These connections express basic characteristics of the three-dimensional world in which we live. To which class of particles do tennis balls, people, trees, mountains and all other macroscopic objects belong?

Challenge 118 s

The size and density of matter

The three spatial dimensions have many consequences for physical systems. We know that all matter is made of fermions, such as electrons, protons and neutrons. The exclusion principle has an interesting consequence for systems made of N *identical* fermions; such systems obey the following expression for momentum p and size l:

$$\Delta p \, \Delta l \gtrsim N^{1/3} \hbar \,. \tag{67}$$

Challenge 119 e

Ref. 92

Challenge 120 e

Can you derive it? This *extended indeterminacy relation* provides a simple way to estimate the spatial size of matter systems. In particular, the extended indeterminacy relation implies that the average energy per quanton increases with quanton density. Can you show this?

The extended indeterminacy relation implies that matter systems whose extension is due to electrons – thus all condensed matter systems – essentially have similar matter and energy densities. The extended indeterminacy relation also implies that nuclei, which are composed of protons and neutrons, all have essentially the same matter density.

For bosons, the components of radiation, there is no extended indeterminacy relation, as the number of components N in a particular quantum state does not have any effect or limits. The indeterminacy relation thus does not limit the power density of laser light; and indeed, the power density of laser beams varies much more than the matter density of solids.

The indeterminacy relation highlights a further difference between matter and radiation. As we saw above, a system of N identical bosons, such as a laser beam, obeys an indeterminacy between the number and the phase which is easily derived from the energy–time indeterminacy relation. The number–phase relation can be written, approximately, as

Page 48

$$\Delta N \, \Delta \varphi \gtrsim 1 \,. \tag{68}$$

It is important in the use of lasers in precision experiments. The relation limits how close a system can get to a pure sine wave; indeed for a pure sine wave, the indeterminacy product would be zero.

For fermions, where the maximum number in the same state is 1, the number–phase uncertainty relation reduces to a total uncertainty on the phase. In other words, we find – again – that we cannot have fermion beams that behave as waves. There are no classical fermion waves, no coherent fermion beams, in nature.

Page 101

A summary on spin and indistinguishability

The quantum of action $\hbar$ implies that physical systems are made of two types of indistinguishable quantum particles: *bosons* and *fermions*. The two possible exchange behaviours are related to the particle spin value, because exchange is related to rotation. The connec-

tion between spin and rotation implies that antiparticles exist. It also implies that spin is intrinsically a three-dimensional phenomenon.

Experiments show that radiation is made of elementary particles that behave as bosons. Bosons have integer spin. Two or more bosons, such as two photons, *can share* the same state. This sharing makes laser light possible.

Experiments show that matter is made of elementary particles that behave as fermions. Fermions have half-integer spin. They obey Pauli's exclusion principle: two fermions *cannot be* in the same state. The exclusion principle between electrons explains the structure and (partly) the size of atoms, as well as the chemical behaviour of atoms, as we will find out later on. Together with the electrostatic repulsion of electrons, the exclusion principle explains the incompressibility of matter and its lack of impenetrability.

Fermions make matter 'hard', bosons allow light beams to cross.

Limits and open questions of quantum statistics

The topic of quantum particle statistics remains a research field in theoretical and experimental physics. In particular, researchers have searched and still are searching for generalizations of the possible exchange behaviours of particles.

In two spatial dimensions, the effect of a particle exchange on the wave function is a continuous phase, in contrast to three dimensions, where the result is a sign. Two-dimensional quantum objects are therefore called *anyons* because they can have 'any' spin. Anyons appear as quasi-particles in various experiments in solid state physics, because the set-up is often effectively two-dimensional. The *fractional quantum Hall effect*, perhaps the most interesting discovery of modern experimental physics, has pushed anyons onto the stage of modern research.

Page 137
Vol. V, page 107

Other theorists generalized the concept of fermions in other ways, introducing parafermions, parabosons, plektons and other hypothetical concepts. Oscar Greenberg has spent most of his professional life on this issue. His conclusion is:

Ref. 93

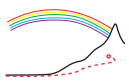
▷ In 3 + 1 space-time dimensions, only fermions and bosons exist.

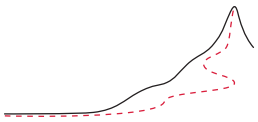
Can you show that this result implies that the ghosts appearing in Scottish tales do not exist?

Challenge 121 s

From a different viewpoint, the belt model of spin 1/2 invites to study the behaviour of braids, open links and knots. (In mathematics, braids and open links are made of strands extending to infinity.) This fascinating part of mathematical physics has become important with in modern unified theories, which all state that particles, especially at high energies, are not point-like, but extended entities. The quest is to understand what happens to permutation symmetry in a unified theory of nature. A glimpse of the difficulties appears already above: how can Figures 60, 65 and 70 be reconciled and combined? We will settle this issue in the final part of our mountain ascent.

Vol. VI, page 174





Chapter 7

SUPERPOSITIONS AND PROBABILITIES – QUANTUM THEORY WITHOUT IDEOLOGY

Ref. 94

> The fact that an adequate philosophical presentation has been so long delayed is no doubt caused by the fact that Niels Bohr brainwashed a whole generation of theorists into thinking that the job was done fifty years ago.
>
> Murray Gell-Mann

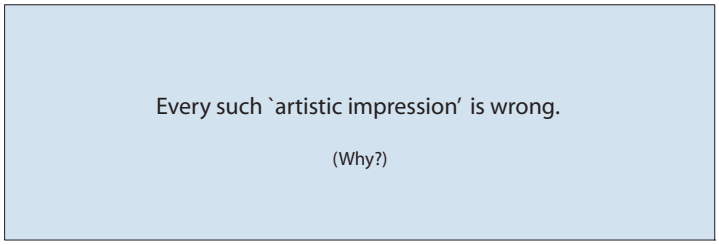
Why is this famous physical issue arousing such strong emotions? In particular, who is brainwashed, Gell-Mann, the discoverer of the quarks, or most of the world's physicists working on quantum theory who follow Niels Bohr's opinion? In the twentieth century, quantum mechanics has thrown many in disarray. We have a simple aim: we want to *understand* quantum theory. Quantum mechanics is unfamiliar for two reasons: it allows *superpositions* and it leads to *probabilities*. In this chapter we explore and clarify these two topics – until we understand quantum theory.

Probabilities appear whenever an aspect of a microscopic system is measured. The quantum of action, the smallest change value found in nature, leads to the appearance of probabilities in measurements.

Superpositions appear because the quantum of action radically changed the two most basic concepts of classical physics: state and system. The *state* is not defined and described any more by the specific values taken by position and momentum, but by the specific wave function 'taken' by the position and momentum operators.** In addition, in classical physics a *system* was described and defined as a set of permanent aspects of nature; permanence was defined as negligible interaction with the environment. Quantum mechanics shows that these definitions have to be modified.

Clarifying the origin of superpositions and probabilities, as well as the concepts of system and state, will help us to avoid getting lost on our way to the top of Motion Mountain. Indeed, quite a number of researchers have lost their way since quantum theory appeared, including important physicists like Murray Gell-Mann and Steven Weinberg.

** It is equivalent, and often conceptually clearer, to say that the state is described by a complete set of commuting operators. In fact, the discussion of states is somewhat simplified in the Heisenberg picture. However, here we study the issue in the Schrödinger picture only, i.e., using wave functions.



Challenge 122 s

FIGURE 72 An artist's impression of a macroscopic superposition is impossible – because such superpositions are not found in our environment.

Why are people either dead or alive?

The evolution equation of quantum mechanics is *linear* in the wave function; the linearity reflects the existence of superpositions. Superpositions imply that we can imagine and try to construct systems where the state ψ is a superposition of two radically distinct situations, such as those of a dead and of a living cat. This famous fictional animal is called *Schrödinger's cat* after the originator of the example. Is it possible to produce it? And how would it evolve in time? We can ask the same two questions in other situations. For example, can we produce a superposition of a state where a car is inside a closed garage with a state where the car is outside? What happens then?

Macroscopic superpositions are strange. Such situations are not observed in everyday life, and only very rarely in the laboratory. The reason for this rareness is an important aspect of what is often called the 'interpretation' of quantum mechanics. In fact, such strange situations *are* possible, and the superposition of macroscopically distinct states has actually been observed in a few cases, though not for cats, people or cars. To get an idea of the constraints, let us specify the situation in more detail.*

Macroscopic superpositions, coherence and incoherence

The object of discussion are linear superpositions of the type $\psi = a\psi_a + b\psi_b$, where ψ_a and ψ_b are macroscopically distinct states of the system under discussion, and where a and b are some complex coefficients. States are called *macroscopically distinct* when each state corresponds to a different macroscopic situation, i.e., when the two states can be distinguished using the concepts or measurement methods of classical physics. In particular, this means that the physical action necessary to transform one state into the other must be much larger than $\hbar$. For example, two different positions of a body composed of a large number of atoms are macroscopically distinct. The state of a cat that is living and of the same cat when it is dead also differ by many quanta of action.

A 'strange' situation is thus a superposition of macroscopically distinct states. Let us work out the essence of such macroscopic superpositions more clearly. Given two macroscopically distinct states ψ_a and ψ_b, any superposition of the type $\psi = a\psi_a + b\psi_b$ is called a *pure state*. Since the states ψ_a and ψ_b can interfere, one also talks about a *(phase) coherent superposition*. In the case of a superposition of macroscopically distinct states,

* Most what can be said about this topic has been said by three important researchers: Niels Bohr, one of the fathers of quantum physics, John von Neumann, who in the nineteen-thirties stressed the differences between evolution and decoherence, and by Heinz Dieter Zeh, who in the nineteen-seventies stressed the importance of baths and the environment in the decoherence process.

Ref. 95
Ref. 96

the scalar product $\psi_a^\dagger \psi_b$ is obviously vanishing. In case of a coherent superposition, the coefficient product $a^* b$ is different from zero. This fact can also be expressed with the help of the *density matrix* ρ of the system, defined as $\rho = \psi \otimes \psi^\dagger$. In the present case it is given by

$$\begin{aligned} \rho_{\text{pure}} = \psi \otimes \psi^\dagger &= |a|^2 \psi_a \otimes \psi_a^\dagger + |b|^2 \psi_b \otimes \psi_b^\dagger + a\, b^* \psi_a \otimes \psi_b^\dagger + a^*\, b\, \psi_b \otimes \psi_a^\dagger \\ &= (\psi_a, \psi_b) \begin{pmatrix} |a|^2 & a\, b^* \\ a^*\, b & |b|^2 \end{pmatrix} \begin{pmatrix} \psi_a^\dagger \\ \psi_b^\dagger \end{pmatrix} . \end{aligned} \tag{69}$$

We can then say that whenever the system is in a pure, or coherent state, then its density matrix, or *density functional*, contains off-diagonal terms of the same order of magnitude as the diagonal ones.* Such a density matrix corresponds to the above-mentioned strange situations that we never observe in daily life. We will shortly understand why.

We now have a look at the opposite situation, a density matrix for macroscopic distinct states with *vanishing* off-diagonal elements. For two states, the example

$$\begin{aligned} \rho_{\text{mixed}} &= |a|^2 \psi_a \otimes \psi_a^\dagger + |b|^2 \psi_b \otimes \psi_b^\dagger \\ &= (\psi_a, \psi_b) \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} \begin{pmatrix} \psi_a^\dagger \\ \psi_b^\dagger \end{pmatrix} \end{aligned} \tag{71}$$

describes a system which possesses *no* phase coherence at all. (Here, $\otimes$ denotes the non-commutative dyadic product or tensor product which produces a tensor or matrix starting from two vectors.) Such a diagonal density matrix cannot be that of a pure state; the density matrix describes a system which is in the state ψ_a with probability $|a|^2$ and which is in the state ψ_b with probability $|b|^2$. Such a system is said to be in a *mixed state*, because its state is *not known*, or equivalently, is in a *(phase) incoherent superposition*: interference effects cannot be observed in such a situation. A system described by a mixed state is always *either* in the state ψ_a *or* in the state ψ_b. In other words, a diagonal density matrix for macroscopically distinct states is not in contrast, but in agreement with everyday experience.

In the picture of density matrices, the non-diagonal elements contain the difference between normal, i.e., incoherent, and unusual or strange, i.e., coherent, superpositions.

The experimental situation is clear: for macroscopically distinct states, only diagonal density matrices are observed in everyday life. Almost all systems in a coherent macroscopic superposition somehow lose their off-diagonal matrix elements. How does this process of *decoherence* – also called *disentanglement* in certain settings – take place? The density matrix itself shows the way.

* Using the density matrix, we can rewrite the evolution equation of a quantum system:

$$\dot{\psi} = -iH\psi \quad \text{becomes} \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{i}{\hbar}[H, \rho] . \tag{70}$$

Both are completely equivalent. (The new expression is sometimes also called the *von Neumann equation*.) We won't actually do any calculations here. The expressions are given so that you recognize them when you encounter them elsewhere.

Decoherence is due to baths

Ref. 97 Challenge 123 ny

In thermodynamics, the density matrix ρ for a large system is used for the definition of its entropy S – and of all its other thermodynamic quantities. These studies show that

$$S = -k \operatorname{tr}(\rho \ln \rho) \tag{72}$$

where tr denotes the *trace*, i.e., the sum of all diagonal elements, and k is the Boltzmann constant. The expression is thus the quantum mechanical definition of entropy.

We now remind ourselves that a physical system with a large and constant entropy is called a *bath*. In simple physical terms, a bath is a system to which we can ascribe a temperature. More precisely,

> A *(physical) bath* – also called a *(thermodynamic) reservoir* – is any large system for which the concept of *equilibrium* can be applied.

Experiments show that in practice, this is equivalent to the condition that a bath consists of many interacting subsystems. For this reason, all macroscopic quantities describing the state of a bath show small, irregular *fluctuations*, a property that will be of central importance shortly.

An everyday bath is also a physical bath: indeed, a thermodynamic bath is similar to an extremely large warm water bath, one for which the temperature does not change even if we add some cold or warm water to it. The physical concept of bath, or reservoir, is thus an abstraction and a generalization of the everyday concept of bath. Other examples of physical baths are: an intense magnetic field, a large amount of gas, or a large solid. (The meanings of 'intense' and 'large' of course depend on the system under study.)

Challenge 124 s

The definition (72) of entropy tells us that the loss of off-diagonal elements corresponds to an increase in entropy. In addition, any increase in entropy of a reversible system, such as the quantum mechanical system in question, is due to an interaction with a bath.

In short, *decoherence is due to interaction with a bath*. In addition, decoherence is a process that increases entropy: decoherence is irreversible. We will now show that baths are everywhere, that decoherence thus takes place everywhere and all the time, and that therefore, macroscopic superpositions are (almost) never observed.

How baths lead to decoherence – scattering

Where is the bath interacting with a typical system? The bath must be outside the system we are talking about, i.e., in its *environment*. Indeed, we know experimentally that a typical environment is large and characterized by a temperature. Some examples are listed in Table 6. In short,

▷ Any environment is a bath.

We can even go further: for every experimental situation, there is a bath *interacting* with the system under study. Indeed, every system which can be observed is not isolated, as it obviously interacts at least with the observer; and every observer by definition contains

TABLE 6 Common and less common baths with their main properties.

Bath type	Temperature	Wavelength	Particle flux	Cross section (atom)	Hit time $1/\sigma\varphi$ for	
	T	λ_{eff}	φ	σ	atom[a]	ball[a]
matter baths						
solid, liquid	300 K	10 pm	$10^{31}\,/\text{m}^2\text{s}$	$10^{-19}\,\text{m}^2$	10^{-12} s	10^{-25} s
air	300 K	10 pm	$10^{28}\,/\text{m}^2\text{s}$	$10^{-19}\,\text{m}^2$	10^{-9} s	10^{-22} s
laboratory vacuum	50 mK	10 µm	$10^{18}\,/\text{m}^2\text{s}$	$10^{-19}\,\text{m}^2$	10 s	10^{-12} s
photon baths						
sunlight	5800 K	900 nm	$10^{23}\,/\text{m}^2\text{s}$		10^{-4} s	10^{-17} s
'darkness'	300 K	20 µm	$10^{21}\,/\text{m}^2\text{s}$		10^{-2} s	10^{-15} s
cosmic microwaves	2.7 K	2 mm	$10^{17}\,/\text{m}^2\text{s}$		10^{2} s	10^{-11} s
terrestrial radio waves						
Casimir effect					very large	
Unruh radiation of Earth	40 zK				very large	
nuclear radiation baths						
radioactivity		10 fm	$1\,/\text{m}^2\text{s}$	$10^{-25}\,\text{m}^2$	10^{25} s	10^{12} s
cosmic radiation	>1000 K	10 fm	$10^{-2}\,/\text{m}^2\text{s}$	$10^{-25}\,\text{m}^2$	10^{27} s	10^{14} s
solar neutrinos	≈ 10 MK	10 fm	$10^{11}\,/\text{m}^2\text{s}$	$10^{-47}\,\text{m}^2$	10^{36} s	10^{15} s
cosmic neutrinos	2.0 K	3 mm	$10^{17}\,/\text{m}^2\text{s}$	$10^{-62}\,\text{m}^2$	10^{45} s	10^{24} s
gravitational baths						
gravitational radiation	$5 \cdot 10^{31}$ K	10^{-35} m			very large	

a. Values are rough estimates. The macroscopic ball is assumed to have a 1 mm size.

a bath, as we will show in more detail shortly. Usually however, the most important baths we have to take into consideration are the atmosphere around a system, the radiation or electromagnetic fields interacting with the system, or, if the system itself is large enough to have a temperature, those degrees of freedom of the system which are not involved in the superposition under investigation.

Since every physical system is in contact with a bath, every density matrix of a macroscopic superposition will lose its diagonal elements eventually. At first sight, this direction of thought is not convincing. The interactions of a system with its environment can be made extremely small by using clever experimental set-ups; that would imply that the time for decoherence can be made extremely large. Thus we need to check how much time a superposition of states needs to decohere. It turns out that there are two standard ways to estimate the *decoherence time*: either by modelling the bath as large number of colliding particles, or by modelling it as a continuous field.

If the bath is described as a set of particles randomly hitting the microscopic system, it is best characterized by the effective wavelength λ_{eff} of the particles and by the average interval t_{hit} between two hits. A straightforward calculation shows that the decoherence

Challenge 125 ny

time t_d is in any case smaller than this time interval, so that

$$t_d \leqslant t_{\text{hit}} = \frac{1}{\varphi\sigma} \,, \tag{73}$$

where φ is the flux of particles and σ the cross-section for the hit.* Typical values are given in Table 6. We easily note that for macroscopic objects, decoherence times are extremely short. (We also note that nuclear and gravitational effects lead to large decoherence times and thus can be neglected.) *Scattering leads to fast decoherence of macroscopic systems.* However, for atoms or smaller systems, the situation is different, as expected. *Microscopic systems can show long decoherence times.*

We note that the quantum of action $\hbar$ appears in the expression for the decoherence time, as it appears in the area σ. Decoherence is a quantum process.

How baths lead to decoherence – relaxation

A second method to estimate the decoherence time is also common. Any interaction of a system with a bath is described by a relaxation time t_r. The term *relaxation* designates any process which leads to the return to the equilibrium state. The terms *damping* and *friction* are also used. In the present case, the relaxation time describes the return to equilibrium of the combination bath and system. Relaxation is an example of an irreversible evolution. A process is called *irreversible* if the reversed process, in which every component moves in opposite direction, is of very low probability.** For example, it is usual that a glass of wine poured into a bowl of water colours the whole water; it is very rarely observed that the wine and the water separate again, since the probability of all water and wine molecules to change directions together at the same time is rather low, a state of affairs making the happiness of wine producers and the despair of wine consumers.

Now let us simplify the description of the bath. We approximate it by a single, unspecified, scalar field which interacts with the quantum system. Due to the continuity of space, such a field has an infinity of degrees of freedom. They are taken to model the many degrees of freedom of the bath. The field is assumed to be in an initial state where its degrees of freedom are excited in a way described by a temperature T. The interaction of the system with the bath, which is at the origin of the relaxation process, can be described by the repeated transfer of small amounts of energy E_{hit} until the relaxation

* The decoherence time is derived by studying the evolution of the density matrix $\rho(x, x')$ of objects localized at two points x and x'. One finds that the off-diagonal elements follow $\rho(x, x', t) = \rho(x, x', 0)\mathrm{e}^{-\Lambda t(x-x')^2}$, where the localization rate Λ is given by

$$\Lambda = k^2 \varphi \sigma_{\text{eff}} \tag{74}$$

where k is the wave number, φ the flux and σ_{eff} the cross-section of the collisions, i.e., usually the size of the macroscopic object. Ref. 98

One also finds the surprising result that a system hit by a particle of energy E_{hit} *collapses* the density matrix roughly down to the de Broglie (or thermal de Broglie) wavelength of the hitting particle. Both results together give the formula above. Ref. 99

** Beware of other definitions which try to make something deeper out of the concept of irreversibility, such as claims that 'irreversible' means that the reversed process is *not at all* possible. Many so-called 'contradictions' between the irreversibility of processes and the reversibility of evolution equations are due to this mistaken interpretation of the term 'irreversible'.

process is completed.

The objects of interest in this discussion, like the mentioned cat, person or car, are described by a mass m. Their main characteristic is the maximum energy E_r which can be transferred from the system to the environment. This energy describes the interactions between system and environment. The superpositions of macroscopic states we are interested in are solutions of the Hamiltonian evolution of these systems.

The initial coherence of the superposition, so disturbingly in contrast with our every-
Ref. 100 day experience, disappears exponentially within a *decoherence time* t_d given by*

$$t_d = t_r \frac{E_{\text{hit}}}{E_r} \frac{e^{E_{\text{hit}}/kT} - 1}{e^{E_{\text{hit}}/kT} + 1} \tag{77}$$

where k is again the *Boltzmann constant* and like above, E_r is the maximum energy which can be transferred from the system to the environment. Note that one always has $t_d \leqslant t_r$. After the decoherence time t_d is elapsed, the system has evolved from the coherent to the incoherent superposition of states, or, in other words, the density matrix has lost its off-diagonal terms. One also says that the phase coherence of this system has been destroyed. Thus, after a time t_d, the system is found either in the state ψ_a or in the state ψ_b, respectively with the probability $|a|^2$ or $|b|^2$, and not any more in a coherent superposition which is so much in contradiction with our daily experience. Which final state is selected depends on the precise state of the bath, whose details were eliminated from the calculation by taking an *average* over the states of its microscopic constituents.

The important result is that for all *macroscopic* objects, the decoherence time t_d is extremely small. In order to see this more clearly, we can study a special simplified case. A macroscopic object of mass m, like the mentioned cat or car, is assumed to be at the same time in two locations separated by a distance l, i.e., in a superposition of the two corresponding states. We further assume that the superposition is due to the object moving as a quantum mechanical oscillator with frequency ω between the two locations; this is the simplest possible system that shows superpositions of an object located in two different positions. The energy of the object is then given by $E_r = m\omega^2 l^2$, and the smallest transfer energy $E_{\text{hit}} = \hbar\omega$ is the difference between the oscillator levels. In a macroscopic situation, this last energy is much smaller than kT, so that from the preceding expression we
Ref. 102 get

$$t_d = t_r \frac{E_{\text{hit}}^2}{2E_r\, kT} = t_r \frac{\hbar^2}{2mkTl^2} = t_r \frac{\lambda_T^2}{l^2} \tag{78}$$

* This result is derived as in the above case. A system interacting with a bath always has an evolution given
Ref. 101 by the general form

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] - \frac{1}{2t_o}\sum_j [V_j\rho, V_j^\dagger] + [V_j, \rho V_j^\dagger]\,, \tag{75}$$

where ρ is the density matrix, H the Hamiltonian, V the interaction, and t_o the characteristic time of the
Challenge 126 ny interaction. Are you able to see why? Solving this equation, one finds for the elements far from the diagonal $\rho(t) = \rho_0 e^{-t/t_0}$. In other words, they disappear with a characteristic time t_o. In most situations one has a relation of the form

$$t_0 = t_r \frac{E_{\text{hit}}}{E_r} = t_{\text{hit}} \tag{76}$$

or some variations of it, as in the example above.

in which the frequency ω has disappeared. The quantity $\lambda_T = \hbar/\sqrt{2mkT}$ is called the *thermal de Broglie wavelength* of a particle.

We note again that the quantum of action $\hbar$ appears in the expression for the decoherence time. Decoherence is a quantum process.

It is straightforward to see that for practically all macroscopic objects the typical decoherence time t_d is extremely short. For example, setting $m = 1\,\mathrm{g}$, $l = 1\,\mathrm{mm}$ and $T = 300\,\mathrm{K}$ we get $t_d/t_r = 1.3 \cdot 10^{-39}$. Even if the interaction between the system and the environment would be so weak that the system would have as relaxation time the age of the universe, which is about $4 \cdot 10^{17}\,\mathrm{s}$, the time t_d would still be shorter than $5 \cdot 10^{-22}\,\mathrm{s}$, which is over a million times faster than the oscillation time of a beam of light (about 2 fs for green light). For Schrödinger's cat, the decoherence time would be even shorter. These times are so short that we cannot even hope to *prepare* the initial coherent superposition, let alone to observe its decay or to measure its lifetime.

For microscopic systems however, the situation is different. For example, for an electron in a solid cooled to liquid helium temperature we have $m = 9.1 \cdot 10^{-31}\,\mathrm{kg}$, and typically $l = 1\,\mathrm{nm}$ and $T = 4\,\mathrm{K}$; we then get $t_d \approx t_r$ and therefore the system can stay in a coherent superposition until it is relaxed, which confirms that for this case coherent effects can indeed be observed if the system is kept isolated. A typical example is the behaviour of electrons in superconducting materials. We will mention a few more below.

Ref. 103

In 1996 the first actual measurement of decoherence times was published by the Paris team led by Serge Haroche. It confirmed the relation between the decoherence time and the relaxation time, thus showing that the two processes have to be distinguished at microscopic scale. In the meantime, many other experiments confirmed the decoherence process with its evolution equation, both for small and large values of t_d/t_r. A particularly beautiful experiment has been performed in 2004, where the disappearance of two-slit interference for C_{70} molecules was observed when a bath interacts with them.

Ref. 104

Ref. 105

Ref. 106

Summary on decoherence, life and death

Our exploration showed that *decoherence results from coupling to a bath in the environment.* Decoherence is a quantum statistical effect, i.e., a thermodynamic effect. Decoherence follows from quantum theory, is an irreversible process, and thus occurs automatically. Above all, decoherence is a process that has been observed in experiments.

The estimates of decoherence times in everyday life told us that both the preparation and the survival of superpositions of macroscopically different states is made *impossible* by the interaction with any bath found in the environment. This is the case even if the usual measure of this interaction, given by the friction of the motion of the system, is very small. Even if a macroscopic system is subject to an extremely low friction, leading to a very long relaxation time, its decoherence time is still vanishingly short. Only carefully designed microscopic systems in expensive laboratory set-ups can reach substantial decoherence times.

Our everyday environment is full of baths. Therefore,

▷ Coherent superpositions of macroscopically distinct states never appear in everyday life, due to the rapid decoherence times induced by baths in the environment.

Cars cannot be in and out of a garage at the same time. We cannot be dead and alive at the same time. An illustration of a macroscopic superposition – see Figure 122 – is impossible. In agreement with the explanation, coherent superpositions of macroscopic states appear in some special laboratory situations.

Page 155

What is a system? What is an object?

In classical physics, a system is a part of nature that can be isolated from its environment. However, quantum mechanics tells us that isolated systems do not exist, since interactions cannot be made vanishingly small. The contradiction can be solved with the results above: they allow us to define the concept of system with more accuracy.

▷ A *system* is any part of nature that interacts *incoherently* with its environment.

This implies:

▷ An *object* is a part of nature interacting with its environment only through baths.

In particular, we get:

▷ A system is called *microscopic* or *quantum mechanical* and can described by a wave function ψ whenever
  — it is almost isolated, with $t_{\text{evol}} = \hbar/\Delta E < t_{\text{r}}$, and
  — it is in *incoherent* interaction with its environment.

Ref. 107

In short, a microscopic or quantum mechanical system can be described by a wave function only if it interacts incoherently and weakly with its environment. (For such a system, the energy indeterminacy ΔE is larger than the relaxation energy.) In contrast, a bath is never isolated in the sense just given, because the evolution time of a bath, the time scale during which its properties change, is always much larger than its relaxation time. Since all macroscopic bodies are in contact with baths – or even contain one – they cannot be described by a wave function. In particular, it is impossible to describe any measuring apparatus with the help of a wave function.

We thus conclude:

▷ A *macroscopic system* is a system with a decoherence time much shorter than any other evolution time of its constituents.

Obviously, macroscopic systems also interact incoherently with their environment. Thus cats, cars and television news speakers are all macroscopic systems.

Entanglement

One possibility is left over by the two definitions of system: what happens in the situation in which the interactions with the environment are *coherent*? We will encounter some ex-

amples shortly. Following the definitions, they are neither microscopic nor macroscopic systems.

▷ A 'system' in which the interaction with its environment is coherent is called *entangled*.

Such 'systems' are not described by a wave function, and strictly speaking, they are *not* systems. In these situations, when the interaction is coherent, one speaks of *entanglement*. For example, one says that a particle or set of particles is said to be *entangled* with its environment.

Entangled, i.e., coherently interacting systems can be divided, but must be disentangled when doing so. The act of division leads to detached entities; detached entities interact incoherently. Quantum theory shows that nature is not made of detached entities, but that it is made of *detachable* entities. In quantum theory, the criterion of detachment is the incoherence of interaction. Coherent superpositions imply the surprising consequence that there are systems which, even though they look being made of detached parts, are not. Entanglement poses a limit to detachment. All surprising properties of quantum mechanics, such as Schrödinger's cat, are consequences of the classical prejudice that a system made of two or more parts can obviously be detached into two subsystems without disturbance. But coherent superpositions, or entangled systems, do not allow detachment without disturbance. Whenever we assume to be able to detach entangled systems, we get strange or incorrect conclusions, such as apparent faster-than-light propagation, or, as one says today, non-local behaviour. Let us have a look at a few typical examples.

Entangled situations are observed in many experiments. For example, when an electron and a positron annihilate into two photons, the polarisations of these two photons are entangled, as measured already in 1949. Also when an excited atom decays in steps, emitting two photons, the photon polarisations are entangled, as was first shown in 1966 with the help of calcium atoms. Similarly, when an unstable molecule in a singlet state, i.e., in a spin 0 state, decays or splits into debris, the spins of the debris are entangled, as observed in the 1970s. Also the spontaneous parametric down-conversion of photons produces entanglement. In a non-linear optical material, an incoming photon is converted into two outgoing photons whose added energies correspond to the energy of the incoming photon. In this case, the two outgoing photons are entangled both in their polarisation and in their direction. In 2001, the spins of two extremely cold caesium gas samples, with millions of atoms each and located a few millimetres apart, have been entangled. Also position entanglement has been regularly observed, for example for closely spaced ions inside ion traps.

Is quantum theory non-local? A bit about the Einstein–Podolsky–Rosen paradox

> "[Mr. Duffy] lived a little distance away from his body ..."
>
> James Joyce, *A Painful Case*

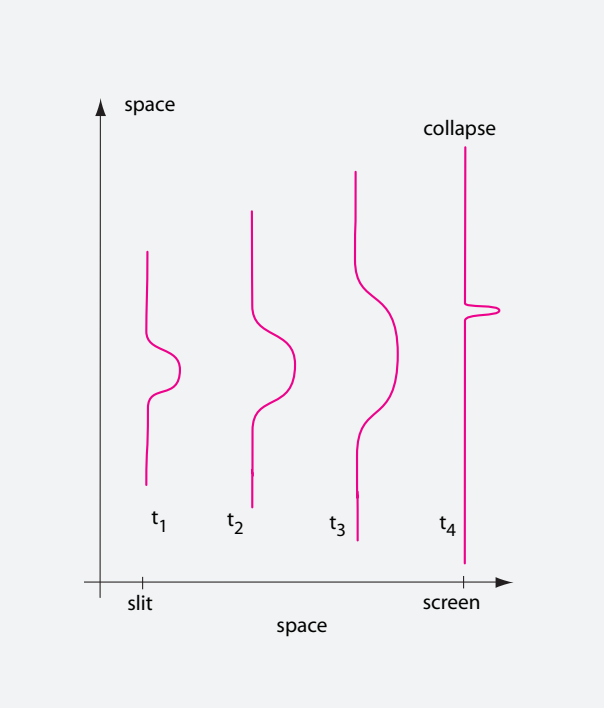


FIGURE 73 Quantum mechanical motion: an electron wave function (actually its module squared) from the moment it passes a slit until it hits a screen.

It is often suggested, incorrectly, that wave function collapse or quantum theory are non-local.* The issue needs clarification.

We start by imagining an electron hitting a screen after passing a slit. Following the description just deduced, the collapse process proceeds schematically as depicted in Figure 73. An animation that includes another example of a collapse process – inspired by Bohm's thought experiment – can be seen in the lower left corners on these pages, starting at page 115. The collapse process has a surprising side: due to the shortness of the decoherence time, during this (and any other) wave function collapse the maximum of the wave function usually changes position faster than light. Is this reasonable?

Page 154

A situation is called *acausal* or *non-local* if energy is transported faster than light. Using Figure 73 you can determine the energy velocity involved, using the results on signal propagation. The result is a value smaller than c. A wave function whose maximum moves faster than light does *not automatically* imply that energy moves faster than light.

Challenge 127 s
Vol. III, page 133

In other words, quantum theory contains speeds greater than light, but no *energy* speeds greater than light. In classical electrodynamics, the same happens with the scalar and the vector potentials if the Coulomb gauge is used. We have also encountered speeds faster than that of light in the motion of shadows and scissors, and in many other observations. Any physicist now has two choices: he can be straight, and say that there is no non-locality in nature; or he can be less straight, and claim there is. In the latter

Ref. 108
Vol. II, page 58

* This continues a topic that we know already: we have explored a different type of non-locality, in general relativity, earlier on.

Vol. II, page 284

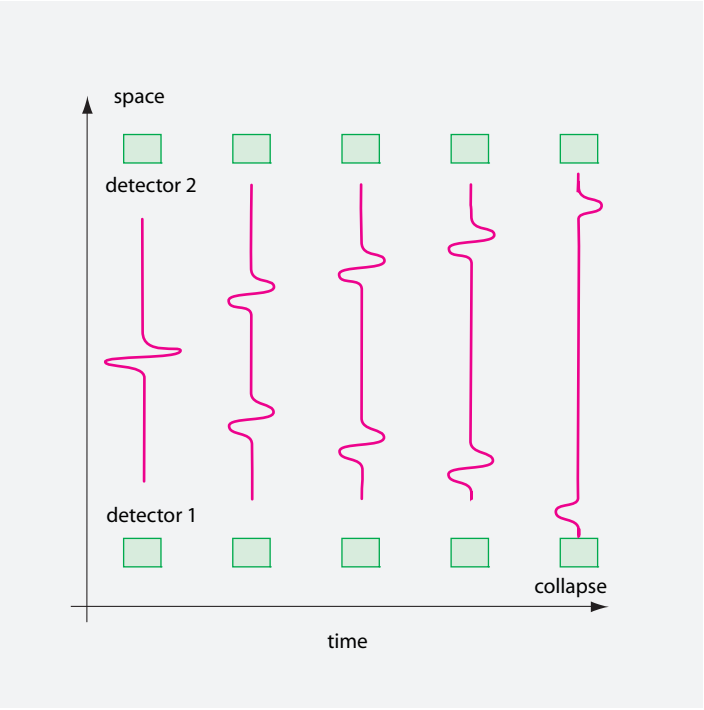


FIGURE 74 Bohm's thought experiment.

case, he has to claim that even classical physics is non-local. However, nobody dares to claim this. In fact, there is a danger in this more provoking usage of the term 'non-local': a small percentage of those who claim that the world is non-local after a while start to believe that there really is faster-than-light energy transport in nature. These people become prisoners of their muddled thinking. On the other hands, muddled thinking helps to get more easily into newspapers. In short, even though the definition of non-locality is not unanimous, here we stick to the stricter one, and define non-locality as *energy* transport faster than light.

Ref. 109, Ref. 110

An often cited thought experiment that shows the pitfalls of non-locality was proposed by Bohm* in the discussion around the so-called Einstein–Podolsky–Rosen paradox. In the famous EPR paper the three authors tried to find a contradiction between quantum mechanics and common sense. Bohm translated their rather confused paper into a clear thought experiment that is shown schematically in Figure 74. When two particles in a spin 0 state move apart, measuring one particle's spin orientation implies an *immediate* collapse also of the other particle's spin, namely in the exactly opposite direction. This happens instantaneously over the whole separation distance; no speed limit is obeyed. In other words, entanglement seems to lead to faster-than-light communication.

However, in Bohm's experiment, no energy is transported faster than light. No non-locality is present, despite numerous claims of the contrary by certain authors. The two

* David Joseph Bohm (1917–1992), was an influential physicist. He codiscovered the Aharonov–Bohm effect and spent a large part of his later life investigating the connections between quantum physics and philosophy.

entangled electrons belong to one system: assuming that they are separate only because the wave function has two distant maxima is a conceptual mistake. In fact, no signal can be transmitted with this method; the decoherence is a case of prediction which looks like a signal without being one. Bohm's experiment, like any other EPR-like experiment, does not allow communication faster than light. We already discussed such cases in the section on electrodynamics.

Vol. III, page 136

Bohm's experiment has actually been performed. The first and most famous realization was due, in 1982, by Alain Aspect; he used photons instead of electrons. Like all latter tests, it has fully confirmed quantum mechanics.

Ref. 111

In fact, experiments such as the one by Aspect confirm that it is impossible to treat either of the two particles as a system by itself; it is impossible to ascribe any physical property, such as a spin orientation, to either of them alone. (The Heisenberg picture would express this restriction even more clearly.) Only the two electrons together form a physical system, because only the pair interacts incoherently with the environment.

The mentioned two examples of apparent non-locality can be dismissed with the remark that since obviously no energy flux faster than light is involved, no problems with causality appear. Therefore the following example is more interesting. Take two identical atoms, one in an excited state, one in the ground state, and call l the distance that separates them. Common sense tells that if the first atom returns to its ground state emitting a photon, the second atom can be excited only after a time $t = l/c$ has been elapsed, i.e., after the photon has travelled to the second atom.

Surprisingly, this conclusion is wrong. The atom in its ground state has a non-zero probability to be excited at the same moment in which the first is de-excited. This has been shown most simply by Gerhard Hegerfeldt. The result has also been confirmed experimentally.

Ref. 112

More careful studies show that the result depends on the type of superposition of the two atoms at the beginning: coherent or incoherent. For incoherent superpositions, the intuitive result is correct; the counter-intuitive result appears only for coherent superpositions. Again, a careful discussion shows that no real non-locality of energy is involved.

In summary, faster-than-light speeds in wave function collapse do not contradict the limit on energy speed of special relativity. Collapse speeds are phase velocities. In nature, phase velocities are unlimited; unlimited phase velocities never imply energy transport faster than light. In addition, we recover the result that physical systems are only clearly defined if they interact incoherently with their environment.

Curiosities and fun challenges about superpositions

Some people wrongly state that atoms in a superposition of two states centred at different positions can be photographed. (This lie is even used by some sects to attract believers.) Why is this not true?

Challenge 128 s

* *

In a few cases, the superposition of different macroscopic states can actually be observed by lowering the temperature to sufficiently small values and by carefully choosing suitably small masses or distances. Two well-known examples of coherent superpositions are those observed in gravitational wave detectors and in Josephson junctions. In the

Ref. 102 first case, one observes a mass as heavy as 1000 kg in a superposition of states located at different points in space: the distance between them is of the order of 10^{-17} m. In the second case, in superconducting rings, superpositions of a state in which a macroscopic current of the order of 1 pA flows in clockwise direction with one where it flows Ref. 113 in counter-clockwise direction have been produced.

* *

Ref. 114 Superpositions of magnetization in up and down direction at the same time have been observed for several materials.

* *

Since the 1990s, the sport of finding and playing with new systems in coherent mac- Ref. 115 roscopic superpositions has taken off across the world. The challenges lie in the clean experiments necessary. Experiments with single atoms in superpositions of states are Ref. 116 among the most popular ones.

* *

Ref. 117 In 1997, coherent atom waves were extracted from a cloud of sodium atoms.

* *

Macroscopic objects usually are in incoherent states. This is the same situation as for light. The world is full of 'macroscopic', i.e., incoherent light: daylight, and all light from lamps, from fire and from glow-worms is incoherent. Only very special and carefully constructed sources, such as lasers or small point sources, emit coherent light. Only these sources allow studying interference effects. In fact, the terms 'coherent' and 'incoherent' originated in optics, since for light the difference between the two, namely the capacity to interfere, had been observed centuries before the case of matter.

Coherence and incoherence of light and of matter manifest themselves differently, because matter can stay at rest but light cannot and because matter is made of fermions, Page 139 but light is made of bosons. Coherence can be observed easily in systems composed of bosons, such as light, sound in solids, or electron pairs in superconductors. Coherence is less easily observed in systems of fermions, such as systems of atoms with their electron clouds. However, in both cases a decoherence time can be defined. In both cases coherence in many particle systems is best observed if all particles are in the same state (superconductivity, laser light) and in both cases the transition from coherent to incoherent is due to the interaction with a bath. A beam is thus incoherent if its particles arrive randomly in time and in frequency. In everyday life, the rarity of observation of coherent matter superpositions has the same origin as the rarity of observation of coherent light.

* *

We will discuss the relation between the environment and the *decay* of unstable systems Vol. V, page 47 later on. The phenomenon is completely described by decoherence.

* *

Challenge 129 ny Can you find a method to measure the *degree* of entanglement? Can you do so for a system made of many particles?

* *

The study of entanglement leads to a simple conclusion: *teleportation contradicts correlation*. Can you confirm the statement?

Challenge 130 ny

* *

Are ghost images in TV sets, often due to spurious reflections, examples of interference?

Challenge 131 s

* *

What happens when two monochromatic electrons overlap?

Challenge 132 d

* *

Some people say that quantum theory could be used for quantum computing, by using coherent superpositions of wave functions. Can you give a general reason that makes this aim very difficult – even though not impossible – even without knowing how such a quantum computer might work, or what the so-called *qubits* might be?

Ref. 118

Challenge 133 s

Why do probabilities and wave function collapse appear in measurements?

