144, we obtain 12. Since the square root of 144 is an integer, 144 is a perfect square.

YOUR TURN 3.51

Determine if the following are perfect squares:

1. 94
2. 441

VIDEO

[Introduction to Fractions (https://openstax.org/r/Equivalent_Fractions)](https://openstax.org/r/Equivalent_Fractions)

EXAMPLE 3.52

Identifying Rational Numbers

Determine which of the following are rational numbers:

1. $\sqrt{73}$
2. 4.556
3. $3\frac{1}{5}$
4. $\frac{41}{17}$
5. $5.\overline{64}$

Solution

1. Since 73 is not a perfect square, its square root is not a rational number. This can also be seen when a calculator is used. Entering $\sqrt{73}$ into a calculator results in 8.544003745317 (and then more decimal values after that). There is no repeated pattern, so this is not a rational number.
2. Since 4.556 is a decimal that terminates, this is a rational number.

3. $3\frac{1}{5}$ is a mixed number, so it is a rational number.
4. $\frac{41}{17}$ is an integer divided by an integer, so it is a rational number.
5. $5.646464\ldots$ is a decimal that repeats a pattern, so it is a rational number.

YOUR TURN 3.52

Determine which of the following are rational numbers:

1. $\sqrt{13}$
2. $-13.\overline{21}$
3. $\frac{-48}{-16}$
4. $-4\frac{18}{19}$
5. 14.1131

Simplifying Rational Numbers and Expressing in Lowest Terms

A rational number is one way to express the division of two integers. As such, there may be multiple ways to express the same value with different rational numbers. For instance, $\frac{4}{5}$ and $\frac{12}{15}$ are the same value. If we enter them into a calculator, they both equal 0.8. Another way to understand this is to consider what it looks like in a figure when two fractions are equal.

In Figure 3.25, we see that $\frac{3}{5}$ of the rectangle and $\frac{9}{15}$ of the rectangle are equal areas.

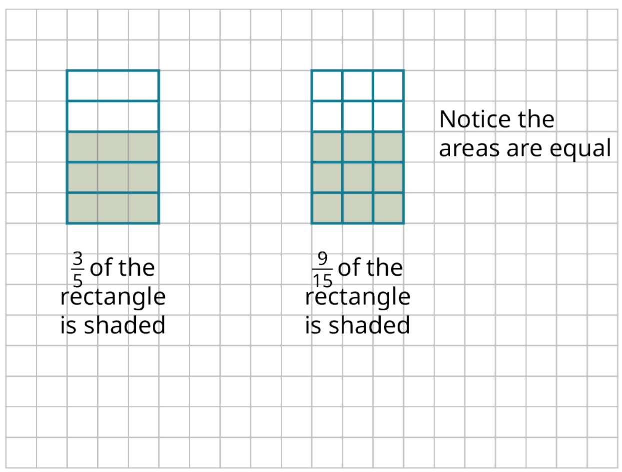


Figure 3.25 Two Rectangles with Equal Areas

They are the same proportion of the area of the rectangle. The left rectangle has 5 pieces, three of which are shaded. The right rectangle has 15 pieces, 9 of which are shaded. Each of the pieces of the left rectangle was divided equally into three pieces. This was a multiplication. The numerator describing the left rectangle was 3 but it becomes 3×3, or 9, as each piece was divided into three. Similarly, the denominator describing the left rectangle was 5, but became 5×3, or 15, as each piece was divided into 3. The fractions $\frac{3}{5}$ and $\frac{9}{15}$ are **equivalent** because they represent the same portion (often loosely referred to as equal).

This understanding of equivalent fractions is very useful for conceptualization, but it isn't practical, in general, for determining when two fractions are equivalent. Generally, to determine if the two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent, we check to see that $a \times d = b \times c$. If those two products are equal, then the fractions are equal also.

EXAMPLE 3.53

Determining If Two Fractions Are Equivalent

Determine if $\frac{12}{30}$ and $\frac{14}{35}$ are equivalent fractions.

Solution

Applying the definition, $a = 12, b = 30, c = 14$ and $d = 35$. So $a \times d = 12 \times 35 = 420$. Also, $b \times c = 30 \times 14 = 420$. Since these values are equal, the fractions are equivalent.

YOUR TURN 3.53

1. Determine if $\frac{8}{14}$ and $\frac{12}{26}$ are equivalent fractions.

That $a \times d = b \times c$ indicates the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent is due to some algebra. One property of natural numbers, integers, and rational numbers (also irrational numbers) is that for any three numbers a, b, and c with $c \neq 0$, if $a = b$, then $a/c = b/c$. In other words, when two numbers are equal, then dividing both numbers by the same non-zero number, the two newly obtained numbers are also equal. We can apply that to $a \times d$ and $b \times c$, to show that $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if $a \times d = b \times c$.

If $a \times d = b \times c$, and $c \neq 0, d \neq 0$, we can divide both sides by and obtain the following: $\frac{a \times d}{c} = \frac{b \times c}{c}$. We can divide out the c on the right-hand side of the equation, resulting in $\frac{a \times d}{c} = b$. Similarly, we can divide both sides of the equation by d and obtain the following: $\frac{a \times d}{c \times d} = \frac{b}{d}$. We can divide out d the on the left-hand side of the equation, resulting in $\frac{a}{c} = \frac{b}{d}$. So, the rational numbers $\frac{a}{c}$ and $\frac{b}{d}$ are equivalent when $a \times d = b \times c$.

VIDEO

Equivalent Fractions (https://openstax.org/r/Equivalent_Fractions)

Recall that a **common divisor** or **common factor** of a set of integers is one that divides all the numbers of the set of numbers being considered. In a fraction, when the numerator and denominator have a common divisor, that common divisor can be **divided out**. This is often called **canceling the common factors** or, more colloquially, as **canceling**.

To show this, consider the fraction $\frac{36}{63}$. The numerator and denominator have the common factor 3. We can rewrite the fraction as $\frac{36}{63} = \frac{12 \times 3}{21 \times 3}$. The common divisor 3 is then divided out, or canceled, and we can write the fraction as $\frac{12 \times \cancel{3}}{21 \times \cancel{3}} = \frac{12}{21}$. The 3s have been crossed out to indicate they have been divided out. The process of dividing out two factors is also referred to as **reducing the fraction**.

If the numerator and denominator have no common positive divisors other than 1, then the rational number is in **lowest terms**.

The process of dividing out common divisors of the numerator and denominator of a fraction is called **reducing the fraction**.

One way to reduce a fraction to lowest terms is to determine the GCD of the numerator and denominator and divide out the GCD. Another way is to divide out common divisors until the numerator and denominator have no more common factors.

EXAMPLE 3.54

Reducing Fractions to Lowest Terms

Express the following rational numbers in lowest terms:

1. $\frac{36}{48}$
2. $\frac{100}{250}$
3. $\frac{51}{136}$

Solution

1. One process to reduce $\frac{36}{48}$ to lowest terms is to identify the GCD of 36 and 48 and divide out the GCD. The GCD of 36 and 48 is 12.
   Step 1: We can then rewrite the numerator and denominator by factoring 12 from both.

$\frac{36}{48} = \frac{12\times3}{12\times4}$

Step 2: We can now divide out the 12s from the numerator and denominator.

$\frac{36}{48} = \frac{\cancel{12}\times3}{\cancel{12}\times4} = \frac{3}{4}$

So, when $\frac{36}{48}$ is reduced to lowest terms, the result is $\frac{3}{4}$.

Alternately, you could identify a common factor, divide out that common factor, and repeat the process until the remaining fraction is in lowest terms.

Step 1: You may notice that 4 is a common factor of 36 and 48.

Step 2: Divide out the 4, as in $\frac{36}{48} = \frac{4\times9}{4\times12} = \frac{\cancel{4}\times9}{\cancel{4}\times12} = \frac{9}{12}$.

Step 3: Examining the 9 and 12, you identify 3 as a common factor and divide out the 3, as in $\frac{9}{12} = \frac{\cancel{3}\times3}{\cancel{3}\times4} = \frac{3}{4}$. The 3 and 4 have no common positive factors other than 1, so it is in lowest terms.

So, when $\frac{36}{48}$ is reduced to lowest terms, the result is $\frac{3}{4}$.

2. **Step 1:** To reduce $\frac{100}{250}$ to lowest terms, identify the GCD of 100 and 250. This GCD is 50.
   Step 2: We can then rewrite the numerator and denominator by factoring 50 from both.

   $\frac{100}{250} = \frac{50\times2}{50\times5}$.

   Step 3: We can now divide out the 50s from the numerator and denominator.

   $\frac{100}{250} = \frac{\cancel{50}\times2}{\cancel{50}\times5} = \frac{2}{5}$

   So, when $\frac{100}{250}$ is reduced to lowest terms, the result is $\frac{2}{5}$.

3. **Step 1:** To reduce $\frac{51}{136}$ to lowest terms, identify the GCD of 51 and 136. This GCD is 17.
   Step 2: We can then rewrite the numerator and denominator by factoring 17 from both.

   $\frac{51}{136} = \frac{17\times3}{17\times8}$

   Step 3: We can now divide out the 17s from the numerator and denominator.

   $\frac{51}{136} = \frac{\cancel{17}\times3}{\cancel{17}\times8} = \frac{3}{8}$

   So, when $\frac{51}{136}$ is reduced to lowest terms, the result is $\frac{3}{8}$.

YOUR TURN 3.54

1. Express $\frac{252}{840}$ in lowest terms.

VIDEO

[Reducing Fractions to Lowest Terms (https://openstax.org/r/Reducing_Fractions_to_Lowest_Terms)](https://openstax.org/r/Reducing_Fractions_to_Lowest_Terms)

TECH CHECK

Using Desmos to Find Lowest Terms

Desmos is a free [online calculator (https://openstax.org/r/calculator)](https://openstax.org/r/calculator). Desmos supports reducing fractions to lowest terms. When a fraction is entered, Desmos immediately calculates the decimal representation of the fraction.

However, to the left of the fraction, there is a button that, when clicked, shows the fraction in reduced form.

VIDEO

Using Desmos to Reduce a Fraction (https://openstax.org/r/Using_Desmos_to_Reduce_a_Fraction)

Adding and Subtracting Rational Numbers

Adding or subtracting rational numbers can be done with a calculator, which often returns a decimal representation, or by finding a common denominator for the rational numbers being added or subtracted.

TECH CHECK

Using Desmos to Add Rational Numbers in Fractional Form

To create a fraction in Desmos, enter the numerator, then use the division key (/) on your keyboard, and then enter the denominator. The fraction is then entered. Then click the right arrow key to exit the denominator of the fraction. Next, enter the arithmetic operation (+ or –). Then enter the next fraction. The answer is displayed dynamically (calculates as you enter). To change the Desmos result from decimal form to fractional form, use the fraction button (Figure 3.26) on the left of the line that contains the calculation:

Figure 3.26 Fraction button on the Desmos keyboard

EXAMPLE 3.55

Adding Rational Numbers Using Desmos

Calculate $\frac{23}{42} + \frac{9}{56}$ using Desmos.

Solution

Enter $\frac{23}{42} + \frac{9}{56}$ in Desmos. The result is displayed as 0.70833333333 (which is $0.708\overline{3}$). Clicking the fraction button to the left on the calculation line yields $\frac{17}{24}$.

YOUR TURN 3.55

1. Calculate $\frac{124}{297} + \frac{3}{125}$.

Performing addition and subtraction without a calculator may be more involved. When the two rational numbers have a common denominator, then adding or subtracting the two numbers is straightforward. Add or subtract the numerators, and then place that value in the numerator and the common denominator in the denominator. Symbolically, we write this as $\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$. This can be seen in the Figure 3.27, which shows $\frac{3}{20} + \frac{4}{20} = \frac{7}{20}$.

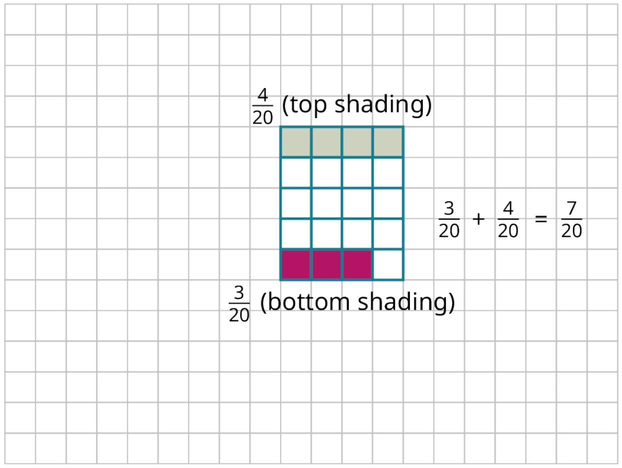


Figure 3.27 Partially Shaded Rectangle

It is customary to then write the result in lowest terms.

FORMULA

If c is a non-zero integer, then $\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$.

EXAMPLE 3.56

Adding Rational Numbers with the Same Denominator

Calculate $\frac{13}{28} + \frac{7}{28}$.

Solution

Since the rational numbers have the same denominator, we perform the addition of the numerators, $13 + 7$, and then place the result in the numerator and the common denominator, 28, in the denominator. $\frac{13}{28} + \frac{7}{28} = \frac{13+7}{28} = \frac{20}{28}$

Once we have that result, reduce to lowest terms, which gives $\frac{20}{28} = \frac{4 \times 5}{4 \times 7} = \frac{\cancel{4} \times 5}{\cancel{4} \times 7} = \frac{5}{7}$.

YOUR TURN 3.56

1. Calculate $\frac{38}{73} + \frac{7}{73}$.

EXAMPLE 3.57

Subtracting Rational Numbers with the Same Denominator

Calculate $\frac{45}{136} - \frac{17}{136}$.

Solution

Since the rational numbers have the same denominator, we perform the subtraction of the numerators, $45 - 17$, and then place the result in the numerator and the common denominator, 136, in the denominator.
$\frac{45}{136} - \frac{17}{136} - \frac{45-17}{136} = \frac{28}{136}$

Once we have that result, reduce to lowest terms, this gives $\frac{28}{136} = \frac{4 \times 7}{4 \times 34} = \frac{\cancel{4} \times 7}{\cancel{4} \times 34} = \frac{7}{34}$.

YOUR TURN 3.57

1. Calculate $\frac{21}{40} - \frac{8}{40}$.

When the rational numbers do not have common denominators, then we have to transform the rational numbers so that they do have common denominators. The common denominator that reduces work later in the problem is the LCM of the numerator and denominator. When adding or subtracting the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, we perform the following steps.

Step 1: Find $LCM(b, d)$.

Step 2: Calculate $n = \frac{LCM(b,d)}{b}$ and $m = \frac{LCM(b,d)}{d}$.

Step 3: Multiply the numerator and denominator of $\frac{a}{b}$ by n, yielding $\frac{a \times n}{b \times n}$.

Step 4: Multiply the numerator and denominator of $\frac{c}{d}$ by m, yielding $\frac{c \times m}{d \times m}$.

Step 5: Add or subtract the rational numbers from Steps 3 and 4, since they now have the common denominators.

You should be aware that the common denominator is $LCM(b, d)$. For the first denominator, we have $b \times n = b \times \frac{LCM(b,d)}{b} = LCM(b, d)$, since we multiply and divide $LCM(b, d)$ by the same number. For the same reason, $d \times m = d \times \frac{LCM(b,d)}{b} = LCM(b, d)$.

EXAMPLE 3.58

Adding Rational Numbers with Unequal Denominators

Calculate $\frac{11}{18} + \frac{2}{15}$.

Solution

The denominators of the fractions are 18 and 15, so we label $b = 18$ and $d = 15$.

Step 1: Find LCM(18,15). This is 90.

Step 2: Calculate n and m. $n = \frac{90}{18} = 5$ and $m = \frac{90}{15} = 6$.

Step 3: Multiplying the numerator and denominator of $\frac{11}{18}$ by $n = 5$ yields $\frac{11 \times 5}{18 \times 5} = \frac{55}{90}$.

Step 4: Multiply the numerator and denominator of $\frac{2}{15}$ by $m = 6$ yields $\frac{2 \times 6}{15 \times 6} = \frac{12}{90}$.

Step 5: Now we add the values from Steps 3 and 4: $\frac{55}{90} + \frac{12}{90} = \frac{67}{90}$.

This is in lowest terms, so we have found that $\frac{11}{18} + \frac{2}{15} = \frac{67}{90}$.

YOUR TURN 3.58

1. Calculate $\frac{4}{9} + \frac{7}{12}$.

EXAMPLE 3.59

Subtracting Rational Numbers with Unequal Denominators

Calculate $\frac{14}{25} - \frac{9}{70}$.

Solution

The denominators of the fractions are 25 and 70, so we label $b = 25$ and $d = 70$.

Step 1: Find LCM(25,70). This is 350.

Step 2: Calculate n and m: $n = \frac{350}{25} = 14$ and $m = \frac{350}{70} = 5$.

Step 3: Multiplying the numerator and denominator of $\frac{14}{25}$ by $n = 14$ yields $\frac{14 \times 14}{25 \times 14} = \frac{196}{350}$.

Step 4: Multiplying the numerator and denominator of $\frac{9}{70}$ by $m = 5$ yields $\frac{9 \times 5}{70 \times 5} = \frac{45}{350}$.

Step 5: Now we subtract the value from Step 4 from the value in Step 3: $\frac{196}{350} - \frac{45}{350} = \frac{151}{350}$.

This is in lowest terms, so we have found that $\frac{14}{25} - \frac{9}{70} = \frac{151}{350}$.

YOUR TURN 3.59

1. Calculate $\frac{10}{99} - \frac{17}{300}$.

VIDEO

[Adding and Subtracting Fractions with Different Denominators (https://openstax.org/r/Adding_and_Subtracting_Fractions)](https://openstax.org/r/Adding_and_Subtracting_Fractions)

Converting Between Improper Fractions and Mixed Numbers

One way to visualize a fraction is as parts of a whole, as in $\frac{5}{12}$ of a pizza. But when the numerator is larger than the denominator, as in $\frac{23}{12}$, then the idea of parts of a whole seems not to make sense. Such a fraction is an **improper fraction.** That kind of fraction could be written as an integer plus a fraction, which is a **mixed number**. The fraction $\frac{23}{12}$ rewritten as a mixed number would be $1\frac{11}{12}$. Arithmetically, $1\frac{11}{12}$ is equivalent to $1 + \frac{11}{12}$, which is read as "one and 11 twelfths."

Improper fractions can be rewritten as mixed numbers using division and remainders. To find the mixed number representation of an improper fraction, divide the numerator by the denominator. The quotient is the integer part, and the remainder becomes the numerator of the remaining fraction.

EXAMPLE 3.60

Rewriting an Improper Fraction as a Mixed Number

Rewrite $\frac{48}{13}$ as a mixed number.

Solution

When 48 is divided by 13, the result is 3 with a remainder of 9. So, we can rewrite $\frac{48}{13}$ as $3\frac{9}{13}$.

YOUR TURN 3.60

1. Rewrite $\frac{95}{26}$ as a mixed number.

VIDEO

[Converting an Improper Fraction to a Mixed Number Using Desmos (https://openstax.org/r/Improper_Fraction_to_Mixed_Number)](https://openstax.org/r/Improper_Fraction_to_Mixed_Number)

Similarly, we can convert a mixed number into an improper fraction. To do so, first convert the whole number part to a fraction by writing the whole number as itself divided by 1, and then add the two fractions.

Alternately, we can multiply the whole number part and the denominator of the fractional part. Next, add that product to the numerator. Finally, express the number as that product divided by the denominator.

EXAMPLE 3.61

Rewriting a Mixed Number as an Improper Fraction
Rewrite $5\frac{4}{9}$ as an improper fraction.

Solution
Step 1: Multiply the integer part, 5, by the denominator, 9, which gives $5 \times 9 = 45$.

Step 2: Add that product to the numerator, which gives $45 + 4 = 49$.

Step 3: Write the number as the sum, 49, divided by the denominator, 9, which gives $\frac{49}{9}$.

YOUR TURN 3.61

1. Rewrite $9\frac{5}{14}$ as an improper fraction.

TECH CHECK

Using Desmos to Rewrite a Mixed Number as an Improper Fraction

Desmos can be used to convert from a mixed number to an improper fraction. To do so, we use the idea that a mixed number, such as $5\frac{6}{11}$, is another way to represent $5 + \frac{6}{11}$. If $5 + \frac{6}{11}$ is entered in Desmos, the result is the decimal form of the number. However, clicking the fraction button to the left will convert the decimal to an improper fraction, $\frac{61}{11}$. As an added bonus, Desmos will automatically reduce the fraction to lowest terms.

Converting Rational Numbers Between Decimal and Fraction Forms

Understanding what decimals represent is needed before addressing conversions between the fractional form of a number and its **decimal form**, or writing a number in **decimal notation**. The decimal number 4.557 is equal to $4\frac{557}{1{,}000}$. The decimal portion, .557, is 557 divided by 1,000. To write any decimal portion of a number expressed as a terminating decimal, divide the decimal number by 10 raised to the power equal to the number of decimal digits. Since there were three decimal digits in 4.557, we divided 557 by $10^3 = 1000$.

Decimal representations may be very long. It is convenient to **round off** the decimal form of the number to a certain number of decimal digits. To round off the decimal form of a number to n (decimal) digits, examine the $(n + 1)$st decimal digit. If that digit is 0, 1, 2, 3, or 4, the number is rounded off by writing the number to the nth decimal digit and no further. If the $(n + 1)$st decimal digit is 5, 6, 7, 8, or 9, the number is rounded off by writing the number to the nth digit, then replacing the nth digit by one more than the nth digit.

EXAMPLE 3.62

Rounding Off a Number in Decimal Form to Three Digits
Round 5.67849 to three decimal digits.

Solution
The third decimal digit is 8. The digit following the 8 is 4. When the digit is 4, we write the number only to the third digit. So, 5.67849 rounded off to three decimal places is 5.678.

YOUR TURN 3.62

1. Round 5.1082 to three decimal places.

EXAMPLE 3.63

Rounding Off a Number in Decimal Form to Four Digits

Round 45.11475 to four decimal digits.

Solution

The fourth decimal digit is 7. The digit following the 7 is 5. When the digit is 5, we write the number only to the fourth decimal digit, 45.1147. We then replace the fourth decimal digit by one more than the fourth digit, which yields 45.1148. So, 45.11475 rounded off to four decimal places is 45.1148.

YOUR TURN 3.63

1. Round 18.6298 to two decimal places.

To convert a rational number in fraction form to decimal form, use your calculator to perform the division.

EXAMPLE 3.64

Converting a Rational Number in Fraction Form into Decimal Form

Convert $\frac{47}{25}$ into decimal form.

Solution

Using a calculator to divide 47 by 25, the result is 1.88.

YOUR TURN 3.64

1. Convert $\frac{48}{30}$ into decimal form.

Converting a terminating decimal to the fractional form may be done in the following way:

Step 1: Count the number of digits in the decimal part of the number, labeled n.

Step 2: Raise 10 to the nth power.

Step 3: Rewrite the number without the decimal.

Step 4: The fractional form is the number from Step 3 divided by the result from Step 2.

This process works due to what decimals represent and how we work with mixed numbers. For example, we could convert the number 7.4536 to fractional from. The decimal part of the number, the .4536 part of 7.4536, has four digits. By the definition of decimal notation, the decimal portion represents $\frac{4{,}536}{10^4} = \frac{4{,}536}{10{,}000}$. The decimal number 7.4536 is equal to the improper fraction $7\frac{4{,}536}{10{,}000}$. Adding those to fractions yields $\frac{74{,}536}{10{,}000}$.

EXAMPLE 3.65

Converting from Decimal Form to Fraction Form with Terminating Decimals

Convert 3.2117 to fraction form.

Solution

Step 1: There are four digits after the decimal point, so $n = 4$.

Step 2: Raise 10 to the fourth power, $10^4 = 10{,}000$.

Step 3: When we remove the decimal point, we have 32,117.

Step 4: The fraction has as its numerator the result from Step 3 and as its denominator the result of Step 2, which is the fraction $\frac{32{,}117}{10{,}000}$.

> YOUR TURN 3.65

1. Convert 17.03347 to fraction form.

The process is different when converting from the decimal form of a rational number into fraction form when the decimal form is a repeating decimal. This process is not covered in this text.

Multiplying and Dividing Rational Numbers

Multiplying rational numbers is less complicated than adding or subtracting rational numbers, as there is no need to find common denominators. To multiply rational numbers, multiply the numerators, then multiply the denominators, and write the numerator product divided by the denominator product. Symbolically, $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$. As always, rational numbers should be reduced to lowest terms.

FORMULA

If b and d are non-zero integers, then $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$.

EXAMPLE 3.66

Multiplying Rational Numbers

Calculate $\frac{12}{25} \times \frac{10}{21}$.

Solution

Multiply the numerators and place that in the numerator, and then multiply the denominators and place that in the denominator.

$$\frac{12}{25} \times \frac{10}{21} = \frac{12 \times 10}{25 \times 21} = \frac{120}{525}$$

This is not in lowest terms, so this needs to be reduced. The GCD of 120 and 525 is 15.

$$\frac{120}{525} = \frac{15 \times 8}{15 \times 35} = \frac{8}{35}$$

> YOUR TURN 3.66

1. Calculate $\frac{45}{88} \times \frac{28}{75}$.

VIDEO

Multiplying Fractions (https://openstax.org/r/Multiplying_Fractions)

As with multiplication, division of rational numbers can be done using a calculator.

EXAMPLE 3.67

Dividing Decimals with a Calculator

Calculate 3.45 ÷ 2.341 using a calculator. Round to three decimal places if necessary.

Solution

Using a calculator, we obtain 1.473729175565997. Rounding to three decimal places we have 1.474.

YOUR TURN 3.67

1. Calculate $45.63 \div 17.13$ using a calculator. Round to three decimal places, if necessary.

Before discussing division of fractions without a calculator, we should look at the reciprocal of a number. The **reciprocal** of a number is 1 divided by the number. For a fraction, the reciprocal is the fraction formed by switching the numerator and denominator. For the fraction $\frac{a}{b}$, the reciprocal is $\frac{b}{a}$. An important feature for a number and its reciprocal is that their product is 1.

When dividing two fractions by hand, find the reciprocal of the **divisor** (the number that is being divided into the other number). Next, replace the divisor by its reciprocal and change the division into multiplication. Then, perform the multiplication. Symbolically, $\frac{b}{a} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c}$. As before, reduce to lowest terms.

FORMULA

If b, c and d are non-zero integers, then $\frac{b}{a} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c}$.

EXAMPLE 3.68

Dividing Rational Numbers

1. Calculate $\frac{4}{21} \div \frac{6}{35}$.
2. Calculate $\frac{1}{8} \div \frac{5}{28}$.

Solution

1. **Step 1:** Find the reciprocal of the number being divided by $\frac{6}{35}$. The reciprocal of that is $\frac{35}{6}$.

   Step 2: Multiply the first fraction by that reciprocal.

   $\frac{4}{21} \div \frac{6}{35} = \frac{4}{21} \times \frac{35}{6} = \frac{140}{126}$

   The answer, $\frac{140}{126}$ is not in lowest terms. The GCD of 140 and 126 is 14. Factoring and canceling gives $\frac{140}{126} = \frac{14 \times 10}{14 \times 9} = \frac{10}{9}$.

2. **Step 1:** Find the reciprocal of the number being divided by, which is $\frac{5}{28}$. The reciprocal of that is $\frac{28}{5}$.

   Step 2: Multiply the first fraction by that reciprocal: $\frac{1}{8} \div \frac{5}{28} = \frac{1}{8} \times \frac{28}{5} = \frac{28}{40}$

   The answer, $\frac{28}{40}$, is not in lowest reduced form. The GCD of 28 and 40 is 4. Factoring and canceling gives $\frac{28}{40} = \frac{4 \times 7}{4 \times 10} = \frac{7}{10}$.

YOUR TURN 3.68

1. Calculate $\frac{46}{175} \div \frac{69}{285}$.
2. Calculate $\frac{3}{40} \div \frac{42}{55}$.

VIDEO

Dividing Fractions (https://openstax.org/r/Dividing_Fractions)

Applying the Order of Operations to Simplify Expressions

The order of operations for rational numbers is the same as for integers, as discussed in Order of Operations. The order of operations makes it easier for anyone to correctly calculate and represent. The order follows the well-known acronym PEMDAS:

P	Parentheses
E	Exponents
M/D	Multiplication and division
A/S	Addition and subtraction

The first step in calculating using the order of operations is to perform operations inside the parentheses. Moving down the list, next perform all exponent operations moving from left to right. Next (left to right once more), perform all multiplications and divisions. Finally, perform the additions and subtractions.

EXAMPLE 3.69

Applying the Order of Operations with Rational Numbers

Correctly apply the rules for the order of operations to accurately compute $\left(\frac{5}{7}-\frac{2}{7}\right)\times 2^3$.

Solution

Step 1: To calculate this, perform all calculations within the parentheses before other operations.

$$\left(\frac{5}{7}-\frac{2}{7}\right)\times 2^3=\left(\frac{3}{7}\right)\times 2^3$$

Step 2: Since all parentheses have been cleared, we move left to right, and compute all the exponents next.

$$\left(\frac{3}{7}\right)\times 2^3=\left(\frac{3}{7}\right)\times 8$$

Step 3: Now, perform all multiplications and divisions, moving left to right.

$$\left(\frac{3}{7}\right)\times 8=\frac{24}{7}$$

YOUR TURN 3.69

1. Correctly apply the rules for the order of operations to accurately compute $\left(\frac{3}{16}+\frac{7}{16}\right)^2+\frac{1}{5}\div\frac{3}{10}$.

EXAMPLE 3.70

Applying the Order of Operations with Rational Numbers

Correctly apply the rules for the order of operations to accurately compute $4+\frac{2}{3}\div\left(\left(\frac{5}{9}\right)^2-\left(\frac{2}{3}+5\right)\right)^2$.

Solution

To calculate this, perform all calculations within the parentheses before other operations. Evaluate the innermost parentheses first. We can work separate parentheses expressions at the same time.

Step 1: The innermost parentheses contain $\frac{2}{3}+5$. Calculate that first, dividing after finding the common denominator.

$$4+\frac{2}{3}\div\left(\left(\frac{5}{9}\right)^2-\left(\mathbf{\frac{2}{3}+5}\right)\right)^2$$
$$=4+\frac{2}{3}\div\left(\left(\frac{5}{9}\right)^2-\left(\mathbf{\frac{2}{3}+\frac{5}{1}}\right)\right)^2$$
$$=4+\frac{2}{3}\div\left(\left(\frac{5}{9}\right)^2-\left(\mathbf{\frac{2}{3}+\frac{15}{3}}\right)\right)^2$$
$$=4+\frac{2}{3}\div\left(\left(\frac{5}{9}\right)^2-\left(\mathbf{\frac{17}{3}}\right)\right)^2$$

Step 2: Calculate the exponent in the parentheses, $\left(\frac{5}{9}\right)^2$.

$$4 + \frac{2}{3} \div \left(\left(\frac{5}{9} \right)^{\mathbf{2}} - \left(\frac{17}{3} \right) \right)^2$$
$$= 4 + \frac{2}{3} \div \left(\left(\frac{\mathbf{25}}{\mathbf{81}} \right) - \left(\frac{17}{3} \right) \right)^2$$

Step 3: Subtract inside the parentheses is done, using a common denominator.

$$4 + \frac{2}{3} \div \left(\left(\frac{\mathbf{25}}{\mathbf{81}} \right) - \left(\frac{\mathbf{17}}{\mathbf{3}} \right) \right)^2$$
$$4 + \frac{2}{3} \div \left(\left(\frac{\mathbf{25}}{\mathbf{81}} \right) - \left(\frac{\mathbf{17 \times 27}}{\mathbf{3 \times 27}} \right) \right)^2$$
$$4 + \frac{2}{3} \div \left(\left(\frac{\mathbf{25}}{\mathbf{81}} \right) - \left(\frac{\mathbf{459}}{\mathbf{81}} \right) \right)^2$$
$$4 + \frac{2}{3} \div \left(\left(\frac{\mathbf{-434}}{\mathbf{81}} \right) \right)^2$$

Step 4: At this point, evaluate the exponent and divide.

$$4 + \frac{2}{3} \div \left(\left(\frac{\mathbf{-434}}{\mathbf{81}} \right) \right)^2$$
$$4 + \frac{2}{3} \div \left(\frac{\mathbf{188{,}356}}{\mathbf{6{,}561}} \right)$$
$$= 4 + \frac{\mathbf{2}}{\mathbf{3}} \times \left(\frac{\mathbf{6{,}561}}{\mathbf{188{,}356}} \right)$$
$$= \mathbf{4} + \frac{\mathbf{2{,}187}}{\mathbf{94{,}178}}$$

Step 5: Add.

$$\mathbf{4} + \frac{\mathbf{2{,}187}}{\mathbf{94{,}178}}$$
$$= \frac{378{,}899}{94{,}178}$$

Had this been done on a calculator, the decimal form of the answer would be 4.0232 (rounded to four decimal places).

YOUR TURN 3.70

1. Correctly apply the rules for the order of operations to accurately compute $\left(\frac{3}{5} + 2\right) \times \left(\frac{4}{5} - \frac{1}{2}\right)^2 \div \frac{11}{15}$.

VIDEO

Order of Operations Using Fractions (https://openstax.org/r/Operations_Using_Fractions)

Applying the Density Property of Rational Numbers

Between any two rational numbers, there is another rational number. This is called the **density property** of the rational numbers.

Finding a rational number between any two rational numbers is very straightforward.

Step 1: Add the two rational numbers.

Step 2: Divide that result by 2.

The result is always a rational number. This follows what we know about rational numbers. If two fractions are added, then the result is a fraction. Also, when a fraction is divided by a fraction (and 2 is a fraction), then we get another fraction. This two-step process will give a rational number, provided the first two numbers were rational.

EXAMPLE 3.71

Applying the Density Property of Rational Numbers

Demonstrate the density property of rational numbers by finding a rational number between $\frac{4}{11}$ and $\frac{7}{12}$.

Solution

To find a rational number between $\frac{4}{11}$ and $\frac{7}{12}$:

Step 1: Add the fractions.

$$\frac{4}{11} + \frac{7}{12} = \frac{4\times12}{11\times12} + \frac{7\times11}{12\times11} = \frac{48}{132} + \frac{77}{132} = \frac{125}{132}$$

Step 2: Divide the result by 2. Recall that to divide by 2, you multiply by the reciprocal of 2. The reciprocal of 2 is $\frac{1}{2}$, as seen below.

$$\frac{125}{132} \div 2 = \frac{125}{132} \times \frac{1}{2} = \frac{125}{264}$$

So, one rational number between $\frac{4}{11}$ and $\frac{7}{12}$ is $\frac{125}{264}$.

We could check that the number we found is between the other two by finding the decimal representation of the numbers. Using a calculator, the decimal representations of the rational numbers are 0.363636..., 0.473484848..., and 0.5833333.... Here it is clear that $\frac{125}{264}$ is between $\frac{4}{11}$ and $\frac{7}{12}$.

YOUR TURN 3.71

1. Demonstrate the density property of rational numbers by finding a rational number between $\frac{27}{13}$ and $\frac{21}{10}$.

Solving Problems Involving Rational Numbers

Rational numbers are used in many situations, sometimes to express a portion of a whole, other times as an expression of a ratio between two quantities. For the sciences, converting between units is done using rational numbers, as when converting between gallons and cubic inches. In chemistry, mixing a solution with a given concentration of a chemical per unit volume can be solved with rational numbers. In demographics, rational numbers are used to describe the distribution of the population. In dietetics, rational numbers are used to express the appropriate amount of a given ingredient to include in a recipe. As discussed, the application of rational numbers crosses many disciplines.

EXAMPLE 3.72

Mixing Soil for Vegetables

James is mixing soil for a raised garden, in which he plans to grow a variety of vegetables. For the soil to be suitable, he determines that $\frac{2}{5}$ of the soil can be topsoil, but $\frac{2}{5}$ needs to be peat moss and $\frac{1}{5}$ has to be compost. To fill the raised garden bed with 60 cubic feet of soil, how much of each component does James need to use?

Solution

In this example, we know the proportion of each component to mix, and we know the total amount of the mix we need. In this kind of situation, we need to determine the appropriate amount of each component to include in the mixture. For each component of the mixture, multiply 60 cubic feet, which is the total volume of the mixture we want, by the fraction required of the component.

Step 1: The required fraction of topsoil is $\frac{2}{5}$, so James needs $60 \times \frac{2}{5}$ cubic feet of topsoil. Performing the multiplication, James needs $60 \times \frac{2}{5} = \frac{120}{5} = 24$ (found by treating the fraction as division, and 120 divided by 5 is 24) cubic feet of topsoil.

Step 2: The required fraction of peat moss is also $\frac{2}{5}$, so he also needs $60 \times \frac{2}{5}$ cubic feet, or $60 \times \frac{2}{5} = \frac{120}{5} = 24$ cubic feet of peat moss.

Step 3: The required fraction of compost is $\frac{1}{5}$. For the compost, he needs $60 \times \frac{1}{5} = \frac{60}{5} = 12$ cubic feet.

YOUR TURN 3.72

1. Ashley wants to study for 10 hours over the weekend. She plans to spend half the time studying math, a quarter

of the time studying history, an eighth of the time studying writing, and the remaining eighth of the time studying physics. How much time will Ashley spend on each of those subjects?

EXAMPLE 3.73

Determining the Number of Specialty Pizzas

At Bella's Pizza, one-third of the pizzas that are ordered are one of their specialty varieties. If there are 273 pizzas ordered, how many were specialty pizzas?

Solution

One-third of the whole are specialty pizzas, so we need one-third of 273, which gives $\frac{1}{3} \times 273 = \frac{273}{3} = 91$, found by dividing 273 by 3. So, 91 of the pizzas that were ordered were specialty pizzas.

