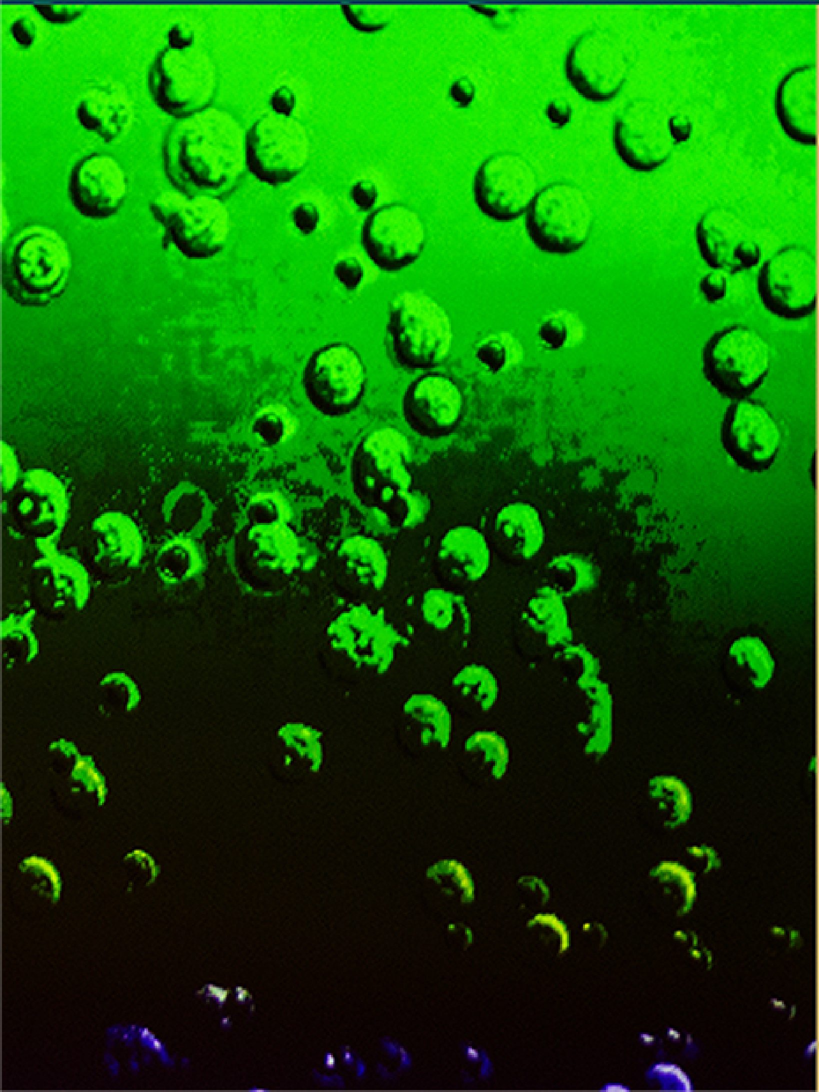
Mathematics for Biomedical Physics



Jogindra M. Wadehra

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Preface

A few years ago, our department started a new undergraduate degree program in Biomedical Physics. The prerequisites to enter this program included two semesters of physics and two semesters of calculus, both differential and integral. A group of participating educators (including some members from the departments of physics, mathematics, and biology as well as from the School of Medicine) developed the courses and their contents for this new program. For one of these courses, Mathematics for Biomedical Physics, all the required topics were not covered in any single textbook available. As the instructor of this course, I had to prepare my own personal notes for some of the topics that I shared with students. I felt that it would be great if a single textbook can be produced which covered all the mathematical topics that were needed for the biomedical physics program. I wanted to make this textbook freely available to all students everywhere using internet. Access to education is a basic human right which should not be hindered by any lack of money or dearth of educational resources. With this thought in mind, I started turning the course materials and lectures that I had been using in my course into an open educational resource (OER). After transcribing all the notes in Word, I gave the notes away to students in my next class, seeking their feedback and comments. I have carefully included responses to these comments in the book.

Even though the prerequisites for this textbook are two semester-long courses in calculus, which cover functions of single variables, the first two chapters of this textbook are devoted to differential calculus and integral calculus. These chapters lead the reader from calculus of functions of a single variable to calculus of multivariable functions. In this scheme, the ideas related to partial derivatives as well as multiple integrals are revealed quite naturally. Throughout the textbook I have attempted to start with something that a typical student may be familiar with and end up with something that will be entirely new for the reader. I would appreciate receiving (at wadehra@wayne.edu) constructive feedback, from students and faculty as well as other readers, who are using this book as a whole or in parts.

The textbook is geared to introduce several mathematical topics at the rudimentary level so that students can appreciate the applications of mathematics to the interdisciplinary field of biomedical physics. Most of the topics are presented in their simplest but rigorous form so that students can easily understand the advanced form of these topics when the need arises. Several end-of-chapter problems and chapter examples relate the applications of mathematics to biomedical physics. After mastering the topics of this book, the students would be ready to embark on quantitative thinking in various topics of biology and medicine. The famous renaissance philosopher and astronomer, Galileo Galilei, is quoted to say, “Mathematics is the language in which Nature has written the book of Universe”. This textbook is an endeavor to teach the language of Nature in a careful, yet simple, manner to undergraduate students.

Chapter 1: Differential Calculus

We will start with a review of differential calculus, assuming that the reader has already seen derivatives for functions of a single variable. After a brief review of simple derivatives, we will introduce the derivatives of functions of multiple variables, also known as partial derivatives.

1.1 DERIVATIVE OF A FUNCTION OF A SINGLE VARIABLE

$F(x)$ is a function of a single *independent* variable x as shown in Figure 1.1. An independent variable means that its value can be assigned at will, without any constraining conditions. By definition, the derivative of $F(x)$ is $\frac{dF}{dx}$, given by

$$\frac{dF}{dx} = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} \qquad Eq. (1.1)$$

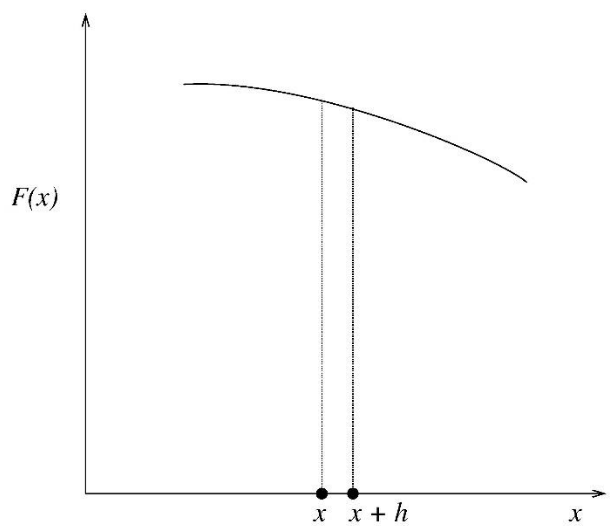

Figure 1.1. $\boldsymbol{F(x)}$ is a function of a single independent variable $\boldsymbol{x}$.

As a simple example, consider the derivative of $F(x) = \sin(x)$,

$$\frac{dF}{dx} = \lim_{h\to 0}\frac{\sin(x+h) - \sin(x)}{h} .$$

In the numerator, we use the trigonometric identity

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

to get

$$\frac{d\,\sin(x)}{dx} = \lim_{h\to 0} \frac{2\sin\left(\frac{h}{2}\right)\cos\left(x + \frac{h}{2}\right)}{h} = \lim_{h\to 0} \frac{\sin(h/2)}{h/2} \lim_{h\to 0} \cos\left(x + \frac{h}{2}\right) .$$

Or,

$$\frac{d\,\sin(x)}{dx} = \cos(x) \lim_{h\to 0} \frac{\sin(h/2)}{h/2} .$$

The limiting value of this expression can be easily obtained, using simple geometry and trigonometry (see Appendix A), to be 1. Using this limiting value, we get the result

$$\frac{d\sin(x)}{dx} = \cos(x) .$$

A similar procedure can be used to determine derivatives of other well-known functions. Here is a compilation of derivatives of several commonly used functions:

$$\frac{dC}{dx} = 0, \quad C \ \text{ is a constant}$$

$$\frac{d\,x^n}{dx} = n\,x^{n-1}$$

$$\frac{d\,\exp(x)}{dx} = \exp(x)$$

$$\frac{d\ln(x)}{dx} = \frac{1}{x}$$

$$\frac{d\cos(x)}{dx} = -\sin(x)$$

$$\frac{d\tan(x)}{dx} = \sec^2 x$$

$$\frac{d\arcsin x}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d\arccos x}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d\arctan x}{dx} = \frac{1}{1 + x^2}$$

Higher Order Derivatives

Assume that the derivative $\frac{dF}{dx}$ of the function $F(x)$ is represented by a new function $F_1(x)$; that is, $F_1(x) = \frac{dF}{dx}$. Then, $\frac{dF_1}{dx}$ is the second derivative of $F(x)$, namely, $\frac{d^2F}{dx^2}$. Similarly, if $F_2(x) = \frac{dF_1}{dx}$, then $\frac{dF_2}{dx}$ is the third derivative of $F(x)$, namely, $\frac{d^3F}{dx^3}$. Using this procedure, higher derivatives of any order can be determined for a function of a single variable.

Chain Rule

When a function F of variable x can be expressed as a function of another function as $F[g(x)]$, then the derivative of F can be most conveniently evaluated using the chain rule as

$$\frac{dF}{dx} = \frac{dF}{dg}\frac{dg}{dx} \quad . \qquad Eq.(1.2)$$

Example: Using the chain rule, determine the derivative of the function $\boldsymbol{F(x) = \sqrt{x^2 + a^2}}$.

Solution: To determine the derivative $\frac{dF}{dx}$, first define a new function $g(x) = x^2 + a^2$. Then, $F(x)$ can be written as a function of $g(x)$ as $F(x) = \sqrt{g(x)} = g^{1/2}$. Using chain rule,

$$\frac{dF}{dx} = \frac{dF}{dg}\frac{dg}{dx} = \frac{1}{2}\, g^{-1/2}\,(2x) = \frac{x}{\sqrt{x^2 + a^2}} \; .$$

Product and Quotient Rules

When a function F of variable x can be expressed as a product of two simpler functions as $F(x) = f(x)\, g(x)$, then the derivative of $F(x)$ is given by

$$\frac{dF}{dx} = f(x)\frac{dg(x)}{dx} + \frac{df(x)}{dx}g(x) \; . \qquad Eq.(1.3)$$

In words, if $F(x) = (FIRST)(\, SECOND)$, then

$$\frac{dF}{dx} = (FIRST)\frac{d\,(SECOND)}{dx} + \frac{d\,(FIRST)}{dx}(SECOND).$$

Example: Using the product rule, determine the derivative of $F(x) = x^2 \ln x$.

Solution: Derivative of $F(x)$ using the product rule is

$$\frac{dF}{dx} = x^2 \frac{d \ln x}{dx} + \ln x \frac{dx^2}{dx} = x^2 \frac{1}{x} + \ln x \; (2x) = x + 2x \ln x \; .$$

A special case of the product rule is the quotient rule, which is used when the function $F(x)$ can be expressed as $F(x) = \frac{f(x)}{g(x)}$. In this case,

$$\frac{dF}{dx} = \frac{1}{g^2}\left[g(x) \frac{df}{dx} - f(x) \frac{dg}{dx}\right] . \qquad Eq. (1.4)$$

Example: Using the quotient rule, determine the derivative of $F(x) = \exp(ax)/x$.

Solution: The derivative of $F(x)$ using the quotient rule is

$$\frac{dF}{dx} = \frac{1}{x^2}\left[x \frac{d \exp{(ax)}}{dx} - \exp(ax) \frac{dx}{dx}\right] = \frac{a}{x}\exp(ax) - \frac{1}{x^2}\exp(ax) = \exp(ax)\frac{ax-1}{x^2} \; .$$

Interpretations of a Derivative

The derivative $\frac{dF}{dx}$ of the function $F(x)$ has two distinct interpretations.

First, the value of a derivative, $\frac{dF(x)}{dx}$, at $x = a$ is the slope of the line that is tangent to the function $F(x)$ at $x = a$. Now, since the slope of a function at a point where the function has its minimum or maximum value is zero, it follows that the derivative of the function at its extremum points (that is, points with minimum or maximum values) is also zero. So, the extremum (minimum or maximum) points of a function can be determined by setting its derivative equal to zero. The equation $\frac{dF(x)}{dx} = 0$ can be solved to obtain the values of $x = x_{min}$ where the function is minimum or $x = x_{max}$ where the function is maximum. In the vicinity of x_{min}, the values of $\frac{dF(x)}{dx}$ are negative for $x < x_{min}$, zero for $x = x_{min}$, and positive for $x > x_{min}$. In other words, $\frac{d^2F}{dx^2}$ is positive at $x = x_{min}$. Similarly, in the vicinity of x_{max}, the values of $\frac{dF(x)}{dx}$ are positive for $x < x_{max}$, zero for $x = x_{max}$, and negative for $x > x_{max}$. In other words, $\frac{d^2F}{dx^2}$ is negative at $x = x_{max}$.

Second, the derivative $\frac{dF(x)}{dx}$ represents the rate of variation of function $F(x)$ with x. Suppose that when independent variable x changes by a small amount Δx, then the corresponding change in the value of the function is ΔF. Using this interpretation of the derivative,

$$(Change\ in\ F) = (Rate\ of\ change\ of\ F\ with\ x) \cdot (Change\ in\ x)$$

or,

$$(\Delta F) = \frac{dF}{dx}\,(\Delta x)\ . \qquad\qquad Eq. (1.5)$$

The idea about the rate of change of a quantity appears in many diverse areas of knowledge. For example, in physiology we talk about the rate at which blood flows in veins; in geography we talk about the rate at which population grows in a certain area; in meteorology we talk about the rate at which pressure varies with altitude; in medicine we talk about the rate at which a cancerous tumor grows or a contagious virus spreads; in sociology we talk about the rate at which a news or some rumor spreads; in psychology we talk about the rate at which different people learn certain skills (that is, the learning curve); in physics we talk about the rate at which the velocity of an accelerating automobile changes, etc. No matter in which context we talk about the rate of change of a quantity, its mathematical description would always be presented in the form of a derivative.

Example: As an example of the first interpretation of a derivative, let us look at the case of a young mom, sitting on a bench on a concrete patio, watching her small child playing on the grass nearby, as shown in the Figure 1.2. The lengths d_p and d_g are the shortest distances of mom and child, respectively, from the patio-grass boundary. Mom can run on the patio with speed v_p and on the grass with speed v_g. When an emergency arises, the mom would like to reach the child in the shortest possible time. Which path should she take during an emergency?

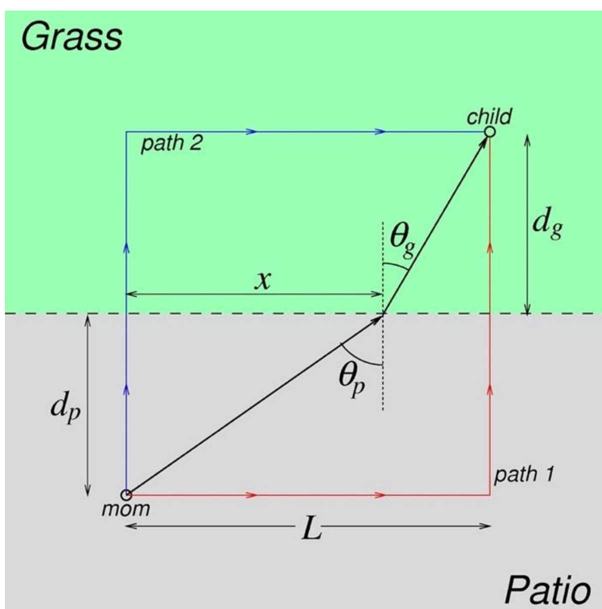

Figure 1.2. The quickest path a mom can take to reach her child in distress.

Solution: The mom can either take path 1 or path 2 (shown in the figure) or any path in between. Consider an arbitrary path, as shown in the figure, taken by mom to rush to her child. Total time taken by mom using this path is

$$t = \frac{\sqrt{d_p^2 + x^2}}{v_p} + \frac{\sqrt{d_g^2 + (L - x)^2}}{v_g} \ .$$

Here L is the separation between the mom and the child along the patio-grass boundary. The distance x, shown in the figure, will vary depending on the specific path taken by the rushing mom. The path with the shortest time, according to the above interpretation of the derivative, will be the one for which $\frac{dt}{dx}$ is equal to zero. Thus,

$$\frac{dt}{dx} = \frac{2x}{2v_p \sqrt{d_p^2 + x^2}} + \frac{2(L - x)(-1)}{2v_g \sqrt{d_g^2 + (L - x)^2}} = 0 \ ,$$

or

$$\frac{1}{v_p} \frac{x}{\sqrt{d_p^2 + x^2}} = \frac{1}{v_g} \frac{(L - x)}{\sqrt{d_g^2 + (L - x)^2}} \ ,$$

or, in terms of the angles θ_p and θ_g shown in the Figure 1.2,

$$\frac{\sin \theta_p}{v_p} = \frac{\sin \theta_g}{v_g} \ .$$

This relationship describes the path that mom should take to reach the child in the shortest possible time. It is, essentially, Snell's law of refraction in optics that concerns the bending of light as it travels from one medium (number 1) into another medium (number 2). Mathematically, Snell's law is stated as $n_1 \sin \theta_1 = n_2 \sin \theta_2$ where n_1 and n_2 are the indices of refraction of the two media. Light travels with different speeds in media with different indices of refraction with $v_1 = c/n_1$ and $v_2 = c/n_2$ (c being the speed of light in vacuum). Thus, Snell's law can be expressed as $\sin \theta_1 / v_1 = \sin \theta_2 / v_2$. Stated differently, Snell's law implies that when light travels from a point in the first medium to another point in the second medium, it takes a path that minimizes its time of travel.

1.2 FUNCTIONS OF MULTIPLE VARIABLES: PARTIAL DERIVATIVES

Recall that for an independent variable, any value can be assigned to it without any constraining conditions. If F is a function of two or more *independent* variables, then the derivative of F with respect to one of the variables, while holding the other variable fixed, is called the partial derivative. Formally, if $F(x, y)$ is a function of two independent variables x and y, then

$$\frac{\partial F(x,y)}{\partial x} = \lim_{h\to 0}\frac{F(x+h,y) - F(x,y)}{h} \qquad Eq.(1.6a)$$

$$\frac{\partial F(x,y)}{\partial y} = \lim_{h\to 0}\frac{F(x,y+h) - F(x,y)}{h} \qquad Eq.(1.6b)$$

are the partial derivatives of F. Note the use of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ for partial derivatives versus $\frac{d}{dx}$ for derivatives of a function of only one independent variable. A common notation for writing partial derivatives also includes $\left(\frac{\partial F}{\partial x}\right)_y$ and $\left(\frac{\partial F}{\partial y}\right)_x$ in which the fixed variable is written as a subscript.

Example: Determine the partial derivatives of the function $F(x,y) = x^3 - x\,y^2 + y$.

Solution: Since x and y are two independent variables, the partial derivatives of $F(x,y)$ with respect to x and y are:

$$\frac{\partial F(x,y)}{\partial x} = 3\,x^2 - y^2 \ ,$$

$$\frac{\partial F(x,y)}{\partial y} = -2xy + 1 \ .$$

As before, $\frac{\partial F}{\partial x}$ (or $\frac{\partial F}{\partial y}$) represents the rate of variation of F with x (or y). If both x and y vary independently, then

$$(Change\ in\ F) = (Rate\ of\ change\ of\ F\ with\ x)\cdot(Change\ in\ x) + (Rate\ of\ change\ of\ F\ with\ y)\cdot(Change\ in\ y)\ ,$$

or

$$(\Delta F) = \frac{\partial F}{\partial x}\,(\Delta x) + \frac{\partial F}{\partial y}\,(\Delta y)\ . \qquad Eq.(1.7)$$

Higher Order Partial Derivatives: Clairaut's Theorem

If $F_x = \frac{\partial F}{\partial x}$ and $F_y = \frac{\partial F}{\partial y}$, then $\frac{\partial F_x}{\partial x} = \frac{\partial^2 F}{\partial x^2}, \frac{\partial F_x}{\partial y} = \frac{\partial^2 F}{\partial y \partial x}, \frac{\partial F_y}{\partial x} = \frac{\partial^2 F}{\partial x \partial y}$ and $\frac{\partial F_y}{\partial y} = \frac{\partial^2 F}{\partial y^2}$ are the second derivatives of $F(x,y)$. Higher order partial derivatives can be defined in an analogous manner. The $Clairaut's\ theorem$ states that

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \quad \text{or} \quad \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)\ .$$

In words, it means that for a function of several independent variables, the multiple partial derivatives can be carried out in any order.

Example: Verify Clairaut's theorem for the function $F(x, y) = x^3 - x\, y^2 + \, y$.

Solution: In the previous example, we calculated $\frac{\partial F}{\partial x} = 3\, x^2 - \, y^2$ and $\frac{\partial F}{\partial y} = -2xy + 1$. So now

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \, -2y \ ,$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \, -2y \ .$$

so the Clairaut's theorem is indeed satisfied.

Example: Check whether the Clairaut's theorem for the function $F(x, y) = \exp(ax)\sin(by)$ is satisfied.

Solution: In this case, there are two independent variables x and y. So, first partial derivatives are

$$\frac{\partial F(x, y)}{\partial x} = a\, \exp(ax)\, \sin(by) \ ,$$

$$\frac{\partial F(x, y)}{\partial y} = \exp(ax)\, b\cos(by) \ .$$

The second derivative gives

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \ a\, \exp(ax)\, b\cos(by) \ ,$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \ a\, \exp(ax)\, b\cos(by) \ .$$

that verifies Clairaut's theorem.

If F is a function of n independent variables, that is, $F = F(x_1, x_2, x_3, \ldots x_n)$, then $\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \ldots \frac{\partial F}{\partial x_n}$ represent the rates of variation of F with $x_1, x_2, \ldots x_n$, respectively. Since all n variables can change independently, then

$$(Change\ in\ F) = (Rate\ of\ change\ of\ F\ with\ x_1) \cdot (Change\ in\ x_1) + \ (Rate\ of\ change\ of\ F\ with\ x_2) \cdot (Change\ in\ x_2) + \cdots + (Rate\ of\ change\ of\ F\ with\ x_n) \cdot (Change\ in\ x_n) \ ,$$

or,

$$(\Delta F) = \frac{\partial F}{\partial x_1}(\Delta x_1) + \frac{\partial F}{\partial x_2}(\Delta x_2) + \cdots + \frac{\partial F}{\partial x_n}(\Delta x_n) \; . \qquad Eq. (1.8)$$

(ΔF) is the total change in function F when all n variables are changed independently. It is called *total differential*. At a point where F is extremum (namely, a minimum or a maximum), $\Delta F = 0$. Since $x_1, x_2, \dots x_n$ are independent variables, they may be chosen so that all but one of the Δx, in turn, are zero. It follows that

$$\frac{\partial F}{\partial x_1} = 0, \quad \frac{\partial F}{\partial x_2} = 0, \quad \dots \frac{\partial F}{\partial x_{n-1}} = 0 \text{ and } \frac{\partial F}{\partial x_n} = 0 \; . \qquad Eq. (1.9)$$

These n algebraic equations can be solved to find the values of variables $x_1, x_2, \dots x_n$ that make the function F extremum.

The idea of a total differential, introduced above, is very useful in estimating the largest possible error in the measurement of a physical quantity that is a function of several variables. An example will help in understanding the concept.

Example: According to the Poiseuille's equation, the total volume of blood flowing through a blood vessel of radius R and length L per unit time is given by $F = k\frac{R^4}{L}$, where constant k is independent of the geometry of the blood vessel. If the relative error, ΔR, in the measurement of the radius is 2% and the relative error, ΔL, in the measurement of length is 4%, then what is the largest possible relative error, ΔF, in the measurement of flux of blood, F, through this blood vessel?

Solution: The relative error in the measurement of a quantity means error in measuring that quantity divided by the actual value of that quantity. In other words, the ratios $\frac{\Delta R}{R}, \frac{\Delta L}{L}$ and $\frac{\Delta F}{F}$ are, respectively, the relative errors in the measurements of the radius, length and flux of the blood through a blood vessel. These ratios are determined by first taking the natural log of the Poiseuille's equation and then taking the derivatives of both sides as

$$\ln F = \ln k + 4 \ln R - \ln L$$

and

$$\frac{\Delta F}{F} = 4\frac{\Delta R}{R} - \frac{\Delta L}{L} \; .$$

Since error in the measurement of radius is 2%, it means that the value of $\frac{\Delta R}{R}$ ranges between -0.02 and $+0.02$. Similarly, the value of $\frac{\Delta L}{L}$ ranges between -0.04 and $+0.04$. Thus, the largest possible value of $\frac{\Delta F}{F}$ is $4(+0.02) - (-0.04) = 0.12$. Or, the largest possible relative error in the measurement of flux of blood is 12%.

Now, going back to the function $F = F(x_1, x_2, x_3, \dots x_n)$ of n variables, if $x_1, x_2, \dots x_n$ are constrained by one or more relationships of the form,

$$\Phi(x_1, x_2, \dots x_n) = \text{ constant } ,$$

then not all n variables are independent. In fact, if F is a function of n variables and if there are m $(m < n)$ constraining relationships among variables, then the number of independent variables is only $n - m$. In this case one can use the *method of Lagrange multipliers* to find the extremum value of the function F.

Method of Lagrange Multipliers

Instead of considering a general function of n variables, let us focus our attention on a function $F(x, y, z)$ of three variables, x, y and z. These variables are constrained by the relation

$$\Phi(x, y, z) = \text{ constant } .$$

Because of this constraint only two out of three variables are independent. Let us choose x and y to be the independent variables. Since we wish to determine the extremum value of $F(x, y, z)$, we take its derivative and set it equal to zero:

$$\frac{\partial F}{\partial x}\,\Delta x + \frac{\partial F}{\partial y}\,\Delta y + \frac{\partial F}{\partial z}\,\Delta z = 0 \ . \qquad Eq.(1.10a)$$

From the constraint equation, $\Phi =$ constant, we get

$$\frac{\partial \Phi}{\partial x}\,\Delta x + \frac{\partial \Phi}{\partial y}\,\Delta y + \frac{\partial \Phi}{\partial z}\,\Delta z = 0 \ . \qquad Eq.(1.10b)$$

We multiply Eq. (1.10b) by a multiplier, $-\lambda$, and add it to Eq. (1.10a) to get

$$\left(\frac{\partial F}{\partial x} - \lambda\,\frac{\partial \Phi}{\partial x}\right)\Delta x + \left(\frac{\partial F}{\partial y} - \lambda\,\frac{\partial \Phi}{\partial y}\right)\Delta y + \left(\frac{\partial F}{\partial z} - \lambda\,\frac{\partial \Phi}{\partial z}\right)\,\Delta z = 0 \ .$$

Since the multiplier λ is yet undetermined and can be chosen at will, we choose it so that

$$\frac{\partial F}{\partial z} - \lambda\,\frac{\partial \Phi}{\partial z} = 0 \ .$$

This choice of λ will remove the term containing Δz, leaving only the two independent variables x and y. As x and y are independent variables, we can choose them, in turn, so that $\Delta x = 0$ and $\Delta y = 0$ separately. It then follows that

$$\frac{\partial F}{\partial x} - \lambda \frac{\partial \Phi}{\partial x} = 0 \ ,$$

and

$$\frac{\partial F}{\partial y} - \lambda \frac{\partial \Phi}{\partial y} = 0 \ .$$

The last three equations, along with the equation of constraint, provide the values of the multiplier λ and the optimal values of the variables x, y and z that make the function F extremum. The constant λ is called Lagrange's undetermined multiplier.

Example: A box (sides a, b, c) of fixed volume V is to be designed so that its surface area is minimum. Find the optimal values of a, b, c.

Solution:

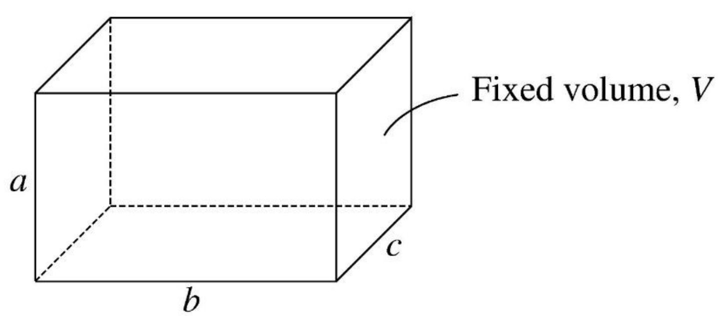

Figure 1.3. A box of fixed volume V whose surface area is minimized.