Measurements in quantum mechanics are puzzling also because they lead to statements in which *probabilities* appear. For example, we speak about the probability of finding an electron at a certain distance from the nucleus of an atom. Statements like this belong to the general type 'when the observable A is measured, the probability to find the outcome a is p.' In the following we will show that the probabilities in such statements are inevitable for any measurement, because, as we will show, (1) any measurement and any observation is a special case of decoherence or disentanglement process and (2) all decoherence processes imply the quantum of action. (Historically, the process of measurement was studied *before* the more general process of decoherence. That explains in part why the topic is so confused in many peoples' minds.)

What is a measurement? As already mentioned earlier on, a measurement is any interaction which produces a record or a memory. (Any effect of everyday life is a record; but this is not true in general. Can you give some examples of effects that are records and some effects which are not?) Measurements can be performed by machines; when they are performed by people, they are called *observations*. In quantum theory, the process of measurement is not as straightforward as in classical physics. This is seen most strikingly when a quantum system, such as a single electron, is first made to pass a diffraction slit, or better – in order to make its wave aspect become apparent – a double slit and then is made to hit a photographic plate, in order to make also its particle aspect appear. Experiment shows that the blackened dot, the spot where the electron has hit the screen, cannot be determined in advance. (The same is true for photons or any other particle.) However, for large numbers of electrons, the spatial distribution of the black dots, the so-called *diffraction pattern*, can be calculated in advance with high precision.

Vol. III, page 267

Challenge 134 s

The outcome of experiments on microscopic systems thus forces us to use probabilities for the description of microsystems. We find that the probability distribution $p(\boldsymbol{x})$ of the spots on the photographic plate can be calculated from the wave function ψ of the electron at the screen surface and is given by $p(\boldsymbol{x}) = |\psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x})|^2$. This is in fact a special

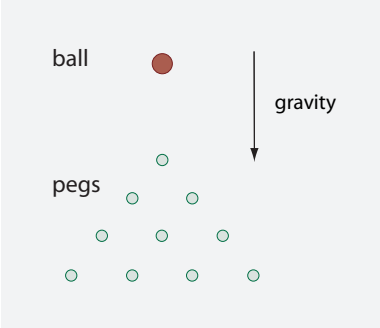


FIGURE 75 A system showing probabilistic behaviour: ball falling through an array of pegs.

case of the general *first property of quantum measurements*:

▷ The measurement of an observable A for a system in a state ψ gives as result one of the eigenvalues a_n, and the probability P_n to get the result a_n is given by

$$P_n = |\varphi_n^\dagger \psi|^2 \,, \tag{79}$$

where φ_n is the eigenfunction of the operator A corresponding to the eigenvalue a_n.*

Experiments also show a *second property of quantum measurements:*

▷ After a measurement, the observed quantum system is in the state φ_n corresponding to the measured eigenvalue a_n. One also says that during the measurement, the wave function has *collapsed* from ψ to φ_n.

Ref. 119

These two experimental properties can also be generalized to the more general cases with degenerate and continuous eigenvalues.

Obviously, the experimental results on the measurement process require an explanation. At first sight, the sort of probabilities encountered in quantum theory are different from the probabilities we encounter in everyday life. Take roulette, dice, the system shown in Figure 75, pachinko machines or the direction in which a pencil on its tip falls: all have been measured experimentally to be random (assuming no cheating by the designer or operators) to a high degree of accuracy. These everyday systems do not puzzle us. We unconsciously assume that the random outcome is due to the small, but uncontrollable variations of the starting conditions or the environment every time the experiment is repeated.**

* All linear transformations transform some special vectors, called *eigenvectors* (from the German word *eigen* meaning 'self') into multiples of themselves. In other words, if T is a transformation, e a vector, and

$$T(e) = \lambda e \tag{80}$$

where λ is a scalar, then the vector e is called an *eigenvector* of T, and λ is associated *eigenvalue*. The set of all eigenvalues of a transformation T is called the *spectrum* of T.

** To get a feeling for the limitations of these unconscious assumptions, you may want to read the already

But microscopic systems seem to be different. The two properties of quantum measurements just mentioned express what physicists observe in every experiment, even if the initial conditions are taken to be *exactly* the same every time. But why then is the position for a single electron, or most other observables of quantum systems, not predictable? In other words, what happens during the collapse of the wave function? How long does the collapse take? In the beginning of quantum theory, there was the perception that the observed unpredictability is due to the lack of information about the state of the particle. This lead many to search for so-called 'hidden variables'. All these attempts were doomed to fail, however. It took some time for the scientific community to realize that the unpredictability is *not* due to the lack of information about the state of the particle, which is indeed described *completely* by the state vector ψ.

In order to uncover the origin of probabilities, let us recall the nature of a measurement, or better, of a general observation.

▷ Any observation is the production of a record.

The record can be a visual or auditive memory in our brain, or a written record on paper, or a tape recording, or any such type of object. As explained in the previous volume, an object is a record if it cannot have arisen or disappeared by chance. To avoid the influence of chance, all records have to be protected as much as possible from the *external* world; e.g. one typically puts archives in earthquake safe buildings with fire protection, keeps documents in a safe, avoids brain injury as much as possible, etc.

Vol. III, page 265

On top of this, records have to be protected from their *internal* fluctuations. These internal fluctuations are due to the many components any recording device is made of. If the fluctuations were too large, they would make it impossible to distinguish between the possible contents of a memory. Now, fluctuations decrease with increasing size of a system, typically with the square root of the size. For example, if a hand writing is too small, it is difficult to read if the paper gets brittle; if the magnetic tracks on tapes are too small, they demagnetize and lose the stored information. In other words, a record is rendered stable against internal fluctuations by making it of sufficient size. Every record thus consists of many components and shows small fluctuations.

The importance of size can be expressed in another way: every system with memory, i.e., every system capable of producing a record, contains a *bath*. In summary, the statement that any observation is the production of a record can be expressed more precisely as:

▷ Any observation of a system is the result of an interaction between that system and a bath in the recording apparatus.

By the way, since baths imply friction, we can also say: memory needs friction. In addition, any observation measuring a physical quantity uses an interaction *depending* on that same quantity. With these seemingly trivial remarks, we can describe in more detail

mentioned story of those physicists who built a machine that could predict the outcome of a roulette ball from the initial velocity imparted by the croupier.

Vol. I, page 127

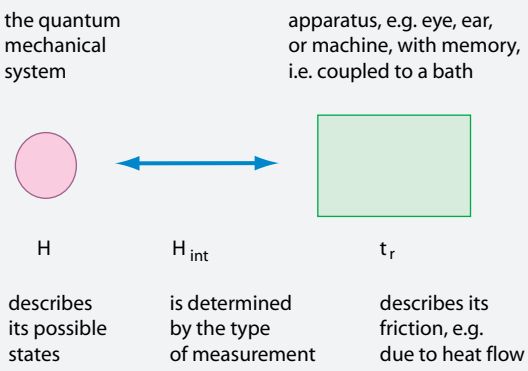


FIGURE 76 The concepts used in the description of measurements.

the process of observation, or, as it is usually called in the quantum theory, the measurement process.

Any measurement *apparatus*, or *detector*, is characterized by two main aspects, shown in Figure 76: the interaction it has with the microscopic system, and the bath it contains to produce the record. Any description of the measurement process thus is the description of the evolution of the microscopic system *and* the detector; therefore one needs the Hamiltonian for the particle, the interaction Hamiltonian, and the bath properties (such as the relaxation time t_r). The interaction specifies what is measured and the bath realizes the memory.

Ref. 120

We know that only classical thermodynamic systems can be irreversible; quantum systems are not. We therefore conclude: a measurement system *must* be described classically: otherwise it would have no memory and would not be a measurement system: it would not produce a record! Memory is a classical effect. (More precisely, memory is an effect that only appears in the classical limit.) Nevertheless, let us see what happens if we describe the measurement system quantum mechanically.

Let us call A the observable which is measured in the experiment and its eigenfunctions φ_n. We describe the quantum mechanical system under observation – often a particle – by a state ψ. The full state of the system can always be written as

$$\psi = \psi_p \, \psi_{\text{other}} = \sum_n c_n \varphi_n \, \psi_{\text{other}} \,. \tag{81}$$

Here, ψ_p is the aspect of the (particle or system) state that we want to measure, and ψ_{other} represents all other degrees of freedom, i.e., those not described – *spanned*, in mathematical language – by the operator A corresponding to the observable we want to measure. The numbers $c_n = |\varphi_n^\dagger \psi_p|$ give the expansion of the state ψ_p, which is taken to be normalized, in terms of the basis φ_n. For example, in a typical position measurement, the functions φ_n would be the position eigenfunctions and ψ_{other} would contain the information about the momentum, the spin and all other properties of the particle.

How does the system–detector interaction look like? Let us call the state of the apparatus before the measurement χ_{start}. The measurement apparatus itself, by definition, is a device which, when it is hit by a particle in the state $\varphi_n\psi_{\text{other}}$, changes from the state χ_{start} to the state χ_n. One then says that the apparatus has *measured* the eigenvalue a_n corresponding to the eigenfunction φ_n of the operator A. The index n is thus the record of the measurement; it is called the *pointer* index or variable. This index tells us in which state the microscopic system was before the interaction. The important point, taken from our previous discussion, is that the states χ_n, being records, are macroscopically distinct, precisely in the sense of the previous section. Otherwise they would not be records, and the interaction with the detector would not be a measurement.

Of course, during measurement, the apparatus sensitive to φ_n changes the part ψ_{other} of the particle state to some other situation $\psi_{\text{other},n}$, which depends on the measurement and on the apparatus; we do not need to specify it in the following discussion.* But let us have an intermediate check of our reasoning. Do apparatuses as described here exist? Yes, they do. For example, any photographic plate is a detector for the position of ionizing particles. A plate, and in general any apparatus measuring position, does this by changing its momentum in a way depending on the measured position: the electron on a photographic plate is stopped. In this case, χ_{start} is a white plate, φ_n would be a particle localized at spot n, χ_n is the function describing a plate blackened at spot n and $\psi_{\text{other,n}}$ describes the momentum and spin of the particle after it has hit the photographic plate at the spot n.

Now we are ready to look at the measurement process itself. For the moment, let us disregard the bath in the detector, and let us just describe it with a state as well, which we call χ_{start}. In the time before the interaction between the particle and the detector, the combined system (including the detector) was in the initial state ψ_i given simply by

$$\psi_i = \psi_p\chi_{\text{start}} = \sum_n c_n\varphi_n\psi_{\text{other}}\chi_{\text{start}} \,, \tag{84}$$

where ψ_p is the (particle or system) state. After the interaction, using the just mentioned, experimentally known characteristics of the apparatus, the combined state ψ_a is

$$\psi_a = \sum_n c_n\varphi_n\psi_{\text{other},n}\chi_n \,. \tag{85}$$

This evolution from ψ_i to ψ_a follows from the evolution equation applied to the particle–detector combination. Now, the combined state ψ_a is a superposition of macroscopically

* How does the interaction look like mathematically? From the description we just gave, we specified the final state for every initial state. Since the two density matrices are related by

$$\rho_{\text{f}} = T\rho_{\text{i}}T^{\dagger} \tag{82}$$

Challenge 135 ny we can deduce the Hamiltonian from the matrix T. Are you able to see how?

By the way, one can say in general that an apparatus measuring an observable A has a system interaction Hamiltonian depending on the pointer variable A, and for which one has

$$[H + H_{\text{int}}, A] = 0 \,. \tag{83}$$

distinct states: it is a superposition of distinct macroscopic states of the detector. In our example ψ_a could correspond to a superposition of one state where a spot on the left upper corner is blackened on an otherwise white plate with another state where a spot on the right lower corner of the otherwise white plate is blackened. Such a situation is never observed. Let us see why.

The density matrix ρ_a of the combined state ψ_a after the measurement given by

$$\rho_a = \psi_a \otimes \psi_a^\dagger = \sum_{n,m} c_n c_m^* (\varphi_n \psi_{\text{other},n} \chi_n) \otimes (\varphi_m \psi_{\text{other},m} \chi_m)^\dagger \,, \tag{86}$$

contains large non-diagonal terms, i.e., terms for $n \neq m$, whose numerical coefficients are different from zero. Now let us take the bath back in. From the previous section we know the effect of a bath on such a macroscopic superposition. We found that a density matrix such as ρ_a decoheres extremely rapidly. We assume here that the decoherence time is negligibly small.* After decoherence, the off-diagonal terms vanish, and only the final, diagonal density matrix ρ_f, given by

$$\rho_f = \sum_n |c_n|^2 (\varphi_n \psi_{\text{other},n} \chi_n) \otimes (\varphi_n \psi_{\text{other},n} \chi_n)^\dagger \tag{87}$$

remains and has experimental relevance. As explained above, such a density matrix describes a mixed state, and the numbers $P_n = |c_n|^2 = |\varphi_n^\dagger \psi_p|^2$ give the probability of measuring the value a_n and of finding the particle in the state $\varphi_n \psi_{\text{other,n}}$ as well as the detector in the state χ_n. But this is precisely what the two properties of quantum measurements state.

We therefore find that describing a measurement as an evolution of a quantum system interacting with a macroscopic detector, itself containing a bath, we can *deduce* the two properties of quantum measurements, probabilistic outcomes and the collapse of the wave function, from the quantum mechanical evolution equation. The decoherence time t_d of the previous section becomes the time of collapse for the case of a measurement; in addition we find

$$t_{\text{collapse}} = t_d < t_r \,. \tag{88}$$

In other words, the collapse time is always smaller than the relaxation time of the bath. We thus have a formula for the time the wave function takes to collapse. All experimental measurements of the time of collapse have confirmed this result.

Ref. 104

Why is $\hbar$ necessary for probabilities?

At first sight, one could argue that the two properties of quantum measurements do not contain $\hbar$, and thus are not consequences of quantum theory. However, this argument is incorrect.

* Note however, that an *exactly* vanishing decoherence time, which would mean a *strictly* infinite number of degrees of freedom of the bath or the environment, is in contradiction with the evolution equation, and in particular with unitarity, locality and causality. It is essential in the whole argument not to confuse the logical consequences of a extremely small decoherence time with those of an exactly vanishing decoherence time.

Decoherence is a quantum process, because $\hbar$ appears in the expression of the decoherence time. Since the collapse of the wave function is based on decoherence, it is a quantum process as well. Also probabilities are due to the quantum of action.

In addition, we have seen that the concept of wave function appears only because the quantum of action $\hbar$ is not zero. Wave functions, their collapse and probabilities are due to the quantum of change $\hbar$.

Page 87

These results recall a statement made earlier on: probabilities appear whenever an experiment attempts to detect changes, i.e., action values, smaller than $\hbar$. Most puzzles around measurement are due to such attempts. However, nature does not allow such measurements; in every such attempt, probabilities appear.

Page 32

Challenge 136 e

Hidden variables

A large number of people are not satisfied with the explanation of probabilities in the quantum world. They long for more mystery in quantum theory. They do not like the idea that probabilities are due to baths and to the quantum of action. The most famous prejudice such people cultivate is the idea that the probabilities are due to some hidden aspect of nature which is still unknown to humans. Such imagined, unknown aspects are called *hidden variables*.

The beautiful thing about quantum mechanics is that it allows both conceptual and experimental tests on whether such hidden variables exist – without the need of knowing them. Obviously, hidden variables controlling the evolution of microscopic system would contradict the statement that action values below $\hbar$ cannot be detected. The smallest observable action value is the reason for the random behaviour of microscopic systems. The smallest action thus *excludes* hidden variables. But let us add some more detailed arguments.

Historically, the first, somewhat abstract argument against hidden variables was given by John von Neumann.* An additional no-go theorem for hidden variables was published by Kochen and Specker in 1967, and independently by John Bell in 1969. The theorem states:

Ref. 121

▷ Non-contextual hidden variables are impossible, if the Hilbert space has a dimension equal or larger than three.

The theorem is about *non-contextual* variables, i.e., about hidden variables *inside* the quantum mechanical system. The Kochen–Specker theorem thus states that there is no non-contextual hidden variables model, because mathematics forbids it. This result essentially eliminates all possibilities for hidden variables, because usual quantum mechanical systems have Hilbert space dimensions larger than three.

* János Neumann (b. 1903 Budapest, d. 1957 Washington DC) influential mathematician. One of the greatest and clearest scientific minds of the twentieth century, he settled many issues, especially in applied mathematics and quantum theory, that others still struggle with today. He then worked on the atomic and the hydrogen bomb, on ballistic missiles, and on general defence problems. For the bomb research, he strongly influenced the building of the earliest electronic computers, extending the ideas of Konrad Zuse. At the end of his life, he wanted to change the weather with nuclear bombs. He died of a cancer that was due to his exposure to nuclear radiation when watching bomb tests.

We cannot avoid noting that there are no restricting theorems about *contextual* hidden variables, i.e., variables in the environment and in particular, in the baths contained in it. Indeed, their necessity was shown above!

Also common sense eliminates hidden variables, without any recourse to mathematics, with a simple argument. If a quantum mechanical system had internal hidden variables, the measurement apparatus would have zillions of them.* And this would mean that it could not work as a measurement system.

Despite all arguments, researchers have always been looking for experimental tests on hidden variables. Most tests are based on the famed *Bell's inequality*, a beautifully simple relation published by John Bell** in the 1960s.

Can we distinguish quantum theory and locally realistic theories that use hidden variables? Bell's starting idea is to do so by measuring the polarizations of two correlated photons. Quantum theory says that the polarization of the photons is fixed only at the time it is measured, whereas local realistic models – the most straightforward type of hidden variable models – claim that the polarization is fixed already in advance by a hidden variable. As Bell found out, experiments can be used to decide which alternative is correct.

Imagine that the polarization is measured at two distant points A and B. Each observer can measure 1 or −1 in each of his favourite direction. Let each observer choose two directions, 1 and 2, and call their results a_1, a_2, b_1 and b_2. Since the measurement results all are either 1 or −1, the value of the specific expression $(a_1 + a_2)b_1 + (a_2 - a_1)b_2$ has always the value ±2.

Ref. 122 Imagine that you repeat the experiment many times, assuming that the hidden variables appear statistically. You then can deduce (a special case of) Bell's inequality for two Challenge 137 e *hidden variables*; it predicts that

$$|(a_1 b_1) + (a_2 b_1) + (a_2 b_2) - (a_1 b_2)| \leqslant 2 \,. \tag{89}$$

Here, the expressions in brackets are the averages of the measurement products over a large number of samples. This hidden variable prediction holds independently of the directions of the involved polarizers.

On the other hand, for the case that the polarizers 1 and 2 at position A and the corresponding ones at position B are chosen with angles of $\pi/4$, *quantum theory* predicts that

$$|(a_1 b_1) + (a_2 b_1) + (a_2 b_2) - (a_1 b_2)| = 2\sqrt{2} > 2 \,. \tag{90}$$

This prediction is in complete contradiction with the hidden variable prediction.

Now, all experimental checks of Bell's inequality have confirmed standard quantum mechanics and falsified hidden variables. There are no exceptions.

Another measurable contradiction between quantum theory and locally realistic theories has been predicted by Greenberger, Horn and Zeilinger in systems with three en- Ref. 123 tangled particles. Again, quantum theory has been confirmed in all experiments.

* Which leads to the definition: one zillion is 10^{23}.

** John Stewart Bell (1928–1990), theoretical physicist who worked mainly on the foundations of quantum theory.

In summary, *hidden variables do not exist*. Of course, this is not really surprising. The search for hidden variables is based on a misunderstanding of quantum mechanics or on personal desires on how the world should be, instead of taking it as it is: there is a smallest measurable action value, $\hbar$, in nature.

Summary on probabilities and determinism

> Geometrica demonstramus quia facimus; si physica demonstrare possemus, faceremus.
>
> Giambattista Vico*

We draw a number of conclusions which we need for the rest of our mountain ascent. Note that these conclusions, even though in agreement with all experiments, are not yet shared by all physicists! The whole topic is a problem for people who prefer ideology to facts.

In everyday life, probabilities often do not appear or are not noted. Quantum theory shows:

- ▷ Probabilities appear whenever a process tries to distinguish between situations that differ by about one quantum of action $\hbar$.
- ▷ The precise mechanism for the appearance of probabilities is due to the involved baths.

In short: probabilities appear whenever an experiment tries to distinguish between *close situations*. In more detail:

— Probabilities do not appear in measurements because the state of the quantum system is unknown or fuzzy, but because the detailed state of an interacting bath in the environment is unknown. Quantum mechanical probabilities are of statistical origin and are due to baths in the environment or in the measurement apparatus, in combination with the quantum of action $\hbar$. The probabilities are due to the large number of degrees of freedom contained in the bath. These large numbers make the outcome of experiments – especially those whose possible outcomes differ by about $\hbar$ – unpredictable. If the state of the involved bath were known, the outcome of an experiment could be predicted. The probabilities of quantum theory are due to the quantum of action and are 'thermodynamic' in origin.

In other words, there are *no fundamental* probabilities in nature. All probabilities in nature are due to decoherence; in particular, all probabilities are due to the statistics of the many particles – some of which may even be virtual – that are part of the baths in the environment. Modifying well-known words by Albert Einstein, we can agree on the following: 'nature does not play dice.' Therefore we called ψ the *wave function* – instead of 'probability amplitude', as is often done. An even better term would be *state function*.

* 'We are able to demonstrate geometrical matters because we make them; if we could prove physical matters we would be able to make them.' Giovanni Battista Vico (b. 1668 Napoli, d. 1744 Napoli) important philosopher and thinker. In this famous statement he points out a fundamental distinction between mathematics and physics.

— Every observation in everyday life is a special case of decoherence. What is usually called the 'collapse of the wave function' is a decoherence process due to the interaction with the baths present in the environment or in the measuring apparatus. Because humans are warm-blooded and have memory, humans themselves are measurement apparatuses. The fact that our body temperature is 37 °C is thus the reason that we see only a single world, and no superpositions. (Actually, there are many additional reasons; can you name a few?)

Challenge 138 s

— Every measurement is complete when the microscopic system has interacted with the bath in the measuring apparatus. Quantum theory as a description of nature does not require detectors; the evolution equation describes all examples of motion. However, *measurements* do require the existence of detectors. A detector, or measurement apparatus, is a machine that records observations. Therefore, it has to include a bath, i.e., has to be a classical, macroscopic object. In this context one speaks also of a *classical apparatus*. This necessity of the measurement apparatus to be classical had been already stressed in the very early stages of quantum theory.

— All measurements, being decoherence processes that involve interactions with baths, are irreversible processes and increase entropy.

— Every measurement, like every example of decoherence, is a special case of quantum mechanical evolution, namely the evolution for the combination of a quantum system, a macroscopic detector and a bath. Since the evolution equation is relativistically invariant, no causality problems appear in measurements; neither do locality problems or logical problems appear.

— Since both the evolution equation and the measurement process do not involve quantities other than space-time, Hamiltonians, baths and wave-functions, no other quantity plays a role in measurement. In particular, no human observer nor any consciousness is involved or necessary. Every measurement is complete when the microscopic system has interacted with the bath in the apparatus. The decoherence inherent in every measurement takes place even if nobody is looking. This trivial consequence is in agreement with the observations of everyday life, for example with the fact that the Moon is orbiting the Earth even if nobody looks at it.* Similarly, a tree falling in the middle of a forest makes noise even if nobody listens. Decoherence is independent of human observation, of the human mind and of human existence.

Vol. III, page 339

— In every measurement the quantum system interacts with the detector. Since there is a minimum value for the magnitude of action, every *observation influences the observed.* Therefore every measurement *disturbs* the quantum system. Any precise description of observations must also include the description of this disturbance. In the present section the disturbance was modelled by the change of the state of the system from ψ_{other} to $\psi_{\text{other,n}}$. Without such a change of state, without a disturbance of the quantum system, a measurement is impossible.

— Since the complete measurement process is described by quantum mechanics, unitarity is and remains the basic property of evolution. There are no non-unitary processes in quantum mechanics.

— The description of the collapse of the wave function as a decoherence process is an

* The opposite view is sometimes falsely attributed to Niels Bohr. The Moon is obviously in contact with many radiation baths. Can you list a few?

Challenge 139 s

explanation exactly in the sense in which the term 'explanation' was defined earlier on; it describes the relation between an observation and all the other aspects of reality, in this case the bath in the detector or the environment. The collapse of the wave function has been measured, calculated and *explained*. The collapse is not a question of 'interpretation', i.e., of opinion, as unfortunately often is suggested.*

Vol. III, page 334

— It is not useful to speculate whether the evolution for a *single* quantum measurement could be determined if the state of the environment around the system were known. Measurements need baths. But a bath is, to an excellent approximation, irreversible and thus cannot be described by a wave function, which behaves reversibly.**

In short:

- ▷ Quantum mechanics is deterministic.
- ▷ Baths are probabilistic.
- ▷ Baths are probabilistic because of the quantum of action.

In summary, there is *no* irrationality in quantum theory. Whoever uses quantum theory as argument for superstitions, irrational behaviour, new age beliefs or ideologies is guilty of disinformation. The statement by Gell-Mann at the beginning of this chapter is such an example. Another is the following well-known, but incorrect statement by Richard Feynman:

Page 143

Ref. 125

> ... nobody understands quantum mechanics.

Nobel Prizes obviously do not prevent views distorted by ideology. The correct statement is:

- ▷ The quantum of action and decoherence are the key to understanding quantum theory.

In fact, $\hbar$ and decoherence allow clarifying many other issues. We explore a few interesting ones.

What is the difference between space and time?

Space and time differ. Objects are localized in space but not in time. Why is this the case? In nature, most bath–system interactions are mediated by a potential. All potentials are by definition position dependent. Therefore, every potential, being a function of the position $\boldsymbol{x}$, commutes with the position observable (and thus with the interaction Hamiltonian). The decoherence induced by baths – except if special care is taken – thus first of all destroys the non-diagonal elements for every superposition of states centred

* This implies that the so-called 'many worlds' interpretation is wishful thinking. The conclusion is confirmed when studying the details of this religious approach. It is a belief system, not based on facts.

Ref. 124

** This very strong type of determinism will be very much challenged in the last part of this text, in which it will be shown that time is not a fundamental concept, and therefore that the debate around determinism looses most of its interest.

at different locations. In short, *objects are localized because they interact with baths via potentials.*

For the same reason, objects also have only one spatial orientation at a time. If the system–bath interaction is spin-dependent, the bath leads to 'localization' in the spin variable. This occurs for all microscopic systems interacting with magnets. As a result, macroscopic superpositions of magnetization are almost never observed. Since electrons, protons and neutrons have a magnetic moment and a spin, this conclusion can even be extended: everyday objects are never seen in superpositions of different rotation states because their interactions with baths are spin-dependent.

As a counter-example, most systems are not localized in time, but on the contrary exist for very long times, because practically all system–bath interactions do *not* commute with time. In fact, this is the way a bath is defined to begin with. In short, *objects are permanent because they interact with baths.*

Are you able to find an interaction which is momentum-dependent instead of position-dependent? What is the consequence for macroscopic systems?

Challenge 140 s

In other words, in contrast to general relativity, quantum theory produces a distinction between space and time. In fact, we can *define* position as the observable that commutes with interaction Hamiltonians. This distinction between space and time is due to the properties of matter and its interactions. We could not have deduced this distinction in general relativity.

Are we good observers?

Are humans classical apparatuses? Yes, they are. Even though several prominent physicists claim that free will and probabilities are related, a detailed investigation shows that this in not the case. Our senses are classical machines because they obey their definition: human senses record observations by interaction with a bath. Our brain is also a classical apparatus: the neurons are embedded in baths. Quantum probabilities do *not* play a determining role in the brain.

Ref. 126

Any observing entity, be it a machine or a human being, needs a bath and a memory to record its observations. This means that observers have to be made of matter; an observer cannot be made of radiation. Our description of nature is thus severely biased: we describe it from the standpoint of matter. That is a bit like describing the stars by putting the Earth at the centre of the universe: we always put matter at the centre of our description. Can we eliminate this basic anthropomorphism? We will find out as we continue our adventure.

Challenge 141 e

Vol. VI, page 83

What relates information theory, cryptology and quantum theory?

Physics means talking about observations of nature. Like any observation, also measurements produce information. It is thus possible to translate much (but not all) of quantum theory into the language of information theory. In particular, the existence of a smallest change value in nature implies that the information about a physical system can never be complete, that information transport has its limits and that information can never be fully trusted. The details of these studies form a fascinating way to look at the microscopic world.

The analogy between quantum theory and information theory becomes even more interesting when the statements are translated into the language of cryptology. Cryptology is the science of transmitting hidden messages that only the intended receiver can decrypt. In our modern times of constant surveillance, cryptology is an important tool to protect personal freedom.*

Ref. 127

The quantum of action implies that messages can be sent in an (almost) safe way. Listening to a message is a measurement process. Since there is a smallest action $\hbar$, we can detect whether somebody has tried to listen to a message that we sent. To avoid a man-in-the-middle attack – somebody who pretends to be the receiver and then sends a copy of the message to the real, intended receiver – we can use *entangled* systems as signals or messages to transmit the information. If the entanglement is destroyed, somebody has listened to the message. Usually, quantum cryptologists use communication systems based on entangled photons.

The major issue of *quantum cryptology*, a large modern research field, is the key distribution problem. All secure communication is based on a secret key that is used to decrypt the message. Even if the communication channel is of the highest security – like entangled photons – one still has to find a way to send the communication partner the secret key necessary for the decryption of the messages. Finding such methods is the main aspect of quantum cryptology. However, close investigation shows that all key exchange methods are limited in their security.

In short, due to the quantum of action, nature provides limits on the possibility of sending encrypted messages. The statement of these limits is (almost) equivalent to the statement that change in nature is limited by the quantum of action.

Is the universe a computer?

The quantum of action provides a limit to secure information exchange. This connection allows us to brush aside several incorrect statements often found in the media. Stating that 'the universe is information' or that 'the universe is a computer' is as reasonable as saying that the universe is an observation or a chewing-gum dispenser. Any expert of motion should beware of these and similarly fishy statements; people who use them either deceive themselves or try to deceive others.

Vol. VI, page 109

Does the universe have a wave function? And initial conditions?

The wave function of the universe is frequently invoked in discussions about quantum theory. Various conclusions are deduced from this idea, for example on the irreversibility of time, on the importance of initial conditions, on changes required to quantum theory and much more. Are these arguments correct?

The first thing to clarify is the meaning of 'universe'. As explained already, the term can have two meanings: either the collection of all matter and radiation, or this collection *plus* all of space-time. Let us also recall the meaning of 'wave function': it describes the

Vol. II, page 223

* Cryptology consists of the field of *cryptography*, the art of coding messages, and the field of *cryptoanalysis*, the art of deciphering encrypted messages. For a good introduction to cryptology, see the text by Albrecht Beutelspacher, Jörg Schwenk & Klaus-Dieter Wolfenstätter, *Moderne Verfahren der Kryptographie*, Vieweg 1995.

state of a system. The state distinguishes two otherwise identical systems; for example, position and velocity distinguish two otherwise identical ivory balls on a billiard table. Alternatively and equivalently, the state describes changes in time.

Does the universe have a state? If we take the wider meaning of universe, it does not. Vol. I, page 27 Talking about the state of the universe is a contradiction: by definition, the concept of state, defined as the non-permanent aspects of an object, is applicable only to *parts* of the universe.

We then can take the narrower sense of 'universe' – the sum of all matter and radiation only – and ask the question again. To determine the state of all matter and radiation, we need a possibility to measure it: we need an environment. But the environment of matter and radiation is space-time only; initial conditions cannot be determined since we need measurements to do this, and thus an apparatus. An apparatus is a material system with a bath attached to it; however, there is no such system outside the universe.

In short, quantum theory does not allow for measurements of the universe.

▷ The universe has no state.

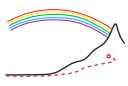
Beware of anybody who claims to know something about the wave function of the universe. Just ask him Wheeler's question: If you know the wave function of the universe, why aren't you rich?

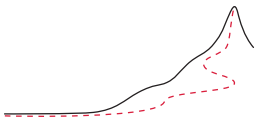
Despite this conclusion, several famous physicists have proposed evolution equations for the wave function of the universe. (The best-known is, ironically, the Wheeler–Ref. 128 DeWitt equation.) It seems a silly point, but not one prediction of these equations has been compared to experiment; the arguments just given even make this impossible in principle. Exploring such equations, so interesting it may seem at first sight, must therefore be avoided if we want to complete our adventure and avoid getting lost in false beliefs.

There are many additional twists to this story. One is that space-time itself, even without matter, might be a bath. This speculation will be shown to be correct in the last volume of this adventure. The result seems to allow speaking of the wave function of the universe. But then again, it turns out that time is *undefined* at the scales where space-time is an effective bath; this again implies that the concept of state is not applicable there.

A lack of 'state' for the universe is a strong statement. It also implies a lack of initial conditions! The arguments are precisely the same. This is a tough result. We are so used to think that the universe has initial conditions that we never question the term. (Even in this text the mistake might appear every now and then.) But there are no initial conditions for the universe.

We can retain as summary, valid even in the light of the latest research: The universe is *not* a system, has *no* wave function and *no* initial conditions – independently of what is meant by 'universe'.





Chapter 8

COLOURS AND OTHER INTERACTIONS BETWEEN LIGHT AND MATTER

> Rem tene; verba sequentur.**
>
> Cato

Stones and all other objects have colours. Why? In other words, what is the specific way in which charged quantum particles that are found inside stones and inside all other objects interact with electromagnetic fields? In this chapter, we first give an overview of the various ways that colours in nature result from the quantum of action, i.e., from the interaction between matter quantons and photons. Then we explore the simplest such system: we show how the quantum of action leads to the various colours produced by hydrogen atoms. After this, we discover that the interaction between matter and radiation leads to other surprising effects, especially when special relativity is taken into account.

The causes of colour

Quantum theory explains all colours in nature. Indeed, all the colours that we observe are due to electrically *charged* particles. More precisely, colours are due to the interactions of charged particles with photons. All colours are thus quantum effects.

So far, we have explored the motion of quantons that are described by mass only. Now we study the motion of particles that are electrically charged. The charged particles at the basis of most colours are electrons and nuclei, including their composites, from ions, atoms and molecules to fluids and solids. Many colour issues are still topic of research. For example, until recently it was unclear why exactly asphalt is black. The exact structure of the chemical compounds, the *asphaltenes*, that produce the very dark brown colour was unknown. Only recent research has settled this question. In fact, the development of new colourants and colour effects is an important part of modern industry.

Ref. 130

An overview of the specific mechanisms that generate colour is given in the following table. The table includes all colours that appear in everyday life. (Can you find one that is missing?)

Ref. 129

Challenge 142 s

** 'Know the subject and the words will follow.' Marcus Porcius Cato, (234–149 BCE) or Cato the elder, Roman politician famous for his speeches and his integrity.

TABLE 7 Causes of colour.

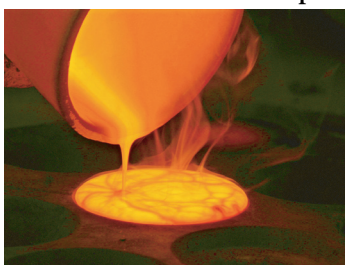
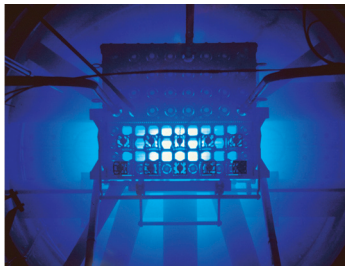
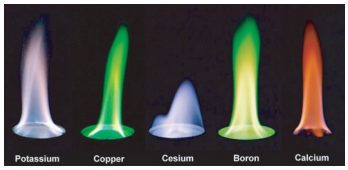
Colour type	Example	Details
Class I: Colours due to simple excitations		
	1. Incandescence and free charge radiation	
	Carbon arc lamp, hot steel, lightbulb wire, most stars, magma, lava, hot melts	Colours are due to continuous spectrum emitted by all hot matter; colour sequence, given by Wien's rule, is black, red, orange, yellow, white, blue-white (molten lead and silver © Graela)
	Wood fire, candle	Wood and wax flames are yellow due to incandescence if carbon-rich and oxygen-poor
	White fireworks, flashlamp, sparklers	Due to metals burning to oxide at high temperature, such as magnesium, zinc, iron, aluminium or zirconium (sparkler © Sarah Domingos)
	Nuclear reactors, synchroton light sources, free electron lasers	Due to fast free charges: Vavilov–Čerenkov radiation is due to speed of particle larger than the speed of light in matter, Bremsstrahlung is due to the deceleration of charged particles (nuclear reactor core under water, courtesy NASA)
	2. Atomic gas excitations	
	Red neon lamp, blue argon lamp, UV mercury lamp, yellow sodium street lamps, most gas lasers, metal vapour lasers, some fluorescence	Colours are due to transitions between atomic energy levels (gas discharges © Pslawinski)
	Aurora, triboluminescence in scotch tape, crystalloluminescence in strontium bromate	In air, blue and red colours are due to atomic and molecular energy levels of nitrogen, whereas green, yellow, orange colours are due to oxygen (aurora © Jan Curtis)
	Lightning, arcs, sparks, coloured fireworks, most coloured flames, some electroluminescence	Colour lines are due to energy levels of highly excited atoms (flames of K, Cu, Cs, B, Ca © Philip Evans)

TABLE 7 Causes of colour (continued).

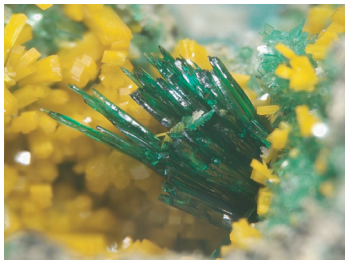
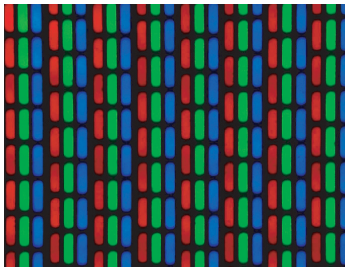
Colour type	Example	Details
	3. Vibrations and rotations of molecules	
	Bluish water, blue ice when clear, violet iodine, red-brown bromine, yellow-green chlorine, red flames from CN or blue-green flames from CH, some gas lasers, blue ozone leading to blue and grey evening sky	Colours are due to quantized levels of rotation and vibrations in molecules (blue iceberg © Marc Shandro)
Class II: Colours due to ligand field effects		
	4. Transition metal compounds	
 	Green malachite $Cu_2CO_3(OH)_2$, blue cobalt oxide, blue azurite $Cu_3(CO_3)_2(OH)_2$, red to brown hematite Fe_2O_3, green MnO, white $Mn(OH)_2$, brown manganite, chrome green Cr_2O_3, green praesodymium, pink europium and yellow samarium compounds, piezochromic and thermochromic $Cr_2O_3 - Al_2O_3$ UV and electron phosphors, scintillation, some fluorescence, some lasers	Colours are due to electronic states of the ions; phosphors are used in cathodes tubes for TV/computer displays and on fluorescent lamp tubes (green malachite on yellow kasolite, a uranium mineral, picture width 5 mm, found in Kolwezi, Zaire/Congo, © Stephan Wolfsried, television shadow mask photo © Planemad)
	5. Transition metal impurities	
	Ruby, emerald, alexandrite, perovskites, corresponding lasers	Electronic states of transition metal ions are excited by light and thus absorb specific wavelengths (ruby on calcite from Mogok, Myanmar, picture width 3 cm, © Rob Lavinsky)

TABLE 7 Causes of colour (continued).

Colour type	Example	Details
Class III: Colours due to molecular orbitals		
	6. Organic compounds	
   	Red haemoglobin in blood, blue blood haemocyanin, green chlorophyll in plants, yellow or orange carotenes in carrots, flowers and yellow autumn leaves, red or purple anthocyanins in berries, flowers and red autumn leaves, blue indigo, red lycopene in tomatoes, red meat from iron-containing myoglobin, brown glucosamine in crust of baked food, brown tannins, black eumelanin in human skin, hair and eye, iron-rich variation pheomelanin in redheads, black melanin also in cut apples and bananas as well as in movable sacks in chameleons, brown-black asphalt, some fluorescence, chemiluminescence, phosphorescence, halochromism, electrochromism and thermochromism, dye lasers	Colours are due to conjugated π-bonds, i.e. to alternating single and double bonds in molecules; floral pigments are almost all anthocyanins, betalains or carotenes; used in colourants for foods and cosmetics, in textile dyes, in electrochromic displays, in inks for colour printers, in photosensitizers (narcissus © Thomas Lüthi, blood on finger © Ian Humes, berries © Nathan Wall, hair courtesy dusdin)
	Glow-worms, some bacteria and fungi, most deep-sea fish, octopi, jellyfish, and other deep-sea animals	Bioluminescence is due to excited molecules, generally called *luciferines* (angler fish, length 4.5 cm, © Steve Haddock)

TABLE 7 Causes of colour (continued).

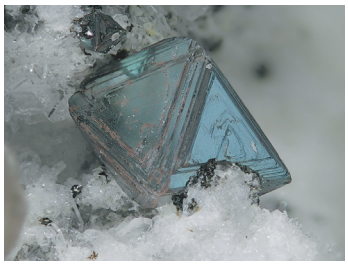
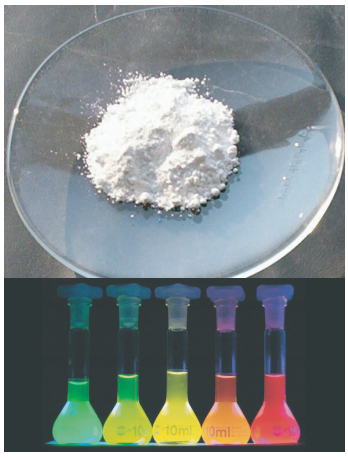
Colour type	Example	Details
 	**7. Inorganic charge transfer** Blue sapphire, blue lapis lazuli, green amazonite, brown-black magnetite Fe_3O_4 and most other iron minerals (colouring basalt black, beer bottles brown, quartz sand yellow, and many other rocks with brown or red tones), black graphite, purple permanganate, orange potassium dichromate, yellow molybdates, red hematite Fe_2O_3, some fluorescence	Light induces change of position of an electron from one atom to another; for example, in blue sapphire the transition is between Ti and Fe impurities; many paint pigments use charge transfer colours; fluorescent analytical reagents are used in molecular medicine and biology (magnetite found in Laach, Germany, picture width 10 mm, © Stephan Wolfsried, sand desert Evelien Willemsen)
Class IV: Colours due to energy band effects		
	**8. Metallic bands** Gold (green in transmission), pyrite, iron, brass, alloys, silver, copper, ruby glass	Colours in reflection and in transmission are due to transitions of electrons between overlapping bands (saxophone © Selmer)
	**9. Pure semiconductor bands** Silicon, GaAs, black galena PbS, red cinnabar HgS, cadmium yellow CdS, black CdSe, red CdS_xSe_{1-x}, white ZnO, orange vermillion HgS, colourless diamond, black to gold piezochromic SmS	Colours are due to electron transitions between separate bands; colour series is black, red, orange, yellow, white/colourless; some used as pigments (zinc oxide courtesy Walkerma)
	10. Doped semiconductor bands Blue, yellow, green and black diamond; LEDs; semiconductor lasers; solar cells; ZnS and $Zn_xCd_{1-x}S$ based and other phosphors	Colours are due to transitions between dopants and semiconductor bands (e.g. blue diamond: boron accepters, black diamond: nitrogen donors) (quantum dots © Andrey Rogach)

TABLE 7 Causes of colour (continued).