YOUR TURN 3.73

1. Danny, a nutritionist, is designing a diet for her client, Callum. Danny determines that Callum's diet should be 30% protein. If Callum consumes 2,400 calories per day, how many calories of protein should Danny tell Callum to consume?

VIDEO

[Finding a Fraction of a Total (https://openstax.org/r/Finding_Fraction_of_Total)](https://openstax.org/r/Finding_Fraction_of_Total)

Using Fractions to Convert Between Units

A common application of fractions is called **unit conversion**, or **converting units**, which is the process of changing from the units used in making a measurement to different units of measurement.

For instance, 1 inch is (approximately) equal to 2.54 cm. To convert between units, the two equivalent values are made into a fraction. To convert from the first type of unit to the second type, the fraction has the second unit as the numerator, and the first unit as the denominator.

From the inches and centimeters example, to change from inches to centimeters, we use the fraction $\frac{2.54 \text{ cm}}{1 \text{ in}}$. If, on the other hand, we wanted to convert from centimeters to inches, we'd use the fraction $\frac{1 \text{ in}}{2.54 \text{ cm}}$. This fraction is multiplied by the number of units of the type you are converting *from*, which means the units of the denominator are the same as the units being multiplied.

EXAMPLE 3.74

Converting Liters to Gallons

It is known that 1 liter (L) is 0.264172 gallons (gal). Use this to convert 14 liters into gallons.

Solution

We know that 1 liter = 0.264172 gal. Since we are converting from liters, when we create the fraction we use, make sure the liter part of the equivalence is in the denominator. So, to convert the 14 liters to gallons, we multiply 14 by $\frac{1 \text{ gal}}{0.264172 \text{ gal }/1 \text{ liter}}$. Notice the gallon part is in the numerator since we're converting *to* gallons, and the liter part is in the denominator since we are converting *from* liters. Performing this and rounding to three decimal places, we find that 14 liters is $14 \text{ liter} \times \frac{0.264172 \text{ gal}}{1 \text{ liter}} = 3.69841 \text{ gal}$.

YOUR TURN 3.74

1. One mile is equal to 1.60934 km. Convert 200 miles to kilometers. Round off the answer to three decimal places.

EXAMPLE 3.75

Converting Centimeters to Inches

It is known that 1 inch is 2.54 centimeters. Use this to convert 100 centimeters into inches.

Solution

We know that 1 inch = 2.54 cm. Since we are converting from centimeters, when we create the fraction we use, make sure the centimeter part of the equivalence is in the denominator, $\frac{1 \text{ in}}{2.54 \text{ cm}}$. To convert the 100 cm to inches, multiply 100 by $\frac{1 \text{ in}}{2.54 \text{ cm}}$. Notice the inch part is in the numerator since we're converting *to* inches, and the centimeter part is in the denominator since we are converting *from* centimeters. Performing this and rounding to three decimal places, we obtain $100 \text{ cm} \times \frac{1 \text{ in}}{2.54 \text{ cm}} = 39.370 \text{ in}$. This means 100 cm equals 39.370 in.

YOUR TURN 3.75

1. It is known that 4 quarts equals 3.785 liters. If you have 25 quarts, how many liters do you have? Round off to three decimal places.

VIDEO

Converting Units (https://openstax.org/r/Converting_Units)

Defining and Applying Percent

A **percent** is a specific rational number and is literally per 100. n percent, denoted n%, is the fraction $\frac{n}{100}$.

EXAMPLE 3.76

Rewriting a Percentage as a Fraction

Rewrite the following as fractions:

1. 31%
2. 93%

Solution

1. Using the definition and $n = 31$, 31% in fraction form is $\frac{31}{100}$.
2. Using the definition and $n = 93$, 93% in fraction form is $\frac{93}{100}$.

YOUR TURN 3.76

Rewrite the following as fractions:

1. 4%
2. 50%

EXAMPLE 3.77

Rewriting a Percentage as a Decimal

Rewrite the following percentages in decimal form:

1. 54%
2. 83%

Solution

1. Using the definition and $n = 54$, 54% in fraction form is $\frac{54}{100}$. Dividing a number by 100 moves the decimal two

places to the left; 54% in decimal form is then 0.54.

2. Using the definition and $n = 83$, 83% in fraction form is $\frac{83}{100}$. Dividing a number by 100 moves the decimal two places to the left; 83% in decimal form is then 0.83.

YOUR TURN 3.77

Rewrite the following percentages in decimal form:

1. 14%
2. 7%

You should notice that you can simply move the decimal two places to the left without using the fractional definition of percent.

Percent is used to indicate a fraction of a total. If we want to find 30% of 90, we would perform a multiplication, with 30% written in either decimal form or fractional form. The 90 is the **total**, 30 is the **percentage**, and 27 (which is 0.30×90) is the **percentage of the total**.

FORMULA

$n\%$ of x items is $\frac{n}{100} \times x$. The x is referred to as the **total**, the n is referred to as the **percent** or **percentage**, and the value obtained from $\frac{n}{100} \times x$ is the **part** of the total and is also referred to as the **percentage of the total**.

EXAMPLE 3.78

Finding a Percentage of a Total

1. Determine 40% of 300.
2. Determine 64% of 190.

Solution

1. The total is 300, and the percentage is 40. Using the decimal form of 40% and multiplying we obtain $0.40 \times 300 = 120$.
2. The total is 190, and the percentage is 64. Using the decimal form of 64% and multiplying we obtain $0.64 \times 190 = 121.6$.

YOUR TURN 3.78

1. Determine 25% of 1,200.
2. Determine 53% of 1,588.

In the previous situation, we knew the total and we found the percentage of the total. It may be that we know the percentage of the total, and we know the percent, but we don't know the total. To find the total if we know the percentage the percentage of the total, use the following formula.

FORMULA

If we know that n% of the total is x, then the total is $\frac{100 \times x}{n}$.

EXAMPLE 3.79

Finding the Total When the Percentage and Percentage of the Total Are Known

1. What is the total if 28% of the total is 140?
2. What is the total if 6% of the total is 91?

Solution

1. 28 is the percentage, so $n = 28$. 28% of the total is 140, so $x = 140$. Using those we find that the total was $\frac{100 \times 140}{28} = 500$.
2. 6 is the percentage, so $n = 6$. 6% of the total is 91, so $x = 91$. Using those we find that the total was $\frac{100 \times 91}{6} = 1{,}516.6$.

YOUR TURN 3.79

1. What is the total if 25% of the total is 30?
2. What is the total if 45% of the total is 360?

The percentage can be found if the total and the percentage of the total is known. If you know the total, and the percentage of the total, first divide the part by the total. Move the decimal two places to the right and append the symbol %. The percentage may be found using the following formula.

FORMULA

The percentage, n, of b that is a is $\frac{a}{b} \times 100\%$.

EXAMPLE 3.80

Finding the Percentage When the Total and Percentage of the Total Are Known

Find the percentage in the following:

1. Total is 300, percentage of the total is 60.
2. Total is 440, percentage of the total is 176.

Solution

1. The total is 300; the percentage of the total is 60. Calculating yields 0.2. Moving the decimal two places to the right gives 20. Appending the percentage to this number results in 20%. So, 60 is 20% of 300.
2. The total is 440; the percentage of the total is 176. Calculating yields 0.4. Moving the decimal two places to the right gives 40. Appending the percentage to this number results in 40%. So, 176 is 40% of 440.

YOUR TURN 3.80

Find the percentage in the following:

1. Total is 1,000, percentage of the total is 70.
2. Total is 500, percentage of the total is 425.

Solve Problems Using Percent

In the media, in research, and in casual conversation percentages are used frequently to express proportions. Understanding how to use percent is vital to consuming media and understanding numbers. Solving problems using percentages comes down to identifying which of the three components of a percentage you are given, the total, the percentage, or the percentage of the total. If you have two of those components, you can find the third using the methods outlined previously.

EXAMPLE 3.81

Percentage of Students Who Are Sleep Deprived

A study revealed that 70% of students suffer from sleep deprivation, defined to be sleeping less than 8 hours per night. If the survey had 400 participants, how many of those participants had less than 8 hours of sleep per night?

Solution

The percentage of interest is 70%. The total number of students is 400. With that, we can find how many were in the percentage of the total, or, how many were sleep deprived. Applying the formula from above, the number who were sleep deprived was $0.70 \times 400 = 280$; 280 students on the study were sleep deprived.

YOUR TURN 3.81

1. Riley has a daily calorie intake of 2,200 calories and wants to take in 20% of their calories as protein. How many calories of protein should be in their daily diet?

EXAMPLE 3.82

Amazon Prime Subscribers

There are 126 million users who are U.S. Amazon Prime subscribers. If there are 328.2 million residents in the United States, what percentage of U.S. residents are Amazon Prime subscribers?

Solution

We are asked to find the percentage. To do so, we divide the percentage of the total, which is 126 million, by the total, which is 328.2 million. Performing this division and rounding to three decimal places yields $\frac{126}{328.2} = 0.384$. The decimal is moved to the right by two places, and a % sign is appended to the end. Doing this shows us that 38.4% of U.S. residents are Amazon Prime subscribers.

YOUR TURN 3.82

1. A small town has 450 registered voters. In the primaries, 54 voted. What percentage of registered voters in that town voted in the primaries?

EXAMPLE 3.83

Finding the Percentage When the Total and Percentage of the Total Are Known

Evander plays on the basketball team at their university and 73% of the athletes at their university receive some sort of scholarship for attending. If they know 219 of the student-athletes receive some sort of scholarship, how many student-athletes are at the university?

Solution

We need to find the total number of student-athletes at Evander's university.

Step 1: Identify what we know. We know the percentage of students who receive some sort of scholarship, 73%. We also know the number of athletes that form the part of the whole, or 219 student-athletes.

Step 2: To find the total number of student-athletes, use $\frac{100 \times x}{n}$, with $x = 219$ and $n = 73$. Calculating with those values yields $\frac{100 \times 219}{73} = 300$.

So, there are 300 total student-athletes at Evander's university

> **YOUR TURN 3.83**
>
> 1. A store declares a deep discount of 40% for an item, which they say will save \$30. What was the original price of the item?

Check Your Understanding

17. Identify which of the following are rational numbers.
$-41,\ \sqrt{13},\ \frac{4}{3},\ 2.75,\ 0.2\overline{13}$

18. Express $\frac{18}{30}$ in lowest terms.

19. Calculate $\frac{3}{8}+\frac{5}{12}$ and express in lowest terms.

20. Convert 0.34 into fraction form.

21. Convert $\frac{47}{12}$ into a mixed number.

22. Calculate $\frac{2}{9}\times\frac{21}{22}$ and express in lowest terms.

23. Calculate $\frac{2}{5}\div\frac{3}{10}+\frac{1}{6}$.

24. Identify a rational number between $\frac{7}{8}$ and $\frac{20}{21}$.

25. Convert $\frac{47}{12}$ into a mixed number.

26. Lina decides to save $\frac{1}{8}$ of her take-home pay every paycheck. Her most recent paycheck was for \$882. How much will she save from that paycheck?

27. Determine 38% of 600.

28. A microchip factory has decided to increase its workforce by 10%. If it currently has 70 employees, how many new employees will the factory hire?

SECTION 3.4 EXERCISES

For the following exercises, identify which of the following are rational numbers.

1. 4.598
2. $\sqrt{144}$
3. $\sqrt{131}$

For the following exercises, reduce the fraction to lowest terms

4. $\frac{8}{10}$
5. $\frac{30}{105}$
6. $\frac{36}{539}$
7. $\frac{231}{490}$
8. $\frac{750}{17{,}875}$

For the following exercises, do the indicated conversion. If it is a repeating decimal, use the correct notation.

9. Convert $\frac{25}{6}$ to a mixed number.
10. Convert $\frac{240}{53}$ to a mixed number.
11. Convert $2\frac{3}{8}$ to an improper fraction.
12. Convert $15\frac{7}{30}$ to an improper fraction.
13. Convert $\frac{4}{9}$ to decimal form.
14. Convert $\frac{13}{20}$ to decimal form.
15. Convert $\frac{27}{625}$ to decimal form.
16. Convert $\frac{11}{14}$ to decimal form.
17. Convert 0.23 to fraction form and reduce to lowest terms.

18. Convert 3.8874 to fraction form and reduce to lowest terms.

For the following exercises, perform the indicated operations. Reduce to lowest terms.

19. $\frac{3}{5} + \frac{3}{10}$

20. $\frac{3}{14} + \frac{8}{21}$

21. $\frac{13}{36} - \frac{14}{99}$

22. $\frac{13}{24} - \frac{4}{117}$

23. $\frac{3}{7} \times \frac{21}{48}$

24. $\frac{48}{143} \times \frac{77}{120}$

25. $\frac{14}{27} \div \frac{7}{12}$

26. $\frac{44}{75} \div \frac{484}{285}$

27. $\left(\frac{3}{5} + \frac{2}{7}\right) \times \frac{10}{21}$

28. $\frac{3}{8} \times \left(\frac{13}{12} - \frac{35}{36}\right)$

29. $\left(\frac{3}{7} + \frac{5}{16}\right)^2 - \frac{5}{12}$

30. $\frac{3}{8} \times \left(\frac{4}{9} - \frac{1}{8}\right)^2$

31. $\left(\frac{2}{5} \times \left(\frac{7}{8} - \frac{2}{3}\right)\right)^2 \div \left(\frac{4}{9} + \frac{5}{6}\right) + \frac{7}{12}$

32. $\left(\frac{1}{5} \div \left(\frac{3}{10} + \frac{11}{15}\right)\right) \times \left(\frac{2}{21} + \frac{5}{9}\right) - \left(\frac{8}{15} \div \frac{4}{33}\right)^2$

33. Find a rational number between $\frac{8}{17}$ and $\frac{15}{28}$

34. Find a rational number between $\frac{3}{50}$ and $\frac{13}{98}$.

35. Find two rational numbers between $\frac{3}{10}$ and $\frac{19}{45}$.

36. Find three rational numbers between $\frac{5}{12}$ and $\frac{175}{308}$.

37. Convert 24% to fraction form and reduce completely.

38. Convert 95% to fraction form and reduce completely.

39. Convert 0.23 to a percentage.

40. Convert 1.22 to a percentage.

41. Determine 30% of 250.

42. Determine 75% of 600.

43. If 25% of a group is 41 members, how many members total are in the group?

44. If 80% of the total is 60, how much is in the total?

45. 13 is what percent of 20?

46. 80 is what percent of 320?

47. Professor Donalson's history of film class has 60 students. Of those students, $\frac{2}{5}$ say their favorite movie genre is comedy. How many of the students in Professor Donalson's class name comedy as their favorite movie genre?

48. Naia's dormitory floor has 80 residents. Of those, $\frac{3}{8}$ play *Fortnight* for at least 15 hours per week. How many students on Naia's floor play *Fortnight* at least 15 hours per week?

49. In Tara's town there are 24,000 people. Of those, $\frac{13}{100}$ are food insecure. How many people in Tara's town are food insecure?

50. Roughly $\frac{4}{5}$ of air is nitrogen. If an enclosure holds 2,000 liters of air, how many liters of nitrogen should be expected in the enclosure?

51. To make the dressing for coleslaw, Maddie needs to mix it with $\frac{3}{5}$ mayonnaise and $\frac{2}{5}$ apple cider vinegar. If Maddie wants to have 8 cups of dressing, how many cups of mayonnaise and how many cups of apple cider vinegar does Maddie need?

52. Malika is figuring out their schedule. They wish to spend $\frac{4}{15}$ of their time sleeping, $\frac{1}{3}$ of their time studying and going to class, $\frac{1}{5}$ of their time at work, and $\frac{2}{15}$ of their time doing other activities, such as entertainment or exercising. There are 168 hours in a week. How many hours in a week will Malika spend:

a. Sleeping?
b. Studying and going to class?
c. Not sleeping?

53. Roughly 20.9% of air is oxygen. How much oxygen is there in 200 liters of air?

54. 65% of college students graduate within 6 years of beginning college. A first-year cohort at a college contains 400 students. How many are expected to graduate within 6 years?

55. A 20% discount is offered on a new laptop. How much is the discount if the new laptop originally cost $700?
56. Leya helped at a neighborhood sale and was paid 5% of the proceeds. If Leya is paid $171.25, what were the total proceeds from the neighborhood sale?
57. **Unit Conversion.** 1 kilogram (kg) is equal to 2.20462 pounds. Convert 13 kg to pounds. Round to three decimal places, if necessary.
58. **Unit Conversion.** 1 kilogram (kg) is equal to 2.20462 pounds. Convert 200 pounds to kilograms. Round to three decimal places, if necessary.
59. **Unit Conversion.** There are 12 inches in a foot, 3 feet in a yard, and 1,760 yards in a mile. Convert 10 miles to inches. To do so, first convert miles to yards. Next, convert the yards to feet. Last, convert the feet to inches.
60. **Unit Conversion.** There are 1,000 meters (m) in a kilometer (km), and 100 centimeters (cm) in a meter. Convert 4 km to centimeters.
61. **Markup.** In this exercise, we introduce the concept of **markup**. The **markup** on an item is the difference between how much a store sells an item for and how much the store paid for the item. Suppose Wegmans (a northeastern U.S. grocery chain) buys cereal at $1.50 per box and sells the cereal for $2.29.
    a. Determine the markup in dollars.
    b. The markup is what percent of the original cost? Round the percentage to one decimal place.
62. In this exercise, we explore what happens when an item is marked up by a percentage, and then marked down using the same percentage.
    Wegmans purchases an item for $5 per unit. The markup on the item is 25%.
    a. Calculate the markup on the item, in dollars.
    b. What is the price for which Wegmans sells the item? This is the price Wegmans paid, plus the markup.
    c. Suppose Wegmans then offers a 25% discount on the sale price of the item (found in part b). In dollars, how much is the discount?
    d. Determine the price of the item after the discount (this is the sales price of the item minus the discount). Round to two decimal places.
    e. Is the new price after the markup and discount equal to the price Wegmans paid for the item? Explain.
63. **Repeated Discounts.** In this exercise, we explore applying more than one discount to an item.
    Suppose a store cuts the price on an item by 50%, and then offers a coupon for 25% off any sale item. We will find the price of the item after applying the sale price and the coupon discount.
    a. The original price was $150. After the 50% discount, what is the price of the item?
    b. The coupon is applied to the discount price. The coupon is for 25%. Find 25% of the sale price (found in part a).
    c. Find the price after applying the coupon (this is the value from part a minus the value from part b).
    d. The total amount saved on the item is the original price after all the discounts. Determine the total amount saved by subtracting the final price paid (part c) from the original price of the item.
    e. Determine the effective discount percentage, which is the total amount saved divided by the original price of the item.
    f. Was the effective discount percentage equal to 75%, which would be the 50% plus the 25%? Explain.

Converting Repeating Decimals to a Fraction

It was mentioned in the section that repeating decimals are rational numbers. To convert a repeating decimal to a rational number, perform the following steps:

Step 1: Label the original number S.
Step 2: Count the number of digits, n, in the repeating part of the number.
Step 3: Multiply S by 10^n , and label this as $10^n \times S$.
Step 4: Determine $10^n - 1$.
Step 5: Calculate $10^n \times S - S$. If done correctly, the repeating part of the number will cancel out.
Step 6: If the result from Step 5 has decimal digits, count the number of decimal digits in the number from Step 5. Label this m.
Step 7: Remove the decimal from the result of Step 5.
Step 8: Add m zeros to the end of the number from Step 4.
Step 9: Divide the result from Step 7 by the result from Step 8. This is the fraction form of the repeating decimal.

64. Convert $0.\overline{7}$ to fraction form.
65. Convert $0.\overline{45}$ to fraction form.
66. Convert $3.1\overline{5}$ to fraction form.
67. Convert $2.71\overline{94}$ to fraction form.

3.5 Irrational Numbers

Figure 3.28 The Pythagoreans were a philosophical sect of ancient Greece, often associated with mathematics. (credit: Fedor Andreevich Bronnikov (1827-1902) “Hymn of the Pythagoreans to the Rising Sun,” 1877, oil on canvas/Wikimedia, public domain)

Learning Objectives

After completing this section, you should be able to:

1. Define and identify numbers that are irrational.
2. Simplify irrational numbers and express in lowest terms.
3. Add and subtract irrational numbers.
4. Multiply and divide irrational numbers.
5. Rationalize fractions with irrational denominators.

The Pythagoreans were a philosophical sect in ancient Greece. Their philosophy included reincarnation and purifying the mind through the study and contemplation of mathematics and science. One of their principles was the cosmos is ruled by order, specifically mathematics and music. They even held mystic beliefs about specific numbers and figures. For example, the number 1 was associated with the mind and essence. Four represented justice, as it is the first product of two even numbers. Most famously, though, is the association with the Pythagorean Theorem, which states that in a right triangle, the sum of the squares of the shorter sides of the triangle (the legs) equals the square of the longer side (the hypotenuse). Even the ancient Egyptians used this relationship, as triangles with side measures 3, 4, and 5 were often used in surveying following the flooding of the Nile.

There is a story of a Pythagorean, Hippasus, discovering that not all numbers could be expressed as fractions. In other words, not all numbers were rational numbers. The story ends with Hippasus, who shared this, or in some versions discovered it, put to death by drowning for sharing this fact, that not all quantities could be expressed as the ratio of two natural numbers.

As colorful as that story may be, it is most likely false, as there are no contemporary sources to corroborate it. But it does seem to mark the discovery that not all quantities or measures were fractions of numbers. And so, **irrational numbers** were discovered.

VIDEO

The Philosophy of the Pythagoreans (https://openstax.org/r/Philosophy_of_Pythagoreans)

Defining and Identifying Numbers That Are Irrational

We defined rational numbers in the last section as numbers that could be expressed as a fraction of two integers. **Irrational numbers** are numbers that cannot be expressed as a fraction of two integers. Recall that rational numbers could be identified as those whose decimal representations either terminated (ended) or had a repeating pattern at some point. So irrational numbers must be those whose decimal representations do not terminate or become a repeating pattern.

One collection of irrational numbers is **square roots** of numbers that aren't **perfect squares**. x is the square root of the number a, denoted $\sqrt{a}$, if $x^2 = a$. The number a is the perfect square of the integer n if $a = n^2$. The rational number $\frac{a}{b}$ is a perfect square if both a and b are perfect squares.

One method of determining if an integer is a perfect square is to examine its prime factorization. If, in that factorization, all the prime factors are raised to even powers, the integer is a perfect square. Another method is to attempt to factor the integer into an integer squared. It is possible that you recognize the number as a perfect square (such as 4 or 9). Or, if you have a calculator at hand, use the calculator to determine if the square root of the integer is an integer.

EXAMPLE 3.84

Identifying Perfect Squares

Determine which of the following are perfect squares.

1. 45
2. 81
3. $\frac{9}{28}$
4. $\frac{144}{400}$

Solution

1. The prime factorization of 45 is $45 = 3^2 \times 5$. Since the 5 is not raised to an even power, 45 is not a perfect square.
2. The prime factorization of 81 is 3^4. All the prime factors are raised to even powers, so 81 is a perfect square.
3. We must determine if both the numerator and denominator of $\frac{9}{28}$ are perfect squares for the rational number to be a perfect square. The numerator is 9, and as mentioned above, 9 is a perfect square (it is 3 squared). Now we check the prime factorization of the denominator, 28, which is $28 = 2^2 \times 7$. Since 7 is not raised to an even power, 28 is not a perfect square. Since the denominator is not a perfect square, $\frac{9}{28}$ is not a perfect square.
4. We must determine if both the numerator and denominator of $\frac{144}{400}$ are perfect squares for the rational number to be a perfect square. The numerator is 144. The prime factorization of 144 is $144 = 2^4 \times 3^2$. Since all the prime factors of 144 are raised to even powers, 144 is a perfect square. Now we check the prime factorization of the denominator, 400, which is $400 = 2^4 \times 5^2$. Since all the prime factors of 400 are raised to even powers, 400 is a perfect square. Since the numerator and denominator of $\frac{144}{400}$ are perfect squares, $\frac{144}{400}$ is a perfect square.

YOUR TURN 3.84

Determine which of the following are perfect squares.

1. 36
2. 27
3. $\frac{9}{49}$
4. $\frac{12}{221}$

TECH CHECK

Using Desmos to Determine if a Number Is a Perfect Square

Desmos may be used to determine if a number is a perfect square by using its square root function. When Desmos is opened, there is a tab in the lower left-hand corner of the Desmos screen. This tab opens the Desmos keypad, shown in Figure 3.29.

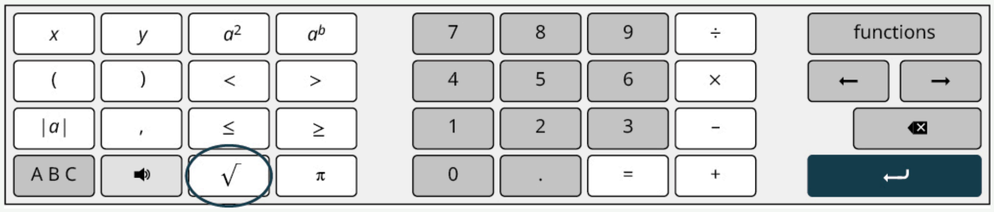


Figure 3.29 Desmos keyboard with square root key circles

There you find the key for the square root, which is circled in Figure 3.29. To find the square root of a number, click the square root key, which begins a calculation, and then enter the value for which you want a square root. If the result is an integer, then the number is a perfect square.

VIDEO

Using Desmos to Find the Square Root of a Number (https://openstax.org/r/square_root_of_a_number)

Another collection of irrational numbers is based on the special number, **pi**, denoted by the Greek letter π, which is the ratio of the circumference of the diameter of the circle (Figure 3.30).

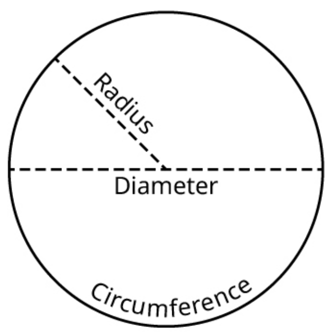


Figure 3.30 Circle with radius, diameter, and circumference labeled

Any multiple or power of π is an irrational number.

Any number expressed as a rational number times an irrational number is an irrational number also. When an irrational number takes that form, we call the rational number the **rational part**, and the irrational number the **irrational part**. It should be noted that a rational number plus, minus, multiplied by, or divided by any irrational number is an irrational number.

EXAMPLE 3.85

Identifying Irrational Numbers

Identify which of the following numbers are irrational.

1. $\sqrt{35}$
2. $0.\overline{15}$
3. $\sqrt{121}$
4. 4π

Solution

1. 35 can be factored as 5×7, showing that 35 is not the square of an integer or a rational number. This mean its square root is an irrational number.
2. Since $0.\overline{15}$ is a decimal with a repeating pattern, it is rational, so it is not an irrational number.
3. $121 = 11^2$. Since 121 is the square of an integer, its square root is a rational number.
4. Since 4π is a multiple of pi, it is irrational. In this case, the rational part of the number is 4, while the irrational part is π.

YOUR TURN 3.85

Identify which of the following numbers are irrational.

1. $\sqrt{225}$
2. $3\sqrt{5}$
3. $\sqrt{80}$
4. $20 - 3\pi$

WHO KNEW?

Euler-Mascheroni Constant

Determining if a number is rational or irrational is not trivial. There are numbers that defied such classification for quite a long time. One such is the Euler-Mascheroni constant. The Euler-Mascheroni constant is used in mathematics, and is primarily associated with the natural logarithm, which is a mathematical function. The constant has been around since around 1790. However, it was unknown if this constant was rational or irrational until 2013, at which point it was proven to be irrational.

Simplifying Square Roots and Expressing Them in Lowest Terms

To **simplify a square root** means that we rewrite the square root as a rational number times the square root of a number that has no perfect square factors. The act of changing a square root into such a form is simplifying the square root.

The number inside the square root symbol is referred to as the **radicand**. So in the expression $\sqrt{a}$ the number a is referred to as the radicand.

Before discussing how to simplify a square root, we need to introduce a rule about square roots. The square root of a product of numbers equals the product of the square roots of those number. Written symbolically, $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$.

FORMULA

For any two numbers a and b, $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$.

Using this formula, we can factor an integer inside a square root into a perfect square times another integer. Then the square root can be applied to the perfect square, leaving an integer times the square root of another integer. If the number remaining under the square root has no perfect square factors, then we've simplified the irrational number into lowest terms. To simplify the irrational number into lowest terms when n is an integer:

Step 1: Determine the largest perfect square factor of n, which we denote a^2.

Step 2: Factor n into $a^2 \times b$.

Step 3: Apply $\sqrt{a^2 \times b} = \sqrt{a^2} \times \sqrt{b}$.

Step 4: Write $\sqrt{n}$ in its simplified form, $a\sqrt{b}$.

When a square root has been simplified in this manner, a is referred to as the rational part of the number, and $\sqrt{b}$ is referred to as the irrational part.

EXAMPLE 3.86

Simplifying a Square Root

Simplify the irrational number $\sqrt{180}$ and express in lowest terms. Identify the rational and irrational parts.

Solution

Begin by finding the largest perfect square that is a factor of 180. We can do this by writing out the factor pairs of 180:

$$1 \times 180 \quad 2 \times 90 \quad 3 \times 60 \quad 4 \times 45 \quad 5 \times 36 \quad 6 \times 30 \quad 9 \times 20 \quad 10 \times 18 \quad 12 \times 15$$

Looking at the list of factors, the perfect squares are 4, 9, and 36. The largest is 36, so we factor the into $36 \times 5 = 6^2 \times 5$. In the formula, $a = 6$ and $b = 5$. Apply $\sqrt{a^2 \times b} = \sqrt{a^2} \times \sqrt{b}$.

$$\sqrt{6^2 \times 5} = \sqrt{6^2} \times \sqrt{5}$$

The simplified form of $\sqrt{180}$ is $6\sqrt{5}$. In this example, the 6 is the rational part, and the $\sqrt{5}$ is the irrational part.

YOUR TURN 3.86

1. Simplify the irrational number $\sqrt{550}$ and express in lowest terms. Identify the rational and irrational parts.

VIDEO

Simplifying Square Roots (https://openstax.org/r/Simplifying_Square_Roots)

EXAMPLE 3.87

Simplifying a Square Root

Simplify the irrational number $\sqrt{330}$ and express in lowest terms. Identify the rational and irrational parts.

Solution

Begin by finding the largest perfect square that is a factor of 330. We can do this by writing out the factor pairs of 330:

$$1 \times 330 \quad 2 \times 165 \quad 3 \times 110 \quad 5 \times 66 \quad 6 \times 55 \quad 10 \times 33 \quad 11 \times 30 \quad 15 \times 22$$

Looking at the list of factors, there are no perfect squares other than 1, which means $\sqrt{330}$ is already expressed in lowest terms. In this case, 1 is the rational part, and $\sqrt{330}$ is the irrational part. **Though we could write this as $1\sqrt{330}$, but the product of 1 and any other number is just the number.**

YOUR TURN 3.87

1. Simplify the irrational number $\sqrt{733}$ and express in lowest terms. Identify the rational and irrational parts.

EXAMPLE 3.88

Simplifying a Square Root

Simplify the irrational number $\sqrt{2{,}548}$ and express in lowest terms. Identify the rational and irrational parts.

Solution

Begin by finding the largest perfect square that is a factor of 2,548. We can do this by writing out the factor pairs of 2,548:

$$1 \times 2548 \quad 2 \times 1274 \quad 4 \times 637 \quad 7 \times 364 \quad 13 \times 196 \quad 14 \times 182 \quad 26 \times 98 \quad 28 \times 91 \quad 49 \times 52$$

Looking at the list of factors, the perfect squares are 4, 49, and 196. The largest is 196, so we factor the 2,548 into $196 \times 13 = 14^2 \times 13$. In the formula, $a = 14$ and $b = 5$. Apply $\sqrt{a^2 \times b} = \sqrt{a^2} \times \sqrt{b}$.

$$\sqrt{14^2 \times 13} = \sqrt{14^2} \times \sqrt{13}$$

The simplified form of $\sqrt{2{,}548}$ is $14\sqrt{13}$. In this example, 14 is the rational part, and $\sqrt{13}$ is the irrational part.

YOUR TURN 3.88

1. Simplify the irrational number $\sqrt{1{,}815}$.

VIDEO

[Simplifying Square Roots (https://openstax.org/r/Simplifying_Square_Roots)](https://openstax.org/r/Simplifying_Square_Roots)

Adding and Subtracting Irrational Numbers

Just like any other number we've worked with, irrational numbers can be added or subtracted. When working with a calculator, enter the operation and a decimal representation will be given. However, there are times when two irrational numbers may be added or subtracted without the calculator. This can happen only when the irrational parts of the irrational numbers are the same.

To add or subtract two irrational numbers that have the same irrational part, add or subtract the rational parts of the numbers, and then multiply that by the common irrational part.

FORMULA

Let our first irrational number be $a \times x$, where a is the rational and x the irrational parts.

Let the other irrational number be $b \times x$, where b is the rational and x the irrational parts.

Then $a \times x \pm b \times x = (a \pm b) \times x$.

EXAMPLE 3.89

Subtracting Irrational Numbers with Similar Irrational Parts

If possible, subtract the following irrational numbers without using a calculator. If this is not possible, state why.

$3\sqrt{7} - 8\sqrt{7}$

Solution

Since these two irrational numbers have the same irrational part, $\sqrt{7}$, we can subtract without using a calculator. The rational part of the first number is 3. The rational part of the second number is 8. Using the formula yields $3\sqrt{7} - 8\sqrt{7} = (3 - 8) \times \sqrt{7} = -5\sqrt{7}$.

YOUR TURN 3.89

1. If possible, subtract the following irrational numbers without using a calculator. If this is not possible, state why.

   $41\sqrt{15} - 23\sqrt{15}$

EXAMPLE 3.90

Adding Irrational Numbers with Similar Irrational Parts

If possible, add the following irrational numbers without using a calculator. If this is not possible, state why.

$35\pi + 17\pi$

Solution

Since these two irrational numbers have the same irrational part, π, the addition can be performed without using a calculator. The rational part of the first number is 35. The rational part of the second number is 17. Using the formula yields $35\pi + 17\pi = (35 + 17) \times \pi = 52\pi$.

YOUR TURN 3.90

1. If possible, add the following irrational numbers without using a calculator. If this is not possible, state why. $4.1\pi + 3.2\pi$

EXAMPLE 3.91

Subtracting Irrational Numbers with Different Irrational Parts

If possible, subtract the following irrational numbers without using a calculator. If this is not possible, state why.

$19\sqrt{3} - 5.6\sqrt{7}$

Solution

The two numbers being subtracted do not have the same irrational part, so the operation cannot be performed.

YOUR TURN 3.91

1. If possible, subtract the following irrational numbers without using a calculator. If this is not possible, state why. $2.1\sqrt{45} - 3.7\sqrt{5}$

Multiplying and Dividing Irrational Numbers

Just like any other number that we've worked with, irrational numbers can be multiplied or divided. When working with a calculator, enter the operation and a decimal representation will be given. Sometimes, though, you may want to retain the form of the irrational number as a rational part times an irrational part. The process is similar to adding and subtracting irrational numbers when they are in this form. We do not need the irrational parts to match. Even though they need not match, they do need to be similar, such as both irrational parts are square roots, or both irrational parts are multiples of pi. Also, if the irrational parts are square roots, we may need to reduce the resulting square root to lowest terms.

When multiplying two square roots, use the following formula. It is the same formula presented during the discussion of simplifying square roots.

FORMULA

For any two positive numbers a and b, $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$.

When dividing two square roots, use the following formula.

FORMULA

For any two positive numbers a and b, with b not equal to 0, $\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.

To multiply or divide irrational numbers with similar irrational parts, do the following:

Step 1: Multiply or divide the rational parts.

Step 2: If necessary, reduce the result of Step 1 to lowest terms. This becomes the rational part of the answer.

Step 3: Multiply or divide the irrational parts.

Step 4: If necessary, reduce the result from Step 3 to lowest terms. This becomes the irrational part of the answer.

Step 5: The result is the product of the rational and irrational parts.

EXAMPLE 3.92

Dividing Irrational Numbers with Similar Irrational Parts

Perform the following operations without a calculator. Simplify if possible.

1. $3\sqrt{15} \div \left(8\sqrt{3}\right)$
2. $14.7\sqrt{135} \div \left(3\sqrt{5}\right)$.

Solution

1. In this division problem, $3\sqrt{15} \div \left(8\sqrt{3}\right)$, notice that the irrational parts of these numbers are similar. They are both square roots, so follow the steps given above.
   Step 1: Divide the rational parts. $3 \div 8 = \frac{3}{8}$

   Step 2: If necessary, reduce the result of Step 1 to lowest terms. The 3 and 8 have no common factors, so $\frac{3}{8}$ is already in lowest terms.

   Step 3: Divide the irrational parts. $\sqrt{15} \div \sqrt{3} = \frac{\sqrt{15}}{\sqrt{3}} = \sqrt{\frac{15}{3}}$

   Step 4: If necessary, reduce the result from Step 3 to lowest terms. The radicand can be reduced, which yields $\sqrt{5}$.

   Step 5: The result is the product of the rational and irrational parts, which is $\frac{3}{8}\sqrt{5}$.

2. In this division problem, $14.7\sqrt{135} \div \left(3\sqrt{5}\right)$, notice that the irrational parts of these numbers are similar. They are both square roots, so follow the steps given above.
   Step 1: Divide the rational parts. $14.7 \div 3 = 4.9$

   Step 2: If necessary, reduce the result of Step 1 to lowest terms. This rational number is expressed as a decimal so will not be reduced.

   Step 3: Divide the irrational parts. $\sqrt{135} \div \sqrt{5} = \frac{\sqrt{135}}{\sqrt{5}} = \sqrt{\frac{135}{5}}$

   Step 4: If necessary, reduce the result from Step 3 to lowest terms. The radicand can be reduced, which yields $\sqrt{\frac{135}{5}} = \sqrt{27} = \sqrt{9 \times 3} = 3\sqrt{3}$.