In this case, the equation of constraint is

$$V = a\,b\,c = \text{constant} \ .$$

Because of this constraint, only two sides of the box can be changed independently—the third side will be determined by the equation of constraint. We choose a and b as the independent variables.

We need to find the minimum value of the surface area of this box, which is

$$S = 2(ab + bc + ca) \ .$$

Setting the derivative of S equal to zero, we get

$$\Delta S = \frac{\partial S}{\partial a}\,\Delta a + \frac{\partial S}{\partial b}\,\Delta b + \frac{\partial S}{\partial c}\,\Delta c = 0 \ ,$$

or, after dividing by the common factor of 2,

$$(b + c)\,\Delta a + \ (c + a)\,\Delta b + (a + b)\,\Delta c = 0 \ . \qquad Eq. (1.11a)$$

Also, the volume V is constant. So,

$$\Delta V = \frac{\partial V}{\partial a}\,\Delta a + \frac{\partial V}{\partial b}\,\Delta b + \frac{\partial V}{\partial c}\,\Delta c = 0 \ ,$$

or

$$(bc)\,\Delta a + (ca)\,\Delta b + (ab)\,\Delta c = 0 \ . \qquad Eq. (1.11b)$$

Multiply Eq. (1.11b) by undetermined multiplier $-\lambda$ and add it to Eq. (1.11a) to get

$$[(b + c) - \lambda(bc)]\,\Delta a + [(c + a) - \lambda(ca)]\,\Delta b + [(a + b) - \lambda(ab)]\,\Delta c = 0 \ . \qquad Eq. (1.11c)$$

The multiplier λ is still undetermined and can be chosen at will. We choose $\lambda = (a + b)/(ab)$, or

$$(a + b) - \lambda(ab) = 0$$

so that term containing Δc is removed in Eq. (1.11c). Also, since a and b are chosen as independent variables, it follows from Eq. (1.11c) that

$$(b + c) - \lambda(bc) = 0 \ ,$$

and

$$(c + a) - \lambda(ca) = 0 \ .$$

From these three relationships, we get

$$\lambda(abc) = a(b + c) = b(c + a) = c(a + b) \ .$$

Or

$$a = b = c \ .$$

Thus, a box of a fixed volume V and a minimum surface area is a cube.

Example: Determine the ratio of radius, r, and height, h, of a right circular cylinder of fixed volume V, that will make the surface area, S, of the cylinder a minimum.

Solution:

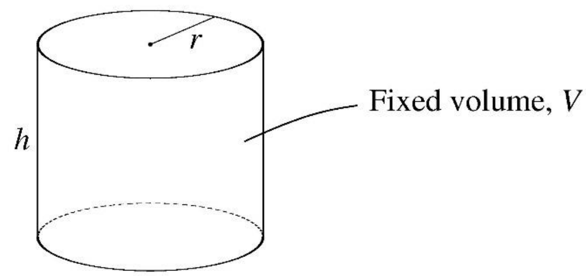

Figure 1.4. A cylinder of fixed volume V whose surface area is minimized.

In this case, $V = \pi r^2 h =$ constant is the equation of constraint. Both r and h can vary, but only one of them is independent. Let us choose r to be the independent variable. The surface area, S, of the cylinder consists of two end circles and the curved surface of the cylinder. Thus,

$$S = 2\pi rh + 2\pi r^2 \ .$$

From the constraint condition

$$\Delta V = \frac{\partial V}{\partial r}\,\Delta r + \frac{\partial V}{\partial h}\,\Delta h = 0 \ ,$$

or

$$2\pi rh\,\Delta r + \pi r^2\,\Delta h = 0 \ ,$$

or

$$2h\,\Delta r + r\,\Delta h = 0 \ . \qquad Eq. (1.12a)$$

Also, when the surface area, S, is a minimum, then

$$\Delta S = \frac{\partial S}{\partial r}\,\Delta r + \frac{\partial S}{\partial h}\,\Delta h = 0 \ ,$$

or

$$(2\pi h + 4\pi r)\,\Delta r + \,2\pi r\,\Delta h = 0\ ,$$

or

$$(h + 2r)\,\Delta r + \,r\,\Delta h = 0\ . \qquad\qquad Eq.(1.12b)$$

On multiplying Eq. (1.12b) by undetermined multiplier $-\lambda$ and adding it to Eq. (1.12a) we get

$$[2h - \lambda(h + 2r)]\,\Delta r + \,[r - \lambda r]\,\Delta h = 0\ . \qquad\qquad Eq.(1.12c)$$

The multiplier λ, which can be chosen at will, is taken as $\lambda = 1$. This choice of λ removes the term containing Δh in Eq. (1.12c). Also, since r is an independent variable, we can choose $\Delta r \neq 0$ which makes

$$2h - (h + 2r) = \ 0\ ,$$

or

$$h = 2r\ .$$

Thus, the right circular cylinder of a fixed volume V will have the least surface area when the height of the cylinder is equal to its diameter.

Let us try to construct some common solids of different shapes, using playdough of a fixed volume V, such that the solid has a minimum surface area, S.

If we construct a playdough cube of side length L, then $V = \ L^3$ and $S = 6\,L^2 = 6\,V^{2/3}$.

On the other hand, a playdough cylinder with a minimum surface area will have a height equal to its diameter. Thus, if R is the radius of this cylinder, then $V = \ \pi R^2(2R) = \ 2\pi R^3$. The surface area is $S = 2(\pi R^2) + (2\pi R)(2R) = \ 6\pi R^2 = 6\pi\left(\frac{V}{2\pi}\right)^{2/3} = \ 5.54\,V^{2/3}$.

Finally, a playdough sphere of radius R will have its volume as $V = \left(\frac{4\pi}{3}\right)R^3$ and its surface area as $S = 4\pi R^2 = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3} = 4.84\,V^{2/3}$.

Thus, among these common solids, all of the same volume V, the sphere will have the smallest surface area. This may partly explain why all the celestial bodies – stars, planets, and moons – are spherical in shape. It also explains why soap bubbles and raindrops tend to be spheres. In other words, Mother Nature prefers to make shapes with least surface area, for a fixed volume of its own playdough.

Chain Rule for Partial Derivatives

Recall that chain rule of calculus is applicable when we need to take derivative of a function of another function.

Case I: The function $F(x)$ is a function of a single variables x which itself is a function of two other variables, s and t, that is, $x(s,t)$. Then, in principle, F is a function of two variables, s and t. The derivatives of F with respect to s and t are

$$\frac{\partial F}{\partial s} = \frac{dF}{dx}\frac{\partial x}{\partial s} \ ,$$

and

$$\frac{\partial F}{\partial t} = \frac{dF}{dx}\frac{\partial x}{\partial t} \ .$$

Example: Given the function $F(x) = x^2$ with $x(s,t) = s + t$, determine the derivatives $\frac{\partial F}{\partial s}$ and $\frac{\partial F}{\partial t}$.

Solution:

$$\frac{\partial F}{\partial s} = (2x)(1) = 2(s+t) \ ,$$

and

$$\frac{\partial F}{\partial t} = (2x)(1) = 2(s+t) \ .$$

Alternatively,

$$F(s,t) = (s+t)^2$$

and, directly, the partial derivatives of $F(s,t)$ with respect to s or t provide the same results as those obtained by using the chain rule for partial derivatives.

Case II: $F(x,y)$ is a function of two variables x and y, and x and y themselves are functions of a single variable t. Then, in principle, F is a function of a single variable t and the derivative of F with respect to t is,

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} \ .$$

Example: Given the function $F(x,y) = \ln(2x+3y)$ with $x = t^2$ and $y = \sin(3t)$, determine the derivative $\frac{dF}{dt}$.

Solution: Using the chain rule for partial derivatives,

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} = \frac{2}{2x+3y}\,(2t) + \frac{3}{2x+3y}\,[3\cos(3t)] = \frac{4t + 9\cos(3t)}{2t^2 + 3\sin(3t)} \ .$$

Alternatively, we could write

$$F(x,y) = \ln[2t^2 + 3\sin(3t)]$$

and then use the rule, for taking derivative of a function of a single variable, to get the same result.

Case III: The function $F(x,y)$ is a function of two variables x and y, and the two variables x and y themselves are functions of two other variables, s and t; that is, $x(s,t)$ and $y(s,t)$. Then, in principle, F is a function of two variables, s and t. The derivatives of F with respect to s and t are

$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} \ ,$$

and

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} \ .$$

Example: Given the function $F(x,y) = x^2 + xy + y^2$ with $x(s,t) = s + t$ and $y(s,t) = st$, determine the derivatives $\frac{\partial F}{\partial s}$ and $\frac{\partial F}{\partial t}$.

Solution:

Using the chain rule for partial derivatives,

$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} = (2x+y)(1) + (x+2y)(t) = (2s + 2t + st) + (s + t + 2st)t \ ,$$

and

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} = (2x + y)(1) + (x + 2y)(s) = (2s + 2t + st) + (s + t + 2st)s \ .$$

Alternatively, one could express $F(x, y)$ as $F(s, t)$ in the form

$$F(s, t) = (s + t)^2 + (s + t)st + s^2t^2 \ .$$

Taking the partial derivatives of $F(s, t)$ with respect to s or t will lead to the same results as those obtained by using the chain rule for partial derivatives.

1.3 WAVE EQUATION

As an application of partial derivatives, we will derive an equation, the wave equation, which describes a periodic function. A function is periodic when it repeats itself either in time or in space. Periodic phenomena are very common in a variety of fields including biology, physics, chemistry, astronomy, and so on. In biology, periodic phenomena include cell cycle, cardiac cycle, circadian rhythms, ovarian cycle, and metabolic cycle. In physics, examples of periodic phenomena include a swinging pendulum, mass on a spring, and water waves (or ripples). In chemistry, periodicity is found in the properties of chemical elements and in oscillating chemical reactions. In astronomy, there are abundant examples of periodic phenomena such as motion of satellites and the Moon around the Earth, motion of planets around the Sun, and the associated periodic occurrences of seasons, tides, and the day-and-night cycle.

One of the simplest mathematical functions that repeats itself is the sine (or cosine) function. For example, $\sin(kx)$ is a periodic function that repeats itself in spatial coordinate x. In fact, the value of this function at some point x_0 is same as its value at $x_0 + \frac{2\pi}{k}$. In other words, the wavelength [or, the length over which the function repeats itself] of this function is $\lambda = \frac{2\pi}{k}$. Similarly, $\sin(\omega t)$ is a periodic function that repeats itself in time t. The value of this function at some time t_0 is the same as its value at $t_0 + \frac{2\pi}{\omega}$. In other words, the period [or, the time over which the function repeats itself] of this function is $T = \frac{2\pi}{\omega}$. Note that $\frac{1}{T}$ measures the number of times the function repeats itself in a unit time. Therefore, $\frac{1}{T}$, that measures how frequently the function repeats itself, is called the frequency, f, of the wave. Thus, $f = \frac{1}{T} = \frac{\omega}{2\pi}$.

Now, a wave, according to its dictionary meaning, refers to “a disturbance on the surface of a liquid body, as the sea or a lake.” We can set up a water wave (commonly called a ripple) by throwing a large stone in a lake. This wave will look like a series of surges which are progressively moving outwardly away from the point where stone touched the water. If we throw a bottle cork in the disturbed water, we will observe that the cork will be simply bobbing up and down at its fixed location, without moving horizontally with the wave, as surges pass by it. It indicates that, in case of ripples in water, the wave is moving outward while the water itself is not moving with the wave. We can investigate the behavior of this wave either as a function of spatial coordinate x or as a function of time t. Using a camera, if we take the photograph of this wave, it will look like Figure 1.5 which shows, at a fixed time [the time at which photograph was taken], the wave as a function of x. On the other hand, we could focus our attention at some fixed point, say the bobbing cork with spatial coordinate x, and measure its displacement with respect to the horizontal level of calm water as a function of time t. Figure 1.6 shows this displacement, for a fixed value of x [location of cork], as a function of time t. From Figures 1.5 and 1.6 we note that this wave is periodic in both x and t. If we represent this wave mathematically by a sine function, we get a sinusoidal or a harmonic wave.

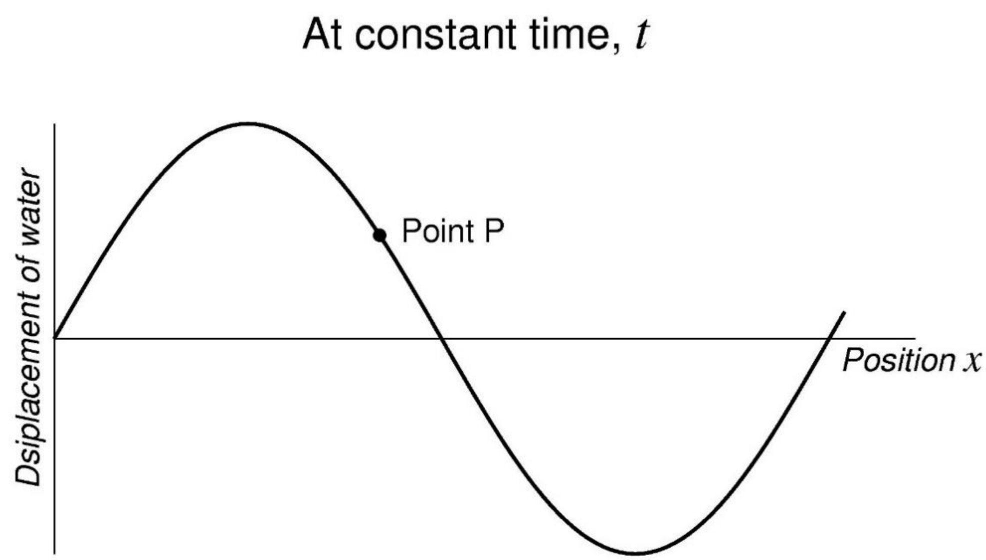

Figure 1.5. Periodic function $sin(kx - \omega t)$ as a function of x for a constant value of t.

We note, in passing, that because of the identity

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos(\theta) \ ,$$

for a given angle θ, the sine function and the cosine function look the same except the sine function is ahead of cosine function by a phase of $\pi/2$. In general, if a function looks like a sine or a cosine function, we will refer to it as a **sinusoidal function**.

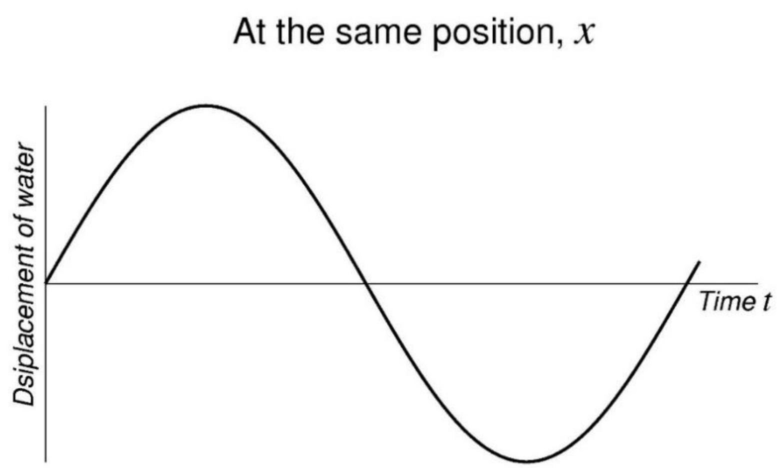

Figure 1.6. Periodic function $sin(kx - \omega t)$ as a function of t for a constant value of x.

The mathematical function

$$F(x, t) = A \sin(kx - \omega t) \qquad Eq. \ (1.13a)$$

represents the sinusoidal wave of figures 1.5 and 1.6. The function $F(x, t)$ represents the displacement, at time t, of a point in water located at position x. The largest value of the displacement is A, which is called the amplitude of the wave. The wavenumber k is related to the wavelength λ as $k = \frac{2\pi}{\lambda}$. The angular frequency ω is related to the period T as $\omega = \frac{2\pi}{T}$. The argument of the sine function, namely $(kx - \omega t)$, is called the phase of the wave. Each point on the wave, such as point P in Figure 1.5, has a fixed constant value of phase which does not change as the point P moves along with the wave. From Eq. (1.13a), as t increases, x also must increase to keep the phase constant. Thus, this wave travels along the $+x$ direction. Similarly, a wave of the form

$$F(x, t) = A \sin(kx + \omega t) \qquad Eq. (1.13b)$$

travels along the $-x$ direction.

The phase velocity refers to the speed of an arbitrary point, like P of some fixed phase, on the wave. For point P,

$$\text{phase} = kx - \omega t = \text{constant} \ .$$

Thus,

$$k\frac{dx}{dt} - \omega = 0$$

or

$$\frac{dx}{dt} = \frac{\omega}{k} = v \ . \qquad Eq. (1.14)$$

Here v is called the phase velocity of the wave. Thus,

$$F_+(x,t) = A\sin(kx - \omega t)$$

and

$$F_-(x,t) = A\sin(kx + \omega t)$$

are the sinusoidal (or harmonic) waves travelling along $+x$ and $-x$ direction, respectively. These waves are moving with a (phase) velocity of

$$v = \frac{\omega}{k} = \left(\frac{\lambda}{2\pi}\right)\left(\frac{2\pi}{T}\right) = \frac{\lambda}{T} \ .$$

Using partial derivatives of

$$F(x,t) = A\sin(kx - \omega t)$$

we get

$$\frac{\partial F}{\partial x} = kA\cos(kx - \omega t)$$

$$\frac{\partial^2 F}{\partial x^2} = -k^2 A\sin(kx - \omega t) = -k^2 F$$

$$\frac{\partial F}{\partial t} = -\,\omega A\cos(kx - \omega t)$$

$$\frac{\partial^2 F}{\partial t^2} = -\omega^2 A\sin(kx - \omega t) = -\omega^2 F$$

Combining these equations, we get

$$-k^2\omega^2 F = \omega^2 \frac{\partial^2 F}{\partial x^2} = k^2 \frac{\partial^2 F}{\partial t^2} \ .$$

Finally, using $\omega = kv$, where v is the phase velocity, we have

$$\frac{\partial^2 F}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2} = 0 \ . \qquad Eq. (1.15)$$

This is known as the $wave\ equation$. Equations of this kind, which relate various partial derivatives, are called *partial differential equations.*

In the above discussion, we derived the wave equation by starting from sinusoidal (or, sine- and cosine-like) functions. However, the general solutions of the wave equation are not necessarily sinusoidal. We will now show that general solutions of the wave equation are not merely functions of variables x and t, but are functions of the combinations $x + vt$ and $x - vt$. To show this, we make a change of variables from x and t to $r = x + vt$ and $s = x - vt$. Using chain rule for partial derivatives,

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial F}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial F}{\partial r} + \frac{\partial F}{\partial s} = \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right) F \ ,$$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial F}{\partial s}\frac{\partial s}{\partial t} = \frac{\partial F}{\partial r} v + \frac{\partial F}{\partial s}(-v) = v\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right) F \ .$$

Or, in operator notation,

$$\frac{\partial}{\partial x} \equiv \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right) \ ,$$

$$\frac{\partial}{\partial t} \equiv v\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right) \ .$$

Then,

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right) = \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right)\left(\frac{\partial F}{\partial r} + \frac{\partial F}{\partial s}\right) = \frac{\partial^2 F}{\partial r^2} + 2\frac{\partial^2 F}{\partial r \partial s} + \frac{\partial^2 F}{\partial s^2} \ ,$$

$$\frac{\partial^2 F}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial t}\right) = v\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial s}\right) v\left(\frac{\partial F}{\partial r} - \frac{\partial F}{\partial s}\right) = v^2\left(\frac{\partial^2 F}{\partial r^2} - 2\frac{\partial^2 F}{\partial r \partial s} + \frac{\partial^2 F}{\partial s^2}\right) \ .$$

Thus, the wave equation becomes,

$$\frac{\partial^2 F}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2} = 4\ \frac{\partial^2 F}{\partial r \partial s} = 0 \ .$$

It implies that

$$\frac{\partial}{\partial r}\left(\frac{\partial F}{\partial s}\right) = 0 \ ,$$

as well as

$$\frac{\partial}{\partial s}\left(\frac{\partial F}{\partial r}\right) = 0 \ .$$

In words, it implies that $\frac{\partial F}{\partial s}$ is independent of r and $\frac{\partial F}{\partial r}$ is independent of s. Thus, the function F can be expressed as a sum of two separate functions, g and h, such that g is a function of variable r only and h is a function of variable s only. So, a general solution of the wave equation is of the form,

$$F = g(r) + h(s) \ ,$$

or

$$F(x,t) = g(x + vt) + h(x - vt) \ .$$

We note in passing that the two sinusoidal functions that we encountered previously are indeed functions of variables $r = x + vt$ and $s = x - vt$. Explicitly, using $\omega = kv$,

$$F_+(x,t) = A\sin(kx - \omega t) = A\sin k(x - vt) \ , \qquad Eq. (1.16a)$$

and

$$F_-(x,t) = A\sin(kx + \omega t) = A\sin k(x + vt) \ . \qquad Eq. (1.16b)$$

1.4 IMPLICIT DERIVATIVES

Consider the relationship $y^2 + \sin x \sin y = x$ between two variables x and y. Both x and y can vary, though not independently since a change in x leads to a change in y and vice versa. Thus, either y can be treated as a function of a single variable x, or x can be treated as a function of the variable y. We can determine $\frac{dy}{dx}$, the rate of change of y with x, or $\frac{dx}{dy}$, the rate of change of x with y. We note that it is not easy to write either x as a function of y or y as a function of x. Starting with the relationship between x and y, we first differentiate it with respect to x to get

$$2y\frac{dy}{dx} + \cos x \sin y + \sin x \cos y \ \frac{dy}{dx} = 1 \ ,$$

or

$$(2y + \sin x \cos y) \ \frac{dy}{dx} = 1 - \cos x \sin y \ ,$$

or

$$\frac{dy}{dx} = \frac{1 - \cos x \sin y}{2y + \sin x \cos y} \quad .$$

Again, starting with the relationship between x and y, we next differentiate it with respect to y to get

$$2y + \cos x \sin y \frac{dx}{dy} + \sin x \cos y \ = \frac{dx}{dy} \ ,$$

or

$$(1 - \cos x \sin y) \ \frac{dx}{dy} = 2y + \sin x \cos y \ ,$$

or

$$\frac{dx}{dy} = \frac{2y + \sin x \cos y}{1 - \cos x \sin y} \quad .$$

These kinds of derivatives, which contain both variables x and y, are called implicit derivatives. Note in passing, in this example,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \ , \qquad Eq. (1.17)$$

which is a general property of derivatives of a function of a single variable.

As another **example**, consider the relationship $x + y = \exp(xy)$ between variables x and y. We first differentiate this relationship with respect to x to get

$$1 + \frac{dy}{dx} = \exp(xy) \left\{ x \frac{dy}{dx} + y \right\} \ ,$$

or

$$\{x \exp(xy) - 1\} \ \frac{dy}{dx} = 1 - y \exp(xy) \ ,$$

or

$$\frac{dy}{dx} = \frac{1 - y \exp(xy)}{x \exp(xy) - 1} \ .$$

Next, differentiate the original relationship with respect to y to get

$$\frac{dx}{dy}+1=\exp(xy)\left\{x+\frac{dx}{dy}y\right\} ,$$

or

$$\{1-y\exp(xy)\}\frac{dx}{dy}=x\exp(xy)-1 ,$$

or

$$\frac{dx}{dy}=\frac{x\exp(xy)-1}{1-y\exp(xy)} .$$

Again, the reciprocity relationship of Eq. (1.17), namely,

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}} ,$$

is satisfied since we are dealing with a function of a single variable here. Now, let us explore whether this reciprocity relationship also works for partial derivatives. As an example, consider the coordinates of a point in a plane. The Cartesian (x, y) and the plane polar (ρ,ϕ) coordinates of the point are related to each other. The x and y coordinates can be expressed as functions of variables ρ and ϕ,

$$x=\rho\cos\phi,\ y=\rho\sin\phi .$$

Conversely, the ρ and ϕ coordinates can be considered as functions of variables x and y,

$$\rho=\sqrt{x^2+y^2},\phi=\tan^{-1}\left(\frac{y}{x}\right) .$$

From here,

$$\frac{\partial\phi}{\partial x}=\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}=-\frac{y}{x^2+y^2} ,$$

and

$$\frac{\partial x}{\partial\phi}=-\rho\sin\phi=-y .$$

Clearly, $\frac{\partial \phi}{\partial x} \neq \frac{1}{\frac{\partial x}{\partial \phi}}$. The reason is that the variables being held fixed are different for each of the two cases. In the above example $\frac{\partial \phi}{\partial x}$ is actually $\left(\frac{\partial \phi}{\partial x}\right)_y$, that is, variable y is held fixed while evaluating this derivative. Similarly, $\frac{\partial x}{\partial \phi}$ is $\left(\frac{\partial x}{\partial \phi}\right)_\rho$ since variable ρ is fixed during evaluation of this derivative.

Legendre Transformation

Suppose $F(x, y)$ is a function of two independent variables x and y. It is possible to define two new variables u and v, which are combinations of x and y [for example, $u = x + y$ and $v = x - y$], and, inversely, x and y are functions of variables u and v. Now if x and y in F are replaced by u and v, then F will become a function of two new variables u and v. Thus,

$$F(x, y) \rightarrow F[x(u, v), y(u, v)] \equiv G(u, v) \ .$$

The procedure for replacing one set of independent variables in a function by another set of independent variables is accomplished, in general, by *Legendre transformation*. The change of independent variables is helpful in classical mechanics in discussion of canonically conjugate variables, and in thermodynamics in the discussion of Maxwell relations. The transformation procedure is described below.

For $F(x, y)$, we can write

$$dF = \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy = p\, dx + q\, dy \ ,$$

where $p(x, y) = \frac{\partial F}{\partial x}$ and $q(x, y) = \frac{\partial F}{\partial y}$. Using Clairaut theorem

$$\left(\frac{\partial p}{\partial y}\right)_x = \left(\frac{\partial q}{\partial x}\right)_y$$

Now, define three new functions,

$$f = F(x, y) - p(x, y)\, x \ ,$$

$$g = F(x, y) - q(x, y)\, y \ ,$$

and $$h = F(x, y) - p(x, y)x - q(x, y)y \ .$$

Then,

$$df = dF - p\, dx - x\, dp = -x\, dp + q\, dy \ ,$$

$$dg = dF - q\,dy - y\,dq = p\,dx - y\,dq \ ,$$

$$dh = dF - p\,dx - x\,dp - q\,dy - y\,dq = -x\,dp - y\,dq \ .$$

Thus, f is a function of variables p and y, g is a function of variables x and q, and h is a function of variables p and q. The partial derivatives of $f(p,y)$, $g(x,q)$ and $h(p,q)$ are

$$\frac{\partial f(p,y)}{\partial p} = -x \quad , \qquad \frac{\partial f(p,y)}{\partial y} = +q \ ,$$

$$\frac{\partial g(x,q)}{\partial x} = +p \quad , \qquad \frac{\partial g(x,q)}{\partial q} = -y \ ,$$

$$\frac{\partial h(p,q)}{\partial p} = -x \quad , \qquad \frac{\partial h(p,q)}{\partial q} = -y \ .$$

Finally, using the Clairaut theorem,

$$-\left(\frac{\partial x}{\partial y}\right)_p = +\left(\frac{\partial q}{\partial p}\right)_y \ , \qquad Eq.(1.18a)$$

$$+\left(\frac{\partial p}{\partial q}\right)_x = -\left(\frac{\partial y}{\partial x}\right)_q \ , \qquad Eq.(1.18b)$$

$$-\left(\frac{\partial x}{\partial q}\right)_p = -\left(\frac{\partial y}{\partial p}\right)_q \ . \qquad Eq.(1.18c)$$

These three relations along with the original relation

$$+\left(\frac{\partial p}{\partial y}\right)_x = +\left(\frac{\partial q}{\partial x}\right)_y \ , \qquad Eq.(1.18d)$$

are the basis of Maxwell's relations in thermodynamics. Appendix B describes an easy mnemonic device to remember Maxwell's relations with correct signs of terms.

PROBLEMS FOR CHAPTER 1

1. Given $F = \exp(x)\sin y$, $x = uv^2, y = u^2v$, use the chain rule to find

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} \ .$$

Write the result in terms of variables x and y only.