Colour type	Example	Details
	11. Colour centres	
	Amethyst, smoky quartz, fluorite, green diamonds, blue, yellow and brown topaz, brown salt, purple colour of irradiated glass containing Mn^{2+}, lyoluminescence, some fluorescence, F-centre lasers	Colours are due to colour centres, i.e. to electrons or to holes bound at crystal vacancies; colour centres are usually created by radiation (amethyst © Rob Lavinsky)
	Some light-dependent sunglasses	The photochromic colouring is due to colour centres formed by the UV light of the Sun
Class V: Colours due to physical and geometrical optics		
	12. Dispersive refraction and polarization	
	Cut diamond, cut zirconia, halos and sun dogs formed by ice crystals in the air	Spectral decomposition (sparkle or 'fire' of gemstones) is due to dispersion in crystals (zirconia photo © Gregory Phillips)
	Rainbow	Colours of primary and secondary bow are due to dispersion in water droplets
	Green flash	dispersion in the atmosphere shifts the sun colours
	13. Scattering	
  	Blue sky, blue colouring of distant mountains, red sunset; colour intensification by pollution; blue quartz	Blue light is scattered more than red light by Rayleigh scattering, when scatterers (molecules, dust) are smaller than the wavelength of light (Tokyo sunset © Altus Plunkett, blue quartz © David Lynch)

TABLE 7 Causes of colour (continued).

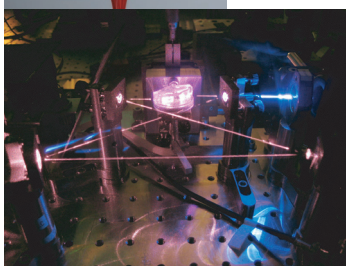
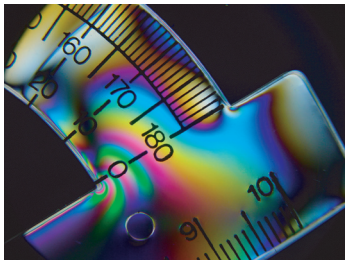
Colour type	Example	Details
	White colour of hair, milk, beer foam, clouds, fog, cigarette smoke coming out of lungs, snow, whipped cream, shampoo, stars in gemstones	The white colour is due to wavelength-independent Mie scattering, i.e. scattering at particles larger than the wavelength of light (snow man © Andreas Kostner)
	Blue human skin colour in cold weather, blue and green eyes in humans, blue monkey skin, blue turkey necks, most blue fish, blue reptiles, blue cigarette smoke	Tyndall blue colours are due to scattering on small particles in front of a dark background (blue poison frog *Dendrobates azureus* © Lee Hancock)
	Ruby glass	The red colour of Murano glass is due to scattering by tiny colloidal gold particles included in the glass in combination with the metallic band structure of gold (ruby glass © murano-glass-shop.it)
	Nonlinearities, Raman effect, potassium dihydrogen phosphate (KDP)	Frequency-shifting scattering, second harmonic generation and other nonlinearities of certain materials change the colour of light impinging with high intensities (800 nm to 400 nm frequency doubling ring laser © Jeff Sherman)
	14. Interference (without diffraction)	
	Nacre, oil films, soap bubbles, coatings on camera lenses, eyes of cats in the dark, wings of flies and dragonflies, fish scales, some snakes, pearls, tempering colours of steel	Thin film interference produces a standard colour sequence that allows precise thickness determination (abalone shell © Anne Elliot)
	Polarization colours of thin layers of birefringent crystals or thicker layers of stressed polymers	Colours are due to interference, as shown by the dependence on layer thickness (photoelasticity courtesy Nevit Dilmen)

TABLE 7 Causes of colour (continued).

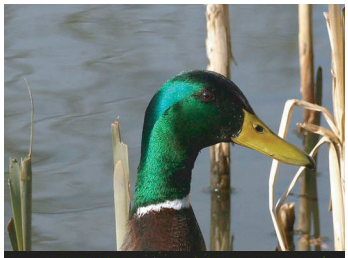
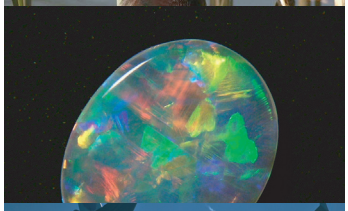
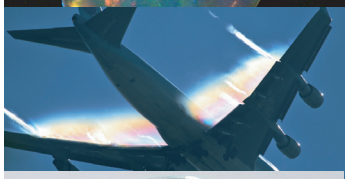
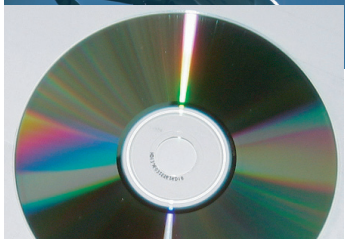
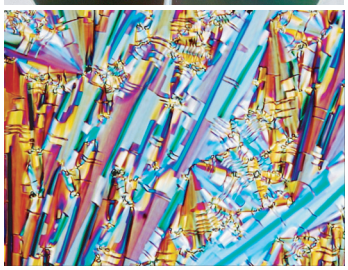
Colour type	Example	Details
	Supernumerary rainbows (see page 102 in volume III)	Due to interference, as shown by the dependence on drop size
	Iridescent beetles, butterflies and bird feathers, iridescent colours on banknotes and on cars	Due to scattering at small structures or at nanoparticles, as shown by the angular dependence of the colour (mallard duck © Simon Griffith)
15. Diffraction (with interference)		
	Opal	Colours are due to the tiny spheres included in the water inside the opal; colours can change if the opal dries out (polished Brazilian opal © Opalsnopals)
	Aureole, glory, corona	Colours are due to diffraction at the tiny mist droplets (aeroplane condensation cloud iridescence © Franz Kerschbaum)
	Diffraction gratings, CDs, vinyl records, some beetles and snakes	Colours are due to diffraction and interference at tiny, regular pits (CD illuminated by flashlamp © Alfons Reichert)
	Photonic crystals	A modern research topic
	Cholesteric liquid crystals, certain beetles	Colours are due to diffraction and interference in internal material layers (liquid crystal colours © Ingo Dierking)
Class VI: Colours due to eye limitations		
Fechner colours, as on lite.bu.edu/vision/applets/Color/Benham/Benham.html	Benham's wheel or top	Colours are due to different speed response of different photoreceptors

TABLE 7 Causes of colour (continued).

COLOUR TYPE	EXAMPLE	DETAILS
Internal colour production when eyes are stimulated	Phosphenes	Occur through pressure (rubbing, sneeze), or with electric or magnetic fields
Polarization colours	Haidinger's brush	See page 113 in volume III
Colour illusions, as on www.psy.ritsumei.ac.jp/~akitaoka/color9e.html	Appearing and disappearing colours	Effects are due to combinations of brain processing and eye limitations
False colour output of eye, as described on page 196 in volume III	Red light can be seen as green	Observable with adaptive optics, if red light is focused on a green-sensitive cone
Colour-blind or 'daltonic' person, see page 209 in volume III, with reduced colour spectrum	Protan, deutan or tritan	Each type limits colour perception in a different way

Colours fascinate. Fascination always also means business; indeed, a large part of the chemical industry is dedicated to synthesizing colourants for paints, inks, clothes, food and cosmetics. Also evolution uses the fascination of colours for its own business, namely propagating life. The specialists in this domain are the flowering plants. The chemistry of colour production in plants is extremely involved and at least as interesting as the production of colours in factories. Practically all flower colourants, from white, yellow, orange, red to blue, are from three chemical classes: the *carotenoids*, the *anthocyanins* (flavonoids) and the *betalains*. These colourants are stored in petals inside dedicated containers, the *vacuoles*. There are many good review articles providing the details. Ref. 131

Even though colours are common in plants and animals, most higher animals do not produce many colourants themselves. For example, humans produce only one colourant: *melanin*. (*Hemoglobin*, which colours blood red, is not a dedicated colourant, but transports the oxygen from the lungs through the body. Also the pink *myoglobin* in the muscles is not a dedicated colourant.) Many higher animals, such as birds, need to *eat* the colourants that are so characteristic for their appearance. The yellow colour of legs of pigeons is an example. It has been shown that the connection between colour and nutrition is regularly used by potential mates to judge from the body colours whether a proposing partner is sufficiently healthy, and thus sufficiently attractive. Ref. 132

Above all, the previous table distinguished six main classes among the causes of colours. The study of the first class, the colours of incandescence, led Max Planck to discover the quantum of action. In the meantime, research has confirmed that in each class, all colours are due to the quantum of action $\hbar$. The relation between the quantum of action and the material properties of atoms, molecules, liquids and solids are so well known that colourants can now be designed on the computer.

In summary, an exploration of the causes of colours found in nature confirms that all colours are due to quantum effects. We show this by exploring the simplest example: the colours of atomic gas excitations.

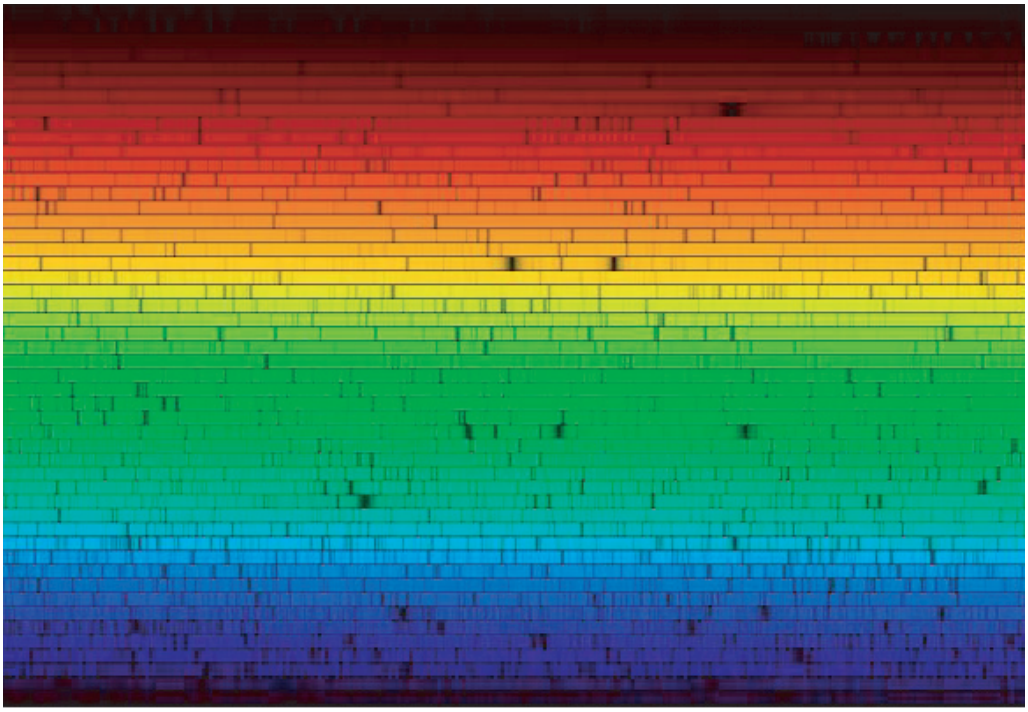


FIGURE 77 The spectrum of daylight: a *stacked* image of an extended rainbow, showing its Fraunhofer lines (© Nigel Sharp, NOAO, FTS, NSO, KPNO, AURA, NSF).

Using the rainbow to determine what stars are made of

Near the beginning of the eighteenth century, Bavarian instrument-maker Joseph Fraunhofer* and the English physicist William Wollaston noted that the rainbow *lacks* certain colours. These colours appear as black lines when the rainbow is spread out in sufficient breadth. Figure 77 shows the lines in detail; they are called *Fraunhofer lines* today. In 1860, Gustav Kirchhoff and Robert Bunsen showed that the colours *missing* in the rainbow were exactly those colours that certain elements *emit* when heated. In this way they managed to show that sodium, calcium, barium, nickel, magnesium, zinc, copper and iron are present in the Sun. Looking at the rainbow thus tells us what the Sun is made of.

* Joseph Fraunhofer (b. 1787 Straubing, d. 1826 Munich), having been orphaned at the age of 11, learned lens-polishing. He taught himself optics from books. He entered an optical company at the age of 19, ensuring the success of the business by producing the best available lenses, telescopes, micrometers, optical gratings and optical systems of his time. He invented the spectroscope and the heliometer. He discovered and counted 476 lines in the spectrum of the Sun; these lines are now named after him. (Today, Fraunhofer lines are still used as measurement standards: the second and the metre are defined in terms of them.) Physicists from all over the world would buy their equipment from him, visit him, and ask for copies of his publications. Even after his death, his instruments remained unsurpassed for generations. With his telescopes, in 1837 Bessel was able to make the first measurement of parallax of a star, and in 1846 Johann Gottfried Galle discovered Neptune. Fraunhofer became a professor in 1819. He died young, from the consequences of the years spent working with lead and glass powder.

Vol. II, page 312

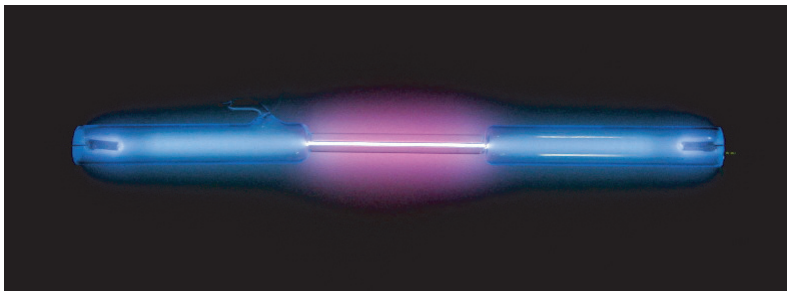


FIGURE 78 A low-pressure hydrogen discharge in a 20 cm long glass tube (© Jürgen Bauer at www.smart-elements.com).

Of the 476 Fraunhofer lines that Kirchhoff and Bunsen observed, 13 did not correspond to any known element. In 1868, Jules Janssen and Joseph Lockyer independently predicted that these unknown lines were from an unknown element. The element was eventually found on Earth, in an uranium mineral called cleveite, in 1895. The new element was called *helium*, from the Greek word ἥλιος 'helios' – Sun.

In 1925, using an equation developed by Saha and Langmuir, the young physicist Cecilia Payne (b. 1900 Wendover, England, d. 1979 Cambridge, Massachusetts) taught the world how to deduce the mass percentage of each element from the light spectrum of a star. She did so in her brilliant PhD thesis. Above all, she found that hydrogen and helium were the two most abundant elements in the Sun, in stars, and thus in the whole universe. This went completely against the ideas of the time, but is now common knowledge. Payne had completed the study of physics in Cambridge, UK, but had not received a degree there because she was a woman. So she left for the United States, where the situation was somewhat better, and where she worked on her PhD thesis; eventually, she became professor at Harvard University, and later head of its astronomy department. Above all, Payne became an important role model for many female scientists.

Despite being the second most common element in the universe, helium is rare on Earth because it is a light noble gas that does not form chemical compounds. Helium atoms on Earth thus rise in the atmosphere and finally escape into space.

Understanding the colour lines produced by each element had started to become interesting already before the discovery of helium; but afterwards the interest increased further, thanks to the increasing number of applications of colour knowledge in chemistry, physics, technology, crystallography, biology and lasers. Colours are big business, as the fashion industry, the media and the advertising business show.

Vol. III, page 125

In summary, colours are specific mixtures of light frequencies. Light is an electromagnetic wave and is emitted by moving charges. For a physicist, colours thus result from the interaction of charged matter with the electromagnetic field. Now, *sharp* colour lines cannot be explained by classical electrodynamics. We need quantum theory to explain them.

What determines the colours of atoms?

The simplest colours to study are the sharp colour lines emitted or absorbed by single atoms. Single atoms are found in gases. The simplest atom to study is that of hydrogen. As shown in Figure 78, hot hydrogen gas emits light. The light consists of a handful of

sharp spectral lines that are shown on the left of Figure 79. Already in 1885, the Swiss schoolteacher Johann Balmer (1828–1898) had discovered that the wavelengths of visible hydrogen lines obey the formula:

$$\frac{1}{\lambda_m} = R\left(\frac{1}{4} - \frac{1}{m^2}\right) \quad \text{for} \quad m = 3, 4, 5, \ldots \,. \tag{91}$$

Careful measurements, which included the hydrogen's spectral lines in the infrared and in the ultraviolet, allowed Johannes Rydberg (1854–1919) to generalize this formula to:

$$\frac{1}{\lambda_{mn}} = R\left(\frac{1}{n^2} - \frac{1}{m^2}\right) \,, \tag{92}$$

where n and $m > n$ are positive integers, and the so-called *Rydberg constant* R has the value $10.97\,\mu\text{m}^{-1}$; easier to remember, the inverse value is $1/R = 91.16\,\text{nm}$. All the colour lines emitted by hydrogen satisfy this simple formula. Classical physics cannot explain this result at all. Thus, quantum theory has a clearly defined challenge here: to explain the formula and the value of R.

Incidentally, the transition λ_{21} for hydrogen is called the *Lyman-alpha line*. Its wavelength, 121.6 nm, lies in the ultraviolet. It is easily observed with telescopes, since most of the visible stars consist of excited hydrogen. The Lyman-alpha line is routinely used to determine the speed of distant stars or galaxies, since the *Doppler effect* changes the wavelength when the speed is large. The record in 2004 was a galaxy with a Lyman-alpha line shifted to 1337 nm. Can you calculate the speed with which it is moving away from the Earth?

Ref. 133

Challenge 143 s

From the start, it was clear that the colours of hydrogen are due to the motion of its electron. (Why?) The first way to deduce Balmer's formula from the quantum of action was found by Niels Bohr in 1903. Bohr understood that in contrast to planets circling the Sun, the electron moving around the proton has only a discrete number of possible motion states: the angular momentum of the electron is quantized. Assuming that the angular momentum of the electron is an integer multiple of $\hbar$ directly yields Balmer's formula and explains the numerical value of the Rydberg constant R. This calculation is so famous that it is found in many secondary school books. The result also strengthened Bohr's decision to dedicate his life to the exploration of the structure of the atom.

Challenge 144 e

Page 81

Challenge 145 e

Twenty years time later, in 1926, Erwin Schrödinger solved his equation of motion for an electron moving in the electrostatic potential $V(r) = e^2/4\pi\varepsilon_0 r$ of a point-like proton. By doing so, Schrödinger reproduced Bohr's result, deduced Balmer's formula and became famous in the world of physics. However, this important calculation is long and complex. It can be simplified.

In order to understand hydrogen colours, it is not necessary to solve an equation of motion for the electron; it is sufficient to compare the energies of the initial and final states of the electron. This can be done most easily by noting that a specific form of the action must be a multiple of $\hbar/2$. This approach, a generalization of Bohr's explanation, was developed by Einstein, Brillouin and Keller, and is now named EBK quantization. It

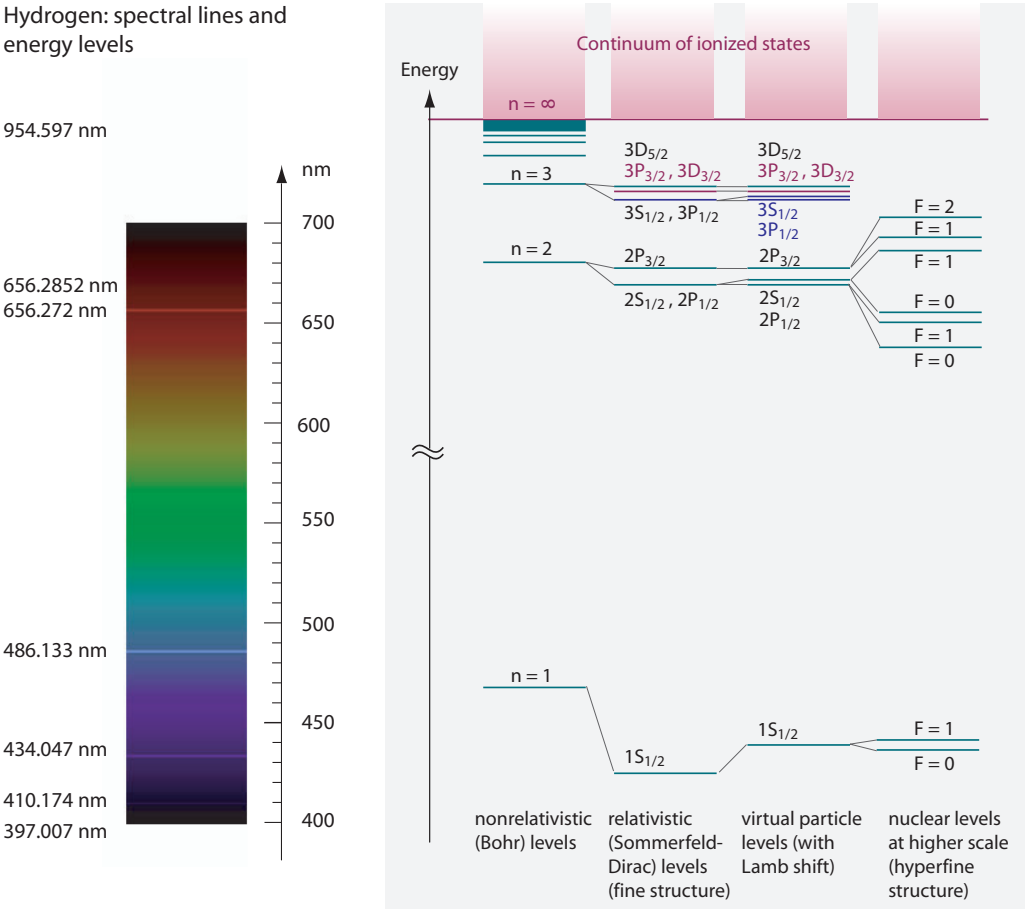


FIGURE 79 Atomic hydrogen: the visible spectrum of hydrogen (NASA) and its calculated energy levels, in four approximations of increasing precision. Can you associate the visible lines to the correct level transitions?

Ref. 134 relies on the fact that the action S of any quantum system obeys

$$S = \frac{1}{2\pi}\oint \mathrm{d}q_i p_i = \left(n_i + \frac{\mu_i}{4}\right)\hbar \tag{93}$$

for every coordinate q_i and its conjugate momentum p_i. The expression reflects the similarity between angular momentum and action. Here, n_i can be zero or any positive integer, and μ_i is the so-called *Maslov index*, an even integer, which in the case of atoms has the value 2 for the radial and azimuthal coordinates r and θ, and 0 for the rotation angle φ. The integral is to be taken along a full orbit. In simple words, *the action S is a half-integer multiple of the quantum of action*. This result can be used to calculate the energy levels of periodic quantum systems. Let us do so for hydrogen atoms.

Any rotational motion in a spherical potential $V(r)$ is characterized by a constant energy E and constant angular momenta L and L_z. Therefore the conjugate momenta

Challenge 146 ny for the coordinates r, θ and φ are

$$p_r = \sqrt{2m(E - V(r)) - \frac{L^2}{r^2}}$$
$$p_\theta = \sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}}$$
$$p_\varphi = L_z \,. \tag{94}$$

Using these expressions in equation (93) and setting $n = n_r + n_\theta + n_\varphi + 1$, we get* the result

$$E_n = -\frac{1}{n^2}\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2} = -\frac{Rhc}{n^2} = -\frac{c^2 m\alpha^2}{2n^2} \approx -\frac{2.19\,\text{aJ}}{n^2} \approx -\frac{13.6\,\text{eV}}{n^2} \,. \tag{97}$$

These *energy levels* E_n, the non-relativistic Bohr levels, are shown in Figure 79. Using the idea that a hydrogen atom emits a single photon when its electron changes from state E_n to E_m, we get exactly the formula deduced by Balmer and Rydberg from observations! Challenge 148 e The match between observation and calculation is about four digits. For (almost) the first time ever, a material property, the colour of hydrogen atoms, had been explained from a fundamental principle of nature. Key to this explanation was the quantum of action $\hbar$. (This whole discussion assumes that the electrons in hydrogen atoms that emit light are Challenge 149 s in eigenstates. Can you argue why this is the case?)

In short, the quantum of action implies that only certain specific energy values for an electron are allowed inside an atom. The lowest energy level, for $n = 1$, is called the *ground state*. Its energy value 2.19 aJ is the *ionization energy* of hydrogen; if that energy is added to the ground state, the electron is no longer bound to the nucleus. The ionization energy thus plays the same role for electrons around atoms as does the *escape velocity*, or better, the escape *energy*, for satellites or rockets shot from planets.

In the same way that the quantum of action determines the colours of the hydrogen Page 180 atom, it determines the colours of all other atoms. All Fraunhofer lines, whether observed in the infrared, visible or ultraviolet, are due to the quantum of action. In fact, every colour in nature is due to a mixture of colour lines, so that all colours, also those of solids and liquids, are determined by the quantum of action.

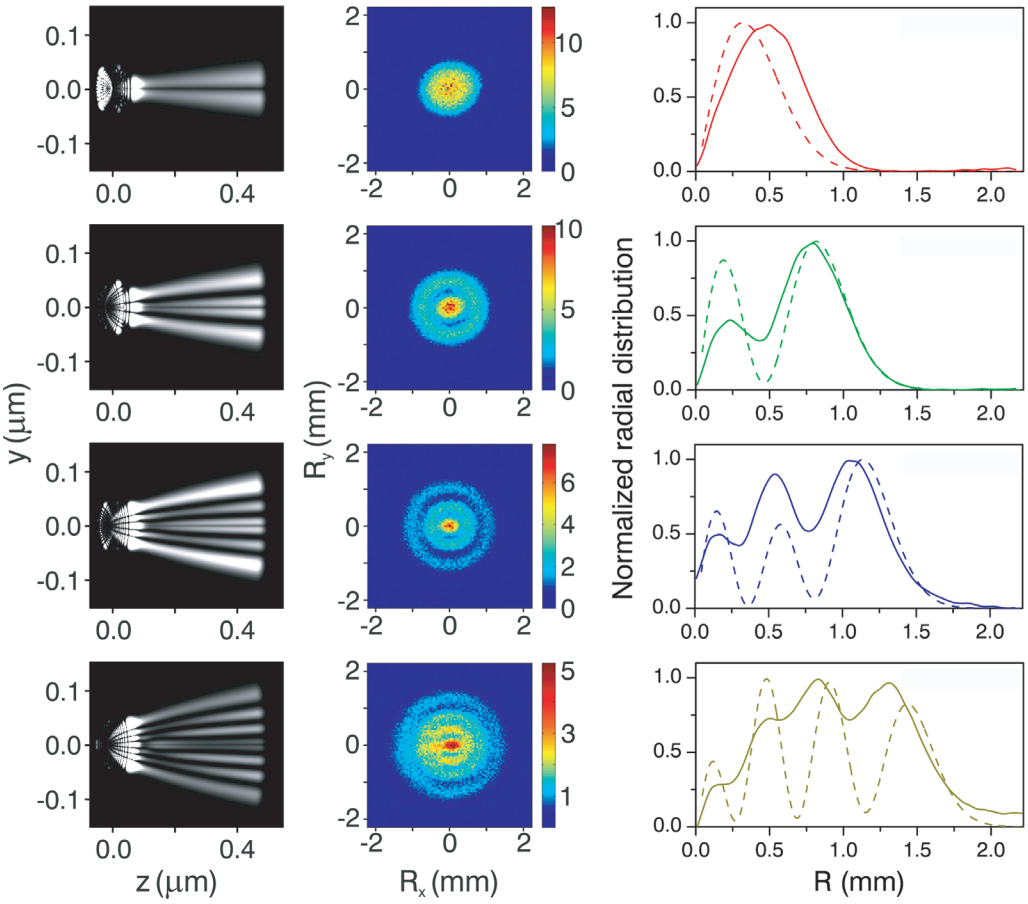


FIGURE 80 The figure shows the calculated and the measured nodal structure of the hydrogen atom in a weak external electric field, magnified by an electrostatic lens. The patterns are two-dimensional interference shadows of the wave functions. Left column: how the wave function is projected from the atoms to the macroscopic screen; central column: the measured nodal structure; right column: comparison of the measured (solid) and calculated (dashed) electron densities. (© Aneta Stodolna/APS, from Ref. 138).

The shape of atoms

Free atoms are spherical. Atoms in external fields are deformed. Whatever the situation, the shape of atoms is due to the shape of the wave function. The simplest case is the

Challenge 147 ny

* The calculation is straightforward. After insertion of $V(r) = e/4\pi\varepsilon_0 r$ into equation (94) one needs to perform the (tricky) integration. Using the general result

$$\frac{1}{2\pi}\oint \frac{\mathrm{d}z}{z}\sqrt{Az^2+2Bz-C} = -\sqrt{C}+\frac{B}{\sqrt{-A}} \tag{95}$$

one gets

$$\left(n_r+\frac{1}{2}\right)\hbar + L = n\hbar = \frac{e^2}{4\pi\varepsilon_0}\sqrt{\frac{m}{-2E}}\,. \tag{96}$$

This leads to the energy formula (97).

hydrogen atom. Its wave functions – more precisely, the eigenfunctions for the first few energy levels – are illustrated on the right hand side of Figure 81. These functions had been calculated by Erwin Schrödinger already in 1926 and are found in all textbooks. We do not perform the calculation here, and just show the results.

The square of the wave function is the probability density of the electron. This density quickly decreases with increasing distance from the nucleus. Like for a real cloud, the density is never zero, even at large distances. We could thus argue that all atoms have infinite size; in practice however, chemical bonds or the arrangement of atoms in solids show that it is much more appropriate to imagine atoms as clouds of finite size.

Surprisingly, the first measurement of the wave function of an atom dates only from the year 2013; it was performed with a clever photoionization technique by Aneta Stodolna and her team. The beautiful experimental result is shown in Figure 80. The figures confirm that wave functions, in contrast to probability densities, have nodes, i.e. lines – or better, surfaces – where their value is zero.

Ref. 138

In summary, all experiments confirm that the electron in the hydrogen atom forms wave functions in exactly the way that is predicted by quantum theory. In particular, the shape of atoms is found to agree with the calculation from quantum mechanics.

The size of atoms

The calculation of the hydrogen energy levels also yields the effective radius of the electron orbits. It is given by

$$r_n = n^2 \frac{\hbar^2 4\pi\varepsilon_0}{m_e e^2} = \frac{\hbar}{m_e c\alpha} = n^2 a_0 \approx n^2\, 52.918\,937\,\text{pm} \text{ , with } n = 1, 2, 3, \ldots \qquad (98)$$

We again see that, in contrast to classical physics, quantum theory allows only certain specific orbits around the nucleus. (For more details about the fine-structure constant α, see below.) To be more precise, these radii are the average sizes of the electron clouds surrounding the nucleus. The smallest orbital radius value, 53 pm for $n = 1$, is called the *Bohr radius*, and is denoted by a_0.

Page 188, page 196

In a gas of hydrogen atoms, most atoms are in the ground state described by $r_1 = a_0$ and E_1. On the other hand, quantum theory implies that a hydrogen atom excited to the level $n = 500$ is about 12 µm in size: larger than many bacteria! Such blown-up atoms, usually called *Rydberg atoms*, have indeed been observed in the laboratory, although they are extremely sensitive to perturbations.

Ref. 135

In short, the quantum of action determines *the size of atoms*. The result thus confirms the prediction by Arthur Erich Haas from 1910. In other words

Page 21

▷ The quantum of action determines the size of all things.

In 1915, Arnold Sommerfeld understood that the analogy of electron motion with orbital gravitational motion could be continued in two ways. First of all, electrons can move, on average, on *ellipses* instead of circles. The quantization of angular momentum then implies that only selected eccentricities are possible. The higher the angular momentum, the larger the number of possibilities: the first few are shown in Figure 81. The

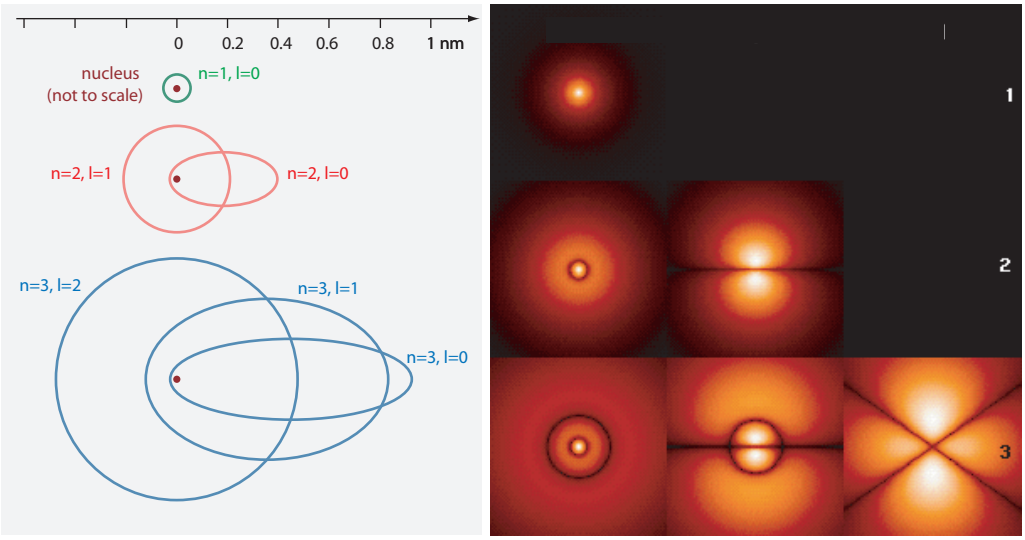


FIGURE 81 The imagined, but not existing and thus false electron orbits of the Bohr–Sommerfeld model of the hydrogen atom (left) and the correct description, using the probability density of the electron in the various states (right) (© Wikimedia).

highest eccentricity corresponds to the minimum value $l = 0$ of the so-called azimuthal quantum number, whereas the case $l = n - 1$ correspond to circular orbits. Furthermore, the ellipses can have different orientations in space.

Ref. 137

The second point Sommerfeld noted was that the speeds of the electron in hydrogen are – somewhat – relativistic: the speed values are not negligible compared to the speed of light. Indeed, the orbital frequency of electrons in hydrogen is

$$f_n = \frac{1}{n^3}\frac{e^4 m_e}{4\varepsilon_0^2 h^3} = \frac{1}{n^3}\frac{m_e c^2 \alpha^2}{h} \approx \frac{6.7\,\mathrm{PHz}}{n^3} \tag{99}$$

and the electron speed is

$$v_n = \frac{1}{n}\frac{e^2}{4\pi\varepsilon_0 \hbar} = \frac{\alpha c}{n} \approx \frac{2.2\,\mathrm{Mm/s}}{n} \approx \frac{0.007\,c}{n}\,. \tag{100}$$

As expected, the further the electron's orbit is from the nucleus, the more slowly it moves. This result can also be checked by experiment: exchanging the electron for a muon allows us to measure the time dilation of its lifetime. Measurements are in excellent agreement with the calculations.

Ref. 136

In short, Sommerfeld noted that Bohr's calculation did not take into account relativistic effects. And indeed, high-precision measurements show slight differences between the Bohr's non-relativistical energy levels and the measured ones. The calculation must be improved.

Relativistic hydrogen

Measuring atomic energy levels is possible with a much higher precision than measuring wave functions. In particular, energy level measurements allow to observe relativistic effects.

Also in the relativistic case, the EBK action has to be a multiple of $\hbar/2$. From the re-
Ref. 134 lativistic expression for the kinetic energy of the electron in a hydrogen atom

$$E + c^2 m = \sqrt{p^2 c^2 + m^2 c^4} - \frac{e^2}{4\pi\varepsilon_0 r} \tag{101}$$

Challenge 150 e we get the expression

$$p_r^2 = 2mE\left(1 + \frac{E}{2c^2 m}\right) + \frac{2me^2}{4\pi\varepsilon_0 r}\left(1 + \frac{E}{c^2 m}\right) . \tag{102}$$

We now introduce, for convenience, the so-called *fine-structure constant*, as $\alpha = e^2/(4\pi\varepsilon_0\hbar c) = \sqrt{4\pi\hbar R/mc^3} \approx 1/137.036$. ($\alpha$ is a dimensionless constant; $R = 10.97\,\mu\text{m}^{-1}$
Challenge 151 e is the Rydberg constant.) The radial EBK action then implies that

$$E_{nl} + c^2 m = \frac{c^2 m}{\sqrt{1 + \frac{\alpha^2}{\left(n - l - \frac{1}{2} + \sqrt{(l+\frac{1}{2})^2 - \alpha^2}\right)^2}}} . \tag{103}$$

This result, first found by Arnold Sommerfeld in 1915, is correct for point-like, i.e., non-rotating electrons. In reality, the electron has spin 1/2; the correct relativistic energy levels thus appear when we set $l = j \pm 1/2$ in the above formula. The result can be approximated by

$$E_{nj} = -\frac{R}{n^2}\left(1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j + \frac{1}{2}} - \frac{3}{4}\right) + \dots\right) . \tag{104}$$

It reproduces the hydrogen spectrum to an extremely high accuracy. If we compare the result with the non-relativistic one, we note that each non-relativistic level n is split in
Page 183 n different levels. This splitting is illustrated in Figure 79. In precision experiments, the splitting of the lines of the hydrogen spectrum is visible as the so-called *fine structure*. The magnitude of the fine structure depends on α, a fundamental constant of nature. Since Arnold Sommerfeld discovered the importance of this fundamental constant in this context, the name he chose, the *fine-structure constant*, has been taken over across the world.
Page 196 The fine-structure constant describes the *strength* of the electromagnetic interaction; the fine-structure constant is the electromagnetic coupling constant.

Modern high-precision experiments show additional effects that modify the colours
Page 183 of atomic hydrogen. They are also illustrated in Figure 79. Virtual-particle effects and the coupling of the proton spin give additional corrections. But that is still not all: isotope effects, Doppler shifts and level shifts due to environmental electric or magnetic fields

FIGURE 82 Paul Dirac (1902–1984)

also influence the hydrogen spectrum. The final effect on the hydrogen spectrum, the famous Lamb shift, will be a topic later on.

Vol. V, page 125

Relativistic wave equations – again

> "The equation was more intelligent than I was."
> Paul Dirac about his equation, repeating a statement made by Heinrich Hertz.

What is the evolution equation for the wave function in the case that relativity, spin and interactions with the electromagnetic field are taken into account? We could try to generalize the representation of relativistic motion given by Foldy and Wouthuysen to the case of particles with electromagnetic interactions. Unfortunately, this is not a simple matter. The simple identity between the classical and quantum-mechanical descriptions is lost if electromagnetism is included.

Page 106

Charged quantum particles are best described by another, equivalent representation of the Hamiltonian, which was discovered much earlier, in 1926, by the British physicist Paul Dirac.* Dirac found a neat trick to take the square root appearing in the relativistic energy operator. In Dirac's representation, the Hamilton operator is given by

$$H_{\text{Dirac}} = \beta m + \boldsymbol{\alpha} \cdot p \,. \tag{105}$$

The quantities β and the three components $(\alpha_1, \alpha_2, \alpha_3) = \boldsymbol{\alpha}$ turn out to be complex 4×4 matrices.

In Dirac's representation, the position operator x is not the position of a particle, but has additional terms; its velocity operator has only the eigenvalues plus or minus the velocity of light; the velocity operator is not simply related to the momentum operator;

* Paul Adrien Maurice Dirac (b. 1902 Bristol, d. 1984 Tallahassee), bilingual physicist, studied electrotechnics in Bristol, then went to Cambridge, where he later became a professor, holding the chair that Newton had once held. In the years from 1925 to 1933 he published a stream of papers, of which several were worth a Nobel Prize; he received it in 1933. Dirac unified special relativity and quantum theory, predicted antimatter, worked on spin and statistics, predicted magnetic monopoles, speculated on the law of large numbers, and more besides. His introversion, friendliness and shyness, and his deep insights into nature, combined with a dedication to beauty in theoretical physics, made him a legend all over the world during his lifetime. For the latter half of his life he tried, unsuccessfully, to find an alternative to quantum electrodynamics, of which he was the founder, as he was repelled by the problems of infinities. He died in Florida, where he lived and worked after his retirement from Cambridge.

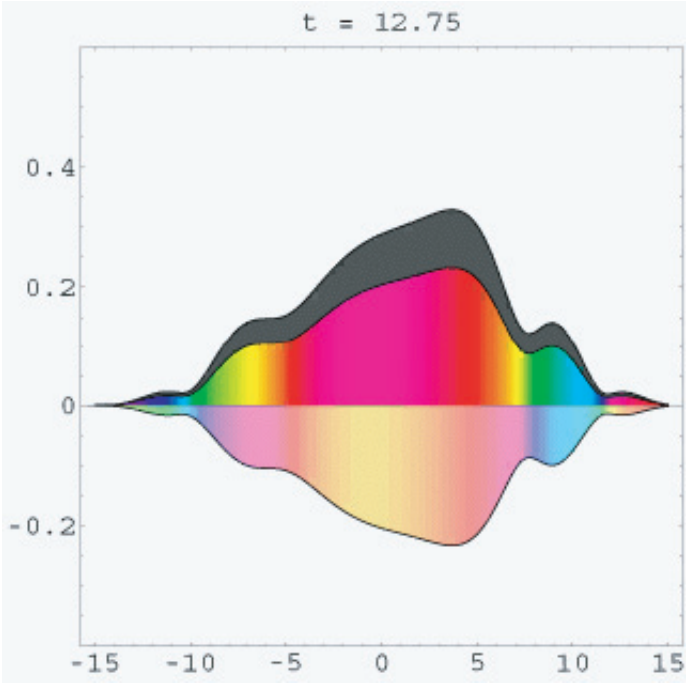


FIGURE 83 The famous Zitterbewegung: the superposition of positive and negative energy states leads to an oscillation around a mean vale. Colour indicates phase; two coloured curves are shown, as the Dirac equation in one dimension has only two components (not four); the grey curve is the probability density. (QuickTime film © Bernd Thaller)

the equation of motion contains the famous 'Zitterbewegung' term; orbital angular momentum and spin are not separate constants of motion.