   Step 5: The result is the product of the rational and irrational parts, which is $4.9 \times 3\sqrt{3} = 14.7\sqrt{3}$.

YOUR TURN 3.92

Perform the following operations without a calculator. Simplify if possible.

1. $84\sqrt{132} \div \left(14\sqrt{11}\right)$
2. $57\sqrt{792} \div \left(25\sqrt{2}\right)$

EXAMPLE 3.93

Multiplying Irrational Numbers with Similar Irrational Parts

Perform the following operations without a calculator. Simplify if possible.

1. $\left(19\sqrt{3}\right) \times \left(5.6\sqrt{12}\right)$
2. $13\pi \times 8\pi$

Solution

1. In this multiplication problem, $\left(19\sqrt{3}\right) \times \left(5.6\sqrt{12}\right)$, notice that the irrational parts of these numbers are similar. They are both square roots. Follow the process above.
   Step 1: Multiply the rational parts. $19 \times 5.6 = 106.4$

   Step 2: If necessary, reduce the result of Step 1 to lowest terms. This rational number is expressed as a decimal and will not be reduced.

   Step 3: Multiply the irrational parts. $\sqrt{3} \times \sqrt{12} = \sqrt{3 \times 12} = \sqrt{36}$

   Step 4: If necessary, reduce the result from Step 3 to lowest terms. The radicand is 36, which is the square of 6. The irrational part reduces to $\sqrt{36} = 6$.

   Step 5: The result is the product of the rational and irrational parts, which is $106.4 \times 6 = 638.4$.

   Notice that sometimes multiplying or dividing irrational numbers can result in a rational number.

2. In this multiplication problem, $13\pi \times 8\pi$, notice that the irrational parts of these numbers are the same, π. Follow the process above.

   Step 1: Multiply the rational parts. $13 \times 8 = 104$

   Step 2: If necessary, reduce the result of Step 1 to lowest terms. That result is an integer.

   Step 3: Multiply the irrational parts. $\pi \times \pi = \pi^2$

   Step 4: If necessary, reduce the result from Step 3 to lowest terms. This cannot be reduced.

   Step 5: The result is the product of the rational and irrational parts, which is $104\pi^2$.

YOUR TURN 3.93

Perform the following operations without a calculator. Simplify if possible.

1. $\left(1.2\sqrt{21}\right) \times \left(4.5\sqrt{14}\right)$
2. $38\pi \div (2\pi)$

Rationalizing Fractions with Irrational Denominators

Fractions often represent that some amount is being equally divided into some number of parts. But to conceptualize a fraction in that manner, the denominator needs to be an integer. An irrational number in the denominator interferes with that interpretation of a fraction. Fractions that have denominators that are just the square root of an integer can be altered into fractions with integer denominators using a process called **rationalizing the denominator**. The process relies on the following property of square roots: $\sqrt{a} \times \sqrt{a} = a$ and the following property of fractions: $\frac{a}{b} = \frac{ac}{bc}$ for any non-zero number c.

Using these two properties, when a fraction has a square root in the denominator, we can eliminate that square root. Multiply the numerator and denominator by that square root from the denominator, $\frac{a}{\sqrt{b}} = \frac{a\sqrt{b}}{\sqrt{b} \times \sqrt{b}}$. Then apply $\sqrt{a} \times \sqrt{a} = a$ to the denominator, yielding $\frac{a\sqrt{b}}{\sqrt{b} \times \sqrt{b}} = \frac{a\sqrt{b}}{b}$. Notice that there is no longer a square root in the denominator, which allows for interpreting the fraction as dividing a whole into equal parts.

VIDEO

[Rationalizing the Denominator (https://openstax.org/r/Rationalizing_Denominator)](https://openstax.org/r/Rationalizing_Denominator)

EXAMPLE 3.94

Rationalizing the Denominator

Rationalize the denominator of the following:

1. $\frac{5}{\sqrt{7}}$
2. $\frac{3\sqrt{6}}{2\sqrt{10}}$

Solution

1. The square root in the denominator is $\sqrt{7}$. In order to rationalize the denominator of $\frac{5}{\sqrt{7}}$, we need to multiply the numerator and denominator by $\sqrt{7}$ and simplify.
   $\frac{5}{\sqrt{7}} = \frac{5\sqrt{7}}{\sqrt{7} \times \sqrt{7}} = \frac{5\sqrt{7}}{7}$
   The square root is in simplified form, so the final answer is $\frac{5\sqrt{7}}{7}$.
2. The square root in the denominator is $\sqrt{10}$.
   Step 1: In order to rationalize the denominator of $\frac{3\sqrt{6}}{2\sqrt{10}}$, we need to multiply the numerator and denominator by $\sqrt{10}$ and simplify.
   $\frac{3\sqrt{6}}{2\sqrt{10}} = \frac{3\sqrt{6} \times \sqrt{10}}{2\sqrt{10} \times \sqrt{10}} = \frac{3\sqrt{60}}{2 \times 10} = \frac{3\sqrt{60}}{20}$
   Step 2: The 60 under the square root can be factored into the following factor pairs:
   $1 \times 60 \quad 2 \times 30 \quad 3 \times 20 \quad 4 \times 15 \quad 5 \times 12 \quad 6 \times 10$
   Step 3: The largest square factor of 60 is 4, so we simplify the $\sqrt{60}$ in the numerator into $2\sqrt{15}$. We also cancel any common factors.
   $\frac{3\sqrt{60}}{20} = \frac{3 \times 2\sqrt{15}}{20} = \frac{6\sqrt{15}}{20} = \frac{3\sqrt{15}}{10}$
   This is completely simplified.

YOUR TURN 3.94

Rationalize the denominator of the following:

1. $\frac{24}{\sqrt{15}}$
2. $\frac{11\sqrt{14}}{6\sqrt{21}}$

There are occasions when the denominator is irrational but is the sum of two numbers where one or both involve square roots. For instance, $\frac{5}{4+\sqrt{3}}$. The process used earlier required that the denominator was the square root of a number and would not work here. However, this type of denominator can be rationalized. In order to rationalize such a denominator, we will multiply the numerator and denominator of the fraction by the **conjugate** of the denominator. The conjugate of $a + b$ is $a - b$. We say that $a + b$ and $a - b$ are **conjugate numbers**.

So, the conjugate of $-3 + \sqrt{10}$ is just $-3 - \sqrt{10}$. But why is this of interest? The reason is because it leads to the **difference of squares** formula, which is used to factor the difference of two squares. Or, for our purposes, in reverse it allows us to eliminate a square root.

FORMULA

For any two numbers, a and b, $a^2 - b^2 = (a - b)(a + b)$.

Looking at that formula, you should see that the two factors on the right-hand side of the equals sign are conjugates of one another. So, for our purposes, we're interested in $(a - b)(a + b) = a^2 - b^2$. This tells us that when we multiply $a + b$ by its conjugate, we get a squared minus b squared, or $a^2 - b^2$. But how is this useful? Let's return to the fraction above, $\frac{5}{4 + \sqrt{3}}$. The denominator is $4 + \sqrt{3}$. Its conjugate is $4 - \sqrt{3}$. According to the formula, and letting $a = 4$ and $b = \sqrt{3}$, we see that $\left(4 + \sqrt{3}\right)\left(4 - \sqrt{3}\right) = 4^2 - \left(\sqrt{3}\right)^2$. But $\left(\sqrt{3}\right)^2$ is just 3. That means the product is $16 - 3$ or 13. This no longer has a square root. We use this to rationalize the denominator.

We will also need the **distributive property** of numbers.

FORMULA

For any three numbers a, b, and c, $a \times (b \pm c) = a \times b \pm a \times c$. This is called the distributive property.

EXAMPLE 3.95

Rationalizing the Denominator Using Conjugates

Rationalize the denominator of $\frac{4}{6 + \sqrt{10}}$.

Solution

Step 1: We recognize that the denominator is the sum of two numbers where one or both involve square roots. This means the conjugate can be used to remove the square root from the denominator.

Step 2: To do so, we multiply the numerator and the denominator each by the conjugate of the denominator. Since the denominator is $6 + \sqrt{10}$, the conjugate we will use is $6 - \sqrt{10}$.

Step 3: The conjugate is multiplied by the numerator and the denominator.

$$\frac{4}{6 + \sqrt{10}} \times \frac{6 - \sqrt{10}}{6 - \sqrt{10}}$$

Step 4: Remembering how a number times its conjugate works, this becomes

$$\frac{4}{6 + \sqrt{10}} \times \frac{6 - \sqrt{10}}{6 - \sqrt{10}} = \frac{4 \times (6 - \sqrt{10})}{6^2 - \left(\sqrt{10}\right)^2}.$$

Step 5: In the numerator, we apply the distributive property. Using it yields

$$\frac{4 \times \left(6 - \sqrt{10}\right)}{6^2 - \left(\sqrt{10}\right)^2} = \frac{24 - 4\sqrt{10}}{36 - 10} = \frac{24 - 4\sqrt{10}}{26}.$$

Step 6: Notice that the denominator no longer contains a square root. It has been rationalized. If desired, this can then be written as a rational number minus an irrational number, by recalling that $\frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}$.

Applying that to the answer, we have $\frac{4}{6 + \sqrt{10}} = \frac{24 - 4\sqrt{10}}{26} = \frac{24}{26} - \frac{4\sqrt{10}}{26}$.

Step 7: With a bit of cancellation, this reduces to $\frac{4}{6 + \sqrt{10}} = \frac{24}{26} - \frac{4\sqrt{10}}{26} = \frac{12}{13} - \frac{2\sqrt{10}}{13}$.

YOUR TURN 3.95

1. Rationalize the denominator of $\frac{15}{5 - \sqrt{13}}$.

VIDEO

Rationalizing the Denominator (https://openstax.org/r/Rationalizing_Denominator)

Check Your Understanding

29. Simplify the following square root: $\sqrt{500}$.

30. Perform the following operation: $3\sqrt{7} - 10\sqrt{7}$.

31. Perform the following operation: $8\sqrt{10} \times 3\sqrt{2}$.

32. Rationalize the denominator of the following: $\frac{4}{\sqrt{7}}$.

SECTION 3.5 EXERCISES

1. Identify which of the following numbers are irrational:
$\sqrt{441}$, 4.33, $\sqrt{70}$, $5 + 9\pi$

2. Identify which of the following numbers are irrational:
$\frac{13}{\sqrt{46}}$, $4 + 13\pi$, $\sqrt{144}$, $\frac{5}{9}$

For the following exercises, simplify the square root by expressing it in lowest terms.

3. $\sqrt{12}$
4. $\sqrt{75}$
5. $\sqrt{605}$
6. $\sqrt{45}$
7. $\sqrt{112}$
8. $\sqrt{396}$
9. $\sqrt{2{,}940}$
10. $\sqrt{2{,}400}$
11. $\sqrt{3{,}240}$
12. $\sqrt{5{,}472}$

For the following exercises, perform the arithmetic operations without a calculator, if possible. If it is not possible, state why.

13. $4\sqrt{3} + 2\sqrt{3}$
14. $8\sqrt{5} + 3\sqrt{5}$
15. $9\sqrt{7} - 15\sqrt{7}$
16. $\sqrt{13} - 15\sqrt{13}$
17. $8\pi - 13\sqrt{2}$
18. $7\sqrt{5} + 6\sqrt{14}$
19. $7.2\pi + 8.6\pi$
20. $14.5\pi - 5.8\pi$
21. $19.8\sqrt{12} - 6.1\sqrt{3}$
22. $7.3\sqrt{45} - 6.8\sqrt{20}$
23. $\left(4\sqrt{15}\right) \times \left(3\sqrt{10}\right)$
24. $\left(7\sqrt{33}\right) \times \left(8\sqrt{66}\right)$
25. $\left(4.5\sqrt{154}\right) \div \left(3\sqrt{77}\right)$
26. $\left(70\sqrt{30}\right) \div \left(14\sqrt{6}\right)$

For the following exercises, rationalize the denominators of the fractions, and then simplify.

27. $\frac{3}{\sqrt{2}}$
28. $\frac{8}{\sqrt{6}}$
29. $\frac{5}{\sqrt{13}}$
30. $\frac{9}{\sqrt{35}}$

31. $\frac{12}{\sqrt{40}}$
32. $\frac{20}{\sqrt{150}}$
33. Determine the conjugate of $5+\sqrt{6}$.
34. Determine the conjugate of $10-\sqrt{13}$.
35. Determine the conjugate of $4+3\sqrt{5}$.
36. Determine the conjugate of $5\sqrt{15}+8\sqrt{13}$.
37. Find the product of $3+2\sqrt{7}$ and its conjugate.
38. Find the product of $2\sqrt{3-5}$ and its conjugate.

For the following exercises, rationalize the denominator of the fraction, and then simplify the fraction.

39. $\frac{4}{1+\sqrt{3}}$
40. $\frac{6}{5+\sqrt{7}}$
41. $\frac{-4}{5-\sqrt{10}}$
42. $\frac{10}{\sqrt{6}+7}$

3.6 Real Numbers

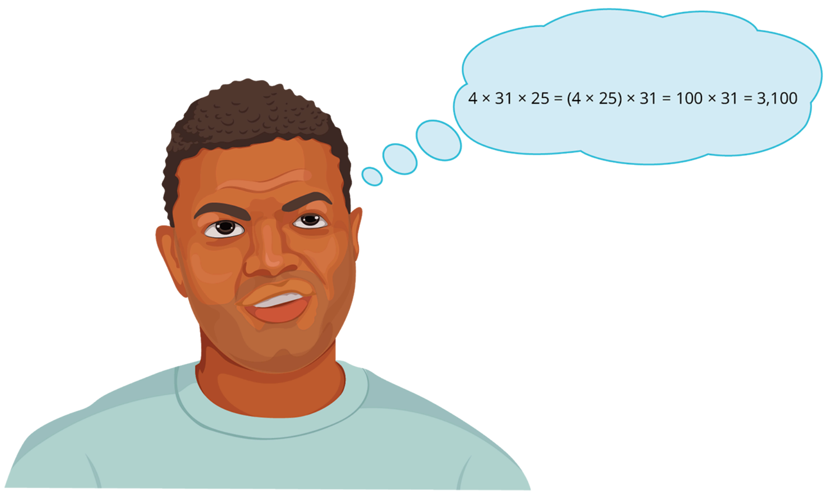


Figure 3.31 Quick mental math involves using the known properties of real numbers.

Learning Objectives

After completing this section, you should be able to:

1. Define and identify numbers that are real numbers.
2. Identify subsets of the real numbers.
3. Recognize properties of real numbers.

Have you ever been impressed by the speed at which someone can do math in their head? Most of us at one time or another have witnessed a person speed through mental math, an impressive feat that often bests calculators. One such person is Neelkantha Bhanu Prakash. As of September 20, 2020, he is considered the world's fastest human calculator. He currently holds four world records. How does someone do that, though? Have they memorized lots of arithmetic facts? Are they simply brilliant?

The answer isn't simple so much as it is about knowledge. Real numbers behave in some very regular ways, following rules that can be learned. In this section, those rules are explored.

Watch the video of Arthur Benjamin's TED Talk to learn about another mathematician with remarkable mental abilities.

VIDEO

[Arthur Benjamin TED Talk, Faster Than a Calculator (https://openstax.org/r/Arthur_Benjamin_TED_talk,_Faster_than_a_Calculator)]

Defining and Identifying Real Numbers

Real numbers are the rational and irrational numbers combined. The real numbers represent the collection of all physical distances that exist, along with 0 and the negatives of those physical distances. For example, if you take a measure of three units, and divide that distance into eight (8) equal lengths, the distance you have formed is $\frac{3}{8}$ units. Also, if you draw a **right triangle** (a triangle with one angle equal to 90 degrees) with one side length of 1, and the other side length of 3, the long side of the triangle will have length $\sqrt{10}$ units, as shown in Figure 3.32.

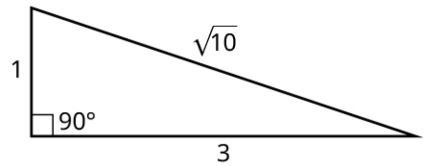


Figure 3.32 Right triangle

Of course, if we name something the real numbers, there must be numbers that aren't real. Otherwise, they'd just be called the numbers. One such not real number, one that cannot be a length, is $\sqrt{-1}$. It is part of a collection of numbers called the complex numbers, it is denoted with the letter i. As an extension, the square root of any negative number is not a real number, but instead a complex number.

To determine if a number is real, check to see if there are any negatives under a square root or any i's. If there are any present, the number is not real.

EXAMPLE 3.96

Identifying Real Numbers

Determine if each of the following are real numbers:

1. $\frac{4\sqrt{3}}{7}$
2. 13.3381
3. $17\sqrt{-8}$

Solution

1. $\frac{4\sqrt{3}}{7}$ is a real number, as there are no negatives under the square roots, nor is there any factor of i.
2. 13.3381 is a rational number, and so it is a real number.
3. $17\sqrt{-8}$ is not a real number, as there is a negative number under the square root.

YOUR TURN 3.96

Determine if each of the following are real numbers:

1. $\frac{8}{15}$
2. $4 + 17i$
3. $-17\sqrt{46}$

Identifying Subsets of Real Numbers

The real numbers were built out of pieces, including integers, rational numbers, and irrational numbers. As such, the real numbers have named subsets, as shown in the table below.

Set Name	Set Symbol	Set Description
Natural Numbers	$\mathbb{N}$	The counting numbers
Whole Numbers		The counting numbers and 0
Integers	$\mathbb{Z}$	The natural numbers, their negatives, and 0
Rational Numbers	$\mathbb{Q}$	Fractions of integers
Irrational Numbers	$\mathbb{P}$	Numbers that cannot be written as a fraction of integers
Real Numbers	$\mathbb{R}$	The union of the rational and irrational numbers, all possible physical lengths, and their negatives

When we categorize numbers using these sets, we use the smallest set that they belong to. For instance, −7 is an integer, and a rational number, and a real number. The smallest set to which −7 belongs is integer, so we'd say it belongs to the integers.

We can also represent the relationships between the different sets of real numbers using set notation. All natural numbers are integers, but there are integers that are not natural numbers, so $\mathbb{N} \subset \mathbb{Z}$. Similarly, every integer is a rational number, but there are rational numbers that are not integers, so $\mathbb{Z} \subset \mathbb{Q}$. The same is true of the rational numbers and the real numbers, so $\mathbb{Q} \subset \mathbb{R}$.

There is no agreed-upon symbol for the irrational numbers. If we represent the irrationals as the set A, we should note that the following are true: $\mathbb{Q} \cup A = \mathbb{R}$ and $\mathbb{Q} \cap A = \varnothing$. Recall that this means the irrationals are the complement of the rational numbers in the universal set of real numbers.

EXAMPLE 3.97

Categorizing Numbers

Identify all subsets of the real numbers to which the following real numbers belong:

1. 14
2. -14.223
3. $\sqrt{17}$

Solution

1. 14 is a natural number, integer, and rational number.
2. -14.223 is a rational number.
3. $\sqrt{17}$ is an irrational number.

YOUR TURN 3.97

Identify all subsets of the real numbers to which the following real numbers belong:

1. $14\sqrt{3}$
2. -147
3. $\frac{37}{150}$

EXAMPLE 3.98

Categorizing Numbers within a Venn Diagram

Place the following numbers correctly in the Venn diagram (Figure 3.33).

$$-4\sqrt{2} \quad -10 \quad \frac{37}{150} \quad 41 \quad \frac{1}{20} \quad 4\pi$$

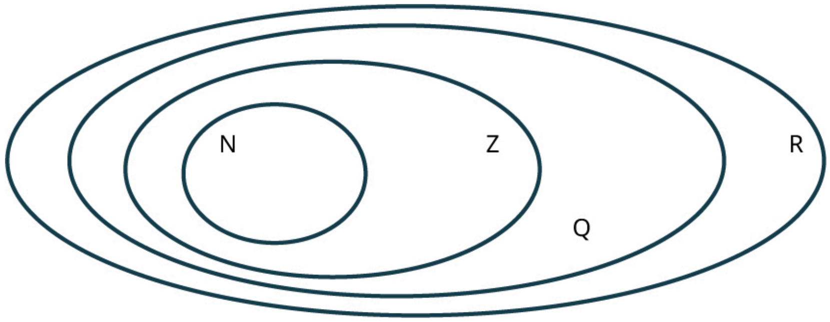


Figure 3.33

Solution

Since $-4\sqrt{2}$ is irrational, it belongs in the real numbers, but outside the rational numbers (Figure 3.34).

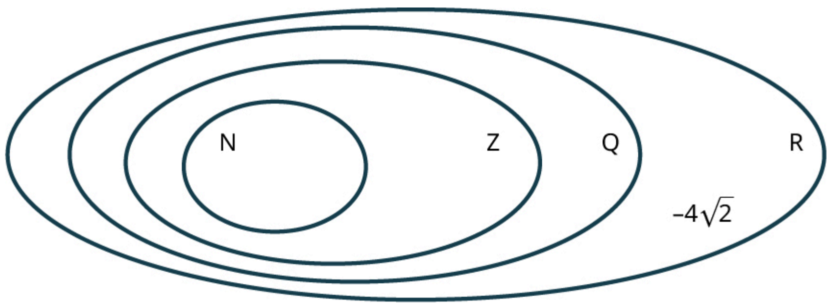


Figure 3.34

Since −10 is an integer, it belongs in the integers but outside the natural numbers (Figure 3.35).

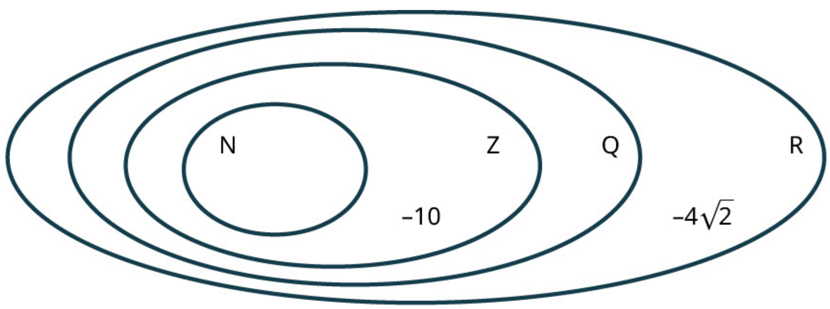


Figure 3.35

Since $\frac{37}{150}$ is a rational number, it belongs in the rational numbers but not in the integers (Figure 3.36).

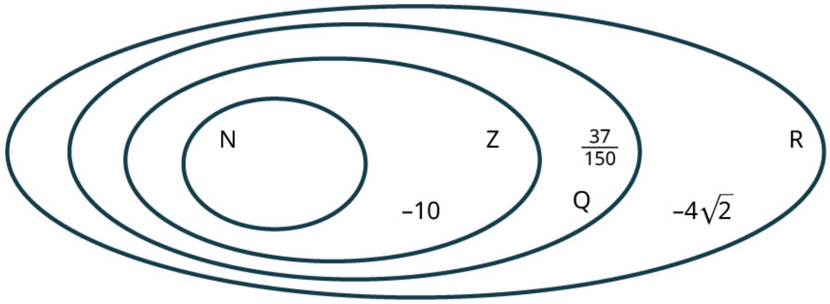


Figure 3.36

Since 41 is a natural number, it belongs in the natural numbers circle (Figure 3.37).

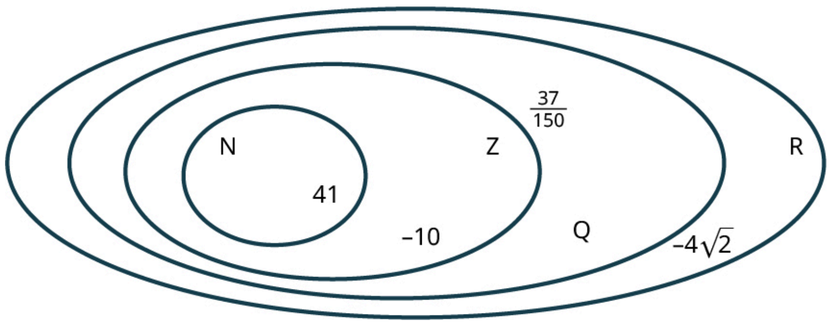


Figure 3.37

Since $\frac{1}{20}$ is a rational number, it belongs in the rational numbers but not in the integers (Figure 3.38).

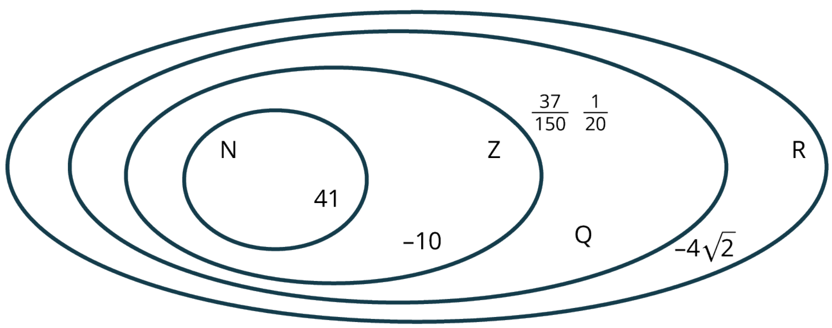


Figure 3.38

Since 4π is irrational, it belongs in the real numbers, but outside the rational numbers (Figure 3.39).

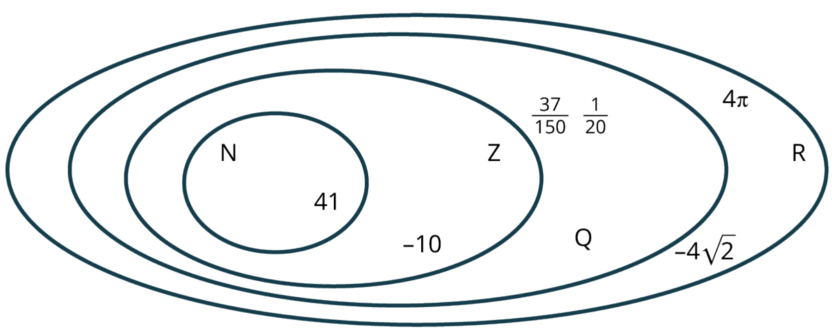
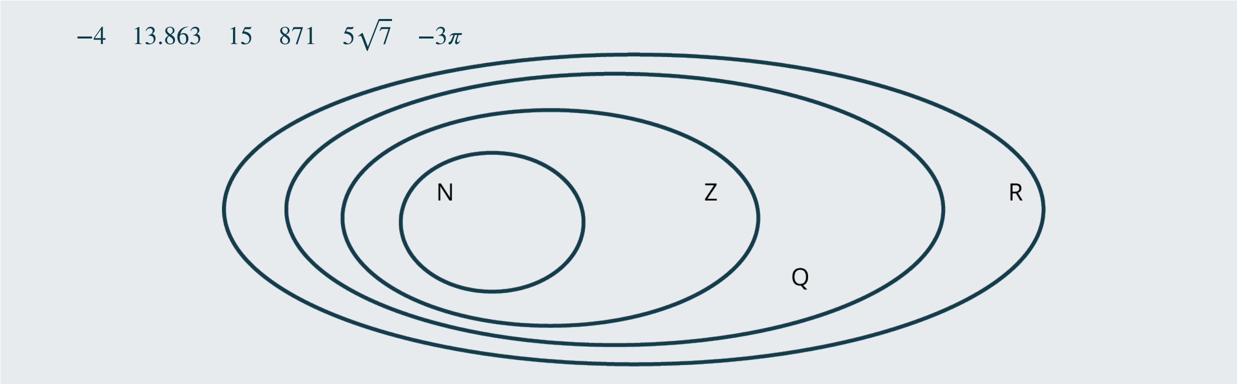


Figure 3.39

YOUR TURN 3.98

1. Place the following numbers correctly into the Venn diagram.



VIDEO

[Identifying Sets of Real Numbers (https://openstax.org/r/Sets_of_Real_Numbers)](https://openstax.org/r/Sets_of_Real_Numbers)

Recognizing Properties of Real Numbers

The real numbers behave in very regular ways. These behaviors are called the **properties of the real numbers**. Knowing these properties helps when evaluating formulas, working with equations, or performing algebra. Being familiar with these properties is helpful in all settings where numbers are used and manipulated. For example, when multiplying $4 \times 13 \times 25$, you could multiply the 4 and 25 first. If you know that product is 100, it makes the multiplication easier.

The table below is a partial list of properties of real numbers.

Property	Example	In Words
Distributive property $a \times (b + c) = a \times b + a \times c$	$5 \times (3 + 4) = 5 \times 3 + 5 \times 4$	Multiplication distributes across addition
Commutative property of addition $a + b = b + a$	$3 + 7 = 7 + 3$	Numbers can be added in any order
Commutative property of multiplication $a \times b = b \times a$	$10 \times 4 = 4 \times 10$	Numbers can be multiplied in any order
Associative property of addition $a + (b + c) = (a + b) + c$	$4 + (3 + 8) = (4 + 3) + 8$	Doesn't matter which pair of numbers is added first
Associative property of multiplication $a \times (b \times c) = (a \times b) \times c$	$2 \times (5 \times 7) = (2 \times 5) \times 7$	Doesn't matter which pair of numbers is multiplied first
Additive identity property $a + 0 = a$	$17 + 0 = 17$	Any number plus 0 is the number
Multiplicative identity property $a \times 1 = a$	$21 \times 1 = 21$	Any number times one is the number
Additive inverse property $a + (-a) = 0$	$14 + (-14) = 0$	Every number plus its negative is 0
Multiplicative inverse property $a \times \left(\frac{1}{a}\right) = 1$, provided $a \neq 0$	$3 \times \left(\frac{1}{3}\right) = 1$	Every non-zero number times its reciprocal is 1

The names of the properties are suggestive. The **commutative properties**, for example, suggest commuting, or moving. **Associative properties** suggest which items are associated with others, or if order matters in the computation. The **distributive property** addresses how a number is distributed across parentheses.

EXAMPLE 3.99

Identifying Properties of Real Numbers

In each of the following, identify which property of the real numbers is being applied.

1. $4 + (8 + 13) = (4 + 8) + 13$
2. $34 \times \left(\frac{1}{34}\right) = 1$
3. $14 + 27 = 27 + 14$

Solution

1. Here, the pair of numbers that is added first is switched. This is the associative property of addition.
2. Here, a number is multiplied by its reciprocal, resulting in 1. This is the multiplicative inverse property.
3. Here, the order in which numbers are added is switched. This is the commutative property of addition.

YOUR TURN 3.99

In each of the following, identify which property of the real numbers is being applied.

1. $5 \times (6 + 19) = 5 \times 6 + 5 \times 19$
2. $41.7 + (-41.7) = 0$

Using these properties to perform arithmetic quickly relies on spotting easy numbers to work with. Look for numbers that add to a multiple of 10, or multiply to a multiple of 10 or 100.

EXAMPLE 3.100

Using Properties of Real Numbers in Calculations

Use properties of the real numbers and mental math to calculate the following:

1. $2 \times 13 \times 50$
2. $13 + 84 + 27$
3. $9 \times 16 \times 11$

Solution

1. Notice that $2 \times 50 = 100$, so that becomes the multiplication to do first. Use the commutative property of multiplication to change the order of the numbers being multiplied.
$$2 \times 13 \times 50 = 2 \times 50 \times 13 = 100 \times 13 = 1{,}300$$
2. Notice that $13 + 27 = 40$, so that becomes the addition to do first. Use the commutative property of addition to change the order in which the numbers are added.
$$13 + 84 + 27 = 13 + 27 + 84 = 40 + 84 = 124$$
3. Notice that $9 \times 11 = 99$. Using that, the problem can be changed to 99×16. That, however, doesn't look easy at all. But $99 = (100 - 1)$. Using the distributive property, we rewrite and expand this as $99 \times 16 = (100 - 1) \times 16 = 100 \times 16 - 1 \times 16 = 1{,}600 - 16$. The last step is subtraction, so the final answer is 1,584. So, multiplying by 99 is the same as multiplying by 100, and then subtracting the other number once.

YOUR TURN 3.100

1. Use properties of real numbers and mental math to calculate the following:

   $9 \times 8 \times 11$

Check Your Understanding

33. Which of the following are real numbers: $4 + i,\ \sqrt{77},\ -19,\ 38.902$?

34. Indicate which of the sets are subsets of the others: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$.

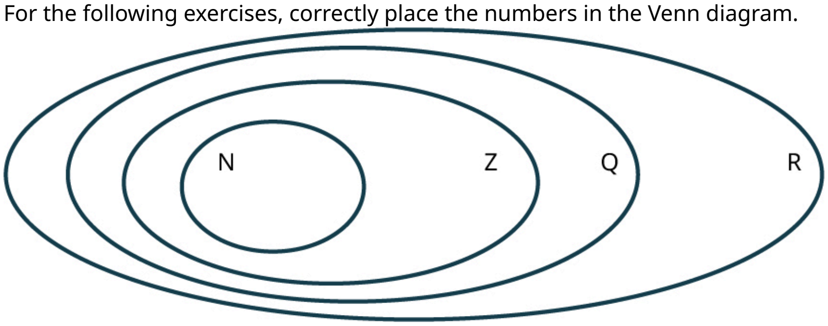
35. Which property is demonstrated here: $3 \times 6 + 3 \times 15 = 3 \times (6 + 15)$?

SECTION 3.6 EXERCISES

For the following exercises, identify each number as a natural number, an integer, a rational number, or a real number.

1. $\frac{1}{3}$
2. 16
3. $\sqrt{19}$
4. $-\frac{3}{7}$
5. π
6. -47
7. 13
8. $\frac{31}{7}$

For the following exercises, correctly place the numbers in the Venn diagram.



9. $-5.6 \quad -7 \quad \sqrt{26} \quad 41 \quad -\frac{13}{50} \quad -46$
10. $-2\pi \quad -\sqrt{11} \quad -17 \quad 679 \quad \frac{14}{37} \quad \frac{57}{151}$

For the following exercises, identify the property of real numbers that is being illustrated.

11. $14 + 38.9 = 38.9 + 14$
12. $37.12 + 98 = 98 + 37.12$
13. $38 \times 16 = 16 \times 38$
14. $6.3 \times \sqrt{14} = \sqrt{14} \times 6.3$
15. $4 + \left(8 + \sqrt{7}\right) = (4 + 8) + \sqrt{7}$
16. $13 \times \left(\frac{6}{11} \times 14\right) = \left(13 \times \frac{6}{11}\right) \times 14$
17. $(5.6 \times 8.7) \times 6 = 5.6 \times (8.7 \times 6)$
18. $\left(3.8 + \sqrt{5}\right) + 13 = 3.8 + \left(\sqrt{5} + 13\right)$
19. $4 \times \left(5 + \sqrt{3}\right) = 4 \times 5 + 4 \times \sqrt{3}$
20. $\sqrt{74} + \left(-\sqrt{74}\right) = 0$
21. $-3.4\pi \times \left(-\frac{1}{3.4\pi}\right) = 1$
22. $17 \times \left(\frac{1}{17}\right) = 1$
23. $4\sqrt{35} + 0 = 4\sqrt{35}$
24. $\sqrt{17} \times (5 + 3.6) = \sqrt{17} \times 5 + \sqrt{17} \times 3.6$
25. $(-10\pi) + 10\pi = 0$
26. $\frac{3}{\sqrt{7}} \times 1 = \frac{3}{\sqrt{7}}$
27. $3\pi + 7\sqrt{21} = 7\sqrt{21} + 3\pi$
28. $14 \times \left(3\sqrt{2} + 11.6\right) = 14 \times 3\sqrt{2} + 14 \times 11.6$

For the following exercises, use properties of real numbers and mental math to calculate the expression.

29. $43 + 62 + 17$
30. $106 + 75 + 94$
31. $5 \times 13 \times 4$
32. $4 \times 72 \times 5$
33. $46 + 77 + 23 + 24 + 103$
34. $23 + 98 + 75 + 12 + 77$
35. $4 \times 13 \times 25$
36. $50 \times 23 \times 2$
37. 13×99
38. 15×39
39. 43×101
40. 16×999

3.7 Clock Arithmetic

Figure 3.40 If a credit card number is entered incorrectly, error checking algorithms will often catch the mistake. (credit: modification of work "Senior couple at home checking finance on credit card from above" by Nenad Stojkovic/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Add, subtract, and multiply using clock arithmetic.
2. Apply clock arithmetic to calculate real-world applications.

Online shopping requires you to enter your credit card number, which is then sent electronically to the vendor. Using an ATM involves sliding your bank card into a reader, which then reads, sends, and verifies your card. Swiping or tapping for a purchase in a brick–and-mortar store is how your card sends its information to the machine, which is then communicated to the store's computer and your credit card company. This information is read, recorded, and transferred many times. Each instance provides one more opportunity for error to creep into the process, a misrecorded digit, transposed digits, or missing digits. Fortunately, these card numbers have a built-in error checking system that relies on modular arithmetic, which is often referred to as clock arithmetic. In this section, we explore clock, or modular, arithmetic.

VIDEO

Determining the Day of the Week for Any Date in History (https://openstax.org/r/Determining_the_Day_of_the_Week_for_Any_Date_in_History)

Adding, Subtracting, and Multiplying Using Clock Arithmetic

When we do arithmetic, numbers can become larger and larger. But when we work with time, specifically with clocks, the numbers cycle back on themselves. It will never be 49 o'clock. Once 12 o'clock is reached, we go back to 1 and repeat the numbers. If it's 11 AM and someone says, "See you in four hours," you know that 11 AM plus 4 hours is 3 PM, not 15 AM (ignoring military time for now). Math worked on the clock, where numbers restart after passing 12, is called **clock arithmetic**.