2. Consider a function of a single variable x, $F(x) = x^3 - 6x^2 + 9x + 4$.

(a) Determine the values of x at which the function $F(x)$ is an extremum.

(b) At the extremum points, does the function $F(x)$ have a minimum or a maximum value?

3. **Biomedical Physics Application.** A common inhabitant of human intestines is the bacterium Escherichia coli. A single cell of this bacterium in a nutrient-broth medium divides into two cells every twenty minutes. The initial population of a culture is 250 cells.

(a) Find an expression for the number of cells after t hours.

(b) Find the number of cells after 5 hours.

(c) When will the population reach 128,000 cells?

4. **Biomedical Physics Application.** At 8:00 AM in the morning, a biologist starts a six-hour experiment with bacterium Escherichia coli. As mentioned in problem 3, a cell of this bacterium in a nutrient-broth medium divides into two cells every twenty minutes. At 12:00 PM the biologist measures the cell population to be 2,048,000.

(a) What was the initial population of the bacterium cells in the beginning of the experiment at 8:00 AM?

(b) What will be the final population of the bacterium cells at the end of the experiment at 2:00 PM?

5. If $V(r,\theta) = (\alpha r^n + \beta r^{-n})\cos(n\theta)$ where α, β and n are constants, find

$$\frac{\partial V}{\partial r}, \frac{\partial^2 V}{\partial r^2} \text{ and } \frac{\partial^2 V}{\partial \theta^2} \ .$$

Hence show that

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0 \ .$$

6. The pressure P, volume V. and temperature T of a certain gas are related by the van der Waals' equation of state,

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT \ ,$$

where a, b and R are all constants. Find the values of P, V and T for which $\left(\frac{\partial P}{\partial V}\right)_T = 0$ and $\left(\frac{\partial^2 P}{\partial V^2}\right)_T = 0$ are satisfied simultaneously. These particular values of pressure, volume and temperature are called critical values (P_c, V_c and T_c). One can define reduced parameters as

$$P_r = \frac{P}{P_c}, V_r = \frac{V}{V_c} \text{ and } T_r = \frac{T}{T_c} \ .$$

Show that the van der Waals' equation can be recast, in terms of the reduced parameters, in the following invariant form:

$$\left(P_r + \frac{3}{V_r^2}\right)\left(V_r - \frac{1}{3}\right) = \frac{8T_r}{3} \ .$$

7. **Biomedical Physics Application.** In all mammalians, including human beings, the rate of growth of skull is known to be different from the rate of growth of backbone. The allometric relationship between skull size S(t) and backbone length B(t), at age t, is

$$S(t) = a\, B(t)^b \ ,$$

where $a = 1.16$ and $b = 0.93$.

(a) Determine the relationship between the relative growth rates, $\frac{1}{S}\frac{dS}{dt}$ and $\frac{1}{B}\frac{dB}{dt}$, of skull and backbone, respectively.

(b) Which part of the body, skull or backbone, grows faster than the other?

8. If $f(x, y, z) = 1/(x^2 + y^2 + z^2)$, calculate $\boldsymbol{\nabla}^2 f$, where $\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplacian operator.

9. For $f(x, y) = x^3 - y^3 - 2xy + 6$, find the values of $\partial^2 f/\partial x^2$ and $\partial^2 f/\partial y^2$ at the points where $\frac{\partial f}{\partial x} = 0$ as well as $\frac{\partial f}{\partial y} = 0$.

10. Show explicitly that arbitrary functions $F(x - vt)$ and $F(x + vt)$ are solutions of the wave equation

$$\frac{\partial^2 F}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2} = 0 \ .$$

11. **Biomedical Physics Application.** All human beings belong to one of the four main blood groups (types of blood) – A, B, AB and O. Blood group O is the most common type of blood in the world. Mixing blood groups can lead to a life-threatening situation. The blood type of a person is determined by three alleles (a variant form of a gene), A, B and O that are inherited from parents, one from father and other from mother. The inherited genes join to form the blood groups A (joining A with A or A with O), B (joining B with B or B with O), AB (joining A with B) and O (joining O with O). According to the Hardy-Weinberg Law of genetics, the fraction of population that carries two different alleles is

$$P = 2xy + 2yz + 2zx$$

where x, y and z are the fractions of alleles A, B and O in the population. Using the fact that $x + y + z = 1$, show using the Method of Lagrange Multipliers that P can be at most 2/3.

12. Assume that a and b are the side lengths of a right-angle triangle. The size of the hypotenuse, h, of this triangle is fixed at $5\sqrt{2}$ m. Using the Method of Lagrange Multipliers, determine the lengths of the sides a and b when the area of the triangle is extremum.

13. **Biomedical Physics Application.** The drug response function $R(t)$, describing the level of medication in the bloodstream after a drug is administered, can be represented as

$$R(t) = R_0 \, t^{-3/2} \exp(-a/t)$$

where R_0 and a depend on the nature of the drug. For a particular drug $R_0 = 0.01$ and $a = 0.138$, and t is measured in minutes. Determine the time, t, at which the level of the medication in the bloodstream is maximum.

14. **Biomedical Physics Application.** In the angioplasty procedure, a "balloon" is inflated inside a partially clogged artery to restore the normal blood flow. The volume of blood flowing per unit time past a given point, F, is proportional to the fourth power of the radius, R, of the artery carrying the blood (Poiseuille's equation),

$$F = k\, R^4 \ .$$

What will be the relative change in F (that is, $\frac{dF}{F}$) when an artery is constricted by a 2% change in its radius due to clogging?

Chapter 2: Integral Calculus

In this chapter we will first review integral calculus, assuming that the reader has already seen integrals of functions of a single variable. After a short review, we will introduce the multiple integrals.

2.1 INDEFINITE INTEGRALS

An indefinite integral is defined as an antiderivative in the following sense. If a known function, $F(x)$, is represented as the derivative of an unknown function, $f(x)$, that is, $F(x) = \frac{df}{dx}$, then

$$\int F(x)\, dx = f(x) \ .$$

The function $f(x)$ is the antiderivative, or integral, of $F(x)$. In this case x is the $variable\ of\ integration$ and the known function $F(x)$ is the $integrand$. Since $\frac{dC}{dx} = 0$ if C is a constant, it is customary to write

$$\int F(x)\, dx = f(x) + C \ , \qquad Eq. (2.1)$$

where C is called the constant of integration. Some well-known indefinite integrals are

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C \ ,$$

$$\int \exp(x)\ dx = \exp(x) + C \ ,$$

$$\int \frac{1}{x}\, dx = \ln x + C \ ,$$

$$\int \sin x\ dx = -\cos x + C \ ,$$

$$\int \cos x\ dx = \sin x + C \ ,$$

$$\int \sec^2 x\ dx = \tan x + C \ ,$$

$$\int \frac{1}{1+x^2}\, dx = \arctan x + C \ ,$$

$$\int \frac{1}{\sqrt{1-\ x^2}}\, dx = \arcsin x +\ C\ .$$

The product rule of differential calculus [see Eq. (1.3)] corresponds to $integration\ by\ parts$ in integral calculus. In differential calculus, if $F(x)\ =\ f(x)\, g(x)$, then

$$\frac{dF}{dx} =\ f(x)\frac{dg(x)}{dx} +\ \frac{df(x)}{dx}g(x) =\ \frac{d[f(x)g(x)]}{dx}\ .$$

Using the definition of an indefinite integral as an antiderivative, it can be written as

$$\int f(x)\frac{dg(x)}{dx}\ dx + \int \frac{df(x)}{dx}g(x)\, dx =\ f(x)g(x)\ .$$

On rewriting this equation as

$$\int f(x)\frac{dg(x)}{dx}\ dx =\ f(x)g(x) - \int \frac{df(x)}{dx}g(x)\, dx\ \ , \qquad Eq.(2.2a)$$

we get the rule for *integration by parts,* which can be written in verbose form as

$$\int [FIRST][SECOND]\ dx$$

$$= [FIRST][Integral\ of\ SECOND] - \int [Derivative\ of\ FIRST][Integral\ of\ SECOND]\ dx\ \ .$$

Whenever the integrand can be expressed as a product of two functions, $FIRST$ and $SECOND$, it is convenient to identify as $FIRST$ the function whose derivatives are comparatively simpler than the function itself. Similarly, it is convenient to identify as $SECOND$ that function whose integrals are simpler than the function itself.

An alternate way of writing the $integration\ by\ parts$ procedure is

$$\int f(x)\, dg(x) =\ f(x)g(x) - \int g(x)\, df(x)\ \ . \qquad Eq.(2.2b)$$

Example: Evaluate $I =\ \int x^2 \exp x\ \ dx$ using integration by parts.

Solution: In this case, the integrand is a product of two functions, x^2 and $\exp x$. Since x^2 becomes simpler on differentiation while $\exp x$ stays the same on differentiation or integration, we choose $FIRST$ as x^2 and $SECOND$ as $\exp x$. Then,

$$I =\ x^2 \exp x - \int (2x)\ \exp x\ \ dx\ \ .$$

Now, I is converted into a simpler integral, which can be evaluated using integration by parts one more time,

$$I = x^2 \exp x - 2\left[x \exp x - \int \exp x \; dx\right] = x^2 \exp x - 2x \exp x + 2 \exp x \; .$$

Finally, after including the constant of integration, the integral is evaluated as

$$I = (x^2 - 2x + 2) \exp x + C \; .$$

Example: Evaluate $I = \int x^2 \ln x \; dx$ using integration by parts.

Solution: In this case, $\ln x$ becomes simpler, compared to x^2, on taking derivatives. So, we choose $FIRST$ as $\ln x$ and $SECOND$ as x^2. Then,

$$I = \ln x \, \frac{x^3}{3} - \int \frac{1}{x}\frac{x^3}{3} \, dx = \ln x \, \frac{x^3}{3} - \frac{1}{3}\int x^2 \, dx = \ln x \, \frac{x^3}{3} - \frac{x^3}{9} \; .$$

After including the constant of integration, the integral is evaluated as

$$I = \frac{x^3}{9} \, [3 \ln x \; - 1] + C \; .$$

2.2 DEFINITE INTEGRALS

In a definite integral the range of the values of x, the variable of integration, is provided. If the range is $a \le x \le b$, then

$$\int_a^b F(x) \, dx = f(x)|_{x=a}^{x=b} = f(b) - f(a) \; , \qquad Eq. (2.3)$$

where $F(x) = \frac{df}{dx}$. Thus, we first determine the function $f(x)$ which is antiderivative is $F(x)$. Then, the value of the definite integral is the difference between the values of $f(x)$ at the upper limit and at the lower limit.

There is an alternate way of interpreting a definite integral. If $F(x)$ is a continuous function in the whole range of x, $a \le x \le b$, as shown in Figure 2.1, then we can divide the range into n equal intervals, each of width $\Delta x = (b - a)/n$. Also, if x_i, for $i = 1,2, \dots n$ is the location of the midpoint of the ith interval, then the definite integral is defined as a limit of a sum as follows:

$$\int_a^b F(x) \, dx \to \lim_{n \to \infty} \sum_{i=1}^{n} F(x_i) \Delta x \; .$$

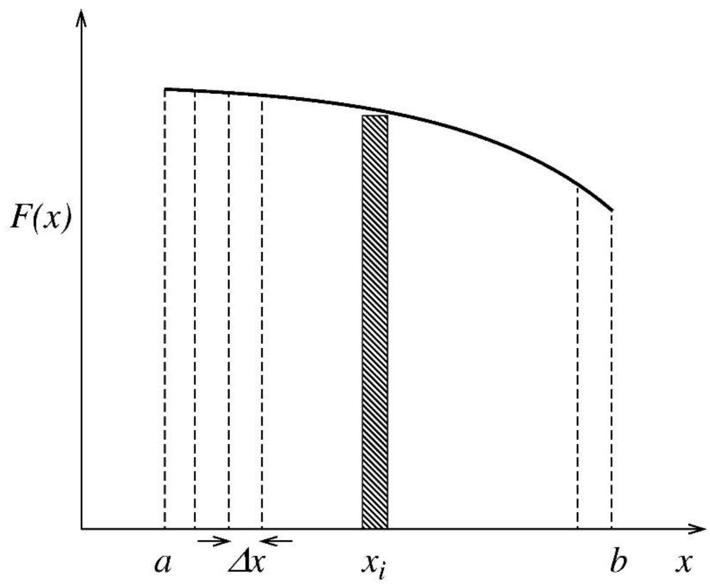

Figure 2.1. Area under the $\boldsymbol{F(x)}$ versus $\boldsymbol{x}$ curve is broken into strips.

In Figure 2.1 the area under the $F(x)$ versus x curve is broken into n strips, each of width Δx. Note that $F(x_i)\Delta x$ is the area of the cross-hatched ith strip shown in the figure. If the number of strips $n \to \infty$, then the width of each individual strip becomes vanishingly small and the sum of areas of all strips becomes equal to the area under the $F(x)$ versus x curve. The definite integral $\int_a^b F(x)\,dx$ thus represents the area under the $F(x)$ curve from $x = a$ to $x = b$.

Note that $\int_a^b F(x)\,dx$ is a number and its value does not depend on x. In fact,

$$\int_a^b F(x)\,dx = \int_a^b F(y)\,dy = \int_a^b F(z)dz\ .$$

Thus, in a definite integral, the symbol for variable of integration, x or y or z, is only a placeholder and it disappears after the integral is evaluated. So, in a definite integral the variable of integration is called the $dummy\ variable$. From the definition of a definite integral, it follows that

$$\int_a^b F(x)\,dx = f(b) - f(a) = -\,[f(a) - f(b)] = -\int_b^a F(x)\,dx$$

and

$$\int_a^a F(x)\,dx = 0\ .$$

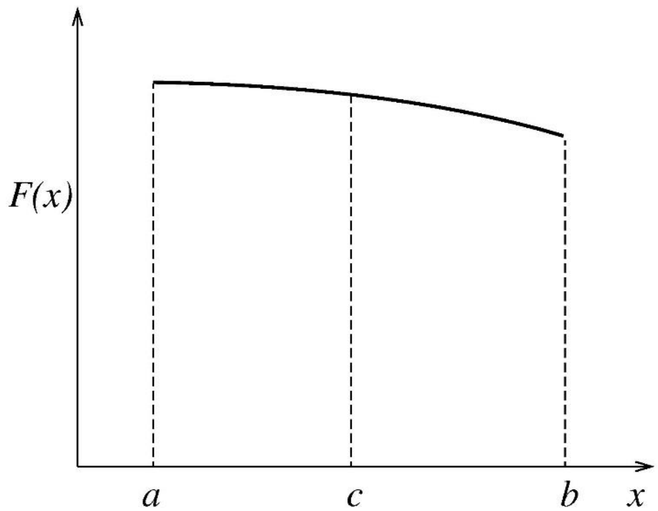

Figure 2.2. The point $\boldsymbol{x = c}$ lies inside the range, $\boldsymbol{a \le x \le b}$, of the integrand.

If the point $x = c$ lies somewhere in the middle of the range of x, then

$$\int_a^b F(x)\,dx = \text{area under the curve from } a \text{ to } b$$

$$= \text{area under the curve from } a \text{ to } c + \text{area under the curve from } c \text{ to } b$$

$$= \int_a^c F(x)\,dx + \int_c^b F(x)\,dx \;.$$

Also, if $F(x) \ge G(x)$ for $a \le x \le b$, then

$$\int_a^b F(x)\,dx \;\ge\; \int_a^b G(x)\,dx \;.$$

Similarly, if $F(x) \le G(x)$ for $a \le x \le b$, then

$$\int_a^b F(x)\,dx \;\le\; \int_a^b G(x)\,dx \;.$$

If $F(x)$ is positive for part of a range and negative for the remaining range of x, as shown in Figure 2.3, then

$$\int_a^b F(x)\,dx = \int_a^c F(x)\,dx + \int_c^b F(x)\,dx \equiv I_1 + I_2 \;,$$

where $I_1 > 0$ and $I_2 < 0$. Note that even in this case the net area under the curve of a function $F(x)$ between $x = a$ and $x = b$ is $\int_a^b F(x)\, dx$. However, the magnitude of the area under the curve of $F(x)$ is $\int_a^b |F(x)| dx$.

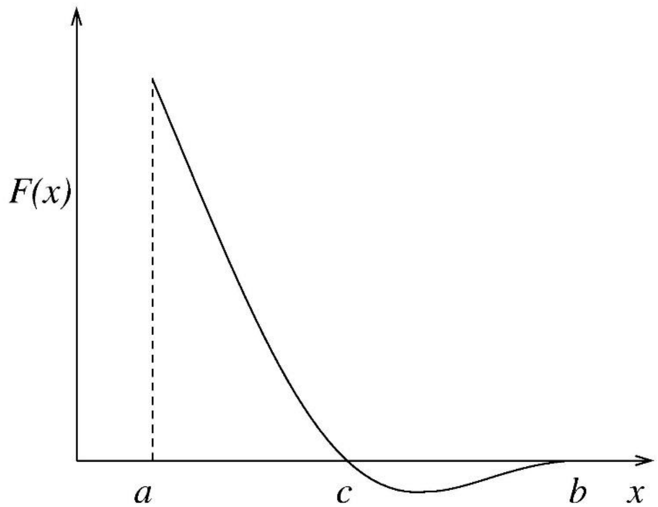

Figure 2.3. The function $\boldsymbol{F(x)}$ is positive as well as negative for parts of the range of $\boldsymbol{x}$.

Example: Find the net area and the magnitude of the area enclosed by function $\boldsymbol{F(x) = x^2 - 5x + 6 = (x-3)(x-2)}$ between $\boldsymbol{x = 0}$ and $\boldsymbol{x = 3}$ and the $\boldsymbol{x}$-axis.

Solution: In this case $F(x) \geq 0$ for $0 \leq x \leq 2$, and $F(x) \leq 0$ for $2 \leq x \leq 3$. The net area is

$$\int_0^3 F(x)\, dx = \int_0^3 (x^2 - 5x + 6)\, dx = \left(\frac{x^3}{3} - \frac{5x^2}{2} + 6x\right)\Bigg|_{x=0}^{x=3} = \frac{9}{2} - 0 = \frac{9}{2}\ .$$

To find the magnitude of the area under the curve, we break the integral from 0 to 2 and from 2 to 3. Then,

$$I_1 = \int_0^2 (x^2 - 5x + 6)\, dx = \left(\frac{x^3}{3} - \frac{5x^2}{2} + 6x\right)\Bigg|_0^2 = \frac{14}{3}$$

and

$$I_2 = \int_2^3 (x^2 - 5x + 6)\, dx = \left(\frac{x^3}{3} - \frac{5x^2}{2} + 6x\right)\Bigg|_2^3 = -\frac{1}{6}\ .$$

Clearly, $I_1 + I_2 = \frac{14}{3} - \frac{1}{6} = \frac{9}{2}$, same as the net area, but the magnitude of the area is

$$|I_1| + |I_1| = \left|\frac{14}{3}\right| + \left|-\frac{1}{6}\right| = \frac{29}{6}\ .$$

2.3 DIFFERENTIATION OF INTEGRALS: LEIBNIZ'S RULE

If one or both limits of a definite integral are functions of a variable, x, or if the integrand contains x as a parameter, then it is possible to differentiate the integral with respect to x. The example of such a definite integral is

$$I(x) = \int_{u(x)}^{v(x)} F(x,t)\, dt \ ,$$

where $u(x)$ and $v(x)$ are functions of x and t is the dummy variable. The derivative $\frac{dI}{dx}$ is evaluated using a method developed by Leibniz. To understand Leibniz's method of differentiating $I(x)$, we first look at three simpler cases of this integral, namely, $I_1(x), I_2(x)$, and $I_3(x)$. In integral $I_1(x)$, the variable x appears only in the upper limit; in integral $I_2(x)$, the variable x appears only in the lower limit; and in integral $I_3(x)$, the variable x appears only in the integrand. If

$$I_1(x) = \int_a^x F(t)\, dt = f(x) - f(a) \ ,$$

then

$$\frac{dI_1}{dx} = \frac{df}{dx} = F(x) \ .$$

On generalizing, if

$$I_1(x) = \int_a^{v(x)} F(t)\, dt = f(v(x)) - f(a) \ ,$$

then

$$\frac{dI_1}{dx} = \frac{df(v(x))}{dx} = \frac{df(v(x))}{dv}\frac{dv(x)}{dx} = F(v(x))\ \frac{dv(x)}{dx} \ .$$

Next, if

$$I_2(x) = \int_x^b F(t)\, dt = f(b) - f(x) \ ,$$

then

$$\frac{dI_2}{dx} = -\frac{df}{dx} = -F(x) \ .$$

Again, on generalizing, if

$$I_2(x) = \int_{u(x)}^{b} F(t)\, dt = \ f(b) - f(u(x)) \ ,$$

then

$$\frac{dI_2}{dx} = -\frac{df\big(u(x)\big)}{dx} = -\frac{df\big(u(x)\big)}{du}\frac{du(x)}{dx} = -F\big(u(x)\big)\ \frac{du(x)}{dx} \ .$$

Finally, if

$$I_3(x) = \int_{a}^{b} F(x,t)\, dt \ ,$$

then

$$\frac{dI_3}{dx} = \int_{a}^{b} \frac{\partial F(x,t)}{\partial x}\ dt \ .$$

These three results of integrals $I_1(x), I_2(x),$ and $I_3(x)$ can be combined and written in a single form, known as Leibniz's rule,

$$\frac{dI(x)}{dx} = \frac{d}{dx}\left(\int_{u(x)}^{v(x)} F(x,t)\, dt\right) = F\big(x, v(x)\big)\frac{dv}{dx} - \ F\big(x, u(x)\big)\frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial F(x,t)}{\partial x}\ dt \ . \qquad Eq. (2.4)$$

Example: Determine $\frac{dI(x)}{dx}$ if

$$\boldsymbol{I(x) = \int_{x}^{2x} \frac{\exp(xt)}{t}\ dt} \ .$$

Solution: Using Leibniz's rule, we have

$$\frac{dI(x)}{dx} = \frac{\exp[x(2x)]}{2x}(2) - \frac{\exp[x(x)]}{x}(1) + \int_{x}^{2x} \frac{t\ \exp(xt)}{t}\ dt \ ,$$

or

$$\frac{dI(x)}{dx} = \frac{\exp[2x^2]}{x} - \frac{\exp[x^2]}{x} + \frac{\exp(xt)}{x}\bigg|_{t=x}^{t=2x} = \frac{2}{x}\left(\exp[2x^2] - \exp[x^2]\right) \; .$$

2.4 MULTIPLE INTEGRALS

Multiple integrals are generalizations of single integrals with one variable. For example, a double integral, I_2, with two variables, x and y, can be perceived as two single integrals.

$$I_2 = \int_{x=a_1}^{x=b_1} \int_{y=a_2}^{y=b_2} F(x,y)dy\,dx = \int_{x=a_1}^{x=b_1} f(x)dx \;, \qquad Eq.(2.5)$$

where

$$f(x) = \int_{y=a_2}^{y=b_2} F(x,y)dy \; .$$

In evaluating the last integral for $f(x)$, the variable x inside the integrand is treated as a constant. Similarly, a triple integral, I_3, with three variables, x, y, and z, can be perceived as three single integrals.

$$I_3 = \int_{x=a_1}^{x=b_1} \int_{y=a_2}^{y=b_2} \int_{z=a_3}^{z=b_3} F(x,y,z)dz\,dy\,dx = \int_{a_1}^{b_1} f(x)dx \;, \qquad Eq.(2.6)$$

where

$$f(x) = \int_{a_2}^{b_2} g(x,y)dy \;\; \text{and} \;\; g(x,y) = \int_{a_3}^{b_3} F(x,y,z)dz \; .$$

Again, in evaluating the integral $g(x,y)$, the variables x and y inside the integrand are treated as constants and in evaluating the integral for $f(x)$, the variable x inside its integrand is treated as a constant. Finally, if $F(x,y,z)$ is a separable function, it means that F can be expressed as a product of three separate functions with each function containing only one variable as $F(x,y,z) = h_1(x)h_2(y)h_3(z)$. So, for a separable integrand $F(x,y,z)$,

$$I_3 = \int_{a_1}^{b_1} h_1(x)dx \int_{a_2}^{b_2} h_2(y)dy \int_{a_3}^{b_3} h_3(z)dz$$

is also separable.

Example: Evaluate the double integral $\int_0^1 \int_0^1 (x-y)^2 dy\,dx$.

Solution: Consider the following integral,

$$\int_0^1 \int_0^1 (x-y)^2 dy\, dx = \int_0^1 \int_0^1 (x^2 - 2xy + y^2)\, dy\, dx$$

$$= \int_0^1 \left| x^2 y - 2x\frac{y^2}{2} + \frac{y^3}{3} \right|_{y=0}^{y=1} dx = \int_0^1 \left(x^2 - x + \frac{1}{3} \right) dx$$

$$= \left| \frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{3} \right|_0^1 = \frac{1}{6}\ .$$

2.5 BAG OF TRICKS

In this section we will introduce some tricks for evaluating somewhat complicated integrals by starting from a bit simpler integral which is commonly known. First, we look at some examples of indefinite integrals and follow it up by definite integrals (exponential and Gaussian integrals), which are useful in several areas of physics.

Trick Number 1: Sometimes a complicated integral, which may appear quite intractable, can be manipulated so that it is written as the derivative of a simpler $basic$ integral. In fact, it is possible to evaluate several new integrals by introducing a temporary parameter in the basic integral, and then differentiating both sides of the resulting integral with respect to this parameter to get the new integral.

Trick Number 1a: As the first example, indefinite integrals of the type

$$\int \frac{dx}{(x+a)^2}, \int \frac{dx}{(x+a)^3}, \dots \qquad Eq.(2.7a)$$

are evaluated by starting from the $basic$ integral $\int \frac{dx}{x} = \ln(x)$. We introduce a parameter a in the basic integral by changing x to $x + a$ as

$$\int \frac{dx}{x+a} = \ln(x+a)\ . \qquad Eq.(2.7b)$$

After having introduced the parameter a, we can now differentiate both sides of the resulting integral with respect to a to evaluate the required integrals.

Example: Evaluate the indefinite integral $\int \frac{dx}{(x+a)^3}$.

Solution: Differentiate both sides of Eq. (2.7b) twice with respect to parameter a to get $(x+a)^3$ in the denominator. The first differentiation gives

$$-\int \frac{dx}{(x+a)^2} = \frac{1}{x+a} \ .$$

The second differentiation leads to

$$-(-2)\int \frac{dx}{(x+a)^3} = -\frac{1}{(x+a)^2} \ ,$$

which simplifies to the required integral

$$\int \frac{dx}{(x+a)^3} = -\frac{1}{2\,(x+a)^2} \ .$$

Trick Number 1b: Next, we can evaluate integrals of the type

$$\int \frac{1}{(a^2-\ x^2)^{3/2}}\ dx\,, \int \frac{1}{(a^2-\ x^2)^{5/2}}\ dx\,, \dots \qquad Eq.\,(2.8a)$$

by starting with the $basic$ integral $\int \frac{1}{\sqrt{1-x^2}}\ dx = \arcsin x$. On replacing x by x/a in the basic integral, we get the integral with parameter a,

$$\int \frac{1}{\sqrt{a^2-\ x^2}}\ dx = \arcsin\left(\frac{x}{a}\right) \ . \qquad Eq.\,(2.8b)$$

By differentiating both sides of this integral with respect to parameter a, we can evaluate integrals appearing in Eq. (2.8a).

Next, there are similar indefinite integrals of the type,

$$\int \frac{x}{(a^2-\ x^2)^{3/2}}\ dx\,, \int \frac{x}{(a^2-\ x^2)^{5/2}}\ dx\,, \dots \qquad Eq.\,(2.9a)$$

In this case, the $basic$ integral is

$$\int \frac{x}{\sqrt{1-\ x^2}}\ dx = -\sqrt{1-\ x^2} \ ,$$

which can be evaluated easily using substitution $u^2 = 1-\ x^2$. On replacing x by x/a in the basic integral, where a is a parameter, we get

$$\int \frac{x}{\sqrt{a^2 - x^2}}\, dx = -\sqrt{a^2 - x^2}\ . \qquad Eq. (2.9b)$$

Differentiating both sides of this integral with respect to parameter a leads to evaluation of integrals of Eq. (2.9a).