So why use this horrible Hamiltonian? Because only the Dirac Hamiltonian can easily be used for charged particles. Indeed, it is transformed to the Hamiltonian coupled to the electromagnetic field by the so-called *minimal coupling*, i.e., by the substitution

Vol. III, page 85

$$\boldsymbol{p} \to \boldsymbol{p} - q\boldsymbol{A}\,, \tag{106}$$

that treats electromagnetic momentum like particle momentum. With this prescription, Dirac's Hamiltonian describes the motion of charged particles interacting with an electromagnetic field $\boldsymbol{A}$. The minimal coupling substitution is not possible in the Foldy–Wouthuysen Hamiltonian. In the Dirac representation, particles are pure, point-like, structureless electric charges; in the Foldy–Wouthuysen representation they acquire a charge radius and a magnetic-moment interaction. (We will come to the reasons below, in the section on QED.)

Ref. 139

In more detail, the simplest description of an electron (or any other elementary, stable,

electrically-charged particle of spin 1/2) is given by the action S and Lagrangian

$$S = \int \mathcal{L}_{\text{QED}} d^4x \quad \text{where} \tag{107}$$
$$\mathcal{L}_{\text{QED}} = \overline{\psi}\,(i\hbar c \not{D} - c^2 m)\,\psi - \frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} \quad \text{and}$$
$$\not{D}_\mu = \gamma^\mu(\partial_\mu - ieA_\mu)$$

The first, matter term in the Lagrangian leads to the Dirac equation: it describes how elementary, charged, spin 1/2 particles are moved by electromagnetic fields. The second, radiation term leads to Maxwell's equations, and describes how electromagnetic fields are moved by the charged particle wave function. Together with a few calculating tricks, these equations describe what is usually called *quantum electrodynamics*, or QED for short.

As far as is known today, the relativistic description of the motion of charged matter and electromagnetic fields given the QED Lagrangian (107) is *perfect*: no differences between theory and experiment have ever been found, despite intensive searches and despite a high reward for anybody who would find one. All known predictions fully agree with all measurements. In the most spectacular cases, the correspondence between theory and measurement extends to more than thirteen digits. But even more interesting than the precision of QED are certain of its features that are missing in classical electrodynamics. Let's have a quick tour.

Getting a first feeling for the Dirac equation

The QED Lagrangian implies that the wave function of a charged particle in a potential follows the Dirac equation:

$$i\hbar\gamma^\mu(\partial_\mu - ieA_\mu)\psi = mc\psi\,. \tag{108}$$

The many indices should not make us forget that this equation simply states that the eigenvalue of the energy–momentum operator is the rest mass (times the speed of light c). In other words, the equation states that the wave ψ moves with a phase velocity c.

The wave function ψ has four complex components. Two describe the motion of particles, and two the motion of antiparticles. Each type of particle needs two complex components, because the equation describes spin and particle density. Spin is a rotation, and a rotation requires three real parameters. Spin and density thus require four real parameters; they can be combined into two complex numbers, both for particles and for antiparticles.

Each of the four components of the wave function of a relativistic spinning particle
Challenge 152 e follows the relativistic Schrödinger–Klein–Gordon equation. This means that the relativistic energy–momentum relation is followed by each component separately.

The relativistic wave function ψ has the important property that a rotation by 2π
Challenge 153 e changes its sign. Only a rotation by 4π leaves the wave function unchanged. This is the typical behaviour of spin 1/2 particles. For this reason, the four-component wave function of a spin 1/2 particle is called a *spinor*.

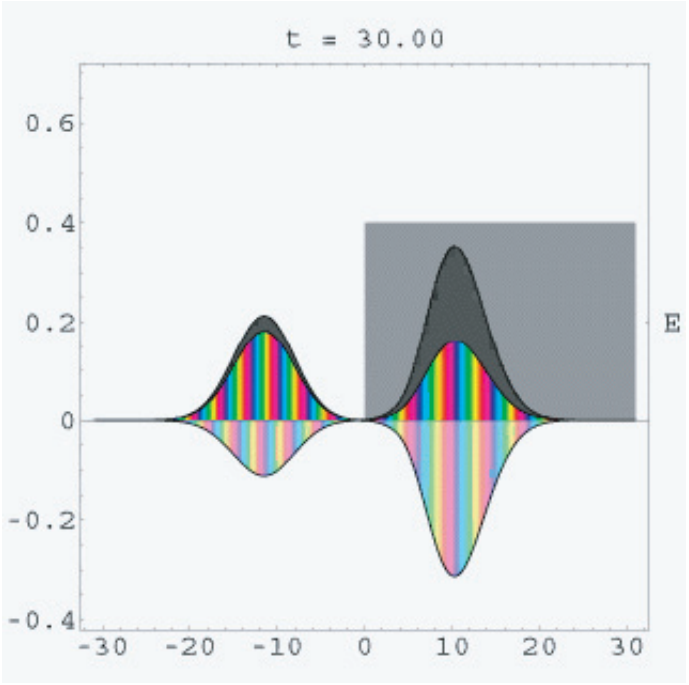


FIGURE 84 Klein's paradox: the motion of a relativistic wave function that encounters a very steep potential. Part of the wave function is transmitted; this part is antimatter, as the larger lower component shows. (QuickTime film © Bernd Thaller)

Antimatter

'Antimatter' is now a household term. Interestingly, the concept appeared *before* there was any experimental evidence for it. The relativistic expression for the energy E of an electron with charge e in the field of a charge Q is

$$E + \frac{Qe}{4\pi\varepsilon_0 r} = \sqrt{m^2c^4 + p^2c^2}\,. \tag{109}$$

This expression also allows solutions with *negative* energy and opposite charge $-e$, if the negative root is used. Quantum theory shows that this is a general property, and these solutions correspond to what is called *antimatter*.

Indeed, the antimatter companion of the electron was predicted in the 1920s by Paul Dirac from his equation (108), which is based on the above relativistic energy relation (109). Unaware of this prediction, Carl Anderson discovered the antielectron in 1932, and called it the *positron*. (The correct name would have been 'positon', without the 'r'. This correct form is used in the French language.) Anderson was studying cosmic rays, and noticed that some 'electrons' were turning the wrong way in the magnetic field he had applied to his apparatus. He checked his apparatus thoroughly, and finally deduced that he had found a particle with the same mass as the electron but with positive electric charge.

The existence of positrons has many strange implications. Already in 1928, before their discovery, the Swedish theorist Oskar Klein had pointed out that Dirac's equation for

electrons makes a strange prediction: when an electron hits a sufficiently steep potential wall, the reflection coefficient is larger than unity. Such a wall will reflect *more* than is thrown at it. In addition, a large part of the wave function is transmitted through the wall. In 1935, after the discovery of the positron, Werner Heisenberg and Hans Euler explained the paradox. They found that the Dirac equation predicts that whenever an electric field exceeds the critical value of

Ref. 140

$$E_c = \frac{m_e c^2}{e\lambda_e} = \frac{m_e^2 c^3}{e\hbar} = 1.3\,\mathrm{EV/m}\,, \tag{110}$$

the vacuum will spontaneously generate electron–positron pairs, which are then separated by the field. As a result, the original field is reduced. This so-called *vacuum polarization* is the reason for the reflection coefficient greater than unity found by Klein. Indeed, steep potentials correspond to high electric fields.

Vacuum polarization shows that, in contrast to everyday life, the *number* of particles is not a constant in the microscopic domain. Only the *difference* between particle number and antiparticle number turns out to be conserved. Vacuum polarization thus limits our possibility to count particles in nature!

Vacuum polarization is a weak effect. It has been only observed in particle collisions of high energy. In those case, the effect even increases the fine-structure constant! Later on we will describe truly gigantic examples of vacuum polarization that are postulated around charged black holes.

Vol. V, page 153

Of course, the generation of electron–positron pairs is not a *creation* out of nothing, but a *transformation* of energy into matter. Such processes are part of every relativistic description of nature. Unfortunately, physicists have a habit of calling this transformation 'pair creation', thus confusing the issue somewhat. The transformation is described by quantum field theory, which we will explore in the next volume.

Virtual particles

Despite what was said so far, action values smaller than the smallest action value do have a role to play. We have already encountered one example: in a collision between two electrons, there is an exchange of virtual photons. We learned that the exchanged virtual photon cannot be observed. Indeed, the action S for this exchange obeys

Page 64

$$S \leqslant \hbar\,. \tag{111}$$

In short, *virtual* particles appear only as mediators in interactions. They cannot be observed. Virtual particles, in contrast to ordinary, real particles, do not obey the relation $E^2 - p^2c^2 = m^2c^4$. For example, the kinetic energy can be negative. Indeed, virtual particles are the opposite of 'free' or real particles. They may be observed in a vacuum if the measurement time is very short. They are intrinsically short-lived.

Virtual photons are the cause for electrostatic potentials, for magnetic fields, for the Casimir effect, for spontaneous emission, for the van der Waals force, and for the Lamb shift in atoms. A more detailed treatment shows that in every situation with virtual photons there are also, with even lower probability, virtual electrons and virtual

positrons.

Massive virtual particles are essential for vacuum polarization, for the limit in the number of the elements, for black-hole radiation and for Unruh radiation. Massive virtual particles also play a role in the strong interaction, where they hold the nucleons together in nuclei, and in weak nuclear interaction, where they explain why beta decay happens and why the Sun shines.

In particular, virtual particle–antiparticle pairs of matter and virtual radiation particles together form what we call the *vacuum*. In addition, virtual radiation particles form what are usually called static fields. Virtual particles are needed for a full description of all interactions. In particular, virtual particles are responsible for every decay process.

Curiosities and fun challenges about colour and atoms

Where is the sea bluest? Sea water, like fresh water, is blue because it absorbs red and green light. The absorption is due to a vibrational band of the water molecule that is due to a combination of symmetric and asymmetric molecular stretches. The absorption is weak, but noticeable. At 700 nm (red), the $1/e$ absorption length of water is 1 m.

Ref. 141

Sea water can also be of bright colour if the sea floor reflects light. In addition, sea water can be green, if it contains small particles that scatter or absorb blue light. Most often, these particles are soil or plankton. (Satellites can determine plankton content from the 'greenness' of the sea.) Thus the sea is especially blue if it is deep, quiet and cold; in that case, the ground is distant, soil is not mixed into the water, and the plankton content is low. The Sargasso Sea is 5 km deep, quiet and cold for most of the year. It is often called the *bluest* of the Earth's waters.

Lakes can also be blue if they contain small mineral particles. The particles scatter light and lead to a blue colour for reasons similar to the blue colour of the sky. Such blue lakes are found in many places on Earth.

* *

On modern high-precision measurements of the hydrogen spectra, listen to the undisputed master of the field: enjoy the 2012 talk by Theodor Hänsch, who has devoted a large part of his life to the topic, at www.mediatheque.lindau-nobel.org.

* *

The hydrogen atom bears many fascinating aspects. In 2015, Friedmann and Hagen showed that a well-known formula for $\pi = 3.14159265...$ can be extracted from the colour spectrum. Quantum mechanics and colours are beautiful subjects indeed.

Ref. 142

* *

If atoms contain orbiting electrons, the rotation of the Earth, via the Coriolis acceleration, should have an effect on their motion, and thus on the colour of atoms. This beautiful prediction is due to Mark Silverman; the effect is so small, however, that it has not yet been measured.

Ref. 136

* *

Light is diffracted by material gratings. Can matter be diffracted by light gratings? Surprisingly, it actually can, as predicted by Dirac and Kapitza in 1937. This was accomRef. 143 plished for the first time in 1986, using atoms. For free electrons, the feat is more difficult; the clearest confirmation came in 2001, when new laser technology was used to perform a beautiful measurement of the typical diffraction maxima for electrons diffracted by a light grating.

* *

Light is totally reflected when it is directed to a dense material at a large enough angle so that it cannot enter the material. A group of Russian physicists have shown that if the Ref. 136 dense material is excited, the intensity of the totally-reflected beam can be *amplified*. It is unclear whether this will ever lead to applications.

* *

The ways people handle *single* atoms with electromagnetic fields provide many beautiful examples of modern applied technologies. Nowadays it is possible to levitate, to trap, to Vol. I, page 344 excite, to photograph, to deexcite and to move single atoms just by shining light onto Ref. 144 them. In 1997, the Nobel Prize in Physics has been awarded to the originators of the field, Steven Chu, Claude Cohen-Tannoudji and William Philips.

* *

Ref. 145 Given two mirrors and a few photons, it is possible to capture an atom and keep it floating between the two mirrors. This feat, one of several ways to isolate single atoms, is now Challenge 154 s standard practice in laboratories. Can you imagine how it is done?

* *

An example of modern research is the study of *hollow* atoms, i.e., atoms missing a number of inner electrons. They have been discovered in 1990 by J.P. Briand and his group. They appear when a completely ionized atom, i.e., one without any electrons, is brought in contact with a metal. The acquired electrons then orbit on the outside, leaving the inner shells empty, in stark contrast with usual atoms. Such hollow atoms can also be Ref. 146 formed by intense laser irradiation.

* *

Relativistic quantum effects can be seen with the unaided eye. The two most important ones concern gold and mercury. The yellow colour of gold – which has atomic number 79 – is due to the transition energy between 5d and 6s electrons, which absorbs blue light. Without relativistic effects, this transition would lie in the ultraviolet, similar to the transition between 4d and 5s electrons for silver, and gold would be colourless. The yellow colour of gold is thus a relativistic effect.

Mercury – which has atomic number 80 – has a filled 6s shell. Due to the same relativistic effects that appear in gold, these shells are contracted and do not like to form bonds. For this reason, mercury is still liquid a room temperature, in contrast to all other metals. Relativity is thus the reason that mercury is liquid, and that thermometers work.

* *

Challenge 155 s Is phosphorus phosphorescent?

* *

It is possible to detect the passage of a single photon through an apparatus without absorbing it. How would you do this?
Challenge 156 ny

Material properties

Like the size of hydrogen atoms, also the size of all other atoms is fixed by the quantum of action. Indeed, the quantum of action determines to a large degree the interactions among electrons. By doing so, the quantum of change determines all the interactions between atoms in everyday matter; therefore it determines all other material properties. The elasticity, the plasticity, the brittleness, the magnetic and electric properties of materials are equally fixed by the quantum of action. Only $\hbar$ makes electronics possible! We will study some examples of material properties in the next volume. Various details of the general connection between $\hbar$ and material properties are still a subject of research, though none is in contradiction with the quantum of action. Material research is among the most important fields of modern science, and most advances in our standard of living result from it. We will explore some aspects in the next volume.

In summary, materials science has confirmed that quantum physics is also the correct description of all materials; quantum physics has confirmed that all material properties of everyday life are of electromagnetic origin; and quantum physics has confirmed that all material properties of everyday life are due to interactions that involve electrons.

A tough challenge: the strength of electromagnetism

The great physicist Wolfgang Pauli used to say that after his death, the first thing he would ask god would be to explain Sommerfeld's fine-structure constant. (Others used to comment that after god will have explained it to him, he will think a little, and then snap: 'Wrong!')

The *fine-structure constant*, introduced by Arnold Sommerfeld, is the dimensionless constant of nature whose value is measured to be
Ref. 147

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \approx \frac{1}{137.035\,999\,679(94)} \approx 0.007\,297\,352\,5376(50)\ . \tag{112}$$

This number first appeared in explanations of the fine structure of atomic colour spectra; hence its strange name. Sommerfeld was the first to understand its general importance.
Ref. 148 It is central to quantum electrodynamics for several reasons.

First of all, the fine-structure constant describes the *strength* of electromagnetism. The number α results from the interaction of two electric charges e. Writing Coulomb's relation for the force F between two electrons as

$$F = \alpha\frac{\hbar c}{r^2} \tag{113}$$

it becomes clear that the fine-structure constant describes the strength of electromagnet-

ism. A higher value for the fine-structure constant α would mean a stronger attraction or repulsion between charged bodies. Thus the value of α determines the sizes of atoms, and indeed of all things, as well as all colours in nature.

Secondly, it is only because the fine-structure constant α is so small that we are able to talk about particles at all. Indeed, only because the fine-structure constant is much smaller than 1 it is possible to distinguish particles from each other. If the number α were near to or larger than 1, particles would interact so strongly that it would not be possible to observe them separately or to talk about particles at all.

This leads on to the third reason for the importance of the fine-structure constant. Since it is a dimensionless number, it implies some yet-unknown mechanism that fixes its value. Uncovering this mechanism is one of the challenges remaining in our adventure. As long as the mechanism remains unknown – as was the case in 2016 – we do not understand the colour and size of a single thing around us!

Small changes in the strength of electromagnetic attraction between electrons and protons would have numerous important consequences. Can you describe what would happen to the size of people, to the colour of objects, to the colour of the Sun, or to the workings of computers, if the strength were to double? And what if it were to gradually drop to half its usual value?

Challenge 157 s

Since the 1920s, explaining the value of α has been seen as one of the toughest challenges facing modern physics. That is the reason for Pauli's fantasy. In 1946, during his Nobel Prize lecture, he repeated the statement that a theory that does not determine this number cannot be complete. Since that time, physicists seem to have fallen into two classes: those who did not dare to take on the challenge, and those who had no clue. This fascinating story still awaits us.

Ref. 149

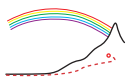
The problem of the fine-structure constant is so deep that it leads many astray. For example, it is sometimes claimed that it is impossible to change physical units in such a way that $\hbar$, c and e are all equal to 1 at the same time, because to do so would change the number $\alpha = 1/137.036...(1)$. Can you explain why the argument is wrong?

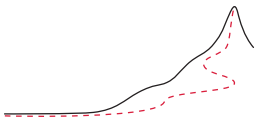
Challenge 158 s

A SUMMARY ON COLOURS AND MATERIALS

In summary, the quantum of action $\hbar$ – together with the interaction between electromagnetic fields and the electrons inside atoms, molecules, liquids and solids – determines the size, the shape, the colour and the material properties of all things around us. The quantum of action determines mechanical properties such as hardness or elasticity, magnetic properties, thermal properties such as heat capacity or heat of condensation, optical properties such as transparency, and electrical properties such as metallic shine. In addition, the quantum of action determines all chemical and biological aspects of matter. This connection is the topic of the next volume.

The strength of the electromagnetic interaction is described by the fine-structure constant $\alpha \approx 1/137.036$. Its value is yet unexplained.





Chapter 9

QUANTUM PHYSICS IN A NUTSHELL

Compared to classical physics, quantum theory is definitely more complex. The basic idea however, is simple: in nature there is a smallest change, or a smallest action with the value $\hbar = 1.1 \cdot 10^{-34}$ Js. More precisely, all of quantum theory can be resumed in one sentence:

▷ In nature, actions or changes smaller than $\hbar = 1.054\,571\,800(13) \cdot 10^{-34}$ Js are not observed.

This smallest action value, the *quantum of action*, leads to all the strange observations made in the microscopic domain, such as the wave behaviour of matter, indeterminacy relations, decoherence, randomness in measurements, indistinguishability, quantization of angular momentum, tunnelling, pair creation, decay and particle reactions.

The essence of quantum theory is thus the lack of infinitely small change. The mathematics of quantum theory is abstract and involved, though. Was this part of our walk worth the effort? It was: the results are profound, and the accuracy of the description is *complete*. We first give an overview of these results and then turn to the questions that are still left open.

Physical results of quantum theory

The existence of a smallest action value in nature leads directly to the main lesson we learned about motion in the quantum part of our adventure:

▷ If it moves, it is made of quantons, or quantum particles.

This statement applies to every physical system, thus to all objects and to all images, i.e., to all matter and radiation. Moving stuff is made of *quantons*. Stones, water waves, light, sound waves, earthquakes, gelatine and everything else we can interact with is made of quantum particles.

In our exploration of relativity we discovered that also horizons and the vacuum can move. If all moving entities are made of quantum particles, what does this imply for horizons and empty space? We can argue that no fundamental problems are expected for horizons, because one way to describe horizons is as an extreme state of matter. But the details for the quantum aspects of vacuum are not simple; they will be the topic of the last part of this adventure.

Vol. II, page 293
Vol. II, page 322

Earlier in our adventure we asked: what are matter, radiation and interactions? Now we know: they all are composites of elementary quantum particles. In particular, interactions are exchanges of elementary quantum particles.

An *elementary quantum particle* is a countable entity that is smaller than its own Compton wavelength. All elementary particles are described by energy–momentum, mass, spin, C, P and T parity. However, as we will see in the next volume, this is not yet the complete list of particle properties. About the *intrinsic* properties of quantum particles, i.e., those that do not depend on the observer, quantum theory makes a simple statement:

> ▷ In nature, all intrinsic properties of quantons, or quantum particles – with the exception of mass – such as spin, electric charge, strong charge, parities etc., appear as integer multiples of a basic unit. Since all physical systems are made of quantons, in composed systems all intrinsic properties – with the exception of mass* – either add or multiply.

Page 128

In summary, all moving entities are made of quantum particles described by *discrete* intrinsic properties. To see how deep this result is, you can apply it to all those moving entities for which it is usually forgotten, such as ghosts, spirits, angels, nymphs, daemons, devils, gods, goddesses and souls. You can check yourself what happens when their particle nature is taken into account.

Challenge 159 e

> “Deorum injuriae diis curae.**”
> Tiberius, as reported by Tacitus.

Results on the motion of quantum particles

Quantons, or quantum particles, differ from everyday particles: quantum particles *interfere*: they behave like a mixture of particles and waves. This property follows directly from the existence of $\hbar$, the smallest possible action in nature. From the existence of $\hbar$, quantum theory deduces all its statements about quantum particle motion. We summarize the main ones.

There is *no rest* in nature. All objects obey the indeterminacy relation, which states that the indeterminacies in position x and momentum p follow

$$\Delta x \Delta p \geqslant \hbar/2 \quad \text{with} \quad \hbar = 1.1 \cdot 10^{-34}\,\text{Js} \tag{114}$$

and making rest an impossibility. The state of quantum particles is defined by the same observables as in classical physics, with the difference that observables do not commute. Classical physics appears in the limit that the Planck constant $\hbar$ can effectively be set to zero.

Quantum theory introduces a *probabilistic element* into motion. Probabilities result from the quantum of action through the interactions with the baths that are part of the

* More precisely, together with mass, also mixing angles are not quantized. These properties are defined in the next volume.

** ‘Offenses of gods are care of the gods.’

environment of every physical system. Equivalently, probabilities result in every experiment that tries to induce a change that is smaller than the quantum of action.

Quantum particles *behave like waves*. The associated de Broglie wavelength λ is given by the momentum p through

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar}{p} \tag{115}$$

both in the case of matter and of radiation. This relation is the origin of the wave behaviour of light and matter. The light particles are called photons; their observation is now standard practice. Quantum theory states that particle waves, like all waves, interfere, refract, disperse, dampen, can be dampened and can be polarized. This applies to photons, electrons, atoms and molecules. All waves being made of quantum particles, all waves can be seen, touched and moved. Light for example, can be 'seen' in photon-photon scattering in vacuum at high energies, can be 'touched' using the Compton effect, and can be 'moved' by gravitational bending. Matter particles, such as molecules or atoms, can be seen in electron microscopes and can be touched and moved with atomic force microscopes. The interference and diffraction of wave particles is observed daily in the electron microscope.

Matter waves can be imagined as clouds that *rotate locally*. In the limit of negligible cloud size, quantum particles can be imagined as rotating little arrows. Equivalently, quantons have a *phase*.

Particles *cannot be enclosed forever.* Even though matter is impenetrable, quantum theory shows that tight boxes or insurmountable obstacles do not exist. Enclosure is never forever. Waiting long enough always allows us to overcome any boundary, since there is a finite probability to overcome any obstacle. This process is called *tunnelling* when seen from the spatial point of view and is called *decay* when seen from the temporal point of view. Tunnelling explains the working of television tubes as well as radioactive decay.

All particles and all particle beams *can be rotated.* Particles possess an intrinsic angular momentum called *spin*, specifying their behaviour under rotations. Bosons have integer spin, fermions have half integer spin. An even number of bound fermions or any number of bound bosons yield a composite boson; an odd number of bound fermions yield a low-energy fermion. Solids are impenetrable because of the fermion character of its electrons in the atoms.

Identical particles are *indistinguishable*. Radiation is made of indistinguishable particles called bosons, matter of fermions. Under exchange of two fermions at space-like separations, the wave function changes sign, whereas for two bosons the wave function remains unchanged. All other properties of quantum particles are the same as for classical particles, namely countability, interaction, mass, charge, angular momentum, energy, momentum, position, as well as impenetrability for matter and penetrability for radiation. Perfect copying machines do not exist.

In collisions, particles *interact locally*, through the exchange of other particles. When matter particles collide, they interact through the exchange of virtual bosons, i.e., off-shell bosons. Motion change is thus due to particle exchange. Exchange bosons of even spin mediate only attractive interactions. Exchange bosons of odd spin mediate repulsive interactions as well.

The properties of collisions imply the non-conservation of particle number. In collisions, particles can appear – i.e., can be 'created' – or disappear – i.e., can be 'annihilated'. This is valid both for bosons and for fermions.

The properties of collisions imply the existence of *antiparticles*, which are regularly observed in experiments. Elementary fermions, in contrast to many elementary bosons, differ from their antiparticles; they can be created and annihilated only in pairs. Elementary fermions have non-vanishing mass and move slower than light.

Particles can *decay* and be *transformed*. Detailed investigations show that collisions imply the non-conservation of particle type. In collisions, selected particles can change their intrinsic properties. This observation will be detailed in the next volume. Equivalently, the quantum of action implies that things *break* and living beings *die*.

Images, made of radiation, are described by the same observables as matter: position, phase, speed, mass, momentum etc. – though their values and relations differ. Images can only be localized with a precision of the wavelength λ of the radiation producing them.

The appearance of Planck's constant $\hbar$ implies that *length scales* and *time scales* exist in nature. Quantum theory introduces a fundamental jitter in every example of motion. Thus the infinitely small is eliminated. In this way, lower limits to structural dimensions and to many other measurable quantities appear. In particular, quantum theory shows that it is impossible that on the electrons in an atom small creatures live in the same way that humans live on the Earth circling the Sun. Quantum theory shows the impossibility of Lilliput.

Clocks and metre bars have *finite precision*, due to the existence of a smallest action and due to their interactions with baths. On the other hand, all measurement apparatuses must contain baths, since otherwise they would not be able to record results.

Ref. 150 Quantum effects leave no room for cold fusion, astrology, teleportation, telekinesis, supernatural phenomena, creation out of nothing, multiple universes, or faster than light phenomena – the EPR paradox notwithstanding.

Page 153

Achievements in accuracy and precision

Apart from the conceptual changes, quantum theory improved the accuracy of predictions from the few – if any – digits common in classical mechanics to the full number of digits – sometimes thirteen – that can be measured today. The limited precision is usually *not* given by the inaccuracy of theory, it is given by the measurement accuracy. In other words, the agreement is only limited by the amount of money the experimenter is willing to spend. Table 8 shows this in more detail.

TABLE 8 Selected comparisons between classical physics, quantum theory and experiment.

Observable	Classical prediction	Prediction of quantum theory[a]	Measurement	Cost estimate
Simple motion of bodies				
Indeterminacy	0	$\Delta x \Delta p \geqslant \hbar/2$	$(1 \pm 10^{-2})\,\hbar/2$	10 k€

Observable	Classical prediction	Prediction of quantum theory[a]	Measurement	Cost estimate
Matter wavelength	none	$\lambda p = 2\pi\hbar$	$(1 \pm 10^{-2})\,\hbar$	10 k€
Compton wavelength	none	$\lambda_c = h/m_e c$	$(1 \pm 10^{-3})\,\lambda$	20 k€
Pair creation rate	0	σE	agrees	100 k€
Radiative decay time in hydrogen	none	$\tau \sim 1/n^3$	(1 ± 10^{-2})	5 k€
Smallest angular momentum	0	$\hbar/2$	$(1 \pm 10^{-6})\,\hbar/2$	10 k€
Casimir effect/pressure	0	$p = (\pi^2\hbar c)/(240 r^4)$	(1 ± 10^{-3})	30 k€
Colours of objects				
Spectrum of hot objects	diverges	$\lambda_{max} = hc/(4.956\,kT)$	$(1 \pm 10^{-4})\ \Delta\lambda$	10 k€
Lamb shift	none	$\Delta\lambda = 1057.86(1)$ MHz	$(1 \pm 10^{-6})\,\Delta\lambda$	50 k€
Rydberg constant	none	$R_\infty = m_e c\alpha^2/2h$	$(1 \pm 10^{-9})\,R_\infty$	50 k€
Stefan–Boltzmann constant	none	$\sigma = \pi^2 k^4/60\hbar^3 c^2$	$(1 \pm 3 \cdot 10^{-8})\,\sigma$	20 k€
Wien's displacement constant	none	$b = \lambda_{max} T$	$(1 \pm 10^{-5})\,b$	20 k€
Refractive index of water	none	1.34	within a few %	1 k€
Photon-photon scattering	0	from QED: finite	agrees	50 M€
Electron gyromagnetic ratio	1 or 2	2.002 319 304 365(7)	2.002 319 304 361 53(53)	30 M€
Muon anomalous magnetic moment	0	$11\,659\,1827(63) \cdot 10^{-11}$	$11\,659\,2080(60) \cdot 10^{-11}$	100 M€
Composite matter properties				
Atom lifetime	≈ 1 µs	∞	$> 10^{20}$ a	1 €
Muonium hyperfine splitting	none	4 463 302 542(620) Hz	4 463 302 765(53) Hz	1 M€
Molecular size and shape	none	from QED	within 10^{-3}	20 k€

Page 214 *a*. All these predictions are calculated from the basic physical constants given in Appendix A.

We notice that the predicted values *do not differ* from the measured ones. If we remember that classical physics does not allow us to calculate any of the measured values, we get an idea of the progress quantum physics has achieved. This advance in understanding is due to the introduction of the quantum of action $\hbar$. Equivalently, we can state: no description of nature without the quantum of action is complete.

In summary, quantum theory is precise and accurate. In the microscopic domain quantum theory is in *perfect* correspondence with nature; despite prospects of fame and riches, despite the largest number of researchers ever, no contradiction between observation and theory has been found yet. On the other hand, explaining the measured value

of the fine-structure constant, $\alpha = 1/137.035\,999\,074(44)$, remains an open problem of the electromagnetic interaction.

Is quantum theory magic?

Studying nature is like experiencing magic. Nature often looks different from what it is. During magic we are fooled – but only if we forget our own limitations. Once we start to see ourselves as part of the game, we start to understand the tricks. That is the fun of magic. The same happens in quantum motion.

* *

Nature seems irreversible, even though it isn't. We never remember the future. We are fooled because we are macroscopic.

* *

Nature seems decoherent, even though it isn't. We are fooled again because we are macroscopic.

* *

There are no clocks in nature. We are fooled by those of everyday life because we are surrounded by a huge number of particles.

* *

Motion often seems to disappear, even though it is eternal. We are fooled again, because our senses cannot experience the microscopic domain.

* *

Objects seem distinguishable, even though the statistical properties of their components show that they are not. We are fooled because we live at low energies.

* *

Matter seems continuous, even though it isn't. We are fooled because of the limitations of our senses.

* *

Motion seems deterministic in the classical sense, even though it is random. We are fooled again because we are macroscopic.

* *

In short, our human condition permanently fools us. The answer to the title question is: classical physics is like magic, and the tricks are uncovered by quantum theory. That is its main attraction.

Quantum theory is exact, but can do more

We can summarize this part of our adventure with a simple statement:

▷ Quantum physics is the description of matter and radiation *without* the concept of infinitely small.

All change in nature, in fact, everything is described by *finite* quantities, and above all, by the smallest change possible in nature, the quantum of action $\hbar = 1.054\,571\,800(13) \cdot 10^{-34}$ Js.

All experiments, without exception, show that the quantum of action $\hbar$ is the smallest observable change. The description of nature with the quantum of action is thus *exact* and *final*. The smallest measurable action $\hbar$, like the maximum energy speed c, is a fundamental property of nature. One could also call both of them *fundamental truths*.

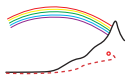
Since quantum theory follows logically and completely from the smallest measurable action $\hbar$, the simplest way – and the only way – to disprove quantum theory is to find an observation that contradicts the smallest change value $\hbar$. Try it!

Challenge 160 e

Even though we have deduced a fundamental property of nature, if we turn back to the start of our exploration of quantum theory, we cannot hide a certain disappointment. We know that classical physics cannot explain life. Searching for the details of microscopic motion, we encountered so many interesting aspects that we have *not yet* achieved the explanation of life. For example, we know what determines the speed of electrons in atoms, but we do not know what determines the running speed of an athlete. In fact, we have not even discussed the properties of any solid or liquid, let alone those of more complex structures like living beings.

Page 15

In other terms, after this introduction into quantum theory, we must still connect quantum processes to our everyday world. Therefore, the topic of the next volume will be the exploration of the motion of and inside living things – and of the motion inside all kind of matter, from solids to stars, using the quantum of action as a foundation. After that, we will explore the motion of empty space.



Appendix A

UNITS, MEASUREMENTS AND CONSTANTS

Measurements are comparisons with standards. Standards are based on *units*. Many different systems of units have been used throughout the world. Most of these standards confer power to the organization in charge of them. Such power can be misused; this is the case today, for example in the computer industry, and was so in the distant past. The solution is the same in both cases: organize an independent and global standard. For measurement units, this happened in the eighteenth century: in order to avoid misuse by authoritarian institutions, to eliminate problems with differing, changing and irreproducible standards, and – this is not a joke – to simplify tax collection and to make it more just, a group of scientists, politicians and economists agreed on a set of units. It is called the *Système International d'Unités*, abbreviated *SI*, and is defined by an international treaty, the 'Convention du Mètre'. The units are maintained by an international organization, the 'Conférence Générale des Poids et Mesures', and its daughter organizations, the 'Commission Internationale des Poids et Mesures' and the 'Bureau International des Poids et Mesures' (BIPM). All originated in the times just before the French revolution. Ref. 151

SI units

All SI units are built from seven *base units*. Their simplest definitions, translated from French into English, are the following ones, together with the dates of their formulation and a few comments:

- 'The *second* is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom.' (1967) The 2019 definition is equivalent, but much less clear.*
- 'The *metre* is the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second.' (1983) The 2019 definition is equivalent, but much less clear.*
- 'The *kilogram*, symbol kg, is the SI unit of mass. It is defined by taking the fixed numerical value of the Planck constant h to be $6.626\,070\,15 \cdot 10^{-34}$ when expressed in the unit $J \cdot s$, which is equal to $kg \cdot m^2 \cdot s^{-1}$.' (2019)*
- 'The *ampere*, symbol A, is the SI unit of electric current. It is defined by taking the fixed numerical value of the elementary charge e to be $1.602\,176\,634 \cdot 10^{-19}$ when expressed in the unit C, which is equal to $A \cdot s$.' (2019)* This definition is equivalent to: One ampere is $6.241\,509\,074... \cdot 10^{18}$ elementary charges per second.
- 'The *kelvin*, symbol K, is the SI unit of thermodynamic temperature. It is defined by

taking the fixed numerical value of the Boltzmann constant k to be $1.380649 \cdot 10^{-23}$ when expressed in the unit $J \cdot K^{-1}$.' (2019)*

- 'The mole, symbol mol, is the SI unit of amount of substance. One mole contains exactly $6.02214076 \cdot 10^{23}$ elementary entities.' (2019)*
- 'The *candela* is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency $540 \cdot 10^{12}$ hertz and has a radiant intensity in that direction of (1/683) watt per steradian.' (1979) The 2019 definition is equivalent, but much less clear.*

We note that both time and length units are defined as certain properties of a standard example of motion, namely light. In other words, also the Conférence Générale des Poids et Mesures makes the point that the observation of motion is a *prerequisite* for the definition and construction of time and space. *Motion is the fundament of every observation and of all measurement.* By the way, the use of light in the definitions had been proposed already in 1827 by Jacques Babinet.**

From these basic units, all other units are defined by multiplication and division. Thus, all SI units have the following properties:

- SI units form a system with *state-of-the-art precision*: all units are defined with a precision that is higher than the precision of commonly used measurements. Moreover, the precision of the definitions is regularly being improved. The present relative uncertainty of the definition of the second is around 10^{-14}, for the metre about 10^{-10}, for the kilogram about 10^{-9}, for the ampere 10^{-7}, for the mole less than 10^{-6}, for the kelvin 10^{-6} and for the candela 10^{-3}.
- SI units form an *absolute* system: all units are defined in such a way that they can be reproduced in every suitably equipped laboratory, independently, and with high precision. This avoids as much as possible any error or misuse by the standard-setting organization. In fact, the SI units are as now as near as possible to Planck's natural units, which are presented below. In practice, the SI is now an international standard defining the numerical values of the seven constants $\Delta\nu_{Cs}$, c, $\hbar$, e, k, N_A and K_{cd}. After over 200 years of discussions, the CGPM has little left to do.
- SI units form a *practical* system: the base units are quantities of everyday magnitude. Frequently used units have standard names and abbreviations. The complete list includes the seven base units just given, the supplementary units, the derived units and the admitted units.

The *supplementary* SI units are two: the unit for (plane) angle, defined as the ratio of arc length to radius, is the *radian* (rad). For solid angle, defined as the ratio of the subtended area to the square of the radius, the unit is the *steradian* (sr).

The *derived* units with special names, in their official English spelling, i.e., without capital letters and accents, are:

* The symbols of the seven units are s, m, kg, A, K, mol and cd. The full offical definitions are found at Ref. 152 www.bipm.org. For more details about the levels of the caesium atom, consult a book on atomic physics. The Celsius scale of temperature θ is defined as: $\theta/°C = T/K - 273.15$; note the small difference with the number appearing in the definition of the kelvin. In the definition of the candela, the frequency of the light corresponds to 555.5 nm, i.e., green colour, around the wavelength to which the eye is most sensitive.

** Jacques Babinet (1794–1874), French physicist who published important work in optics.

Name	Abbreviation	Name	Abbreviation
hertz	$\mathrm{Hz} = 1/\mathrm{s}$	newton	$\mathrm{N} = \mathrm{kg\,m/s^2}$
pascal	$\mathrm{Pa} = \mathrm{N/m^2} = \mathrm{kg/m\,s^2}$	joule	$\mathrm{J} = \mathrm{Nm} = \mathrm{kg\,m^2/s^2}$
watt	$\mathrm{W} = \mathrm{kg\,m^2/s^3}$	coulomb	$\mathrm{C} = \mathrm{As}$
volt	$\mathrm{V} = \mathrm{kg\,m^2/As^3}$	farad	$\mathrm{F} = \mathrm{As/V} = \mathrm{A^2s^4/kg\,m^2}$
ohm	$\Omega = \mathrm{V/A} = \mathrm{kg\,m^2/A^2s^3}$	siemens	$\mathrm{S} = 1/\Omega$
weber	$\mathrm{Wb} = \mathrm{Vs} = \mathrm{kg\,m^2/As^2}$	tesla	$\mathrm{T} = \mathrm{Wb/m^2} = \mathrm{kg/As^2} = \mathrm{kg/Cs}$
henry	$\mathrm{H} = \mathrm{Vs/A} = \mathrm{kg\,m^2/A^2s^2}$	degree Celsius	°C (see definition of kelvin)
lumen	$\mathrm{lm} = \mathrm{cd\,sr}$	lux	$\mathrm{lx} = \mathrm{lm/m^2} = \mathrm{cd\,sr/m^2}$
becquerel	$\mathrm{Bq} = 1/\mathrm{s}$	gray	$\mathrm{Gy} = \mathrm{J/kg} = \mathrm{m^2/s^2}$
sievert	$\mathrm{Sv} = \mathrm{J/kg} = \mathrm{m^2/s^2}$	katal	$\mathrm{kat} = \mathrm{mol/s}$

We note that in all definitions of units, the kilogram only appears to the powers of 1, 0 and −1. Can you try to formulate the reason? Challenge 161 s

The *admitted* non-SI units are *minute*, *hour*, *day* (for time), *degree* $1° = \pi/180$ rad, *minute* $1' = \pi/10\,800$ rad, *second* $1'' = \pi/648\,000$ rad (for angles), *litre*, and *tonne*. All other units are to be avoided.

All SI units are made more practical by the introduction of standard names and abbreviations for the powers of ten, the so-called *prefixes*:*

Power	Name		Power	Name		Power	Name		Power	Name	
10^{1}	deca	da	10^{-1}	deci	d	10^{18}	Exa	E	10^{-18}	atto	a
10^{2}	hecto	h	10^{-2}	centi	c	10^{21}	Zetta	Z	10^{-21}	zepto	z
10^{3}	kilo	k	10^{-3}	milli	m	10^{24}	Yotta	Y	10^{-24}	yocto	y
10^{6}	Mega	M	10^{-6}	micro	μ	unofficial:			Ref. 153		
10^{9}	Giga	G	10^{-9}	nano	n	10^{27}	Xenta	X	10^{-27}	xenno	x
10^{12}	Tera	T	10^{-12}	pico	p	10^{30}	Wekta	W	10^{-30}	weko	w
10^{15}	Peta	P	10^{-15}	femto	f	10^{33}	Vendekta	V	10^{-33}	vendeko	v
						10^{36}	Udekta	U	10^{-36}	udeko	u

▪ SI units form a *complete* system: they cover in a systematic way the full set of observables of physics. Moreover, they fix the units of measurement for all other sciences as well.

* Some of these names are invented (yocto to sound similar to Latin *octo* 'eight', zepto to sound similar to Latin *septem*, yotta and zetta to resemble them, exa and peta to sound like the Greek words ἑξάκις and πεντάκις for 'six times' and 'five times', the unofficial ones to sound similar to the Greek words for nine, ten, eleven and twelve); some are from Danish/Norwegian (atto from *atten* 'eighteen', femto from *femten* 'fifteen'); some are from Latin (from *mille* 'thousand', from *centum* 'hundred', from *decem* 'ten', from *nanus* 'dwarf'); some are from Italian (from *piccolo* 'small'); some are Greek (micro is from μικρός 'small', deca/deka from δέκα 'ten', hecto from ἑκατόν 'hundred', kilo from χίλιοι 'thousand', mega from μέγας 'large', giga from γίγας 'giant', tera from τέρας 'monster').