Clock arithmetic hinges on the number 12. Each cycle of 12 hours returns to the original time (Figure 3.41). Imagine going around the clock one full time. Twelve hours pass, but the time is the same. So, if it is 3:00, 14 hours later and two hours later both read the same on the clock, 5:00. Adding 14 hours and adding 2 hours are identical. As is adding 26 hours. And adding 38 hours.

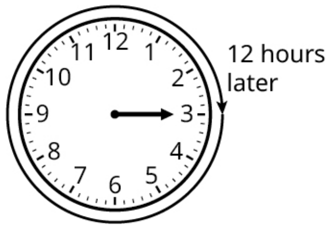


Figure 3.41 Clock showing 3:00 with arrow going around the clock one full time, or 12 hours

What do 2, 14, 26, and 38 have in common in relation to 12? When they are divided by 12, they each have a remainder of 2. That's the key. When you add a number of hours to a specific time on the clock, first divide the number of hours being added by 12 and determine the remainder. Add that remainder to the time on the clock to know what time it will be.

A good visualization is to wrap a number line around the clock, with the 0 at the starting time. Then each time 12 on the number line passes, the number line passes the starting spot on the clock. This is referred to as **modulo 12** arithmetic. Even though the process says to divide the number being added by 12, first perform the addition; the result will be the same if you add the numbers first, and then divide by 12 and determine the remainder.

In general terms, let n be a positive integer. Then n modulo 12, written (n mod 12), is the remainder when n is divided by 12. If that remainder is x, we would write $n = x$ (mod 12).

Caution: 12 mod 12 is 0. So, if a mod 12 problem ends at 0, that would be 12 on the clock.

EXAMPLE 3.101

Determining the Value of a Number modulo 12

Find the value of the following numbers modulo 12:

1. 34
2. 539
3. 156

Solution

To determine the value of a number modulo 12, divide the number by 12 and record the remainder.

1. To find the value 34 modulo 12:
   Step 1: Determine the remainder when 34 is divided by 12 using long division. The largest multiple of 12 that is less than or equal to 34 is 24, which is the product of 12 and 2.

$$\begin{array}{r} 2 \\ 12\overline{)34} \\ \underline{24} \end{array}$$

   Step 2: Performing the subtraction yields 10.

$$\begin{array}{r} 2 \\ 12\overline{)34} \\ \underline{24} \\ 10 \end{array}$$

   Since that subtraction resulted in a number less than 12, that is the remainder, 10. The value of 34 modulo 12 is 10, or 34 = 10 (mod 12).

2. To find the value 539 modulo 12:
   Step 1: Determine the remainder when 539 is divided by 12 using long division. We first look to the first two digits of 539, 53. The largest multiple of 12 that is less than or equal to 53 is 48, which is the product of 12 and 4.

$$\begin{array}{r} 4 \\ 12\overline{)539} \\ \underline{48} \end{array}$$

Step 2: Performing the subtraction results in 5.

$$\begin{array}{r} 4 \\ 12\overline{)539} \\ \underline{48} \\ 5 \end{array}$$

Step 3: Now, the 9 is brought down.

$$\begin{array}{r} 4 \\ 12\overline{)539} \\ \underline{48} \\ 59 \end{array}$$

Step 4: The largest multiple of 12 that is less than or equal to 59 is once more 48 itself, which is 12×4.

$$\begin{array}{r} 44 \\ 12\overline{)539} \\ \underline{48} \\ 59 \\ \underline{48} \end{array}$$

Step 5: Finishing the process, the 48 is subtracted from the 59, yielding 11.

$$\begin{array}{r} 44 \\ 12\overline{)539} \\ \underline{48} \\ 59 \\ \underline{48} \\ 11 \end{array}$$

We've used all the digits of 539, and the last subtraction resulted in a number less than 12, so that number, 11, is the remainder. The value of 539 modulo 12 is 11, or, 539 = 11 (mod 12).

3. To find the value 156 modulo 12:
   Step 1: Determine the remainder when 156 is divided by 12 using long division. We first look to the first two digits of 156, 15. The largest multiple of 12 that is less than or equal to 15 is 12 itself, which is the product of 12 and 1.

$$\begin{array}{r} 1 \\ 12\overline{)156} \\ \underline{12} \end{array}$$

   Step 2: Performing the subtraction results in 3.

$$\begin{array}{r} 1 \\ 12\overline{)156} \\ \underline{12} \\ 3 \end{array}$$

   Step 3: Now, the 6 is brought down.

$$\begin{array}{r} 1 \\ 12\overline{)156} \\ \underline{12} \\ 36 \end{array}$$

   Step 4: The largest multiple of 12 that is less than or equal to 36 is 36 itself, which is 12×3.

$$\begin{array}{r} 13 \\ 12\overline{)156} \\ \underline{12} \\ 36 \\ \underline{36} \end{array}$$

Step 5: Finishing the process, the 36 is subtracted from the 36, yielding 0.

$$\begin{array}{r} 13 \\ 12\overline{)156} \\ \underline{12} \\ 36 \\ \underline{36} \\ 0 \end{array}$$

We've used all the digits of 156, and the last subtraction resulted in a number less than 12, so that number, 0, is the remainder. The value of 156 modulo 12 is 0, or, 156 = 0 (mod 12).

We should note here that, had we been speaking of time, the 0 would be interpreted as 12:00.

YOUR TURN 3.101

Find the value of the following numbers modulo 12.

1. 93
2. 387

TECH CHECK

Using Desmos to Determine the Value of a Number module 12

Desmos may be used to determine the value of a number modulo 12. It is flexible enough to find the value of a number modulo of any other integer you want. To determine the value of n modulo 12, type **mod(n,12)** into Desmos. The result will be displayed immediately. This can be used to find 539 modulo 12, as shown in the Figure 3.42.

Figure 3.42 Display of 539 modulo 12

Clock arithmetic is modulo 12 arithmetic but applied to time. As time is divided into 12 hours that repeat a cycle, we use modulo 12 for clock arithmetic.

VIDEO

Clock Arithmetic (https://openstax.org/r/Clock_Arithmetic)

EXAMPLE 3.102

Adding with Clock Arithmetic

If it's 3:00, what time will it be in 89 hours?

Solution

To find that future time, we may determine the value of 89 (mod 12), either by long division or by using a calculator, such as Desmos. Then add the result to 3:00. Entering **mod(89,12)** in Desmos results in 5. Adding 5 hours, which was 89 (mod12), to 3:00 results in 8:00.

YOUR TURN 3.102

1. If it is 9:00 now, what time will it be in 43 hours?

Subtracting time on the clock works in much the same way as addition. Find the value of the number of hours being subtracted modulo 12, then subtract that from the original time.

EXAMPLE 3.103

Subtracting with Clock Arithmetic

If it is 4:00 now, what time was it 67 hours ago?

Solution

To find that past time, we may determine the value of 67 (mod 12), either by long division or by using a calculator, such as Desmos. Then subtract the result to 4:00. Entering **mod(67,12)** in Desmos results in 7. Subtracting 7 hours from 4:00 results in −3:00. We know, though, that time is not represented with negative times. This value, −3:00, indicates three hours before 12:00, which is 9:00. So, 67 hours before 4:00 was 9:00. We see this in the Figure 3.43.

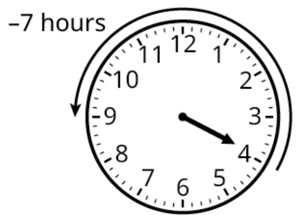


Figure 3.43 Clock showing 7 hours subtracted from 4:00

YOUR TURN 3.103

1. If it is 7:00 now, what time was it 34 hours ago?

Recall that clock arithmetic was referred to as modulo 12 arithmetic. Multiplying in modulo 12 also relies on the remainder when dividing by 12. To multiply modulo 12 is just to multiply the two numbers, and then determine the remainder when divided by 12.

EXAMPLE 3.104

Multiplying modulo 12

What is the product of 11 and 45 modulo 12?

Solution

We begin by multiplying 11 and 45, which is 495. Next, we find 495 modulo 12, either by dividing the result by 12 to determine the remainder, or by using a calculator. Entering **mod(495,12)** in Desmos yields 3. Had long division been used, the remainder would be 3. So $11 \times 45 = 3$ modulo 12.

YOUR TURN 3.104

1. What is the product of 4 and 19 modulo 12?

Calculating Real-World Applications with Clock Arithmetic

EXAMPLE 3.105

Applying Clock Arithmetic

Suppose it is 3:00, and you decide to check your email every 5 hours. What time will it be when you check your email the ninth time?

Solution

If you check your email every 5 hours nine times, that ninth check will occur 45 hours after 3:00, which is an addition of 45 hours to 3:00. So, we find 45 modulo 12, which is 9. Nine hours after 3:00 is 12:00. It will be 12:00 when you check your email the ninth time.

YOUR TURN 3.105

1. You have agreed to text your friend every 3 hours while driving across the country. You began your trip at 8 AM. What time will it be when you text your friend the 15th time?

Clock arithmetic processes can be applied to days of the week. Every 7 days the day of the week repeats, much like every 12 hours the time on the clock repeats. The only difference will be that we work with remainders after dividing by 7. In technical terms, this is referred to as **modulo 7**. More generally, let n be a positive integer. Then n modulo 7, written n mod 7, is the remainder when n is divided by 7. If that value is x, we may write $n = x$ (mod 7).

EXAMPLE 3.106

Applying Clock Arithmetic to Days of the Week

Your family has a cat, and no one wants to empty the litter box. However, it has to be done daily. The six of you agree to take turns, so everyone has to empty the litter box every 6 days. You empty the box on a Thursday. What day will you empty the box for the 10th time?

Solution

The first time you emptied the litter box was on a Thursday. So,the 10th time you empty the litter box will be 9 times later (you've already had your first turn, so 9 turns left!). This will happen 54 (9 times 6) days later. Finding the value of 54 modulo 7, using division to determine the remainder or using a calculator to find the value of 54 modulo 7 gives the answer 5. Five days after a Thursday is Tuesday.

YOUR TURN 3.106

1. Your family shares the cooking duties in the home. You've agreed to prepare the meal every 5 days. The last time you prepared dinner was a Tuesday. What day of the week will it be after you've prepared the meals 20 more times?

Check Your Understanding

For the following exercises, use clock arithmetic to perform the following:

36. $7 + 19$

37. $8 - 31$

38. 5×37

39. Calene calls her mother every fourth day. She calls on a Monday. What day of the week will it be on Calene's eighth time calling after that?

SECTION 3.7 EXERCISES

1. Explain what modulo 12 means.

2. Explain what modulo 7 means.
3. What is 75 modulo 12?
4. What is 139 modulo 12?
5. What is 38 modulo 7?
6. What is 83 modulo 7?

For the following exercises, use clock arithmetic (mod 12), to perform the indicated calculation.

7. $7 + 13$
8. $8 + 19$
9. $4 + 27$
10. $3 + 100$
11. $9 - 15$
12. $6 - 27$
13. 4×18
14. 7×29
15. 11×38
16. 6×23
17. It is 8:00. What time will it be in 70 hours?
18. It is a Thursday. What day of the week will it be in 100 days?
19. It is Monday. What day of the week will it be in 58 days?
20. It is 3:00. What time of the day will it be in 150 hours?
21. It is 6:00. What time was it 34 hours ago?
22. It is 2:00. What time was it 100 hours ago?
23. A trucker passes through Kokomo, Indiana, once every 9 days. They come through Kokomo on a Wednesday. What day of the week will the driver pass through Kokomo after 8 more visits?
24. Jason checks his email every 5 hours. He checks it at 6 PM one day. What time of the day will it be when he checks his mail the 50th time after that 6 PM check?
25. Mickey gets a new prescription of a drug that she needs to take every day. The prescription is for 250 days. She takes the first pill of the new bottle on a Friday. What day of the week will her prescription run out?
26. Zainab visits the nursing home every 5 days. She visits on a Sunday. What day of the week will it be when she visits it for the 7th time after that?
27. Micaela has to check in with her boss every 14 hours. If she checks in at 3:00, what time will it be when she checks in the 10th time after that?
28. Tracy has an alarm set for every 4 hours. It goes off at 3:00. What time will it be when the alarm goes off the 20th time after that?
29. Dejan must check his blood sugar every 5 hours. He checks his blood sugar at 4:00. What time will it be when Dejan checks their his sugar the 40th time after that?
30. Latanjana is in the hospital, where her blood pressure is checked every 3 hours. If her blood pressure is checked at 5:00, what time will it be when her blood pressure is checked the 13th time after that?

Months come in twelves, just as hours do. This means that months can be calculated using modulo 12, just like hours. For the following exercises, calculate what month it will be for each exercise.

31. Micaela works for a sprinkler maintenance company and runs a routine check on the Harris's sprinkler system every third month. If Micaela checks the system in an April, what month will it be when Micaela returns to the Harris's for the 11th time after that?
32. Dana runs a half marathon every 5 months. She runs one in a May. What month will it be when she runs her 8th marathon after that?

3.8 Exponents

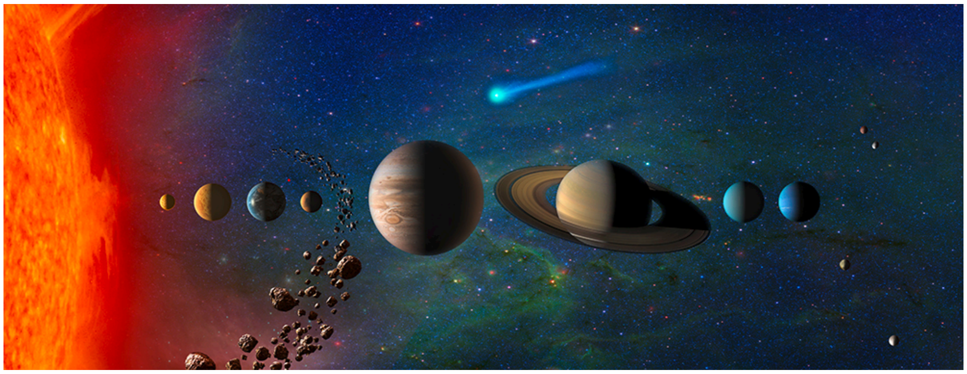


Figure 3.44 Astronomical distances are written using exponents. (credit: "Our Solar System (Artist's Concept)" by NASA/ Jet Propulsion Laboratory-Caltech/Public Domain)

Learning Objectives

After completing this section, you should be able to:

1. Apply the rules of exponents to simplifying expressions.

Sometimes, we look for shorthand when writing or expressing something that simply takes too long. The use of LOL and tl;dr. This shorthand only works if everyone reading the shorthand knows what it stands for. Using exponents is a similar instance. Writing out a long string of a number times itself over and over takes too much time, and eventually one would forget how many of the value has been written or read. For example, $8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8$. There has to be a shorter and more efficient way to write 8 times itself 1, 2, 3....hmmmm, 19 times.

And that's the role that exponents play in mathematics. They are shorthand for multiplying a number by itself a number of times. Without it, calculations would become a mess and we'd have to write a lot more.

Applying the Rules of Exponents to Simplify Expressions

Squaring a number is multiplying it by itself, and has that name because it is the area of a square with that side length. Cubing a number is finding the volume of a cube with that length of sides. That's why we refer to 5^2 as five squared, or 10^3 as ten cubed. Exponents represent that multiplication.

Let's remind ourselves of the terminology associated with exponents and what exponents represent. Suppose you want to multiply a number, let's label that number a, by itself some number of times. Let's label the number of times b. We denote that as a^b. We say a raised to the bth power. When we write or see 7^5, we call the 7 the **base** and we call 5 the **exponent**. What it represents is 7 multiplied by itself 5 times. This means exponents are used as a shorthand for repeated multiplications, where we write $7^5 = 7 \times 7 \times 7 \times 7 \times 7$. We would write 7^5 and say seven to the fifth power.

VIDEO

Exponential Notation (https://openstax.org/r/Exponential_Notation)

The definitions of base and exponent make it possible to understand the exponent rules.

Product Rule for Exponents

The first rule we examine is the product rule, $a^n a^m = a^{n+m}$. This rule means that when we multiply a base raised to a power times the same base to another power, the result is the base raised to the sum of the powers. To demonstrate, consider $9^3 \times 9^5$. If we apply the product rule to that we get $9^3 \times 9^5 = 9^{3+5} = 9^8$. This can be tested by looking at the multiplications that are represented. The 9^3 is 9 times itself 3 times, while 9^5 is 9 times itself 5 times. Substituting those into $9^3 \times 9^5$ we see $9^3 \times 9^5 = (9 \times 9 \times 9) \times (9 \times 9 \times 9 \times 9 \times 9) = 9^8$, which is what the formula told us would happen.

Caution: The product rule only applies when the bases are the same. If the bases are different, we do not apply this rule.

FORMULA

If a number, a, raised to a power, n, is then multiplied by a raised to another power, m, the result is $a^n a^m = a^{n+m}$.

EXAMPLE 3.107

Using the Product Rule for Exponents

If possible, use the product rule to simplify the following:

1. $21^9 \times 21^{15}$
2. $5^9 \times 8^4$

Solution

1. We can apply the product rule to simplify the expression because the bases are the same and we are multiplying.
$$21^9 \times 21^{15} = 21^{(9+15)} = 21^{24}$$
2. Since the bases are not the same (one is 5, the other 8), this cannot be simplified using the product rule for exponents.

YOUR TURN 3.107

If possible, use the product rule to simplify the following:

1. $12^{13} \times 12^8$
2. $3^6 \times 4^{10}$

These rules can be applied to unknowns too.

EXAMPLE 3.108

Using the Product Rule for Exponents of Unknowns

Use the product rule to simplify $a^4 \times a^{10}$.

Solution

The bases are the same, and we are multiplying, so we apply the multiplication rule to simplify the expression.

$$a^4 \times a^{10} = a^{(4+10)} = a^{14}$$

YOUR TURN 3.108

1. Use the product rule to simplify $b^6 \times b^3$.

Quotient Rule for Exponents

The next rule we examine is the quotient, or division, rule.

FORMULA

When a number, a, raised to a power, n, is divided by a raised to another power, m, then the result is $\frac{a^n}{a^m} = a^{(n-m)}$.

This rule means that when we divide a base raised to a power by the same base to another power, the result is the base raised to the difference of the powers. To demonstrate, consider $\frac{14^{13}}{14^6}$. If we apply the quotient rule to that, we get $\frac{14^{13}}{14^6} = 14^{13-6} = 14^7$. This can be tested by looking at the division that is represented. Remember, 14^{13} is 14 multiplied to itself 13 times, while 14^6 is 14 multiplied to itself 6 times. Substituting those into $\frac{14^{13}}{14^6}$ gives the following:

$$\frac{4^{13}}{4^6} = \frac{4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4}{4 \times 4 \times 4 \times 4 \times 4 \times 4}$$

We see here that there are a LOT of fours to be divided out.

$$= \frac{\cancel{4} \times \cancel{4} \times \cancel{4} \times \cancel{4} \times \cancel{4} \times \cancel{4} \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4}{\cancel{4} \times \cancel{4} \times \cancel{4} \times \cancel{4} \times \cancel{4} \times \cancel{4}} = \frac{4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4}{1}$$
$$= 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4$$

What remains is 4 to the 7th power, $4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^7$.

All of the work above confirmed what the formula told us would be the result.

Caution: The quotient rule only applies when the bases are the same. If the bases are different, we do not apply this rule.

EXAMPLE 3.109

Using the Quotient Rule for Exponents

Use the quotient rule to simplify $\frac{5^{19}}{5^{11}}$.

Solution

We can apply the quotient rule to simplify the expression since the bases are the same and we are dividing.

$$\frac{5^{19}}{5^{11}} = 5^{(19-11)} = 5^8$$

YOUR TURN 3.109

1. Use the quotient rule to simplify $\frac{b^6}{b^4}$.

VIDEO

[Product and Quotient Rule for Exponents (https://openstax.org/r/Product_and_Quotient_Rule_for_Exponents)](https://openstax.org/r/Product_and_Quotient_Rule_for_Exponents)

A natural consequence of the quotient rule is what it means to raise a non-zero number to the zeroth power. Let's look at the simplification when the exponents are equal.

$$\frac{3^6}{3^6} = 3^{(6-6)} = 3^0$$

We know that a number divided by itself is 1, so $\frac{3^6}{3^6} = 1$. From that is must be that $\frac{3^6}{3^6} = 3^0 = 1$. This provides the rule for a number raised to the power 0: $a \neq 0$.

FORMULA

If you have a non-zero number a, then $a^0 = 1$.

Distributive Rule for Exponents

The next rule we look to is a distributive rule for exponents.

FORMULA

If you have a product, $(a \times b)$, and raise it to an exponent, n, then $(a \times b)^n = a^n \times b^n$.

This means that when we have two numbers multiplied together, and that is raised to a power, it is the same as raising each of the numbers to the same power first, then multiplying. For example, $(3 \times 7)^4 = 3^4 \times 7^4$. This can be explained using the definition of exponents and multiplying all the factors.

$$(3 \times 7)^4 = (3 \times 7) \times (3 \times 7) \times (3 \times 7) \times (3 \times 7)$$

We may change the order in which numbers are multiplied. This is the commutative property of the real numbers. This can be written as $3 \times 3 \times 3 \times 3 \times 7 \times 7 \times 7 \times 7$. Using exponents, that shortens to $3^4 \times 7^4$.

This also works in the other direction, $a^n \times b^n = (a \times b)^n$. Read this way, if we have one base raised to an exponent, and another base raised to the same exponent, we can multiply the bases and raise that product to the shared exponent. For instance, $7^8 \times 11^8 = (7 \times 11)^8 = 77^8$.

Caution: The exponent distributive rule, $a^n \times b^n = (a \times b)^n$, only works if the exponents are the same.

EXAMPLE 3.110

Using the Distributive Rule for Exponents

Use the exponent distributive rule to expand $(6 \times 13)^7$.

Solution

Applying the distributive rule to the product, we get $(6 \times 13)^7 = 6^7 \times 13^7$.

YOUR TURN 3.110

1. Use the exponent distributive rule to expand $(2 \times 19)^{14}$.

EXAMPLE 3.111

Using the Distributive Rule for Exponents

Use the exponent distributive rule to expand $(c \times d)^{10}$.

Solution

Applying the distributive rule to the product, we get $(c \times d)^{10} = c^{10} \times d^{10}$.

YOUR TURN 3.111

1. Use the exponent distributive rule to expand $(a \times b)^6$.

This distribution also works for quotients. A fraction raised to an exponent equals the numerator raised to the exponent divided by the denominator raised to the exponent. For example, $\left(\frac{3}{5}\right)^7 = \frac{3^7}{5^7}$. Demonstrating this is similar to the previous rule.

FORMULA

When you have a fraction, $\frac{a}{b}$, raised to an exponent, n, then $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

EXAMPLE 3.112

Using the Distributive Rule for Exponents with Fractions

Use the exponent distributive rule to expand the following:

1. $\left(\frac{4}{9}\right)^6$
2. $\left(\frac{3}{b}\right)^{11}$

Solution

1. Applying the distributive rule to the quotient, we get $\left(\frac{4}{9}\right)^6 = \frac{4^6}{9^6}$.
2. Applying the distributive rule to the quotient, we get $\left(\frac{3}{b}\right)^{11} = \frac{3^{11}}{b^{11}}$.

YOUR TURN 3.112

Use the exponent distributive rule to expand the following:

1. $\left(\frac{14}{5}\right)^9$
2. $\left(\frac{a}{18}\right)^5$

VIDEO

[Fraction Raised to a Power (https://openstax.org/r/Fraction_Raised_to_a_Power)](https://openstax.org/r/Fraction_Raised_to_a_Power)

Power Rule

In the previous two sets of rules, we've seen exponents applied to products and quotients. Now we look to exponents applied to other exponents. For example, $\left(3^6\right)^4 = 3^{(6 \times 4)} = 3^{24}$. This can be explained by examining what the outer exponent does. We raise 3^6 to the fourth power, so we multiply 3^6 by itself 4 times, $\left(3^6\right)^4 = 3^6 \times 3^6 \times 3^6 \times 3^6$. Now if we apply the product rule for exponents, this becomes $3^{(6+6+6+6)} = 3^{24}$.

FORMULA

If you raise a non-zero base, say a, to an exponent n, and raise that to another exponent, m, you get the base raised to the product of the exponents, which is $(a^n)^m = a^{(n \times m)}$.

EXAMPLE 3.113

Raising an Exponent to an Exponent

Expand the following:

1. $\left(6^7\right)^3$
2. $\left(b^{12}\right)^4$

Solution

1. Using the power rule of exponents, $\left(6^7\right)^3 = 6^{(7 \times 3)} = 6^{21}$.
2. Using the power rule of exponents, $\left(b^{12}\right)^4 = b^{(12 \times 4)} = b^{48}$.

YOUR TURN 3.113

Expand the following:

1. $\left(11^4\right)^{12}$
2. $\left(a^7\right)^6$

Negative Exponent Rule

Up until now, we've only looked at positive exponents. The last exponent rule we look at is what negative exponents represent. Recall the quotient rule: $\frac{a^n}{a^m} = a^{(n+m)}$. What would happen if the exponent in the denominator was larger than that in the numerator? For example, $\frac{4^5}{4^7}$. If we apply the quotient rule, we obtain $\frac{4^5}{4^7} = 4^{5-7} = 4^{-2}$. We need to make sense of that negative exponent. To do so, we can expand the quotient and see what happens: $\frac{4^5}{4^7} = \frac{4\times4\times4\times4\times4}{4\times4\times4\times4\times4\times4\times4}$. When we divide out common factors, only two factors of 4 are left in the denominator, as we see here: $\frac{1}{4\times4}$. Using exponent notation, this is $\frac{1}{4^2}$. Since 4^{-2} and $\frac{1}{4^2}$ represent the same number, $\frac{4^5}{4^7}$, they are equal. This demonstrates how negative exponents are defined.

FORMULA

$a^{-n} = \frac{1}{a^n}$ provided that $a \neq 0$.

Similarly, $\frac{1}{a^{-n}} = a^n$.

EXAMPLE 3.114

Eliminating Negative Exponents

Convert the following to expressions with no negative exponent:

1. $3^4 \times 5^{-8}$
2. $a^{-9} \times b^5$
3. $\frac{7}{c^{-2}}$

Solution

1. Using the negative exponent rule on the 5^{-8} and multiplying, $3^4 \times 5^{-8} = 3^4 \times \frac{1}{5^8} = \frac{3^4}{5^8}$.
2. Using the negative exponent rule on the a^{-9} and multiplying, $a^{-9} \times b^5 = \frac{1}{a^9} \times b^5 = \frac{b^5}{a^9}$.
3. Begin by rewriting the expression as $\frac{7}{c^{-2}} = \frac{7}{1} \times \frac{1}{c^{-2}}$. Apply the negative exponent rule to $\frac{1}{c^{-2}}$ in the expression, which becomes $\frac{7}{1} \times \frac{1}{c^{-2}} = 7 \times c^2$, which has no negative exponents.

YOUR TURN 3.114

Convert the following to expressions with no negative exponent:

1. $12^{-3} \times 7^5$
2. $c^{-7} \times 5^3$

EXAMPLE 3.115

Eliminating Denominators by Using Negative Exponents

Use negative exponents to rewrite the following expressions with no denominator:

1. $\frac{7^3}{13^9}$
2. $\frac{c^4}{d^8}$

Solution

1. Rewrite the expression $\frac{7^3}{13^9}$ as $\frac{7^3}{1} \times \frac{1}{13^9}$. Then use the definition of negative exponents to rewrite the $\frac{1}{13^9}$ as 13^{-9}. Last, multiply, yielding $\frac{7^3}{1} \times \frac{1}{13^9} = 7^3 \times 13^{-9}$.
2. Rewrite the expression $\frac{c^4}{d^8}$ as $\frac{c^4}{1} \times \frac{1}{d^8}$. Then use the definition of negative exponents to rewrite the $\frac{1}{d^8}$ as d^{-8}. Last, multiply, yielding $\frac{c^4}{1} \times \frac{1}{d^8} = c^4 \times d^{-8}$.

YOUR TURN 3.115

Use negative exponents to rewrite the following expressions with no denominator:

1. $\frac{6^3}{13^8}$

2. $\frac{c^5}{2^9}$

The table below shows a summary of the exponent rules from this section.

Rule	Example	In Words
Product Rule $a^n a^m = a^{n+m}$	$8^2 \times 8^5 = 8^7$	A base raised to a power, times the same based raised to another power, is the base raised to the sum of the powers.
Quotient Rule $\frac{a^n}{a^m} = a^{(n-m)}$	$\frac{11^{14}}{11^{12}} = 11^{12}$	A base raised to a power, divided by the same based raised to another power, is the base raised to the difference of the powers.
Zero Power Rule $a^0 = 1$ provided that $a \neq 1$	$412^0 = 1$	Any non-zero number raised to the zeroth power equals 1.
Distributive Rule, Multiplication $(a \times b)^n = a^n \times b^n$	$(14 \times 31)^9 = 14^9 \times 31^9$	Exponents distribute across multiplication.
Distributive Rule, Division $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{62}{91}\right)^8 = \frac{62^8}{91^8}$	Exponents distribute across division.
Power Rule $(a^n)^m = a^{(n \times m)}$	$\left(5^9\right)^{15} = 5^{135}$	A base raised to a power, raised to another power, is the base raised to the first power times the second power.
Negative Exponent Rule $a^{-n} = \frac{1}{a^n}$ provided that $a \neq 0$	$6^{-8} = \frac{1}{6^8}$ $\frac{1}{12^7} = 12^{-7}$	A base raised to a negative exponent is 1 divided by the base raised to the positive power, and vice versa.

These rules often occur in tandem with each other, but it requires that you carefully apply the rules.

EXAMPLE 3.116

Simplifying Expressions Using Exponent Rules

Simplify the following:

1. $\left(\frac{4^2 \times 7}{9^3}\right)^5$
2. $\left(\frac{5a^4}{b^9}\right)^6$

Solution

1. **Step 1:** To simplify this, we start by distributing the power 5 across the quotient:

$$\left(\frac{4^2 \times 7}{9^3}\right)^5 = \frac{\left(4^2 \times 7\right)^5}{\left(9^3\right)^5}$$

Step 2: We distribute the power 5 in the numerator across that multiplication:

$$\left(\frac{4^2 \times 7}{9^3}\right)^5 = \frac{\left(4^2 \times 7\right)^5}{\left(9^3\right)^5} = \frac{\left(4^2\right)^5 \times 7^5}{\left(9^3\right)^5}$$

Step 3: We apply the power rule where indicated:

$$\left(\frac{4^2 \times 7}{9^3}\right)^5 = \frac{\left(4^2 \times 7\right)^5}{\left(9^3\right)^5} = \frac{\left(4^2\right)^5 \times 7^5}{\left(9^3\right)^5} = \frac{4^{(2\times5)} \times 7^5}{9^{(3\times5)}} = \frac{4^{10} \times 7^5}{9^{15}}$$

2. **Step 1:** To simplify this, we start by distributing the power 6 across the quotient:

$$\left(\frac{5a^4}{b^9}\right)^9 = \frac{\left(5 \times a^4\right)^6}{\left(b^9\right)^6}$$

Step 2: We distribute the power 5 in the numerator across that multiplication:

$$\frac{\left(5 \times a^4\right)^6}{\left(b^9\right)^6} = \frac{(5)^6 \times \left(a^4\right)^6}{\left(b^9\right)^6}$$

Step 3: We apply the power rule where indicated:

$$\frac{(5)^6 \times \left(a^4\right)^6}{\left(b^9\right)^6} = \frac{5^6 a^{24}}{b^{54}}$$

YOUR TURN 3.116

Simplify the following:

1. $\left(\frac{7^9}{10^5 \times 6^3}\right)^8$
2. $\left(\frac{4}{a^9 b^6}\right)^2$

VIDEO

[Simplifying Expressions with Exponents (https://openstax.org/r/Simplifying_Expressions_with_Exponents)](https://openstax.org/r/Simplifying_Expressions_with_Exponents)

Check Your Understanding

40. Simplify $a^3 \times a^5$.

41. Simplify $\frac{5^4}{5^8}$.

42. Simplify $(6b)^9$.

43. Simplify $\left(\frac{c}{7}\right)^3$.

44. Simplify $\left(\frac{3a^2}{4b^5}\right)^6$.

SECTION 3.8 EXERCISES

For the following exercises, simplify the expression.

1. $4^5 \times 4^2$
2. $3^3 \times 3^6$
3. $a^2 \times a^7$
4. $b^7 \times b^{12}$
5. $\frac{4^6}{4^2}$
6. $\frac{15^{14}}{15^{11}}$
7. $\frac{c^9}{c^4}$
8. $\frac{a^{14}}{a^5}$
9. $\frac{11^3}{11^7}$
10. $\frac{7^{14}}{7^{23}}$
11. $\frac{b^6}{b^{19}}$
12. $\frac{d^8}{d^{17}}$
13. $(4 \times 3)^4$
14. $(5 \times 8)^7$
15. $(3c)^6$
16. $(n \times m)^9$
17. $\left(\frac{7}{2}\right)^8$
18. $\left(\frac{a}{6}\right)^{11}$
19. $\left(\frac{4}{c}\right)^9$
20. $\left(6^4\right)^8$
21. $\left(12^5\right)^3$
22. $\left(x^6\right)^2$
23. $\left(b^6\right)^{11}$
24. $\left(\frac{3b}{5}\right)^4$
25. $\left(\frac{a}{2b}\right)^5$
26. $\left(\frac{3a^4}{7}\right)^3$
27. $\left(\frac{6x^7}{11}\right)^8$
28. $\left(\frac{2x^4}{b^7}\right)^3$
29. $\left(\frac{7r^9}{a^4}\right)^{12}$

For the following exercises, rewrite the expression without a denominator.

30. $\frac{3}{a^4}$
31. $\frac{5}{b^6}$
32. $\frac{4}{b^2c^3}$
33. $\frac{9}{x^4y^5}$

For the following exercises, rewrite the expression without negative exponents.

34. $12^4 \times 5^{-3}$
35. $3b^{-12}$

36. $\frac{15}{a^{-10}}$

37. $\frac{6x^{-5}}{y^{-7}}$

3.9 Scientific Notation

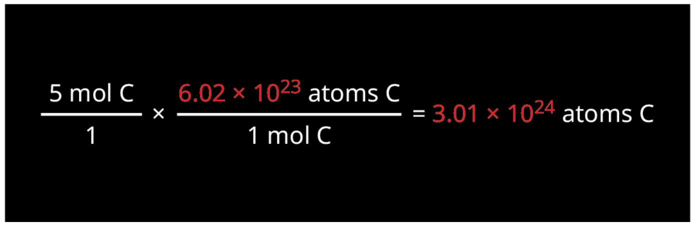


Figure 3.45 Calculations in the sciences often involve numbers in scientific notation form.

Learning Objectives

After completing this section, you should be able to:

1. Write numbers in standard or scientific notation.
2. Convert numbers between standard and scientific notation.
3. Add and subtract numbers in scientific notation.
4. Multiply and divide numbers in scientific notation.
5. Use scientific notation in computing real-world applications.

The amount of information available on the Internet is simply incomprehensible. One estimate for the amount of data that will be on the Internet by 2025 is 175 Zettabytes. A single zettabyte is one billion trillion. Written out, it is 1,000,000,000,000,000,000,000. One estimate is that we're producing 2.5 quintillion bytes of data per day. A quintillion is a trillion trillion, or, written out, 1,000,000,000,000,000,000. To determine how many days it takes to increase the amount of information that is on the Internet by 1 zettabyte, divide these two numbers, a zettabyte being 1,000,000,000,000,000,000,000, and 2.5 quintillion, being 2,500,000,000,000,000,000, shows it takes 400 days to generate 1 zettabyte of information. But doing that calculation is awkward with a calculator. Keeping track of the zeros can be tedious, and a mistake can easily be made.

On the other end of the scale, a human red blood cell has a diameter of 7.8 micrometers. One micrometer is one millionth of a meter. Written out, 7.8 micrometers is 0.0000078 meters. Smaller still is the diameter of a virus, which is about 100 nanometers in diameter, where a nanometer is a billionth of a meter. Written out, 100 nanometers is 0.0000001 meters. To compare that to engineered items, a single transistor in a computer chip can be 14 nanometers in size (0.000000014 meters). Smaller yet is the diameter of an atom, at between 0.1 and 0.5 nanometers.

Sometimes we have numbers that are incredibly big, and so have an incredibly large number of digits, or sometimes numbers are incredibly small, where they have a large number of digits after the decimal. But using those representations of the names of the sizes makes comparing and computing with these numbers problematic. That's where scientific notation comes in.

Writing Numbers in Standard or Scientific Notation Form

When we say that a number is in **scientific notation**, we are specifying the form in which that number is written. That form begins with an integer with an absolute value between 1 and 9, then perhaps followed the decimal point and then some more digits. This is then multiplied by 10 raised to some power. When the number only has one non-zero digit, the scientific notation form is the digit multiplied by 10 raised to an exponent. When the number has more than one non-zero digit, the scientific notation form is a single digit, followed by a decimal, which is then followed by the remaining digits, which is then multiplied by 10 to a power.

The following numbers are written in scientific notation:

1.45×10^3

-8.345×10^{-4}

3×10^2

3.14159×10^0

The following numbers are not written in scientific notation:

1.45 because it isn't multiplied by 10 raised to a power

-50.053×10^7 because the absolute value of −50.053 is not at least 1 and less than 10

41.7×10^9 because 41.7 is not at least 1 and less than 10

0.036×10^{-3} because 0.036 is not at least one and less than 10

EXAMPLE 3.117

Identifying Numbers in Scientific Notation

Which of the following numbers are in scientific notation? If the number is not in scientific notation, explain why it is not.