Example: Evaluate the indefinite integral $\int \frac{x}{(a^2 - x^2)^{3/2}}\, dx$ as well as the definite integral $\int_0^{a/2} \frac{x}{(a^2 - x^2)^{3/2}}\, dx$.

Solution: Differentiating both sides of Eq. (2.9b) with respect to a gives

$$\int \left[-\frac{1}{2}\frac{2a}{(a^2 - x^2)^{3/2}}\right] x\, dx = -\frac{1}{2}\frac{2a}{(a^2 - x^2)^{\frac{1}{2}}}\ ,$$

or

$$\int \frac{x}{(a^2 - x^2)^{3/2}}\, dx = \frac{1}{\sqrt{a^2 - x^2}}\ .$$

The definite integral is obtained by substituting the upper and the lower limits,

$$\int_0^{a/2} \frac{x}{(a^2 - x^2)^{3/2}}\, dx = \frac{2}{\sqrt{3}\, a} - \frac{1}{a}\ .$$

Trick Number 1c: Several indefinite integrals of the type

$$\int \frac{1}{(x^2 + a^2)^2}\, dx\, , \int \frac{1}{(x^2 + a^2)^3}\, dx\, , \ldots \qquad Eq. (2.10a)$$

can be evaluated by starting from the $basic$ integral $\int \frac{1}{x^2+1}\, dx = \arctan x$. Introduce a parameter a by replacing x by x/a in the basic integral to get

$$\int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)\ . \qquad Eq. (2.10b)$$

Now, multiple differentiations of both sides of this integral with respect to parameter a lead to the required integrals of Eq. (2.10a).

Example: Using the integral $\int_0^{\infty} \frac{1}{1+x^2}\, dx = \frac{\pi}{2}$ as a guide, introduce a parameter and then differentiate both sides with respect to this parameter to evaluate the integral

$$\int_0^\infty \frac{\mathbf{1}}{(\boldsymbol{x}^2+\boldsymbol{y}^2)^3}\,\boldsymbol{dx}\ .$$

Solution: In the guiding integral $\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$, introduce a parameter y by changing x to x/y,

$$\int_0^\infty \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{dx}{y} = \frac{\pi}{2} \quad \text{or} \quad \int_0^\infty \frac{dx}{x^2+y^2} = \frac{\pi}{2y}\ .$$

Now, to get $(x^2+y^2)^3$ in the dominator, we differentiate both sides of the integral with respect to parameter y twice. The first differentiation gives

$$\int_0^\infty \left[-\frac{2y}{(x^2+y^2)^2}\right] dx = \frac{\pi}{2}\left[-\frac{1}{y^2}\right],$$

or

$$\int_0^\infty \frac{dx}{(x^2+y^2)^2} = \frac{\pi}{4}\frac{1}{y^3}\ .$$

The second differentiation gives

$$\int_0^\infty \left[-\frac{2(2y)}{(x^2+y^2)^3}\right] dx = \frac{\pi}{4}\left[-\frac{3}{y^4}\right],$$

or

$$\int_0^\infty \frac{dx}{(x^2+y^2)^3} = \frac{3\pi}{16}\frac{1}{y^5}\ ,$$

which is the value of the required integral.

Next, somewhat similar integrals of the type

$$\int \frac{x}{(x^2+\ a^2)^2}\,dx\,, \int \frac{x}{(x^2+\ a^2)^3}\,dx\,, \ldots \qquad Eq.(2.11a)$$

can be evaluated by starting from the $basic$ integral

$$\int \frac{x}{x^2+\ 1}\,dx = \frac{1}{2}\ln(x^2+\ 1)\ .$$

This integral is easily evaluated by using the substitution $u = x^2 +\ 1$. On replacing x by x/a, where a is a parameter, we get

$$\int \frac{x}{x^2 + a^2}\, dx = \frac{1}{2}\ln(x^2 + a^2) - \ln a \quad . \qquad Eq. (2.11b)$$

Differentiating both sides of this integral with respect to parameter a provides the integrals of Eq. (2.11a).

Trick Number 2: Continuing with our bag of tricks to solve indefinite integrals, let us evaluate integrals of the form $\int \frac{P(x)}{Q(x)} dx$, where $P(x)$ and $Q(x)$ are polynomials of x.

Trick Number 2a: Our first case is the one in which degree of polynomial $P(x)$ is less than the degree of $Q(x)$. Examples of such integrals are

$$\int \frac{dx}{x^2 - x - 6} \qquad \text{or} \qquad \int \frac{x\, dx}{x^2 - 7x + 12} \; .$$

If the denominator can be factored, then use partial fractions to break the integrand of the first integral as

$$\frac{1}{x^2 - x - 6} = \frac{1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} \; .$$

To determine A and B, multiply both sides by $(x^2 - x - 6)$ to get

$$1 = A(x+2) + B(x-3) \; .$$

Since this is true for any values of x, set $x = 3$ and $x = -2$ successively, to get $A = \frac{1}{5}$ and

$B = -\frac{1}{5}$. Thus,

$$\frac{1}{x^2 - x - 6} = \frac{1}{5}\left[\frac{1}{x-3} - \frac{1}{x+2}\right] \; .$$

The original integral then becomes

$$\int \frac{dx}{x^2 - x - 6} = \frac{1}{5}\int \frac{dx}{x-3} - \frac{1}{5}\int \frac{dx}{x+2} = \frac{1}{5}\ln(x-3) - \frac{1}{5}\ln(x+2) = \frac{1}{5}\ln\left(\frac{x-3}{x+2}\right) \; .$$

Similarly, for the second integral,

$$\frac{x}{x^2 - 7x + 12} = \frac{x}{(x-4)(x-3)} = \frac{A}{x-4} + \frac{B}{x-3} \; ,$$

or, on multiplying both sides by $(x^2 - 7x - 12)$, we get

$$x = A(x-3) + B(x-4) \; .$$

On setting $x = 3$ and $x = 4$ separately, we get $A = 4$ and $B = -3$. So,

$$\int \frac{x\,dx}{x^2 - 7x + 12} = 4\int \frac{dx}{x-4} - 3\int \frac{dx}{x-3} = 4\ln(x-4) - 3\ln(x-3)\ .$$

Trick Number 2b: There are situations in which the denominator of the integrand cannot be factored easily. In this case, one can try to complete the square and make a substitution that will turn the integral into a simpler well-known integral. For example,

$$\int \frac{dx}{3x^2 - 6x + 7} = \int \frac{dx}{3(x-1)^2 + 4}\ .$$

Define a new variable, $u = \sqrt{3}(x-1)/2$, or $dx = 2\,du/\sqrt{3}$. With this substitution,

$$\int \frac{dx}{3x^2 - 6x + 7} = \frac{1}{2\sqrt{3}}\int \frac{du}{u^2+1} = \frac{1}{2\sqrt{3}}\arctan u = \frac{1}{2\sqrt{3}}\arctan\left[\frac{\sqrt{3}(x-1)}{2}\right]\ .$$

Trick Number 2c: If the numerator contains a polynomial $P(x)$ of degree one less than the degree of polynomial $Q(x)$ in the denominator, then write the numerator as

$$numerator = a(derivative\ of\ denominator) + b\ ,$$

where a and b are some numbers. In this manner the integral becomes tractable. As an example, consider the integral

$$\int \frac{x+4}{x^2+2x+5}\,dx\ .$$

The numerator can be expressed as

$$x + 4 = \frac{1}{2}\frac{d}{dx}[x^2 + 2x + 5] + 3\ ,$$

so that

$$\int \frac{x+4}{x^2+2x+5}\,dx = \int \frac{dx}{x^2+2x+5}\left\{\frac{1}{2}\frac{d}{dx}[x^2+2x+5] + 3\right\} = \frac{1}{2}\ln[x^2+2x+5] + \int \frac{3\,dx}{x^2+2x+5}\ .$$

The remaining integral can be determined using "complete the square" technique as described in *Trick Number 2b*.

Trick Number 2d: Next, we consider the case in which the degree of polynomial $P(x)$ in the numerator is more than the degree of polynomial $Q(x)$ in the denominator. In this situation, it is best to use long division to simplify the integrand.

Example: Evaluate the indefinite integral,

$$\int \frac{\mathbf{3x^3 - 7x^2 + 17x - 3}}{\mathbf{x^2 - 2x + 5}}\, \boldsymbol{dx} \ .$$

Solution: Using long division,

$$\frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x + 5} = 3x - 1 + \frac{2}{x^2 - 2x + 5} \ .$$

Thus,

$$I = \int \frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x + 5} dx = \int 3x\, dx - \int 1 dx + \int \frac{2\, dx}{x^2 - 2x + 5}$$

$$= \frac{3x^2}{2} - x + 2I_1 \ ,$$

where I_1 can be evaluated using "complete the square" technique as described in *Trick Number 2b*.

Trick Number 3: Integrals involving trigonometric functions are simplified using standard identities such as,

$$\cos^2 x + \sin^2 x = 1 \ ,$$

$$\sin(2x) = 2 \sin x \cos x \ ,$$

$$\cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x\, , \text{etc} \ .$$

Example: Evaluate $= \int \mathbf{\sin^3 x \ \cos^2 x \ } \boldsymbol{dx}$**.**

Solution: The given integral is

$$I = \int \sin^2 x \ \cos^2 x \ [\sin x\, dx] = -\int (1 - \cos^2 x) \ \cos^2 x \ d(\cos x) \ .$$

On substituting $u = \cos x$, we get

$$I = -\int (1 - u^2)\, u^2\, du = -\frac{u^3}{3} + \frac{u^5}{5} == -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} \ .$$

Example: Evaluate $\boldsymbol{I = \int \sin^2 x\ \cos^2 x\ dx}$.

Solution: Since $\sin^2 x\ \cos^2 x = \frac{1}{4}\sin^2(2x) = \frac{1}{8}[1-\cos(4x)]$, we get

$$I = \frac{1}{8}\int [1-\cos(4x)]\ dx = \frac{1}{8}\left[x - \frac{\sin(4x)}{4}\right] .$$

2.6 EXPONENTIAL AND GAUSSIAN INTEGRALS

Continuing with our bag of tricks, now we look at some examples of definite integrals. Two types of integrals that we commonly encounter in physics are exponential integrals and Gaussian integrals. As the names suggest, the exponential integrals contain the exponential function, $\exp(-x)$, as a part of the integrand. The Gaussian integrals contain the Gaussian function, $\exp(-x^2)$, as a part of the integrand. The exponential integrals, I_e, are defined as

$$I_e^n = \int_0^\infty x^n \exp(-ax)\ dx \quad , \qquad Eq. (2.12)$$

and the Gaussian integrals, I_g, are,

$$I_g^n = \int_0^\infty x^n \exp(-ax^2)\ dx \quad . \qquad Eq. (2.13)$$

Sometimes we also encounter Gaussian integrals, I_{gg}, over an extended range of variable x,

$$I_{gg}^n = \int_{-\infty}^\infty x^n \exp(-ax^2)\ dx \quad . \qquad Eq. (2.14)$$

In these integrals n, a positive integer, is called the order of the integral and a is a positive constant. In each case we will try to set up a *reduction formula* that relates the integral I^n to similar integrals of lower order, such as I^{n-1} and/or I^{n-2}. Using this reduction formula successively, one can relate I^n to simpler integrals I^1 and/or I^0. In case of the exponential integral, we have, for $n = 0$,

$$I_e^0 = \int_0^\infty \exp(-ax)\ dx = \frac{1}{a} .$$

Also, for $n \geq 1$, we can integrate by parts to get the reduction formula

$$I_e^n = \int_0^\infty x^n \exp(-ax)\ dx = \left. x^n \frac{\exp(-ax)}{-a} \right|_0^\infty - \int_0^\infty \frac{\exp(-ax)}{-a} \cdot n x^{n-1}\, dx = 0 + \frac{n}{a} I_e^{n-1}\ .$$

The first integrated term vanishes at both limits. At the upper limit, as $x \to \infty$, the factor $\exp(-ax)$ approaches zero faster than any increase in x^n. At the lower limit, x^n goes to zero since $n \geq 1$. This reduction formula can be used successively to obtain

$$I_e^n = \frac{(n)}{a}\frac{(n-1)}{a} I_e^{n-2} = \frac{(n)}{a}\frac{(n-1)}{a}\frac{(n-2)}{a} I_e^{n-3} = \cdots = \frac{(n)}{a}\frac{(n-1)}{a} \cdots \frac{(2)}{a}\frac{(1)}{a} I_e^0\ ,$$

or, on substituting the value of I_e^0 here,

$$I_e^n = \frac{n!}{a^{n+1}}\ . \qquad Eq. (2.15)$$

In case of Gaussian integrals, I_g^n, we need I_g^0 and I_g^1 to use the reduction formula. We postpone the evaluation of I_g^0 until after we have discussed curvilinear coordinates and multiple integrals in Chapter 9. The value of this integral, however, is,

$$I_g^0 = \int_0^\infty \exp(-ax^2)\ dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\ .$$

The integral I_g^1, on the other hand, can be evaluated easily by substituting $u = ax^2$,

$$I_g^1 = \int_0^\infty x \exp(-ax^2)\ dx = \frac{1}{2a}\int_0^\infty \exp(-u)\ du = \frac{1}{2a}\ .$$

To set up the reduction formula for Gaussian integrals, we use the fact

$$\frac{d}{dx}\exp(-ax^2) = -2ax \exp(-ax^2) \quad \text{or } x \exp(-ax^2) = -\frac{1}{2a}\frac{d}{dx}\exp(-ax^2)\ .$$

Thus, for $n \geq 2$, using integration by parts we obtain

$$I_g^n = \int_0^\infty x^{n-1} \cdot x \exp(-ax^2)\ dx = -\frac{1}{2a}\int_0^\infty x^{n-1} \cdot \frac{d}{dx}\exp(-ax^2)\ dx$$

$$= -\frac{1}{2a} x^{n-1} \cdot \exp(-ax^2)|_0^\infty + \frac{1}{2a}\int_0^\infty (n-1)x^{n-2} \exp(-ax^2)\ dx \qquad Eq. (2.16)$$

$$= 0 + \frac{(n-1)}{2a}\int_0^\infty x^{n-2}\exp(-ax^2)\ dx = \frac{(n-1)}{2a} I_g^{n-2}\ .$$

Using this reduction formula, we can write $I_g^2, I_g^4, I_g^6\ \cdots$ in terms of I_g^0, and $I_g^3, I_g^5, I_g^7 \cdots$ in terms of I_g^1. For $n = 2m$, where m is an integer,

$$I_g^{2m} = \frac{(2m)!}{(4a)^m (m)!} I_g^0\ .$$

Similarly, for $n = 2m + 1$,

$$I_g^{2m+1} = \frac{(m)!}{a^m} I_g^1\ .$$

Finally, the Gaussian integral, I_{gg}, over an extended range of variables is

$$I_{gg}^n = \int_{-\infty}^{\infty} x^n \exp(-ax^2)\ dx = \int_0^\infty x^n \exp(-ax^2)\ dx + \int_{-\infty}^{0} x^n \exp(-ax^2)\ dx\ .$$

In the second integral, change the variable from x to y via $x = -y$. Then,

$$I_{gg}^n = \int_0^\infty x^n \exp(-ax^2)\ dx - \int_\infty^0 (-y)^n \exp(-ay^2)\ dy = I_g^n + (-1)^n\ I_g^n\ .$$

Thus, if n is an odd integer, then $I_{gg}^n = 0$, and if n is an even integer, then $I_{gg}^n = 2\ I_g^n$.

PROBLEMS FOR CHAPTER 2

1. Evaluate using partial fractions

$$\int_2^4 \frac{2x+1}{x^2-1}\,dx \ .$$

2. Evaluate the indefinite integral

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx \ ,$$

using the bag of tricks.

3. Use Leibniz's rule to find dI/dx for

$$I = \int_{a-x}^{x^2} (x-t)\,dt$$

with $a > 0$.

Next, evaluate the integral I explicitly, and then find dI/dx.

4. Use Leibniz's rule to find the value of θ that provides the extremum value of the integral

$$I(\theta) = \int_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}+\theta} \left(t-\frac{\pi}{4}\right)\frac{\sin(t)}{t}\,dt \ .$$

5. Define an integral $I_n = \int (\ln x)^n \; dx$. Using integration by parts, derive the following reduction formula:

$$I_n = x\,(\ln x)^n - \,n\,I_{n-1} \ .$$

Use this reduction formula to determine

$$I_3 = \int (\ln x)^3 \; dx \ .$$

6. Consider the integral

$$I_n = \int_0^{\pi/2} \sin^{\mathrm{n}} x\,dx \ .$$

Explicitly evaluate the first two integrals I_0 and I_1.

Using integration by parts and the identity $\cos^2 x = 1 - \sin^2 x$, derive the reduction formula for $n \geq 2$:

$$I_n = \frac{n-1}{n} I_{n-2} \ .$$

Using the reduction formula, evaluate the integrals

$$I_5 = \int_0^{\pi/2} \sin^5 x \, dx \quad \text{and} \quad I_6 = \int_0^{\pi/2} \sin^6 x \, dx \ .$$

7. Consider the integral

$$I_n = \int_{-a}^{a} (x^2 - a^2)^n \, dx \ .$$

Explicitly evaluate the first integral I_0.

Using integration by parts, derive the reduction formula

$$I_n = -\frac{2na^2}{2n+1} I_{n-1}$$

for $n \geq 1$. Use the reduction formula to determine the value of

$$I_3 = \int_{-a}^{a} (x^2 - a^2)^3 \, dx \ .$$

8. Introduce a parameter in the standard integral

$$\int \frac{1}{\sqrt{1 - x^2}} \, dx = \arcsin x$$

and then differentiate both sides with respect to the parameter to determine the value of

$$I = \int_0^{a/2} \frac{dx}{(a^2 - x^2)^{3/2}} \ .$$

9. **Biomedical Physics Application.** A gunshot wound victim is bleeding at the rate given by

$$r(t) = r_0 t^2 \exp(bt) \ ,$$

where $r_0 = 0.000008 \, ml/s^3$, and $b = 0.004 \, s^{-1}$. The ambulance takes five minutes after the shooting to arrive at the crime scene. How much total blood has left victim's body before the medical attention is provided to the victim?

A healthy adult has about 5 liters of blood circulating in the human body. Most adults can lose up to 14% of their blood before their vital signs begin to deteriorate. What would be the outcome of the victim?

10. **Biomedical Physics Application.** The West Nile Virus is a disease that is contracted from infected mosquitoes. Scientists at the Centers for Disease Control and Prevention monitor the population growth of mosquitoes in a controlled environment. The mosquito population in this environment starts with 1000 mosquitoes at the start of the summer and grows at the rate of

$$r(t) = (1000)\exp(0.1\, t)$$

mosquitoes per day. What is the total mosquito population at the end of the seventh week of summer?

11. **Biomedical Physics Application.** In physiology we learn that a typical healthy adult human being breathes 12 to 15 times per minute. The volume of air inhaled in a single breath is about 500 mL. Assuming that breathing is a periodic process taking 5 seconds from the beginning of inhalation to the end of exhalation, the *rate* of air flow into the lungs can be represented by the function

$$f(t) = 100\,\pi \sin\left(\frac{2\pi t}{5}\right) \ \frac{mL}{s} \ .$$

Determine the total volume of the air inhaled at any time t starting at $t = 0$. Provide the volume of air inhaled by a healthy human being at $t = 1s, 2s, 3s, 4s, \text{and } 5\ s$.

Chapter 3: Infinite Series

In this chapter the convergent or divergent behavior of an infinite series is investigated. Several tests for checking this behavior are outlined. Taylor series and Maclaurin series are described and are used to obtain an infinite series expansion of several simple functions.

3.1 AN INFINITE SERIES

Consider an infinite series

$$a_1 + a_2 + a_3 \dots \dots + a_n + \cdots$$

where each element, a_n, of the series is a separate constant. For this series we define the partial sum, $S(N)$, as the sum of its first N terms

$$S(N) = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n \ .$$

The infinite series is said to converge if the partial sum $S(N)$ has a finite limit as $N \to \infty$. This limit is the value of the converging series; that is, if

$$\lim_{N\to\infty} [S(N)] = S \ ,$$

then we write

$$S = a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n \ . \qquad Eq. (3.1)$$

If the partial sum does not have a finite limit as $N \to \infty$, then the series is divergent. Note in passing that in the summation notation, the running variable n can be replaced by any other symbol as follows:

$$S = \sum_{n=1}^{\infty} a_n \ \text{ or } \ S = \sum_{m=1}^{\infty} a_m \ \text{ or } \ S = \sum_{p=1}^{\infty} a_p \ .$$

Therefore, the running variable n or m or p in the summation notation is referred to as the $dummy\ variable$ in the same sense as the dummy variable of a definite integral; that is, it serves as a placeholder that disappears after the sum is evaluated. To determine the convergence of a series, we compare it with another series whose convergence or divergence is known. For the comparison purpose we use either a geometric series or a harmonic series.

Geometric Series

The series in which the ratio of two successive terms is constant, namely

$$a + ax + ax^2 + ax^3 + \cdots ax^{n-1} + \cdots \qquad Eq.(3.2a)$$

is known as a geometric series. Here a is the *first term* and x is the $common\ ratio$. The partial sum of first N terms of this series is

$$S(N) = a + ax + ax^2 + \cdots ax^{(N-1)} \ . \qquad Eq.(3.2b)$$

On multiplying this partial sum by the common ratio x, we get

$$xS(N) = ax + ax^2 + \cdots ax^{(N-1)} + ax^N \ . \qquad Eq.(3.2c)$$

Subtracting Eq. $(3.2c)$ from Eq. $(3.2b)$ results in

$$(1 - x)S(N) = a - ax^N \ ,$$

or

$$S(N) = a\frac{1 - x^N}{1 - x} \ . \qquad Eq.(3.2d)$$

Now if $x \geq 1$, then $S(N) \to \infty$ as $N \to \infty$, so that geometric series is divergent for $x \geq 1$. For $x < 1$, $x^N \to 0$ for $N \to \infty$ so that

$$\lim_{N\to\infty}[S(N)] \to \frac{a}{1 - x} \ . \qquad Eq.(3.2e)$$

Therefore, the geometric series, $a + ax + ax^2 + \cdots$, converges to $\frac{a}{1-x}$ for $x < 1$ and diverges for $x \geq 1$. Note that in a converging geometric series, the common ratio x can take any value as long as this value is less than 1.

Harmonic Series

Next, we consider a harmonic series. Unlike a geometric series, which can take several values depending on the value of x, there is only one harmonic series. Successive terms of the harmonic series are reciprocals of the natural integers as follows:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots$$

Even though each successive term in this series is smaller than the previous term, the series is divergent. The divergent nature of the series can be seen by grouping the terms as

$$1+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\cdots+\frac{1}{16}\right)+\cdots$$

A typical pair of parentheses will include terms like

$$\underbrace{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}}_{n\ terms}>\underbrace{\frac{1}{n+n}+\frac{1}{n+n}+\cdots+\frac{1}{n+n}}_{n\ terms}=\frac{n}{2n}=\frac{1}{2}\ .$$

Thus, for harmonic series,

$$1+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\cdots+\frac{1}{16}\right)+\cdots>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$$

Now, the series on the right-hand side of this inequality is clearly divergent since the partial sum of this series is

$$S(N)=1+\frac{1}{2}(N-1)=\frac{N+1}{2}\ ,$$

and

$$\lim_{N\to\infty}[S(N)]\to\infty\ .$$

Thus, the harmonic series is known to be a divergent series.

3.2 TESTS FOR CONVERGENCE

We can use several different tests to check the convergence of an unknown series. Three common tests are the comparison test, the ratio test, and the integral test.

Comparison Test

Now, to test the convergent or divergent nature of a series, we compare it term-by-term with another series whose convergence or divergence is known. For example, if the infinite series $a_1+a_2+a_3+\cdots$ is known to be convergent, then consider an unknown series, $b_1+b_2+b_3$ .... If $b_n\leq a_n$ for all values of n from 1 to infinity, then $\sum_n b_n\leq\sum_n a_n$ and, therefore, $\sum_n b_n$ is also convergent.

On the other hand, suppose the infinite series $u_1 + u_2 + u_3 \ldots$ is known to be divergent. Then, consider an unknown series $v_1 + v_2 + v_3 \ldots$ If $v_n \geq u_n$ for all values of n from 1 to infinity, then $\sum_n v_n \geq \sum_n u_n$ and, therefore, $\sum_n v_n$is also divergent. For this comparison test, a standard geometric series will serve as a prototype convergent series, and the harmonic series will serve as a prototype divergent series.

Example: Check the convergence of the following series.

$$\boldsymbol{S = \sum_{n=1}^{\infty} \frac{1}{2 + 3^n}} \ .$$

Solution: Since $\frac{1}{2+3^n} < \frac{1}{3^n}$ for any n, and $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series with the common ratio $1/3$, the unknown series S is also convergent.

Example: Check whether the following series converges or diverges.

$$\boldsymbol{S = \sum_{n=1}^{\infty} \frac{\ln(n)}{n}} \ .$$

Solution: We compare this series with the harmonic series. Since $\frac{\ln(n)}{n} > \frac{1}{n}$ for $n \geq 3$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series, so S is also a divergent series.

Ratio test of Cauchy and D'Alembert

The Cauchy and D'Alembert ratio test provides the ability to compare an unknown infinite series with the geometric series $1 + x + x^2 + \cdots$. We know that the above geometric series converges only if x, the ratio of two successive terms, is less than 1. Thus, for the unknown series $a_1 + a_2 + a_3 \ldots + a_n + a_{n+1} + \cdots$, with all $a_i > 0$, we look for the ratio of two successive terms, $\frac{a_{n+1}}{a_n}$. If

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1, \text{the series is convergent,}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1, \text{the series is divergent, and} \qquad Eq. (3.3)$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1, \text{the nature of the series is indeterminate.}$$

Example: Determine the converging or diverging behavior of the infinite series,

$$\boldsymbol{\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} + \cdots = \sum_{n=1}^{\infty} \frac{n}{(n+1)!}} \ .$$

Solution: In this case,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{n+1}{(n+2)!}\frac{(n+1)!}{n}=\frac{n+1}{(n+2)\,n}\to 0\ .$$

Thus, according to the ratio test, the series converges. The value of this series is 1 [see Example 3 at the end of this chapter].

Example: Check the convergence of the following series,

$$\boldsymbol{\left(\frac{1}{3}\right)1!+\left(\frac{1}{3}\right)^2 2!+\left(\frac{1}{3}\right)^3 3!\ldots+\left(\frac{1}{3}\right)^n n!+\cdots=\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n n!}\ .$$

Solution: In this case,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{\left(\frac{1}{3}\right)^{n+1}(n+1)!}{\left(\frac{1}{3}\right)^{n}n!}=\frac{1}{3}(n+1)\to\infty\ .$$

Thus, the unknown series is divergent.

Integral test of Cauchy and Maclaurin

Once again, consider an infinite series of constants,

$$a_1+a_2+a_3\ldots\ldots+a_n+\cdots$$

Imagine a function, $f(x)$, that is a continuous and monotonically decreasing function of x with $f(n)=a_n$. Consider its partial sum

$$S(N)=\sum_{n=1}^{N}a_n\ .$$

Now consider a series of unit-width rectangles, with heights a_1, a_2, a_3, ...etc, as shown in Figure 3.1. From the figure

$$a_1+a_2+\cdots+a_{N-1}>\int_1^N f(x)dx \qquad \text{[dashed line]} \qquad ,$$

and

$$a_2 + a_3 + \cdots + a_N < \int_1^N f(x)dx \qquad \text{[solid line]} \qquad .$$

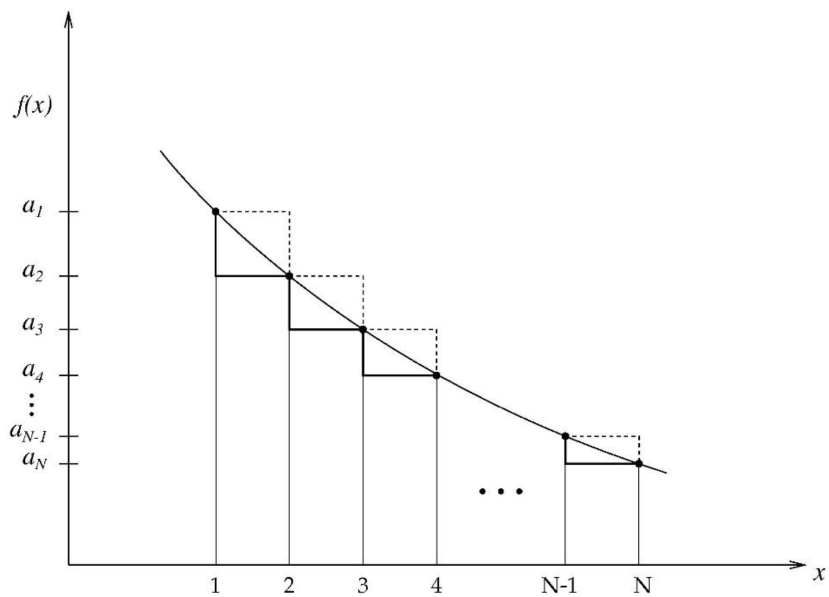

Figure 3.1. Integral test of Cauchy and Maclaurin.