Translate: I was caught in such a traffic jam that I needed a microcentury for a picoparsec and that my car's fuel consumption was two tenths of a square millimetre. Challenge 162 e

- SI units form a *universal* system: they can be used in trade, in industry, in commerce, at home, in education and in research. They could even be used by extraterrestrial civilizations, if they existed.
- SI units form a *self-consistent* system: the product or quotient of two SI units is also an SI unit. This means that in principle, the same abbreviation, e.g. 'SI', could be used for every unit.

The SI units are not the only possible set that could fulfil all these requirements, but they are the only existing system that does so.*

The meaning of measurement

Every measurement is a comparison with a standard. Therefore, any measurement requires *matter* to realize the standard (even for a speed standard), and *radiation* to achieve the comparison. The concept of measurement thus assumes that matter and radiation exist and can be clearly separated from each other.

Challenge 163 e

Every measurement is a comparison. Measuring thus implies that space and time exist, and that they differ from each other.

Every measurement produces a measurement result. Therefore, every measurement implies the *storage* of the result. The process of measurement thus implies that the situation before and after the measurement can be distinguished. In other terms, every measurement is an *irreversible* process.

Every measurement is a process. Thus every measurement takes a certain amount of time and a certain amount of space.

All these properties of measurements are simple but important. Beware of anybody who denies them.

Planck's natural units

Since the exact form of many equations depends on the system of units used, theoretical physicists often use unit systems optimized for producing simple equations. The chosen units and the values of the constants of nature are related. In microscopic physics, the system of *Planck's natural units* is frequently used. They are defined by setting $c = 1$, $\hbar = 1$, $G = 1$, $k = 1$, $\varepsilon_0 = 1/4\pi$ and $\mu_0 = 4\pi$. Planck units are thus defined from combinations of fundamental constants; those corresponding to the fundamental SI units are given in Table 10.** The table is also useful for converting equations written in natural units back to SI units: just substitute every quantity X by X/X_{Pl}.

Challenge 164 e

* Apart from international units, there are also *provincial* units. Most provincial units still in use are of Roman origin. The mile comes from *milia passum*, which used to be one thousand (double) strides of about 1480 mm each; today a nautical mile, once defined as minute of arc on the Earth's surface, is defined as exactly 1852 m. The inch comes from *uncia*/*onzia* (a twelfth – now of a foot). The pound (from *pondere* 'to weigh') is used as a translation of *libra* – balance – which is the origin of its abbreviation lb. Even the habit of counting in dozens instead of tens is Roman in origin. These and all other similarly funny units – like the system in which all units start with 'f', and which uses furlong/fortnight as its unit of velocity – are now officially defined as multiples of SI units.

** The natural units x_{Pl} given here are those commonly used today, i.e., those defined using the constant $\hbar$, and not, as Planck originally did, by using the constant $h = 2\pi\hbar$. The electromagnetic units can also be defined with other factors than $4\pi\varepsilon_0$ in the expressions: for example, using $4\pi\varepsilon_0\alpha$, with the *fine-structure constant* α, gives $q_{\text{Pl}} = e$. For the explanation of the numbers between brackets, see below.

Page 196

TABLE 10 Planck's (uncorrected) natural units.

NAME	DEFINITION				VALUE
Basic units					
the Planck length	l_{Pl}	=	$\sqrt{\hbar G/c^3}$	=	$1.616\,0(12)\cdot 10^{-35}$ m
the Planck time	t_{Pl}	=	$\sqrt{\hbar G/c^5}$	=	$5.390\,6(40)\cdot 10^{-44}$ s
the Planck mass	m_{Pl}	=	$\sqrt{\hbar c/G}$	=	21.767(16) µg
the Planck current	I_{Pl}	=	$\sqrt{4\pi\varepsilon_0 c^6/G}$	=	$3.479\,3(22)\cdot 10^{25}$ A
the Planck temperature	T_{Pl}	=	$\sqrt{\hbar c^5/Gk^2}$	=	$1.417\,1(91)\cdot 10^{32}$ K
Trivial units					
the Planck velocity	v_{Pl}	=	c	=	0.3 Gm/s
the Planck angular momentum	L_{Pl}	=	$\hbar$	=	$1.1\cdot 10^{-34}$ Js
the Planck action	S_{aPl}	=	$\hbar$	=	$1.1\cdot 10^{-34}$ Js
the Planck entropy	S_{ePl}	=	k	=	13.8 yJ/K
Composed units					
the Planck mass density	ρ_{Pl}	=	$c^5/G^2\hbar$	=	$5.2\cdot 10^{96}$ kg/m^3
the Planck energy	E_{Pl}	=	$\sqrt{\hbar c^5/G}$	=	2.0 GJ = $1.2\cdot 10^{28}$ eV
the Planck momentum	p_{Pl}	=	$\sqrt{\hbar c^3/G}$	=	6.5 Ns
the Planck power	P_{Pl}	=	c^5/G	=	$3.6\cdot 10^{52}$ W
the Planck force	F_{Pl}	=	c^4/G	=	$1.2\cdot 10^{44}$ N
the Planck pressure	p_{Pl}	=	$c^7/G\hbar$	=	$4.6\cdot 10^{113}$ Pa
the Planck acceleration	a_{Pl}	=	$\sqrt{c^7/\hbar G}$	=	$5.6\cdot 10^{51}$ m/s^2
the Planck frequency	f_{Pl}	=	$\sqrt{c^5/\hbar G}$	=	$1.9\cdot 10^{43}$ Hz
the Planck electric charge	q_{Pl}	=	$\sqrt{4\pi\varepsilon_0 c\hbar}$	=	1.9 aC = 11.7 e
the Planck voltage	U_{Pl}	=	$\sqrt{c^4/4\pi\varepsilon_0 G}$	=	$1.0\cdot 10^{27}$ V
the Planck resistance	R_{Pl}	=	$1/4\pi\varepsilon_0 c$	=	30.0 Ω
the Planck capacitance	C_{Pl}	=	$4\pi\varepsilon_0\sqrt{\hbar G/c^3}$	=	$1.8\cdot 10^{-45}$ F
the Planck inductance	L_{Pl}	=	$(1/4\pi\varepsilon_0)\sqrt{\hbar G/c^7}$	=	$1.6\cdot 10^{-42}$ H
the Planck electric field	E_{Pl}	=	$\sqrt{c^7/4\pi\varepsilon_0\hbar G^2}$	=	$6.5\cdot 10^{61}$ V/m
the Planck magnetic flux density	B_{Pl}	=	$\sqrt{c^5/4\pi\varepsilon_0\hbar G^2}$	=	$2.2\cdot 10^{53}$ T

The natural units are important for another reason: whenever a quantity is sloppily called 'infinitely small (or large)', the correct expression is 'as small (or as large) as the corresponding corrected Planck unit'. As explained throughout the text, and especially in the final part, this substitution is possible because almost all Planck units provide, within a correction factor of order 1, the extremal value for the corresponding observable – some an upper and some a lower limit. Unfortunately, these correction factors are not yet widely known. The exact extremal value for each observable in nature is obtained

Vol. VI, page 37

when G is substituted by $4G$ and $4\pi\varepsilon_0$ by $4\pi\varepsilon_0\alpha$ in all Planck quantities. These extremal values, or *corrected Planck units*, are the *true natural units*. To exceed the extremal values is possible only for some extensive quantities. (Can you find out which ones?) Challenge 165 s

Other unit systems

A central aim of research in high-energy physics is the calculation of the strengths of all interactions; therefore it is not practical to set the gravitational constant G to unity, as in the Planck system of units. For this reason, high-energy physicists often only set $c = \hbar = k = 1$ and $\mu_0 = 1/\varepsilon_0 = 4\pi$,* leaving only the gravitational constant G in the equations.

In this system, only one fundamental unit exists, but its choice is free. Often a standard length is chosen as the fundamental unit, length being the archetype of a measured quantity. The most important physical observables are then related by

$$\begin{array}{rl} 1/[l^2] = [E]^2 &= [F] = [B] = [E_{\text{electric}}]\,, \\ 1/[l] = [E] &= [m] = [p] = [a] = [f] = [I] = [U] = [T]\,, \\ 1 &= [v] = [q] = [e] = [R] = [S_{\text{action}}] = [S_{\text{entropy}}] = \hbar = c = k = [\alpha]\,, \\ [l] &= 1/[E] = [t] = [C] = [L] \quad \text{and} \\ [l]^2 &= 1/[E]^2 = [G] = [P] \end{array} \tag{116}$$

where we write $[x]$ for the unit of quantity x. Using the same unit for time, capacitance and inductance is not to everybody's taste, however, and therefore electricians do not use this system.**

Often, in order to get an impression of the energies needed to observe an effect under study, a standard energy is chosen as fundamental unit. In particle physics the most common energy unit is the *electron volt* eV, defined as the kinetic energy acquired by an electron when accelerated by an electrical potential difference of 1 volt ('proton volt' would be a better name). Therefore one has $1\,\text{eV} = 1.6 \cdot 10^{-19}\,\text{J}$, or roughly

$$1\,\text{eV} \approx \tfrac{1}{6}\,\text{aJ} \tag{117}$$

which is easily remembered. The simplification $c = \hbar = 1$ yields $G = 6.9 \cdot 10^{-57}\,\text{eV}^{-2}$ and allows one to use the unit eV also for mass, momentum, temperature, frequency, time and length, with the respective correspondences $1\,\text{eV} \equiv 1.8 \cdot 10^{-36}\,\text{kg} \equiv 5.4 \cdot 10^{-28}\,\text{Ns} \equiv 242\,\text{THz} \equiv 11.6\,\text{kK}$ and $1\,\text{eV}^{-1} \equiv 4.1\,\text{fs} \equiv 1.2\,\mu\text{m}$. Challenge 166 e

* Other definitions for the proportionality constants in electrodynamics lead to the Gaussian unit system often used in theoretical calculations, the Heaviside–Lorentz unit system, the electrostatic unit system, and the electromagnetic unit system, among others. Ref. 154

** In the list, l is length, E energy, F force, E_{electric} the electric and B the magnetic field, m mass, p momentum, a acceleration, f frequency, I electric current, U voltage, T temperature, v speed, q charge, R resistance, P power, G the gravitational constant.

The web page www.chemie.fu-berlin.de/chemistry/general/units_en.html provides a tool to convert various units into each other.

Researchers in general relativity often use another system, in which the *Schwarzschild radius* $r_S = 2Gm/c^2$ is used to measure masses, by setting $c = G = 1$. In this case, mass and length have the *same* dimension, and $\hbar$ has the dimension of an area.

To get some feeling for the unit eV, the following relations are useful. Room temperature, usually taken as 20°C or 293 K, corresponds to a kinetic energy per particle of 0.025 eV or 4.0 zJ. The highest particle energy measured so far belongs to a cosmic ray with an energy of $3 \cdot 10^{20}$ eV or 48 J. Down here on the Earth, an accelerator able to produce an energy of about 105 GeV or 17 nJ for electrons and antielectrons has been built, and one able to produce an energy of 14 TeV or 2.2 μJ for protons will be finished soon. Both are owned by CERN in Geneva and have a circumference of 27 km.

Ref. 155

The lowest temperature measured up to now is 280 pK, in a system of rhodium nuclei held inside a special cooling system. The interior of that cryostat may even be the coolest point in the whole universe. The kinetic energy per particle corresponding to that temperature is also the smallest ever measured: it corresponds to 24 feV or $3.8\,\mathrm{vJ} = 3.8 \cdot 10^{-33}$ J. For isolated particles, the record seems to be for neutrons: kinetic energies as low as 10^{-7} eV have been achieved, corresponding to de Broglie wavelengths of 60 nm.

Ref. 156

Curiosities and fun challenges about units

The Planck length is roughly the de Broglie wavelength $\lambda_B = h/mv$ of a man walking comfortably ($m = 80$ kg, $v = 0.5$ m/s); this motion is therefore aptly called the 'Planck stroll.'

Ref. 157

* *

The Planck mass is equal to the mass of about 10^{19} protons. This is roughly the mass of a human embryo at about ten days of age.

* *

The most precisely measured quantities in nature are the frequencies of certain millisecond pulsars, the frequency of certain narrow atomic transitions, and the Rydberg constant of *atomic* hydrogen, which can all be measured as precisely as the second is defined. The caesium transition that defines the second has a finite linewidth that limits the achievable precision: the limit is about 14 digits.

Ref. 158

* *

The most precise clock ever built, using microwaves, had a stability of 10^{-16} during a running time of 500 s. For longer time periods, the record in 1997 was about 10^{-15}; but values around 10^{-17} seem within technological reach. The precision of clocks is limited for short measuring times by noise, and for long measuring times by drifts, i.e., by systematic effects. The region of highest stability depends on the clock type; it usually lies between 1 ms for optical clocks and 5000 s for masers. Pulsars are the only type of clock for which this region is not known yet; it certainly lies at more than 20 years, the time elapsed at the time of writing since their discovery.

Ref. 159

Ref. 160

* *

The shortest times measured are the lifetimes of certain 'elementary' particles. In particular, the lifetime of certain D mesons have been measured at less than 10^{-23} s. Such times are measured using a bubble chamber, where the track is photographed. Can you

Ref. 161

Challenge 167 s estimate how long the track is? (This is a trick question – if your length cannot be observed with an optical microscope, you have made a mistake in your calculation.)

* *

The longest times encountered in nature are the lifetimes of certain radioisotopes, over 10^{15} years, and the lower limit of certain proton decays, over 10^{32} years. These times are thus much larger than the age of the universe, estimated to be fourteen thousand million
Ref. 162 years.

* *

Variations of quantities are often much easier to measure than their values. For example, in gravitational wave detectors, the sensitivity achieved in 1992 was $\Delta l/l = 3 \cdot 10^{-19}$ for
Ref. 163 lengths of the order of 1 m. In other words, for a block of about a cubic metre of metal it is possible to measure length changes about 3000 times smaller than a proton radius. These set-ups are now being superseded by ring interferometers. Ring interferometers measuring frequency differences of 10^{-21} have already been built; and they are still being
Ref. 164 improved.

Precision and accuracy of measurements

Measurements are the basis of physics. Every measurement has an *error*. Errors are due to lack of precision or to lack of accuracy. *Precision* means how well a result is reproduced when the measurement is repeated; *accuracy* is the degree to which a measurement corresponds to the actual value.

Lack of precision is due to accidental or *random errors*; they are best measured by the *standard deviation*, usually abbreviated σ; it is defined through

$$\sigma^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \,, \tag{118}$$

where $\bar{x}$ is the average of the measurements x_i. (Can you imagine why $n - 1$ is used in
Challenge 168 s the formula instead of n?)

For most experiments, the distribution of measurement values tends towards a normal distribution, also called *Gaussian distribution*, whenever the number of measurements is increased. The distribution, shown in Figure 85, is described by the expression

$$N(x) \approx \mathrm{e}^{-\frac{(x-\bar{x})^2}{2\sigma^2}} \,. \tag{119}$$

The square σ^2 of the standard deviation is also called the *variance*. For a Gaussian distri-
Challenge 169 e bution of measurement values, 2.35σ is the full width at half maximum.

Lack of accuracy is due to *systematic errors*; usually these can only be estimated. This estimate is often added to the random errors to produce a *total experimental error*, some-
Ref. 165 times also called *total uncertainty*. The *relative* error or uncertainty is the ratio between the error and the measured value.

For example, a professional measurement will give a result such as 0.312(6) m. The

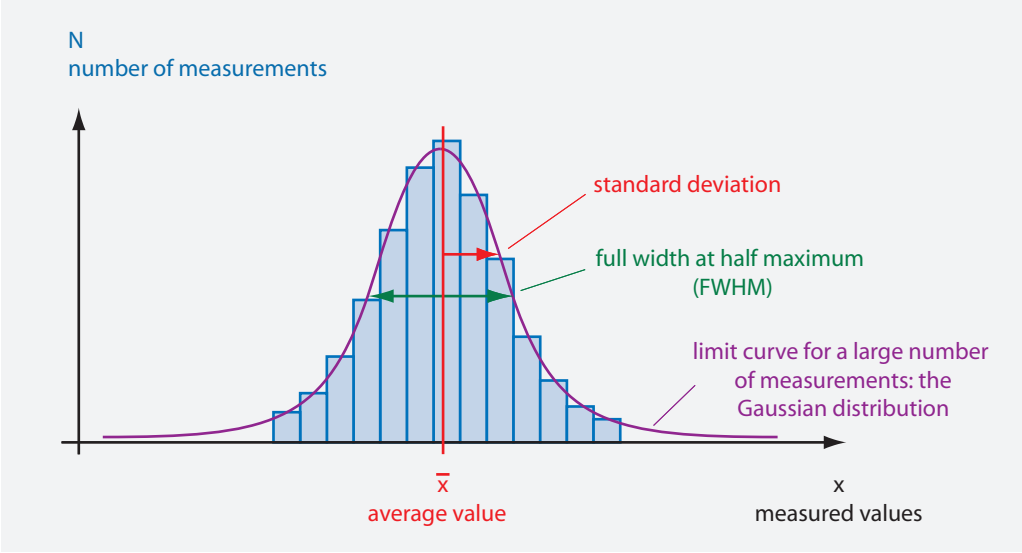


FIGURE 85 A precision experiment and its measurement distribution. The precision is high if the width of the distribution is narrow; the accuracy is high if the centre of the distribution agrees with the actual value.

number between the parentheses is the standard deviation σ, in units of the last digits. As above, a Gaussian distribution for the measurement results is assumed. Therefore, a

Challenge 170 e value of 0.312(6) m implies that the actual value is expected to lie

— within 1σ with 68.3 % probability, thus in this example within 0.312 ± 0.006 m;
— within 2σ with 95.4 % probability, thus in this example within 0.312 ± 0.012 m;
— within 3σ with 99.73 % probability, thus in this example within 0.312 ± 0.018 m;
— within 4σ with 99.9937 % probability, thus in this example within 0.312 ± 0.024 m;
— within 5σ with 99.999 943 % probability, thus in this example within 0.312 ± 0.030 m;
— within 6σ with 99.999 999 80 % probability, thus within 0.312 ± 0.036 m;
— within 7σ with 99.999 999 999 74 % probability, thus within 0.312 ± 0.041 m.

Challenge 171 s (Do the latter numbers make sense?)

Note that standard deviations have one digit; you must be a world expert to use two, and a fool to use more. If no standard deviation is given, a (1) is assumed. As a result, among professionals, 1 km and 1000 m are *not* the same length!

What happens to the errors when two measured values A and B are added or subtracted? If the all measurements are independent – or uncorrelated – the standard deviation of the sum *and* that of difference is given by $\sigma = \sqrt{\sigma_A^2 + \sigma_B^2}$. For both the product or ratio of two measured and uncorrelated values C and D, the result is $\rho = \sqrt{\rho_C^2 + \rho_D^2}$, where the ρ terms are the *relative* standard deviations.

Challenge 172 s Assume you measure that an object moves 1 m in 3 s: what is the measured speed value?

Limits to precision

What are the limits to accuracy and precision? There is no way, even in principle, to measure a length x to a *precision* higher than about 61 digits, because in nature, the ratio between the largest and the smallest measurable length is $\Delta x/x > l_{\mathrm{Pl}}/d_{\mathrm{horizon}} = 10^{-61}$. (Is this ratio valid also for force or for volume?) In the final volume of our text, studies of clocks and metre bars strengthen this theoretical limit.

Challenge 173 e
Vol. VI, page 94

But it is not difficult to deduce more stringent practical limits. No imaginable machine can measure quantities with a higher precision than measuring the diameter of the Earth within the smallest length ever measured, about 10^{-19} m; that is about 26 digits of precision. Using a more realistic limit of a 1000 m sized machine implies a limit of 22 digits. If, as predicted above, time measurements really achieve 17 digits of precision, then they are nearing the practical limit, because apart from size, there is an additional practical restriction: cost. Indeed, an additional digit in measurement precision often means an additional digit in equipment cost.

Physical constants

In physics, general observations are deduced from more fundamental ones. As a consequence, many measurements can be deduced from more fundamental ones. The most fundamental measurements are those of the physical constants.

The following tables give the world's best values of the most important physical constants and particle properties – in SI units and in a few other common units – as published in the standard references. The values are the world averages of the best measurements made up to the present. As usual, experimental errors, including both random and estimated systematic errors, are expressed by giving the standard deviation in the last digits. In fact, behind each of the numbers in the following tables there is a long story which is worth telling, but for which there is not enough room here.

Ref. 166
Ref. 167

In principle, *all* quantitative properties of matter can be calculated with quantum theory – more precisely, equations of the standard model of particle – and a set of *basic* physical constants that are given in the next table. For example, the colour, density and elastic properties of any material can be predicted, in principle, in this way.

Vol. V, page 261

TABLE 11 Basic physical constants.

Quantity	Symbol	Value in SI units	Uncert.[a]
Constants that define the SI measurement units			
Vacuum speed of light [c]	c	299 792 458 m/s	0
Original Planck constant [c]	h	$6.626\,070\,15 \cdot 10^{-34}$ Js	0
Reduced Planck constant, quantum of action	$\hbar$	$1.054\,571\,817\ldots \cdot 10^{-34}$ Js	0
Positron charge [c]	e	0.160 217 6634 aC	0
Boltzmann constant [c]	k	$1.380\,649 \cdot 10^{-23}$ J/K	0
Avogadro's number	N_{A}	$6.022\,140\,76 \cdot 10^{23}$ 1/mol	0
Constant that *should* define the SI measurement units			

TABLE 11 (Continued) Basic physical constants.

QUANTITY	SYMBOL	VALUE IN SI UNITS	UNCERT.[a]
Gravitational constant	G	$6.674\,30(15)\cdot 10^{-11}\,\mathrm{Nm^2/kg^2}$	$2.2\cdot 10^{-5}$
Other fundamental constants			
Number of space-time dimensions		3 + 1	0 [b]
Fine-structure constant [d] or	$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$	1/137.035 999 084(21)	$1.5\cdot 10^{-10}$
e.m. coupling constant	$= g_{\mathrm{em}}(m_e^2c^2)$	= 0.007 297 352 5693(11)	$1.5\cdot 10^{-10}$
Fermi coupling constant [d] or	$G_F/(\hbar c)^3$	$1.166\,3787(6)\cdot 10^{-5}\,\mathrm{GeV^{-2}}$	$5.1\cdot 10^{-7}$
weak coupling constant	$\alpha_w(M_Z) = g_w^2/4\pi$	1/30.1(3)	$1\cdot 10^{-2}$
Strong coupling constant [d]	$\alpha_s(M_Z) = g_s^2/4\pi$	0.1179(10)	$8.5\cdot 10^{-3}$
Weak mixing angle	$\sin^2\theta_W(\overline{MS})$	0.231 22(4)	$1.7\cdot 10^{-4}$
	$\sin^2\theta_W$ (on shell)	0.222 90(30)	$1.3\cdot 10^{-3}$
	$= 1-(m_W/m_Z)^2$		
CKM quark mixing matrix	$\lvert V\rvert$	$\begin{pmatrix} 0.97383(24) & 0.2272(10) & 0.00396(9) \\ 0.2271(10) & 0.97296(24) & 0.04221(80) \\ 0.00814(64) & 0.04161(78) & 0.999100(34) \end{pmatrix}$	
Jarlskog invariant	J	$3.08(18)\cdot 10^{-5}$	
PMNS neutrino mixing m.	$\lvert P\rvert$	$\begin{pmatrix} 0.82(2) & 0.55(4) & 0.150(7) \\ 0.37(13) & 0.57(11) & 0.71(7) \\ 0.41(13) & 0.59(10) & 0.69(7) \end{pmatrix}$	
Electron mass	m_e	$9.109\,383\,7015(28)\cdot 10^{-31}\,\mathrm{kg}$	$3.0\cdot 10^{-10}$
		$5.485\,799\,090\,65(16)\cdot 10^{-4}\,\mathrm{u}$	$2.9\cdot 10^{-11}$
		0.510 998 950 00(15) MeV	$3.0\cdot 10^{-10}$
Muon mass	m_μ	$1.883\,531\,627(42)\cdot 10^{-28}\,\mathrm{kg}$	$2.2\cdot 10^{-8}$
		105.658 3755(23) MeV	$2.2\cdot 10^{-8}$
Tau mass	m_τ	$1.776\,82(12)\,\mathrm{GeV/c^2}$	$6.8\cdot 10^{-5}$
El. neutrino mass	m_{ν_e}	$< 2\,\mathrm{eV/c^2}$	
Muon neutrino mass	m_{ν_μ}	$< 2\,\mathrm{eV/c^2}$	
Tau neutrino mass	m_{ν_τ}	$< 2\,\mathrm{eV/c^2}$	
Up quark mass	u	$21.6(+0.49/-0.26)\,\mathrm{MeV/c^2}$	
Down quark mass	d	$4.67(+0.48/-0.17)\,\mathrm{MeV/c^2}$	
Strange quark mass	s	$93(+11/-5)\,\mathrm{MeV/c^2}$	
Charm quark mass	c	$1.27(2)\,\mathrm{GeV/c^2}$	
Bottom quark mass	b	$4.18(3)\,\mathrm{GeV/c^2}$	
Top quark mass	t	$172.9(0.4)\,\mathrm{GeV/c^2}$	
Photon mass	γ	$< 2\cdot 10^{-54}\,\mathrm{kg}$	
W boson mass	$W^\pm$	$80.379(12)\,\mathrm{GeV/c^2}$	
Z boson mass	Z^0	$91.1876(21)\,\mathrm{GeV/c^2}$	
Higgs mass	H	$125.10(14)\,\mathrm{GeV/c^2}$	
Gluon mass	$g_{1\ldots8}$	$c.\,0\,\mathrm{MeV/c^2}$	

a. Uncertainty: standard deviation of measurement errors.
b. Measured from to 10^{-19} m to 10^{26} m.
c. Defining constant.
d. All coupling constants depend on the 4-momentum transfer, as explained in the section on renormalization. *Fine-structure constant* is the traditional name for the electromagnetic coupling constant g_{em} in the case of a 4-momentum transfer of $Q^2 = m_e^2c^2$, which is the smallest one possible. At higher momentum transfers it has larger values, e.g., $g_{em}(Q^2 = M_W^2c^2) \approx 1/128$. In contrast, the strong coupling constant has lover values at higher momentum transfers; e.g., $\alpha_s(34\,\text{GeV}) = 0.14(2)$.

Page 131

Why do all these basic constants have the values they have? For any basic constant *with a dimension*, such as the quantum of action $\hbar$, the numerical value has only historical meaning. It is $1.054 \cdot 10^{-34}$ Js because of the SI definition of the joule and the second. The question why the value of a *dimensional* constant is not larger or smaller therefore always requires one to understand the origin of some *dimensionless* number giving the ratio between the constant and the corresponding *natural unit* that is defined with c, G, k, N_A and $\hbar$. Details and values for the natural units are given in the dedicated section.

Page 208

In other words, understanding the sizes of atoms, people, trees and stars, the duration of molecular and atomic processes, or the mass of nuclei and mountains, implies understanding the ratios between these values and the corresponding natural units. The key to understanding nature is thus the understanding of all measurement ratios, and thus of all dimensionless constants. This quest, including the understanding of the fine-structure constant α itself, is completed only in the final volume of our adventure.

The basic constants yield the following useful high-precision observations.

TABLE 12 Derived physical constants.

QUANTITY	SYMBOL	VALUE IN SI UNITS	UNCERT.
Vacuum permeability	μ_0	1.256 637 062 12(19) μH/m	$1.5 \cdot 10^{-10}$
Vacuum permittivity	$\varepsilon_0 = 1/\mu_0 c^2$	8.854 187 8128(13) pF/m	$1.5 \cdot 10^{-10}$
Vacuum impedance	$Z_0 = \sqrt{\mu_0/\varepsilon_0}$	376.730 313 668(57) Ω	$1.5 \cdot 10^{-10}$
Loschmidt's number at 273.15 K and 101 325 Pa	N_L	$2.686\,780\,111... \cdot 10^{25}$ 1/m^3	0
Faraday's constant	$F = N_A e$	96 485.332 12... C/mol	0
Universal gas constant	$R = N_A k$	8.314 462 618... J/(mol K)	0
Molar volume of an ideal gas at 273.15 K and 101 325 Pa	$V = RT/p$	22.413 969 54... l/mol	0
Rydberg constant [a]	$R_\infty = m_e c\alpha^2/2h$	10 973 731.568 160(21) m^{-1}	$1.9 \cdot 10^{-12}$
Conductance quantum	$G_0 = 2e^2/h$	77.480 917 29... μS	0
Magnetic flux quantum	$\varphi_0 = h/2e$	2.067 833 848... fWb	0
Josephson frequency ratio	$2e/h$	483.597 8484... THz/V	0
Von Klitzing constant	$h/e^2 = \mu_0 c/2\alpha$	25 812.807 45... Ω	0
Bohr magneton	$\mu_B = e\hbar/2m_e$	9.274 010 0783(28) yJ/T	$3.0 \cdot 10^{-10}$
Classical electron radius	$r_e = e^2/4\pi\varepsilon_0 m_e c^2$	2.817 940 3262(13) fm	$4.5 \cdot 10^{-10}$
Compton wavelength of the electron	$\lambda_C = h/m_e c$	2.426 310 238 67(73) pm	$3.0 \cdot 10^{-10}$
	$\lambda\!\!\!^{-}_C = \hbar/m_e c = r_e/\alpha$	0.386 159 267 96(12) pm	$3.0 \cdot 10^{-10}$

TABLE 12 (Continued) Derived physical constants.

Quantity	Symbol	Value in SI units	Uncert.
Bohr radius [a]	$a_\infty = r_e/\alpha^2$	52.917 721 0903(80) pm	$1.5 \cdot 10^{-10}$
Quantum of circulation	$h/2m_e$	3.636 947 5516(11) cm^2/s	$3.0 \cdot 10^{-10}$
Specific positron charge	e/m_e	175.882 001 076(55) GC/kg	$3.0 \cdot 10^{-10}$
Cyclotron frequency of the electron	$f_c/B = e/2\pi m_e$	27.992 489 872(9) GHz/T	$3.0 \cdot 10^{-10}$
Electron magnetic moment	μ_e	−9.284 764 7043(28) yJ/T	$3.0 \cdot 10^{-10}$
	μ_e/μ_B	−1.001 159 652 181 28(18)	$1.7 \cdot 10^{-13}$
	μ_e/μ_N	$-1\,838.281\,971\,88(11) \cdot 10^3$	$6.0 \cdot 10^{-11}$
Electron g-factor	g_e	−2.002 319 304 362 56(35)	$1.7 \cdot 10^{-13}$
Muon–electron mass ratio	m_μ/m_e	206.768 2830(46)	$2.2 \cdot 10^{-8}$
Muon magnetic moment	μ_μ	$-4.490\,448\,30(10) \cdot 10^{-26}$ J/T	$2.2 \cdot 10^{-8}$
Muon g-factor	g_μ	−2.002 331 8418(13)	$6.3 \cdot 10^{-10}$
Atomic mass unit	$1\,u = m_{^{12}C}/12$	$1.660\,539\,066\,60(50) \cdot 10^{-27}$ kg	$3.0 \cdot 10^{-10}$
Proton mass	m_p	$1.672\,621\,923\,69(51) \cdot 10^{-27}$ kg	$3.1 \cdot 10^{-10}$
		1.007 276 466 621(53) u	$5.3 \cdot 10^{-11}$
		938.272 088 16(29) MeV	$3.1 \cdot 10^{-10}$
Proton–electron mass ratio	m_p/m_e	1 836.152 673 43(11)	$6.0 \cdot 10^{-11}$
Specific proton charge	e/m_p	$9.578\,833\,1560(29) \cdot 10^7$ C/kg	$3.1 \cdot 10^{-10}$
Proton Compton wavelength	$\lambda_{C,p} = h/m_p c$	1.321 409 855 39(40) fm	$3.1 \cdot 10^{-10}$
Nuclear magneton	$\mu_N = e\hbar/2m_p$	$5.050\,783\,7461(15) \cdot 10^{-27}$ J/T	$3.1 \cdot 10^{-10}$
Proton magnetic moment	μ_p	$1.410\,606\,797\,36(60) \cdot 10^{-26}$ J/T	$4.2 \cdot 10^{-10}$
	μ_p/μ_B	$1.521\,032\,202\,30(46) \cdot 10^{-3}$	$3.0 \cdot 10^{-10}$
	μ_p/μ_N	2.792 847 344 63(82)	$2.9 \cdot 10^{-10}$
Proton gyromagnetic ratio	$\gamma_p = 2\mu_p/h$	42.577 478 518(18) MHz/T	$4.2 \cdot 10^{-10}$
Proton g factor	g_p	5.585 694 6893(16)	$2.9 \cdot 10^{-10}$
Neutron mass	m_n	$1.674\,927\,498\,04(95) \cdot 10^{-27}$ kg	$5.7 \cdot 10^{-10}$
		1.008 664 915 95(43) u	$4.8 \cdot 10^{-10}$
		939.565 420 52(54) MeV	$5.7 \cdot 10^{-10}$
Neutron–electron mass ratio	m_n/m_e	1 838.683 661 73(89)	$4.8 \cdot 10^{-10}$
Neutron–proton mass ratio	m_n/m_p	1.001 378 419 31(49)	$4.9 \cdot 10^{-10}$
Neutron Compton wavelength	$\lambda_{C,n} = h/m_n c$	1.319 590 905 81(75) fm	$5.7 \cdot 10^{-10}$
Neutron magnetic moment	μ_n	$-0.966\,236\,51(23) \cdot 10^{-26}$ J/T	$2.4 \cdot 10^{-7}$
	μ_n/μ_B	$-1.041\,875\,63(25) \cdot 10^{-3}$	$2.4 \cdot 10^{-7}$
	μ_n/μ_N	−1.913 042 73(45)	$2.4 \cdot 10^{-7}$
Stefan–Boltzmann constant	$\sigma = \pi^2 k^4/60\hbar^3 c^2$	56.703 744 19... nW/m^2K^4	0
Wien's displacement constant	$b = \lambda_{max} T$	2.897 771 955... mmK	0
		58.789 257 57... GHz/K	0
Electron volt	eV	0.160 217 6634... aJ	0
Bits to entropy conversion const.	$k \ln 2$	10^{23} bit = 0.956 994... J/K	0

TABLE 12 (Continued) Derived physical constants.

QUANTITY	SYMBOL	VALUE IN SI UNITS	UNCERT.
TNT energy content		3.7 to 4.0 MJ/kg	$4 \cdot 10^{-2}$

a. For infinite mass of the nucleus.

Some useful properties of our local environment are given in the following table.

TABLE 13 Astronomical constants.

QUANTITY	SYMBOL	VALUE
Tropical year 1900 [a]	a	31 556 925.974 7 s
Tropical year 1994	a	31 556 925.2 s
Mean sidereal day	d	$23^h56'4.090\,53''$
Average distance Earth–Sun [b]		149 597 870.691(30) km
Astronomical unit [b]	AU	149 597 870 691 m
Light year, based on Julian year [b]	al	9.460 730 472 5808 Pm
Parsec	pc	30.856 775 806 Pm = 3.261 634 al
Earth's mass	$M_{\earth}$	$5.973(1) \cdot 10^{24}$ kg
Geocentric gravitational constant	GM	$3.986\,004\,418(8) \cdot 10^{14}$ m^3/s^2
Earth's gravitational length	$l_{\earth} = 2GM/c^2$	8.870 056 078(16) mm
Earth's equatorial radius [c]	$R_{\earth\mathrm{eq}}$	6378.1366(1) km
Earth's polar radius [c]	$R_{\earth\mathrm{p}}$	6356.752(1) km
Equator–pole distance [c]		10 001.966 km (average)
Earth's flattening [c]	$e_{\earth}$	1/298.25642(1)
Earth's av. density	$\rho_{\earth}$	5.5 Mg/m^3
Earth's age	$T_{\earth}$	4.50(4) Ga = 142(2) Ps
Earth's normal gravity	g	9.806 65 m/s^2
Earth's standard atmospher. pressure	p_0	101 325 Pa
Moon's radius	$R_{\moon\,\mathrm{v}}$	1738 km in direction of Earth
Moon's radius	$R_{\moon\,\mathrm{h}}$	1737.4 km in other two directions
Moon's mass	$M_{\moon}$	$7.35 \cdot 10^{22}$ kg
Moon's mean distance [d]	$d_{\moon}$	384 401 km
Moon's distance at perigee [d]		typically 363 Mm, historical minimum 359 861 km
Moon's distance at apogee [d]		typically 404 Mm, historical maximum 406 720 km
Moon's angular size [e]		average 0.5181° = 31.08′, minimum 0.49°, maximum 0.55°
Moon's average density	$\rho_{\moon}$	3.3 Mg/m^3
Moon's surface gravity	$g_{\moon}$	1.62 m/s^2
Moon's atmospheric pressure	$p_{\moon}$	from 10^{-10} Pa (night) to 10^{-7} Pa (day)
Jupiter's mass	$M_{♃}$	$1.90 \cdot 10^{27}$ kg

TABLE 13 (Continued) Astronomical constants.

Quantity	Symbol	Value
Jupiter's radius, equatorial	$R_{♃}$	71.398 Mm
Jupiter's radius, polar	$R_{♃}$	67.1(1) Mm
Jupiter's average distance from Sun	$D_{♃}$	778 412 020 km
Jupiter's surface gravity	$g_{♃}$	24.9 m/s^2
Jupiter's atmospheric pressure	$p_{♃}$	from 20 kPa to 200 kPa
Sun's mass	$M_\odot$	1.988 43(3) · 10^{30} kg
Sun's gravitational length	$2GM_\odot/c^2$	2.953 250 08(5) km
Heliocentric gravitational constant	$GM_\odot$	132.712 440 018(8) · 10^{18} m^3/s^2
Sun's luminosity	$L_\odot$	384.6 YW
Solar equatorial radius	$R_\odot$	695.98(7) Mm
Sun's angular size		0.53° average; minimum on fourth of July (aphelion) 1888″, maximum on fourth of January (perihelion) 1952″
Sun's average density	$\rho_\odot$	1.4 Mg/m^3
Sun's average distance	AU	149 597 870.691(30) km
Sun's age	$T_\odot$	4.6 Ga
Solar velocity around centre of galaxy	$v_{\odot g}$	220(20) km/s
Solar velocity against cosmic background	$v_{\odot b}$	370.6(5) km/s
Sun's surface gravity	$g_\odot$	274 m/s^2
Sun's lower photospheric pressure	$p_\odot$	15 kPa
Distance to Milky Way's centre		8.0(5) kpc = 26.1(1.6) kal
Milky Way's age		13.6 Ga
Milky Way's size		*c.* 10^{21} m or 100 kal
Milky Way's mass		10^{12} solar masses, *c.* 2 · 10^{42} kg
Most distant galaxy cluster known	SXDF-XCLJ 0218-0510	9.6 · 10^9 al

a. Defining constant, from vernal equinox to vernal equinox; it was once used to define the second. (Remember: π seconds is about a nanocentury.) The value for 1990 is about 0.7 s less, corresponding to a slowdown of roughly 0.2 ms/a. (Watch out: why?) There is even an empirical formula for the change of the length of the year over time.

Challenge 174 s

Ref. 168

b. The truly amazing precision in the average distance Earth–Sun of only 30 m results from time averages of signals sent from Viking orbiters and Mars landers taken over a period of over twenty years. Note that the International Astronomical Union distinguishes the average distance Earth–Sun from the *astronomical unit* itself; the latter is defined as a fixed and exact length. Also the *light year* is a unit defined as an exact number by the IAU. For more details, see www.iau.org/public/measuring.

c. The shape of the Earth is described most precisely with the World Geodetic System. The last edition dates from 1984. For an extensive presentation of its background and its details, see the

www.wgs84.com website. The International Geodesic Union refined the data in 2000. The radii and the flattening given here are those for the 'mean tide system'. They differ from those of the 'zero tide system' and other systems by about 0.7 m. The details constitute a science in itself.
d. Measured centre to centre. To find the precise position of the Moon at a given date, see the www.fourmilab.ch/earthview/moon_ap_per.html page. For the planets, see the page www.fourmilab.ch/solar/solar.html and the other pages on the same site.
e. Angles are defined as follows: 1 degree = $1^\circ = \pi/180$ rad, 1 (first) minute = $1' = 1°/60$, 1 second (minute) = $1'' = 1'/60$. The ancient units 'third minute' and 'fourth minute', each 1/60th of the preceding, are not in use any more. ('Minute' originally means 'very small', as it still does in modern English.)

Challenge 175 s

Some properties of nature at large are listed in the following table. (If you want a challenge, can you determine whether any property of the universe itself is listed?)

TABLE 14 Cosmological constants.