1. -9.67×10^{20}
2. 145×10^{-8}
3. 1.45

Solution

1. The number -9.67×10^{20} is in scientific notation because the absolute value of −9.67 is at least 1 and less than 10.
2. The number 145×10^{-8} is not in scientific notation because 145 is not at least 1 and less than 10.
3. The number 1.45 is not in scientific notation form. Even though it is at least 1 but less than 10, it is not multiplied by 10 raised to a power.

YOUR TURN 3.117

Which of the following numbers are in scientific notation? If the number is not in scientific notation, explain why it is not.

1. 42.67×10^{13}
2. 7.113×10^{-2}
3. -80.91

Some numbers are so large or so small that it is impractical to write them out fully. Avogadro's number is important in chemistry. It represents the number of units in 1 mole of any substance. The substance many be electrons, atoms, molecules, or something else. Written out, the number is: 602,214,076,000,000,000,000,000. Another example of a number that is impractical to write out fully is the length of a light wave. The wavelength of the color blue is about 0.000000450 to 0.000000495 meters. Such numbers are awkward to work with, and so scientific notation is often used. We need to discuss how to convert numbers into scientific notation, and also out of scientific notation.

Recall that multiplying a number by 10 adds a 0 to the end of the number or moves the decimal one place to the right, as in $43 \times 10 = 430$ or $3.89 \times 10 = 38.9$. And if you multiply by 100, it adds two zeros to the end of the number or moves the decimal two places to the right, and so on. For example, $38 \times 100{,}000 = 3{,}800{,}000$ and $32.998 \times 10{,}000 = 329{,}980$. Multiplying a number by 1 followed by some number of zeros just adds that many zeros to the end of the number or moves the decimal place that many places to the right. Numbers written as 1 followed by some zeros are just powers of 10, as in $10^1 = 10$, $10^2 = 100$, $10^3 = 1{,}000$, etc. Generally, $10^n = 1\underbrace{0...0}_{n}$.

We can use this to write very large numbers. For instance, Avogadro's number is 602,214,076,000,000,000,000,000, which can be written as $6.02214076 \times 10^{23}$. The multiplication moves the decimal 23 places to the right.

Similarly, when we divide by 10, we move the decimal one place to the left, as in $\frac{46.7}{10} = 4.67$. If we divide by 100, we move the decimal two places to the left, as in $\frac{3.456}{100} = 0.03456$. In general, when you divide a number by a 1 followed by n zeros, you move the decimal n places to the left, as in $\frac{8{,}244.902}{1{,}000{,}000} = 0.008244902$. This denominator could be written as 10^6. If we use that in the expression and allow for negative exponents, rewrite the number as $\frac{8{,}244.902}{1{,}000{,}000} = \frac{8{,}244.902}{10^6} = 8{,}244.902 \times 10^{-6}$. With this, we can write division by a 1 followed by n zeros as multiplication by 10 raised to $-n$.

Using that information, we can demonstrate how to convert from a number in standard form into scientific notation form.

Case 1: The number is a single-digit integer.

In this case, the scientific notation form of the number is $digit \times 10^1$.

Case 2: The absolute value of the number is less than 1.

Follow the process below.

- **Step 1:** Count the number of zeros between the decimal and the first non-zero digit. Label this n.
- **Step 2:** Starting with the first non-zero digit of the number, write the digits. If the number was negative, include the negative sign.
- **Step 3:** If there is more than one digit, place the decimal after the first digit from Step 2.
- **Step 4:** Multiply the number from Step 3 by 10^{n+1}.

Case 3: The absolute value of the number is 10 or larger.

Follow the process below.

- **Step 1:** Count the number of digits that are to the left of the decimal point. Label this n.
- **Step 2:** Write the digits of the number without the decimal place, if one was present. If the number was negative, include the negative sign.
- **Step 3:** If there is more than one digit, place the decimal point after the first digit.
- **Step 4:** Multiply the number from Step 3 by 10^{n-1}.

EXAMPLE 3.118

Writing a Number in Scientific Notation

Write the following numbers in scientific notation form:

1. 428.9
2. −0.00000981
3. 8

Solution

1. Since the absolute value of 428.9 is 10 or larger, so we use the process from Case 3, above.
   Step 1: There are three digits to the left of the decimal point, so $n = 3$.

   Step 2: Write the digits of the number without the decimal place, which is 4289.

   Step 3: Since there is more than one digit, place the decimal point after the first digit. We now have 4.289.

   Step 4: Since $n = 3$, we multiply 4.289 by 10 raised to the second power, 4.289×10^2.

   The scientific notation form of 428.9 is 4.289×10^2.

2. Since the absolute value of −0.00000981 is less than 1, we use the process from Case 2.
   Step 1: The number of zeros between the decimal and the first non-zero digit is 5, so $n = 5$.

   Step 2: We write the non-zero digits, including the negative sign, yielding −981.

   Step 3: The decimal gets placed to the right of the first digit, resulting in −9.81.

   Step 4: Since $n = 5$, we multiply −9.81 by 10 raised to the fourth power, -9.81×10^{-6}.

   The scientific notation form of −0.00000981 is 9.81×10^{-6}.

3. Since 8 is a single-digit integer, apply Case 1. The scientific notation form of 8 is 8×10^1.

YOUR TURN 3.118

Write the following numbers in scientific notation form:

1. −38300
2. 0.0045
3. 1

When we write numbers in scientific notation form, we can manipulate the representation of the number by moving the decimal around, and making an appropriate change to the exponent of the 10. For instance, let's look at 145.8141×10^8. If we wanted to move the decimal one place to the left, we'd have to increase the power of 10, as shown here: $145.8141 \times 10^8 = 14.58141 \times 10^9$. Since we moved the decimal one to the left, we balance that with moving the exponent up by one. Similarly, if we move the decimal one place to the right, we have to balance that by moving the exponent one to the left, or subtracting one from the exponent, as shown here: $145.8141 \times 10^8 = 1458.141 \times 10^7$. Generally, for a number in the form $number \times 10^n$:

- If you move the decimal to the left by k digits, you increase the exponent by k.
- If you move the decimal to the right by k digits, you decrease the exponent by k digits.

EXAMPLE 3.119

Increasing the Exponent

Change 456.142×10^5 by moving the decimal two places to the left.

Solution

Since we are moving the decimal to the left by two places, we increase the exponent of 10 by 2, so that the exponent is now 7. This gives us $456.142 \times 10^5 = 4.56142 \times 10^7$.

YOUR TURN 3.119

1. Change 46.113×10^8 by moving the decimal four places to the left.

EXAMPLE 3.120

Decreasing the Exponent

Change 12.3×10^2 by moving the decimal five places to the right.

Solution

Since we are moving the decimal to the right by five places, we decrease the exponent of 10 by 5, so that exponent is now −3. This give us $12.3 \times 10^2 = 1230000.0 \times 10^{-3}$.

YOUR TURN 3.120

1. Change 149.11×10^{-4} by moving the decimal two places to the right.

Converting Numbers from Scientific Notation to Standard Form

In the previous section, converting a number from standard form to scientific notation was explored. Now, we explore converting from scientific notation back into standard form. Doing so involves moving the decimal according to the power of the 10. The decimal is moved a number of steps equal to the exponent of the 10. As demonstrated previously, when the exponent of the 10 is negative, the decimal is moved to the left and when the exponent of the 10 is positive, the decimal is moved to the right.

EXAMPLE 3.121

Converting from Scientific Notation to Standard Form

Convert the following into standard form:

1. 2.78×10^9
2. 9.04×10^{-8}

Solution

1. Since the exponent is positive, the decimal moves nine places to the right, so 2.78×10^9 is 2,780,000,000.

2. Since the exponent is negative, the decimal moves eight places to the left, so 9.04×10^{-8} is 0.0000000904.

YOUR TURN 3.121

Convert the following into standard form:

1. 1.02×10^{6}
2. 4.09×10^{-5}

VIDEO

[Converting from Standard Form to Scientific Notation Form (https://openstax.org/r/Converting_from_Standard_Form_to_Scientific_Notation_Form)](https://openstax.org/r/Converting_from_Standard_Form_to_Scientific_Notation_Form)

[Converting from Scientific Notation Form to Standard Form (https://openstax.org/r/Converting_from_Scientific_Notation_Form_to_Standard_Form)](https://openstax.org/r/Converting_from_Scientific_Notation_Form_to_Standard_Form)

TECH CHECK

Scientific Notation on a Calculator

Most scientific and graphing calculators come with the ability to directly convert from standard form to scientific notation. On the TI-83, it is accessed through the MODE menus. For a commonly used, free phone scientific calculator, the calculator can be forced to work in scientific notation mode through its settings.

Some calculators, such as the Desmos online calculator, display scientific notation as a number times 10 to a power as you've seen in this section. However, some calculators indicate scientific notation by replacing the $\times 10^{n}$ with an E (or EE) followed by the exponent. For example, Figure 3.46 shows what you may see on a TI-84.

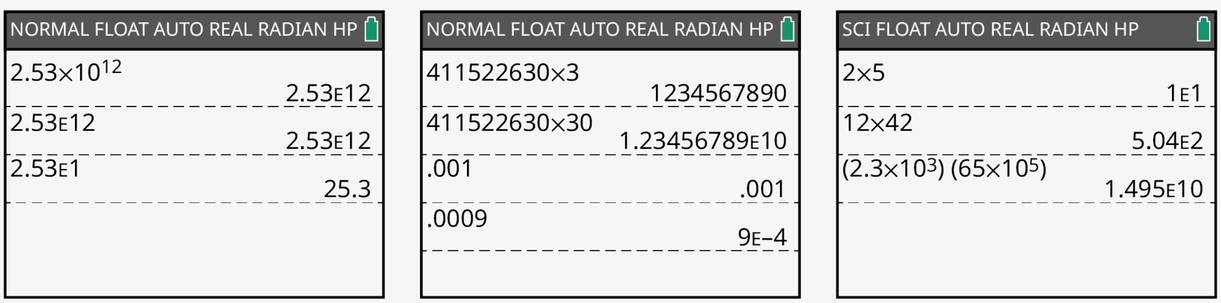


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Figure 3.46 Calculator screens

Adding and Subtracting Numbers in Scientific Notation

To add or subtract numbers in scientific notation, the numbers first need to have the same exponent for the 10s. It is possible to add the following since the powers of 10 match: $4.5 \times 10^{4} + 3.15 \times 10^{4} = 7.65 \times 10^{4}$

Notice that the number parts were added, but the exponent part remained the same. This is due to the distributive property of the real numbers. The 10^{4} is factored from the two terms, as shown:
$4.5 \times 10^{4} + 3.15 \times 10^{4} = (4.5 + 3.15) \times 10^{4} = 7.65 \times 10^{4}$

Numbers in scientific notation can be added or subtracted directly using a calculator. Simply enter the values in scientific form and set your calculator to display scientific notation.

EXAMPLE 3.122

Adding and Subtracting Numbers in Scientific Notation with the Same Powers of 10

Calculate the following:

1. $3.8 \times 10^{-3} + 1.006 \times 10^{-3}$
2. $9.61 \times 10^{8} - 3.85 \times 10^{8}$

Solution

1. Since the powers of 10 match, we use the distributive property of real numbers to factor 10^{-3} from the numbers. We then add the number parts separately to get 4.806.
$3.8 \times 10^{-3} + 1.006 \times 10^{-3} = (3.8 + 1.006) \times 10^{-3} = 4.806 \times 10^{-3}$
2. Since the powers of 10 match, we use the distributive property of real numbers to factor 10^{8} from the numbers. We then subtract the number parts separately to get 5.76.
$$9.61 \times 10^{8} - 3.85 \times 10^{8} = (9.61 - 3.85) \times 10^{8} = 5.76 \times 10^{8}$$

YOUR TURN 3.122

Calculate the following:

1. $7.57 \times 10^{13} + 2.031 \times 10^{13}$
2. $3.03 \times 10^{-6} - 1.5 \times 10^{-6}$

Adding and subtracting in scientific notation is straightforward when the exponents are the same. There are two issues that can arise. The first issue is what to do if after adding or subtracting the result is not in scientific notation.

EXAMPLE 3.123

Correcting an Answer to Scientific Notation After Adding or Subtracting

Calculate the following:

1. $7.03 \times 10^{13} + 8.5 \times 10^{13}$
2. $4.3 \times 10^{21} - 4.613 \times 10^{21}$

Solution

1. Since the powers of 10 match, we add the number parts and multiply that by 10^{13}:
$7.03 \times 10^{13} + 8.5 \times 10^{13} = (7.03 + 8.5) \times 10^{13} = 15.53 \times 10^{13}$.

   However, 15.53×10^{13} is not in scientific notation because the absolute value of 15.53 is more than 10. To put this number in scientific notation, the decimal needs to move one to the left. To balance that move, the power of 10 must be increased by 1. So, the answer in scientific notation is 1.553×10^{14}.

2. Since the powers of 10 match, we add the number parts:
$4.3 \times 10^{21} - 4.613 \times 10^{21} = (4.3 - 4.613) \times 10^{21} = -0.313 \times 10^{21}$

   However, -0.313×10^{21} is not in scientific notation because it is less than 1. To put it in scientific notation, the decimal needs to move one to the right. To balance that move, the power of 10 must be decreased by 1. So, the answer in scientific notation is -3.13×10^{20}.

YOUR TURN 3.123

Calculate the following:

1. $5.08 \times 10^{3} + 6.9 \times 10^{3}$
2. $8.968 \times 10^{-38} - 8.761 \times 10^{-38}$

The second issue that might be encountered when adding or subtracting is that the powers of 10 do not match. In that case, one of the numbers must be changed so that the powers of 10 match. It is easiest to make the smaller power of 10 larger to match the other power of 10.

For example, to perform the following, $4.5 \times 10^5 + 3.9 \times 10^3$, we'd change the 3.9×10^3 so that the power of 10 is 5. To do so, we need to increase the power of 10 and move the decimal in the number part two places to the left. That would alter 3.9×10^3 into 0.039×10^5. We would use 0.039×10^5 in the addition problem, so that the exponents match, allowing the addition to occur. $4.5 \times 10^5 + 3.9 \times 10^3 = 4.5 \times 10^5 + 0.039 \times 10^5 = (4.5 + 0.039) \times 10^5 = 4.539 \times 10^5$

The steps to take when the exponents of the 10s are not equal are:

Step 1: Increase the smaller exponent to equal the larger exponent. Label the amount increased as n.

Step 2: For the number with the smaller power of 10, move the decimal point of the number part to the left n places.

Step 3: Perform the addition or subtraction.

Step 4: If the result is not in scientific notation, adjust the number to be in scientific notation.

EXAMPLE 3.124

Adding Numbers in Scientific Notation with Different Powers of 10

Calculate the following:

$6.1 \times 10^4 + 4.8 \times 10^5$

Solution

Step 1: The lower exponent is 4. To make this equal to the larger exponent, we increased it by 1.

Step 2: Since the smaller exponent was increased by 1, move the decimal one to the left, so the addition become $6.1 \times 10^4 + 4.8 \times 10^5 = 0.61 \times 10^5 + 4.8 \times 10^5$.

Step 3: Now add the numbers, $0.61 \times 10^5 + 4.8 \times 10^5 = 5.41 \times 10^5$

Step 4: The result is in scientific notation, so no additional adjustment is necessary.

$6.1 \times 10^4 + 4.8 \times 10^5 = 5.41 \times 10^5$

YOUR TURN 3.124

1. Calculate the following:
   $1.14 \times 10^{-43} + 2.56 \times 10^{-46}$

EXAMPLE 3.125

Subtracting Numbers in Scientific Notation with Different Powers of 10

Calculate the following:
$7.9 \times 10^{-15} - 6.8 \times 10^{-13}$

Solution

Step 1: The lower exponent is −15 and the larger is −13. To make −15 equal to the larger exponent, we increased it by 2.

Step 2: Since the smaller exponent increased by 2, move the decimal two to the left. The subtraction changes to $7.9 \times 10^{-15} - 6.8 \times 10^{-13} = 0.079 \times 10^{-13} - 6.8 \times 10^{-13}$.

Step 3: Subtract the numbers, $0.079 \times 10^{-13} - 6.8 \times 10^{-13} = -6.721 \times 10^{-13}$.

Step 4: The result is in scientific notation, so no additional adjustment is necessary.
$7.9 \times 10^{-15} - 6.8 \times 10^{-13} = -6.721 \times 10^{-13}$

YOUR TURN 3.125

1. Calculate the following:

$9.15 \times 10^{28} - 7.23 \times 10^{26}$

Multiplying and Dividing Numbers in Scientific Notation

Multiplying and dividing numbers in scientific notation is somewhat easier than adding or subtracting, because the exponents of the 10s do not have to match. However, it is much more likely that the result will not be in scientific notation, and so that will have to be adjusted at the end. Generally, we multiply or divide the number parts of the two values, and then apply exponent rules to the 10 raised to the powers.

To multiply two numbers in scientific notation:

Step 1: Multiply the number parts.

Step 2: Add the exponents of the 10s.

Step 3: The result is the answer from Step 1 times 10 raised to the answer from Step 2.

Step 4: If the number is not in scientific notation, adjust it appropriately.

EXAMPLE 3.126

Multiplying Numbers in Scientific Notation

Calculate the following:

1. $(4.3 \times 10^3) \times (1.8 \times 10^7)$
2. $(5 \times 10^{-13}) \times (7.3 \times 10^6)$

Solution

1. **Step 1:** Multiply the number parts to get $4.3 \times 1.8 = 7.74$.

   Step 2: Add the exponents of the 10s to get $3 + 7 = 10$.

   Step 3: The result is then 7.74×10^{10}.

   Step 4: This number is already in scientific notation, so no additional adjustment is necessary, $(4.3 \times 10^3) \times (1.8 \times 10^7) = 7.74 \times 10^{10}$.

2. **Step 1:** Multiply the number parts to get $5 \times 7.3 = 36.5$.

   Step 2: Add the exponents of the 10s to get $-13 + 6 = -7$.

   Step 3: The result then is 36.5×10^{-7}.

   Step 4: Since the number is not in scientific notation, it must be adjusted. To put 36.5×10^{-7} into scientific notation, the decimal moves one to the left, so the exponent would be increased by 1, giving 3.65×10^{-6}.

   $(5 \times 10^{-13}) \times (7.3 \times 10^6) = 3.65 \times 10^{-6}$

YOUR TURN 3.126

Calculate the following:

1. $(2.29 \times 10^3) \times (3 \times 10^4)$
2. $(6.91 \times 10^{-3}) \times (9.1 \times 10^5)$

VIDEO

Multiplying Numbers in Scientific Notation (https://openstax.org/r/Multiplying_Numbers_in_Scientific_Notation)

Dividing Numbers in Scientific Notation

To divide two numbers that are in scientific notation:

Step 1: Divide the number parts.

Step 2: Subtract the exponent of the denominator from the exponent of the numerator.

Step 3: The answer is the result from Step 1 times 10 raised to the result from Step 2.

Step 4: If the number is not in scientific notation, adjust it appropriately.

EXAMPLE 3.127

Dividing Numbers in Scientific Notation

Calculate the following:

1. $(8.4 \times 10^{31}) \ / \ (2.1 \times 10^{7})$
2. $(4.14 \times 10^{-13}) \ / \ (8.28 \times 10^{9})$

Solution

1. $(8.4 \times 10^{31}) \ / \ (2.1 \times 10^{7})$

   Step 1: Divide the number parts to get $8.4 \div 2.1 = 4$.

   Step 2: Subtract the exponent of the denominator from the exponent of the numerator to get $37 - 7 = 24$.

   Step 3: The result is then 4×10^{24}.

   Step 4: This number is already in scientific notation, so no adjustment is necessary.
   $(8.4 \times 10^{31}) \ / \ (2.1 \times 10^{7}) = 4 \times 10^{24}$

2. $(4.14 \times 10^{-13}) \ / \ (8.28 \times 10^{9})$

   Step 1: Divide the number parts to get $4.14 \div 8.28 = 0.5$.

   Step 2: Subtract the exponent of the denominator from the exponent of the numerator to get $-13 - 9 = -22$.

   Step 3: The result then is 0.5×10^{-22}.

   Step 4: Since this number is not in scientific notation, it must be adjusted. To put 0.5×10^{-22} into scientific notation, the decimal needs to move one to the right, so the exponent is decreased by 1, giving 5×10^{-23}.
   $(4.14 \times 10^{-13}) \ / \ (8.28 \times 10^{9}) = 5 \times 10^{-23}$

YOUR TURN 3.127

Calculate the following:

1. $(3.6 \times 10^{-2})/(1.5 \times 10^{3})$
2. $(1.8 \times 10^{4})/(4.8 \times 10^{3})$

VIDEO

Dividing Numbers in Scientific Notation (https://openstax.org/r/Dividing_Numbers_in_Scientific_Notation)

Using Scientific Notation in Computing Real-World Applications

As noted at the start of this section, scientific notation is useful when the standard representation of a number is awkward or impractical, which occurs when the numbers being used are extremely large or extremely small. For example, Venus is 67,667,000 miles from the sun. In scientific notation, this is 6.7667×10^{7}. Planetary and galaxy distances is one set of numbers that is easier to express using scientific notation.

EXAMPLE 3.128

Calculating Distances

How much farther from the sun is Earth compared to Venus if Venus is 6.7667×10^{7} miles from the sun and Earth is 9.1692×10^{7} miles from the sun?

Solution

To determine how much farther Earth is compared to Venus, we'd subtract the distances.
$9.1692 \times 10^7 - 6.7667 \times 10^7 = 2.4025 \times 10^7$.

So, Earth is 2.4025×10^7 miles farther from the sun than Venus.

YOUR TURN 3.128

1. Earlier we saw that a single transistor in a computer chip 0.000000014 meters, or 1.4×10^{-8} m, in size, and that the diameter of an atom could be 0.2 nanometers, or 2×10^{-10} m in size. How much larger is the transistor than the atom?

EXAMPLE 3.129

Calculating Probability

The probability of winning the Mega Millions lottery is published as 3.304693×10^{-9}. The probability of being hit by lightning is approximated to be 2×10^{-6}. How many times more likely are you to be hit by lightning than win the Mega Millions?

Solution

To find out how many times more likely you are to be hit by lightning, divide the probability of being hit by lightning by the probability of winning the Mega Millions.

$$(2 \times 10^{-6}) \,/\, (3.304693 \times 10^{-9}) = 6.609386 \times 10^3$$

Step 1: Divide the number parts to get = 0.6052 (rounded to the fourth digit).

Step 2: Subtract the exponent of the denominator from the exponent of the numerator to get $-6 - (-9) = 3$.

Step 3: The result then is 0.6052×10^3.

Step 4: Since this number is not in scientific notation, it must be adjusted. To put 0.6052×10^3 into scientific notation, the decimal needs to move one place to the right, so the exponent is decreased by 1, giving 6.052×10^2.

You are 6.052×10^2, or 605.2, times more likely to be hit by lightning than you are to win the Mega Millions.

YOUR TURN 3.129

1. Mercury is about 3.114×10^7 miles from the sun. Neptune is about 2.781×10^9 miles from the sun. How many times further is Neptune from the sun than Mercury?

EXAMPLE 3.130

Calculating Time and Length

Sometimes it is entertaining to determine the time it takes for something to happen. Fingernails grow about 8.032×10^{-11} km per minute. How many kilometers long would fingernails be after 6×10^4 minutes?

Solution

To find the length of the fingernails after the specified time, we multiply their rate of growth and the time they've grown.
$(8.032 \times 10^{-11}) \times (6 \times 10^4) = 48.192 \times 10^{-7} = 4.8192 \times 10^{-6}$

So, after 6×10^4 minutes, the fingernails would be 4.8192×10^{-6} km long. To put this in perspective, 1×10^{-6} km is a millimeter, and 6×10^4 minutes is about 4.16 days. So, after about 4.16 days, fingernails have grown about 4.8 millimeters.

YOUR TURN 3.130

1. There are approximately 1×10^{12} grains of sand in a cubic meter. If the number of grains of sand on the Australian coastline is roughly 7.5×10^{21} grains, roughly how many cubic meters of sand is there on the Australian coastline?

EXAMPLE 3.131

Calculating Data Generated

As mentioned in the opening to this section, it is estimated that we're producing 2.5 quintillion bytes of data per day. A good estimate is that there are 7.674 billion people on the planet. Convert both of those numbers to scientific notation, and then determine how much data is being generated per person each day.

Solution

Written in standard form, 2.5 quintillion is 2,500,000,000,000,000,000. Changing that to scientific notation, move the decimal 18 places, so 2.5 quintillion bytes = 2.5×10^{18} bytes. Writing 7.647 billion in scientific notation would be 7.647×10^{9} because a billion is 1,000,000,000 = 10^{9}. So, to find out how much data is being produced daily per person, we would divide these two numbers. $\frac{2.5 \times 10^{18}}{7.647 \times 10^{9}} = 0.327 \times 10^{9} = 3.27 \times 10^{8}$

In standard form, that's 327,000,000 bytes per person, so 327 million bytes of data daily are being produced per person.

YOUR TURN 3.131

1. Humans collectively exhale approximately 6.4235×10^{12} pounds of carbon dioxide per year. There are approximately 7.647×10^{9} humans currently living on Earth. How many pounds of carbon dioxide does a single human, on average, exhale per year?

VIDEO

[Application of Scientific Notation (https://openstax.org/r/Application_of_Scientific_Notation)](https://openstax.org/r/Application_of_Scientific_Notation)

What Numbers Could Be Considered "Too Big" or "Too Small"?

One wonders when the numbers we represent become too large or small for consideration. Perhaps the following examples put limits on what is meaningful. The number of particles in the known universe has been estimated at 4×10^{80} particles. The smallest distance that has been measured is 1×10^{-18} m, though the theoretical smallest measurable value is 1×10^{-35} m. The distance across the universe is 4.4×10^{26} m. Considering what those numbers represent, the extreme largest and extreme smallest, they might be numbers that constrain what we should reasonably be expected to deal with.

Check Your Understanding

45. Write 0.00456 in scientific notation.

46. Write 5.67×10^{8} in standard form.

47. Calculate $4.5 \times 10^{3} + 9.8 \times 10^{2}$.

48. Calculate $2.5 \times 10^{5} - 9.8 \times 10^{6}$.

49. Calculate $(7.4 \times 10^{4}) \times (4.8 \times 10^{3})$.

50. Calculate $\frac{4.6 \times 10^{-4}}{8 \times 10^{-8}}$.

51. The distance from Earth to the moon is 1.514×10^{10} inches. The thickness of a dollar bill is 4.3×10^{-3} inches. How many dollar bills must be stacked so the pile reaches the moon?

SECTION 3.9 EXERCISES

For the following exercises, convert numbers to scientific notation.

1. 0.0134
2. 0.0000761
3. $3,400$
4. $8,980,000$

For the following exercises, convert numbers to standard form.

5. 9.01×10^5
6. 3.78×10^7
7. 4.32×10^{-3}
8. 5.781×10^{-5}

For the following exercises, the numbers are not in scientific notation. Convert them to scientific notation.

9. 37.65×10^4
10. 0.0034×10^6
11. 0.0834×10^{-7}
12. 14.56×10^{-3}

For the following exercises, make the conversions required.

13. The distance from the sun to the star Polaris is about 3,056,000,000,000,000 km. Express that distance in scientific notation.
14. The distance from us to the next-closest galaxy is about 662,000,000,000,000,000 km. Express that distance in scientific notation.
15. The mass of a grain of sand is about 6.66×10^{-4} g. Convert that mass to standard form.
16. The diameter of a cell is about 2×10^{-6} m. Convert that diameter to standard form.
17. The equatorial circumference of Earth is approximately 4.007×10^4 km. Convert that circumference to standard form.
18. The straight-line distance from Buffalo, NY, to Buenos Aires, Argentina, is approximately 8.86×10^6 m. Convert that distance to standard form.
19. The mass of a proton is approximately 1.67×10^{-27} kg. Convert that mass to standard form.
20. The diameter of a housefly egg is approximately 1.2×10^{-3} m. Convert that diameter to standard form.
21. The tallest building in the world, the Burj Khalifa in Dubai, stands at 829.8 m tall. Convert that height to scientific notation.
22. Using the rings of the shell, the age of an Icelandic clam is 507 years. Express that age in scientific notation.

Calculate the following:

23. $1.3 \times 10^2 + 3.8 \times 10^2$
24. $7.8 \times 10^{12} + 1.1 \times 10^{12}$
25. $3.36 \times 10^4 + 2.71 \times 10^4$
26. $4.58 \times 10^9 + 1.93 \times 10^9$
27. $8.1 \times 10^{-17} + 1.6 \times 10^{-17}$
28. $4.506 \times 10^{-3} + 3.908 \times 10^{-3}$
29. $8.602 \times 10^{-25} + 1.096 \times 10^{-25}$
30. $2.0557 \times 10^{-6} + 1.001 \times 10^{-6}$
31. $5.2 \times 10^4 - 4.1 \times 10^4$
32. $9.48 \times 10^{15} - 6.78 \times 10^{15}$
33. $7.81 \times 10^{-7} - 4.62 \times 10^{-7}$
34. $4.53 \times 10^{-3} - 2.79 \times 10^{-3}$
35. $4.6 \times 10^{-5} + 9.1 \times 10^{-5}$
36. $6.7 \times 10^8 + 5.7 \times 10^8$
37. $4.13 \times 10^{-4} + 7.93 \times 10^{-4}$
38. $5.671 \times 10^{-8} + 9.073 \times 10^{-8}$
39. $4.513 \times 10^6 + 7.856 \times 10^4$
40. $7.135 \times 10^8 + 5.143 \times 10^6$
41. $3.17 \times 10^{-3} + 5.92 \times 10^{-5}$
42. $4.503 \times 10^{-6} + 3.119 \times 10^{-4}$

43. $(4.5 \times 10^4) \times (1.2 \times 10^6)$
44. $(3.45 \times 10^7) \times (2.81 \times 10^{-3})$
45. $(3.1 \times 10^8) \times (2.7 \times 10^{-5})$
46. $(6.32 \times 10^{-4}) \times (1.31 \times 10^{-5})$
47. $(3.91 \times 10^6) \times (8.13 \times 10^2)$
48. $(7.12 \times 10^{-11}) \times (6.61 \times 10^{-5})$
49. $(3.45 \times 10^4) \div (1.5 \times 10^6)$
50. $(1.4 \times 10^3) \div (5.6 \times 10^{-5})$

For the following exercises, apply your understanding of scientific notation to real-world applications.

51. When stretched out, a strand of human DNA is, on average, 2.066×10^2 cm. One centimeter, or 1 cm, is 1×10^{-5} km. Determine the average length of a strand of human DNA in kilometers.
52. One approximation of the average number of cells in the human body is 3×10^{13} cells (30 trillion!!!). If the DNA of each cell were stretched out and laid end to end, what would be the total length of the DNA in km? Use your answer from Exercise 51 for the length, in kilometers, of DNA.
53. The equatorial circumference of Earth is approximately 4.007×10^4 km. Use the answer from Exercise 52 to determine the number of times that the stretched out human DNA would encircle Earth.
54. The average stride length of a 1.651 m tall woman is 6.6×10^{-1} meters. If such a person could walk from Buffalo, NY, to Buenos Aires, Argentina, in a straight line, how many steps would that person need to take? See Exercise 18 for the distance from Buffalo, NY, to Buenos Aires, Argentina.

3.10 Arithmetic Sequences

Learning Objectives

After completing this section, you should be able to:

1. Identify arithmetic sequences.
2. Find a given term in an arithmetic sequence.
3. Find the nth term of an arithmetic sequence.
4. Find the sum of a finite arithmetic sequence.
5. Use arithmetic sequences to solve real- world applications

As we saw in the previous section, we are adding about 2.5 quintillion bytes of data per day to the Internet. If there are 550 quintillion bytes of data today, then there will be 552.5 quintillion bytes tomorrow, and 555 quintillion bytes in 2 days. This is an example of an arithmetic sequence. There are many situations where this concept of fixed increases comes into play, such as raises or table arrangements.

Identifying Arithmetic Sequences

A **sequence** of numbers is just that, a list of numbers in order. It can be a short list, such as the number of points earned on each assignment in a class, such as {10, 10, 8, 9, 10, 6, 10}. Or it can be a longer list, even infinitely long, such as the list of prime numbers. For example, here's a sequence of numbers, specifically, the squares of the first 12 natural numbers.

{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144}

Each value in the sequence is called a **term**. Terms in the list are often referred to by their location in the sequence, as in the nth term. For the sequence {1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144}, the first term of the sequence is 1, the fourth term is 16, and so on. In the sequence of assignment scores {10, 10, 8, 9, 10, 6, 10}, the first term is 10 and the third term is 8 (Figure 3.47).

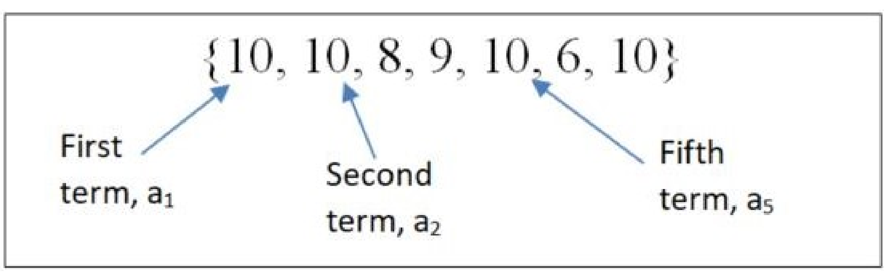


Figure 3.47 Sequence showing first, second, and fifth terms

The notation we use with sequences is a letter, which represents a term in the sequence, and a subscript, which indicates what place the term is in the sequence. For the sequence {1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144}, we will use the letter a as a value in the sequence, and so a_5 would be the term in the sequence at the fifth position. That number is 25, so we can write $a_5 = 25$.

In this section, we focus on a special kind of sequence, one referred to as an **arithmetic sequence**. Arithmetic sequences have terms that increase by a fixed number or decrease by a fixed number, called the **constant difference** (denoted by d), provided that value is not 0. This means the next term is always the previous term plus or minus a specified, constant value. Another way to say this is that the difference between any consecutive terms of the sequence is always the same value.

To see a constant difference, look at the following sequence: {7, 15, 23, 31, 39, 47, 55, 63, 71, 79, 87}. Figure 3.48 illustrates that each term of the sequence is the previous term plus 8. Eight is the constant difference here.

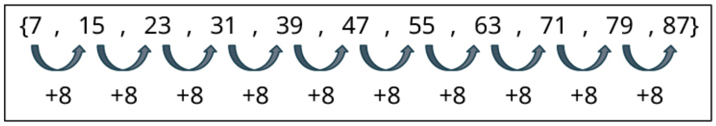


Figure 3.48 Sequence of numbers with 8 added to each term

EXAMPLE 3.132

Identifying Arithmetic Sequences

Determine if the following sequences are arithmetic sequences. Explain your reasoning.

1. $\{4, 7, 10, 13, 16, 19, 22, 25, ...\}$
2. $\{20, 40, 80, 160, 320, 640\}$
3. $\{7, 1, -5, -11, -17, -23, -29, -34, -40\}$

Solution

1. In the sequence $\{4, 7, 10, 13, 16, 19, 22, 25, ...\}$, every term is the previous term plus 3. The ellipsis indicates that the pattern continues, which means keep adding 3 to the previous term to get the new term. Therefore, this is an infinite arithmetic sequence.
2. In the sequence $\{20, 40, 80, 160, 320, 640\}$, terms increase by various amounts, for instance from term 1 to term 2, the sequence increases by 20, but from term 2 to term 3 the sequence increases by 40. So, this is not an arithmetic sequence.
3. In the sequence $\{7, 1, -5, -11, -17, -23, -29, -34, -40\}$, every term is the previous term minus 6, so this is an arithmetic sequence.

YOUR TURN 3.132

Determine if the following sequences are arithmetic sequences. Explain your reasoning.

1. $\{7.6, 5.4, 3.2, 1.0, -1.2, -3.4, -5.6, -7.8, -10.0\}$
2. $\{14, 16, 18, 22, 28, 40, 32, 0\}$
3. $\{14, 20, 26, 32, 38, 44, 50, 56, 62, 68, 74, 80, \ldots\}$

Arithmetic sequences can be expressed with a formula. When we know the first term of an arithmetic sequence, which we label a_1, and we know the constant difference, which is denoted d, we can find any other term of the arithmetic sequence. The formula for the ith term of an arithmetic sequence is $a_i = a_1 + d \times (i - 1)$.

FORMULA

If we have an arithmetic sequence with first term a_1 and constant difference d, then the ith term of the arithmetic sequence is $a_i = a_1 + d \times (i - 1)$.

Let's examine the formula with this arithmetic sequence: $\{4, 7, 10, 13, 16, 19, 22, 25, ...\}$. In this sequence $a_1 = 4$ and $d = 3$. The table below shows the values calculated.

i, Place in Sequence	a_i, i^{th} Term	Value of Term	Term Written as $a_1 + 3 \times (i - 1)$
1	a_1	4	$4 + 3 \times 0$
2	a_2	7	$4 + 3 \times 1$
3	a_3	10	$4 + 3 \times 2$
4	a_4	13	$4 + 3 \times 3$
5	a_5	16	$4 + 3 \times 4$
i	a_i		$4 + 3 \times (i - 1)$

We can see how the ith term can be directly calculated. In this sequence, the formula is $a_1 + 3 \times (i - 1)$ where the first term, a_1, is 4 and the constant difference d is 3. We can then determine the 47th term of this sequence: $a_{47} = 4 + 3 \times (47 - 1) = 4 + 3 \times 46 = 4 + 138 = 142$.