That is,

$$S(N) > \int_1^{N+1} f(x)dx$$

and

$$S(N) < \int_1^N f(x)dx + a_1 \ .$$

On combining both the results, we get

$$\int_1^{N+1} f(x)dx < S(N) < \int_1^N f(x)dx + a_1 \ .$$

Now take the limit $N \to \infty$ to get,

$$\int_1^\infty f(x)dx < \sum_{n=1}^{\infty} a_n < \int_1^\infty f(x)dx + a_1 \ . \qquad Eq. (3.4)$$

Thus, the series $\sum_{n=1}^{\infty} a_n$ converges or diverges just as the integral $\int_1^\infty f(x)dx$ converges or diverges.

Example: Check the converging behavior of the Riemann Zeta function,

$$\boldsymbol{\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}} \ .$$

Solution: We define $f(x) = \frac{1}{x^p}$ so that $f(n) = \frac{1}{n^p}$. Then,

$$\int_1^\infty f(x)dx = \int_1^\infty x^{-p}dx = \begin{cases} \left.\dfrac{x^{-p+1}}{-p+1}\right|_1^\infty, & \text{for } p \neq 1 \\ \ln(x)\,|_1^\infty, & \text{for } p = 1 \end{cases} \ .$$

The integral $\int_1^\infty f(x)dx$, therefore, diverges for $p \leq 1$ and converges for $p > 1$. The series for the Riemann Zeta function is thus convergent for $p > 1$. This is sometimes referred to as the p-series test.

Absolute Convergence

The series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if the related series $\sum_{n=1}^{\infty} |a_n|$ also converges. An absolute convergence implies convergence but not vice versa. As an example, the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges to $\ln 2$ [we will show it later in this chapter; see Eq. (3.12)]. But the corresponding absolute series,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \ ,$$

being the harmonic series, diverges. So, the original series is not absolutely convergent. An absolutely convergent series is important for two reasons: The product of two absolutely convergent series is another absolutely convergent series. The sum of the product of two absolutely convergent series is equal to the product of the sum of two individual series. That is,

$$S_a S_b + S_a S_c = S_a(S_b + S_c) \ .$$

If an infinite series is absolutely convergent, its sum is independent of the order in which the terms are added. Stated differently, if an infinite series is not absolutely convergent, the value of its sum will depend on the order in which the terms are added.

As an example, let us go back to the series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots \ ,$$

which is not absolutely convergent. Now we show explicitly that the value of this series depends on the order in which its terms are added. Separating out the positive and negative parts of this series, we get

$$S = \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} \dots\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots\right) \ .$$

The terms in the second parenthesis can be further separated out as

$$S = \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} \dots\right) - \left(\frac{1}{2} + \frac{1}{6} + \frac{1}{10} + \frac{1}{14} \dots\right) - \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \frac{1}{16} \dots\right) \ .$$

Now, take out the common factor of $\frac{1}{2}$ in the second parenthesis and then easily combine the first and second parenthesis as

$$S = \left(\frac{1}{2} + \frac{1}{6} + \frac{1}{10} + \frac{1}{14} \dots\right) - \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \frac{1}{16} \dots\right) \ .$$

On taking out an overall common factor of $\frac{1}{2}$, we get

$$S = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \dots\right) \ ,$$

or

$$S = \frac{1}{2} S \ ,$$

which, of course, is an absurd result. The absurdity arises since this series is not absolutely convergent.

The second property of absolutely convergent series allows us to rearrange various series. Suppose $S_a = \sum_{n=1}^{\infty} a_n$ and $S_b = \sum_{n=1}^{\infty} b_n$ are two absolutely converging series, then $S = S_a S_b$ is the following double series:

$$S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m \, b_n = \sum_{m,n=1}^{\infty} c_{mn} \ .$$

In this double sum the summation is carried over rows (and columns) of the table shown in Figure 3.2. However, if we define

$$r = n + m - 1 \ ,$$

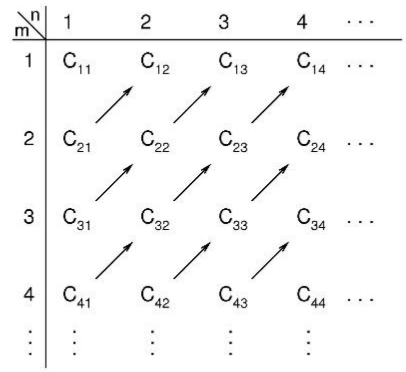

Figure 3.2. Evaluation of a double infinite sum.

then we note that r also takes all integer values over the range from 1 to ∞. So, we can replace the dummy variable m by variable r. Also, note $m = r - n + 1 \geq 1$ or $n \leq r$. Thus,

$$S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn} = \sum_{r=1}^{\infty}\sum_{n=1}^{r} c_{r-n+1,n} \quad . \qquad Eq.(3.5)$$

This summation is carried along the diagonal of the table in Figure 3.2. The advantage of this manipulation is that a double infinite sum is reduced to a single infinite sum plus a finite sum. This works only for absolutely convergent series.

3.3 SERIES OF FUNCTIONS

If each term of an infinite series is a function of a variable x, then the sum of the series, if it exists, is a function of x. Such a series looks like

$$a_0(x) + a_1(x) + a_2(x) + \cdots + a_n(x) + \cdots ,$$

with the sum of this infinite series being

$$S(x) = \sum_{n=0}^{\infty} a_n(x) = \lim_{N\to\infty} S(N;x) \ .$$

Here $S(N;x)$ is the partial sum, $S(N;x) = \sum_{n=0}^{N} a_n(x)$. In case of series of functions, we will start the sum from $n = 0$ instead of $n = 1$, as done previously for series of constant terms.

The dependence of $\lim_{N\to\infty} S(N;x)$ on x is expressed through *uniform convergence* of the series. If for a certain range of values of x, $a \leq x \leq b$, and for any small number $\epsilon > 0$, it is possible to choose a number ν [which is independent of x] such that

$$S(x) - \epsilon < S(N;x) < S(x) + \epsilon \qquad \text{for all } N \geq \nu \ ,$$

or, alternatively,

$$|S(x) - S(N;x)| < \epsilon \qquad \text{for all } N \geq \nu \ ,$$

then the series converges uniformly to $S(x)$ in the range $a \leq x \leq b$. Thus, no matter how small ϵ is, it is always possible to choose a sufficiently large N such that the difference between the full sum $S(x)$ and the partial sum $S(N;x)$ is less than ϵ for all values of x between a and b. A good thing about a uniformly converging series is that it can be differentiated and integrated term by term. A series in x which converges uniformly in the interval $a \leq x \leq b$ can be integrated term by term provided the limits of the integration lie within $[a, b]$. Thus, if $S(x) = \sum_{n=0}^{\infty} a_n(x)$ is uniformly convergent, then

$$\int_p^q S(x)dx = \sum_{n=0}^{\infty} \int_p^q a_n(x)dx \ , \qquad Eq.(3.6a)$$

provided p and q lie within $[a, b]$. Similarly, a uniformly convergent series can be differentiated term by term within its range of convergence; that is, if $S(x) = \sum_{n=0}^{\infty} a_n(x)$, then

$$\frac{dS(x)}{dx} = \sum_{n=0}^{\infty} \frac{da_n(x)}{dx} \ . \qquad Eq.(3.6b)$$

Taylor and Maclaurin Series

The reverse process, in which a given function $S(x)$ is expanded into an infinite series, is called the power series expansion. Two power series expansions of interest to us are the Taylor and the Maclaurin series expansions. To gain more insight into the Taylor series expansion of a given function $S(x)$, consider expanding this function as

$$S(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \ , \qquad Eq.(3.7)$$

where the coefficients c_n are to be determined. This is the power series expansion of $S(x)$ about the point $x = a$. Define

$$S^{(n)}(x) = \frac{d^n S(x)}{dx^n} \text{ and } S^{(n)}(a) = S^n(x)|_{x=a} \ .$$

On setting $x = a$ in Eq. (3.7), we note that $c_0 = S(a)$. On taking the first derivative of the series in Eq. (3.7) and then setting $x = a$, we get

$$\frac{dS(x)}{dx} = 0 + c_1 + 2c_2(x-a) + \cdots$$

and

$$c_1 = \left.\frac{dS}{dx}\right|_{x=a} = S^{(1)}(a) \ .$$

Similarly, on taking the second derivative of the series in Eq. (3.7) and then setting $x = a$, we get

$$\frac{d^2S}{dx^2} = 2 \cdot 1c_2 + 3 \cdot 2c_3(x-a) + \cdots$$

and

$$c_2 = \frac{1}{2!}\left.\frac{d^2S}{dx^2}\right|_{x=a} = \frac{S^{(2)}(a)}{2!} \ .$$

Similarly, on taking the third derivative of the series in Eq. (3.7) and then setting $x = a$, we get

$$c_3 = \frac{1}{3!}\left.\frac{d^3S}{dx^3}\right|_{x=a} = \frac{S^{(3)}(a)}{3!} \ .$$

In general,

$$c_n = \frac{1}{n!}\left.\frac{d^nS}{dx^n}\right|_{x=a} = \frac{S^{(n)}(a)}{n!} \ .$$

Substituting the values of c_n in Eq. (3.7) provides the Taylor series expansion, namely

$$S(x) = \sum_{n=0}^{\infty} \frac{S^{(n)}(a)}{n!}(x-a)^n \ . \qquad Eq. (3.8a)$$

This is the Taylor series expansion of $S(x)$ about $x = a$. In general, the Taylor series can be truncated after N terms as

$$S(x) = \sum_{n=0}^{N-1} \frac{S^{(n)}(a)}{n!}(x-a)^n + R_N$$

where the remainder R_N is $\frac{S^{(N)}(b)}{N!}(x-a)^N$, for any b between a and x. The Taylor series expansion is convergent only if $R_N \underset{N\to\infty}{\longrightarrow} 0$.

For $a = 0$,

$$S(x) = \sum_{n=0}^{\infty} \frac{S^{(n)}(0)}{n!} x^n \quad , \qquad Eq.(3.8b)$$

which is purely a power series expansion of $S(x)$ and is called the Maclaurin series for $S(x)$.

Example: Determine the Maclaurin series expansion of $S(x) = \exp x$.

Solution: In this case, we note that $S^{(n)}(x) = \exp x$ and $S^{(n)}(0) = 1$ for all n. The Maclaurin series becomes

$$\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad Eq.(3.9)$$

The convergence of the series can be checked by the ratio test since $\frac{a_{n+1}}{a_n} = \frac{x}{n+1} \to 0$ as $n \to \infty$ and x is finite. It is worth commenting that we can make expansion of $\exp x$ about any point (other than zero) using Taylor series.

Example: Determine Maclaurin series expansions of $\sin x$ and $\cos x$.

Solution: In this case,

$$f(x) = \sin x\,,\ f(0) = 0 \ ; \qquad g(x) = \cos x\,,\ g(0) = 1\ ,$$

$$f^{(1)}(x) = \cos x\,,\ f^{(1)}(0) = 1 \ ; \qquad g^{(1)}(x) = -\sin x\,,\ g^{(1)}(0) = 0\ ,$$

$$f^{(2)}(x) = -\sin x\,,\ f^{(2)}(0) = 0 \ ; \qquad g^{(2)}(x) = -\cos x\,,\ g^{(2)}(0) = -1\ ,$$

$$f^{(3)}(x) = -\cos x\,,\ f^{(3)}(0) = -1 \ ; \qquad g^{(3)}(x) = \sin x\,,\ g^{(3)}(0) = 0\ ,$$

$$f^{(4)}(x) = \sin x\,,\ f^{(4)}(0) = 0 \ ; \qquad g^{(4)}(x) = \cos x\,, g^{(4)}(0) = 1\ .$$

Thus, in general, for $n = 0,1,2 \cdots$,

$$f^{(2n)}(0) = 0,\ f^{(2n+1)}(0) = (-1)^n \ ; \qquad g^{(2n)}(0) = (-1)^n, g^{(2n+1)}(0) = 0\ .$$

Use these values in

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n$$

to obtain

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \ , \qquad Eq. (3.10)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \ . \qquad Eq. (3.11)$$

Example: Determine Maclaurin series expansion of $S(x) = \ln(1 + x)$.

Solution: In this case, $S(0) = 0$ and

$$S^{(1)}(x) = (1+x)^{-1} \qquad S^{(1)}(0) = 1$$

$$S^{(2)}(x) = -(1+x)^{-2} \qquad S^{(2)}(0) = -1$$

$$S^{(3)}(x) = 2(1+x)^{-3} \qquad S^{(3)}(0) = 2!$$

$$S^{(4)}(x) = -3!\,(1+x)^{-4} \qquad S^{(4)}(0) = -3!$$

$$\vdots$$

$$S^{(n)}(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n} \quad n \geq 1 \qquad S^{(n)}(0) = (-1)^{n-1}(n-1)! \quad n \geq 1 \ .$$

Thus, the series for $\ln(1 + x)$ is

$$S(x) = \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad Eq. (3.12)$$

Note in passing that $S(1) = \ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ as mentioned earlier in this chapter.

Example: Determine Maclaurin series expansion of a binomial function (Binomial Theorem), namely

$$\boldsymbol{f(x) = (1 + x)^m}$$

where m is a real number (positive or negative; integer or non-integer), and $|x| < 1$.

Solution: In this case, various derivatives of $f(x)$ are:

$$f^{(1)}(x) = m(1+x)^{m-1}$$

$$f^{(2)}(x) = m(m-1)(1+x)^{m-2}$$

$$f^{(3)}(x) = m(m-1)(m-2)(1+x)^{m-3} \vdots$$

$$f^{(n)}(x) = m(m-1)(m-2)\dots(m-n+1)(1+x)^{m-n} \ .$$

On setting $x = 0$, we get

$$f^{(n)}(0) = m(m-1)\dots(m-n+1) \ .$$

Thus, the Maclaurin series for the binomial function is

$$f(x) = (1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots \qquad Eq.(3.13)$$

As a particular case, if m is a positive integer, say N, then Nth derivative of $f(x)$ is

$$f^{(N)}(x) = N(N-1)(N-2)\dots(1) = N! \ ,$$

which is independent of x. So, all higher derivatives starting from $f^{(N+1)}(x)$ onwards are zero. In this case, the infinite series expansion of the binomial function $(1+x)^N$ reduces to a finite series. For $N = 2,3,4 \dots$ we have the well-known expansions

$$(1+x)^2 = 1 + 2x + x^2 \ ,$$

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3 \ ,$$

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4 \ ,$$

etc.

Additional Examples

If a series of functions is a uniformly convergent series, then it can be differentiated and/or integrated to reduce an unfamiliar series to a familiar one. A few examples of this trick are discussed here.

Example 1: Consider the series $S(x)$,

$$S(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

where the range of convergence of this series is known to be $-1 < x < +1$. Integrate the series term-by-term to obtain a geometric series,

$$\int_0^x S(x)dx = x + x^2 + x^3 + x^4 + \cdots = \frac{x}{1-x} \ .$$

Now differentiate both sides of this expression to get

$$S(x) = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{1}{1-x} + \frac{x}{(1-x)^2} = \frac{1}{(1-x)^2} \ .$$

Example 2: Consider another series $S(x)$,

$$S(x) = \frac{1}{1.2} + \frac{x}{2.3} + \frac{x^2}{3.4} + \frac{x^3}{4.5} + \cdots$$

then,

$$x^2 S(x) = \frac{x^2}{1.2} + \frac{x^3}{2.3} + \frac{x^4}{3.4} + \frac{x^5}{4.5} + \cdots$$

On taking the second derivative of this series, it reduces to a geometric series, which can be easily summed as

$$\frac{d^2}{dx^2}[x^2 S(x)] = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \ .$$

Now, we integrate this expression twice to get $S(x)$. The first integration gives

$$\frac{d}{dx}\left(x^2 S(x)\right) = -\ln(1-x) + C_1 \ ,$$

and the second integration gives

$$x^2 S(x) = -\int \ln(1-x)\, dx + C_1 x + C_2 \ .$$

Here C_1 and C_2 are two constants of integration. The indefinite integral here can be evaluated using the substitution, $y = \ln(1-x)$, or equivalently, $x = 1 - \exp y$. Then,

$$\int \ln(1-x)\, dx = \int y[-\exp y] dy = -y \exp y + \int \exp y\, dy = (1-y)\exp y = (1-x)[1 - \ln(1-x)] \ .$$

So,

$$x^2 S(x) = -(1-x)[1 - \ln(1-x)] + C_1\, x + C_2 \ .$$

To evaluate the two constants of integration, we set $x = 0$ and recall that

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} \dots \ ,$$

to obtain

$$C_2 = 1 \ .$$

Substitute this value of C_2 in the above expression for $S(x)$ and then divide both sides by x to get

$$xS(x) = 1 + \frac{(1-x)\ln(1-x)}{x} + C_1 \ .$$

Again, set $x = 0$ to obtain $C_1 = 0$. So, finally, the value of the original series $S(x)$ is

$$S(x) = \frac{1}{x} + \frac{1-x}{x^2}\ln(1-x) \ .$$

In some cases, even if the series does not contain a variable, one can still use the trick of differentiation and/or integration by introducing a variable judiciously.

Example 3: Consider the series

$$S = \tfrac{1}{2!} + \tfrac{2}{3!} + \tfrac{3}{4!} + \cdots + \tfrac{n-1}{n!} + \cdots \ .$$

We used this as an example of a converging series earlier; the value of this series was quoted as 1. Let us define

$$f(x) = \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{3x^4}{4!} + \frac{4x^5}{5!} + \cdots + \frac{(n-1)x^n}{n!} + \cdots$$

Clearly, $S = f(1)$ and $f(0) = 0$. Now,

$$\frac{df}{dx} = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots + \frac{x^{n-1}}{(n-2)!} + \cdots$$

$$= x\left\{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^{n-2}}{(n-2)!} + \cdots\right\} = x \exp x \ .$$

On integrating both sides, we get

$$\int_0^x \frac{df}{dx} dx = f(x) - f(0) = \int_0^x x \exp x \, dx \ .$$

Or

$$f(x) = x \exp x \,|_0^x - \int_0^x \exp x \, dx = x \exp x - \exp x + 1 = 1 + (x-1)\exp x \ .$$

Then, the value of the original series of constants is $S = f(1) = 1$.

PROBLEMS FOR CHAPTER 3

1. Find the value of α ($\alpha > 0$) if

$$\sum_{n=0}^{\infty} \frac{1}{(1+\alpha)^n} = \frac{(1+\alpha)^2}{2} \ .$$

2. Use the comparison test to check the convergence of following two series.

$$\sum_{n=1}^{\infty} \frac{5^n}{3^{2n}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \ .$$

3. Use the Cauchy and D'Alembert ratio test to check the convergence of following two series [Useful information, $e = 2.72$].

$$\sum_{n=1}^{\infty} n \exp(-n) \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{10^n}{(n!)^2} \ .$$

4. Use the Cauchy and Maclaurin integral test to check the convergence of following two series.

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n}{1+n^2} \ .$$

5. Show that the first five terms in the Taylor series expansion of $\exp(-x)\cos x$ about $x = 0$ are

$$\exp(-x)\cos x = 1 - x + \frac{x^3}{3} - \frac{x^4}{6} + \frac{x^5}{30} \dots \ .$$

6. Expand the integrand below using binomial theorem

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \int_0^x \frac{du}{1-u^2} \ ,$$

and integrate term by term to obtain the power series expansion of

$$\ln\left(\frac{1+x}{1-x}\right) \quad \text{for} \quad |x| < 1 \ .$$

7. By comparing with other well-known series, evaluate the sum of the following series exactly

$$1 + \ln 3 + \frac{(\ln 3)^2}{2!} + \frac{(\ln 3)^3}{3!} + \cdots$$

8. The classical expression for the kinetic energy of a particle is $KE = \frac{1}{2} m_o v^2$ where m_o is the mass of the particle when it is at rest and v is the speed of the particle. When the particle is moving with speed comparable

to c, where c is speed of light, the motion of the particle is correctly described by the theory of relativity. According to this theory, the mass of a particle moving with speed v is

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} \ ,$$

and the kinetic energy of the particle is the difference between its total energy mc^2 and its energy at rest $m_o c^2$, that is, $KE = mc^2 - \ m_o c^2$.

(a) Using binomial expansion show that when v is very small compared to c, this expression for the kinetic energy of particle agrees with the classical expression.

(b) The leading relativistic correction term for the classical expression for the kinetic energy is of the form $\alpha \ m_o \frac{v^4}{c^2}$. What is the value of α?

9. **Biomedical Physics Application.** In the thirteenth century, Italian mathematician Fibonacci investigated the growth of rabbit population by assuming,

(a) rabbits never die,

(b) rabbits can mate at the age of one month with the gestation period of one month,

(c) the mating pair produces a new pair every month starting at the age of two months.

Starting with a newly born pair of rabbits, what is the number of rabbit pairs at the beginning of *n*th month? Here is the accounting:

At the beginning of first month, there is only one original pair, $a_1 = 1$.

At the beginning of second month, the pair mates, but there is still only one pair, $a_2 = 1$.

At the beginning of third month, there is original pair and a newly born pair, $a_3 = 2$.

At the beginning of fourth month, original pair produces a new pair and the one-month-old pair mates, $a_4 = 3$.

At the beginning of fifth month, original pair and two-month-old pair reproduce while one-month-old pair mates, $a_5 = 5$, etc.

At the beginning of *n*th month, number of pairs, a_n, equals the sum of newly born pairs [which is equal to the number of pairs at the beginning of (*n-2*)th month, a_{n-2}] plus the number of pairs living at the beginning of (*n-1*)th month, a_{n-1}. Explicitly, the first few terms in the Fibonacci series look like

1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 +… ,

with the recursion relation $a_n = a_{n-1} + a_{n-2}$.

(i) If $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lambda$, then show that $\lambda = \frac{1+\sqrt{5}}{2}$.

(ii) Prove the following five relationships for elements of the Fibonacci series,

$$\frac{1}{a_{n-1}a_{n+1}} = \frac{1}{a_{n-1}a_n} - \frac{1}{a_n a_{n+1}} \quad ,$$

$$\sum_{i=0}^{n-1} a_{2i+1} = a_{2n} \quad ,$$

$$\sum_{i=1}^{n} a_{2i} = a_{2n+1} - 1 \quad ,$$

$$\sum_{i=1}^{n} a_i = a_{n+2} - 1 \quad ,$$

$$\sum_{i=1}^{n} a_i^2 = a_n a_{n+1} \quad .$$

The Fibonacci series appears in several biological settings, such as number of leaves on a stem, number of branches in a tree as a function of height, designs of seashell or pinecone or artichoke or pineapple fruitlets, seeds in a sunflower, family tree of honeybees, etc.

Interlude

In this short interlude we introduce a few mathematical bits and pieces which will be helpful in our future journey of mathematics. We will introduce a unit imaginary number i, Kronecker Delta δ_{ij}, Dirac Delta function $\delta(x)$, Levi-Civita symbol ϵ_{ijk} and Euler's formula.

Unit Imaginary Number: $\boldsymbol{i = \sqrt{-1}}$

The square root of some positive numbers is easy to figure out and remember, such as $\sqrt{4} = 2, \sqrt{25} = 5,$ etc. For other positive numbers, we may need a calculator, such as $\sqrt{3} = 1.732, \sqrt{7} = 2.646,$ etc. But, our beloved calculator is not very helpful when we attempt to find out the square root of some negative numbers. For example, what is $\sqrt{-4}$ or $\sqrt{-7}$? For this purpose, we define the square root of -1 as a unit imaginary number, i. In terms of i we can write $\sqrt{-4} = \sqrt{(-1)(4)} = 2\,i$ or $\sqrt{-7} = \sqrt{(-1)(7)} = 2.646\,i$. Various powers of i are $i^2 = -1, i^3 = i^2 i = -i, i^4 = i^2 i^2 = 1$. Using the fact that $i^4 = 1$, we can reduce i^n, where n is a large number, to a much simpler form. As an example, $i^{65} = i^{16(4)+1} = i$ or $i^{135} = i^{33(4)+3} = i^3 = -i$ or $i^{206} = i^{51(4)+2} = i^2 = -1$, etc.

Kronecker Delta: $\boldsymbol{\delta_{ij}}$

Kronecker Delta is a symbol with two indices, i and j, and it can take two possible values, 0 and 1, depending on the values of the indices. The two indices, i and j, themselves can assume all possible discrete integer values from $-\infty$ to $+\infty$. Specifically,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} . \qquad Eq.(I.1a)$$

Using this definition of Kronecker Delta, we can write

$$\sum_{i=-\infty}^{\infty} f(x_i)\,\delta_{ij} = f(x_j) \ . \qquad Eq.(I.1b)$$

Dirac Delta Function

In the definition of the Kronecker Delta above, its two indices, i and j, can assume all possible $discrete$ integer values from $-\infty$ to $+\infty$. Its analog in which these two indices can take all possible $continuous$ values is referred to as Dirac Delta function. To make this connection more understandable, we rewrite the Kronecker Delta in an alternate notation as

$$\delta(i-j) = 0 \;\; if \;\; i-j \neq 0 \quad , \quad [i,j = \cdots -2,-1,0,+1,+2 \cdots] \qquad Eq.(I.2a)$$

along with

$$\sum_{i=-\infty}^{\infty} f(x_i)\,\delta(i-j) = f(x_j) \;. \qquad Eq.(I.2b)$$

Now, if δ is generalized from discrete variables i and j to all possible continuous values, x and y, then the Dirac Delta function should have the properties:

$$\delta(x-y) = 0 \;\; if \; x-y \neq 0 \qquad \text{or simply} \qquad \delta(x) = 0 \;\; if \; x \neq 0 \;, \qquad Eq.(I.3a)$$

as well as

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-y)\,dx = f(y) \qquad \text{or simply} \qquad \int_{-\infty}^{\infty} \delta(x)\,dx = 1 \;. \qquad Eq.(I.3b)$$

The Eqs. ($I.3a$ and $I.3b$) imply that $\delta(x)$ cannot be zero for $x = 0$ because then the integral (which represents the area under the $\delta(x)$ versus x curve) will be zero, not 1. In order to understand the value of $\delta(0)$, consider a step function $F(x)$, of width 2ϵ around $x = 0$ and height $1/(2\epsilon)$,

$$F(x) = \begin{cases} 0 \;\; for & |x| > \epsilon \\ 1/(2\epsilon) \;\; for & \epsilon \geq x \geq -\epsilon \end{cases} \;.$$

The area under this step function (or, the integral of the function $F(x)$ over x from $-\infty$ to $+\infty$) is 1. On taking the limit $\epsilon \to 0$ in $F(x)$, the step function will have value 0 for $x \neq 0$, value ∞ for $x = 0$ and will preserve the area under the $F(x)$ versus x curve to be 1. Thus, our functional definition of the Dirac Delta function, $\delta(x)$, is any function which satisfies the following three conditions:

$$\delta(x) = 0 \;\; if \;\; x \neq 0, \;\; \delta(x) = \infty \;\; if \;\; x = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)\,dx = 1 \;. \qquad Eq.(I.4)$$

It is easy to show that the following limiting relationships, as $\epsilon \to 0$, satisfy all three defining conditions of the Dirac Delta function and, therefore, can be taken as definitions of the Delta function itself:

$$\delta(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\epsilon}{x^2 + \epsilon^2} \;, \qquad Eq.(I.5a)$$

$$\delta(x) = \frac{1}{\sqrt{\pi}} \lim_{\epsilon \to 0} \frac{1}{\epsilon} exp(-x^2/\epsilon^2) \;, \qquad Eq.(I.5b)$$

$$\delta(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{1}{x} sin\left(\frac{x}{\epsilon}\right) \;. \qquad Eq.(I.5c)$$

There are two comments worth mentioning here. First, unlike the Kronecker Delta which is a dimensionless number, the Dirac Delta function $\delta(x)$ has dimensions of $1/x$ for the integral of $Eq. (I.4)$ to be meaningful. Second, the Dirac Delta function defined in Eq. (I.4) is a one-dimensional Delta function which can be generalized to two-dimensional and three-dimensional Dirac Delta functions.