Quantity	Symbol	Value
Cosmological constant	Λ	*c.* $1 \cdot 10^{-52}\,\mathrm{m}^{-2}$
Age of the universe [a]	t_0	$4.333(53) \cdot 10^{17}\,\mathrm{s} = 13.8(0.1) \cdot 10^9\,\mathrm{a}$
(determined from space-time, via expansion, using general relativity)		
Age of the universe [a]	t_0	over $3.5(4) \cdot 10^{17}\,\mathrm{s} = 11.5(1.5) \cdot 10^9\,\mathrm{a}$
(determined from matter, via galaxies and stars, using quantum theory)		
Hubble parameter [a]	H_0	$2.3(2) \cdot 10^{-18}\,\mathrm{s}^{-1} = 0.73(4) \cdot 10^{-10}\,\mathrm{a}^{-1}$
	$= h_0 \cdot 100\,\mathrm{km/s\,Mpc} = h_0 \cdot 1.0227 \cdot 10^{-10}\,\mathrm{a}^{-1}$	
Reduced Hubble parameter [a]	h_0	0.71(4)
Deceleration parameter [a]	$q_0 = -(\ddot{a}/a)_0/H_0^2$	−0.66(10)
Universe's horizon distance [a]	$d_0 = 3ct_0$	$40.0(6) \cdot 10^{26}\,\mathrm{m} = 13.0(2)\,\mathrm{Gpc}$
Universe's topology		trivial up to $10^{26}\,\mathrm{m}$
Number of space dimensions		3, for distances up to $10^{26}\,\mathrm{m}$
Critical density	$\rho_c = 3H_0^2/8\pi G$	$h_0^2 \cdot 1.878\,82(24) \cdot 10^{-26}\,\mathrm{kg/m}^3$
of the universe		$= 0.95(12) \cdot 10^{-26}\,\mathrm{kg/m}^3$
(Total) density parameter [a]	$\Omega_0 = \rho_0/\rho_c$	1.02(2)
Baryon density parameter [a]	$\Omega_{B0} = \rho_{B0}/\rho_c$	0.044(4)
Cold dark matter density parameter [a]	$\Omega_{CDM0} = \rho_{CDM0}/\rho_c$	0.23(4)
Neutrino density parameter [a]	$\Omega_{\nu 0} = \rho_{\nu 0}/\rho_c$	0.001 to 0.05
Dark energy density parameter [a]	$\Omega_{X0} = \rho_{X0}/\rho_c$	0.73(4)
Dark energy state parameter	$w = p_X/\rho_X$	−1.0(2)
Baryon mass	m_b	$1.67 \cdot 10^{-27}\,\mathrm{kg}$
Baryon number density		$0.25(1)\,/\mathrm{m}^3$
Luminous matter density		$3.8(2) \cdot 10^{-28}\,\mathrm{kg/m}^3$
Stars in the universe	n_s	$10^{22\pm1}$
Baryons in the universe	n_b	$10^{81\pm1}$
Microwave background temperature [b]	T_0	2.725(1) K
Photons in the universe	n_γ	10^{89}
Photon energy density	$\rho_\gamma = \pi^2 k^4/15T_0^4$	$4.6 \cdot 10^{-31}\,\mathrm{kg/m}^3$

TABLE 14 (Continued) Cosmological constants.

QUANTITY	SYMBOL	VALUE
Photon number density		$410.89\,/\mathrm{cm}^3$ or $400\,/\mathrm{cm}^3 (T_0/2.7\,\mathrm{K})^3$
Density perturbation amplitude	$\sqrt{S}$	$5.6(1.5) \cdot 10^{-6}$
Gravity wave amplitude	$\sqrt{T}$	$< 0.71\sqrt{S}$
Mass fluctuations on 8 Mpc	σ_8	0.84(4)
Scalar index	n	0.93(3)
Running of scalar index	$\mathrm{d}n/\mathrm{d}\ln k$	−0.03(2)
Planck length	$l_{\mathrm{Pl}} = \sqrt{\hbar G/c^3}$	$1.62 \cdot 10^{-35}\,\mathrm{m}$
Planck time	$t_{\mathrm{Pl}} = \sqrt{\hbar G/c^5}$	$5.39 \cdot 10^{-44}\,\mathrm{s}$
Planck mass	$m_{\mathrm{Pl}} = \sqrt{\hbar c/G}$	21.8 µg
Instants in history [a]	t_0/t_{Pl}	$8.7(2.8) \cdot 10^{60}$
Space-time points inside the horizon [a]	$N_0 = (R_0/l_{\mathrm{Pl}})^3 \cdot (t_0/t_{\mathrm{Pl}})$	$10^{244\pm1}$
Mass inside horizon	M	$10^{54\pm1}\,\mathrm{kg}$

a. The index 0 indicates present-day values.
b. The radiation originated when the universe was 380 000 years old and had a temperature of about 3000 K; the fluctuations ΔT_0 which led to galaxy formation are today about $16 \pm 4\,\mu\mathrm{K} = 6(2) \cdot 10^{-6}\,T_0$.

Vol. II, page 231

USEFUL NUMBERS

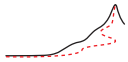
π	$3.14159\,26535\,89793\,23846\,26433\,83279\,50288\,41971\,69399\,37510_5$
e	$2.71828\,18284\,59045\,23536\,02874\,71352\,66249\,77572\,47093\,69995_9$
γ	$0.57721\,56649\,01532\,86060\,65120\,90082\,40243\,10421\,59335\,93992_3$
ln 2	$0.69314\,71805\,59945\,30941\,72321\,21458\,17656\,80755\,00134\,36025_5$
ln 10	$2.30258\,50929\,94045\,68401\,79914\,54684\,36420\,76011\,01488\,62877_2$
$\sqrt{10}$	$3.16227\,76601\,68379\,33199\,88935\,44432\,71853\,37195\,55139\,32521_6$

Ref. 169

If the number π is *normal*, i.e., if all digits and digit combinations in its decimal expansion appear with the same limiting frequency, then every text ever written or yet to be written, as well as every word ever spoken or yet to be spoken, can be found coded in its sequence. The property of normality has not yet been proven, although it is suspected to hold. Does this mean that all wisdom is encoded in the simple circle? No. The property is nothing special: it also applies to the number 0.12345678910111213141516171819202 1... and many others. Can you specify a few examples?

Challenge 176 s

By the way, in the graph of the exponential function e^x, the point (0, 1) is the only point with two rational coordinates. If you imagine painting in blue all points on the plane with two rational coordinates, the plane would look quite bluish. Nevertheless, the graph goes through only one of these points and manages to avoid all the others.



Appendix B

NUMBERS AND VECTOR SPACES

> A mathematician is a machine that transforms coffee into theorems.
> Paul Erdős (b. 1913 Budapest, d. 1996 Warsaw)

Mathematical concepts can all be expressed in terms of 'sets' and 'relations.' Many fundamental concepts were presented in the last chapter. Why does mathematics, given this simple basis, grow into a passion for certain people? How can sets and relations become the center of a person's life? The mathematical appendices present a few more advanced concepts as simply and vividly as possible, for all those who want to understand and to smell the passion for mathematics.

Vol. III, page 285
Ref. 170

Unfortunately, the passion for mathematics is not easy to spot, because like many other professions, also mathematicians *hide* their passions. In mathematics, this is done through formalism and apparent detachment from intuition. Good mathematical teaching however, puts intuition at the beginning. In this appendix we shall introduce the simplest *algebraic* structures. The appendix in the next volume will present some more involved algebraic structures and then the most important *topological* structures; the third basic type of mathematical structures, *order* structures, are not so important in physics – with one exception: the definition of the real numbers contains an order structure.

Mathematicians are concerned not only with the exploration of concepts, but also with their *classification*. Whenever a new mathematical concept is introduced, mathematicians try to classify all the possible cases and types. This has been achieved most spectacularly for the different types of numbers, for finite simple groups and for many types of spaces and manifolds.

Numbers as mathematical structures

> A person who can solve $x^2 - 92y^2 = 1$ in less than a year is a mathematician.
> Brahmagupta (b. 598 Sindh, d. 668) (implied: solve in *integers*)

Challenge 177 ny

Children know: numbers are entities that can be added and multiplied. Mathematicians are more discerning. Any mathematical system with the same basic properties as the *natural* numbers is called a *semi-ring*. Any mathematical system with the same basic properties as the *integers* is called a *ring*. (The terms are due to David Hilbert. Both structures can also be finite rather than infinite.) More precisely, a *ring* $(R, +, \cdot)$ is a set R of ele-

ments with two binary operations, called *addition* and *multiplication*, usually written + and · (the latter may simply be understood, thus without explicit notation), for which the following properties hold for all elements $a, b, c \in R$:

— R is a commutative group with respect to addition, i.e.
  $a + b \in R$, $a + b = b + a$, $a + 0 = a$, $a + (-a) = a - a = 0$ and $a + (b + c) = (a + b) + c$;
— R is closed under multiplication, i.e., $ab \in R$;
— multiplication is associative, i.e., $a(bc) = (ab)c$;
— distributivity holds, i.e., $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

Many authors add the axiom

— a multiplicative unit exists, i.e., $1a = a1 = a$.

Defining properties such as these are called *axioms*. We stress that axioms are *not* basic beliefs, as is often stated or implied; axioms are the basic properties used in the definition of a concept: in this case, of a ring. With the last axiom, one also speaks of a *unital ring*.

A *semi-ring* is a set satisfying all the axioms of a ring, except that the existence of neutral and negative elements for addition is replaced by the weaker requirement that if $a + c = b + c$ then $a = b$. Sloppily, a semi-ring is a ring 'without' negative elements.

To incorporate division and define the rational numbers, we need another concept. A *number field* or *field* K is a ring with

— a multiplicative identity 1, such that all elements a obey $1a = a$;
— at least one element different from zero; and most importantly
— a (multiplicative) inverse a^{-1} for every element $a \neq 0$.

A ring or field is said to be *commutative* if the multiplication is commutative. A non-commutative field is also called a *skew field*. Fields can be finite or infinite. (A field or a ring is characterized by its *characteristic* p. This is the smallest number of times one has to add 1 to itself to give zero. If there is no such number the characteristic is set to 0. p is always a prime number or zero.) All finite fields are commutative. In a field, all equations of the type $cx = b$ and $xc = b$ ($c \neq 0$) have solutions for x; there is a unique solution if $b \neq 0$. To sum up sloppily by focusing on the most important property, a field is a set of elements for which, together with addition, subtraction and multiplication, a *division* (by non-zero elements) is also defined. The *rational numbers* are the simplest field that incorporates the integers.

The system of the *real numbers* is the minimal extension of the rationals which is complete and totally ordered.* Can you show that $\sqrt{2}$ is a real, but not a rational number? Challenge 178 e

* A set is mathematically *complete* if physicists call it continuous. More precisely, a set of numbers is *complete* if every non-empty subset that is bounded above has a least upper bound.

A set is *totally ordered* if there exists a binary relation $\leqslant$ between pairs of elements such that for all elements a and b
— if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$;
— if $a \leqslant b$ and $b \leqslant a$, then $a = b$;
— $a \leqslant b$ or $b \leqslant a$ holds.
In summary, a set is totally ordered if there is a binary relation that allows saying about any two elements which one is the predecessor of the other in a consistent way. This is the fundamental – and also the only – order structure used in physics.

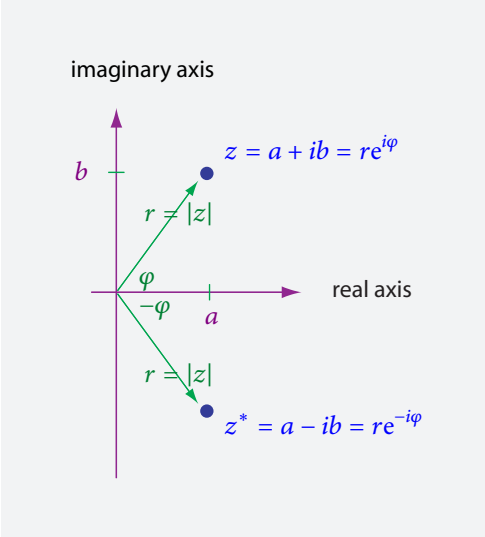


FIGURE 86 Complex numbers are points in the two-dimensional plane; a complex number z and its conjugate z^* can be described in cartesian form or in polar form.

In classical physics and quantum theory, it is always stressed that measurement results are and must be real numbers. But are all real numbers possible measurement results? In other words, are all measurement results just a subset of the reals?

Challenge 179 s

However, the concept of 'number' is not limited to these examples. It can be generalized in several ways. The simplest generalization is achieved by extending the real numbers to manifolds of more than one dimension.

Ref. 171

Complex numbers

In nature, complex numbers are a useful way to describe in compact form systems and situations that contain a *phase*. Complex numbers are thus useful to describe waves of any kind.

Complex numbers form a two-dimensional manifold. A complex number is defined, in its cartesian form, by $z = a + ib$, where a and b are real numbers, and i is a new symbol, the so-called *imaginary unit*. Under multiplication, the generators of the complex numbers, 1 and i, obey

$$\begin{array}{c|cc} \cdot & 1 & i \\ \hline 1 & 1 & i \\ i & i & -1 \end{array} \tag{120}$$

often summarized as $i = +\sqrt{-1}$. In a complex number $z = a + ib$, a is called the *real part*, and b the *complex part*. They are illustrated in Figure 86.

The *complex conjugate* z^*, also written $\bar{z}$, of a complex number $z = a + ib$ is defined as $z^* = a - ib$. The *absolute value* $|z|$ of a complex number is defined as $|z| = \sqrt{zz^*} = \sqrt{z^*z} = \sqrt{a^2 + b^2}$. It defines a *norm* on the vector space of the complex numbers. From $|wz| = |w|\,|z|$ follows the *two-squares theorem*

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) = (a_1b_1 - a_2b_2)^2 + (a_1b_2 + a_2b_1)^2 \tag{121}$$

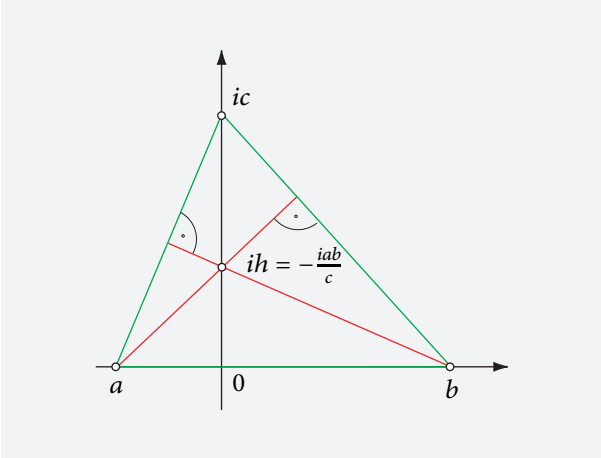


FIGURE 87 A property of triangles easily provable with complex numbers.

valid for all real numbers a_i, b_i. It was already known, in its version for integers, to Diophantus of Alexandria in the third century CE.

Complex numbers can also be written as ordered pairs (a, A) of real numbers, with their addition defined as $(a, A) + (b, B) = (a + b, A + B)$ and their multiplication defined as $(a, A) \cdot (b, B) = (ab - AB, aB + bA)$. This notation allows us to identify the complex numbers with the *points* on a plane or, if we prefer, to *arrows* in a plane. Translating the definition of multiplication into geometrical language allows us to rapidly prove certain geometrical theorems, such as the one of Figure 87.

Challenge 180 e

Complex numbers $a + ib$ can also be represented as 2×2 matrices

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{with} \quad a, b \in \mathbb{R} \,. \tag{122}$$

Matrix addition and multiplication then correspond to complex addition and multiplication. In this way, complex numbers can be represented by a special type of real matrix. What is $|z|$ in matrix language?

Challenge 181 s

The set $\mathbb{C}$ of complex numbers with addition and multiplication as defined above forms both a commutative two-dimensional field and a vector space over $\mathbb{R}$. In the field of complex numbers, quadratic equations $az^2 + bz + c = 0$ for an unknown z always have *two* solutions (for $a \neq 0$ and counting multiplicity).

Page 235

Challenge 182 e

Complex numbers can be used to describe the points of a plane. A rotation around the origin can be described by multiplication by a complex number of unit length. Other two-dimensional quantities can also be described with complex numbers. Electrical engineers use complex numbers to describe quantities with phases, such as alternating currents or electrical fields in space.

Writing complex numbers of unit length as $\cos\theta + i\sin\theta$ is a useful method for remembering angle addition formulae. Since one has $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$, one can easily deduce formulae such as $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$. By the way, the unit complex numbers form the Lie group SO(2)=U(1).

Challenge 183 e

Challenge 184 e

Every complex number can be written as

$$z = re^{i\varphi} \,. \tag{123}$$

This polar form of writing complex numbers is the reason for introducing them in the first place. The angle φ is called the *phase*; the real number $r = |z|$ is called the *absolute value* or the *modulus* or the *magnitude*. When used to describe oscillations or waves, it makes sense to call r the *amplitude*. The complex exponential function is periodic in $2\pi i$; in other words, we have

$$e^1 = e^{1+2\pi i} \,, \tag{124}$$

which shows the property we expect from a phase angle.

If one uses the last equation twice, one may write

$$e^1 = e^{1+2\pi i} = (e^{1+2\pi i})^{1+2\pi i} = e^{(1+2\pi i)(1+2\pi i)} = e^{1-4\pi^2+4\pi i} = e^{1-4\pi^2} \,. \tag{125}$$

Challenge 185 e Oops, that would imply $\pi = 0$! What is wrong here?

Complex numbers can also be used to describe Euclidean plane geometry. Rotations, translations and other isometries, but also reflections, glide reflections and scaling are easily described by simple operations on the complex numbers that describe the coordinate of points.

By the way, there are exactly as many complex numbers as there are real numbers. Challenge 186 s Can you show this?

> Love is complex: it has real and imaginary parts.
> Anonymous

Quaternions

The positions of the points on a line can be described by real numbers. Complex numbers can be used to describe the positions of the points of a plane. It is natural to try to generalize the idea of a number to higher-dimensional spaces. However, it turns out that no useful number system can be defined for *three*-dimensional space. A new number system, the *quaternions*, can be constructed which corresponds the points of *four*-dimensional space, but only if the commutativity of multiplication is sacrificed. No useful number system can be defined for dimensions other than 1, 2 and 4.

The quaternions were discovered by several mathematicians in the nineteenth century, among them Hamilton,* who studied them for much of his life. In fact, Maxwell's theory of electrodynamics was formulated in terms of quaternions before three-dimensional vectors were used. Ref. 173

Vol. V, page 358 Under multiplication, the quaternions $\mathbb{H}$ form a 4-dimensional algebra over the reals

* William Rowan Hamilton (b. 1805 Dublin, d. 1865 Dunsink), child prodigy and famous mathematician, named the quaternions after an expression from the Vulgate (Acts. 12: 4).

with a basis $1, i, j, k$ satisfying

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

(126)

These relations are also often written $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. The quaternions $1, i, j, k$ are also called *basic units* or *generators*. The lack of symmetry across the diagonal of the table shows the non-commutativity of quaternionic multiplication. With the quaternions, the idea of a non-commutative product appeared for the first time in mathematics. However, the multiplication of quaternions *is* associative. As a consequence of non-commutativity, polynomial equations in quaternions have many more solutions than in complex numbers: just search for all solutions of the equation $X^2 + 1 = 0$ to convince yourself of it.

Challenge 187 s

Every quaternion X can be written in the form

$$X = x_0 + x_1 i + x_2 j + x_3 k = x_0 + \boldsymbol{v} = (x_0, x_1, x_2, x_3) = (x_0, \boldsymbol{v}) \,, \tag{127}$$

where x_0 is called the *scalar* part and $\boldsymbol{v}$ the *vector* part. The multiplication is thus defined as $(x, \boldsymbol{v})(y, \boldsymbol{w}) = (xy - \boldsymbol{v}\cdot\boldsymbol{w}, x\boldsymbol{w} + y\boldsymbol{v} + \boldsymbol{v}\times\boldsymbol{w})$. The multiplication of two general quaternions can be written as

$$\begin{aligned}(a_1, b_1, c_1, d_1)(a_2, b_2, c_2, d_2) = (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 \,,\ a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2,\\ a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2, a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2) \,.\end{aligned} \tag{128}$$

The conjugate quaternion $\overline{X}$ is defined as $\overline{X} = x_0 - \boldsymbol{v}$, so that $\overline{XY} = \overline{Y}\ \overline{X}$. The *norm* $|X|$ of a quaternion X is defined as $|X|^2 = X\overline{X} = \overline{X}X = x_0^2 + x_1^2 + x_2^2 + x_3^2 = x_0^2 + \boldsymbol{v}^2$. The norm is multiplicative, i.e., $|XY| = |X|\,|Y|$.

Unlike complex numbers, every quaternion is related to its complex conjugate by

$$\overline{X} = -\tfrac{1}{2}(X + iXi + jXj + kXk) \,. \tag{129}$$

No relation of this type exists for complex numbers. In the language of physics, a complex number and its conjugate are independent variables; for quaternions, this is not the case. As a result, functions of quaternions are less useful in physics than functions of complex variables.

The relation $|XY| = |X|\,|Y|$ implies the *four-squares theorem*

$$\begin{aligned}&(a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2)\\ &= (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4)^2 + (a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3)^2\\ &+ (a_1b_3 + a_3b_1 + a_4b_2 - a_2b_4)^2 + (a_1b_4 + a_4b_1 + a_2b_3 - a_3b_2)^2\end{aligned} \tag{130}$$

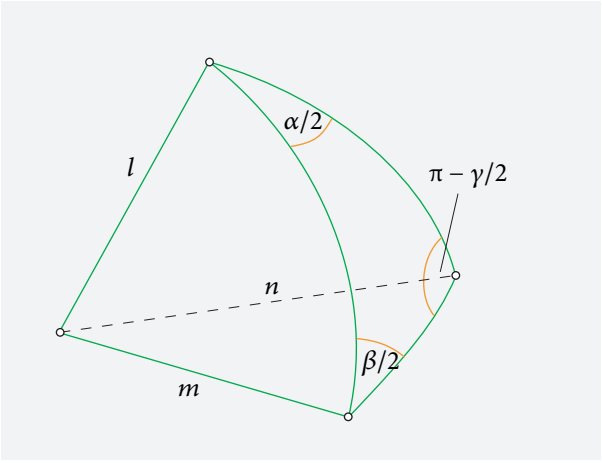


FIGURE 88 Combinations of rotations.

valid for all real numbers a_i and b_i, and thus also for any set of eight integers. It was discovered in 1748 by Leonhard Euler (1707–1783) when trying to prove that each integer is the sum of four squares. (The latter fact was proved only in 1770, by Joseph Lagrange.)

Hamilton thought that a quaternion with zero scalar part, which he simply called a *vector* (a term which he invented), could be identified with an ordinary three-dimensional translation vector; but this is wrong. Such a quaternion is now called a *pure*, or *homogeneous*, or *imaginary* quaternion. The product of two pure quaternions $V = (0, \boldsymbol{v})$ and $W = (0, \boldsymbol{w})$ is given by $VW = (-\boldsymbol{v} \cdot \boldsymbol{w}, \boldsymbol{v} \times \boldsymbol{w})$, where $\cdot$ denotes the scalar product and $\times$ denotes the vector product. Note that any quaternion can be written as the ratio of two pure quaternions.

In reality, a pure quaternion $(0, \boldsymbol{v})$ does not behave like a translation vector under coordinate transformations; in fact, a pure quaternion represents a *rotation* by the angle π or 180° around the axis defined by the direction $\boldsymbol{v} = (v_x, v_y, v_z)$.

Challenge 188 ny

It turns out that in three-dimensional space, a *general* rotation about the origin can be described by a *unit* quaternion Q, also called a *normed* quaternion, for which $|Q| = 1$. Such a quaternion can be written as $(\cos\theta/2, \boldsymbol{n}\sin\theta/2)$, where $\boldsymbol{n} = (n_x, n_y, n_z)$ is the normed vector describing the direction of the rotation axis and θ is the rotation angle. Such a unit quaternion $Q = (\cos\theta/2, \boldsymbol{n}\sin\theta/2)$ rotates a pure quaternion $V = (0, \boldsymbol{v})$ into another pure quaternion $W = (0, \boldsymbol{w})$ given by

$$W = QVQ^* \,. \tag{131}$$

Thus, if we use pure quaternions such as V or W to describe positions, we can use unit quaternions to describe rotations and to calculate coordinate changes. The concatenation of two rotations is then given by the product of the corresponding unit quaternions. Indeed, a rotation by an angle α about the axis $\boldsymbol{l}$ followed by a rotation by an angle β about the axis $\boldsymbol{m}$ gives a rotation by an angle γ about the axis $\boldsymbol{n}$, with the values determined by

$$(\cos\gamma/2, \sin\gamma/2\boldsymbol{n}) = (\cos\beta/2, \sin\beta/2\boldsymbol{m})(\cos\alpha/2, \sin\alpha/2\boldsymbol{l}) \,. \tag{132}$$

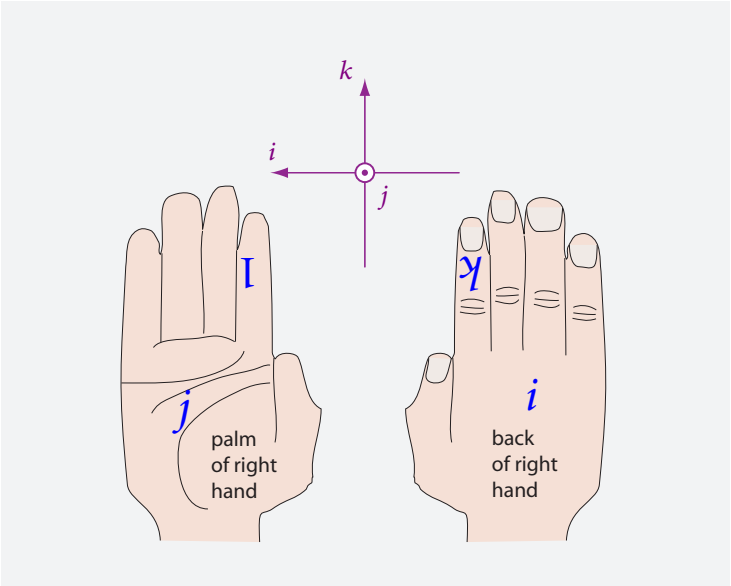


FIGURE 89 The top and back of the right hand, and the quaternions.

One way to show the result graphically is given in Figure 88. By drawing a triangle on a unit sphere, and taking care to remember the factor 1/2 in the angles, the combination of two rotations can be simply determined.

The interpretation of quaternions as rotations is also illustrated, in a somewhat different way, in the motion of any hand. To see this, take a green marker and write the letters 1, i, j and k on your hand as shown in Figure 89. Defining the three possible 180° rotation axes as shown in the figure and taking concatenation as multiplication, the motion of the right hand follows the same 'laws' as those of pure unit quaternions. (One needs to distinguish $+i$ and $-i$, and the same for the other units, by the sense of the arm twist. And the result of a multiplication is that letter that can be read by a person facing you.) You can show that $i^2 = j^2 = k^2 = -1$, that $i^4 = 1$, and conform all other quaternion relations. The model also shows that the rotation angle of the arm is half the rotation angle of the corresponding quaternion. In other words, quaternions can be used to describe the belt trick, if the multiplication VW of two quaternions is taken to mean that rotation V is performed *after* rotation W. Quaternions, like human hands, thus behave like a spin 1/2 particle. Quaternions and spinors are isomorphic.

Ref. 174

Challenge 189 e

Challenge 190 s

Page 130

The reason for the half-angle behaviour of rotations can be specified more precisely using mathematical language. The rotations in three dimensions around a point form the 'special orthogonal group' in three dimensions, which is called SO(3). But the motions of a hand attached to a shoulder via an arm form a different group, isomorphic to the Lie group SU(2). The difference is due to the appearance of half angles in the parametrization of rotations; indeed, the above parametrizations imply that a rotation by 2π corresponds to a multiplication by -1. Only in the twentieth century was it realized that there exist fundamental physical observables that behaves like hands attached to arms: they are called *spinors*. More on spinors can be found in the section on permutation symmetry, where belts are used as an analogy as well as arms. In short, the group SU(2) formed by

Vol. V, page 369

Page 130

Ref. 175 the unit quaternions is the *double cover* of the rotation group SO(3).

The simple representation of rotations and positions with quaternions is used by computer programmes in robotics, in astronomy and in flight simulation. In the software used to create three-dimensional images and animations, visualization software, quaternions are often used to calculate the path taken by repeatedly reflected light rays and thus give surfaces a realistic appearance.

The algebra of the quaternions is the only associative, non-commutative, finite-dimensional normed algebra with an identity over the field of real numbers. Quaternions form a non-commutative field, i.e., a skew field, in which the inverse of a quaternion X is $\overline{X}/|X|$. We can therefore define division of quaternions (while being careful to distinguish XY^{-1} and $Y^{-1}X$). Therefore quaternions are said to form a *division algebra*. In fact, the quaternions $\mathbb{H}$, the complex numbers $\mathbb{C}$ and the reals $\mathbb{R}$ are the only three finite-dimensional associative division algebras. In other words, the skew-field of quaternions is the only finite-dimensional real associative non-commutative algebra without divisors of zero. The *centre* of the quaternions, i.e., the set of quaternions that commute with all other quaternions, is just the set of real numbers.

Quaternions can be represented as matrices of the form

$$\begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix} \quad \text{with} \quad A, B \in \mathbb{C} \quad \text{thus} \quad A = a + ib, B = c + id \,, \tag{133}$$

or, alternatively, as

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} \quad \text{with} \quad a, b, c, d \in \mathbb{R} \,, \tag{134}$$

where the quaternion X then is given as $X = A + Bj = a + ib + jc + kd$. Matrix addition and multiplication then corresponds to quaternionic addition and multiplication.

The generators of the quaternions can be realized as

$$1 : \sigma_0 \quad , \quad i : -i\sigma_1 \quad , \quad j : -i\sigma_2 \quad , \quad k : -i\sigma_3 \tag{135}$$

where the σ_n are the Pauli spin matrices.*

* The *Pauli spin matrices* are the complex Hermitean matrices

$$\sigma_0 = \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{136}$$

all of whose eigenvalues are ± 1; they satisfy the relations $[\sigma_i, \sigma_k]_+ = 2\,\delta_{ik}$ and $[\sigma_i, \sigma_k] = 2i\,\varepsilon_{ikl}\,\sigma_l$. The linear combinations $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm \sigma_2)$ are also frequently used. By the way, another possible representation of the quaternions is $i : i\sigma_3, j : i\sigma_2, k : i\sigma_1$.

Real 4×4 representations are not unique, as the alternative representation

$$\begin{pmatrix} a & b & -d & -c \\ -b & a & -c & d \\ d & c & a & b \\ c & -d & -b & a \end{pmatrix} \tag{137}$$

Challenge 191 ny shows. No representation of quaternions by 3×3 matrices is possible.

These matrices contain real and complex elements, which pose no special problems. In contrast, when matrices with quaternionic elements are constructed, care has to be taken, because quaternionic multiplication is not commutative, so that simple relations such as $\mathrm{tr}AB = \mathrm{tr}BA$ are *not* generally valid.

What can we learn from quaternions about the description of nature? First of all, we see that binary rotations are similar to positions, and thus to translations: all are represented by 3-vectors. Are rotations the basic operations of nature? Is it possible that translations are only 'shadows' of rotations? The connection between translations and rotations is investigated in the last volume of our mountain ascent.

Vol. VI, page 174

When Maxwell wrote down his equations of electrodynamics, he used quaternion notation. (The now usual 3-vector notation was introduced later by Hertz and Heaviside.) The equations can be written in various ways using quaternions. The simplest is achieved when one keeps a distinction between $\sqrt{-1}$ and the units i, j, k of the quaternions. One then can write all of electrodynamics in a single equation:

Vol. III, page 76

Ref. 173

Challenge 192 s

$$\mathrm{d}F = -\frac{Q}{\varepsilon_0} \tag{138}$$

where F is the generalized electromagnetic field and Q the generalized charge. These are defined by

$$\begin{aligned} F &= E + \sqrt{-1}\,cB \\ E &= iE_x + jE_y + kE_z \\ B &= iB_x + jB_y + kB_z \\ \mathrm{d} &= \delta + \sqrt{-1}\,\partial_t/c \\ \delta &= i\partial_x + j\partial_y + k\partial_z \\ Q &= \rho + \sqrt{-1}\,J/c \end{aligned} \tag{139}$$

where the fields E and B and the charge distributions ρ and J have the usual meanings. The content of equation (138) for the electromagnetic field is exactly the same as the usual formulation.

Despite their charm and their four-dimensionality, quaternions do not seem to be useful for the reformulation of special relativity; the main reason for this is the sign in the expression for their norm. Therefore, relativity and space-time are usually described using real numbers. And even if quaternions were useful, they would not provide additional insights into physics or into nature.

Octonions

In the same way that quaternions are constructed from complex numbers, *octonions* can be constructed from quaternions. They were first investigated by Arthur Cayley (1821–1895). Under multiplication, *octonions* (or *octaves*) are the elements of an eight-dimensional algebra over the reals with the generators $1, i_n$ with $n = 1 \dots 7$ satisfying

$$\begin{array}{c|cccccccc}
\cdot & 1 & i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 \\
\hline
1 & 1 & i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 \\
i_1 & i_1 & -1 & i_3 & -i_2 & i_5 & -i_4 & i_7 & -i_6 \\
i_2 & i_2 & -i_3 & -1 & i_1 & -i_6 & i_7 & i_4 & -i_5 \\
i_3 & i_3 & i_2 & -i_1 & -1 & i_7 & i_6 & -i_5 & -i_4 \\
i_4 & i_4 & -i_5 & i_6 & -i_7 & -1 & i_1 & -i_2 & i_3 \\
i_5 & i_5 & i_4 & -i_7 & -i_6 & -i_1 & -1 & i_3 & i_2 \\
i_6 & i_6 & -i_7 & -i_4 & i_5 & i_2 & -i_3 & -1 & i_1 \\
i_7 & i_7 & i_6 & i_5 & i_4 & -i_3 & -i_2 & -i_1 & -1
\end{array} \tag{140}$$

In fact, 479 other, equivalent multiplication tables are also possible. This algebra is called the *Cayley algebra*; it has an identity and a unique division. The algebra is non-commutative, and also non-associative. It is, however, *alternative*, meaning that for all elements x and y, one has $x(xy) = x^2 y$ and $(xy)y = xy^2$: a property somewhat weaker than associativity. It is the only 8-dimensional real alternative algebra without zero divisors. Because it is not associative, the set $\mathbb{O}$ of all octonions does not form a field, nor even a ring, so that the old designation of 'Cayley numbers' has been abandoned. The octonions are the most general hypercomplex 'numbers' whose norm is multiplicative. Its generators obey $(i_n i_m) i_l = \pm i_n (i_m i_l)$, where the minus sign, which shows the non-associativity, is valid for combinations of indices that are not quaternionic, such as 1-2-4.

Octonions can be represented as matrices of the form

$$\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \text{where } A, B \in \mathbb{H}\,, \quad \text{or as real } 8 \times 8 \text{ matrices.} \tag{141}$$

Matrix multiplication then gives the same result as octonionic multiplication.

The relation $|wz| = |w|\,|z|$ allows one to deduce the impressive *eight-squares theorem*

$$\begin{aligned}(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + b_6^2 + b_7^2 + b_8^2) \\ = (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7 - a_8b_8)^2 \\ + (a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3 + a_5b_6 - a_6b_5 + a_7b_8 - a_8b_7)^2 \\ + (a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2 - a_5b_7 + a_6b_8 + a_7b_5 - a_8b_6)^2 \\ + (a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1 + a_5b_8 + a_6b_7 - a_7b_6 - a_8b_5)^2 \\ + (a_1b_5 - a_2b_6 + a_3b_7 - a_4b_8 + a_5b_1 + a_6b_2 - a_7b_3 + a_8b_4)^2 \\ + (a_1b_6 + a_2b_5 - a_3b_8 - a_4b_7 - a_5b_2 + a_6b_1 + a_7b_4 + a_8b_3)^2 \\ + (a_1b_7 - a_2b_8 - a_3b_5 + a_4b_6 + a_5b_3 - a_6b_4 + a_7b_1 + a_8b_2)^2 \\ + (a_1b_8 + a_2b_7 + a_3b_6 + a_4b_5 - a_5b_4 - a_6b_3 - a_7b_2 + a_8b_1)^2 \end{aligned} \tag{142}$$

valid for all real numbers a_i and b_i and thus in particular also for all integers. (There are many variations of this expression, with different possible sign combinations.) The theorem was discovered in 1818 by Carl Ferdinand Degen (1766–1825), and then rediscovered in 1844 by John Graves and in 1845 by Arthur Cayley. There is no generalization to higher numbers of squares, a fact proved by Adolf Hurwitz (1859–1919) in 1898.

The octonions can be used to show that a vector product can be defined in more than three dimensions. A *vector product* or *cross product* is an operation $\times$ satisfying

$$\begin{aligned} u \times v &= -v \times u \quad \textit{anticommutativity} \\ (u \times v)\, w &= u\,(v \times w) \quad \textit{exchange rule.} \end{aligned} \tag{143}$$

Using the definition

$$X \times Y = \frac{1}{2}(XY - YX)\,, \tag{144}$$

the cross products of imaginary quaternions, i.e., of quaternions of the type $(0, \boldsymbol{u})$, are again imaginary, and correspond to the usual, three-dimensional vector product, thus fulfilling (143). Interestingly, it is possible to use definition (144) for *octonions* as well. In that case, the product of imaginary octonions is also imaginary, and (143) is again satisfied. In fact, this is the only other non-trivial example of a vector product.

Ref. 171

Challenge 193 e

In summary: *A vector product exists only in three and in seven dimensions.* Many scholars have conjectured that this relation is connected with a possible ten-dimensionality of nature; however, these speculations have not met with any success.

The symmetries of the forces in nature lead to a well-known question. The unit complex numbers from the Lie group U(1) and the unit quaternions the Lie group SU(2). Do the unit octonions form the Lie group SU(3)?

Challenge 194 s

Other types of numbers

The process of constructing new systems of hypercomplex 'numbers' or real algebras by 'doubling' a given one can be continued ad infinitum. However, octonions, *sedenions* and

all the following doublings are neither rings nor fields, but only non-associative algebras with unity. Other finite-dimensional algebras with unit element over the reals, once called hypercomplex 'numbers', can also be defined: they include the so-called 'dual numbers', 'double numbers', 'Clifford–Lifshitz numbers' etc. They play no role in physics.

Mathematicians have also defined number fields which have 'one and a bit' dimensions, such as algebraic number fields. There is also a generalization of the concept of integers to the complex domain: the *Gaussian integers*, defined as $n + im$, where n and m are ordinary integers. Gauss even defined what are now known as *Gaussian primes*. (Can you find out how?) They are not used in the description of nature, but are important in number theory, the exploration of the properties of integers.

Ref. 176

Challenge 195 s

Physicists used to call quantum-mechanical operators 'q-numbers.' But this term has now fallen out of fashion.

Another way in which the natural numbers can be extended is to include numbers larger than infinite. The most important such classes of *transfinite number* are the *ordinals*, the *cardinals* and the *surreals*. The ordinals are essentially an extension of the integers beyond infinity, whereas the surreals are a continuous extension of the reals, also beyond infinity. Loosely speaking, among the transfinites, the ordinals have a similar role as the integers have among the reals; the surreals fill in all the gaps between the ordinals, like the reals do for integers. Interestingly, many series that diverge in $\mathbb{R}$ converge in the surreals. Can you find one example?

Ref. 177

Vol. III, page 293

Challenge 196 ny

The surreals include infinitely small numbers, as do the numbers of *nonstandard analysis*, also called *hyperreals*. In both number systems, in contrast to real numbers, the numbers 1 and 0.999 999... (where an infinite, but hyperfinite string of nines is implied) do not coincide, but are separated by infinitely many other numbers. We explored surreals earlier on. Nonstandard numbers can be used to define the infinitesimals used in integration and differentiation, even at secondary school level.

Ref. 171

Vol. III, page 295

Ref. 172

From vector spaces to Hilbert spaces

Vector spaces, also called *linear spaces*, are mathematical generalizations of certain aspects of the intuitive three-dimensional space. A set of elements any two of which can be added together and any one of which can be multiplied by a number is called a vector space, if the result is again in the set and the usual rules of calculation hold.

More precisely, a *vector space* over a number field K is a set of elements, called *vectors*, for which a vector addition and a *scalar multiplication* is defined, such that for all vectors a, b, c and for all numbers s and r from K one has

$$\begin{aligned}
(a+b)+c = a+(b+c) = a+b+c \quad & \textit{associativity of vector addition} \\
n + a = a \quad & \textit{existence of null vector} \\
(-a) + a = n \quad & \textit{existence of negative vector} \\
1a = a \quad & \textit{regularity of scalar multiplication} \\
(s+r)(a+b) = sa + sb + ra + rb \quad & \textit{complete distributivity of scalar multiplication}
\end{aligned} \tag{145}$$

If the field K, whose elements are called *scalars* in this context, is taken to be the real (or

complex, or quaternionic) numbers, one speaks of a real (or complex, or quaternionic) vector space. Vector spaces are also called *linear vector spaces* or simply *linear spaces*.

The complex numbers, the set of all real functions defined on the real line, the set of all polynomials, the set of matrices with a given number of rows and columns, all form vector spaces. In mathematics, a vector is thus a more general concept than in physics.

Challenge 197 s (What is the simplest possible mathematical vector space?)

In physics, the term 'vector' is reserved for elements of a more specialized type of vector space, namely normed inner product spaces. To define these, we first need the concept of a metric space.

A *metric space* is a set with a metric, i.e., a way to define distances between elements. A real function $d(a, b)$ between elements is called a metric if

$$\begin{aligned} d(a,b) &\geqslant 0 && \textit{positivity of metric} \\ d(a,b) + d(b,c) &\geqslant d(a,c) && \textit{triangle inequality} \\ d(a,b) = 0 \quad \text{if and only if} \quad a &= b && \textit{regularity of metric} \end{aligned} \tag{146}$$

A non-trivial example is the following. We define a special distance d between cities. If the two cities lie on a line going through Paris, we use the usual distance. In all other cases, we define the distance d by the shortest distance from one to the other travelling

Challenge 198 s via Paris. This strange method defines a metric between all cities in France, the so-called *French railroad distance*.

A *normed* vector space is a linear space with a norm, or 'length', associated to each a vector. A *norm* is a non-negative number $\|a\|$ defined for each vector a with the properties

$$\begin{aligned} \|ra\| &= |r|\,\|a\| && \textit{linearity of norm} \\ \|a+b\| &\leqslant \|a\| + \|b\| && \textit{triangle inequality} \\ \|a\| = 0 \quad \text{only if} \quad a &= 0 && \textit{regularity} \end{aligned} \tag{147}$$

Challenge 199 ny Usually there are many ways to define a norm for a given vector space. Note that a norm can always be used to define a metric by setting

$$d(a,b) = \|a-b\| \tag{148}$$

so that all normed spaces are also metric spaces. This is the *natural* distance definition (in contrast to unnatural ones like that between French cities given above).

The norm is often defined with the help of an inner product. Indeed, the most special class of linear spaces are the *inner product spaces*. These are vector spaces with an *inner product*, also called *scalar product* $\cdot$ (not to be confused with the scalar multiplication!)

which associates a number to each pair of vectors. An inner product space over $\mathbb{R}$ satisfies

$$\begin{aligned} a \cdot b = b \cdot a & \quad \textit{commutativity of scalar product} \\ (ra) \cdot (sb) = rs(a \cdot b) & \quad \textit{bilinearity of scalar product} \\ (a+b) \cdot c = a \cdot c + b \cdot c & \quad \textit{left distributivity of scalar product} \\ a \cdot (b+c) = a \cdot b + a \cdot c & \quad \textit{right distributivity of scalar product} \\ a \cdot a \geqslant 0 & \quad \textit{positivity of scalar product} \\ a \cdot a = 0 \quad \text{if and only if} \quad a = 0 & \quad \textit{regularity of scalar product} \end{aligned} \tag{149}$$

for all vectors a, b, c and all scalars r, s. A *real* inner product space of finite dimension is also called a *Euclidean* vector space. The set of all velocities, the set of all positions, or the set of all possible momenta form such spaces.