EXAMPLE 3.133

Calculating a Term in an Arithmetic Sequence

Identify a_1 and d for the following arithmetic sequence. Use this information to determine the 60th term.

$$\{18, 31, 44, 57, 70, 83, ...\}$$

Solution

Inspecting the sequence shows that $a_1 = 18$ and $d = 13$. We use those values in the formula, with $i = 60$.

$$a_{60} = a_1 + d \times (i - 1) = 18 + 13 \times (60 - 1) = 18 + 13 \times 59 = 18 + 767 = 785$$

YOUR TURN 3.133

1. Identify a_1 and d for the following arithmetic sequence. Use this information to determine the 86th term.

   $\{4.5, 8.1, 11.7, 15.3, 18.9, 22.5, 26.1, \ldots\}$

VIDEO

[Arithmetic Sequences (https://openstax.org/r/Arithmetic_Sequences)](https://openstax.org/r/Arithmetic_Sequences)

If we know two terms of the sequence, it is possible to determine the general form of an arithmetic sequence, $a_i = a_1 + d \times (i - 1)$.

FORMULA

If we have the ith term of an arithmetic sequence, a_i, and the jth term of the sequence, a_j, then the constant difference is $d = \frac{a_j - a_i}{j - i}$ and the first term of the sequence is $a_1 = a_i - d(i - 1)$.

EXAMPLE 3.134

Determining First Term and Constant Difference Using Two Terms

A sequence is known to be arithmetic. Two of its terms are $a_7 = 56$ and $a_{19} = 104$. Use that information to find the constant difference, the first term, and then the 50th term of the sequence.

Solution

To find the constant difference, use $d = \frac{a_j - a_i}{j - i}$. The location of the terms is given by the subscript of the two a terms, $i = 7$ and $j = 19$. So, the constant difference can be calculated as such:

$$d = \frac{104 - 56}{19 - 7} = \frac{48}{12} = 4.$$

The constant difference of 4 is then used to find a_1.

$$a_1 = a_i - d(i - 1) = a_7 - 4(7 - 1) = 56 - 4 \times 6 = 32.$$

So $d = 4$ and $a_1 = 32$.

With this information, the 50th term can be found.

$$a_{50} = a_1 + d \times (i - 1) = 32 + 4 \times (50 - 1) = 32 + 4 \times 49 = 32 + 196 = 228.$$

The 50th term is $a_{50} = 228$.

YOUR TURN 3.134

1. A sequence is known to be arithmetic. Two of the terms are $a_{14} = 41$ and $a_{38} = 161$. Use that information to find the constant difference and the first term. Then determine the 151st term of the sequence.

VIDEO

[Finding the First Term and Constant Difference for an Arithmetic Sequence (https://openstax.org/r/Finding_the_First_Term_and_Constant_Difference_for_an_Arithmetic_Sequence)]

Finding the Sum of a Finite Arithmetic Sequence

Sometimes we want to determine the sum of the numbers of a finite arithmetic sequence. The formula for this is fairly straightforward.

FORMULA

The sum of the first n terms of a finite arithmetic sequence, written s_n, with first and last term a_1 and a_n, respectively, is $s_n = n\left(\frac{a_1 + a_n}{2}\right)$.

EXAMPLE 3.135

Finding the Sum of a Finite Arithmetic Sequence

What is the sum of the first 60 terms of an arithmetic sequence with $a_1 = 4.5$ and $d = 2.5$?

Solution

The formula requires the first and last terms of the sequence. The first term is given, $a_1 = 4.5$. The 60th term is needed. Using the formula $a_1 = a_i + d(i - 1)$ provides the value for the 60th term.

$$a_{60} = 4.5 + 2.5(60 - 1) = 4.5 + 2.5 \times 59 = 4.5 + 147.5 = 152.$$

Applying the formula $s_n = n\left(\frac{a_1 + a_n}{2}\right)$ provides the sum of the first 60 terms.

$$s_{60} = 60\left(\frac{4.5 + 152}{2}\right) = 60 \times \frac{156.5}{2} = 4{,}695.$$

The sum of the first 60 terms is 4,695.

YOUR TURN 3.135

1. What is the sum of the first 101 terms of an arithmetic sequence with $a_1 = 13$ and $d = 2.25$?

VIDEO

[Finding the Sum of a Finite Arithmetic Sequence (https://openstax.org/r/Finding_the_Sum_of_a_Finite_Arithmetic_Sequence)](https://openstax.org/r/Finding_the_Sum_of_a_Finite_Arithmetic_Sequence)

Using Arithmetic Sequences to Solve Real-World Applications

Applications of arithmetic sequences occur any time some quantity increases by a fixed amount at each step. For instance, suppose someone practices chess each week and increases the amount of time they study each week. The first week the person practices for 3 hours, and vows to practice 30 more minutes each week. Since the amount of time practicing increases by a fixed number each week, this would qualify as an arithmetic sequence.

EXAMPLE 3.136

Applying an Arithmetic Sequence

Jordan has just watched *The Queen's Gambit* and decided to hone their skills in chess. To really improve at the game, Jordan decides to practice for 3 hours the first week, and increase their time spent practicing by 30 minutes each week. How many hours will Jordan practice chess in week 20?

Solution

Jordan's practice scheme is an arithmetic sequence, as it increases by a fixed amount each week. The first week there are 3 hours of practice. This means $a_1 = 3$. Jordan increases the time spent practicing by 30 minutes, or half an hour, each week. This means $d = 0.5$. Using those values, and that we want to know the amount of time Jordan will study in week 20, we determine the time in week 20 using $a_i = a_1 + d \times (i - 1)$.

$$a_{20} = 3 + 0.5 \times (20 - 1) = 3 + 0.5 \times 19 = 3 + 9.5 = 12.5$$

So, Jordan will practice 12.5 hours in week 20.

YOUR TURN 3.136

1. Christina decides to save money for after graduation. Christina starts by setting aside \$10. Each week, Christina increases the amount she saves by \$5. How much money will Christina save in week 52?

EXAMPLE 3.137

Finding the Sum of a Finite Arithmetic Sequence

Let's check back in on Jordan. Recall, Jordan had just watched *The Queen's Gambit* and decided to hone their skills, practicing for 3 hours the first week, and increasing the time spent practicing by 30 minutes each week. How many hours total will Jordan have practiced chess after 30 weeks of practice?

Solution

To calculate the total amount of time that Jordan practiced, we need to use $s_n = n\left(\frac{a_1 + a_n}{2}\right)$. The formula requires the first and last terms of the sequence. Since Jordan practiced 3 hours in the first week, the first term is $a_1 = 3$. Because we want the total practice time after 30 weeks, we need the 30th term. Because the constant difference is $d = 0.5$, the 30th

term is $a_{30} = 3 + 0.5(30 - 1) = 3 + 0.5 \times 29 = 3 + 14.5 = 17.5$.

Applying the formula $s_n = n\left(\frac{a_1+a_n}{2}\right)$ provides the sum of the first 30 terms.

$$s_{30} = 30\left(\frac{3 + 17.5}{2}\right) = 60 \times \frac{20.5}{2} = 615.$$

This means that Jordan practiced a total of 615 hours after 30 weeks.

YOUR TURN 3.137

1. In a theater, the first row has 24 seats. Each row after that has 2 more seats. How many total seats are there if there are 40 rows of seat in the theater?

WHO KNEW?

The Fibonacci Sequence

Not all sequences are arithmetic. One special sequence is the **Fibonacci sequence**, which is the sequence that has as its first two terms 1 and 1. Every term thereafter is the sum of the previous two terms. The first nine terms of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21, and 34.

This sequence is found in nature, architecture, and even music! In nature, the Fibonacci sequence describes the spirals of sunflower seeds, certain galaxy spirals, and flower petals. In music, the band Tool used the Fibonacci sequence in the song "Lateralus." The Fibonacci sequence even relates to architecture, as it is closely related to the golden ratio.

VIDEO

Fibonacci Sequence and "Lateralus" (https://openstax.org/r/Fibonacci_Sequence_and_Lateralus)

Check Your Understanding

52. Is the following an arithmetic sequence? Explain.
{3, 6, 9, 15, 25, 39, 90}
53. What is the 7th term of the following sequence?
{1, 5, 7, 100, 4, -17, 8, 100, 19, 7.6, 345}
54. In an arithmetic sequence, the first term is 10 and the constant difference is 4.5. What is the 135th term?
55. If the eighth term of an arithmetic sequence is 35 and the 40th term is 131, what is the constant difference and the first term of the sequence?
56. What is the sum of the first 100 terms of the arithmetic sequence with first term 4 and constant difference 7?
57. A new marketing firm began with 30 people in its survey group. The firm adds 4 people per day. How many people will be in their survey group after 100 days?

SECTION 3.10 EXERCISES

For the following exercises, determine if the sequence is an arithmetic sequence.

1. $\{3, 7, 11, 15, 25, 100, \ldots\}$
2. $\{27, 24, 21, 18, 15, 12, 9, \ldots\}$
3. $\{6, -1, -8, -15, -23, -31, -39, \ldots\}$
4. $\{-5, 4, 13, 22, 31, 40, 49, 58, 67, \ldots\}$
5. $\{14, 19, 24, 29, 34, 50, 60\}$
6. $\{3.9, 2.3, 0.7, -0.9, -2.5, -4.1, -5.7, \ldots\}$

7. $\{4, -8, 12, -16, 20, -24, 28, -32, \ldots\}$
8. $\{1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$

For the following exercises, the sequences given are arithmetic sequences. Determine the constant difference for each sequence. Verify that each term is the previous term plus the constant difference.

9. $\{18, 68, 118, 168, 218, 268, \ldots\}$
10. $\{13, 35, 57, 79, 101, 123, 145, 167, \ldots\}$
11. $\{14, 11, 8, 5, 2, -1, -4, \ldots\}$
12. $\{4.5, 1.9, -0.7, -3.3, -5.9, \ldots\}$
13. $\{-27, -13, 1, 15, 29, 43, 57, 71, \ldots\}$
14. $\{3.8, 10.6, 17.4, 24.2, 31, 37.8, 44.6, \ldots\}$

For the following exercises, the first term and the constant difference of an arithmetic sequence is given. Using that information, determine the indicated term of the sequence.

15. $a_1 = 12$, $d = 11$, find a_{20} .
16. $b_1 = 5$, $d = 8$, find b_{38} .
17. $c_1 = 48$, $d = -7$, find c_{50} .
18. $a_1 = 110$, $d = -16$, find a_{27} .
19. $t_1 = 15.3$, $d = 4.2$, find t_{17} .
20. $b_1 = 23.8$, $d = 11.7$, find b_{120} .
21. $b_1 = 27.45$, $d = -3.67$, find b_{40}.
22. $a_1 = 67.4$, $d = -12.3$, find a_{200} .

For the following exercises, two terms of an arithmetic sequence are given. Using that information, identify the first term and the constant difference.

23. $a_5 = 27$, $a_{15} = 77$
24. $b_{10} = 47$, $b_{25} = 137$
25. $a_9 = 38$, $a_{45} = 189.2$
26. $a_6 = 43$, $a_{41} = -377$
27. $a_4 = -12.3$, $a_{54} = -106.5$
28. $a_{12} = 45.9$, $a_{60} = -563.7$

For the following exercises, the first term and the constant difference is given for an arithmetic sequence. Use that information to find the sum of the first n terms of the sequence, s_n.

29. $a_1 = 15$, $d = 7$, calculate s_{10}
30. $a_1 = 2$, $d = 13$, calculate s_{20} .
31. $a_1 = 105$, $d = 0.3$, calculate s_{15} .
32. $a_1 = 56.2$, $d = 1.1$, calculate a_{35} .
33. $a_1 = 450$, $d = -20$, calculate s_{20}.
34. $a_1 = 1400$, $d = -35$, calculate s_{40} .

For the following exercises, apply your knowledge of arithmetic sequences to these real-world scenarios.

35. A collection is taken up to support a family in need. The initial amount in the collection is $135. Everyone places $20 in the collection. When the 35th person puts their $20 in the collection, how much is present in the collection?
36. There are 50 songs on a playlist. Every minute, 3 more songs are added to the playlist. How many songs are on the playlist after 40 minutes have passed?
37. One genre on Netflix has 1,000 shows. Every week, 20 shows are added to that genre. After 15 weeks, how many shows are in that genre?
38. A new local band has 10 people come to their first show. News of the band spreads afterwards. Each week, 4 more people attend their show than the previous week. After 50 weeks, how many people are at their show?
39. The Jester Comic book store is going out of business and is taking in no new inventory. Its inventory is currently 13,563 titles. Each day after, they sell or give away 250 titles. After 15 days, how many titles are left?
40. Jasmyn has decided to train for a marathon. In week one, Jasmyn runs 5 miles. Each week, Jasmyn increased the running distance by 2 miles. How many miles will Jasmyn run in week 13 of the training schedule?
41. A 42-gallon bathtub sits with 14 gallons in it. The faucet is turned on and is now being filled at the rate of 2.2 gallons per minute, but is draining slowly, at 1.8 gallons per minute. After 20 minutes, how many gallons are in the tub?
42. A trained diver is 250 feet deep. The diver is nearly out of air and needs to surface. However, the diver can only comfortably ascend 30 feet per minute. How deep is the diver after ascending for 5 minutes?

43. Jaclyn, an investor, begins a start-up to revitalize homes in South Bend, Indiana. She begins with $10,000, making her investor 1. Each investor that joins will invest $500 more than the previous investor. How much does the 50th investor invest in the project? With that 50th investor, what is the total amount invested in the project?

44. Jasmyn has decided to train for a marathon. In week one, Jasmyn runs 5 miles. Each week, Jasmyn increased the running distance by 2 miles. After training for 14 weeks, how many total miles will Jasmyn have run?

45. The base of a pyramidal structure has 144 blocks. Each level above has 5 fewer blocks than the previous level. How many total blocks are there if the pyramidal structure has 25 levels?

46. As part of a deal, a friend tells you they will give you $10 on day 1, $20 on day 2, $30 on day 3, for all 30 days of a month. At the end of that month, what is the total amount your friend has given you?

3.11 Geometric Sequences

Figure 3.49 Savings grows in a geometric sequence. (credit: modification of "A big part of financial freedom is having your heart and mind free from worry about the what-ifs of life. – Suze Orman" by Morgan/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Identify geometric sequences.
2. Find a given term in a geometric sequence.
3. Find the nth term of a geometric sequence.
4. Find the sum of a finite geometric sequence.
5. Use geometric sequences to solve real-world applications.

One of the concerns when investing is the **doubling time**, which is length of time it takes for the value of the investment to be twice, or double, that of its starting value. A shorter doubling times means the investment gets bigger, sooner. For example, if you invest $200 in an account with an 8-year doubling time, then in 8 years the value of the account will be double the starting amount, or $2 \times \$200 = \$400$. After another 8 years (for a total of 16 years) the investment would be twice its value after the first 8 years, or $2 \times (2 \times \$400) = 2 \times (\$400) = \$800$. Every 8 years, the investment would double again, so after the third 8-year period, the investment would be worth $2 \times 2 \times (2 \times \$400) = \$1{,}600$. This process exhibits exponential growth, an application of geometric sequences, which is explored in this section.

Identifying Geometric Sequences

We know what a sequence is, but what makes a sequence a geometric sequence? In an arithmetic sequence, each term is the previous term plus the constant difference. So, you add a (possibly negative) number at each step. In a **geometric sequence**, though, each term is the previous term multiplied by the same specified value, called the **common ratio**. In the sequence $\{3, 6, 12, 24, 48, 96, 192, 384, 728, 1456\}$ the common ratio is 2. To see the difference between an arithmetic sequence and geometric sequence, examine these two sequences (Figures 3.52 and 3.53).

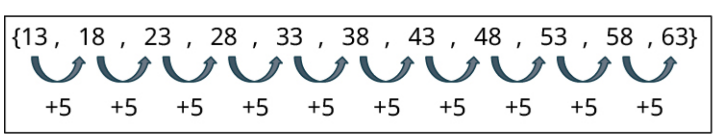


Figure 3.50 Arithmetic sequence

Each term in this arithmetic sequence is the previous term plus 5.

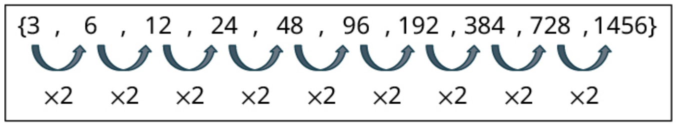


Figure 3.51 Geometric sequence

Each term in this geometric sequence is the previous term times 2.

In the sequence $\{3, 6, 12, 24, 48, 96, 192, 384, 728, 1456\}$, the numbers get big fairly quickly, and stay positive. However, that's not always the case with geometric sequences. Depending on the value of the common ratio, the terms could increase each time (like in the one shown in Figure 3.51), or the terms can get smaller each time, or the terms can alternate between positive and negative values. It all depends on the value of the common ratio, r.

Consider this geometric sequence:

$$\{5, 15, 45, 135, 405, 2025...\}$$

Each term is the previous term times 5, which means the common ratio is 5. This common ratio is larger than 1, and so the terms increase each time. Now, look at this geometric sequence:

$$\{2, -6, 18, -54, 162, -486, 1458...\}$$

Each term is the previous term times −3, and the sign of the terms alternate from positive to negative. Then, there's this geometric sequence:

$$\left\{9, 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}...\right\}$$

Each term is the previous term times $\frac{1}{3}$, and the terms decrease each time. What we should take away from these three examples is if the common ratio is a positive number larger than 1, then the sequence increases. If the common ratio is a negative number, then the sign of the terms alternates between positive and negative. If the common ratio is between 0 and 1, then the terms decrease.

Two special cases of geometric sequences are when the constant ratio is 1 and when the common ratio is 0. When the constant ratio is 1, every term of the sequence is the same, as in $\{3, 3, 3, 3, 3, 3, 3, 3, 3\}$. This is referred to as a **constant sequence**. When the constant ratio is 0, the first term can be any number, but every term after the first term is 0, as in $\{-43.2, 0, 0, 0, 0, 0, 0, 0\}$.

EXAMPLE 3.138

Identifying Geometric Sequences

For each sequence, determine if the sequence is a geometric sequence. If so, identify the common ratio.

1. $\{5, 20, 80, 320, 1{,}280, 5{,}120, 20{,}480, ...\}$
2. $\{-3, 6, -12, 24, 11, 33\}$
3. $\left\{4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...\right\}$

Solution

1. In the sequence $\{5, 20, 80, 320, 1{,}280, 5{,}120, 20{,}480, ...\}$, the jump from 5 to 20 is a multiplication by 4, as is the next jump to 80, and the next to 320. Each term is 4 times the previous term. Since each term is 4 times the previous, this is a geometric sequence. The common ratio is 4.
2. In the sequence $\{-3, 6, -12, 24, 11, 33\}$, notice that 6 is −3 times −2. The jump from 6 to −12 is another multiplication by negative. So, if this is a geometric sequence, each term should be the previous term times −2. But the change from 24 to 11 is not a multiplication by −2, This means the sequence is not a geometric sequence.
3. In the sequence $\left\{4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...\right\}$, the change from 4 to 2 is a multiplication by $\frac{1}{2}$, as is the next jump, from 2 to 1, as is the next from 1 to $\frac{1}{2}$. Each term is $\frac{1}{2}$ times the previous term. Since each term is $\frac{1}{2}$ times the previous, this is a geometric sequence. The common ratio is $\frac{1}{2}$.

YOUR TURN 3.138

For each sequence, determine if the sequence is a geometric sequence. If so, identify the common ratio.

1. $\{-1, -5, -25, -125, -625, -3125, , \ldots\}$
2. $\{-3, 6, -12, 24, 11, 33\}$
3. $\left\{-500, 50, -5, \frac{1}{2}, -\frac{1}{20}, \ldots\right\}$

As with arithmetic sequences, the first term of a geometric sequence is labeled a_1. The number that is multiplied by each term is called the common ratio and is denoted r. So, if the first term is known, a_1, and the common ratio is known, r, then the nth term, a_n, can be calculated with the formula $a_n = a_1 r^{n-1}$.

FORMULA

The nth term of the geometric sequence, a_n, with first term a_1 and common ratio r, is $a_n = a_1 r^{n-1}$.

Return to the sequence $\{3, 6, 12, 24, 48, 96, 192, 384, 728, ...\}$. We observe that the first term is 3, so $a_1 = 3$. We also found that the common ratio is 2, so $r = 2$. The table below shows how any term can be calculated using just a_1 and r.

i, Place in Sequence	a_i, i^{th},Term	Value of Term	Term Written as $a_1 \times r^{i-1}$
1	a_1	3	3×2^0
2	a_2	6	3×2^1
3	a_3	12	3×2^2
4	a_4	24	3×2^3
5	a_5	48	3×2^4
i	a_i		$3 \times 2^{i-1}$

EXAMPLE 3.139

Determining the Value of a Specific Term in a Geometric Sequence

In the following geometric sequences, determine the indicated term of the geometric sequence with a given first term and common ratio.

1. Determine the 9th term of the geometric sequence with $a_1 \times 6$ and $r = 3$.
2. Determine the 11th term of the geometric sequence with $a_1 = 2$ and $r = -5$.

Solution

1. Using $a_n = a_1 r^{n-1}$ with $a_1 = 6$, $r = 3$, and $n = 9$, we calculate
$$a_9 = a_1 r^{9-1} = 6 \times (3)^{9-1} = 6 \times (3)^8 = 6 \times 6561 = 39366$$
.
The 9th term of the geometric sequence with $a_1 = 6$ and $r = 3$ is $a_9 = 39366$.
2. Using $a_n = a_1 r^{n-1}$ with $a_1 = 2$, $r = -5$, and $n = 11$, we calculate
$$a_{11} = a_1 r^{11-1} = 2 \times (-5)^{11-1} = 2 \times (-5)^{10} = 2 \times 9{,}765{,}625 = 19{,}531{,}250$$
.

YOUR TURN 3.139

In the following geometric sequences, determine the indicated term of the geometric sequence with a given first term and common ratio.

1. Determine the 12th term of the geometric sequence with $a_1 = 3072$ and $r = \frac{1}{2}$.
2. Determine the 5th term of the geometric sequence with $a_1 = 0.5$ and $r = 8$.

VIDEO

Geometric Sequences (https://openstax.org/r/Geometric_Sequences)

Finding the Sum of a Finite Geometric Sequence

As with arithmetic sequences, it is possible to add the terms of the geometric sequence. Like arithmetic sequences, the formula for the finite sum of the terms of a geometric sequence has a straightforward formula.

FORMULA

The sum of the first n terms of a finite geometric sequence, written s_n, with first term a_1 and common ratio r, is $s_n = a_1\left(\frac{1-r^{n-1}}{1-r}\right)$ provided that $r \neq 1$.

EXAMPLE 3.140

Calculating the Sum of a Finite Geometric Sequence

1. What is the sum of the first 13 terms of the geometric sequence with first term $a_1 = 5$ and common ratio $r = 3$?
2. What is the sum of the first 7 terms of the geometric sequence with first term $a_1 = 16$ and common ratio $r = \frac{1}{8}$?

Solution

1. Using $a_1 = 5$, $r = 3$, and $n = 13$, we find that the sum is:

$$S_{13} = a_1\left(\frac{1-r^{n-1}}{1-r}\right) = 5 \times \left(\frac{1-3^{13-1}}{1-3}\right) = 5 \times \left(\frac{1-3^{12}}{-2}\right)$$

$$= 5 \times \left(\frac{1-531{,}441}{-2}\right) = 5 \times \left(\frac{1-531{,}441}{-2}\right) = 5 \times \left(\frac{-531{,}440}{-2}\right) = 5 \times 265{,}720 = 1{,}328{,}600$$

The sum of the first 13 terms of this geometric sequence is 1,328,600.

2. Using $a_1 = 16$, $r = \frac{1}{8}$, and $n = 7$, we find that the sum is:

$$s_7 = a_1\left(\frac{1-r^{n-1}}{1-r}\right) = 16 \times \left(\frac{1-\left({}^{1}\!/_{8}\right)^{7-1}}{1-\left({}^{1}\!/_{8}\right)}\right) = 16 \times \left(\frac{1-\left({}^{1}\!/_{8}\right)^{6}}{{}^{7}\!/_{8}}\right) = 16 \times \left(\frac{1-{}^{1}\!/_{262{,}144}}{{}^{7}\!/_{8}}\right) = 16 \times \left(\frac{{}^{262{,}143}\!/_{262{,}144}}{{}^{7}\!/_{8}}\right)$$

$$= 16 \times \left(\frac{262{,}143}{229{,}376}\right) = \frac{262{,}143}{14{,}336} = 18.2856$$

The sum of the first 7 terms of this geometric sequence is 18.2856.

YOUR TURN 3.140

1. What is the sum of the first 10 terms of the geometric sequence with first term $a_1 = 7$ and common ratio $r = 6$?
2. What is the sum of the first 6 terms of the geometric sequence with first term $a_1 = 27$ and common ratio $r = \frac{1}{3}$?

Using Geometric Sequences to Solve Real-World Applications

Geometric sequences have a multitude of applications, one of which is compound interest. Compound interest is something that happens to money deposited into an account, be it savings or an individual retirement account, or IRA. The interest on the account is calculated and added to the account at regular intervals. This means the interest that was earned later gains its own interest. This allows the money to grow faster. If that interest is added every month, we say it is compounded monthly. If the interest is added daily, then we say it is compounded daily. The amount of money that is deposited into the account is called the principal and is denoted P. The account earns money on that principal. The amount it earns is a percentage of the money in the account. The interest rate, expressed as a decimal, is denoted r.

FORMULA

If you deposit P dollars in an account that earns interest compounded yearly, then the amount in the account, A, after t years is calculated with the formula: $A = P(1 + r)^t$. This is a geometric sequence, with constant ratio $(1 + r)$ and first term $a_1 = P$.

EXAMPLE 3.141

Calculating Interest Compounded Yearly

Daryl deposits $1,000 in an account earning 4% interest compounded yearly. How much money is in the account after 25 years?

Solution

Using $A = P(1 + r)^t$ with $P = 1000$, $r = 0.04$, and $t = 25$, we find that $A = P(1 + r)^t = 1{,}000 \times (1 + 0.04)^{25} = 1{,}000 \times (1.04)^{25} = 1{,}000 \times 2.66583633 = 2{,}665.85$. After 25 years, there is $\$2{,}665.84$ in the account.

YOUR TURN 3.141

1. Sophia deposited $4,000 in an account that earns 5.5% interest compounded yearly. After 20 years, Sophia withdrew all the money in the account to pay for her child's college. How much money was in the account when Sophia withdrew the money?

Another application of geometric sequences is exponential growth. This arises in biology quite frequently, especially in relation to bacterial cultures, but also with other organism population models. In bacterial cultures, the time it takes the population to double is often recorded. This time to double is the same, regardless of how big the population gets. So, if the population doubles after 3 hours, it doubles again after another 3 hours, and again after another 3 hours, and so on. Put into geometric sequence language, it has a common ratio of 2.

EXAMPLE 3.142

Doubling a Bacterial Culture

When *Escherichia coli* (*E. coli*) is in a broth culture at 37°C, the population of *E. coli* doubles in number with 30 organisms, how many *E. coli* bacteria are present in the culture after 16 hours?

Solution

Since the population is doubling every 20 minutes, this is a geometric sequence situation with common ratio $r = 2$. The culture begins with 30 organisms, so $a_1 = 30$. The time,16 hours, is 48 twenty-minute periods, so we're looking for the 48th term in the sequence. Using these values in the geometric sequence formula gives

$$a_{48} = a_1 r^{n-1} = 30 \times 2^{48-1} = 30 \times 2^{47} = 30 \times (1.40737 \times 10^{14}) = 4.22212 \times 10^{15}.$$

So, after 16 hours, the culture contains 4.22212×10^{15} *E. coli* organisms. That's more than 4,000 trillion bacteria.

YOUR TURN 3.142

1. When *Streptococcus lactis* (*S. lactis*) is in a milk culture at 37°C, the population of *S. lactis* doubles in number

every 30 minutes. If the culture began with 15 organisms, how many *S. lactis* bacteria are present in the culture after 20 hours?

EXAMPLE 3.143

Applying the Sum of a Finite Geometric Sequence

A player places one grain of rice on the first square of a chess board. On the second square, the player places 2 grains of rice. On the third square, the player places 4 grains of rice. On each successive square of the board, the player doubles the number of grains of rice placed on the chess board. When the player places the last rice on the 64th square, how many total grains of rice have been placed on the board?

Solution

Since the number of grains of rice is doubled at each step, this is a geometric sequence with first term $a_1 = 1$ and common ratio $r = 2$. Rice is placed on 64 total squares, so we want the sum of the first 64 terms. Using this information and the formula, the total number of grains of rice on the board will be:

$$s_{64} = a_1\left(\frac{1-r^{n-1}}{1-r}\right) = 1 \times \left(\frac{1-2^{64-1}}{1-2}\right) = \left(\frac{1-2^{63}}{-1}\right) = -\left(1-2^{63}\right)$$
$$= -(-9.2233720369 \times 10^{18}) = 9.2233720369 \times 10^{18}$$

That's a 20-digit number!

YOUR TURN 3.143

1. You have a square 1 meter on each side. You begin by coloring one half of the square blue. Then you color half the remaining area blue. Then you color half the remaining area blue once more. At each step, you color half the remaining area. What is the total area you have colored blue after performing this process 15 times?

VIDEO

Sum of a Finite Geometric Sequence (https://openstax.org/r/Sum_of_a_Finite_Geometric_Sequence)

Check Your Understanding

58. Is the following a geometric sequence? Explain.
{3, 6, 12, 24, 48, 96, 192}

59. Find the common ratio of the geometric sequence {3, −30, 300, −3,000, ...}.

60. In a geometric sequence, the first term is 10 and the common ratio is 1.5. What is the 15th term?

61. What is the sum of the first 100 terms of the geometric sequence with first term 4 and common ratio 0.3?

62. $15,000 is deposited in an account the yields 4.2% interest compounded annually. How much is in the account after 17 years?

SECTION 3.11 EXERCISES

For the following exercises, determine if the sequence is a geometric sequence.

1. $\{3, 7, 11, 15, 25, 100, \ldots\}$
2. $\{2, 4, 8, 16, 32, \ldots\}$
3. $\{9, 0.9, 0.09, 0.009, 0.00009, \ldots\}$
4. $\{262{,}144, 65{,}536, 16{,}384, 4{,}096, 1{,}024, \ldots\}$
5. $\{14, 19, 24, 29, 34, 50, 60\}$
6. $\{3.9, 2.3, 0.7, -0.9, -2.5, -4.1, -5.7, ..\}$
7. $\{4, -8, 16, -32, 64, -128, 256, \ldots\}$
8. $\{8, -4, 2, -1, 0.5, -0.25, 0.125, -0.0625, \ldots\}$

For the following exercises, the sequences given are geometric sequences. Determine the common ratio for each.

Verify that each term is the previous term times the common ratio.

9. $\{3, 6, 12, 24, 48, 96, \ldots\}$
10. $\{8, 24, 72, 216, 648, 1944, \ldots\}$
11. $\{15, 3, 0.6, 0.12, 0.024, 0.0048, 0.00096, \ldots\}$
12. $\{52, 26, 13, 6.5, 3.25, 1.625, 0.8125, 0.40265...\}$
13. $\{18, -18, 18, -18, 18 - 18, \ldots\}$
14. $\{48, -12, 3 - 0.75, 0.1875, -0.046875, \ldots\}$

For the following exercises, the first term and the common ratio of a geometric sequence is given. Using that information, determine the indicated term of the sequence.

15. $a_1 = 5$, $r = 3$, find a_6.
16. $b_1 = 7$, $r = 9$, find b_5.
17. $c_1 = 11$, $r = 4$, find c_{12}.
18. $a_1 = 2$, $r = 7$, find a_9.
19. $t_1 = 100$, $r = \frac{1}{5}$, find t_{10}.
20. $b_1 = 56$, $r = 0.25$, find b_{15}.
21. $b_1 = 13$, $r = -2$, find b_{10}.
22. $a_1 = 11$, $r = -3$, find a_{12}.
23. $a_1 = 12$, $r = -\frac{1}{3}$, find a_8.
24. $a_1 = 100$, $r = -10$, find a_{15}.

For the following exercises, the first term and the common ratio is given for a geometric sequence. Use that information to find the sum of the first n terms of the sequence, s_n.

25. $a_1 = 3$, $r = 4$, calculate s_5.
26. $a_1 = 5$, $r = 3$, calculate s_9.
27. $a_1 = 4$, $r = 5$, calculate s_8.
28. $a_1 = 48$, $r = 2$, calculate s_{11}.
29. $a_1 = 450$, $r = 0.5$, calculate s_{12}.
30. $a_1 = 300$, $r = 0.25$, calculate s_{10}.
31. $a_1 = 3$, $r = -2$, calculate s_{11}.
32. $a_1 = 5$, $r = -4$, calculate s_8.

For the following exercises, apply your understanding of geometric sequences to real-world applications.

33. *Lactobacilius acidophilus* (*L. acidophilus*) is a bacterium that grows in milk. In optimal conditions, its population doubles every 26 minutes. If a culture starts with 20 *L. acidophilus* bacteria, how many bacteria will there be after 390 minutes? Hint: This means the 26-minute time period has occurred 15 times.
34. *Bacillus megaterium* (*B. megaterium*) is a bacterium that grows in sucrose salts. In optimal conditions, its population doubles every 25 minutes. If a culture starts with 30 *B. megaterium* bacteria, how many bacteria will there be after 1,000 minutes? Hint: This means the 25-minute time period has occurred 40 times.
35. Alex and Jill deposit $4,000 in an account bearing 5% interest compounded yearly. If they do not deposit any more money in that account, how much will it be worth in 30 years?
36. Kerry and Megan deposit $6,000 dollars in and account bearing 4% compounded yearly. If they do not deposit any more money in that account, how much will be in the account after 40 years?
37. You decide to color a square that measures 1 m on each side in a very particular manner. You first cut the square in half vertically. You color one side of the square with purple. On the side of the square that was not colored, you draw a line dividing that region horizontally exactly in half. You color the lower half blue. Now, you cut the remaining quarter of the square precisely in half with a vertical line. You color the left side red. You repeat this process 12 times. After you color that 12th piece, what is the total area you have colored?
38. Consider the geometric sequence with first term 0.9 and common ratio of 0.1. What is the sum of the first 5 terms?
39. Repeat Exercise 38, for the sum of the first 10 terms.

For the following questions, recall that the formula for interest compounded yearly is $A = P(1 + r)^t$, where A is the amount in the account after t years, P is the initial amount deposited, and r is the interest rate per year. However, if the account is compounded monthly, the formula changes to $A = P\left(1 + \frac{r}{12}\right)^{12t}$.

40. Returning to Kerry and Megan (Exercise 36), what would their account be worth if their account was compounded monthly?
41. Returning to Alex and Jill (Exercise 35), what would their account be worth if their account was compounded monthly?

42. Imagine your family tree. You have two parents. Your parents have two parents: your grandparents. And so on. How many great-great-great-great-grandparents do you have? Hint: This would be six generations back.

43. Imagine your family tree. You have two parents. Your parents have two parents: your grandparents. And so on. How many great (20 times) grandparents do you have? Hint: This would be 22 generations back.

Chapter Summary

Key Terms

3.1 Prime and Composite Numbers

- natural numbers
- factor of a number
- multiple of a number
- prime number
- composite number
- prime factorization
- greatest common divisor (GCD)
- least common multiple (LCM)

3.2 The Integers

- integer
- absolute value
- average of a set of numbers

3.3 Order of Operations

- order of operations
- PEMDAS

3.4 Rational Numbers

- density property of rational numbers
- improper fraction
- lowest terms
- mixed number
- rational number
- repeating decimal
- terminating decimal

3.5 Irrational Numbers

- conjugate numbers
- difference of squares
- irrational numbers
- lowest terms
- rationalize the denominator

3.6 Real Numbers

- complex number
- imaginary number
- real number

3.7 Clock Arithmetic

- clock arithmetic
- modulo 7
- modulo 12

3.8 Exponents

- base
- exponent

3.9 Scientific Notation

- scientific notation
- standard notation

3.10 Arithmetic Sequences

- sequence
- term of a sequence
- arithmetic sequence
- first term
- constant difference

3.11 Geometric Sequences

- geometric sequence
- common ratio

3.11 Geometric Sequences

- common ratio
- geometric sequence

Key Concepts

3.1 Prime and Composite Numbers

- The natural numbers can be categorized as 1, prime numbers, and composite numbers.
- Prime numbers have as their only factors 1 and themselves.
- Composite numbers have at least three distinct factors.
- Composite numbers can be written in their prime factorization form, which is found by repeatedly factoring prime factors from the number.
- The greatest common divisor (GCD) of a set of numbers is the largest integer that divides all of the numbers in the set. The prime factorizations of the numbers can be used to identify the greatest common divisor.
- The least common multiple (LCM) of a set of numbers is the smallest integer that is divisible by all of the numbers in the set. The prime factorizations of the numbers can be used to identify the least common multiple.
- There are various ways that the GCD and LCM are applied.

3.2 The Integers

- A set of numbers that can be built from the natural numbers are the integers, which consist of the natural numbers, zero (0), and the negatives of the natural numbers.
- Integers are often graphed on a number line, which helps display the relative positions and values of those numbers.
- The number line can be used to visualize when one integer is larger than or smaller than another integer.
- Arithmetic operations with integers are similar to the operations with natural numbers, except that the sign (positive or negative) of the numbers will determine the sign (positive or negative) of the result.

3.3 Order of Operations

- Establishing shared rules on which arithmetic operations are calculated first is necessary. Without them, different people may find different values for the same expression.
- The highest precedence is with expressions in parentheses. This allows parts of an expression to be calculated in an order different than the basic order of operations.
- The lowest precedence is addition and subtraction, as they are the basis for all other calculations.
- Multiplication and division have precedence over addition and subtraction, as they are representations of repeated addition or subtraction.
- Exponents have precedence over multiplication and division, as they represent repeated multiplication and division.