Levi-Civita symbol: ϵ_{ijk}

Analogous to a Kronecker delta, one can define the Levi-Civita symbol ϵ_{ijk} with three indices (ijk). The three indices (ijk) can take only three values $(1 \text{ or } 2 \text{ or } 3)$ each. The Levi-Civita symbol itself can have one of the three possible values, +1 or 0 or -1. Specifically,

$$\epsilon_{ijk} = \begin{cases} 0 \text{ if any two indices are equal} \\ +1 \text{ if an even exchange of } (ijk) \text{ gives } (123) \\ -1 \text{ if an odd exchange of } (ijk) \text{ gives } (123) \end{cases} \qquad Eq. (I.6)$$

Then, out of 27 (3 times 3 times 3) possible Levi-Civita symbols, only six are nonzero. These are $\epsilon_{123} = +1, \epsilon_{132} = -1, \epsilon_{213} = -1, \epsilon_{231} = +1, \epsilon_{312} = +1, \text{and } \epsilon_{321} = -1$. As we will see later, one can write the cross product of two vectors in a very compact form using the Levi-Civita symbol. Also, the value of a determinant of order 3 can be written using the Levi-Civita symbol.

The Levi-Civita symbol is related to Kronecker delta by

$$\sum_i \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \quad . \qquad Eq. (I.7)$$

A nice mnemonics device to remember the order of indices in this relationship is illustrated in the figure below so that, right-hand-side of the relation is $(first)(second) - (outer)(inner)$.

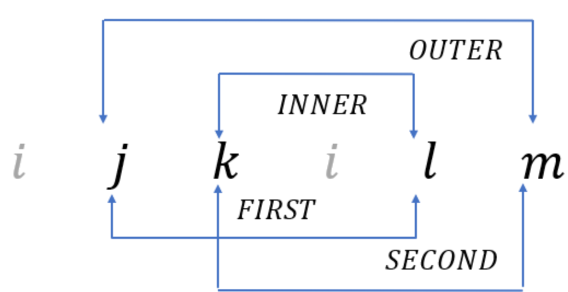

Figure I.1. Mnemonic for the order of indices in the Epsilon-Delta Identity

This relationship between Levi-Civita symbols and Kronecker deltas (called epsilon-delta identity) is very useful in deriving several identities of vector calculus as well as several properties of determinants. A proof of the epsilon-delta identity is given in Appendix C.

Euler's Formula

From Chapter 3, we recall the Maclaurin series expansion of some simple functions, such as:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \ ,$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \ ,$$

$$\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ .$$

Using these expansions, it follows, with $i = \sqrt{-1}$, that

$$\exp(ix) = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0,2,4\ldots} \frac{(ix)^n}{n!} + \sum_{n=1,3,5\ldots} \frac{(ix)^n}{n!} \ .$$

Now, on replacing the dummy index n by $2m$ in the first sum and n by $2m+1$ in the second sum, we get

$$\exp(ix) = \sum_{m=0}^{\infty} \frac{i^{2m} x^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{i^{2m+1} x^{2m+1}}{(2m+1)!}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} \ ,$$

or

$$\exp(ix) = \cos x + i \sin x \ . \qquad Eq. (I.8)$$

This is known as Euler's formula. By many accounts, Euler's formula is the most beautiful equation of mathematics, or the jewel of mathematics. When written in the form $e^{i\pi} + 1 = 0$, it is made up of five different mathematical constants (namely, 0, 1, π, e, and i) each having its own independent value. Euler's formula will be very helpful in our further studies of Fourier series, Fourier transform, and complex variables.

PROBLEMS FOR INTERLUDE

1. Using properties of the Kronecker delta, show that

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij}\, x_i x_j = \sum_{i=1}^{3} x_i^2 \ ,$$

and

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} \ = \sum_{i=1}^{3} \delta_{ii} = 3 \ .$$

2. Using properties of the Levi-Civita symbol, show that

$$\sum_{ij} \epsilon_{ijk}\, \epsilon_{ijm} = \ 2\delta_{km} \ ,$$

and

$$\sum_{ijk} \epsilon_{ijk}\, \epsilon_{ijk} = \ 6 \ .$$

3. Using Euler's formula, show that

$$\cos x = \frac{\exp(ix) + \exp(-ix)}{2} \ ,$$

$$\sin x = \frac{\exp(ix) - \exp(-ix)}{2i} \ .$$

Chapter 4: Fourier Series

Any periodic function can be expressed as an infinite series of sinusoidal (sine or cosine like) functions. This series is called the Fourier series of the periodic function. For a general nonperiodic function, the extension of the Fourier series leads to the Fourier transform. In this chapter, we will use orthogonality relationships of sinusoidal functions to derive coefficients in a Fourier series. We will also show how the Fourier series becomes the Fourier transform on increasing the periodicity of the periodic function.

4.1 PERIODIC FUNCTIONS

In preparation for our discussion of Fourier series and the Fourier transform, let us first get over some mathematical preliminaries. In our earlier discussion we derived an equation, the wave equation, whose solution describes any periodic function. In particular, we noted that a sinusoidal function

$$f(x,t) = \sin[kx \pm \omega t]$$

satisfies the wave equation with $k = 2\pi/\lambda,$ and $\omega = 2\pi/T$. We also noted that a periodic function that repeats itself both in time (t) as well as in a spatial coordinate (x) is not a function of variables x or t separately but is a function of dimensionless variables $kx \pm \omega t$. In general, a periodic function $f(\theta)$, of variable θ, repeats itself as the variable changes. The "periodicity" of such a repeating function is defined as the range of variable θ over which the function repeats itself. For example, if f is a function of time, it repeats itself after an amount of time called its period, T. If f repeats itself over space, then the length over which the function repeats itself is the wavelength, λ. So, we can define periodicity for these periodic functions as either T or λ. As another example, the periodicity of a simple sinusoidal function, $\sin\theta$ or $\cos\theta$, is 2π. The integral of these functions over a complete range of their periodicity is the same no matter what the starting (lower limit) or ending (upper limit) points of the integral are. Specifically, for $\alpha > 0$, we have

$$\int_{\alpha}^{\alpha+2\pi} \sin\theta\, d\theta = \int_{0}^{2\pi} \sin\theta\, d\theta + \int_{2\pi}^{\alpha+2\pi} \sin\theta\, d\theta - \int_{0}^{\alpha} \sin\theta\, d\theta = \int_{0}^{2\pi} \sin\theta\, d\theta \ ,$$

$$\int_{\alpha}^{\alpha+2\pi} \cos\theta\, d\theta = \int_{0}^{2\pi} \cos\theta\, d\theta + \int_{2\pi}^{\alpha+2\pi} \cos\theta\, d\theta - \int_{0}^{\alpha} \cos\theta\, d\theta = \int_{0}^{2\pi} \cos\theta\, d\theta \ ,$$

since in both cases a cancellation of integrals occurs in the middle step (on replacing the dummy variable θ by another variable ψ using $\theta = 2\pi + \psi$). Recall that a product of any two sinusoidal functions can be written as a linear combination of single sinusoidal functions, namely,

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)] \ ,$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \ ,$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \ .$$

Thus, in general (with p and q integers),

$$\int_{\alpha}^{\alpha+2\pi} \sin(p\theta)\sin(q\theta)\,d\theta = \int_{0}^{2\pi} \sin(p\theta)\sin(q\theta)\,d\theta \ ,$$

$$\int_{\alpha}^{\alpha+2\pi} \cos(p\theta)\cos(q\theta)\,d\theta = \int_{0}^{2\pi} \cos(p\theta)\cos(q\theta)\,d\theta \ ,$$

$$\int_{\alpha}^{\alpha+2\pi} \cos(p\theta)\sin(q\theta)\,d\theta = \int_{0}^{2\pi} \cos(p\theta)\sin(q\theta)\,d\theta \ .$$

Orthogonality Relations

Recall that for an even function of θ, $f(-\theta) = f(\theta)$, and for an odd function, $f(-\theta) = -f(\theta)$. In our discussion of Fourier series and the Fourier transform, we will encounter integrals whose integrands are products of either two sine (ss), two cosine (cc), or a cosine and a sine (cs) functions. The complete range of periodicity of these functions, namely 2π, is taken to be from $-\pi$ to $+\pi$ so that $\sin\theta$ is an odd function and $\cos\theta$ is an even function of θ over this complete range. Assuming that p and q are *nonzero* integers, we have

$$I_{ss} = \int_{-\pi}^{\pi} \sin(p\theta)\sin(q\theta)\,d\theta$$

$$= -\frac{1}{2}\int_{-\pi}^{\pi} d\theta\{\cos[(p+q)\theta] - \cos[(p-q)\theta]\} = -\frac{1}{2}\left\{\frac{\sin(p+q)\theta}{p+q} - \frac{\sin(p-q)\theta}{p-q}\right\}\Bigg|_{-\pi}^{\pi}$$

$$= \begin{cases} 0 & \text{if} \quad p \neq q \\ \pi & \text{if} \quad p = q \end{cases}$$

$$\text{or} \qquad I_{ss} = \pi\delta_{pq} \ . \qquad Eq. (4.1a)$$

Similarly,

$$I_{cc} = \int_{-\pi}^{\pi} \cos(p\theta)\cos(q\theta)\, d\theta$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} d\theta\{\cos[(p+q)\theta] + \cos[(p-q)\theta]\} = \frac{1}{2}\left\{\frac{\sin(p+q)\theta}{p+q} + \frac{\sin(p-q)\theta}{p-q}\right\}_{-\pi}^{\pi}$$

$$= \begin{cases} 0 & \text{if} \quad p \neq q \\ \pi & \text{if} \quad p = q \end{cases}$$

or $\quad I_{cc} = \pi\delta_{pq}$. $\quad$ *Eq.* (4.1*b*)

Finally,

$$I_{cs} = \int_{-\pi}^{\pi} \cos(p\theta)\sin(q\theta)\, d\theta$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} d\theta\{\sin[(p+q)\theta] - \sin[(p-q)\theta]\} = \frac{1}{2}\left\{-\frac{\cos(p+q)\theta}{p+q} + \frac{\cos(p-q)\theta}{p-q}\right\}_{-\pi}^{\pi} \qquad Eq.(4.1c)$$

$= 0$ for all integer values of p and q .

We will refer to the three integrals I_{cc}, I_{ss} and I_{cs} as the orthogonality relationships for the sine and cosine functions. As an aside, we recall that Euler's formula relates sine and cosine functions to exponential functions. Thus, we can write an orthogonality relationship for exponential functions. More specifically,

$$I_{ee} = \int_{-\pi}^{\pi} \exp(ip\theta)\exp(-iq\theta)\, d\theta = \int_{-\pi}^{\pi} [\cos(p\theta) + i\sin(p\theta)][\cos(q\theta) - i\sin(q\theta)]\, d\theta$$

$$= \int_{-\pi}^{\pi} \cos(p\theta)\cos(q\theta)\, d\theta + \int_{-\pi}^{\pi} \sin(p\theta)\sin(q\theta)\, d\theta = \pi\,\delta_{pq} + \pi\,\delta_{pq} = 2\pi\delta_{pq} \qquad Eq.(4.2)$$

Periodic Functions of Spatial Coordinate, x, and Time, t.

If the periodic function under consideration is a function of a spatial coordinate x, with periodicity of λ, then we replace θ in the above examples by kx. Substituting $\theta = kx = 2\pi x/\lambda$, we get

$$I_{ss} = \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \sin(pkx)\sin(qkx)\, dx = \frac{\lambda}{2}\delta_{pq} \quad (p, q \neq 0) \ , \qquad Eq.(4.3a)$$

$$I_{cc} = \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \cos(pkx)\cos(qkx)\,dx = \frac{\lambda}{2}\delta_{pq} \ \ (p, q \neq 0) \ , \qquad Eq. (4.3b)$$

and

$$I_{cs} = \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \cos(pkx)\sin(qkx)\,dx = 0 \ , \qquad Eq. (4.3c)$$

for all integer values of p and q.

If the periodic function under consideration is a function of time t, with periodicity of T, then we replace θ by ωt. Substituting $\theta = \omega t = 2\pi t/T$,

$$I_{ss} = \int_{-\frac{T}{2}}^{\frac{T}{2}} \sin(p\omega t)\sin(q\omega t)\,dt = \frac{T}{2}\delta_{pq} \ \ (p, q \neq 0) \ , \qquad Eq. (4.4a)$$

$$I_{cc} = \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(p\omega t)\cos(q\omega t)\,dt = \frac{T}{2}\delta_{pq} \ \ (p, q \neq 0) \ , \qquad Eq. (4.4b)$$

and

$$I_{cs} = \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(p\omega t)\sin(q\omega t)\,dt = 0 \ \ \text{for all integer values of } p \text{ and } q \ . \qquad Eq. (4.4c)$$

4.2 FOURIER SERIES

Now, look at the three functions of variable θ shown in Figure 4.1. Clearly these functions are periodic functions with periodicity of 2π. The first function $f_1(\theta)$ is an even function of θ since $f_1(\theta) = f_1(-\theta)$. The second function $f_2(\theta)$ is an odd function of θ since $f_2(\theta) = -f_2(-\theta)$. Finally, the third function $f_3(\theta)$ is neither even nor odd since there is no direct relationship between $f_3(\theta)$ and $f_3(-\theta)$. According to Fourier, any arbitrary periodic function $f(\theta)$ can be represented as a linear combination of sinusoidal functions as:

$$f(\theta) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\theta) + \sum_{m=1}^{\infty} b_m \sin(m\theta) \qquad Eq. (4.5)$$

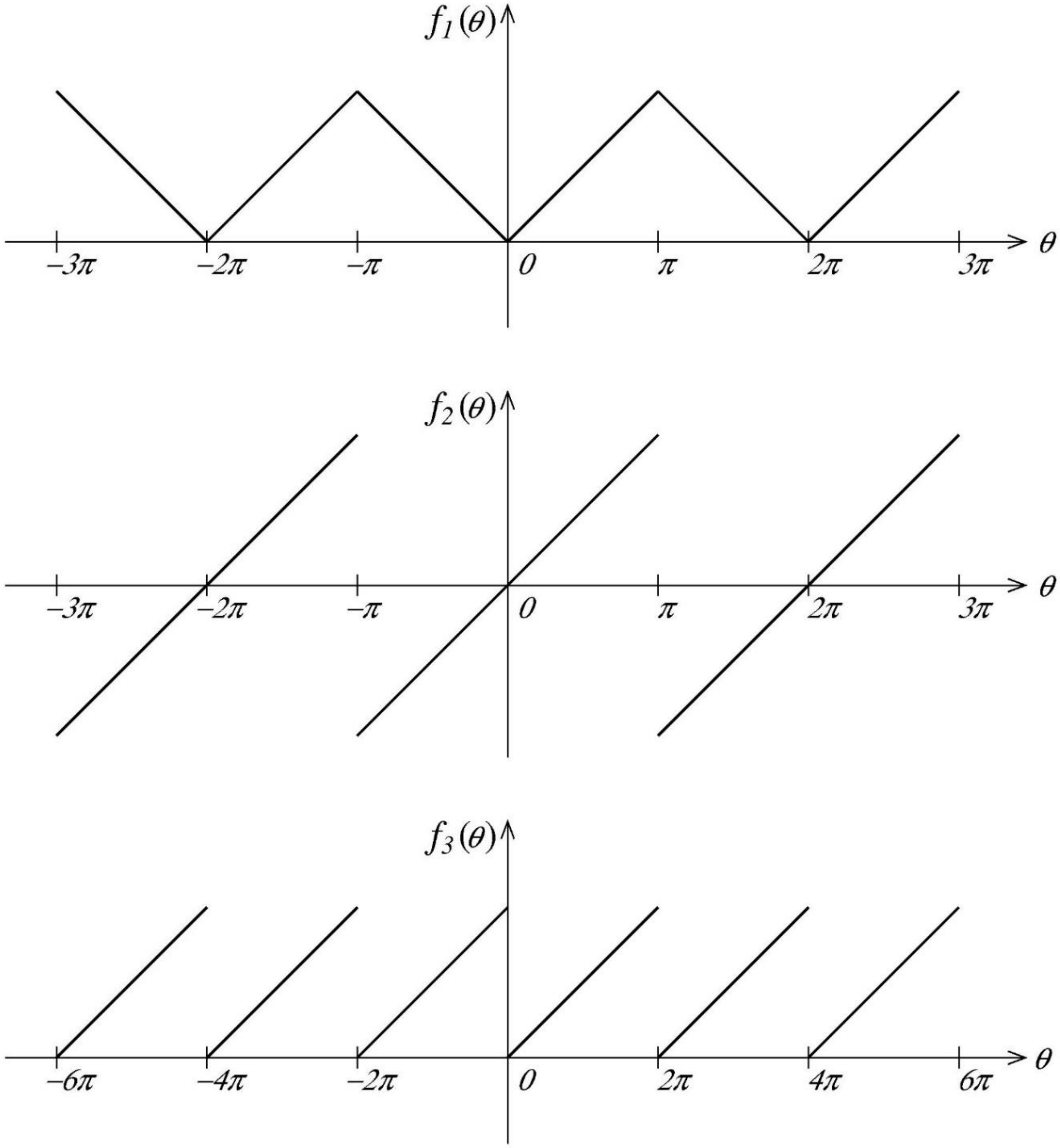

Figure 4.1. Three periodic functions of θ: $f_1(\theta)$ is an even function of θ, $f_2(\theta)$ is an odd function of θ, and $f_3(\theta)$ is neither even nor odd function of θ.

which is known as the Fourier series. Here $a_0, a_m,$ and $b_m$are Fourier coefficients. Note an additional factor of ½ with coefficient a_0. The mystery of this factor will become clear after we evaluate a_0. Given a periodic function $f(\theta)$, we can obtain its Fourier series by simply calculating its Fourier coefficients. To obtain these coefficients, we multiply Eq. (4.5) by either $\cos(p\theta)$ or $\sin(p\theta)$, where p is a nonzero integer, and integrate over θ from $-\pi$ to π to get

$$\int_{-\pi}^{\pi} \cos(p\theta)\, f(\theta) d\theta = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos(p\theta)\, d\theta$$

$$+ \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos(p\theta) \cos(m\theta)\, d\theta + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \cos(p\theta) \sin(m\theta)\, d\theta$$

$$= 0 + \sum_{m=1}^{\infty} a_m\, \pi \delta_{mp} \ + 0 = a_p \pi \ ,$$

or

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(m\theta)\, f(\theta) d\theta \qquad m \geq 1 \ . \qquad Eq.\ (4.6)$$

Similarly, on multiplying Eq. (4.5) by $\sin(p\theta)$ and going through similar steps,

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(m\theta)\, f(\theta) d\theta \qquad m \geq 1 \ . \qquad Eq. (4.7)$$

Finally, integrating Eq. (4.5) directly,

$$\int_{-\pi}^{\pi} f(\theta) d\theta = \frac{a_0}{2} \int_{-\pi}^{\pi} d\theta + 0 + 0 = a_0 \pi$$

or

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) d\theta \qquad , \qquad Eq. (4.8)$$

which is the same as Eq. (4.6) with $m = 0$. The reason for associating an extra factor of ½ with a_0 was to ensure that evaluation of this coefficient using Eq. (4.8) is similar to the evaluation of a_m $(m \geq 1)$ using Eq. (4.6). This completes the evaluation of Fourier coefficients $a_0, a_m,$ and b_m for a general periodic function $f(\theta)$.

Even and Odd Functions

Further simplification of Fourier coefficients occurs if the function $f(\theta)$ is either an even function or an odd function of θ. An even function of variable θ is symmetric about the origin, namely, $f(-\theta) = f(\theta)$, whereas for an odd function, $f(-\theta) = -f(\theta)$. First, consider the Fourier coefficients b_m. Split the integral in Eq. (4.7) into two parts as

$$b_m = \frac{1}{\pi}\int_0^{\pi} \sin(m\theta)\, f(\theta) d\theta + \frac{1}{\pi}\int_{-\pi}^{0} \sin(m\theta)\, f(\theta) d\theta \; = I_1 + I_2 \; .$$

In the second integral, substitute $y = -\theta$ or $\theta = -y$, and note that for an even function

$$I_2 = \frac{1}{\pi}\int_{\pi}^{0} [\sin(-my)] f(-y) d(-y) \; = \frac{1}{\pi}\int_{\pi}^{0} \sin(my)\, f(y) d(y) = -I_1 \; .$$

So, $b_m = 0$ for an even function $f(\theta)$. Similarly, for an odd function, $I_2 = I_1$ so that $b_m \neq 0$ and it is given by

$$b_m = \frac{2}{\pi}\int_0^{\pi} \sin(m\theta)\, f(\theta) d\theta \; .$$

Next, consider the Fourier coefficients a_m. In this case, splitting the integral of Eq. (4.6) into two parts gives

$$a_m = \frac{1}{\pi}\int_0^{\pi} \cos(m\theta)\, f(\theta) d\theta + \frac{1}{\pi}\int_{-\pi}^{0} \cos(m\theta)\, f(\theta) d\theta \; = J_1 + J_2 \; .$$

Again, in the second integral, substitute $y = -\theta$ or $\theta = -y$, and for an odd function, $f(-\theta) = -f(\theta)$, so that

$$J_2 = \frac{1}{\pi}\int_{\pi}^{0} [\cos(-my)] f(-y) d(-y) \; = \frac{1}{\pi}\int_{\pi}^{0} \cos(my)\, f(y) d(y) = -J_1 \; .$$

Thus, for an odd function, $a_m = 0$, including a_0. On the other hand, for an even function, $J_2 = J_1$ so that $a_m \neq 0$ and it is given by

$$a_m = \frac{2}{\pi}\int_0^{\pi} \cos(m\theta)\, f(\theta) d\theta \; .$$

This is really not a very surprising result since it implies that for an odd function $[f(-\theta) = -f(\theta)]$, the Fourier series consists of only odd sine functions:

$$f(\theta) = \sum_{m=1}^{\infty} b_m \sin(m\theta) \quad ,$$

while for an even function $[f(-\theta) = f(\theta)]$, the Fourier series consists of only even cosine functions:

$$f(\theta) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\theta) \ .$$

Here are examples of determining Fourier series for three different periodic functions. The first function is an odd function of θ, the second function is an even function of θ, and the third function is neither an even nor odd function of θ.

Example: Given a periodic function,

$$\boldsymbol{f(\theta) = \theta \ \ \text{for} \ -\pi \le \theta \le \pi} \qquad\qquad [\textbf{Sawtooth function}],$$

determine its Fourier series.

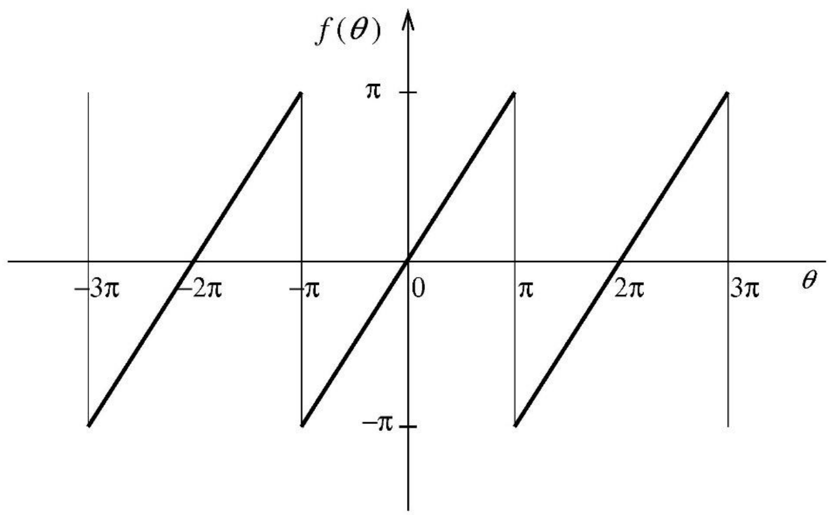

Figure 4.2. The sawtooth function, $f(\theta)$, as a function of θ.

Solution: As seen in Figure 4.2, $f(\theta)$ is an odd function of θ. So, all $a_m = 0$ (including a_0), and

$$f(\theta) = \sum_{m=1}^{\infty} b_m \sin(m\theta)$$

with

$$b_m = \frac{2}{\pi}\int_0^{\pi} \sin(m\theta)\, f(\theta) d\theta = \frac{2}{\pi}\int_0^{\pi} \theta \sin(m\theta)\, d\theta$$

$$= \frac{2}{\pi}\left\{-\frac{\theta\cos(m\theta)}{m}\Bigg|_0^\pi + \int_0^\pi \frac{\cos(m\theta)}{m}\,d\theta\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{\pi}{m}\cos(m\pi) + \frac{\sin(m\theta)}{m^2}\Bigg|_0^\pi\right\}$$

$$= -\frac{2}{m}\cos(m\pi) = (-1)^{m+1}\frac{2}{m} \ .$$

Thus,

$$f(\theta) = \theta = 2\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\sin(m\theta) \ ,$$

$$= 2\left\{\sin(\theta) - \frac{1}{2}\sin(2\theta) + \frac{1}{3}\sin(3\theta) - \frac{1}{4}\sin(4\theta) + \cdots\right\} \ .$$

This is the Fourier series for the sawtooth function. Figure 4.3 shows the sawtooth function as well as the sum of first N terms of the Fourier series (red dotted line) for $N = 2, 5, 10,$ and 100.

Clearly, the depiction of the sawtooth function by the Fourier series becomes more accurate as N increases.