An inner product space over $\mathbb{C}$ satisfies*

$$\begin{aligned} a \cdot b = \overline{b \cdot a} = \overline{b} \cdot \overline{a} & \quad \textit{Hermitean property} \\ (ra) \cdot (sb) = r\overline{s}(a \cdot b) & \quad \textit{sesquilinearity of scalar product} \\ (a+b) \cdot c = a \cdot c + b \cdot c & \quad \textit{left distributivity of scalar product} \\ a \cdot (b+c) = a \cdot b + a \cdot c & \quad \textit{right distributivity of scalar product} \\ a \cdot a \geqslant 0 & \quad \textit{positivity of scalar product} \\ a \cdot a = 0 \quad \text{if and only if} \quad a = 0 & \quad \textit{regularity of scalar product} \end{aligned} \tag{150}$$

for all vectors a, b, c and all scalars r, s. A *complex* inner product space (of finite dimension) is also called a *unitary* or *Hermitean* vector space. If the inner product space is *complete*, it is called, especially in the infinite-dimensional complex case, a *Hilbert space*. The space of all possible states of a quantum system forms a Hilbert space.

Page 224

All inner product spaces are also metric spaces, and thus normed spaces, if the metric is defined by

$$d(a,b) = \sqrt{(a-b) \cdot (a-b)}\,. \tag{151}$$

Only in the context of an inner product spaces we can speak about angles (or phase differences) between vectors, as we are used to in physics. Of course, like in normed spaces, inner product spaces also allows us to speak about the length of vectors and to define a *basis*, the mathematical concept necessary to define a coordinate system. Which vector spaces or inner product spaces are of importance in physics?

Challenge 200 s

The *dimension* of a vector space is the number of linearly independent basis vectors. Can you define these terms precisely?

Challenge 201 s

A *Hilbert space* is a real or complex inner product space that is also a *complete* metric space. In other terms, in a Hilbert space, distances vary continuously and behave as naively expected. Hilbert spaces usually, but not always, have an infinite number of dimensions.

* Two inequivalent forms of the sesquilinearity axiom exist. The other is $(ra) \cdot (sb) = \overline{r}s(a \cdot b)$. The term *sesquilinear* is derived from Latin and means for 'one-and-a-half-linear'.

The definition of Hilbert spaces and vector spaces assume continuous sets to start with. If nature would not be continuous, could one still use the concepts? Challenge 202 s

Mathematical curiosities and fun challenges

Mathematics provides many counter-intuitive results. Reading a good book on the topic, such as Bernard R. Gelbaum & John M. H. Olmsted, *Theorems and Counterexamples in Mathematics*, Springer, 1993, can help you sharpen your mind and make you savour the beauty of mathematics even more.

* *

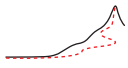
It is possible to draw a curve that meets all points in a square or all points in a cube. This is shown, for example, in the text Hans Sagan, *Space Filling Curves*, Springer Verlag, 1994. As a result, the distinction between one, two and three dimensions is blurred in pure mathematics. In physics however, dimensions are clearly and well-defined; every object in nature has three dimensions. Challenge 203 e

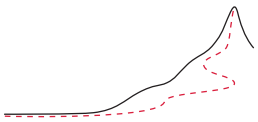
* *

Show that two operators A and B obey Challenge 204 ny

$$\begin{aligned} e^A e^B = \exp(A + B + \frac{1}{2}[A,B] \\ + \frac{1}{12}[[A,B],B] - \frac{1}{12}[[A,B],A] \\ - \frac{1}{48}[B,[A,[A,B]]] - \frac{1}{48}[A,[B,[A,B]]] \\ + \ldots) \end{aligned} \tag{152}$$

for most operators A and B. This result is often called the *Baker–Campbell–Hausdorff formula* or the *BCH formula*.





CHALLENGE HINTS AND SOLUTIONS

> Never make a calculation before you know the answer.
>
> John Wheeler's motto

Challenge 1, page 10: Do not hesitate to be demanding and strict. The next edition of the text will benefit from it.

Challenge 2, page 16: Classical physics fails in explaining any material property, such as colour or softness. Material properties result from nature's interactions; they are inevitably quantum. Explanations of material properties require, without exception, the use of particles and their quantum properties.

Challenge 3, page 17: Classical physics allows any observable to change *smoothly* with time. In classical physics, there is no minimum value for any observable physical quantity.

Challenge 4, page 20: The higher the mass, the smaller the motion fuzziness induced by the quantum of action, because action is mass times speed times distance: For a large mass, the speed and distance variations are small.

Challenge 5, page 20: The simplest time is $\sqrt{G\hbar/c^5}$. The numerical factor is obviously not fixed; it is changed later on. Using $4G$ instead of G the time becomes the shortest time measurable in nature.

Challenge 7, page 21: The electron charge is special to the electromagnetic interactions; it does not take into account the nuclear interactions or gravity. It is unclear why the length defined with the elementary charge e should be of importance for neutral systems or for the vacuum. On the other hand, the quantum of action $\hbar$ is valid for *all* interactions and *all* observations.

In addition, we can argue that the two options to define a fundamental length – with the quantum of action and with the quantum of charge – are not too different: the electron charge is related to the quantum of action by $e = \sqrt{4\pi\varepsilon_0 \alpha c \hbar}$. The two length scales defined by the two options differ only by a factor near 11.7. In fact, both scales are *quantum* scales.

Challenge 8, page 21: On purely dimensional grounds, the radius of an atom must be

$$r \approx \frac{\hbar^2 4\pi\varepsilon_0}{m_e e^2} \,, \tag{153}$$

Page 186 which is about 53 nm. Indeed, this guess is excellent: it is just the Bohr radius.

Challenge 9, page 21: Due to the quantum of action, atoms in all people, be they giants or dwarfs, have the same size. This implies that giants cannot exist, as was shown already by Galileo. Vol. I, page 338 The argument is based on the given strength of materials; and a same strength everywhere is equivalent to the same properties of atoms everywhere. That dwarfs cannot exist is due to a similar reason; nature is not able to make people smaller than usual (even in the womb they differ markedly from adults) as this would require smaller atoms.

Challenge 12, page 27: A disappearance of a mass m in a time Δt is an action change $c^2 m \Delta t$. That is much larger than $\hbar$ for all objects of everyday life.

Challenge 14, page 29: Tunnelling of a lion would imply action values S of the order of $S = 100\,\mathrm{kgm^2/s} \gg \hbar$. This cannot happen spontaneously.

Challenge 15, page 30: Every memory, be it human memory or an electronic computer memory, must avoid decay. And decay can only be avoided through high walls and low tunnelling rates.

Challenge 16, page 30: Yes! Many beliefs and myths – from lottery to ghosts – are due to the neglect of quantum effects.

Challenge 17, page 30: Perfectly continuous flow is in contrast to the fuzziness of motion induced by the quantum of action.

Challenge 18, page 31: The impossibility of following two particles along their path appears when their mutual distance d is smaller than their position indeterminacy due to their relative momentum p, thus when $d < \hbar/p$. Check the numbers with electrons, atoms, molecules, bacteria, people and galaxies.

Challenge 19, page 31: Also photons are indistinguishable. See page 63.

Challenge 21, page 36: In the material that forms the escapement mechanism.

Challenge 22, page 36: Growth is not proportional to light intensity or to light frequency, but shows both intensity and frequency *thresholds*. These are quantum effects.

Challenge 23, page 36: All effects mentioned above, such as tunnelling, interference, decay, transformation, non-emptiness of the vacuum, indeterminacy and randomness, are also observed in the nuclear domain.

Challenge 24, page 37: This is not evident from what was said so far, but it turns out to be correct. In fact, there is no other option, as you will see when you try to find one.

Challenge 25, page 37: Tom Thumb is supposedly as smart as a normal human. But a brain cannot be scaled down. Fractals contradict the existence of Planck's length, and Moore's law contradicts the existence of atoms.

Challenge 26, page 37: The total angular momentum counts, including the orbital angular momentum. The orbital angular momentum $\boldsymbol{L}$ is given, using the radius and the linear momentum, $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$. The total angular momentum is a multiple of $\hbar$.

Challenge 27, page 37: Yes, we could have!

Challenge 28, page 37: That is just the indeterminacy relation. Bohr expanded this idea to all sort of other pairs of concepts, more in the philosophical domain, such as clarity and precision of explanations: both cannot be high at the same time.

Challenge 29, page 39: The big bang cannot have been an event, for example.

Challenge 32, page 45: Charged photons would be deflected by electric of magnetic fields; in particular, they would not cross undisturbed. This is not observed. Massive photons would be deflected by masses, such as the Sun, much more than is observed.

Challenge 34, page 45: To measure momentum, we need a spatially extended measurement device; to measure position, we need a localized measurement device.

Challenge 35, page 47: Photons are elementary because they realize the minimum action, because they cannot decay, because they cannot be deformed or split, because they have no mass, no electric charge and no other quantum number, and because they appear in the Lagrangian of quantum electrodynamics.

Challenge 36, page 50: The measured electric fields and photon distribution are shown in the famous graphs reproduced in Figure 90.

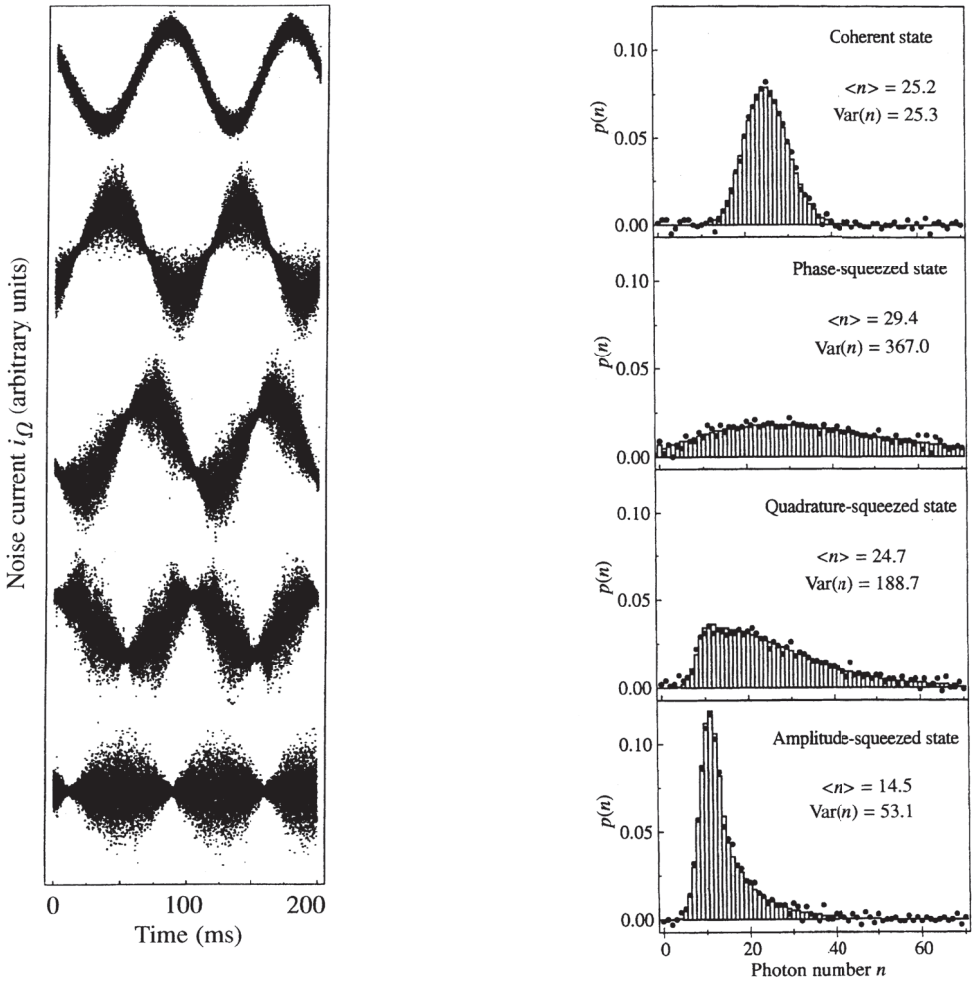


FIGURE 90 Left, from top to bottom: the electric field and its fuzziness measured for a coherent state, for a squeezed vacuum state, for a phase-squeezed state, for a mixed, quadrature-squeezed state and for an amplitude-squeezed state, all with a small number of photons. Right: the corresponding photon number distributions for the uppermost four states. (© G. Breitenbach/Macmillan, from Ref. 19)

Challenge 38, page 50: This is an unclearly posed problem. The radiation is thermal, but the photon number depends on the volume under discussion.

Challenge 40, page 56: Radio photons can be counted using optical pumping experiments in which atomic states are split by a small, 'radio-wavelength' amount, with the help of magnetic fields. Also caesium clocks detect radio photons with optical means. The Josephons effect and magnetic resonance imaging are additional detection methods for radio photons.

Challenge 41, page 57: To be observable to the eye, the interference fringes need to be visible for around 0.1 s. That implies a maximum frequency difference between the two beams of around 10 Hz. This is achievable only if either a single beam is split into two or if the two beams come from high-precision, stabilized lasers.

Challenge 42, page 62: Implicit in the arrow model is the idea that one quantum particle is described by one arrow.

Challenge 48, page 64: Despite a huge number of attempts and the promise of eternal fame, this is the sober conclusion.

Challenge 53, page 68: Yes, the argument is correct. In fact, more detailed discussions show that classical electrodynamics is in contradiction with *all* colours observed in nature.

Ref. 178 **Challenge 57**, page 73: The calculation is not easy, but not too difficult either. For an initial orientation *close* to the vertical, the fall time T turns out to be

$$T = \frac{1}{2\pi} T_0 \ln \frac{8}{\alpha} \tag{154}$$

where α is the starting angle, and a fall through angle π is assumed. Here T_0 is the oscillation time of the pencil for small angles. (Can you determine it?) The indeterminacy relation for the tip of the pencil yields a *minimum* starting angle, because the momentum indeterminacy cannot be made arbitrarily large. You should be able to provide an upper limit. Once this angle is known, you can calculate the maximum time.

Challenge 58, page 74: Use the temperature to calculate the average kinetic energy, and thus the average speed of atoms.

Challenge 59, page 74: At such low temperatures, the atoms cannot be fully distinguished; they form a state of matter with peculiar properties, called a *condensate*. The condensate is not at rest either; but due to its large mass, its fluctuations are greatly reduced, compared to those of a single atom.

Challenge 61, page 78: Only variables whose product has the same units as physical action – Js – can be complementary to each other.

Challenge 62, page 79: Use $\Delta E < E$ and $a\ \Delta t < c$.

Challenge 67, page 86: The quantum of action does not apply only to measurements, it applies to motion itself, and in particular, to all motion. Also effects of the nuclear forces, of nuclear particles and of nuclear radiation particles must comply to the limit. And experiments show that they indeed do. In fact, if they did not, the quantum of action in electrodynamic situations could be circumvented, as you can check.

Challenge 74, page 96: Outside the garage, all atoms need to form the same solid structure again.

Challenge 75, page 97: Terabyte chips would need to have small memory cells. Small cells imply thin barriers. Thin barriers imply high probabilities for tunnelling. Tunnelling implies lack of memory.

Challenge 81, page 108: If a particle were not elementary, its components would be bound by an interaction. But there are no known interactions outside those of the standard model.

Challenge 82, page 109: The difficulties to see hydrogen atoms are due to their small size and their small number of electrons. As a result, hydrogen atoms produce only weak contrasts in X-ray images. For the same reasons it is difficult to image them using electrons; the Bohr radius of hydrogen is only slightly larger than the electron Compton wavelength.

For the first time, in 2008, a research team claimed to have imaged hydrogen atoms adsorbed on graphene with the help of a transmission electron microscope. For details, see J. C. Meyer, C. O. Grit, M. F. Crommle & A. Zetti, *Imaging and dynamics of light atoms and molecules on graphene*, Nature 454, pp. 319–322, 2008. However, it seems that the report has not been confirmed by another group yet.

More hydrogen images have appeared in recent years. You may search for olympicene on the Page 185 internet, for example. For another recent result about hydrogen imaging, see above.

Challenge 84, page 109: This is not easy! Can you use the concept of action to show that there indeed is a fundamental difference between very similar and very different operators?

Challenge 86, page 110: $r = 86\,\mathrm{pm}$, thus $T = 12\,\mathrm{eV}$. This compares to the actual value of 13.6 eV. The trick for the derivation of the formula is to use $\langle\psi\,|\,r_x^2\,|\,\psi\rangle = \frac{1}{3}\langle\psi\,|\,\boldsymbol{rr}\,|\,\psi\rangle$, a relation valid for states with no orbital angular momentum. It is valid for all coordinates and also for the three momentum observables, as long as the system is non-relativistic.

Challenge 87, page 111: A quantum fluctuation would require the universe to exist already. Such statements, regularly found in the press, are utter nonsense.

Challenge 88, page 112: Point particles cannot be marked; nearby point particles cannot be distinguished, due to the quantum of action.

Challenge 89, page 112: The solution is *two* gloves. In the original setting, if two men and two women want to make love without danger, in theory they need only *two* condoms.

Challenge 94, page 114: The Sackur–Tetrode formula is best deduced in the following way. We start with an ideal monoatomic gas of volume V, with N particles, and total energy U. In phase space, *state sum* Z is given by

$$Z = \frac{V^N}{N!}\frac{1}{\Lambda^{3N}}\,. \tag{155}$$

We use Stirling's approximation $N! \approx N^N/e^N$, and the definition of the entropy as $S = \partial(kT\ln Z)/\partial T$. Inserting the definition of Λ, this gives the Sackur–Tetrode equation.

Challenge 96, page 117: To write anything about two particles on paper, we need to distinguish them, even if the distinction is arbitrary.

Challenge 99, page 123: The idea, also called *quantum money*, is not compatible with the size and lifetime requirements of actual banknotes.

Challenge 100, page 124: Twins differ in the way their intestines are folded, in the lines of their hands and other skin folds. Sometimes, but not always, features like black points on the skin are mirror inverted on the two twins.

Challenge 109, page 135: Three.

Challenge 110, page 135: Not for a mattress. This is not easy to picture.

Challenge 111, page 136: Angels can be distinguished by name, can talk and can sing; thus they are made of a large number of fermions. In fact, many angels are human sized, so that they do not even fit on the tip of a pin.

Challenge 117, page 140: A boson can be represented by an object glued to one infinitesimally thin thread whose two tails reach spatial infinity.

Challenge 118, page 141: Trees, like all macroscopic objects, have a spin value that depends on their angular momentum. Being classical objects whose phase can be observed, the spin value is uncertain. It makes no sense to ask whether trees or other macroscopic objects are bosons or fermions, as they are not quantons.

Challenge 121, page 142: Ghosts, like angels, can be distinguished by name, can talk and can be seen; thus they contain fermions. However, they can pass through walls and they are transparent; thus they cannot be made of fermions, but must be images, made of bosons. That is a contradiction.

Challenge 122, page 144: Macroscopic superpositions cannot be drawn, because observation implies interaction with a bath, which destroys macroscopic superposition.

Challenge 124, page 146: The loss of non-diagonal elements leads to an increase in the diagonal elements, and thus of entropy.

Challenge 127, page 153: The energy speed is given by the advancement of the outer two tails; that speed is never larger than the speed of light.

Challenge 128, page 155: No, as taking a photo implies an interaction with a bath, which would destroy the superposition. In more detail, a photograph requires illumination; illumination is a macroscopic electromagnetic field; a macroscopic field is a bath; a bath implies decoherence; decoherence destroys superpositions.

Challenge 131, page 157: It depends. They can be due to interference or to intensity sums. In the case of radio the effect is clearer. If at a particular frequency the signals changes periodically from one station to another, one has a genuine interference effect.

Challenge 132, page 157: They interfere. But this is a trick question; what is a monochromatic electron? Does it occur in the laboratory?

Challenge 133, page 157: Such a computer requires clear phase relations between components; such phase relations are extremely sensitive to outside disturbances. At present, they do not hold longer than a hundred microseconds, whereas long computer programs require minutes and hours to run.

Challenge 134, page 157: A record is an effect of a process that must be *hard to reverse or undo*. The traces of a broken egg are easy to clean on a large glass plate, but hard in the wool of a sheep. Broken teeth, torn clothes, or scratches on large surfaces are good records. Forensic scientists know many additional examples.

Challenge 138, page 166: Any other bath also does the trick, such as the atmosphere, sound vibrations, electromagnetic fields, etc.

Challenge 139, page 166: The Moon is in contact with baths like the solar wind, falling meteorites, the electromagnetic background radiation of the deep universe, the neutrino flux from the Sun, cosmic radiation, etc.

Challenge 140, page 168: Spatially periodic potentials have the property. Decoherence then leads to momentum diagonalization.

Challenge 142, page 171: If so, let the author know.

Challenge 143, page 182: The red shift value is $z = 9.9995$. From the formula for the longitudinal Doppler shift we get $v/c = ((z+1)^2 - 1)/((z+1)^2 + 1)$; this yields 0.984 in the present case. The galaxy thus moves away from Earth with 98.4 % of the speed of light.

Challenge 149, page 184: Hydrogen atoms are in eigenstates for the reasons explained in the chapter on superpositions and probabilities: in a gas, atoms are part of a bath, and thus almost always in energy eigenstates.

Challenge 154, page 195: If several light beams are focused in the space between the mirrors, and if the light beam frequency is properly tuned with respect to the absorption frequencies of the atoms, atoms will experience a restoring force whenever they move away from the focus region. By shining light beams to the focus region from 6 directions, atoms are trapped. The technique of laser cooling is now widely used in research laboratories.

Challenge 155, page 196: No, despite its name, phosphorus is not phosphorescent, but chemoluminescent.

Challenge 157, page 197: This is a trick question. A change in α requires a change in c, $\hbar$, e or ε_0. None of these changes is possible or observable, as all our measurement apparatus are based on these units. Speculations about change of α, despite their frequency in the press and in scientific journals, are idle talk.

Challenge 158, page 197: A change of physical units such that $\hbar = c = e = 1$ would change the value of ε_0 in such a way that $4\pi\varepsilon_o = 1/\alpha \approx 137.036$.

Challenge 161, page 207: Mass is a measure of the amount of energy. The 'square of mass' makes no sense.

Challenge 165, page 210: Planck limits can be exceeded for extensive observables for which many particle systems can exceed single particle limits, such as mass, momentum, energy or electrical resistance.

Challenge 167, page 212: Do not forget the relativistic time dilation.

Challenge 168, page 212: The formula with $n - 1$ is a better fit. Why?

Challenge 171, page 213: No! They are much too precise to make sense. They are only given as an illustration for the behaviour of the Gaussian distribution. Real measurement distributions are not Gaussian to the precision implied in these numbers.

Challenge 172, page 213: About 0.3 m/s. It is *not* 0.33 m/s, it is *not* 0.333 m/s and it is *not* any longer strings of threes.

Challenge 174, page 219: The slowdown goes *quadratically* with time, because every new slowdown adds to the old one!

Challenge 175, page 220: No, only properties of parts of the universe are listed. The universe itself has no properties, as shown in the last volume.

Vol. VI, page 112

Challenge 176, page 221: The double of that number, the number made of the sequence of all even numbers, etc.

Challenge 179, page 225: We will find out in the last volume that all measurement values have upper and lower bounds. We will also find out that two physical measurement results cannot differ just from, say, the 300th decimal place onwards. So indeed, all measurement results are real numbers, but not vice versa. It needs to be stressed that for quantum theory, for relativity and also for Galilean physics this restriction has no consequences whatsoever.

Challenge 181, page 226: $|z|^2$ is the *determinant* of the matrix $z = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Challenge 186, page 227: Use Cantor's diagonal argument, as in challenge 274.

Challenge 187, page 228: Any quaternion $X = ai+bj+ck$ with $a^2+b^2+c^2 = 1$ solves the equation $X^2 + 1 = 0$; the purely imaginary solutions $+i$ and $-i$ are thus augmented by a continuous sphere of solutions in quaternion space.

Challenge 190, page 230: Any rotation by an angle 2π is described by -1. Only a rotation by 4π is described by $+1$; quaternions indeed describe spinors.

Challenge 192, page 232: Just check the result component by component. See also the mentioned reference.

Challenge 194, page 234: No. Because the unit octonions are not associative, they do not form a group at all. Despite its superficial appeal, this line of reasoning has not led to any insight into the nature of the fundamental interactions.

Challenge 195, page 235: For a Gaussian integer $n + im$ to be prime, the integer $n^2 + m^2$ must be prime, and in addition, a condition on n mod 3 must be satisfied; which one and why?

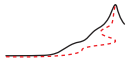
Challenge 197, page 236: The set that contains only the zero vector.

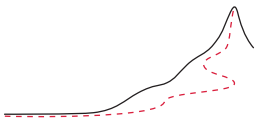
Challenge 198, page 236: The metric is regular, positive definite and obeys the triangle inequality.

Challenge 200, page 237: Essentially only the vector spaces listed in the appendix (or in the book).

Challenge 201, page 237: If you cannot, blame your math teacher at secondary school, and then look up the definitions. It is not a difficult topic.

Challenge 202, page 238: Spaces could exist approximately, as averages of non-continuous structures. This idea is explored in modern research; an example is given in the last volume of this series.





BIBLIOGRAPHY

> No man but a blockhead ever wrote except for money.
>
> Samuel Johnson

> As soon as you write, no time to read remains.
>
> Anonymous

1 Giuseppe Fumagalli, *Chi l'ha detto?*, Hoepli, Milano, 1983. Cited on page 15.

2 The quantum of action was introduced in Max Planck, *Über irreversible Strahlungsvorgänge*, Sitzungsberichte der Preußischen Akademie der Wissenschaften, Berlin pp. 440–480, 1899. In the paper, Planck used the letter *b* for what nowadays is called *h*. Cited on page 17.

3 Bohr explained the indivisibilty of the quantum of action in his famous Como lecture. See N. Bohr, *Atomtheorie und Naturbeschreibung*, Springer, 1931. On page 16 he writes: 'No more is it likely that the fundamental concepts of the classical theories will ever become superfluous for the description of physical experience. The recognition of the indivisibility of the quantum of action, and the determination of its magnitude, not only depend on an analysis of measurements based on classical concepts, but it continues to be the application of these concepts alone that makes it possible to relate the symbolism of the quantum theory to the data of experience.' He also writes: '...the fundamental postulate of the indivisibility of the quantum of action is itself, from the classical point of view, an irrational element which inevitably requires us to forgo a causal mode of description and which, because of the coupling between phenomena and their observation, forces us to adopt a new mode of description designated as complementary in the sense that any given application of classical concepts precludes the simultaneous use of other classical concepts which in a different connection are equally necessary for the elucidation of the phenomena ...' and '...the finite magnitude of the quantum of action prevents altogether a sharp distinction being made between a phenomenon and the agency by which it is observed, a distinction which underlies the customary concept of observation and, therefore, forms the basis of the classical ideas of motion.' Other statements about the indivisibility of the quantum of action can be found in N. Bohr, *Atomic Physics and Human Knowledge*, Science Editions, 1961. See also Max Jammer, *The Philosophy of Quantum Mechanics*, Wiley, first edition, 1974, pp. 90–91. Cited on page 17.

4 For some of the rare modern publications emphasizing the quantum of action see M. B. Mensky, *The action uncertainty principle and quantum gravity*, Physics Letters A **162**, p. 219, 1992, and M. B. Mensky, *The action uncertainty principle in continuous quantum measurements*, Physics Letters A **155**, pp. 229–235, 1991. Schwinger's quantum-action principle is also used in Richard F. W. Bader, *Atoms in Molecules – A Quantum Theory*, Oxford University Press, 1994.

There is a large number of general textbooks on quantum theory. There is one for every taste.

A well-known *conceptual* introduction is Jean-Marc Lévy-Leblond & Françoise Balibar, *Quantique – Rudiments*, Masson, 1997, translated into English as *Quantics*, North-Holland, 1990.

One of the most beautiful books is Julian Schwinger, *Quantum Mechanics – Symbolism of Atomic Measurements*, edited by Berthold-Georg Englert, Springer Verlag, 2001.

A modern approach with a beautiful introduction is Max Schubert & Gerhard Weber, *Quantentheorie – Grundlagen und Anwendungen*, Spektrum Akademischer Verlag, 1993.

A standard beginner's text is C. Cohen-Tannoudji, B. Diu & F. Laloë, *Mécanique quantique I et II*, Hermann, Paris, 1977. It is also available in several translations.

A good text is Asher Peres, *Quantum Theory – Concepts and Methods*, Kluwer, 1995.

For a lively approach, see Vincent Icke, *The Force of Symmetry*, Cambridge University Press, 1994.

New textbooks are published regularly around the world. Cited on pages 17 and 255.

5 The best source for the story about the walk in the forest with Planck's son Erwin is Hans Roos & Armin Hermann, editors, *Max Planck – Vorträge, Reden, Erinnerungen*, Springer, 2001, page 125. As the text explains, the story was told by Erwin Planck to at least two different people. Erwin Planck himself was part of the failed 1944 plot against Hitler and was hanged in January 1945. Cited on page 20.

6 Max Born, *Zur Quantenmechanik der Stoßvorgänge (vorläufige Mitteilung)*, Zeitschrift für Physik **37**, pp. 863–867, 1926, Max Born, *Quantenmechanik der Stoßvorgänge*, Zeitschrift für Physik **38**, pp. 803–827, 1926. Cited on page 24.

7 See for example the papers by Jan Hilgevoord, *The uncertainty principle for energy and time*, American Journal of Physics **64**, pp. 1451–1456, 1996, and by Paul Busch, *On the time–energy uncertainty reaction, parts 1 & 2*, Foundations of Physics **20**, pp. 1–43, 1990. A classic is the paper by Eugene P. Wigner, *On the time–energy uncertainty relation*, in Abdus Salam & Eugene P. Wigner, editors, *Aspects of Quantum Theory*, Cambridge University Press, 1972. Cited on page 25.

8 See also the booklet by Claus Mattheck, *Warum alles kaputt geht – Form und Versagen in Natur und Technik*, Forschungszentrum Karlsruhe, 2003. Cited on page 30.

9 R. Clifton, J. Bub & H. Halvorson, *Characterizing quantum theory in terms of information-theoretic constraints*, arxiv.org/abs/quant-ph/0211089. Cited on page 36.

10 This way to look at cans of beans goes back to the text by Susan Hewitt & Edward Subitzky, *A call for more scientific truth in product warning labels*, Journal of Irreproducible Results **36**, nr. 1, 1991. Cited on page 37.

11 J. Malik, *The yields of the Hiroshima and Nagasaki nuclear explosions*, Technical Report LA-8819, Los Alamos National Laboratory, September 1985. Cited on page 38.

12 The quotes on motion are found in chapter VI of F. Engels, *Herrn Eugen Dührings Umwälzung der Wissenschaft*, Verlag für fremdsprachliche Literatur, 1946. The book is commonly called *Anti-Dühring*. Cited on pages 39 and 74.

13 Rodney Loudon, *The Quantum Theory of Light*, Oxford University Press, 2000. Cited on page 40.

14 E. M. Brumberg & S. I. Vavilov, Izvest. Akad. Nauk. Omen Ser. 7, p. 919, 1933. Cited on page 40.

15 On photon detection in the human eye, see the influential review by F. Rieke &

D. A. Baylor, *Single-photon detection by rod cells of the retina*, Reviews of Modern Physics **70**, pp. 1027–1036, 1998. It can be found on the internet as a pdf file. Cited on page 42.

16 F. Rieke & D. A. Baylor, *Single-photon detection by rod cells of the retina*, Reviews of Modern Physics **70**, pp. 1027–1036, 1998. They also mention that the eye usually works at photon fluxes between $10^8\,/\mu m^2 s$ (sunlight) and $10^{-2}\,/\mu m^2 s$ (starlight). The cones, in the retina detect, in colour, light intensities in the uppermost seven or eight decades, whereas the rods detect, in black and white, the lower light intensities. Cited on page 44.

17 E. Fischbach, H. Kloor, R. A. Langel, A. T. Y. Lui & M. Peredo, *New geomagnetic limit on the photon mass and on long-range forces coexisting with electromagnetism*, Physical Review Letters **73**, pp. 514–517, 1994. Cited on page 45.

18 A. H. Compton, *The scattering of X-rays as particles*, American Journal of Physics **29**, pp. 817–820, 1961. This is a pedagogical presentation of the discoveries he made in 1923. Cited on page 45.

19 The reference paper on this topic is G. Breitenbach, S. Schiller & J. Mlynek, *Measurement of the quantum states of squeezed light*, **387**, pp. 471–475, 1997. It is available freely at gerdbreitenbach.de/publications/nature1997.pdf. Cited on pages 48 and 241.

20 The famous paper is R. Hanbury Brown & R. Q. Twiss, Nature **178**, p. 1046, 1956. They got the idea to measure light in this way from their earlier work, which used the same method with radio waves: R. Hanbury Brown & R. Q. Twiss, Nature **177**, p. 27, 1956. The complete discussion is given in their papers R. Hanbury Brown & R. Q. Twiss, *Interferometry of the intensity fluctuations in light. I. Basic theory: the correlation between photons in coherent beams of radiation*, Proceedings of the Royal Society A **242**, pp. 300–324, 1957, and R. Hanbury Brown & R. Q. Twiss, *Interferometry of the intensity fluctuations in light. II. An experimental test of the theory for partially coherent light*, Proceedings of the Royal Society A **243**, pp. 291–319, 1958. Both are dowloadable for free on the internet and are well worth reading. Cited on page 52.

21 J. Glanz, *First light from a space laser*, Science **269**, p. 1336, 1995. Cited on page 54.

22 A. Einstein, *Über einen die Erzeugung und Umwandlung des Lichtes betreffenden heuristischen Standpunkt*, Annalen der Physik **17**, pp. 132–184, 1905. Cited on page 55.

23 See the summary by P. W. Milonni, *Answer to question 45: What (if anything) does the photoelectric effect teach us?*, American Journal of Physics **65**, pp. 11–12, 1997. Cited on page 55.

24 For a detailed account, See J. J. Prentis, *Poincaré's proof of the quantum discontinuity of nature*, American Journal of Physics **63**, pp. 339–350, 1995. The original papers are Henri Poincaré, *Sur la théorie des quanta*, Comptes Rendus de l'Académie des Sciences (Paris) **153**, pp. 1103–1108, 1911, and Henri Poincaré, *Sur la théorie des quanta*, Journal de Physique (Paris) **2**, pp. 5–34, 1912. Cited on page 55.

25 J. Jacobson, G. Björk, I. Chang & Y. Yamamoto, *Photonic de Broglie waves*, Physical Review Letters **74**, pp. 4835–4838, 1995. The first measurement was published by E. J. S. Fonseca, C. H. Monken & S. de Pádua, *Measurement of the de Broglie wavelength of a multiphoton wave packet*, Physical Review Letters **82**, pp. 2868–2671, 1995. Cited on page 55.

26 For the three-photon state, see M. W. Mitchell, J. S. Lundeen & A. M. Steinberg, *Super-resolving phase measurements with a multiphoton entangled state*, Nature **429**, pp. 161–164, 2004, and for the four-photon state see, in the same edition, P. Walther, J. -W. Pan,

M. Aspelmeyer, R. Ursin, S. Gasparoni & A. Zeilinger, *De Broglie wavelength of a non-local four-photon state*, Nature **429**, pp. 158–161, 2004. Cited on page 55.

27 For an introduction to squeezed light, see L. Mandel, *Non-classical states of the electromagnetic field*, Physica Scripta T **12**, pp. 34–42, 1986. Cited on page 55.

28 Friedrich Herneck, *Einstein und sein Weltbild: Aufsätze und Vorträge*, Buchverlag Der Morgen, 1976, page 97. Cited on page 56.

29 The famous quote on single-photon interference is found on page 9 of famous, beautiful but difficult textbook P. A. M. Dirac, *The Principles of Quantum Mechanics*, Clarendon Press, 1930. It is also discussed, in a somewhat confused way, in the otherwise informative article by H. Paul, *Interference between independent photons*, Reviews of Modern Physics **58**, pp. 209–231, 1986. Cited on pages 59 and 66.

30 The original papers on coherent states are three: R. J. Glauber, *The quantum theory of optical coherence*, Physical Review **130**, pp. 2529–2539, 1963, J. R. Klauder, *Continuous-representation theory*, I and II, Journal of Mathematical Physics **4**, pp. 1055–1058, 1963, and E. C. G. Sudarshan, *Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams*, Physical Review Letters **10**, p. 227, 1963. Cited on page 63.

31 See, for example the wonderful text Richard P. Feynman, *QED – The Strange Theory of Light and Matter*, pp. 73–75, Princeton University Press, 1988, or Richard P. Feynman & Steven Weinberg, *Elementary Particles and the Laws of Physics*, p. 23, Cambridge University Press 1987. Cited on page 63.

32 Wolfgang Tittel, J. Brendel, H. Zbinden & N. Gisin, *Violation of Bell inequalities by photons more than 10 km apart*, Physical Review Letters **81**, pp. 3563–3566, 26 October 1998. Cited on page 64.

33 N. Bohr & L. Rosenfeld, *Zur Frage der Meßbarkeit der elektromagnetischen Feldgrößen*, Mat.-fys. Medd. Danske Vid. Selsk. **12**, p. 8, 1933. The results were later published in English as N. Bohr & L. Rosenfeld, *Field and charge measurements in quantum electrodynamics*, Physical Review **78**, pp. 794–798, 1950. Cited on page 65.

34 Misleading statements are given in the introduction and in the conclusion of the review by H. Paul, *Interference between independent photons*, Review of Modern Physics **58**, pp. 209–231, 1986. However, in the bulk of the article the author in practice retracts the statement, e.g. on page 221. Cited on page 66.

35 G. Magyar & L. Mandel, *Interference fringes produced by superposition of two independent maser light beams*, Nature **198**, pp. 255–256, 1963. Cited on page 67.

36 R. Kidd, J. Aedini & A. Anton, *Evolution of the modern photon*, American Journal of Physics **57**, pp. 27–35, 1989, Cited on page 69.

37 The whole bunch of atoms behaves as one single molecule; one speaks of a Bose–Einstein condensate. The first observations, worthy of a Nobel prize, were by M.H. Anderson & al., *Observation of Bose–Einstein condensation in a dilute atomic vapour*, Science **269**, pp. 198–201, 1995, C. C. Bradley, C. A. Sackett, J. J. Tollett & R. G. Hulet, *Evidence of Bose–Einstein condensation in an atomic gas with attractive interactions*, Physical Review Letters **75**, pp. 1687–1690, 1995, K. B. Davis, M. -O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn & W. Ketterle, *Bose–Einstein condensation in a gas of sodium atoms*, Physical Review Letters **75**, pp. 3969–3973, 1995. For a simple introduction, see W. Ketterle, *Experimental studies of Bose–Einstein condensation*, Physics Today pp. 30–35, December 1999. Cited on page 74.

38 J. L. Costa-Krämer, N. Garcia, P. García-Mochales & P. A. Serena, *Nanowire formation in macroscopic metallic contacts: a universal property of metals*, Surface

Science Letters **342**, pp. L1144–L1152, 1995. See also J. L. Costa-Krämer, N. Garcia, P. A. Serena, P. García-Mochales, M. Marqués & A. Correia, *Conductance quantization in nanowires formed in macroscopic contacts*, Physical Review B p. 4416, 1997. Cited on page 74.

39 The beautiful undergraduate experiments made possible by this discovery are desribed in E. L. Foley, D. Candela, K. M. Martini & M. T. Tuominen, *An undergraduate laboratory experiment on quantized conductance in nanocontacts*, American Journal of Physics **67**, pp. 389–393, 1999. Cited on pages 74 and 75.

40 L. de Broglie, *Ondes et quanta*, Comptes rendus de l'Académie des Sciences **177**, pp. 507–510, 1923. Cited on page 76.

41 C. Jönsson, *Interferenz von Elektronen am Doppelspalt*, Zeitschrift für Physik **161**, pp. 454–474, 1961, C. Jönsson, *Electron diffraction at multiple slits*, American Journal of Physics **42**, pp. 4–11, 1974. Because of the charge of electons, this experiment is not easy to perform: any parts of the set-up that are insulating get charged and distort the picture. That is why the experient was performed much later with electrons than with atoms, neutrons and molecules. Cited on page 77.

42 M. Arndt, O. Nairz, J. Vos-Andreae, C. Keller, G. van der Zouw & A. Zeilinger, *Wave–particle duality of C60 molecules*, Nature **401**, pp. 680–682, 14 October 1999. See also the observation for tetraphenyleprophyrin and $C_{60}F_{48}$ by the same team, published as L. Hackermüller & al., *Wave nature of biomolecules and fluorofullerenes*, Physical Review Letters **91**, p. 090408, 2003.

No phenomoenon of quantum theory has been experimentally studied as much as quantum interference. The transition from interference to non-interference has also been explored, as in P. Facchi, A. Mariano & S. Pascazio, *Mesoscopic interference*, Recent Developments in Physics **3**, pp. 1–29, 2002. Cited on page 77.

43 G. Papini, *Shadows of a maximal acceleration*, arxiv.org/abs/gr-qc/0211011. Cited on page 79.

44 J. Perrin, *Nobel Prize speech*, found at www.nobel.se, and H. Nagaoka, *Kinetics of a system of particles illustrating the line and the band spectrum and the phenomena of radioactivity*, Philosophical Magazine S6, **7**, pp. 445–455, March 1904. Cited on page 79.

45 N. Bohr, *On the constitution of atoms and molecules: Introduction and Part I – binding of electrons by positive nuclei*, Philosophical Magazine **26**, pp. 1–25, 1913, *On the constitution of atoms and molecules: Part II – systems containing only a single nucleus*, ibid., pp. 476–502, *On the constitution of atoms and molecules: Part III*, ibid., pp. 857–875. Cited on page 79.