3.4 Rational Numbers

- Rational numbers are fractions of integers, and can always be written as an integer divided by an integer.
- The numerator and denominator of a fraction may have common factors. In such cases, the fraction can be reduced by canceling common factors. When the numerator and denominator of a fraction have no common factors, the fraction is said to be reduced.
- An improper fraction is one with a numerator larger than the denominator. Such a fraction can be rewritten as an integer plus a proper fraction. This is called a mixed number.
- Using division and remainder, an improper fraction may be written as a mixed number.
- A mixed number can be converted to an improper fraction by reversing the process for changing an improper fraction to a mixed number.

- The arithmetic operations or addition, subtraction, multiplication and division can all be performed on rational numbers.
- Addition and subtraction of rational numbers can be performed after a common denominator has been identified, and the fractions have been converted to forms having the common denominator.
- Multiplication and division of rational numbers can be performed without regard to common denominators.
- Between any two rational numbers, there is always another rational number. This is the density property of the rational numbers.

3.5 Irrational Numbers

- Irrational numbers are numbers that cannot be written as an integer divided by another integer. One example is pi, denoted π. Another collection of irrational numbers are natural numbers that are not perfect squares.
- Some irrational numbers can be written as a rational part multiplied by an irrational part. If two irrational numbers have the same irrational parts, they can be added or subtracted.
- When irrational numbers are similar, on can multiply and divide the numbers without a calculator.
- Since $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$, and $\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$, products and quotients of square roots can be determined.
- Because $\sqrt{a^2} = a$ and $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$, it is possible to simplify square root expressions so the radicand contains no perfect square factors.
- When a fraction has an irrational number as its denominator, it is possible to convert the denominator into a rational number using its conjugate. Doing so involves multiplying the numerator and denominator by the conjugate of the denominator, and then applying the difference of squares formula.
- With a single square root term
- Using conjugate numbers for two term denominators

3.6 Real Numbers

- Real numbers is the collection of all rational and irrational numbers. Conceptually, it is the collection of all values that can be represented on a number line, or, as a length along with sign.
- The subsets of the real numbers include the natural numbers, integers, rational numbers and irrational numbers. The natural numbers are a subset of the integers, which is a subset of the rational numbers. The rational and irrational numbers are disjoint sets.
- The real numbers, due to order of operation rules and that performing arithmetic operations on real number always results in a real number, have arithmetic properties that apply in all cases. There include the distributive property, the commutative property, and the associative property. Also, every real number has an additive inverse and, except for zero (0), have a multiplicative inverse.

3.7 Clock Arithmetic

- Clock arithmetic uses the idea that after 12 o'clock comes 1 o'clock. For clock arithmetic, this means that every time 12 is passed in an arithmetic process, the next number is 1, not 13.
- To determine the clock result of an arithmetic operation, divide the final result by 12 and keep the remainder. If the remainder is 0, then the time is 12 o'clock.
- Clock arithmetic is technically called modulo 12 arithmetic. To perform modulo 12 arithmetic, calculate the expression, then divide the result by 12. The modulo 12 result is the remainder.
- Days, in our system, pass in groups of seven. To calculate in day arithmetic, modulo 7 is used. To perform modulo 7 arithmetic, calculate the expression, then divide the result by 7. The modulo 7 result is the remainder.

3.8 Exponents

- Exponents are used to express multiplying a number by itself a number of times. The number being multiplied by itself is the base. The number of times it is multiplied by itself is the exponent, which is often referred to as the power.
- Understanding that exponents represent repeated multiplication of a base makes it possible to establish some rules for combining exponential expressions, using the product rule, the quotient rule, and the power rule. Additionally, it allows us to formulate distributive rules for exponents.
- Any non-zero number raised to the 0th power is 1. This makes the definition of the 0th power consistent with the division rule for exponents.
- For consistency, negative exponents represent the reciprocal of the base raised to the power, so that $a^{-n} = \frac{1}{a^n}$, provided that $a \neq 0$.

3.9 Scientific Notation

- Some numbers are so large or so small that writing the number out is clumsy and make it difficult to determine the true size of the number. Scientific notation makes the number more readable and make the relative size of the number immediately apparent.
- A number written in scientific notation is a number at least 1 and smaller than 10 multiplied by 10 raised to an exponent. Converting between scientific notation and standard notation involves correctly applying multiplication and division by powers of 10, which in practice equates to understanding how moving the decimal point of a number impacts the exponent of 10.
- Adding and subtracting numbers in base 10 requires the exponent of 10 in each number be the same. Once the numbers are converted to have the same exponent with the ten, then the numbers are added or subtracted as indicated, with the power of 10 remaining the same. If the result is not in scientific notation (for instance, the number has exceeded 10), then then number must be converted into scientific notation.
- Multiplying and dividing numbers in scientific notation is done by multiplying or dividing the number parts, then multiplying or dividing the 10 raised to the power parts, then multiplying those two results. If the new number is not in scientific notation, then the result must be converted into scientific notation.

3.10 Arithmetic Sequences

- A sequence is a list of numbers. Any individual number in that list, or sequence, is a term of the sequence. A specific term of a sequence is denoted by the sequence symbol with a subscript indicating where the term in the sequence is.
- A special form of a sequence is an arithmetic sequence. Each arithmetic sequence is determined by its first term and its constant difference. Any term in an arithmetic sequence is determined by adding the constant difference to the preceding term.
- If the first term and the constant difference of an arithmetic sequence are known, then any term of the sequence can be found directly.
- Because arithmetic sequences follow such a strict pattern, the sum of the first n terms of an arithmetic sequence can be determined with the formula $s_n = n\left(\frac{a_1+a_n}{2}\right)$.

3.11 Geometric Sequences

- A special form of a sequence is a geometric sequence. Each geometric sequence is determined by its first term and its constant ratio. Any term in a geometric sequence is determined by multiplying the constant ratio to the preceding term.
- If the first term and the constant ratio of a geometric sequence are known, then any term of the sequence can be found directly.
- Because geometric sequences follow such a strict pattern, the sum of the first n terms of a geometric sequence can be determined with the formula $s_n = a_1\left(\frac{1-r^{n-1}}{1-r}\right)$.
- Finding the sum of a finite geometric sequence
- Applying arithmetic sequences

3.11 Geometric Sequences

- Geometric sequence.
- Finding an arbitrary term in a geometric sequence.
- Constant ratio.
- Finding the sum of a finite geometric sequence.
- Applying arithmetic sequences.

Videos

3.1 Prime and Composite Numbers

- Divisibility Rules (https://openstax.org/r/Divisibility_Rules)
- Illegal Prime Number (https://openstax.org/r/Illegal_Prime_Number)
- Using a Factor Tree to Find the Prime Factorization (https://openstax.org/r/Using_a_Factor_Tree_to_Find_the_Prime_Factorization)
- Finding the Prime Factorization of 168 (https://openstax.org/r/Finding_the_Prime_Factorization_of_168)
- Using Desmos to find the GCD (https://openstax.org/r/Using_Desmos_to_find_the_GCD)
- Applying the GCD (https://openstax.org/r/Applying_the_GCD)

- Finding the LCM (https://openstax.org/r/Finding_the_LCM)
- Using Desmos to find the LCM (https://openstax.org/r/Using_Desmos_to_find_the_LCM)
- Application of LCM (https://openstax.org/r/Application_of_LCM)

3.2 The Integers

- Graphing Integers on the Number Line (https://openstax.org/r/Graphing_Integers_on_the_Number_Line)
- Comparing Integers Using the Number Line (https://openstax.org/r/Comparing_Integers_Using_the_Number_Line)
- Evaluating the Absolute Value of an Integer (https://openstax.org/r/Evaluating_the_Absolute_Value_of-an-Integer)

3.3 Order of Operations

- Order of Operations 1 (https://openstax.org/r/Order_of_Operations_1)
- Order of Operations 2 (https://openstax.org/r/Order_of_Operations_2)
- Order of Operations 3 (https://openstax.org/r/Order_of_Operations_3)
- Order of Operations 4 (https://openstax.org/r/Order_of_Operations_4)

3.4 Rational Numbers

- Introduction to Fractions (https://openstax.org/r/Introduction_to_Fractions)
- Equivalent Fractions (https://openstax.org/r/Equivalent_Fractions)
- Reducing Fractions to Lowest Terms (https://openstax.org/r/Reducing_Fractions_to_Lowest_Terms)
- Using Desmos to Reduce a Fraction (https://openstax.org/r/Using_Desmos_to_Reduce_a_Fraction)
- Adding and Subtracting Fractions with Different Denominators (https://openstax.org/r/Adding_and_Subtracting_Fractions)
- Converting an Improper Fraction to a Mixed Number Using Desmos (https://openstax.org/r/Improper_Fraction_to_Mixed_Number)
- Multiplying Fractions (https://openstax.org/r/Multiplying_Fractions)
- Dividing Fractions (https://openstax.org/r/Dividing_Fractions)
- Order of Operations Using Fractions (https://openstax.org/r/Operations_Using_Fractions)
- Finding a Fraction of a Total (https://openstax.org/r/Finding_Fraction_of_Total)
- Converting Units (https://openstax.org/r/Converting_Units)

3.5 Irrational Numbers

- The Philosophy of the Pythagoreans (https://openstax.org/r/Philosophy_of_Pythagoreans)
- Using Desmos to Find the Square Root of a Number (https://openstax.org/r/square_root_of_a_number)
- Simplifying Square Roots (https://openstax.org/r/Simplifying_Square_Roots)
- Rationalizing the Denominator (https://openstax.org/r/Rationalizing_Denominator)

3.6 Real Numbers

- Properties of the Real Numbers 1 (https://openstax.org/r/Properties_of_the_Real_Numbers_1)
- Properties of the Real Numbers 2 (https://openstax.org/r/Properties_of_the_Real_Numbers_2)
- Properties of the Real Numbers 3 (https://openstax.org/r/Properties_of_the_Real_Numbers_3)

3.6 Real Numbers

- Arthur Benjamin TED talk, Faster than a Calculator (https://openstax.org/r/Arthur_Benjamin_TED_talk,_Faster_than_a_Calculator)
- Identifying Sets of Real Numbers (https://openstax.org/r/Sets_of_Real_Numbers)
- Properties of the Real Numbers #1 (https://openstax.org/r/Properties_of_the_Real_Numbers_1)
- Properties of the Real Numbers #2 (https://openstax.org/r/Properties_of_the_Real_Numbers_2)
- Properties of the Real Numbers #3 (https://openstax.org/r/Properties_of_the_Real_Numbers_3)

3.7 Clock Arithmetic

- Determining the Day of the Week for Any Date in History (https://openstax.org/r/Determining_the_Day_of_the_Week_for_Any_Date_in_History)
- Clock Arithmetic (https://openstax.org/r/Clock_Arithmetic)

3.8 Exponents

- Exponential Notation (https://openstax.org/r/Exponential_Notation)
- Product and Quotient Rule for Exponents (https://openstax.org/r/Product_and_Quotient_Rule_for_Exponents)
- Fraction Raised to a Power (https://openstax.org/r/Fraction_Raised_to_a_Power)

- Simplifying Expressions with Exponents (https://openstax.org/r/Simplifying_Expressions_with_Exponents)

3.9 Scientific Notation

- Converting from Standard Form to Scientific Notation Form (https://openstax.org/r/Converting_from_Standard_Form_to_Scientific_Notation_Form)
- Converting from Scientific Notation Form to Standard Form (https://openstax.org/r/Converting_from_Scientific_Notation_Form_to_Standard_Form)
- Multiplying Numbers in Scientific Notation (https://openstax.org/r/Multiplying_Numbers_in_Scientific_Notation)
- Dividing Numbers in Scientific Notation (https://openstax.org/r/Dividing_Numbers_in_Scientific_Notation)
- Application of Scientific Notation (https://openstax.org/r/Application_of_Scientific_Notation)

3.10 Arithmetic Sequences

- Arithmetic Sequences (https://openstax.org/r/Arithmetic_Sequences)
- Finding the First Term and Constant Difference for an Arithmetic Sequence (https://openstax.org/r/Finding_the_First_Term_and_Constant_Difference_for_an_Arithmetic_Sequence)
- Finding the Sum of a Finite Arithmetic Sequence (https://openstax.org/r/Finding_the_Sum_of_a_Finite_Arithmetic_Sequence)
- Fibonacci Sequence and “Lateralus” (https://openstax.org/r/Fibonacci_Sequence_and_Lateralus)

3.11 Geometric Sequences

- Geometric Sequences (https://openstax.org/r/Geometric_Sequences)
- Sum of a Finite Geometric Sequence (https://openstax.org/r/Sum_of_a_Finite_Geometric_Sequence)

Formula Review

3.4 Rational Numbers

$\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$
$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$
$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ $\frac{a}{b} \div \frac{c}{d} = \frac{a}{d} \times \frac{d}{c} = \frac{a \times d}{b \times c}$

3.5 Irrational Numbers

$$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$$

$$a \times x \pm b \times x = (a \pm b) \times x$$

$$\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

$$a^2 - b^2 = (a - b)(a + b)$$

3.6 Real Numbers

$a \times (b + c) = a \times b + a \times c$
$a + b = b + a$
$a \times b = b \times a$
$a + (b + c) = (a + b) + c$
$a \times (b \times c) = (a \times b) \times c$
$a + 0 = a$
$a \times 1 = a$
$a + (-a) = 0$
$a \times \left(\frac{1}{a}\right) = 1$

3.8 Exponents

$$a^n a^m = a^{n+m}$$

$$\frac{a^n}{a^m} = a^{(n-m)}$$

$a^0 = 1$, provided that $a \neq 0$

$$(a \times b)^n = a^n \times b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$\left(a^n\right)^m = a^{(n \times m)}$$

$a^{-n} = \frac{1}{a^n}$, provided that $a \neq 0$

3.10 Arithmetic Sequences

$a_i = a_1 + d \times (i - 1)$

$d = \frac{a_j - a_i}{j - i}$

$a_1 = a_i - d(i - 1)$

$s_n = n\left(\frac{a_1 + a_n}{2}\right)$

3.11 Geometric Sequences

$a_n = a_1 r^{n-1}$

$s_n = a_1\left(\frac{1 - r^{n-1}}{1 - r}\right)$

3.11 Geometric Sequences

$a_n = a_1 r^{n-1}$

$s_n = a_1\left(\frac{1 - r^{n-1}}{1 - r}\right)$

Projects

Encryption began at least as far back as the Roman Empire. During the reign of Caesar, a particular cypher was used, fittingly named the Caesar Cypher. This encryption process granted the Romans a great tactical advantage. Even if a message was intercepted, it would not make sense to the person intercepting the message.

Find four instances when encryption was used and cracked over the course of history.

The Golden Ratio in Art and Architecture

The golden ratio has been used in art and architecture as far back as ancient Greece (possibly further). It also appears in South America (Incan architecture). Find five instances of the use of the golden ratio in art or architecture and describe its use in each of those instances.

Your Budget

Budgeting either is, or will shortly be, an important aspect of your life. Managing money well reduces stress in your life, and provides space for planning for future expenses, such as vacations or home improvements.

Imagine your life 10 years from now. Estimate your monthly income. Identify expenses you will encounter monthly (mortgage or rent, car payment, insurance, entertainment, etc.). Decide on an amount you plan to save monthly (this is treated as an expense). Create a spreadsheet with those values. Record your monthly net income (your income minus your expenses). Determine how much money you will have saved over the course of 5 years (ignore interest). Write a reflection on your anticipated financial health.

Estimating Pi

The value of pi is the ratio of the circumference of a circle to the diameter of the circle. It is also equal to the ratio of the area of the circle to the square of the radius of the square.

Research three ways to physically estimate pi.

Estimate pi using all three processes you found.

Present your process and solutions in class.

Design Your Own Shift Cypher

A cypher is a message written in such a way as to mask its contents. Changing a message into its cypher form is called encryption. Decryption or deciphering is the process of changing a cyphertext message back into the original (legible) message. One process of encryption is to scramble the letters, symbols, and punctuation of a message according to a mathematical rule. One rule that could be used for such a cypher is addition in a chosen modulus. In this project, you will create such a cypher, encrypt a message, and then decrypt the message.

Step 1: Choose the letters, symbols, and punctuation marks you want to allow in your messages. This should include at least the uppercase letters and a space character. This is your character set.

Step 2: Count the number of characters you will use. Label this number n.

Step 3: Pair each character of your character set an integer from 0 to $(n - 1)$. Do not assign more than one character to an integer.

Step 4: Choose an integer between 1 and $(n - 1)$. This will be the number used to create the cypher. Label this number s.

Step 5: Write a message using your character set.

Step 6: Replace every character in your message by the integer with which it was paired in Step 3.

Step 7: For every number, x, from Step 7, perform the addition $x + s$ (mod n).

Step 8: Replace every number found in Step 7 with the character with which it was paired in Step 3. This is your cyphertext.

To decrypt your cyphertext, reverse the steps above.

Step 1: Replace the cyphertext characters with the paired values.

Step 2: For each value x, perform the *subtraction* $x– s$ (mod n).

Step 3: Replace the numbers from Step 2 with their paired characters from the character set.

The message is then deciphered.

Design Your Own Cypher Using Multiplication

A cypher is a message written in such a way as to mask its contents. Changing a message into its cypher form is called encryption. Decryption or deciphering is the process of changing a cyphertext message back into the original (legible) message. One process of encryption is to scramble the letters, symbols, and punctuation of a message according to a mathematical rule. One rule that could be used for such a cypher is multiplication in a chosen modulus. In this project, you will create such a cypher, encrypt a message, then decrypt the message.

Step 1: Choose the letters, symbols, and punctuation marks you want to allow in your messages. This should include at least the uppercase letters and a space character. This is your character set.

Step 2: Count the number of characters you will use. Label this number n.

Step 3: Pair each character of your character set an integer from 0 to $(n - 1)$. Do not assign more than one character to an integer.

Step 4: Choose an integer, labeled s, between 1 and $(n - 1)$ so that $\text{GCF}(n, s) = 1$. This will be the number used to create the cypher.

Step 5: Write a message using your character set.

Step 6: Replace every character in your message by the integer with which it was paired in Step 3.

Step 7: For every number, x, from Step 6, perform the multiplication $x - s$ (mod n).

Step 8: Replace every number found in Step 7 with the character with which it was paired in Step 3. This is your cyphertext.

Before beginning to decrypt in this cypher, you need to know the **multiplicative inverse** of the value you chose as *s*.

Step 1: The multiplicative inverse of s is the number that, when multiplied by s in your modulus, equals 1. To find this, you will have to multiply s and every number between 2 and $(n– 1)$ until the product is 1 (mod n). Once this number is found, the message can be decrypted. Call this number r.

Step 2: To decrypt your cyphertext, replace the cyphertext characters with the paired values.

Step 3: For each of the value, x, perform the multiplication $x \times r (\text{mod}\, n)$.

Step 4: Replace the numbers from Step 3 with their paired characters from the character set.

The message is then deciphered.

Chapter Review

Prime and Composite Numbers

1. Identify which of the following numbers are prime, composite, or neither: 201, 34, 17, 1, 37.
2. Find the prime factorization of 500.
3. Find the greatest common divisor of 80 and 340.
4. Find the greatest common divisor of 30, 40, and 70.
5. Find the least common multiple of 45 and 60.
6. Bella and JJ volunteer at the zoo. Bella volunteers every 8 days, while JJ volunteers every 14 days. How many days pass between days they volunteer together?

The Integers

7. Identify all the integers in the following list: $-4, \frac{2}{7}, 15, 97.5, 0, \sqrt{122}$.
8. Plot the following on the same number line: $1, 5, -2, 0$.
9. What two numbers have absolute value of 18?
10. Calculate $14 - (-12)$.
11. Calculate $(-19) \times 4$.
12. Calculate $(-240) \div (-12)$.
13. Six students rent a house together. The total monthly rent (including heat and electricity) is $3,120. If they all pay an equal amount, how much does each student pay?

Order of Operations

14. Calculate $5 \times 3 - (-12)$.
15. Calculate $4^2 \times 6 + 2^3$.
16. $(4 - 6) \times 2$
17. $(8 - 1)^3 - 3 \times 9$
18. $120 - (51 + 12) \div 3^2$

Rational Numbers

19. Reduce $\frac{90}{153}$ to lowest terms.
20. Convert $\frac{430}{25}$ to a mixed number and reduce to lowest terms.
21. Convert $\frac{9}{5}$ to decimal form.
22. Convert $\frac{5}{11}$ to decimal form.
23. Calculate $\frac{4}{15} + \frac{9}{10}$ and reduce to lowest terms.
24. Compute $\frac{3}{10} \div \frac{12}{25}$ and reduce to lowest terms.
25. Determine 30% of 400.
26. 18 is what percent of 40?
27. In Professor Finnegan's Science Fiction course, there are 60 students. Of those, 15% say they've read *A Hitchhiker's Guide to the Galaxy*. How many of the students have read that book?

Irrational Numbers

28. Simplify the square root by expressing it in lowest terms: $\sqrt{275}$.
29. Calculate $4\sqrt{13} - 15\sqrt{13}$ without a calculator. If not possible, explain why.

30. Calculate $\left(2.3\sqrt{5}\right) \times \left(4.2\sqrt{3}\right)$ without a calculator. If not possible, explain why.

31. Rationalize the denominator of $\frac{6}{\sqrt{22}}$, and simplify the fraction.

32. Find the conjugate of $4\sqrt{7} + 3$ and find the product of $4\sqrt{7} + 3$ and its conjugate.

33. Rationalize the denominator of $\frac{3}{8 + \sqrt{34}}$ and simplify the fraction.

Real Numbers

34. Identify the numbers of the following list as a natural number, an integer, a rational number, or a real number: $\sqrt{37}$, −0.43, 18, −43, 12π.

35. Identify the property of real numbers illustrated here: $14 + 19 = 19 + 14$.

36. Identify the property of real numbers illustrated here: $87.4 + 0 = 87.4$.

37. Identify the property of real numbers illustrated here: $2 \times (\sqrt{31} - 12) = 2\sqrt{31} - 2 \times 12$.

38. Identify the property of real numbers illustrated here: $98 \times 41 = 41 \times 98$.

39. Use mental math to calculate 21×99.

Clock Arithmetic

40. Determine 74 modulo 9.

41. Use clock arithmetic to calculate $13 + 25$.

42. Use clock arithmetic to calculate 4×8.

43. It is Wednesday. What day of the week will it be in 44 days?

44. It is 4:00. What time will it be in 100 hours?

45. Security guards with Acuriguard submit a report on campus activity every 4 days. If they make a report on a Monday, what day of the week will it be after 10 more reports?

Exponents

46. Use exponent rules to simplify $17^8 \times 17^3$.

47. Use exponent rules to simplify $\frac{15^9}{15^{-5}}$.

48. Use exponent rules to simplify $(3k)^6$.

49. Use exponent rules to simplify $\left(\frac{y}{4}\right)^{11}$.

50. Use exponent rules to simplify $\left(51^3\right)^6$.

51. Use exponent rules to simplify $\left(\frac{3x^4}{5}\right)^7$.

52. Rewrite $\frac{5^2}{x^4}$ without a denominator.

53. Rewrite $16z^{-9}$ without negative exponents.

Scientific Notation

54. Convert 0.0000452 to scientific notation.

55. Convert 3.01×10^5 to standard notation.

56. Convert 42.9×10^{-9} to scientific notation.

57. Calculate $4.51 \times 10^5 - 9.11 \times 10^4$.

58. Calculate $\left(9.15 \times 10^3\right) \div \left(3 \times 10^8\right)$.

59. The Sextans Dwarf Spheroidal Galaxy has diameter 8,400 light years (ly). Express this in Scientific notation.

60. The Reticulum II Galaxy has diameter 3.78×10^2 light years (ly), while the Andromeda Galaxy has a diameter of

2×10^5. How many times bigger is the Andromeda Galaxy compared to the Reticulum II Galaxy?

Arithmetic Sequences

61. Determine the common difference of the following sequence: {19, 13, 7, 1, -5, ...}.

62. Find the 25th term, a_{25}, of the arithmetic sequence with $a_1 = 13$ and $d = 1.7$.

63. Find the first term and the common difference of the arithmetic sequence with 8th term $b_8 = 50$ and 15th term $b_{15} = 106$.

64. Find the sum of the first 30 terms, s_{30}, for the arithmetic sequence with first term $a_1 = -4$ and common difference $d = 3.5$.

65. Jem makes a stack of 5 pennies. Each day, Jem adds three pennies to the stack. How many pennies are in the stack after 10 days?

Geometric Sequences

66. Determine the common ratio of the following geometric sequence: {6, 18, 54, 162, ...}.

67. Find the 6th term, c_6, of the geometric sequence with $c_1 = 400$ and $r = 0.25$.

68. Find the sum of the first 12 terms, s_{12}, for the geometric sequence with first term $a_1 = -4$ and common ratio $r = -1.25$.

69. Carolann and Tyler deposit $8,500 in an account bearing 5.5% interest compounded yearly. If they do not deposit any more money in that account, how much will be in the account after 15 years?

70. The total number of ebooks sold in 2013 was 242 million ($a_1 = 242$). Each year, the number of ebooks sold has declined by 3% ($r = 0.97$). How many ebooks were sold between 2013 and 2022?

Chapter Test

1. Find the prime factorization of 300.

2. Tiles will be used to cover an area that is 650 cm × 1,200 cm. What is the largest size square tile that can be used so that all the tiles used are full tiles?

3. Calculate the following: $3 \times \left(\frac{4}{3} \times (14 - 8) + 3\right) - 4 \times \left(8 - 2^2\right)$.

4. What does PEMDAS stand for?

5. David's tax return is for $1,560. He decides to spend 20% of that return. How much does David spend?

6. Convert $\frac{305}{26}$ into a mixed number.

7. Convert $0.\overline{659}$ to a fraction of integers.

8. Simplify the following square root: $\sqrt{6237}$.

9. What is the conjugate of $4 - \sqrt{10}$?

10. Rationalize the denominator of $\frac{2}{10 + \sqrt{13}}$.

11. What two sets of numbers comprise the real numbers?

12. Which property of the real numbers is shown: $4(a + 7) = 4a + 4 \times 7$?

13. Georita believes it will take 30 hours for her and her family to drive to their vacation. If they leave at 2:00, what time should they arrive (ignore AM/PM)?

14. Suppose a bacterium in the gut has a generation time (time to divide) of 16 hours. If it first divides at 4:00, what time will it be when they divide the 80th time afterward?

15. Simplify $\left(\frac{9x}{a^2}\right)^5$.

16. The moon is 3.844×10^8 m from Earth. A dollar bill has a thickness of 1.0922×10^{-4} m. If dollar bills could be stacked perfectly, how many would it take to reach the moon?

17. A single long table can seat 8 people, 3 on each side and 1 on each end. If a second table is added to the first, end to end, then 14 people can sit at the table, 6 per side and 1 at each end. Adding another table adds another 6 people. How many people can sit at a table made by placing 10 of these tables end to end?

18. The first row of a theater seats 25 people. Each following row seats 2 more people. If there are 80 rows in the theater, how many people, total, can sit in the theater?

19. Alice deposits $2,500 in a bond yielding 6% interest compounded annually. How much is the bond worth in 20 years?

20. What is the 15th term of a geometric sequence with first term 5 and common ratio 3?

4

NUMBER REPRESENTATION AND CALCULATION

Figure 4.1 Different cultures developed different ways to record quantity. (credit: modification of work "Tally sticks from the Swiss Alps" by Sandstein, Swiss Alpine Museum permanent collection/Wikimedia Commons, CC BY 3.0)

Chapter Outline

4.1 Hindu-Arabic Positional System
4.2 Early Numeration Systems
4.3 Converting with Base Systems
4.4 Addition and Subtraction in Base Systems
4.5 Multiplication and Division in Base Systems

Introduction

Right now, almost all cultures use the familiar Hindu-Arabic numbering system, which uses the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 along with place values based on powers of ten. This is a relatively recent development. The system didn't develop until the 6th or 7th century C.E. and took some time to spread across the world, which means other cultures at other times had to develop their own methods of counting and recording quantity. Being different cultures and different times means there were significant differences in counting systems. Cultures needed to count and measure time for agriculture and for religious observations. It was needed for trade. Some languages had words only for one, two, and many. Other cultures developed more complex ways to represent quantity, with the Oksapmin people of New Guinea using an astonishing 27 words for their system.

Representing these quantities in a recorded form likely began with a simple marking system, where one scratch on a stick or bone represented one of whatever was being counted. We still see this today with tally marks. These systems use repeated symbols to represent more than one. We also have systems where different symbols represent different quantities but still use some repetition, such as in Roman numerals.

Other systems were devised that rely on place values, like the Hindu-Arabic system in use today. Place value systems needed a zero, though, and weren't immediately recognized and took time to develop. And within these positional systems there is variation. Some systems counted in twenties, others in tens, and some in a mix (adding another reason to visit Hawaii). Even now, though we all use and think using tens, computers are designed to work in groups of two, which requires a different perspective on numbering.

In this chapter, we explore different numbering systems and grouping systems, eventually discussing base 2, the language of computers.

4.1 Hindu-Arabic Positional System

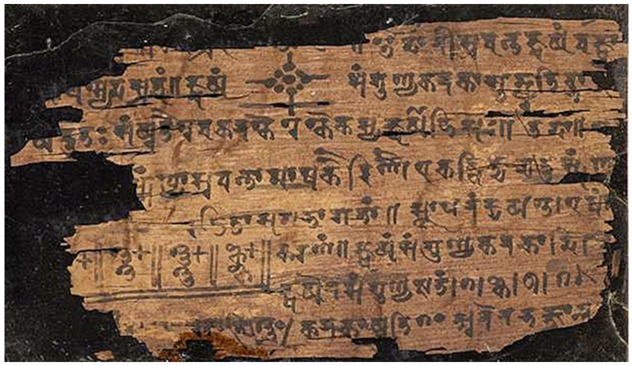


Figure 4.2 This manuscript is an early example of Hindu numerals. (credit: modification of work “Bakshali manuscript”, Bodleian Libraries/ University of Oxford, public domain)

Learning Objectives

After completing this section, you should be able to:

1. Evaluate an exponential expression.
2. Convert a Hindu-Arabic numeral to expanded form.
3. Convert a number in expanded form to a Hindu-Arabic numeral.

The modern system of counting and computing isn’t necessarily natural. That different symbols are used to indicate different quantities or amounts is a relatively new invention. Simple marking by scratches or dots, one for each item being counted, was the norm long into human history. The modern system doesn’t use repeated symbols to indicate more than one of a thing. It uses the place of a digit in a **numeral** to determine what that digit represents. A numeral is a symbol used to represent a number. A **number** is an abstract idea that represents quantity or amount.

Being clear about the difference between numeral and number is important. Just like a person can be called by various names, such as brother, father, husband, uncle, they are all representing the same person, John Smith. The person John Smith is the number, and the names brother, father, husband, and uncle are the numerals.

WHO KNEW?

Hindu-Arabic Numerals

The numerals we currently use are referred to as Hindu-Arabic numerals, although they have changed as time has passed. Early forms of the numerals for 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 began in India, and passed through Persia to the Middle East. Place value was also employed in the early systems of India. Once this system was in north Africa and the Middle East, it spread to Europe, eventually replacing Roman numerals. Over time, the original symbols transformed into our modern ones. Read this article for another perspective on how the symbols began (based on the moon!) (https://openstax.org/r/myindiamyglory.com/2018/09/05).

The system we use for counting and computing uses place values based on powers of 10. In this section, we review exponents and our positional system.

Evaluating Exponential Expressions

Most modern numerical systems depend on place values, where the quantity represented depends not only on the digit, but also on where the digit is in the number. The place value is a power of some specific number, which means most numbering systems are actually exponential expressions. An **exponential expression** is any mathematical expression that includes exponents. So, evaluating such an expression means performing the calculation. In this chapter, we will be using exponents that are positive integer values. Before we do so, let’s remind ourselves about exponents and what they represent. Suppose you want to multiply a number. Let’s label that number a, by itself some number of times. Let’s label the number of times b. We denote that as a^b. We say a, or the **base**, raised to the bth power, or the **exponent**. For example, if we are multiplying 13 by itself eight times, we write 13^8 and say 13 to the eighth power.

When computing exponential expressions, we should be careful to remember the order of operations. Using the order of operation rules, calculations inside the parentheses are done first, then exponents are calculated, then multiplication and division calculations are performed, and then addition and subtraction.

VIDEO

Exponential Notation (https://openstax.org/r/Exponential_Notation)

EXAMPLE 4.1

Evaluating an Exponential Expression

Evaluate the following exponential expressions.

1. $4 \times 5^2 + 2 \times 6^3$
2. $6 \times 8^2 + 3 \times 8^1 + 4 \times 8^0$
3. $3 \times 10^2 + 0 \times 10^1 + 6 \times 10^0$

Solution

1. To evaluate, or calculate, this expression, we use order of operations, which means the exponents are done first, then multiplications, and then additions.
$$4 \times 5^2 + 2 \times 6^3 = 4 \times 5 \times 5 + 2 \times 6 \times 6 \times 6 = 4 \times 25 + 2 \times 216 = 100 + 432 = 532$$
2. To evaluate the expression, we use the order of operations, which means the exponents are done first, then the multiplications, then the additions. Remember that any base raised to the exponent 0 is 1.
$$6 \times 8^2 + 3 \times 8^1 + 4 \times 8^0 = 6 \times 8 \times 8 + 3 \times 8 + 4 \times 1 = 6 \times 64 + 3 \times 8 + 4 \times 1 = 384 + 24 + 4 = 412$$
3. To evaluate the expression, we use the order of operations, which means the exponents are done first, then the multiplications, and then the additions. Remember that any base raised to the exponent 0 is 1.
$$3 \times 10^2 + 0 \times 10^1 + 6 \times 10^0 = 3 \times 100 + 0 \times 10 + 6 \times 1 = 300 + 0 + 6 = 306$$

YOUR TURN 4.1

Evaluate the following exponential expressions.

1. $3 \times 2^5 + 5 \times 8^2$
2. $5 \times 7^3 + 2 \times 7^2 + 5 \times 7^1 + 3 \times 7^0$
3. $1 \times 10^4 + 7 \times 10^3 + 4 \times 10^2 + 8 \times 10^1 + 8 \times 10^0$

Converting Hindu-Arabic Numerals to Expanded Form

When you see the number 738, and you speak the number out loud, what do you say? You probably said “seven hundred thirty-eight” while wondering what point could possibly be made by asking this. What you didn’t say was “seven, and three, and eight.” A pre-K student might say that. Which should make you wonder, why?

The reason is that you’ve been taught **place values**, or the positions of digits in a number that determine the values of those digits. You know that in a three-digit number, the first digit is hundreds, the second digit is tens, and the last digit is ones. These place values rely on powers of 10, which makes this system a **base 10 system**.

This sense of place value is what makes our system of numbers so useful. You’ve also been taught the **Hindu-Arabic numeration system**. This system, which uses the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, and also employs place value based on powers of 10, is in use today.

Writing a number using these place values is writing them in **expanded form**. For a number with n digits, the expanded form is the first digit times 10 raised to one less than n, plus each following digit times 10 raised to one less than the previous power of 10. For example, the number 738 would be written as $7 \times 10^2 + 3 \times 10^1 + 8 \times 10^0$.

What about a four-digit number, like 5,825? Out loud, we’d say five thousand, seven hundred twenty-five. In expanded form, it would be $5 \times 10^3 + 8 \times 10^2 + 2 \times 10^1 + 5 \times 10^0$. Notice that the largest exponent is one less than the number of digits, and that the exponents go down by one as we move through the number.

PEOPLE IN MATHEMATICS

Aryabhata of Kusumapura and Brahmagupta

The Hindu-Arabic numeral system developed in India, and Aryabhata of Kusumapura is credited with the place value notation in the 5th century. However, the system wasn't as complete as it could be, until. roughly a century later, when Brahmagupta introduced the symbol for 0. The 0 is necessary to indicate that a given place value has been skipped, as in 4,098. In 4,098, the 10^2 power is skipped. Without such a symbol, 4,098 and 498 look similar. The value of both the place value notation and the introduction of the symbol 0 cannot be overstated, for math and the sciences.

EXAMPLE 4.2

Writing a Number in Expanded Form

Write the following in expanded form.

1. 563
2. 4,821
3. 903,786

Solution

1. **Step 1:** Since there are three digits in 563, n is 3. So, this is the first digit times 10 raised to the power of 2, so we start with 5×10^2.

   Step 2: Then we add the next digit, 6, multiplied by 10 to a power one less than the previous, at which point we have $5 \times 10^2 + 6 \times 10^1$.

   Step 3: Finally, the last digit is multiplied by 10 to the zeroth power and added to the previous. This results in $5 \times 10^2 + 6 \times 10^1 + 3 \times 10^0$.

2. **Step 1:** Since there are four digits in 4,821, n is 4. We multiply the first digit, 4, by 10 raised to the power of 3, which is 4×10^3.

   Step 2: Then we add the next digit, 8, multiplied by 10 to a power one less than the previous, at which point we have $4 \times 10^3 + 8 \times 10^2$.

   Step 3: We continue to the next digit, lowering the exponent of 10 by one. Now we have $4 \times 10^3 + 8 \times 10^2 + 2 \times 10^1$.

   Step 4: Finally, the last digit is multiplied by 10 to the zeroth power and added to the previous. This results in $4 \times 10^3 + 8 \times 10^2 + 2 \times 10^1 + 1 \times 10^0$.

3. Since there are six digits in 903,786, n is 6. So, we begin the process with 9 times 10 raised to the 5th power and continue through the numbers, reducing the exponent of 10 by one each time. This results in $9 \times 10^5 + 0 \times 10^4 + 3 \times 10^3 + 7 \times 10^2 + 8 \times 10^1 + 6 \times 10^0$.

YOUR TURN 4.2

Write the following in expanded form.

1. 924
2. 1,279
3. 4,130,045

Converting Numbers in Expanded Form to Hindu-Arabic Numerals

Converting from expanded form back into a Hindu-Arabic numeral is the reverse process of expanding a number, and is equivalent to evaluating the exponential expression.