Example: Find the Fourier series for a periodic function $f(\theta)$ which is defined in the interval $-\pi \le \theta \le \pi$ as

$$\boldsymbol{f(\theta) = \theta^2} \ .$$

Solution: Since $f(\theta) = \theta^2$ is an even function of $\theta, b_m = 0$ for $m \geq 1$. Multiply both sides of the above function by $\cos n\theta$ and integrate over θ from $-\pi$ to π to get

$$\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos(n\theta)\,d\theta + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi}\cos(m\theta)\cos(n\theta)\,d\theta$$

$$= 0 + \pi a_n$$

or

$$a_n = \frac{2}{\pi}\int_0^\pi \theta^2\cos(n\theta)\,d\theta = \frac{2\,\theta^2}{\pi}\cdot\frac{\sin(n\theta)}{n}\Bigg|_0^\pi - \frac{2}{\pi}\int_0^\pi 2\theta\cdot\frac{\sin(n\theta)}{n}\,d\theta$$

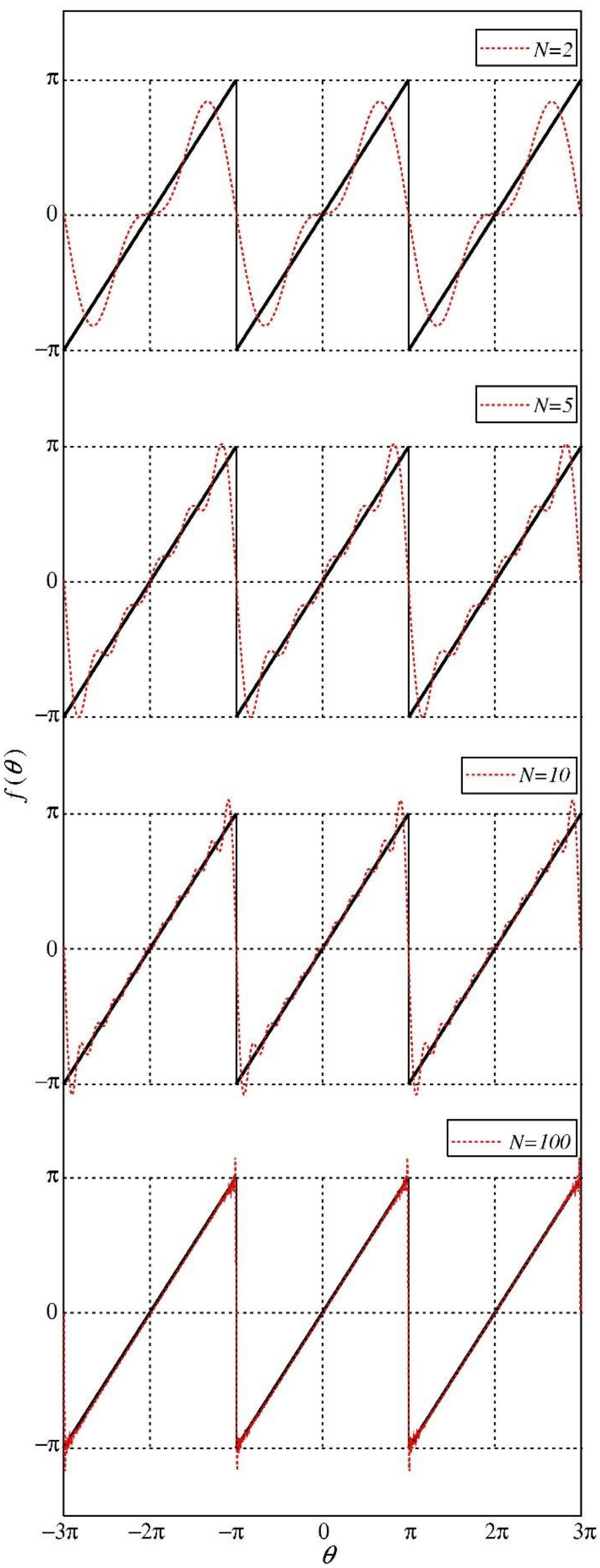

Figure 4.3. The sum of first N terms in the Fourier series of a sawtooth function.

$$= -\frac{4}{n\pi}\int_0^{\pi} \theta \sin(n\theta)\, d\theta = \frac{4}{n\pi}\theta \cdot \frac{\cos(n\theta)}{n}\Bigg|_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi} \frac{\cos(n\theta)}{n} d\theta$$

$$= \frac{4}{\pi n^2}[\pi\cos(n\pi)] - 0 = \frac{4}{n^2}(-1)^n \ .$$

Next, integrate the given function from $-\pi$ to π to get

$$\int_{-\pi}^{\pi} \theta^2 d\theta = \frac{a_0}{2}\int_{-\pi}^{\pi} d\theta + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos(m\theta) d\theta$$

or

$$\frac{\theta^3}{3}\Bigg|_{-\pi}^{\pi} = \frac{a_0}{2}\theta\Bigg|_{-\pi}^{\pi} + 0 \qquad \text{or} \qquad \frac{2}{3}\pi^3 = \frac{a_0}{2}\cdot 2\pi \qquad \text{or} \qquad a_0 = \frac{2}{3}\pi^2 \ .$$

Thus,

$$f(\theta) = \frac{\pi^2}{3} + \sum_{m=1}^{\infty} \frac{4(-1)^m}{m^2}\cos(m\theta) \ .$$

Example: Determine the Fourier series for a unit step function defined as

$$\boldsymbol{f(\theta) = \begin{cases} 0 & \text{for} \quad -\pi < \theta < 0 \\ 1 & \text{for} \quad 0 < \theta < \pi \end{cases}}$$

Solution: The function $f(\theta)$ is neither an odd function nor an even function of θ, so the Fourier series for $f(\theta)$ will consist of all Fourier coefficients, $a_0, a_m,$ and b_m, namely,

$$f(\theta) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\theta) + \sum_{m=1}^{\infty} b_m \sin(m\theta) \quad .$$

We evaluate these coefficients using Eqs. (4.6) to Eq. (4.8),

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) d\theta = \frac{1}{\pi}\int_0^{\pi} (1)\, d\theta = \frac{\pi}{\pi} = 1 \ ,$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(m\theta)\, f(\theta) d\theta = \frac{1}{\pi}\int_0^{\pi} \cos(m\theta)\, d\theta = \frac{1}{\pi}\cdot\frac{\sin(m\theta)}{m}\Bigg|_0^{\pi} = 0 \qquad m \geq 1 \ ,$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(m\theta)\, f(\theta) d\theta = \frac{1}{\pi}\int_{0}^{\pi} \sin(m\theta)\, d\theta = -\frac{1}{\pi}\cdot\frac{\cos(m\theta)}{m}\bigg|_0^{\pi} = \frac{1}{\pi m}[1-(-1)^m] \ ,$$

or,

$$b_m = \begin{cases} 0 & \text{for} \qquad m = 2, 4, 6, 8 \ \ldots \\ \dfrac{2}{\pi m} & \text{for} \qquad m = 1, 3, 5, 7 \ \ldots \end{cases} .$$

So, the Fourier series for $f(\theta)$ is

$$f(\theta) = \frac{1}{2} + \frac{2}{\pi}\left[\frac{\sin(\theta)}{1} + \frac{\sin(3\theta)}{3} + \frac{\sin(5\theta)}{5} + \frac{\sin(7\theta)}{7} + \cdots\right] \quad .$$

Complex Fourier Series

Since Euler's formula relates sine and cosine functions to exponential functions, it is possible to write the Fourier series as a series of exponential functions instead of sine and cosine functions,

$$f(\theta) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \left(\frac{\exp(im\theta) + \exp(-im\theta)}{2}\right) + \sum_{m=1}^{\infty} b_m \left(\frac{\exp(im\theta) - \exp(-im\theta)}{2i}\right)$$

$$= \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(\frac{a_m - ib_m}{2}\right) \exp(im\theta) + \sum_{m=1}^{\infty} \left(\frac{a_m + ib_m}{2}\right) \exp(-im\theta) \ .$$

On changing m to $-m$ in the second sum, we have

$$f(\theta) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(\frac{a_m - ib_m}{2}\right) \exp(im\theta) + \sum_{m=-1}^{-\infty} \left(\frac{a_{-m} + ib_{-m}}{2}\right) \exp(im\theta)$$

Now, we define new coefficients c_m as follows,

$$c_0 = \frac{a_0}{2} \ ,$$

$$c_m = \frac{a_m - ib_m}{2} \text{ for } m = 1, 2, \cdots \infty \ ,$$

and

$$c_m = \frac{a_{-m} + ib_{-m}}{2} \text{ for } m = -1, -2, \cdots - \infty \ .$$

In terms of these new coefficients, the Fourier series becomes

$$f(\theta) = \sum_{m=-\infty}^{\infty} c_m \exp(im\theta) \quad . \qquad Eq. (4.9a)$$

This is the complex (or exponential) form of the Fourier series. The coefficients c_m are obtained by using the orthogonality relation [Eq. 4.2] for exponential functions,

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\psi) \exp(-im\psi)\, d\psi \quad . \qquad Eq. (4.9b)$$

Example: Let us repeat the sawtooth function,

$$\boldsymbol{f(\theta) = \theta \qquad \text{for} \quad -\pi \le \theta \le \pi \ .}$$

using the complex Fourier series.

Solution:

The Fourier coefficients for this complex series are,

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta \exp(-im\theta)\, d\theta = \frac{1}{2\pi} \left\{ \frac{\theta \exp(-im\theta)}{-im} \bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\exp(-im\theta)}{-im} d\theta \right\} for\ m \neq 0 \ ,$$

$$= \frac{1}{2\pi} \left\{ \frac{\theta \exp(-im\theta)}{-im} \bigg|_{-\pi}^{\pi} - \frac{\exp(-im\theta)}{(-im)^2} \bigg|_{-\pi}^{\pi} \right\}$$

$$= \frac{1}{2\pi} \left\{ \frac{\pi \exp(-im\pi)}{-im} - \frac{(-\pi) \exp(im\pi)}{-im} - \frac{\exp(-im\pi)}{(-im)^2} + \frac{\exp(im\pi)}{(-im)^2} \right\}$$

$$= -\frac{1}{im} \cos(m\pi) - \frac{i}{\pi m^2} \sin(m\pi) = -\frac{(-1)^m}{im} - 0 = \frac{i(-1)^m}{m} \ for\ m \neq 0 \ .$$

For $m = 0$,

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta\, d\theta = 0 \ .$$

So, the complex Fourier series looks like

$$f(\theta) = \theta = 0 + \sum_{m=1}^{\infty} \frac{i(-1)^m}{m} \exp(im\theta) + \sum_{m=-1}^{-\infty} \frac{i(-1)^m}{m} \exp(im\theta)$$

On changing m to $-m$ in the second sum, we have

$$f(\theta) = \sum_{m=1}^{\infty} \frac{i(-1)^m}{m} \exp(im\theta) + \sum_{m=1}^{\infty} \frac{i(-1)^{-m}}{-m} \exp(-im\theta)$$

$$= \sum_{m=1}^{\infty} \frac{i(-1)^m}{m} [\exp(im\theta) - \exp(-im\theta)] = 2 \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sin(m\theta) \; ,$$

the same as before.

4.3 FOURIER TRANSFORMS

In our discussion of Fourier series, our motivation was to write any arbitrary periodic function as a series of some well-known periodic functions—namely, sine and cosine functions. A question arises: Can we write something analogous to a Fourier series for an arbitrary function that does not appear to be a periodic function? Let us explore this possibility. Look at an isolated step-like function in space as shown in the Figure 4.4a.

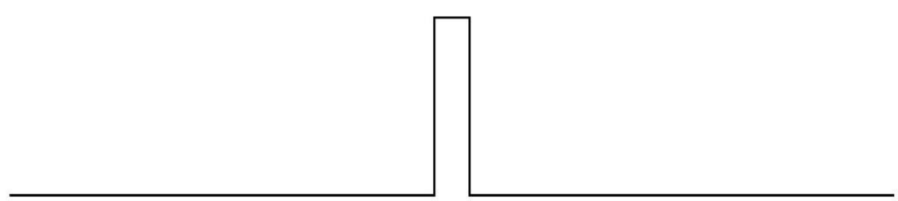

Figure 4.4a. An isolated step-like function in space.

At first sight this function appears as a non-periodic step function in space. Now, imagine starting with a periodic step function, as in part a of Figure 4.4b, of some definite wavelength as indicated. If we were to remove alternate spikes in part a of this figure, we will end up with part b, which still shows up as a periodic function, except with a longer wavelength. We repeat the process of removing alternate spikes in part b and end up with part c, which also shows a periodic function but with a still-longer wavelength. If we continue the process of removing alternate spikes from each new figure ad infinitum, we will end up with the isolated function of Figure 4.4a.

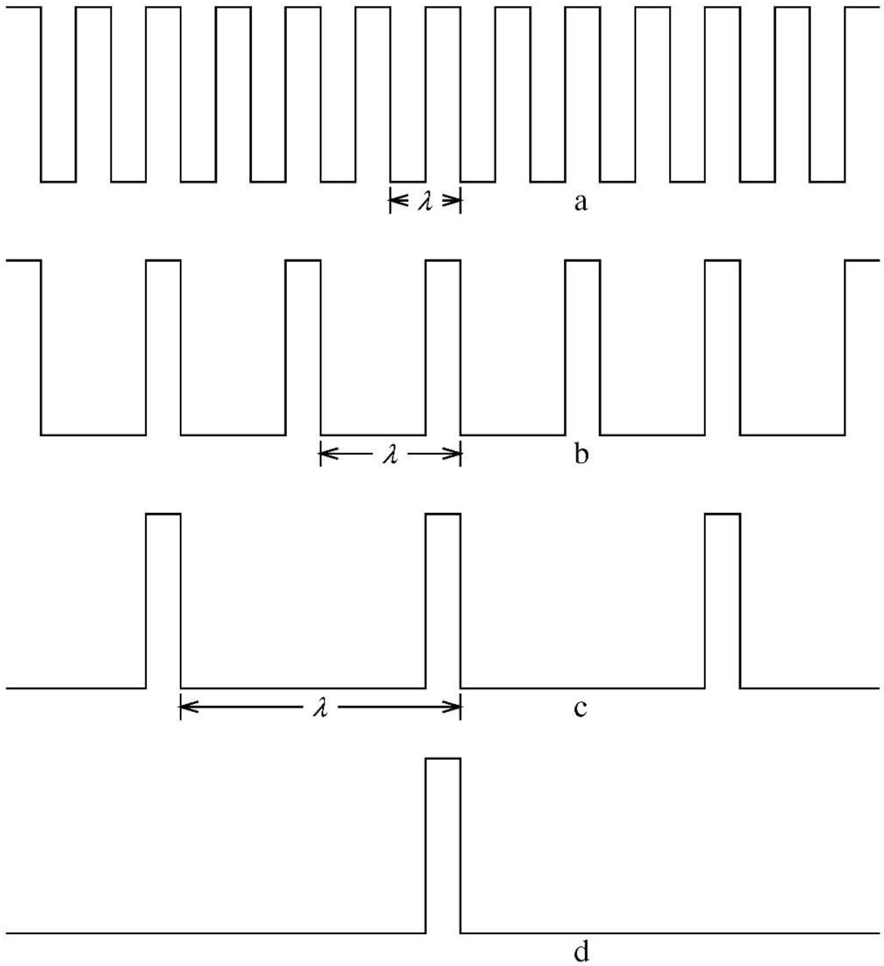

Figure 4.4b. A periodic function with an infinite wavelength appears as an isolated function.

This procedure suggests that an equivalent of the Fourier series of an isolated function, of spatial coordinate x, will be obtained by taking the limit of an infinite wavelength $(\lambda \to \infty)$. Using complex Fourier series, with $\theta = kx$ we have

$$f(x) = \sum_{m=-\infty}^{\infty} c_m \exp(imkx) \ ,$$

where

$$c_m = \frac{1}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(x') \exp(-imkx') \, dx' \ .$$

The dummy variable, m, in the infinite sum above goes from $-\infty$ to $+\infty$ in steps of 1, that is, $\Delta m = 1$. Now we make a change of dummy variable in this infinite sum from m to q, defined as $q = mk = m(2\pi/\lambda)$. Then,

$$\Delta q = \frac{2\pi}{\lambda}\Delta m = \frac{2\pi}{\lambda} \quad \text{or} \quad \frac{1}{\lambda} = \frac{\Delta q}{2\pi} \; .$$

Note that values of q are discrete with step size $2\pi/\lambda$. When $\lambda \to \infty$, the sum over m becomes an integral over q, and $\Delta q \to dq$. Then,

$$f(x) = \lim_{\lambda\to\infty} \int_{m \; or \; q=-\infty}^{\infty} \exp(iqx)\frac{\Delta q}{2\pi} \int_{-\lambda/2}^{\lambda/2} f(x')\exp(-iqx')\,dx' \; ,$$

or

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} f(x')\exp[iq(x-x')]dx' \; .$$

If we define

$$F(q) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')\exp(-iqx')\,dx' \; ,$$

then

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dq\, F(q)\; \exp(iqx) \; .$$

$F(q)$ is called the Fourier transform of $f(x)$ and vice versa. The factor of $1/2\pi$ can be distributed in many ways between *F* and *f*. However, it is a common practice to distribute $1/\sqrt{2\pi}$ with function f and $1/\sqrt{2\pi}$ with function F to preserve symmetry between a function and its Fourier transform.

Let us then summarize the relationship between a function and its Fourier transform. Consider a pair of two physical quantities, u and v, whose product is dimensionless and, therefore, can be measured in radians. Such variables are called conjugate variables. We have seen an example of this in our discussion of the wave equation, namely the pair of wave number k and spatial coordinate x such that the product kx is dimensionless. Another pair consisting of angular frequency ω and time t is such that the product ωt is dimensionless. Since u and v are conjugate variables, the Fourier transform "transforms" the u-dependent function $f(u)$ into a completely equivalent representation $F(v)$, a v-dependent function, in the following way:

$$F(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \exp(-iuv)\, du \ .$$

$F(v)$ is called the Fourier transform of $f(u)$. The inverse Fourier transform creates the opposite transformation, namely,

$$f(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(v) \exp(iuv)\, dv \ .$$

Thus, $f(u)$ is the Fourier transform of $F(v)$. In particular, a function of spatial coordinate x (position in space) is cast into an equivalent function of wavenumber k (momentum space) by the Fourier transformation,

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-ikx)\, dx \ , \qquad Eq.(4.10a)$$

and its inverse,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) \exp(ikx)\, dk \ . \qquad Eq.(4.11a)$$

As an aside, k-space is also called momentum space (or, p-space) since in quantum physics $p = \hbar k$, where $\hbar$ is the universal Planck's constant. Similarly, using the Fourier transformation, a temporal function of time t is cast into an equivalent function of frequency ω as

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \exp(-i\omega t)\, dt \ , \qquad Eq.(4.10b)$$

and its inverse,

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t)\, d\omega \ . \qquad Eq.(4.11b)$$

Example: Determine the Fourier transform of a Gaussian function $f(x)$ given in the coordinate space as

$$\boldsymbol{f(x) = f(0) \exp\left(-\frac{x^2}{a^2}\right) = \frac{1}{a\sqrt{\pi}} exp\left(-\frac{x^2}{a^2}\right)} \ .$$

Solution: The constant in front, namely, $f(0) = 1/(a\sqrt{\pi})$, normalizes the Gaussian function as

$$\int_{-\infty}^{\infty} f(x)dx = 1 \ .$$

The integral here is same as integral I_{gg}^0 of Eq. (2.14). The bell-shaped function $f(x)$ has its maximum value, $f(0)$, for $x = 0$ and it goes to zero as $x \to \pm\infty$. The extent of values of x, namely Δx, over which the function $f(x)$ is appreciable is given by the full width of the function at half of its maximum value (FWHM). Thus,

$$\Delta x = FWHM = 2a\sqrt{ln\,2} \ . \qquad Eq.(4.12a)$$

In other words, the parameter a in the Gaussian function is a measure of the approximate width of the function; the larger the parameter a is, the wider the Gaussian function, and vice versa. Figure 4.5 shows the function $f(x)$ for $a = 1, 2, \text{and } 3$.

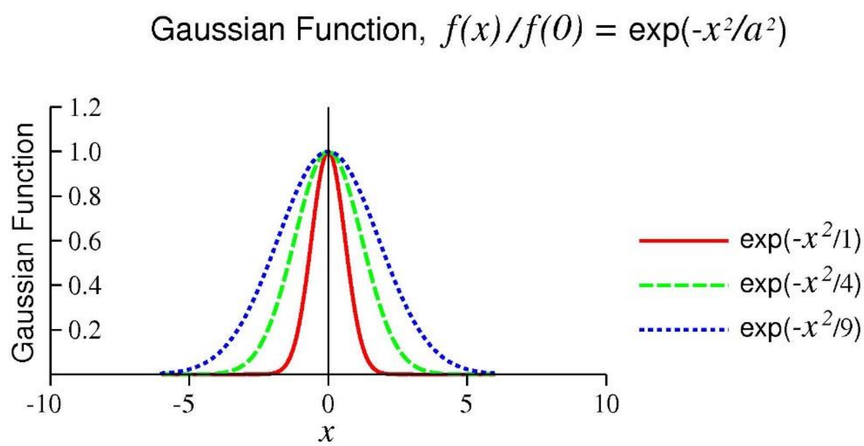

Figure 4.5. The Gaussian function $f(x)/f(0) = \ exp(-x^2/a^2)$ for various values of a.

The Fourier transform of $f(x)/f(0)$ is

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{a^2}\right] \exp[-ikx]\, dx \ .$$

On perfecting the square in the exponent, we get

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{a^2}\left(x^2 + ika^2x + \frac{(ika^2)^2}{4}\right) + \frac{(ika^2)^2}{4a^2}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{k^2a^2}{4}\right] \int_{-\infty}^{\infty} \exp\left[-\frac{(x + ika^2/2)^2}{a^2}\right] dx$$

On making a change of variable from x to u, using $u = \frac{(x+ika^2/2)}{a}$,

$$F(k) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{k^2a^2}{4}\right] \int_{-\infty}^{\infty} \exp(-u^2)\, adu \ ,$$

$$= \frac{a}{\sqrt{2}} \exp\left[-\frac{k^2a^2}{4}\right] \ ,$$

on using the integral I_{gg}^0 of Eq. (2.14). We note that the Fourier transform $F(k)$ of a Gaussian function in the coordinate space is also a Gaussian function, but in the k-space. The extent of values of k, namely Δk, over which the function $F(k)$ is appreciable is again given by the full width of this function at half of its maximum value (FWHM). Thus,

$$\Delta k = FWHM = \frac{4}{a}\sqrt{ln\, 2} \ . \qquad Eq. (4.12b)$$

Now the parameter a has an inverse relationship to the width of the function $F(k)$; the larger the parameter a is, the thinner the function $F(k)$. Note, the product of Δx and of Δk in Eq. (4.12) is independent of a. This essentially is the statement of the Uncertainty Principle in quantum physics.

Example: Determine the Fourier transform of a single rectangular pulse in the coordinate space looking like,

$$\boldsymbol{f(x)} = \begin{cases} \boldsymbol{\frac{1}{(2a)}} & \boldsymbol{-a \le x \le a} \\ \boldsymbol{0}, & \boldsymbol{|x| \ge a} \end{cases} .$$

Solution: For this function,

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{2a} \int_{-a}^{a} dx = 1 \ .$$

Furthermore, the extent of the values of x over which the function is appreciable is,

$$\Delta x = 2a \ . \qquad Eq. (4.13a)$$

The Fourier transform of the single rectangular pulse is

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-ikx)\, dx = \frac{1}{\sqrt{2\pi}} \frac{1}{2a} \int_{-a}^{a} \exp(-ikx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2a} \frac{\exp(-ikx)}{-ik} \Big|_{-a}^{a} = -\frac{1}{2aik} \frac{1}{\sqrt{2\pi}} [\exp(-ika) - \exp(ika)]$$

$$= -\frac{1}{2aik} \frac{1}{\sqrt{2\pi}} [-2i \sin(ka)] = \frac{1}{\sqrt{2\pi}} \frac{\sin(ka)}{ka} \ .$$

Figure 4.6 shows $F(k)$ as a function of the variable k. It is a wiggly function with the central wiggle being the largest one, extending from $k = -\pi/a$ to $+\pi/a$.

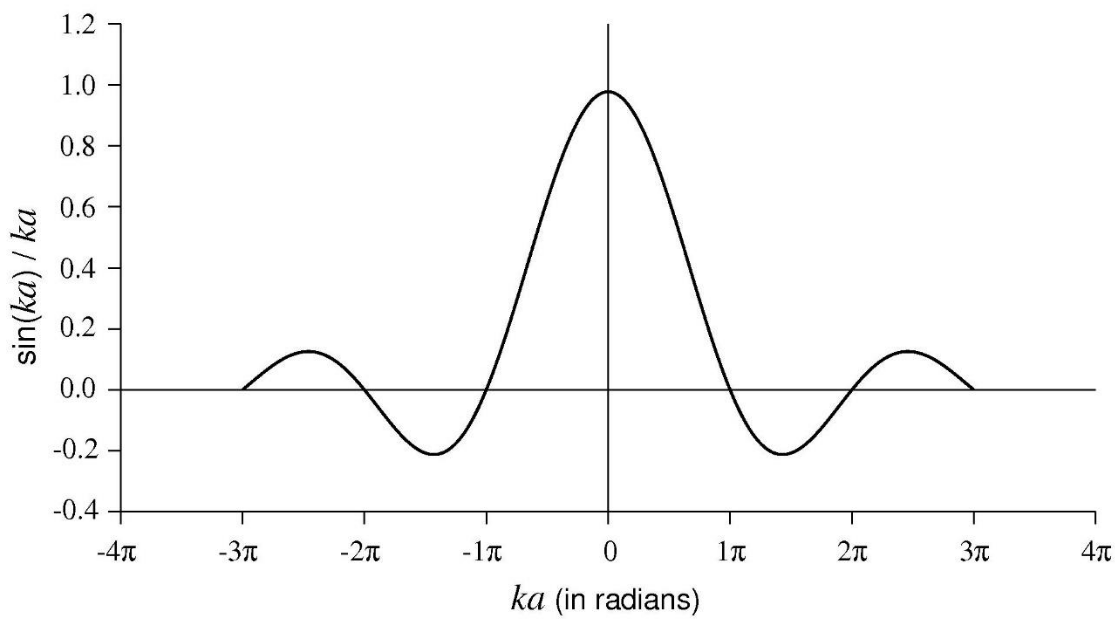

Figure 4.6. The function $\sin(ka)/(ka)$ as a function of ka.

Thus, the extent of the values of k over which $F(k)$ is appreciable is,

$$\Delta k = \frac{2\pi}{a} \ . \qquad Eq. (4.13b)$$

Once again, the product of Δx and of Δk, in Eqs. (4.13), is independent of a, which confirms the statement of the Uncertainty Principle in quantum physics.

From examples of Fourier transforms, it can be concluded that if the extent of a function is very wide, then the extent of its Fourier transform is very narrow and vice versa. This fact is very useful in defining the reciprocal lattices in condensed matter physics.

PROBLEMS FOR CHAPTER 4

1. Show explicitly, for p and q as positive integers and α as an arbitrary constant,

$$\int_{\alpha}^{\alpha+2\pi} \sin(p\theta)\sin(q\theta)\ d\theta = \int_{0}^{2\pi} \sin(p\theta)\sin(q\theta)\ d\theta\ ,$$

$$\int_{\alpha}^{\alpha+2\pi} \cos(p\theta)\cos(q\theta)\ d\theta = \int_{0}^{2\pi} \cos(p\theta)\cos(q\theta)\ d\theta\ ,$$

$$\int_{\alpha}^{\alpha+2\pi} \sin(p\theta)\cos(q\theta)\ d\theta = \int_{0}^{2\pi} \sin(p\theta)\cos(q\theta)\ d\theta\ .$$

2. Find the Fourier series for a periodic function $f(\theta)$ which is defined in the interval $-\pi \le \theta \le \pi$ as

$$f(\theta) = \begin{cases} \pi + \theta & \text{for} \quad -\pi \le \theta \le 0 \\ \pi - \theta & \text{for} \quad 0 \le \theta \le \pi \end{cases} \quad .$$

3. Determine the Fourier series for the “square” wave defined as

$$f(x) = \begin{cases} -1 \ \text{ for } -\dfrac{\lambda}{2} < x < 0 \\ \ \ 1 \ \text{ for } 0 < x < \dfrac{\lambda}{2} \end{cases} \quad .$$

This series appears in discussions of high frequency electronic circuits.

4. Determine the Fourier series for the function defined as

$$f(t) = |\sin \omega t| \ \text{ for } -\pi < \omega t < \pi\ .$$

This series appears in discussions of the full-wave rectifiers in electronics.

5. Determine the Fourier series for the function defined as

$$f(t) = \begin{cases} \sin \omega t & \text{for} \quad 0 < \omega t < \pi \\ 0 & \text{for} \quad -\pi < \omega t < 0 \end{cases}$$

This series appears in discussions of the half-wave rectifiers in electronics.

6. Consider a periodic function $f(\theta)$ which is defined in the interval $-\pi \le \theta \le \pi$ as

$$f(\theta) = \begin{cases} +\cos\theta & \text{for} \quad 0 \le \theta \le \pi \\ -\cos\theta & \text{for} \quad -\pi \le \theta \le 0 \end{cases}$$

(a) Is the function $f(\theta)$ an even function or an odd function or neither?

(b) Determine the Fourier series for $f(\theta)$.

7. Find the Fourier series for a periodic function $f(\theta)$ which is defined as

$$f(\theta) = |\theta| \quad \text{for} \quad -\pi \leq \theta \leq \pi \ .$$

8. Show that the Fourier transform of the function

$$f(x) = \begin{cases} \cos\left(\frac{\pi x}{2a}\right) & \text{for } -a \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$

(where a is a constant) is

$$F(k) = \frac{a}{\sqrt{2\pi}} \frac{\pi}{\left(\frac{\pi}{2}\right)^2 - (ka)^2} \cos(ka) \ .$$

Plot $f(x)$ as a function of x and $F(k)$ as a function of k. Using these plots, estimate the extents Δx and Δk of the functions $f(x)$ and $F(k)$, respectively. Show that Δx times Δk is independent of a.

{Hint: Use $\cos x = [\exp(ix) + \exp(-ix)]/2.$ }

9. Show that the Fourier transform of the parabolic function

$$f(x) = \begin{cases} a^2 - x^2 & \text{for } |x| \leq a \\ 0 & \text{for } |x| > a \end{cases}$$

is

$$F(k) = \frac{a^3}{\sqrt{2\pi}} \left\{ \frac{4}{(ka)^3} [\sin(ka) - (ka)\cos(ka)] \right\} \ .$$

Plot $f(x)$ as a function of x and $F(k)$ as a function of k. Using these plots, estimate the extents Δx and Δk of the functions $f(x)$ and $F(k)$, respectively. Show that Δx times Δk is independent of a.