46 Robert H. Dicke & James P. Wittke, *Introduction to Quantum Theory*, Addison-Wesley, Reading, Massachusetts, 1960. See also Stephen Gasiorowicz, *Quantum Physics*, John Wiley & Sons, 1974. Cited on page 81.

47 P. Carruthers & M. M. Nieto, *Phase and angle variables in quantum mechanics*, Review of Modern Physics **40**, pp. 411–440, 1968. Cited on page 82.

48 The indeterminacy relation for rotational motion is well explained by W. H. Louisell, *Amplitude and phase uncertainty relations*, Physics Letters **7**, p. 60, 1963. Cited on page 82.

49 S. Franke-Arnold, S. M. Barnett, E. Yao, J. Leach, J. Courtial & M. Padgett, *Uncertainty principle for angular position and angular momentum*, New Journal of Physics **6**, p. 103, 2004. This is a freely accessible online journal. Cited on page 82.

50 W. Gerlach & O. Stern, *Der experimentelle Nachweis des magnetischen Moments des Silberatoms*, Zeitschrift für Physik **8**, p. 110, 1921. See also the pedagogical explanation by

M. Hannout, S. Hoyt, A. Kryowonos & A. Widom, *Quantum measurement and the Stern–Gerlach experiment*, American Journal of Physics **66**, pp. 377–379, 1995. Cited on page 83.

51 J. P. Woerdman, G. Nienhuis, I. Kuščer, *Is it possible to rotate an atom?*, Optics Communications **93**, pp. 135–144, 1992. We are talking about atoms rotating around their centre of mass; atoms can of course rotate around other bodies, as discussed by M. P. Silverman, *Circular birefringence of an atom in uniform rotation: the classical perspective*, American Journal of Physics **58**, pp. 310–317, 1990. Cited on page 85.

52 J. Schmiedmayer, M. S. Chapman, C. R. Ekstrom, T. D. Hammond, S. Wehinger & D. E. Pritchard, *Index of refraction of various gases for sodium matter waves*, Physical Review Letters **74**, p. 1043-1046, 1995. Cited on page 85.

53 The original result is due to V. de Sabbata & C. Sivaram, *A minimal time and time-temperature uncertainty principle*, Foundations of Physics Letters **5**, pp. 183–189, 1992. Experimental details are found, for example, in G. T. Gillies & S. W. Allison, *Experimental test of a time-temperature formulation of the uncertainty principle via nanoparticle fluorescence*, Foundations of Physics Letters **18**, pp. 65–74, 2005. Cited on page 86.

54 Albert Einstein & Max Born, *Briefwechsel 1916 bis 1955*, Rowohlt, 1969, as cited on page 34. Cited on page 87.

55 E. Schrödinger, *Quantisierung als Eigenwertproblem I*, Annalen der Physik **79**, pp. 361–376, 1926, and *Quantisierung als Eigenwertproblem II*, Annalen der Physik **79**, pp. 489–527, 1926. Cited on page 92.

56 C. G. Gray, G. Karl & V. A. Novikov, *From Maupertius to Schrödinger. Quantization of classical variational principles*, American Journal of Physics **67**, pp. 959–961, 1999. Cited on page 92.

57 Y. Aharonov & D. Bohm, *Significance of electromagnetic potentials in the quantum theory*, Physical Review **115**, pp. 485–491, 1959. Cited on page 98.

58 R. Colella, A. W. Overhauser & S. A. Werner, *Observation of gravitationally induced quantum interference*, Physical review Letters **34**, pp. 1472–1474, 1975. Cited on page 100.

59 The trend-setting result that started this exploration was Hans-Werner Fink & al., *Atomic resolution in lens-less low-energy electron holography*, Physical Review Letters **67**, pp. 1543–1546, 1991. Cited on page 101.

60 L. Cser, Gy. Török, G. Krexner, I. Sharkov & B. Faragó, *Holographic imaging of atoms using thermal neutrons*, Physical Review Letters **89**, p. 175504, 2002. Cited on page 101.

61 G. E. Uhlenbeck & S. Goudsmit, *Ersetzung der Hypothese vom unmechanischen Zwang durch eine Forderung bezüglich des inneren Verhaltens jedes einzelnen Elektrons*, Naturwissenschaften **13**, pp. 953–954, 1925. Cited on page 104.

62 L. Thomas, *The motion of the spinning electron*, Nature **117**, p. 514, 1926. Cited on page 105.

63 K. von Meyenn & E. Schucking, *Wolfgang Pauli*, Physics Today pp. 43–48, February 2001. Cited on page 105.

64 T. D. Newton & E. P. Wigner, *Localized states for elementary systems*, Review of Modern Physics **21**, pp. 400–406, 1949. L. L. Foldy & S. A. Wouthuysen, *On the Dirac theory of spin 1/2 particles and its nonrelativistic limit*, Physical Review **78**, pp. 29–36, 1950. Both are classic papers. Cited on page 106.

65 J. P. Costella & B. H. J. McKellar, *The Foldy–Wouthuysen transformation*, American Journal of Physics **63**, pp. 1119–1121, 1995. Cited on page 106.

66 For an account of the first measuremnt of the g-factor of the electron, see H. R. Crane, *How we happended to measure g-2: a tale of serendipity*, Physics in Perspective **2**, pp. 135–140, 2000. The most interesting part is how the experimentalists had to overcome the conviction of almost all theorists that the measurement was impossible in principle. Cited on page 107.

67 The g-factors for composite nuclei are explained briefly on en.wikipedia.org/wiki/Nuclear_magnetic_moment and measured values are found at www-nds.iaea.org. See also H. Dehmelt, *Is the electron a composite particle?*, Hyperfine Interactions **81**, pp. 1–3, 1993. Cited on pages 108 and 254.

68 The nearest anyone has come to an image of a hydrogen atom is found in A. Yazdani, *Watching an atom tunnel*, Nature **409**, pp. 471–472, 2001. The experiments on Bose–Einstein condensates are also candidates for images of hydrogen atoms. The company Hitachi made a fool of itself in 1992 by claiming in a press release that its newest electron microscope could image hydrogen atoms. Cited on page 109.

69 A. M. Wolsky, *Kinetic energy, size, and the uncertainty principle*, American Journal of Physics **42**, pp. 760–763, 1974. Cited on page 110.

70 For a fascinating summary, see M. A. Cirone, G. Metikas & W. P. Schleich, *Unusual bound or localized states*, preprint at arxiv.org/abs/quant-ph/0102065. Cited on page 110.

71 See the paper by Martin Gardner, *Science fiction puzzle tales*, Clarkson Potter, **67**, pp. 104–105, 1981, or his book *Aha! Insight*, Scientific American & W.H. Freeman, 1978. Several versions are given A. Hajnal & P. Lovász, *An algorithm to prevent the propagation of certain diseases at minimum cost*, in *Interfaces Between Computer Science and Operations Research*, edited by J. K. Lenstra, A. H. G. Rinnooy Kan & P. Van Emde Boas, Mathematisch Centrum, Amsterdam 1978, whereas the computer euphemism is used by A. Orlitzky & L. Shepp, *On curbing virus propagation*, Technical memorandum, Bell Labs 1989. Cited on page 112.

72 A complete discussion of the problem can be found in chapter 10 of Ilan Vardi, *Computational Recreations in Mathematica*, Addison Wesley, 1991. Cited on page 112.

73 On Gibbs' paradox, see your favourite text on thermodynamics or statistical mechanics. See also W. H. Zurek, *Algorithmic randomness and physical entropy*, Physical Review A **40**, pp. 4731–4751, 1989. Zurek shows that the Sackur–Tetrode formula can be derived from algorithmic entropy considerations. Cited on page 114.

74 S. N. Bose, *Plancks Gesetz und Lichtquantenhypothese*, Zeitschrift für Physik **26**, pp. 178–181, 1924. The theory was then expanded in A. Einstein, *Quantentheorie des einatomigen idealen Gases*, Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin **22**, pp. 261–267, 1924, A. Einstein, *Quantentheorie des einatomigen idealen Gases. Zweite Abhandlung*, Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin **23**, pp. 3–14, 1925, A. Einstein, *Zur Quantentheorie des idealen Gases*, Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin **23**, pp. 18–25, 1925. Cited on page 118.

75 C. K. Hong, Z. Y. Ou & L. Mandel, *Measurement of subpicosecond time intervals between two photons by interference*, Physical Review Letters **59**, pp. 2044–2046, 1987. See also T. B. Pittman, D. V. Strekalov, A. Migdall, M. H. Rubin, A. V. Sergienko & Y. H. Shih, *Can two-photon interference be considered the interference of two photons?*, Physical Review Letters **77**, pp. 1917–1920, 1996. Cited on page 118.

76 An example of such an experiment performed with electrons instead of photons is described in E. Bocquillon, V. Freulon, J.-M. Berroir, P. Degiovanni, B. Plaçais, A. Cavanna, Y. Jin & G. Fève, *Coherence and indistinguishability of single electrons emitted by independent sources*, Science **339**, pp. 1054–1057, 2013. See also the comment C. Schönenberger, *Two indistinguishable electrons interfere in an electronic device*, Science **339**, pp. 1041–1042, 2013. Cited on page 119.

77 M. Schellekens, R. Hoppeler, A. Perrin, J. Viana Gomes, D. Boiron, C. I. Westbrook & A. Aspect, *Hanbury Brown Twiss effect for ultracold quantum gases*, Science **310**, p. 648, 2005, preprint at arxiv.org/abs/cond-mat/0508466. J. Viana Gomes, A. Perrin, M. Schellekens, D. Boiron, C. I. Westbrook & M. Belsley, *Theory for a Hanbury Brown Twiss experiment with a ballistically expanding cloud of cold atoms*, Physical Review A **74**, p. 053607, 2006, preprint at arxiv.org/abs/quant-ph/0606147. T. Jeltes, J. M. McNamara, W. Hogervorst, W. Vassen, V. Krachmalnicoff, M. Schellekens, A. Perrin, H. Chang, D. Boiron, A. Aspect & C. I. Westbrook, *Comparison of the Hanbury Brown-Twiss effect for bosons and fermions*, Nature **445**, p. 402, 2007, preprint at arxiv.org/abs/cond-mat/0612278. Cited on page 120.

78 The experiment is described in E. Ramberg & G. A. Snow, *Experimental limit on a small violation of the Pauli principle*, Physics Letters B **238**, pp. 438–441, 1990. Other experimental tests are reviewed in O. W. Greenberg, *Particles with small violations of Fermi or Bose statistics*, Physical Review D **43**, pp. 4111–4120, 1991. Cited on page 122.

79 The original no-cloning theorem is by D. Dieks, *Communication by EPR devices*, Physics Letters A **92**, pp. 271–272, 1982, and by W. K. Wootters & W. H. Zurek, *A single quantum cannot be cloned*, Nature **299**, pp. 802–803, 1982. For a discussion of photon and multiparticle cloning, see N. Gisin & S. Massar, *Optimal quantum cloning machines*, Physics Review Letters **79**, pp. 2153–2156, 1997. The whole topic has been presented in detail by V. Buzek & M. Hillery, *Quantum cloning*, Physics World **14**, pp. 25–29, November 2001. Cited on page 122.

80 S. J. Wiesner, *Conjugate Coding*, **SIGACT News**, **15**, pp. 78–88, 1983. This widely cited paper was one of starting points of quantum information theory. Cited on page 123.

81 The most recent experimental and theoretical results on physical cloning are described in A. Lamas-Linares, C. Simon, J. C. Howell & D. Bouwmeester, *Experimental quantum cloning of single photons*, Science **296**, pp. 712 – 714, 2002, D. Collins & S. Popescu, *A classical analogue of entanglement*, preprint arxiv.org/abs/quant-ph/0107082, 2001, and A. Daffertshofer, A. R. Plastino & A. Plastino, *Classical no-cloning theorem*, Physical Review Letters **88**, p. 210601, 2002. Cited on page 123.

82 E. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, Annals of Mathematics **40**, pp. 149–204, 1939. This famous paper summarises the work which later brought him the Nobel Prize in Physics. Cited on pages 125 and 137.

83 For a full list of isotopes, see R. B. Firestone, *Table of Isotopes, Eighth Edition, 1999 Update*, with CDROM, John Wiley & Sons, 1999. Cited on page 127.

84 This is deduced from the $g - 2$ measurements, as explained in his Nobel-prize talk by Hans Dehmelt, *Experiments with an isolated subatomic particle at rest*, Reviews of Modern Physics **62**, pp. 525–530, 1990. On this topic, see also his paper Ref. 67. No citations.

and in Hans Dehmelt, *Is the electron a composite particle?*, Hyperfine Interactions **81**, pp. 1–3, 1993.

85 G. Gabrielse, H. Dehmelt & W. Kells, *Observation of a relativistic, bistable hysteresis in the cyclotron motion of a single electron*, Physical Review Letters **54**, pp. 537–540,

1985. No citations.

86 W. Pauli, *The connection between spin and statistics*, Physical Review **58**, pp. 716– 722, 1940. Cited on page 133.

87 The belt trick has been popularized by Dirac, Feynman and many others. An example is R. P. Feynman, *The reason for antiparticles*, in *Elementary Particles and the Laws of Physics: The 1986 Dirac Memorial Lectures*, Cambridge University Press, 1987. The belt trick is also explained, for example, on page 1148 in C. W. Misner, K. S. Thorne & J. A. Wheeler, *Gravitation*, Freeman, 1973. It is called the *scissor trick* on page 43 of volume 1 of R. Penrose & W. Rindler, *Spinors and Spacetime*, 1984. It is also cited and discussed by R. Gould, *Answer to question #7*, American Journal of Physics **63**, p. 109, 1995. Still, some physicists do not like the belt-trick image for spin 1/2 particles; for an example, see I. Duck & E. C. G. Sudarshan, *Toward an understanding of the spin-statistics theorem*, American Journal of Physics **66**, pp. 284–303, 1998. Cited on page 133.

88 M. V. Berry & J. M. Robbins, *Indistinguishability for quantum particles: spin, statistics and the geometric phase*, Proceedings of the Royal Society in London A **453**, pp. 1771–1790, 1997. See also the comments to this result by J. Twamley, *Statistics given a spin*, Nature **389**, pp. 127–128, 11 September 1997. Their newer results are M. V. Berry & J. M. Robbins, *Quantum indistinguishability: alternative constructions of the transported basis*, Journal of Physics A (Letters) **33**, pp. L207–L214, 2000, and M. V. Berry & J. M. Robbins, in *Spin–Statistics*, eds. R. Hilborn & G. Tino, American Institute of Physics, 2000, pp. 3–15. See also Michael Berry's home page at www.phy.bris.ac.uk/people/berry_mv. Cited on page 135.

89 R. W. Hartung, *Pauli principle in Euclidean geometry*, American Journal of Physics **47**, pp. 900–910, 1979. Cited on page 136.

90 The issue is treated in his *Summa Theologica*, in question 52 of the first part. The complete text, several thousand pages, can be found on the www.newadvent.org website. Also present-day angelologists, of which there are only a few across the world, agree with Aquinas. Cited on page 136.

91 The point that spin can be seen as a rotation was already made by F. J. Belinfante, *On the spin angular momentum of mesons*, Physica **6**, p. 887, 1939, and taken up again by Hans C. Ohanian, *What is spin?*, American Journal of Physics **54**, pp. 500–505, 1986. See also E. Duran & A. Erschow, Physikalische Zeitschrift der Sowjetunion **12**, p. 466, 1937. Cited on page 138.

92 See the book by Jean-Marc Lévy-Leblond & Françoise Balibar in Ref. 4. Cited on page 141.

93 Generalizations of bosons and fermions are reviewed in the (serious!) paper by O. W. Greenberg, D. M. Greenberger & T. V. Greenbergest, *(Para)bosons, (para)fermions, quons and other beasts in the menagerie of particle statistics*, at arxiv.org/abs/hep-th/9306225. A newer summary is O. W. Greenberg, *Theories of violation of statistics*, electronic preprint available at arxiv.org/abs/hep-th/0007054. Cited on page 142.

94 Gell-Mann wrote this for the 1976 Nobel Conference (not for the Nobel speech; he is the only winner who never published it.) M. Gell-Mann, *What are the building blocks of matter?*, in D. Huff & O. Prewitt, editors, *The Nature of the Physical Universe*, New York, Wiley, 1979, p. 29. Cited on page 143.

95 See e.g. the reprints of his papers in the standard collection by John A. Wheeler & Wojciech H. Zurek, *Quantum Theory and Measurement*, Princeton University Press, 1983. Cited on page 144.

96 H. D. Zeh, *On the interpretation of measurement in quantum theory*, Foundations of Physics **1**, pp. 69–76, 1970. Cited on page 144.

97 Linda Reichl, *A Modern Course in Statistical Physics*, Wiley, 2nd edition, 1998. An excellent introduction into thermodynamics. Cited on page 146.

98 E. Joos & H. D. Zeh, *The emergence of classical properties through interactions with the environment*, Zeitschrift für Physik B **59**, pp. 223–243, 1985. See also Erich Joos, *Decoherence and the appearance of a classical world in quantum theory*, Springer Verlag, 2003. Cited on page 148.

99 M. Tegmark, *Apparent wave function collapse caused by scattering*, Foundation of Physics Letters **6**, pp. 571–590, 1993, preprint at arxiv.org/abs/gr-qc/9310032. See also his paper that shows that the brain is not a quantum computer, M. Tegmark, *The importance of quantum decoherence in brain processes*, Physical Review E **61**, pp. 4194–4206, 2000, preprint at arxiv.org/abs/quant-ph/9907009. Cited on page 148.

100 The decoherence time is bound from above by the relaxation time. See A. O. Caldeira & A. J. Leggett, *Influence of damping on quantum interference: an exactly soluble model*, Physical Review A **31**, 1985, pp. 1059–1066. This is the main reference about effects of decoherence for a harmonic oscillator. The general approach to relate decoherence to the influence of the environment is due to Niels Bohr, and has been pursued in detail by Heinz Dieter Zeh. Cited on page 149.

101 G. Lindblad, *On the generators of quantum dynamical subgroups*, Communications in Mathematical Physics **48**, pp. 119–130, 1976. Cited on page 149.

102 Wojciech H. Zurek, *Decoherence and the transition from quantum to classical*, Physics Today pp. 36–44, October 1991. An easy but somewhat confusing article. His reply to the numerous letters of response in Physics Today, April 1993, pp. 13–15, and pp. 81–90, exposes his ideas in a clearer way and gives a taste of the heated discussions on this topic. Cited on pages 149 and 156.

103 John Bardeen, explained this regularly in the review talks he gave at the end of his life, such as the one the author heard in Tokyo in the year 1990. Cited on page 150.

104 Collapse times have been measured for the first time by the group of Serge Haroche in Paris. See M. Brune, E. Hagley, J. Dreyer, X. Maitre, A. Maali, C. Wunderlich, J. M. Raimond & S. Haroche, *Observing the progressive decoherence of the "meter" in a quantum measurement*, Physical Review Letters 77, pp. 4887–4890, 1996. See also C. Guerlin, J. Bernu, S. Deléglise, C. Sayrin, S. Gleyzes, S. Kuhr, M. Brune, J.-M. Raimond & S. Haroche, *Progressive field-state collapse and quantum non-demolition photon counting*, Nature **448**, pp. 889–893, 2007. Cited on pages 150 and 162.

105 Later experiments confirming the numerical predictions from decoherence were published by C. Monroe, D. M. Meekhof, B. E. King & D. J. Wineland, *A "Schrödinger cat" superposition state of an atom*, Science **272**, pp. 1131–1136, 1996, W. P. Schleich, *Quantum physics: engineering decoherence*, Nature **403**, pp. 256–257, 2000, C. J. Myatt, B. E. King, Q. A. Turchette, C. A. Sackett, D. Kielpinski, W. M. Itano, C. Monroe & D. J. Wineland, *Decoherence of quantum superpositions through coupling to engineered reservoirs*, Nature **403**, pp. 269–273, 2000. See also the summary by W. T. Strunz, G. Alber & F. Haake, *Dekohärenz in offenen Quantensystemen*, Physik Journal **1**, pp. 47–52, November 2002. Cited on page 150.

106 L. Hackermüller, K. Hornberger, B. Brezger, A. Zeilinger & M. Arndt, *Decoherence of matter waves by thermal emission of radiation*, Nature **427**, pp. 711–714, 2004. Cited on page 150.

107 K. Baumann, *Quantenmechanik und Objektivierbarkeit*, Zeitschrift für Naturforschung **25a**, pp. 1954–1956, 1970. Cited on page 151.

108 See for example D. Styer, Physics Today p. 11, September 2000. Cited on page 153.

109 David Bohm, *Quantum Theory*, Prentice-Hall, 1951, pp. 614–622. Cited on page 154.

110 A. Einstein, B. Podolsky & N. Rosen, *Can quantum-mechanical description of reality be considered complete?*, Physical Review **48**, pp. 696–702, 1935. Cited on page 154.

111 A. Aspect, J. Dalibard & G. Roger, *Experimental tests of Bell's inequalities using time-varying analyzers*, Physical Review Letters **49**, pp. 1804–1807, 1982, Cited on page 155.

112 G. C. Hergerfeldt, *Causality problems for Fermi's two-atom system*, Physical Review Letters **72**, pp. 596–599, 1994. Cited on page 155.

113 An experimental measurement of superpositions of left and right flowing currents with 10^{10} electrons was J. E. Mooij, T. P. Orlando, L. Levitov, L. Tian, C. H. van der Wal & S. Lloyd, *Josephson persistent-current qubit*, Science **285**, pp. 1036–1039, 1999. In the year 2000, superpositions of 1 μA clockwise and anticlockwise have been detected; for more details, see J.R. Friedman & al., *Quantum superposition of distinct macroscopic states*, Nature **406**, p. 43, 2000. Cited on page 156.

114 On the superposition of magnetization in up and down directions there are numerous papers. Recent experiments on the subject of quantum tunnelling in magnetic systems are described in D. D. Awschalom, J. F. Smith, G. Grinstein, D. P. DiVicenzo & D. Loss, *Macroscopic quantum tunnelling in magnetic proteins*, Physical Review Letters **88**, pp. 3092–3095, 1992, and in C. Paulsen & al., *Macroscopic quantum tunnelling effects of Bloch walls in small ferromagnetic particles*, Europhysics Letters **19**, pp. 643–648, 1992. Cited on page 156.

115 For example, superpositions were observed in Josephson junctions by R. F. Voss & R. A. Webb, *Macroscopic quantum tunnelling in 1 mm Nb Josephson junctions*, Physical Review Letters **47**, pp. 265–268, 1981, Cited on page 156.

116 S. Haroche, *Entanglement, decoherence and the quantum-classical transition*, Physics Today **51**, pp. 36–42, July 1998. An experiment putting atom at two places at once, distant about 80 nm, was published by C. Monroe, C. Monroe, D. M. Meekhof, B. E. King & D. J. Wineland, *A 'Schroedinger Cat' Superposition of an Atom*, Science **272**, pp. 1131–1136, 1996. Cited on page 156.

117 M. R. Andrews, C. G. Townsend, H. -J. Miesner, D. S. Durfee, D. M. Kurn & W. Ketterle, *Observations of interference between two Bose condensates*, Science **275**, pp. 637–641, 31 January 1997. See also the www.aip.org/physnews/special.htm website. Cited on page 156.

118 A clear discussion can be found in S. Haroche & J. -M. Raimond, *Quantum computing: dream or nightmare?*, Physics Today **49**, pp. 51–52, 1996, as well as the comments in Physics Today **49**, pp. 107–108, 1996. Cited on page 157.

119 The most famous reference on the wave function collapse is chapter IV of the book by Kurt Gottfried, *Quantum Mechanics*, Benjamin, New York, 1966. It was the favourite reference by Victor Weisskopf, and cited by him on every occasion he talked about the topic. Cited on page 158.

120 The prediction that quantum tunnelling could be observable when the dissipative interaction with the rest of the world is small enough was made by Leggett; the topic is reviewed in A. J. Leggett, S. Chahravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg & W. Zwerger, *Dynamics of dissipative 2-state systems*, Review of Modern Physics **59**, pp. 1–85, 1987. Cited on page 160.

121 S. Kochen & E. P. Specker, *The problem of hidden variables in quantum mechanics*, Journal of Mathematics and Mechanics **17**, pp. 59–87, 1967. Cited on page 163.

122 J. F. Clauser, M. A. Horne, A. Shimony & R. A. Holt, *Proposed experiment to test local hidden-variable theories*, Physical Review Letters **23**, pp. 880–884, 1969. The more general and original result is found in J. S. Bell, *On the Einstein Podolsky Rosen Paradox*, Physics **1**, p. 195, 1964. Cited on page 164.

123 D. M. Greenberger, M. A. Horne & A. Zeilinger, *Going beyond Bell's theorem*, postprint of the 1989 paper at arxiv.org/abs/0712.0912. The first observation was D. Bouwmeester, J.-W. Pan, M. Daniell, H. Weinfurter & A. Zeilinger, *Observation of three-photon Greenberger-Horne–Zeilinger entanglement*, preprint at arxiv.org/abs/quant-ph/9810035. Cited on page 164.

124 Bryce de Witt & Neill Graham, eds., *The Many–Worlds Interpretation of Quantum Mechanics*, Princeton University Press, 1973. This interpretation talks about entities which cannot be observed, namely the many worlds, and often assumes that the wave function of the universe exists. Both habits are beliefs and in contrast with facts. Cited on page 167.

125 'On the other had I think I can safely say that nobody understands quantum mechanics.' From Richard P. Feynman, *The Character of Physical Law*, MIT Press, Cambridge, 1965, p. 129. He repeatedly made this statement, e.g. in the introduction of his otherwise excellent *QED – The Strange Theory of Light and Matter*, Penguin Books, 1990. Cited on page 167.

126 M. Tegmark, *The importance of quantum decoherence in brain processes*, Physical Review D **61**, pp. 4194–4206, 2000, or also arxiv.org/abs/quant-ph/9907009. Cited on page 168.

127 Connections between quantum theory and information theory can be followed in the International Journal of Quantum Information. Cited on page 169.

128 J. A. Wheeler, pp. 242–307, in *Batelle Recontres: 1967 Lectures in Mathematics and Physics*, C. DeWitt & J. A. Wheeler, editors, W.A. Benjamin, 1968. For a pedagogical explanation, see John W. Norbury, *From Newton's laws to the Wheeler-DeWitt equation*, arxiv.org/abs/physics/980604 or European Journal of Physics **19**, pp. 143–150, 1998. Cited on page 170.

129 The most fascinating book on the topic is by Kurt Nassau, *The Physics and Chemistry of Color – the Fifteen Causes of Color*, 1983, and the excellent webexhibits.org/causesofcolour website. Cited on page 171.

130 Y. Ruiz-Morales & O. C. Mullins, *Measured and Simulated Electronic Absorption and Emission Spectra of Asphaltenes*, Energy & Fuels **23**, pp. 1169–1177, 2009. U. Bergmann, H. Groenzin, O. C. Mullins, P. Glatzel, J. Fetzer & S. P. Cramer, *Carbon K-edge X-ray Raman spectroscopy supports simple, yet powerful description of aromatic hydrocarbons and asphaltenes*, Chemical Physics Letters **369**, pp. 184–191, 2003. Cited on page 171.

131 Two excellent reviews with numerous photographs are E. Grotewohl, *The genetics and biochemistry of floral pigments*, Annual Reviews of Plant Biology **57**, pp. 761–780, 2006, and Y. Tanaka, N. Sasaki & A. Ohmiya, *Biosynthesis of plant pigments: anthocyanins, betalains and carotenoids*, The Plant Journal **54**, pp. 733–749, 2008. Cited on page 179.

132 L. Pérez-Rodriguez & J. Viñuda, *Carotenoid-based bill and eye coloration as honest signals of condition: an experimental test in the red-legged partridge (Alectoris rufa)*, Naturwissenschaften **95**, pp. 821–830, 2008, Cited on page 179.

133 R. Pello, D. Schaerer, J. Richard, J.-F. Le Borgne & J.-P. Kneib, *ISAAC/VLT observations of a lensed galaxy at z=10.0*, Astronomy and Astrophysics **416**, p. L35, 2004.

Cited on page 182.

134 A pedagogical introduction is given by L. J. Curtis & D. G. Ellis, *Use of the Einstein–Brillouin–Keller action quantization*, American Journal of Physics **72**, pp. 1521–1523, 2004. See also the introduction of A. Klein, *WKB approximation for bound states by Heisenberg matrix mechanics*, Journal of Mathematical Physics **19**, pp. 292–297, 1978. Cited on pages 183 and 188.

135 J. Neukammer & al., *Spectroscopy of Rydberg atoms at n ~ 500*, Physical Review Letters **59**, pp. 2947–2950, 1987. Cited on page 186.

136 Mark P. Silverman, *And Yet It Moves: Strange Systems and Subtle Questions in Physics*, Cambridge University Press 1993. A beautiful book by an expert on motion. Cited on pages 187, 194, and 195.

137 This is explained by J. D. Hey, *Mystery error in Gamow's Tompkins reappears*, Physics Today pp. 88–89, May 2001. Cited on page 187.

138 The beautiful experiment was first published in A. S. Stodolna, A. Rouzée, F. Lépine, S. Cohen, F. Robicheaux, A. Gijsbertsen, J. H. Jungmann, C. Bordas & M. J. J. Vrakking, *Hydrogen atoms under magnification: direct observation of the nodal structure of Stark states*, Physical Review Letters **110**, p. 213001, 2013. Cited on pages 185 and 186.

139 L. L. Foldy, *The electromagnetic properties of Dirac particles*, Physical Review **83**, pp. 688–693, 1951. L. L. Foldy, *The electron–neutron interaction*, Physical Review **83**, pp. 693–696, 1951. L. L. Foldy, *Electron–neutron interaction*, Review of Modern Physics **30**, pp. 471–481, 1952. Cited on page 190.

140 H. Euler & B. Kockel, *Über die Streuung von Licht an Licht nach der Diracschen Theorie*, Naturwissenschaften **23**, pp. 246–247, 1935, H. Euler, *Über die Streuung von Licht an Licht nach der Diracschen Theorie*, Annalen der Physik **26**, p. 398, 1936, W. Heisenberg & H. Euler, *Folgerung aus der Diracschen Theorie des Electrons*, Zeitschrift für Physik **98**, pp. 714–722, 1936. Cited on page 193.

141 See the simple explanation by L. J. F. Hermans, *Blue skies, blue seas*, Europhysics News **37**, p. 16, 2006, and the detailed explanation by C. L. Braun & S. N. Smirnov, *Why is water blue?*, Journal of Chemical Education **70**, pp. 612–614, 1993. Cited on page 194.

142 The discovery is published in T. Friedmann & C. R. Hagen, *Quantum Mechanical Derivation of the Wallis Formula for* π , Journal of Mathematical Physics **56**, p. 112101, 2015, preprint at arxiv.org/1510.07813. See also I Chashchina & Z. K. Silagadze, *On the quantum mechanical derivation of the Wallis formula for* π, preprint at arxiv.org/1704.06153. Cited on page 194.

143 For the atomic case, see P. L. Gould, G. A. Ruff & D. E. Pritchard, *Diffraction of atoms by light: the near resonant Kapitza–Dirac effect*, Physical Review Letters **56**, pp. 827–830, 1986. Many early experimental attempts to observe the diffraction of electrons by light, in particular those performed in the 1980s, were controversial; most showed only the deflection of electrons, as explained by H. Batelaan, Contemporary Physics **41**, p. 369, 2000. Later on, he and his group performed the newest and most spectacular experiment, demonstrating real diffraction, including interference effects; it is described in D. L. Freimund, K. Aflatooni & H. Batelaan, *Observation of the Kapitza–Dirac effect*, Nature **413**, pp. 142–143, 2001. Cited on page 195.

144 A single–atom laser was built in 1994 by K. An, J. J. Childs, R. R. Dasari & M. S. Feld, *Microlaser: a laser with one atom in an optical resonator*, Physical Review Letters **73**, p. 3375, 1994. Cited on page 195.

145 An introduction is given by P. Pinkse & G. Rempe, *Wie fängt man ein Atom mit einem Photon?*, Physikalische Blätter **56**, pp. 49–51, 2000. Cited on page 195.

146 J.P. Briand & al., *Production of hollow atoms by the excitation of highly charged ions in interaction with a metallic surface*, Physical Review Letters **65**, pp. 159–162, 1990. See also G. Marowsky & C. Rhodes, *Hohle Atome und die Kompression von Licht in Plasmakanälen*, Physikalische Blätter **52**, pp. 991–994, Oktober 1996. Cited on page 195.

147 G. Gabrielse, D. Hanneke, T. Kinoshita, M. Nio & B. Odom, *New determination of the fine structure constant from the electron g value and QED*, Physical Review Letters **97**, p. 030802, 2006. Cited on page 196.

148 A. Sommerfeld, *Zur Quantentheorie der Spektrallinien*, Annalen der Physik **51**, pp. 1–94, 1916, and its continuation with the same title on pp. 125–167 in the same volume. The fine structure constant is introduced in the first paper, but Sommerfeld explains that the paper is a transcript of talks that he gave in 1915. Cited on page 196.

149 Wolfgang Pauli, *Exclusion principle and quantum mechanics*, Nobel lecture, 13 December 1946, in *Nobel Lectures, Physics*, Volume 3, 1942–1962, Elsevier, 1964. Cited on page 197.

150 An informative account of the world of psychokinesis and the paranormal is given by the famous professional magician James Randi, *Flim-flam!*, Prometheus Books, Buffalo 1987, as well as in several of his other books. See also the www.randi.org website. Cited on page 201.

151 *Le Système International d'Unités*, Bureau International des Poids et Mesures, Pavillon de Breteuil, Parc de Saint Cloud, 92310 Sèvres, France. All new developments concerning SI units are published in the journal *Metrologia*, edited by the same body. Showing the slow pace of an old institution, the BIPM launched a website only in 1998; it is now reachable at www.bipm.fr. See also the www.utc.fr/~tthomass/Themes/Unites/index.html website; this includes the biographies of people who gave their names to various units. The site of its British equivalent, www.npl.co.uk/npl/reference, is much better; it provides many details as well as the English-language version of the SI unit definitions. Cited on page 205.

152 The bible in the field of time measurement is the two-volume work by J. Vanier & C. Audoin, *The Quantum Physics of Atomic Frequency Standards*, Adam Hilge, 1989. A popular account is Tony Jones, *Splitting the Second*, Institute of Physics Publishing, 2000.

The site opdafl.obspm.fr/www/lexique.html gives a glossary of terms used in the field. For precision *length* measurements, the tools of choice are special lasers, such as mode-locked lasers and frequency combs. There is a huge literature on these topics. Equally large is the literature on precision *electric current* measurements; there is a race going on for the best way to do this: counting charges or measuring magnetic forces. The issue is still open. On *mass* and atomic mass measurements, see Volume II, on page 71. On high-precision *temperature* measurements, see Volume I, on page 548. Cited on page 206.

153 The unofficial SI prefixes were first proposed in the 1990s by Jeff K. Aronson of the University of Oxford, and might come into general usage in the future. See New Scientist **144**, p. 81, 3 December 1994. Other, less serious proposals also exist. Cited on page 207.

154 For more details on electromagnetic unit systems, see the standard text by John David Jackson, *Classical Electrodynamics*, 3rd edition, Wiley, 1998. Cited on page 210.

155 D.J. Bird & al., *Evidence for correlated changes in the spectrum and composition of cosmic rays at extremely high energies*, Physical Review Letters **71**, pp. 3401–3404, 1993. Cited on page 211.

156 P. J. Hakonen, R. T. Vuorinen & J. E. Martikainen, *Nuclear antiferromagnetism in rhodium metal at positive and negative nanokelvin temperatures*, Physical Review Letters **70**, pp. 2818–2821, 1993. See also his article in Scientific American, January 1994. Cited on page 211.

157 A. Zeilinger, *The Planck stroll*, American Journal of Physics **58**, p. 103, 1990. Can you find another similar example? Cited on page 211.

Challenge 205 e

158 An overview of this fascinating work is given by J. H. Taylor, *Pulsar timing and relativistic gravity*, Philosophical Transactions of the Royal Society, London A **341**, pp. 117–134, 1992. Cited on page 211.

159 The most precise clock built in 2004, a caesium fountain clock, had a precision of one part in 10^{15}. Higher precision has been predicted to be possible soon, among others by M. Takamoto, F.-L. Hong, R. Higashi & H. Katori, *An optical lattice clock*, Nature **435**, pp. 321–324, 2005. Cited on page 211.

160 J. Bergquist, ed., *Proceedings of the Fifth Symposium on Frequency Standards and Metrology*, World Scientific, 1997. Cited on page 211.

161 See the information on $D_s^{\pm}$ mesons from the particle data group at pdg.web.cern.ch/pdg. Cited on page 211.

162 About the long life of tantalum 180, see D. Belic & al., *Photoactivation of* $^{180}Ta^{m}$ *and its implications for the nucleosynthesis of nature's rarest naturally occurring isotope*, Physical Review Letters **83**, pp. 5242–5245, 20 December 1999. Cited on page 212.

163 See the review by L. Ju, D. G. Blair & C. Zhao, *The detection of gravitational waves*, Reports on Progress in Physics **63**, pp. 1317–1427, 2000. Cited on page 212.

164 See the clear and extensive paper by G. E. Stedman, *Ring laser tests of fundamental physics and geophysics*, Reports on Progress in Physics **60**, pp. 615–688, 1997. Cited on page 212.

165 The various concepts are even the topic of a separate international standard, ISO 5725, with the title *Accuracy and precision of measurement methods and results*. A good introduction is John R. Taylor, *An Introduction to Error Analysis: the Study of Uncertainties in Physical Measurements*, 2nd edition, University Science Books, Sausalito, 1997. Cited on page 212.

166 P. J. Mohr, B. N. Taylor & D. B. Newell, *CODATA recommended values of the fundamental physical constants: 2010*, preprint at arxiv.org/abs/1203.5425. This is the set of constants resulting from an international adjustment and recommended for international use by the Committee on Data for Science and Technology (CODATA), a body in the International Council of Scientific Unions, which brings together the International Union of Pure and Applied Physics (IUPAP), the International Union of Pure and Applied Chemistry (IUPAC) and other organizations. The website of IUPAC is www.iupac.org. Cited on page 214.

167 Some of the stories can be found in the text by N. W. Wise, *The Values of Precision*, Princeton University Press, 1994. The field of high-precision measurements, from which the results on these pages stem, is a world on its own. A beautiful introduction to it is J. D. Fairbanks, B. S. Deaver, C. W. Everitt & P. F. Michaelson, eds., *Near Zero: Frontiers of Physics*, Freeman, 1988. Cited on page 214.

168 For details see the well-known astronomical reference, P. Kenneth Seidelmann, *Explanatory Supplement to the Astronomical Almanac*, 1992. Cited on page 219.

169 See the corresponding reference in the first volume. Cited on page 221.

170 A good reference is the *Encyclopedia of Mathematics*, in 10 volumes, Kluwer Academic Publishers, 1988–1993. It explains most concepts used in mathematics. Spending an hour with it looking up related keywords is an efficient way to get an introduction into any part of

mathematics, especially into the vocabulary and the main connections.

The opposite approach, to make things as complicated as possible, is taken in the delightful text by Carl E. Linderholm, *Mathematics Made Difficult*, 1971. Cited on page 223.

171 An excellent introduction into number systems in mathematics, including hyperreal (or nonstandard) numbers, quaternions, octonions, p-adic and surreal numbers, is the book by Heinz-Dieter Ebbinghaus, Hans Hermes, Friedrich Hirzebruch, Max Koecher, Klaus Mainzer, Jürgen Neukirch, Alexander Prestel & Reinhold Remmert, *Zahlen*, 3rd edition, Springer Verlag, 1993. It is also available in English, under the title *Numbers*, Springer Verlag, 1990. Cited on pages 225, 234, and 235.

172 For a book on how to use hyperreals in secondary school, see Helmut Wunderling, *Analysis als Infinitesimalrechnung*, Duden Paetec Schulbuchverlag, 2007. Cited on page 235.

173 A. Waser, *Quaternions in Electrodynamics*, 2001. The text can be downloaded from various websites. Cited on pages 227 and 232.

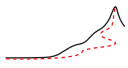
174 S. L. Altman, *Rotations, Quaternions and Double Groups*, Clarendon Press, 1986, and also S. L. Altman, *Hamilton, Rodriguez and the quaternion scandal*, Mathematical Magazine **62**, pp. 291–308, 1988. See also J. C. Hart, G. K. Francis & L. H. Kauffman, *Visualzing quaternion rotation*, ACM Transactions on Graphics **13**, pp. 256–276, 1994. The latter can be downloaded in several places via the internet. Cited on page 230.

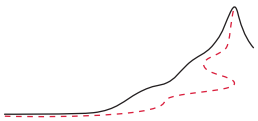
175 See the fine book by Louis H. Kauffman, *Knots and Physics*, World Scientific, 2nd edition, 1994, which gives a clear and visual introduction to the mathematics of knots and their main applications to physics. Cited on page 231.

176 Gaussian integers are explored by G. H. Hardy & E. M. Wright, *An Introduction to the Theory of Numbers*, 5th edition, Clarendon Press, Oxford, 1979, in the sections 12.2 'The Rational Integers, the Gaussian Integers, and the Integers', pp. 178–180, and 12.6 'Properties of the Gaussian Integers' pp. 182–183. For challenges relating to Gaussian integers, look at www.mathpuzzle.com/Gaussians.html. Cited on page 235.

177 About transfinite numbers, see the delightful paperback by Rudy Rucker, *Infinity and the Mind – the Science and Philosophy of the Infinite*, Bantam, 1983. Cited on page 235.

178 E. I. Butikov, *The rigid pendulum – an antique but evergreen physical model*, European Journal of Physics **20**, pp. 429–441, 1999. D. Easton, *The quantum mechanical tipping pencil – a caution for physics teachers*, European Journal of Physics **28**, pp. 1097–1104, 2007, Cited on page 242.





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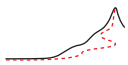


NAME INDEX

B

BUSCH

F

FINK

M

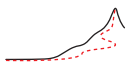
MEWES

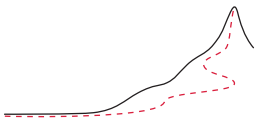
R

ROBERTSON

V

VERBEELEN





SUBJECT INDEX

B

BIPM

C

C

COMPUTER

E
EKERT

G

GRATING

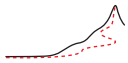
J
Josephson

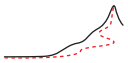
P

PARTICLE

S

SACKUR–TETRODE





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