EXAMPLE 4.3

Converting a Number from Expanded Form to a Hindu-Arabic Numeral
Convert the following into Hindu-Arabic numerals.

1. $3 \times 10^2 + 4 \times 10^1 + 8 \times 10^0$
2. $5 \times 10^3 + 0 \times 10^2 + 9 \times 10^1 + 9 \times 10^0$
3. $6 \times 10^6 + 2 \times 10^5 + 0 \times 10^4 + 9 \times 10^3 + 1 \times 10^2 + 1 \times 10^1 + 7 \times 10^0$

Solution

1. Evaluating the expression results in: $3 \times 10^2 + 4 \times 10^1 + 8 \times 10^0 = 3 \times 100 + 4 \times 10 + 8 \times 1 = 300 + 40 + 8 = 348$
2. Evaluating the expression results in:
$5 \times 10^3 + 0 \times 10^2 + 9 \times 10^1 + 9 \times 10^0 = 5 \times 1000 + 0 \times 100 + 9 \times 10 + 9 \times 1 = 5000 + 0 + 90 + 9 = 5099$
3. Evaluating the expression results in:
$6 \times 10^6 + 2 \times 10^5 + 0 \times 10^4 + 9 \times 10^3 + 1 \times 10^2 + 1 \times 10^1 + 7 \times 10^0$
$= 6 \times 1{,}000{,}000 + 2 \times 100{,}000 + 0 \times 10{,}000 + 9 \times 1{,}000 + 1 \times 100 + 1 \times 10 + 7 \times 1$
$= 6{,}000{,}000 + 200{,}000 + 0 + 9{,}000 + 100 + 10 + 7$
$= 6{,}209{,}117$

YOUR TURN 4.3

Convert the following to Hindu-Arabic Numerals.

1. $6 \times 10^2 + 2 \times 10^1 + 1 \times 10^0$
2. $3 \times 10^3 + 2 \times 10^2 + 0 \times 10^1 + 3 \times 10^0$
3. $4 \times 10^7 + 0 \times 10^6 + 6 \times 10^5 + 3 \times 10^4 + 0 \times 10^3 + 8 \times 10^2 + 9 \times 10^1 + 1 \times 10^0$

Check Your Understanding

1. What is meant by a place value system?
2. Evaluate the following exponential expression: $4 \times 8^2 + 2 \times 8^1 + 7 \times 8^0$.
3. Express the following number in expanded form: 45,209.
4. What number provides the value of a digit in our system of numeration?
5. How are numerals and numbers different?
6. Express as a Hindu-Arabic number: $6 \times 10^5 + 0 \times 10^4 + 1 \times 10^3 + 9 \times 10^2 + 4 \times 10^1 + 7 \times 10^0$.

SECTION 4.1 EXERCISES

1. What does it mean for a system to be a place value system?
2. In the system we use today, what number are the place values based on?
3. How are numerals and numbers different?
4. What relates numerals to numbers?

For the following exercises, evaluate the exponential expression.

5. $3 \times 4^2 + 5 \times 2^3$
6. $5 \times 6^3 + 7 \times 3^2$
7. $7 \times 5^2 + 2 \times 4^5$
8. $10 \times 11^2 + 7 \times 3^4$
9. $5 \times 6^2 + 3 \times 6^1 + 4 \times 6^0$
10. $4 \times 12^2 + 11 \times 12^1 + 2 \times 12^0$
11. $14 \times 8^3 + 19 \times 5^4 + 2 \times 3^1$
12. $8 \times 10^4 + 3 \times 5^3 + 9 \times 4^2$
13. $1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

14. $5 \times 8^4 + 1 \times 8^3 + 0 \times 8^2 + 7 \times 8^1 + 3 \times 8^0$

For the following exercises, express the Hindu-Arabic number in expanded form.

15. 13
16. 25
17. 82
18. 99
19. 131
20. 408
21. 651
22. 3,901
23. 5,098
24. 12,430

For the following exercises, express the expanded number as a Hindu-Arabic number.

25. $3 \times 10^1 + 2 \times 10^0$
26. $5 \times 10^1 + 7 \times 10^0$
27. $2 \times 10^2 + 4 \times 10^1 + 9 \times 10^0$
28. $6 \times 10^2 + 0 \times 10^1 + 1 \times 10^0$
29. $1 \times 10^3 + 4 \times 10^2 + 4 \times 10^1 + 0 \times 10^0$
30. $7 \times 10^3 + 0 \times 10^2 + 1 \times 10^1 + 8 \times 10^0$
31. $6 \times 10^4 + 7 \times 10^3 + 0 \times 10^2 + 0 \times 10^1 + 0 \times 10^0$
32. $9 \times 10^4 + 8 \times 10^3 + 7 \times 10^2 + 3 \times 10^1 + 4 \times 10^0$
33. $7 \times 10^7 + 3 \times 10^6 + 4 \times 10^5 + 0 \times 10^4 + 4 \times 10^3 + 1 \times 10^2 + 5 \times 10^1 + 1 \times 10^0$
34. $8 \times 10^8 + 0 \times 10^7 + 4 \times 10^6 + 9 \times 10^5 + 9 \times 10^4 + 2 \times 10^3 + 2 \times 10^2 + 6 \times 10^1 + 0 \times 10^0$

4.2 Early Numeration Systems

Figure 4.3 Babylonians used clay tablets for writing and record keeping. (credit: modification of work by Osama Shukir Muhammed Amin FRCP(Glasg), CC BY 4.0 International)

Learning Objectives

After completing this section, you should be able to:

1. Understand and convert Babylonian numerals to Hindu-Arabic numerals.
2. Understand and convert Mayan numerals to Hindu-Arabic numerals.
3. Understand and convert between Roman numerals and Hindu-Arabic numerals.

Each culture throughout history had to develop its own method of counting and recording quantity. The system used in Australia would necessarily differ from the system developed in Babylon that would, in turn, differ from the system developed in sub-Saharan Africa. These differences arose due to cultural differences. In nearly all societies, knowing the difference between one and two would be useful. But it might not be useful to know the difference between 145 and 167, as those quantities never had a practical use. For example, a shepherd likely didn't manage more than 100 sheep, so quantities larger than 100 might never have been encountered. This can even be seen in our use of the term few, which is an inexact quantity that most would agree means more than two. However, as societies became more complex, as commerce arose, as military bodies developed, so did the need for a system to handle large numbers. No matter the system, the issues of representing multiple values and how many symbols to use had to be addressed. In this section, we explore how the Babylonians, Mayans, and Romans addressed these issues.

Understand and Convert Babylonian Numerals to Hindu-Arabic Numerals

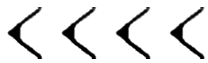
The Babylonians used a mix of an **additive system of numbers** and a **positional system of numbers**. An additive system is a number system where the value of repeated instances of a symbol is added the number of times the symbol appears. A positional system is a system of numbers that multiplies a “digit” by a number raised to a power, based on the position of the “digit.”

The Babylonian place values didn’t use powers of 10, but instead powers of 60. They didn’t use 60 different symbols though. For the value 1, they used the following symbol:

For values up to 9, that symbol would be repeated, so three would be written as

To represent the quantity 10, they used

For 20, 30, 40, and 50, they repeated the symbol for 10 however many times it was needed, so 40 would be written



When they reached 60, they moved to the next place value. The complete list of the Babylonian numerals up to 59 is in Table 4.1.

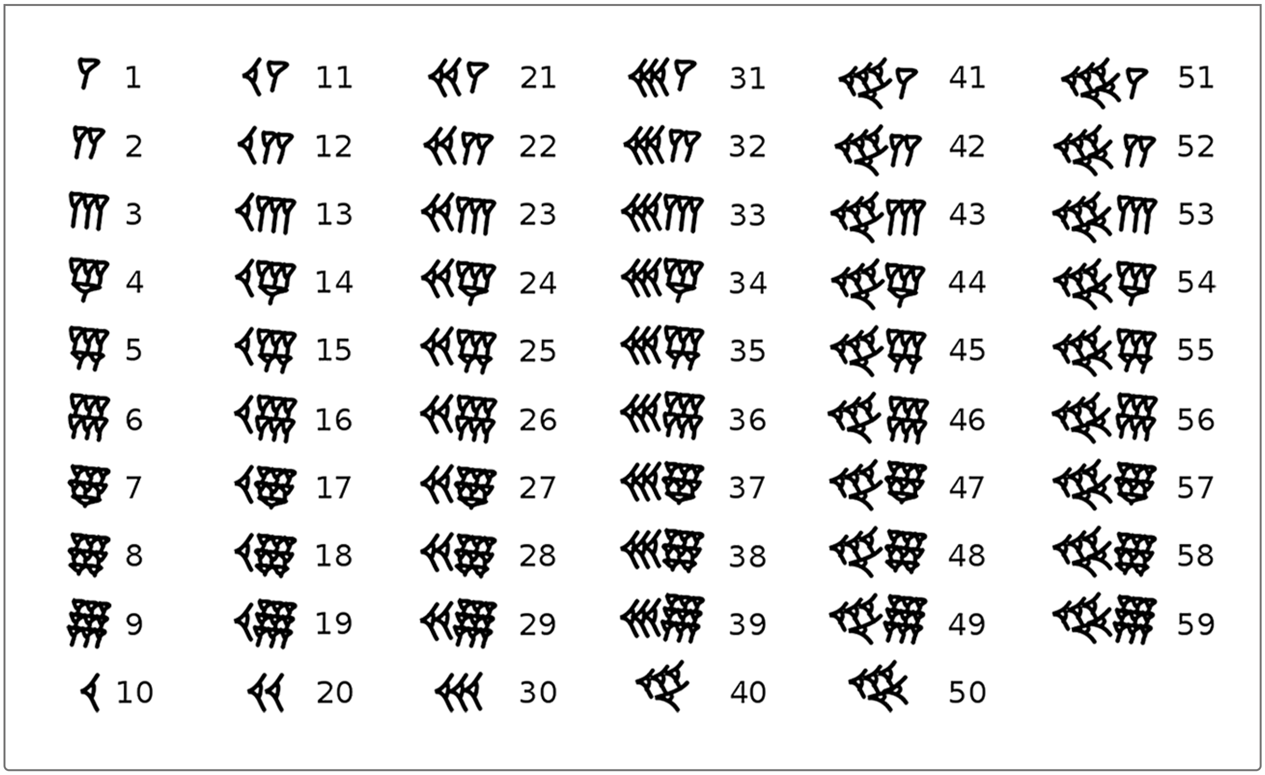


Table 4.1 Babylonian Numerals

You can see how Babylonians repeated the symbols to indicate multiples of a value. The number 6 is 6 of the symbol for 1 grouped together. The symbol for 30 is three of the symbols for 10 grouped together. However, their system doesn't go past 59. To go past 59, they used place values. As opposed to the Hindu-Arabic system, which was based on powers of 10, the Babylonian positional system was based on powers of 60. You should also notice there is no symbol for 0, which has some impact on the number system. Since the Babylonian number system lacked a 0, they didn't have a placeholder when a power of 60 was absent. Without a 0, 101, 110, and 11 all look the same. However, there is some evidence that the Babylonians left a small space between "digits" where we would use a 0, allowing them to represent the absence of that place value. To summarize, the **Babylonian system of numbers** used repeating a symbol to indicate more than one, used place values, and lacked a 0.

WHO KNEW?

Invention of 0

The idea of 0 is not a natural one. Most cultures failed to recognize the need for a 0. If someone asked a farmer in 300 B.C.E. how many cows they had, but they had none, they would not answer "zero." They'd say "I don't have any" and be done with it. It wasn't until roughly 3 B.C.E. that 0 appeared in Mesopotamia. It was independently discovered (or invented!) in the Mayan culture around 4 C.E. it made its appearance in India in the 400s C.E., and began to spread at that point. It wasn't developed earlier mostly because positional systems were not yet fully developed. Once positional systems arose, the need to represent a missing power had to be addressed.

So how do we convert from Babylonian numbers to Hindu-Arabic numbers? To do so, we need to use the symbols from Table 4.1, and then place values based on powers of 60. If you have n digits in the Babylonian number, you multiply the first "digit" by 60 raised to one less than the number of "digits." You then continue through the "digits," multiplying each by 60 raised to a power that is one smaller. However, be careful of spaces, since they represent a zero in that place.

EXAMPLE 4.4

Converting Two-Digit Babylonian Numbers to Hindu-Arabic Numbers

Convert the Babylonian number

into a Hindu-Arabic number.

Solution

has two digits:

and

Step 1: So the first symbol,

represents 4 in the Babylonian system. This is multiplied by 60 to the first power (just as would happen in a two digit number), which gives us 4×60^1.

Step 2: The next symbol is

which represents 27 in the Babylonian system. This is multiplied by 60 raised to 0, which gives $4 \times 60^1 + 27 \times 60^0$.

Step 3: Calculating that yields $4 \times 60^1 + 27 \times 60^0 = 240 + 27 = 267$. So the Babylonian number

equals 267 in the Hindu-Arabic number system.

YOUR TURN 4.4

1. Convert the Babylonian number  into a Hindu-Arabic number.

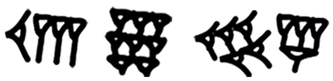
EXAMPLE 4.5

Converting Three-Digit Babylonian Numbers to Hindu-Arabic Numbers

Convert the Babylonian number

into a Hindu-Arabic number.

Solution



has three digits:

and

and

Step 1: So the first symbol,

represents 13 in the Babylonian system. This is multiplied by 60 to the second power (since there are 3 digits), which gives us 13×60^2.

Step 2: The next symbol is

which represents 8 in the Babylonian system, is multiplied by 60 raised to the first power, which gives us $13 \times 60^2 + 8 \times 60^1$.

Step 3: The last digit is

representing 54, which is multiplied by 60 raised to 0, which gives $13 \times 60^2 + 8 \times 60^1 + 54 \times 60^0$.

Step 4: Calculating that yields
$13 \times 60^2 + 8 \times 60^1 + 54 \times 60^0 = 13 \times 3{,}600 + 8 \times 60 + 54 \times 1 = 46{,}800 + 480 + 54 = 47{,}334$.

So, the Babylonian number

equals 47,334 in the Hindu-Arabic number system.

YOUR TURN 4.5

1. Convert the Babylonian number  into a Hindu-Arabic number.

EXAMPLE 4.6

Converting Four-Digit Babylonian Numbers to Hindu-Arabic Numbers

Convert the Babylonian number

into a Hindu-Arabic number.

Solution

It appears that

has three digits, but there is a space in between

and

Remember, the Babylonian system has no 0, it instead employs a space where we expect a zero. This means this is a four digit number.

Step 1: The first symbol,

represents 12 in the Babylonian system. This is multiplied by 60 to the third power since there are four digits, which gives us 12×60^3.

Step 2: The next symbol is a blank, which for us is a 0, representing 0×10^2, giving us $12 \times 60^3 + 0 \times 10^2$.

Step 3: The next symbol is

which represents 42 in the Babylonian system, is multiplied by 60 raised to the first power, which gives us $12 \times 60^3 + 0 \times 10^2 + 42 \times 60^1$.

Step 4: The last Babylonian digit,

represents 39 in the Babylonian system. This is multiplied by 60 raised to 0, which gives $12 \times 60^3 + 0 \times 10^2 + 42 \times 60^1 + 39 \times 60^0$.

Step 5: Calculating that yields

$$\begin{aligned}&12 \times 60^3 + 0 \times 10^2 + 42 \times 60^1 + 39 \times 60^0\\&= 12 \times 216{,}000 + 0 \times 10^2 + 42 \times 60 + 39 \times 1\\&= 2{,}592{,}000 + 0 + 2{,}520 + 39\\&= 2{,}594{,}559\end{aligned}$$

So the Babylonian number

equals 2,594,559 in the Hindu-Arabic number system.

YOUR TURN 4.6

1. Convert the Babylonian number  into a Hindu-Arabic number.

WHO KNEW?

The Legacy of Babylonian System

The Babylonian system can still be seen today. An hour is 60 minutes, and a minute is 60 seconds. Additionally, when measuring angles in degrees, each degree can be split into 60 minutes (1/60th of a degree) and 60 seconds (1/60th of a minute).

VIDEO

[Converting Between Babylonian and Hindu-Arabic Numbers (https://openstax.org/r/Babylonian_to_Hindu-Arabic_Numbers)](https://openstax.org/r/Babylonian_to_Hindu-Arabic_Numbers)

Understand and Convert Mayan Numerals to Hindu-Arabic Numerals

The Mayans employed a positional system just as we do and the Babylonians did, but they based their position values on powers of 20 and they had a dedicated symbol for zero. Similar to the Babylonians, the Mayans would repeat symbols to indicate certain values. A single dot was a 1, two dots were a 2, up to four dots. Then a five was a horizontal bar. The horizontal bars could be used three times, since the fourth horizontal bar would make a 20, which was a new position in the number. The 0 was a special picture, which appears like a turtle lying on its back. The shell would then be "empty," so maybe that's why the symbol was 0. The complete list is provided in Table 4.2. Another feature of Mayan numbers was that they were written vertically. The powers of 20 increased from bottom to top.

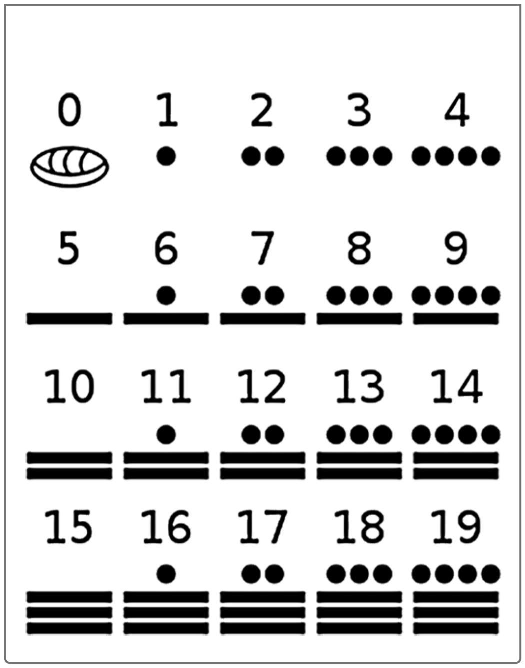


Table 4.2 Mayan Numerals

To summarize, the **Mayan system of numbers** used repeating symbol to indicate more than one, used place values, and employed a 0. So how do we convert from Mayan numbers to Hindu-Arabic numbers? To do so, we need to use the symbols from Table 4.2 and then place values based on powers of 20. If you have n digits in the Mayan number, you multiply the first "digit" by 20 raised to one less than the number of "digits." You then continue through the "digits," multiplying each by 20 raised to a power that is one smaller than the previous power. Fortunately, there is an explicit 0, so there is no ambiguity about numbers like 110, 101, and 11.

EXAMPLE 4.7

Converting Two-Digit Mayan Numbers to Hindu-Arabic Numbers

Convert the Mayan number

into a Hindu-Arabic number.

Solution

has two digits:

and

Step 1: So, the first symbol,

represents 15 in the Mayan system. This is multiplied by 20 to the first power, which gives us 15×20^1.

Step 2: The next symbol is

which represents 9 in the Mayan system. This is multiplied by 20 raised to 0, which gives $15 \times 20^1 + 9 \times 20^0$.

Step 3: Calculating that yields $15 \times 20^1 + 9 \times 20^0 = 300 + 9 = 309$. So

equals 309 in the Hindu-Arabic number system.

YOUR TURN 4.7

1. Convert the Mayan number into a Hindu-Arabic number.

EXAMPLE 4.8

Converting Three-Digit Mayan Numbers to Hindu-Arabic Numbers

Convert the Mayan number

into a Hindu-Arabic number.

Solution

has three digits:


and

and

Step 1: So the first symbol,

represents 6 in the Mayan system. This is multiplied by 20 to the second power (since there are 3 digits), which gives us 6×20^2.

Step 2: The next symbol is

which represents 8 in the Mayan system, is multiplied by 20 raised to the first power, which gives us $6 \times 20^2 + 8 \times 20^1$.

Step 3: The last digit is

representing 4, which is multiplied by 20 raised to 0, which gives $6 \times 20^2 + 8 \times 20^1 + 4 \times 20^0$.

Step 4: Calculating that yields $6 \times 20^2 + 8 \times 20^1 + 4 \times 20^0 = 6 \times 400 + 8 \times 20 + 4 \times 1 = 2{,}400 + 160 + 4 = 2{,}564$. So the Mayan number

equals 2,564 in the Hindu-Arabic number system.

YOUR TURN 4.8

1. Convert the Mayan number into a Hindu-Arabic number.

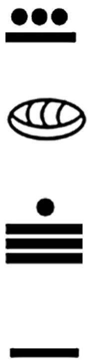

EXAMPLE 4.9

Converting Four-Digit Mayan Numbers to Hindu-Arabic Numbers

Convert the Mayan number


into a Hindu-Arabic number.

Solution



has four digits, so the first power of 20 that is used is 3.

Step 1: The first symbol,

represents 8 in the Mayan system. This is multiplied by 20 to the third power (since there are four digits), which gives us 8×20^3.

Step 2: The next symbol is

which is a 0, representing 0×20^2, giving us $8 \times 20^3 + 0 \times 20^2$.

Step 3: The next symbol is

which represents 16 in the Mayan system, is multiplied by 20 raised to the first power, which gives us $8 \times 20^3 + 0 \times 20^2 + 16 \times 20^1$.

Step 4: The last Mayan digit,

represents 5 in the Mayan system. This is multiplied by 20 raised to 0, which gives $8 \times 20^3 + 0 \times 20^2 + 16 \times 20^1 + 5 \times 20^0$.

Step 5: Calculating that yields

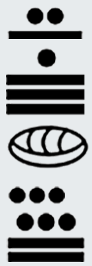
$$\begin{aligned}&8 \times 20^3 + 0 \times 20^2 + 16 \times 20^1 + 5 \times 20^0\\&= 8 \times 8000 + 0 \times 400 + 16 \times 20 + 5 \times 1\\&= 64{,}000 + 0 + 320 + 5\\&= 64{,}325\end{aligned}$$

So the Mayan number

equals 64,325 in the Hindu-Arabic number system.

YOUR TURN 4.9

1. Convert the Mayan number into a Hindu-Arabic number.



WHO KNEW?

The Mayan Calendar

The Mayans used this base 20 system for everyday situations. But their culturally important, and extremely accurate, calendar system used a slightly different system. For their calendars, they used a system where the place values were 1, 20, then 20*18, then 20*18*18. The reason for this is 20*18 is 360, which is closer to the number of days in a year. Had they used a purely base 20 system for their calendar, they'd be very far off with 400 days in a year.

Three hundred sixty days still left the Mayans a bit short, as there are 365 days in a year (ignoring leap years). The Mayan calendar also included 5 days, called Wayeb days, which brings their calendar to 365 days. As it happens, Wayeb is the Mayan god of misfortune, so these 5 days were considered the bad luck days.

VIDEO

[Converting Mayan Numbers to Hindu-Arabic Numbers (https://openstax.org /r/Mayan_to_Hindu-Arabic_Numbers)](https://openstax.org/r/Mayan_to_Hindu-Arabic_Numbers)

Understand and Convert Between Roman Numerals and Hindu-Arabic Numerals

The Mayan and Babylonian systems shared two features, one of which we are familiar with (place value) and one that we don't use (repeated symbols). The **Roman system of numbers** used repeated symbols, but does not employ a place value. It also lacks a 0. The Roman system is built on the following symbols in Table 4.3.

Roman Numeral	Hindu-Arabic Value
I	1
V	5
X	10
L	50

Table 4.3 Roman Numerals

Roman Numeral	Hindu-Arabic Value
C	100
D	500
M	1,000

Table 4.3 Roman Numerals

As in the Mayan and Babylonian systems, a symbol may be repeated to indicate a larger value. However, at 4, they did not use IIII. They instead used IV. Since the I came before the V, the number stands for "one before five." A similar process was used for 9, which was written IX, or "one before ten." The value 40 was written XL, or "ten before fifty," while 49 was written XLIX, or "forty plus nine."

The following are the rules for writing and reading Roman numerals.

- The representations for bigger values precede those for smaller values.
- Up to three symbols may be grouped together; for example, III for 3, or XXX for 30, or CC for 200.
- A larger value followed by a smaller value indicated addition; for example, VII for 7, XIII for 13, LV for 55, and MCC for 1200.
- I can be placed before V to indicate 4, or before X, to indicate 9. These are the only ways I is used as a subtraction.
- X can be placed before L to indicate 40, and before C to indicate 90. These are the only ways X is used as a subtraction.
- C can be placed before D to indicate 400, and before M to indicate 900. These are the only ways C is used as a subtraction.
- If multiple symbols are used, and a subtraction involving that symbol, the subtraction part comes after the multiple symbols. For example, XXIX for 29 and CCXC for 290.

WHO KNEW?

Legacy of Roman Numerals

The Roman numbering system is still used today in some situations. Many cornerstones of buildings have the year written in Roman numerals. Movie titles often represent the year the movie was produced as Roman numerals. The most recognizable might be that the Super Bowl is numbered using Roman numerals.

EXAMPLE 4.10

Converting Roman Numerals to Hindu-Arabic Numbers

Convert the following Roman numerals into Hindu-Arabic numerals.

1. XXVII
2. XXXIV
3. MMCMXLVIII

Solution

1. The numeral XXVII begins with two X's, which is then followed by a V. So, the two X's combine to be 20. The V is followed by two I's, so the V indicates the addition of 5. The two I's that follow indicate addition of two. That ends the symbols, so the value is 20 plus 5 plus 2, or 27 in Hindu-Arabic numerals.
2. The numeral XXXIV begins with three X's, which is then followed by an I. So, the three X's combine to be 30. The I is followed by a V, which indicates 4. That ends the symbols, so the value is 30 plus 4, or 34 in Hindu-Arabic numerals.
3. The numeral MMCMXLVIII begins with two M's, which is then followed by a C. So, the two M's combine to make 2000. The C is followed by an M, which indicates 900. The CM is followed by XL, which indicates 40. The L is followed by V, which indicates 5. The V is followed by three I's, indicating 3. Adding those values yields 2,948.

YOUR TURN 4.10

Convert the following Roman numerals into Hindu-Arabic numerals.

1. LXXVII
2. CCXL
3. MMMCDXLVII

VIDEO

[Converting From Roman Numbers to Hindu-Arabic Numbers (https://openstax.org/r/Roman_to_Hindu-Arabic_Numbers)]

Of course, we can convert from Hindu-Arabic numerals, to Roman numerals, too.

EXAMPLE 4.11

Converting Hindu-Arabic Numbers to Roman Numerals

Convert the following Hindu-Arabic numerals into Roman numerals.

1. 38
2. 94
3. 846
4. 2,987

Solution

1. Thirty is represented as three X's, and the 8 is represented with VIII, so 38 in Roman numerals is XXXVIII.
2. Ninety is represented by XC, and four is represented by IV, so 94 in Roman numerals is XCIV.
3. The number is less than 900 and more than 500, so the first symbol to be used is D, which is 500. To get to 800, we need 300 more, which is represented with three C's. Forty is represented with XL, and the six. The Roman numerals are DCCCXLVI.
4. The two thousand is represented by two M's. The 900 is represented by CM. The 80 is represented by LXXX (50 plus 30). Finally, the 7 is represented by VII. We have that 2,987 in Roman numerals is MMCMLXXXVII.

YOUR TURN 4.11

Convert the following Hindu-Arabic numerals into Roman numerals.

1. 27
2. 49
3. 739
4. 3,647

VIDEO

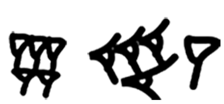
[Converting From Hindu-Arabic Numbers to Roman Numbers (https://openstax.org/r/Hindu-Arabic_to_Roman_Numbers)]

Check Your Understanding

7. What is the place value for Babylonian numerals?

8. What place value is used in the Mayan numeration system?

9. What place value is used for Roman numerals?

10. Convert the Babylonian numeral  into a Hindu-Arabic numeral.

11. Convert the Mayan numeral into a Hindu-Arabic numeral.

12. Convert the Roman numeral CCXLVII into a Hindu-Arabic numeral.

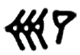
13. Convert 479 into a Roman numeral.

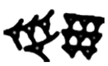
SECTION 4.2 EXERCISES

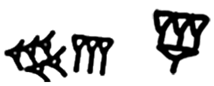
For the following exercises, convert the Babylonian numeral into a Hindu-Arabic numeral.

1. 

2. 

3. 

4. 

5. 

6. 

7. 

8. 

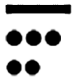
For the following exercises, express the Mayan numeral as a Hindu-Arabic numeral. Use the common system, which is based on powers of 20 only.

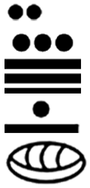
9. 

10. 

11. 

12. 

13. 

14. 

15.

16. 

For the following exercises, express the Roman numeral as a Hindu-Arabic numeral.

17. VII
18. XI
19. IX
20. XXIV
21. MCXLII
22. CXXII
23. DCCXLIV
24. MCMLIX

For the following exercises, express the Hindu-Arabic numeral as a Roman numeral.

25. 8
26. 14
27. 27
28. 94
29. 274
30. 487
31. 936
32. 2,481

33. What uses a place value system for numbers: Roman, Babylonian, Egyptian, Greek?

34. What uses an additive system: Roman, Mayan, Egyptian, Greek?

35. What uses a 0: Roman, Mayan, Egyptian, Greek?

4.3 Converting with Base Systems

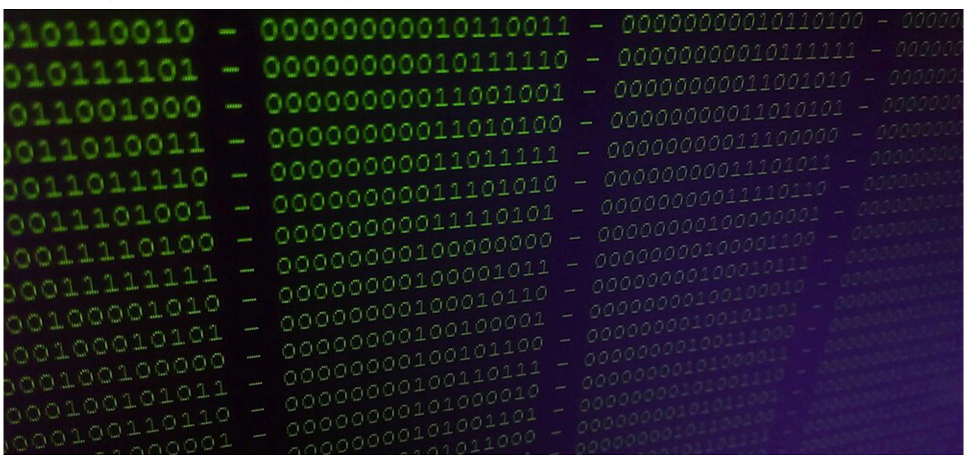


Figure 4.4 Computers use Base 2, which only uses 0's and 1's, to represent quantity. (credit: modification of work "IMAG0933" by yvanhou/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Convert another base to base 10.
2. Write numbers in different base systems.
3. Convert base 10 to other bases.
4. Determine errors in converting between bases.

In our system of numbers, we use base 10, but using base 10 was not a given within other systems. There were other systems that used bases other than 10, as we saw with the Mayans and the Babylonians. The base 10 system comes down to grouping objects in sets of 10, but grouping in sets of 10 only happens if the culture values grouping by that many. We feel 10 is natural because we have 10 fingers. There are other systems using other grouping values, such as 4 or 20.

One good reason for examining other bases is to remind ourselves how we had to learn arithmetic when we were young, memorizing rules for our base 10 system. We had to learn why those arithmetic rules made sense, such as why $1 + 1 = 2$ and $1 + 2 = 3$. Another good reason for learning other base systems is due to computers; their circuitry instead uses base 2.

In this section, we explore other base systems and how to convert between them.

Conversion of Another Base into Base 10 and Other Bases

We saw in Hindu-Arabic Positional System that our Hindu-Arabic system uses **base 10**, which is a system using place values of digits that depend on powers of 10 (or, are based on powers of 10). We've already worked with bases other than base 10: The Babylonian system was base 60, while the Mayan system was base 20.

To explore how our base 10 system is used, answer the following question: What's the following quantity: 4,572? You probably said four thousand five hundred seventy-two (no, there is no “and” between hundred and seventy). But why do you think that 4 means four thousand? A very young person when learning their numbers might say that's a four five seven and two. But you added the context of thousands to the four. Why?

Place value, that's why. You learned early on that where the numeral was gave it different meanings. Ten thousands, thousands, hundreds, tens, and ones. So, you translate that symbol string (4,572) into “four thousand five hundred seventy-two.” As we saw in Hindu-Arabic Positional System, expanding a Hindu-Arabic number involved writing the number using each digit times its appropriate power of 10. So, we could write 4,572 as $4 \times 10^3 + 5 \times 10^2 + 7 \times 10^1 + 2 \times 10^0 = 4 \times 1000 + 5 \times 100 + 7 \times 10 + 2 \times 1$.

One possible reason we use base 10 is that we have 10 fingers, and in the cultures where the Hindu-Arabic system developed, that became the standard. Other cultures may have used other ways of organizing numbers, perhaps using 20 by including toes, or using 60 because 60 has many divisors. Mathematically though, base 10 is an awkward base to work in since 10 has limited divisors. But we think it is easy and simple because that's what we've been taught to use.

Using a base 10 system means we need 10 symbols to make our numbering system work: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Now imagine that we all only had 6 fingers instead of 10 and our counting system was based on those 6 fingers. We would be counting in groups of 6, not groups of 10. How would this change how we work with quantity?

First, we'd need only six symbols. Let's use 0, 1, 2, 3, 4, 5. Second, our place values would be based on powers of 6, not powers of 10. For instance, the number 3,024 in base 6 would be $3 \times 6^3 + 0 \times 6^2 + 2 \times 6^1 + 4 \times 6^0$. That is how you can translate a base 6 number into a base 10 number. When we calculate that expression we get $3 \times 6^3 + 0 \times 6^2 + 2 \times 6^1 + 4 \times 6^0 = 3 \times 216 + 0 \times 36 + 2 \times 6 + 4 \times 1 = 648 + 0 + 12 + 4 = 664$.

This means the base 6 number 3,024 is equal to the base 10 number 664.

From now on, if we are using a base 6 number, we will follow it with the subscript 6, like the following: $3{,}024_6$ means the number is in base 6.

A base 10 number gets no subscript (it's the standard). So, 3,024 is a base 10 number. A base 13 number would be $4{,}672_{13}$.

So, a base 6 system uses only the symbols 0, 1, 2, 3, 4, and 5. Also, the place values use powers of 6. However, we still don't know how to count in base 6. In order to do so, we'd have to know how to represent the quantities larger than five in base 6. Let's review how our base 10 system works by counting from 0 to 100, which shows how larger values are represented.

In writing the base 10 numbers, you start with these first 10 values:

0	1	2	3	4	5	6	7	8	9

But you've run out of symbols. So, we use two digits:

10	11	12	13	14	15	16	17	18	19

The 1 out front means you've run out of digits one time.

But now you've run out twice. Continuing with those numbers gives:

20	21	22	23	24	25	26	27	28	29

And so on,

30	31	32	33	34	etc....

Eventually, you hit the 90s,

90	91	92	93	94	95	96	97	98	99

And you've run out of the digits again! So, we say we've run out of digits in the tens place one time, hence:

100	101	102	103	104	105	106	107	108	109

That's the pattern we use in base 10. We write out the symbols until we've used all the symbols, then add a digit in front that counts how many times we've used the digits. Knowing the numbers, or being able to count higher and higher, is necessary to understand how all the arithmetic works, as it all goes back to counting.

The counting pattern is the same for any other base, including base 6. So, let's start:

0	1	2	3	4	5

But we've run out of symbols! Just like in base 10, we use a second digit, where the first digit will tell us we've run out of symbols one time.

10	11	12	13	14	15

And we use the same pattern:

20	21	22	23	24	25
30	31	32	33	34	35
40	41	42	43	44	45
50	51	52	53	54	55

But we've run out of symbols for that front digit. So, we indicate it the same way as in base 10...by adding a third digit in front, indicating we've run out of symbols once in the second place:

100	101	102	103	104	105
110	111	etc.			

The symbol pattern is the same, but truncated. We only use the six symbols. So that is how we represent base 6. Being able to write out these numbers is important when working with addition in the base.

When using a base larger than 10, though, we need more symbols. Instead of creating new symbols, we use capital letters, with A representing the digit for "10," B representing the digit for "11," and so on.

EXAMPLE 4.12

Determining Digits of a Base with Less Than 10 Digits

What are the digits used for base 7?

Solution

Since this is base 7, we need only 7 symbols: 0, 1, 2, 3, 4, 5, 6.

YOUR TURN 4.12

1. What are the digits for base 4?

EXAMPLE 4.13

Determining Digits of a Base with More Than 10 Digits

What are the digits used for base 14?

Solution

Since this is base 14, we need 14 symbols. We don't have single character numbers for 10, 11, 12, and 13, so, in a fit of inspired creativity, we use capital letters A, B, C, D to represent those quantities. So, the digits in base 14 are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D.

YOUR TURN 4.13

1. What are the digits used for base 12?

WHO KNEW?

Using Base 12

As mentioned in the text, working in base 10 is mathematically awkward. Ten has only two natural number divisors: 2 and 5. This means dividing into groups is not easy. However, 12, or a dozen, has more divisors: 2, 3, 4, and 6. The Dozenal Society recognizes this more mathematically pleasant detail. It advocates for a switch to using base 12 for numbers. Their argument is based on the divisibility of the number 12. But has there ever been a society that used such a system? The answer is yes. A dialect of the Gwandara language in Nigeria uses the base 12 system. It is unlikely, though, that the Dozenal Society will achieve their goal, as the base 10 system is so entrenched in our society.