10. Show that the Fourier transform of the triangular pulse function,

$$f(x) = \begin{cases} 1 - |x|/a & \text{for } |x| \leq a \\ 0 & \text{for } |x| > a \end{cases}$$

(where a is a positive constant) is

$$F(k) = \frac{a}{\sqrt{2\pi}} \left(\frac{\sin(ka/2)}{ka/2} \right)^2 \ .$$

Plot $f(x)$ as a function of x and $F(k)$ as a function of k. Using these plots, estimate the extents Δx and Δk of the functions $f(x)$ and $F(k)$, respectively. Show that Δx times Δk is independent of a.

Chapter 5: Complex Variables

In this chapter we will describe algebra related to complex numbers. Representations of complex numbers in two-dimensional planes will be explained with several examples. Algebraic properties of addition, subtraction, multiplication, and division of complex variables will be outlined. Finally, DeMoivre's formula will be derived and applied to obtain some useful trigonometric identities.

5.1 COMPLEX NUMBERS AND COMPLEX ALGEBRA

A complex number is defined as an *ordered* pair of two real numbers (x, y) in which the first number, x, is called the real part of the complex number and the second number, y, is called the imaginary part of the complex number. It is customary to reserve the letter z for complex numbers and write the complex number as

$$z = x + iy \ . \qquad Eq. (5.1a)$$

The real part of z, namely $Re(z)$, is x and the imaginary part of z, namely $Im(z)$, is y. Complex numbers can be represented as points in a two-dimensional plane, which is analogous to the common $x - y$ plane and is called the complex plane or the z-plane. Because of its similarity with Cartesian coordinates, the representation of a complex number as $z = x + iy$ is called the Cartesian representation of the complex number. In a plane one can switch from Cartesian coordinates (x, y) to plane polar coordinates (r, θ). From Figure 5.1,

$$x = r\cos\theta\,, y = r\sin\theta \ . \qquad Eq. (5.2a)$$

Thus, a complex number can be written as

$$z = r(\cos\theta + i\sin\theta) = r\exp(i\theta) \ . \qquad Eq. (5.1b)$$

Here r is the magnitude of z and θ is the argument or phase of z. The magnitude of a complex number is also commonly written as $|z|$ so that $r = |z|$. Note,

$$r = \sqrt{x^2 + y^2}, \theta = \arg(z) = \arctan\left(\frac{y}{x}\right) \ . \qquad Eq. (5.2b)$$

The representation of a complex number as $z = |z|\exp(i\theta) = r\exp(i\theta)$ is called the polar or exponential representation of the complex number.

The complex conjugate of a complex number is obtained by replacing i by $-i$ and is denoted by placing a bar on top of the number. Thus $\bar{z} = r\exp(-i\theta)$ or $\bar{z} = x - iy$. Note $|z| = r = |\bar{z}|$. Also $z\bar{z} = r^2 = |z|^2 = x^2 + y^2$. Thus,

$$|Re(z)| \leq |z| \text{ and } |Im(z)| \leq |z| \ .$$

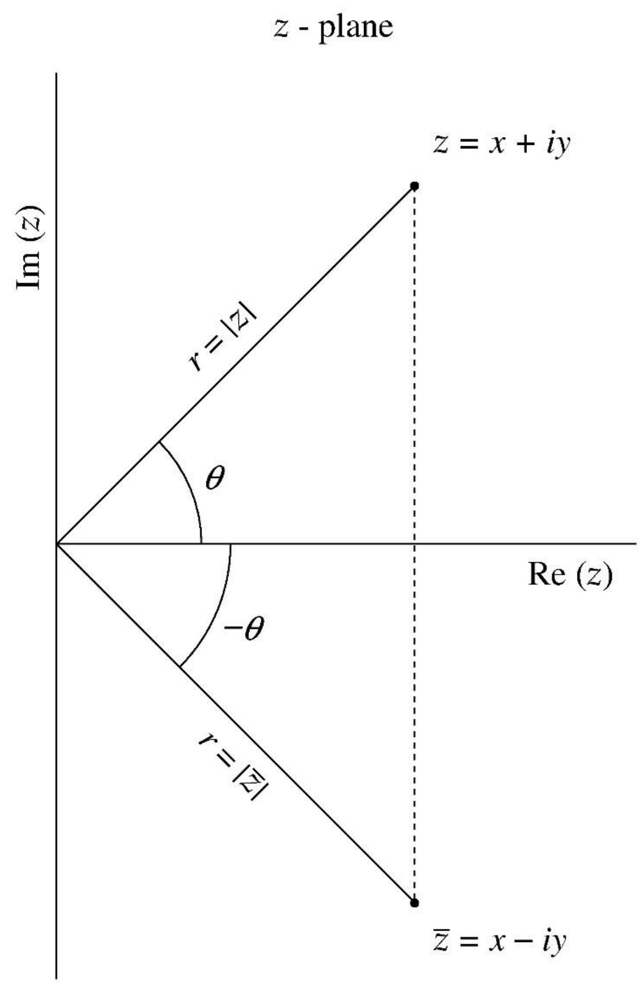

Figure 5.1. Cartesian and polar (or exponential) representation of a complex number.

Example: Given, $z_1 = 2 + 5i$, determine $Re(z_1)$, $Im(z_1)$, $|z_1|$, phase angle θ, complex conjugate $\overline{z_1}$ and exponential representation of z_1.

Solution: Looking at the complex number $z_1 = 2 + 5i$, we note $Re(z_1) = 2$, $Im(z_1) = 5$, $|z_1| = \sqrt{2^2 + 5^2} = \sqrt{29}$, $\tan\theta = \frac{5}{2} = 2.5$, $\theta = 68.2° = 1.19$ radians, $\overline{z_1} = 2 - 5i$, and $z_1 = \sqrt{29}\exp(1.19\ i)$.

Example: Given $z_2 = 4 + 3i$, determine $Re(z_2)$, $Im(z_2)$, $|z_2|$, phase angle θ, complex conjugate $\overline{z_2}$, and exponential representation of z_2.

Solution: Again, looking at the complex number $z_2 = 4 + 3i$, we note $Re(z_2) = 4$, $Im(z_2) = 3$, $|z_2| = \sqrt{4^2 + 3^2} = 5$, $\tan\theta = \frac{3}{4} = 0.75$, $\theta = 36.9° = 0.64$ radians, $\overline{z_2} = 4 - 3i$, and $z_2 = 5\exp(0.64\ i)$.

5.2 PROPERTIES OF COMPLEX NUMBERS

Here is a list of some general properties of complex numbers.

Property Number 1: Addition or subtraction of two complex numbers is achieved by adding or subtracting the real parts and the imaginary parts separately. If

$$z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2 \ ,$$

then

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2) \ .$$

The addition of two complex numbers is both commutative as well as associative,

$$z_1 + z_2 = z_2 + z_1 \ ,$$

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \ .$$

The product of two complex numbers is treated as simple polynomial multiplication,

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \ .$$

The product of two complex numbers is commutative and associative as well as distributive,

$$z_1 z_2 = z_2 z_1 \ ,$$

$$z_1(z_2 z_3) = (z_1 z_2) z_3 \ ,$$

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \ .$$

Property Number 2: If two complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, are equal, then their real and imaginary parts are separately equal. In other words, $z_1 = z_2$ implies $x_1 = x_2$ and $y_1 = y_2$. If $z = x + iy = 0$, then it implies that both $x = 0$ and $y = 0$ simultaneously.

Property Number 3: If $z_1 z_2 = 0$ it implies that either $z_1 = 0$ or $z_2 = 0$ or both are zero. To prove this fact, we note

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = 0 \ .$$

Thus $x_1 x_2 - y_1 y_2 = 0 \quad \text{and} \quad x_1 y_2 + x_2 y_1 = 0$. Therefore,

$$(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2 = 0 \ ,$$

or

$$x_1^2(x_2^2 + y_2^2) + y_1^2(x_2^2 + y_2^2\ ) = 0 \ ,$$

or

$$(x_1^2 + y_1^2)(x_2^2 + y_2^2) = 0 \ .$$

Thus, either $x_1^2 + y_1^2 = 0,$ which implies both $x_1 = 0$ and $y_1 = 0,$ that is, $z_1 = 0$, or $x_2^2 + y_2^2 = 0,$ which implies both $x_2 = 0$ and $y_2 = 0,$ that is, $z_2 = 0$, or both z_1 and z_2 are zero. Note in passing,

$$|z_1z_2| = \{(x_2^2 + y_1^2)(x_2^2 + y_2^2)\}^{1/2} = |z_1||z_2| \ .$$

Property Number 4: Division of one complex number by another complex number works as follows. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 + y_2^2} = \frac{(x_1x_2 + y_1y_2)}{x_2^2 + y_2^2} + i\frac{(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2} \ .$$

Property Number 5: The properties of complex conjugation imply

$$\overline{z_1 \pm z_2} = \bar{z}_1 \ \pm \bar{z}_2 \ ,$$

$$\overline{z_1z_2} = \bar{z}_1\ \bar{z}_2 \ ,$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \qquad (z_2 \neq 0) \ ,$$

$$\overline{(\bar{z})} = z \ .$$

If $\bar{z} = z$ then z is pure real, that is, $Im(z) = 0$ and if $\bar{z} = -z$ then z is pure imaginary, that is, $Re(z) = 0$. In fact,

$$Re(z) = \frac{z + \bar{z}}{2}, \qquad Im(z) = \frac{z - \bar{z}}{2\,i} \ .$$

Property Number 6: If $z = r\exp(i\theta) = x + iy$ then $\theta = \arg(z) = \arctan(y/x)$, and

$$\arg(z_1z_2) = \arg(z_1) + \arg(z_2) \ ,$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \ .$$

Example: In this example, we take the two complex numbers that we used in the previous two examples, namely, $z_1 = 2 + 5i$ and $z_2 = 4 + 3i$, and illustrate some of the properties that we discussed above.

Solution:

$$z_1 + z_2 = 6 + 8i, \qquad |z_1 + z_2| = \sqrt{6^2 + 8^2} = \sqrt{100} = 10 \ ,$$

$$z_1 - z_2 = -2 + 2\,i \qquad |z_1 - z_2| = \sqrt{(-2)^2 + 2^2} = \sqrt{8} \ ,$$

$$z_1 z_2 = (2 + 5i)(4 + 3i) = 8 + 20i + 6i + i^2 15 = -7 + 26i \ ,$$

$$|z_1 z_2| = \sqrt{(-7)^2 + 26^2} = \sqrt{49 + 676} = \sqrt{725} = \sqrt{25 * 29} = 5\sqrt{29} = |z_1||z_2| \ ,$$

$$\frac{1}{z_1} = \frac{1}{2 + 5i}\frac{2 - 5i}{2 - 5i} = \frac{2 - 5i}{29} = \frac{2}{29} - \frac{5}{29}i \ ,$$

$$\frac{1}{z_2} = \frac{1}{4 + 3i}\frac{4 - 3i}{4 - 3i} = \frac{4 - 3i}{25} = \frac{4}{25} - \frac{3}{25}i \ .$$

Here is a list of some items which are so important that one should commit them to memory. The useful things to remember, for *k* an integer (positive or negative) or zero, are:

$$\exp(i\pi/2) = i$$

$$\exp(-i\pi/2) = -i$$

$$\exp(ik\pi) = (-1)^k$$

$$\exp(i2k\pi) = +1$$

Based on these useful items to remember, Property Number 2 for two complex numbers, in exponential form, can be written as follows. If $z_1 = r_1 \exp(i\theta_1)$ and $z_2 = r_2 \exp(i\theta_2)$, then $z_1 = z_2$ implies $r_1 = r_2$ and $\theta_1 = \theta_2 + 2k\pi$, where k is an integer including zero.

In some situations, dividing by a complex number or finding the reciprocal of a complex number is best accomplished by using the exponential form of the complex number instead of using Property Number 4. The following example will illustrate this point.

Example: Write the complex number $z = 4i/(\sqrt{3}+i)$ in its Cartesian form.

Solution: We write the numerator and denominator of the complex number z in exponential form as

$$z = \frac{4i}{\sqrt{3}+i} = \frac{4\exp(i\pi/2)}{2\exp(i\pi/6)} = 2\exp\left(\frac{i\pi}{2} - \frac{i\pi}{6}\right) = 2\exp(i\pi/3) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + i\sqrt{3}\ .$$

5.3 POWERS OF COMPLEX NUMBERS

In order to calculate powers of a complex number, z, it is best to work with exponential notation, namely,

$$z^n = (x+iy)^n = r^n \exp(in\theta)\ .$$

Example: Given $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, determine all higher powers of z.

Solution: First, we write z in its polar or exponential form. Since $r = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$ and $\theta = \arctan\sqrt{3} = \frac{\pi}{3}$, it follows that in exponential form

$$z = 1\exp\left(\frac{i\pi}{3}\right)\ .$$

Then,

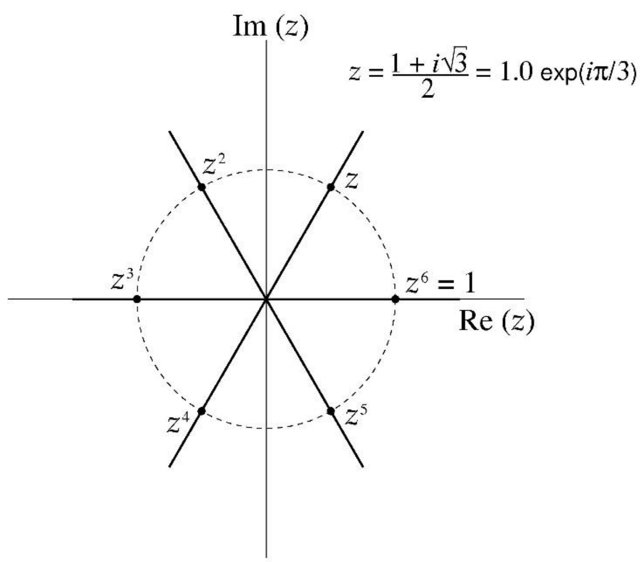

Figure 5.2. Powers of a complex number $z = 1\exp\left(\frac{i\pi}{3}\right)$.

$$z^2 = 1\exp\left(i\frac{2\pi}{3}\right) ,$$

$$z^3 = 1\exp(i\pi) ,$$

$$z^4 = 1\exp\left(i\frac{4\pi}{3}\right) ,$$

$$z^5 = 1\exp\left(i\frac{5\pi}{3}\right) ,$$

$$z^6 = 1\exp\left(i\frac{6\pi}{3}\right) = 1\exp(i2\pi) = 1 .$$

Since $z^6 = 1$, all higher powers of z, starting with z^7 onwards, take one of the six values above. These six values are shown in a complex plane in Figure 5.2. They lie on a circle of unit radius. If the magnitude r is different from 1, then we get a spiral instead of a circle. For example, for $r = 1.1$,

$$z = 1.1\exp\left(i\frac{\pi}{2}\right) ,$$

$$z^2 = 1.21\exp(i\pi) ,$$

$$z^3 = 1.331\exp\left(i\frac{3\pi}{2}\right) ,$$

$$z^4 = 1.4641\exp(2i\pi), \quad \text{etc} .$$

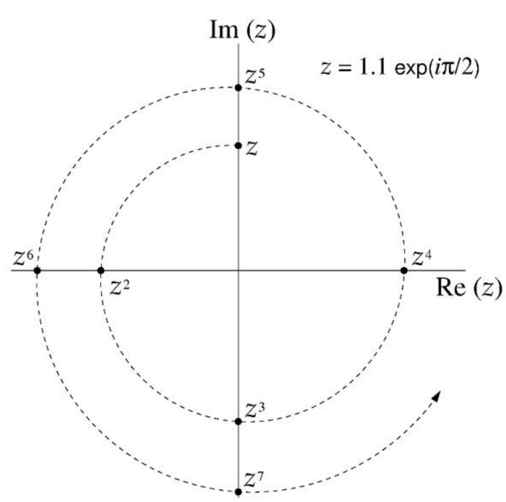

Figure 5.3. Powers of a complex number $z = 1.1\exp\left(i\frac{\pi}{2}\right)$.

In the complex plane these values appear on an ever-expanding spiral, as seen in Figure 5.3. Similarly, if the magnitude of a complex number z is less than 1, that is $r < 1$, then all higher powers of z lie in the complex plane on an ever-contracting spiral. Also, for $r \neq 1$, all higher powers of z have a separate distinct value.

Example: Given $z = 1 + i$, what is z^8?

Solution: First, write the complex number z in polar or exponential form. Note $r = \sqrt{1+1} = \sqrt{2}$ and $\tan\theta = \frac{1}{1}$ so that $\theta = \pi/4$. So,

$$z = 1 + i = \sqrt{2}\exp\left(i\frac{\pi}{4}\right) .$$

Then, taking any higher power of z is straightforward,

$$z^8 = \left(\sqrt{2}\right)^8 \exp\left(8\frac{i\pi}{4}\right) = 2^4\exp(i2\pi) = 16 .$$

However, the long method would be to multiply $1 + i$ by itself eight times:

$$(1+i)^2 = 1 + 2i + i^2 = 1 + 2i - 1 = 2i ,$$

$$(1+i)^4 = (1+i)^2(1+i)^2 = (2i)(2i) = 4i^2 = -4 ,$$

$$(1+i)^8 = (1+i)^4(1+i)^4 = (-4)^2 = 16 .$$

5.4 ROOTS OF A COMPLEX NUMBER

Roots, such as square root or cube root, of a complex number are also evaluated most conveniently by using exponential form of the complex number. As a practice run, let us evaluate roots of $+1$ and -1. From algebra we know that a polynomial of order n,

$$1 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0 ,$$

where coefficients a_i can be real or complex, has exactly n roots. Some roots may be repeated. Thus, roots of a simple polynomial $1 - z^n = 0$ are the n roots of unity or +1. To obtain these roots, write

$$z^n = +1 = \exp(i2\pi k) \quad k = 0,1,2 \ldots$$

Then, $z = \exp\left(i\frac{2\pi k}{n}\right)$, which for different values of k provides n roots of unity. For $k = 0, 1, \dots (n-1)$, these roots are, $1, \exp\left(\frac{i2\pi}{n}\right), \exp\left(\frac{i4\pi}{n}\right), \exp\left(\frac{i6\pi}{n}\right), \dots \exp\left(\frac{i2\pi(n-1)}{n}\right)$. The next value of k, namely $k = n$, gives back the first root, that is, $z = 1$.

Similarly, from $z^n + 1 = 0$ or $z^n = -1 = \exp(i\pi)\ \exp(i2\pi k) = \exp[i\pi(2k+1)]$, the n roots of -1 are $\exp\left(\frac{i\pi}{n}\right), \exp\left(\frac{i3\pi}{n}\right), \exp\left(\frac{i5\pi}{n}\right), \dots \exp\left(\frac{i(2n-1)\pi}{n}\right)$. The next value of k, namely $k = n$, gives back the first root, that is, $z = \exp\left(\frac{i\pi}{n}\right)$.

Example: Determine all possible values of square roots of +1.

Solution: In this case,

$$z^2 - 1 = 0 \quad \text{or} \quad z^2 = 1 = \exp(i2\pi k)$$

$$\text{or } z = \exp(i\pi k) \equiv z_k \text{ for } k = 0,1,2, \dots.$$

The mathematical symbol ≡ means that it is an equality for any value of variable k. Thus, for $k = 0$ and 1,

$$z_0 = +1 \text{ and } z_1 = -1 \ .$$

Higher values of k simply keep repeating values of z_0 and z_1, that is,

$$z_k = +1 \ \text{ for } k = 2,4,6, \dots \text{ and } \ z_k = -1 \text{ for } k = 3,5,7, \dots$$

Thus, there are only two independent square roots of +1 and their values are ± 1. In the complex plane, these two roots of +1 lie on a unit circle as seen in Figure 5.4.

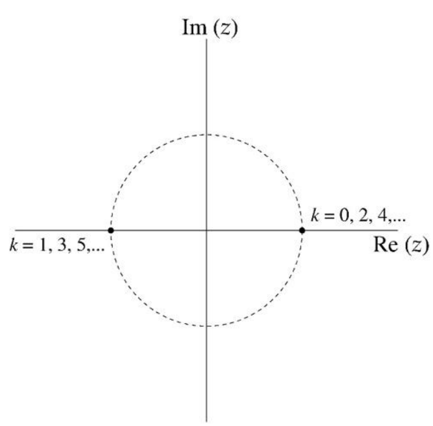

Figure 5.4. Square roots of +1.

Example: Determine all possible values of cube roots of +1.

Solution: In this case

$$z^3 - 1 = 0 \quad \text{or} \quad z^3 = 1 = \exp(i2\pi k)$$

$$\text{or,} \quad z = \exp\left(i\frac{2\pi}{3}k\right) \equiv z_k \text{ for } k = 0,1,2, \dots$$

$$z_0 = +1,$$

$$z_1 = \exp\left(i\frac{2\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \frac{-1 + i\sqrt{3}}{2},$$

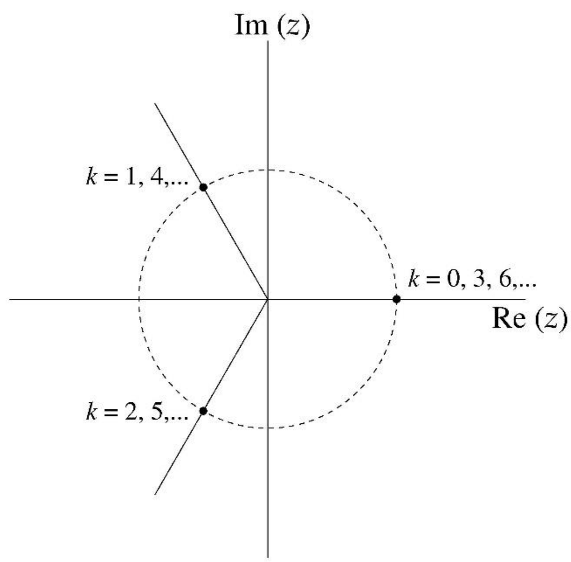

Figure 5.5. Cube roots of +1.

$$z_2 = \exp\left(i\frac{4\pi}{3}\right) = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = \frac{-1 - i\sqrt{3}}{2}.$$

Since $z_3 = \exp(i2\pi) = +1$, which is same as z_0, the values of all z_k for $k \geq 3$ are repeated. So, there are only three independent cube roots of +1, namely, $z_0, z_1,$ and z_2**.** Again, these three cube roots of +1 are seen to lie on a unit circle in the complex plane as in Figure 5.5.

Example: Given $z = 4i$, what is $z^{1/2}$?

Solution: First, convert z into its exponential form, $r = \sqrt{0 + 16} = 4$ and $\tan\theta = \frac{4}{0} = \infty$ or $\theta = \pi/2$, so that

$$z = 4\exp\left(i\frac{\pi}{2}\right)\ \exp(i2\pi k)$$

$$z^{1/2}\ = 2\exp(i\pi/4)\exp(i\pi k) \equiv z_k \text{ for } k = 0,1,2, \ldots.$$

For $k = 0$ and $k = 1$, we get

$$z_0 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2}(1+i)\ \ ,$$

$$z_1 = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -\sqrt{2}(1+i)\ \ .$$

For higher values of k, $k = 2, 3, 4, \ldots$, the roots z_0 and z_1 are repeated. Thus, $z = 4i$ has only two independent square roots, $\pm\sqrt{2}(1+i)$.

Example: Given $\boldsymbol{z = 1 - \sqrt{3}i}$, what is $\boldsymbol{\sqrt{z}}$?

Solution: Again, converting the complex number from its Cartesian form to its polar or exponential form, we get

$$z = 1 - \sqrt{3}i = 2\exp(-i\pi/3)\ \exp(i2\pi k)\ \ .$$

Then,

$$z^{1/2} = \sqrt{2}\exp(-i\pi/6)\exp(i\pi k) \equiv z_k \text{ for } k = 0,1,2, \ldots.$$

For $k = 0$ and $k = 1$, we have

$$z_0 = \sqrt{2}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \sqrt{\frac{3}{2}} - i\ \frac{1}{\sqrt{2}}\ ,$$

$$z_1 = \sqrt{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = -\sqrt{\frac{3}{2}} + i\ \frac{1}{\sqrt{2}}\ .$$

Again, for higher values of k, $k = 2, 3, 4, \ldots$, the roots z_0 and z_1 are repeated. Thus,

$z = 1 - \sqrt{3}i$ has only two independent square roots, $\pm\sqrt{\frac{3}{2}} \mp i\ \frac{1}{\sqrt{2}}$.

5.5 DEMOIVRE'S FORMULA

In the Interlude section we introduced the Euler's formula, namely,

$$\exp(ix) = \cos x + i \sin x \ .$$

If we put $x = n\theta$ in Euler's formula, we get

$$\exp(in\theta) = \cos(n\theta) + i \sin(n\theta) \ .$$

Also,

$$\exp(in\theta) = (\exp(i\theta))^n = (\cos\theta + i\sin\theta)^n \ .$$

So,

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \qquad Eq. (5.3)$$

which is known as DeMoivre's formula.

Applications of DeMoivre's Formula

In some situations, it becomes useful to write powers of a sine or a cosine function, such as $\sin^{\mathrm{n}}\theta$ or $\cos^{\mathrm{n}}\theta$ [integer n], in terms of sine or cosine of various multiple angles like $\sin(m\theta)$ or $\cos(m\theta)$ [integer m]. This can be accomplished by a simple application of DeMoivre's formula. This formula also allows us to write $\sin(n\theta)$ or $\cos(n\theta)$ [integer n] in terms of multiple powers of sine or cosine, as $\sin^{\mathrm{m}}\theta$ or $\cos^{\mathrm{m}}\theta$ [integer m]. For convenience, write

$$z = \cos\theta + i\sin\theta = \exp(i\theta) \ ,$$

then

$$z^{-1} = [\exp(i\theta)]^{-1} = \cos\theta - i\sin\theta \ ,$$

and $z^n = \exp(in\theta)\,, z^{-n} = \exp(-in\theta)$.

Then,

$$z^n + z^{-n} = \exp(in\theta) + \exp(-in\theta) = 2\cos(n\theta) \ , \qquad Eq. (5.4a)$$

and

$$z^n - z^{-n} = \exp(in\theta) - \exp(-in\theta) = 2i\sin(n\theta) \quad . \qquad Eq. (5.4b)$$

These relationships along with DeMoivre's formula allow us to accomplish what we set out to do.

Example: Write $\cos^3\theta$ and $\sin^4\theta$ in terms of sine or cosine of angles which are multiples of θ.

Solution: Using Eqs. (5.4),

$$\cos^3\theta = \left(\frac{z+z^{-1}}{2}\right)^3 = \frac{1}{8}(z^3 + 3z + 3z^{-1} + z^{-3}) = \frac{1}{8}([z^3 + z^{-3}] + 3[z + z^{-1}])$$

$$= \frac{1}{8}[2\cos(3\theta) + 6\cos\theta] = \frac{1}{4}[\cos(3\theta) + 3\cos\theta]$$

and

$$\sin^4\theta = \left(\frac{z-z^{-1}}{2i}\right)^4 = \frac{1}{16}(z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4})$$

$$= \frac{1}{16}([z^4 + z^{-4}] - 4[z^2 + z^{-2}] + 6)$$

$$= \frac{1}{16}[2\cos(4\theta) - 4 \cdot 2\cos(2\theta) + 6]$$

$$= \frac{1}{8}[\cos(4\theta) - 4\cos(2\theta) + 3] \ .$$

Example: Write $\cos(4\theta)$, $\sin(4\theta)$, $\cos(2\theta)$, and $\sin(2\theta)$ in terms of multiples of $\sin\theta$ and $\cos\theta$.

Solution: Starting with DeMoivre's formula,

$$\cos(4\theta) + i\sin(4\theta) = (\cos\theta + i\sin\theta)^4$$

$$= \cos^4\theta + 4\cos^3\theta\,(i\sin\theta) + 6\cos^2\theta\,(i\sin\theta)^2 + 4\cos\theta\,(i\sin\theta)^3 + (i\sin\theta)^4$$

$$= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + 4i\,(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)$$

Separating out the real and imaginary parts on both sides leads to

$$\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \ ,$$

and

$$\sin(4\theta) = 4\cos\theta\sin\theta\ (\cos^2\theta - \sin^2\theta)\ .$$

Similarly,

$$\cos(2\theta) + i\sin(2\theta) = (\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2\cos\theta\ (i\sin\theta) + (i\sin\theta)^2$$

$$= (\cos^2\theta - \sin^2\theta) + i2\sin\theta\cos\theta\ .$$

Separating out real and imaginary parts, we get well-known relations

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta\ ,$$

$$\sin(2\theta) = 2\sin\theta\cos\theta\ .$$