

beginning assets and equity and report the results as return on assets (ROA) and returns on equity (ROE) measures.

AIS earnings estimates make no effort to find the contributions of a single capital asset or subset of the firm's assets. However, isolating earnings from a single capital asset and possibly its supporting assets is exactly what we do when constructing PV models for firm investments—we attempt to isolate the return from the investment(s) added to the production unit or the firm. Otherwise, how can we decide to invest or not in the challenger?

However, we face a difficulty when we attempt to measure the returns from an incremental investment. The difficulty is that the firm's optimal response to a new investment is often to change the levels of nondurable assets and services extracted from other durables operated by the firm making it difficult to identify what part of the output change we can attribute to the new investment versus output changes that should be attributed to changes in the use of other inputs employed by the firm. This problem, measuring the contributions of incremental investments will occupy much of our attention in this chapter.

What follows. In what follows, we describe how to analyze incremental investments using incremental budgeting analysis. We introduce the incremental investment analysis using simplified mathematical methods and graphical analysis. Then we illustrate incremental budgeting analysis using Green & White Services example and this book's generalized PV template described in Appendix to Chapter 8.

Measuring Contributions from Incremental Investments

An example of an incremental investment. To illustrate an incremental investment, suppose that Hi Quality Nursery (HQN) considers investing in a sprinkler system to reduce its demand for seasonal labor. HQN asks: will investing in the new sprinkler system improve my firm's financial well-being?

How do we approach incremental investments differently compared to stand-alone investments? Or, in our example, how do we measure the financial consequences of the sprinkler investment and the reduction in hired labor? To find the answer: we find what changes in revenue and expenses will the sprinkler investment and the hired labor disinvestment produce for HQN?

We know that the new sprinkler system means that HQN can reduce its dependence on hourly workers to water its plants during the growing season. Furthermore, this change in HQN's operating procedures will likely continue for many periods in the future. Some new expenses may be incurred associated with operating and maintaining the new sprinkler system, but these are likely to be much less than the reduction in hired labor resulting from the sprinkler investment. It's not clear what changes in revenue would result from adopting a new watering system. It may be that a reduction in operating costs means that some unprofitable plants and shrubs may now be profitable because operating costs to produce them are less.

Because the consequences of the sprinkler/hired labor changes will likely have long-term consequences, this problem can't be analyzed using a single period financial tool such as CFS. Instead the problem requires that we employ multi-period PV model templates. The answers that can be supplied by PV models include: how will an incremental investment change the capital structure of the firm? How will incremental investment change the equity of the firm? How will the incremental investment change the internal rate of return on the firm's assets and equity?

Contributions from existing investments. Suppose that we have a profit function π that depends on the revenue and costs associated with an independent profit generating plant described as $f(x)$ where the variable x represents the existing asset and liability configuration of the firm:

$$(10.1) \quad \pi = f(x)$$

Analyzing the firm as a function of its current asset and liability configuration described as x in equation (10.1) is what we do when we construct coordinated financial statements (CFS). Rates of return on assets and equity as well as our other ratio calculations all reflect the current financial condition of the firm. To find rates of return on assets or equity, we divide equation one by its assets or by its equity.

Now suppose that we want to analyze how profits will respond to a change in x . Using the delta notation, Δ to reflect change, we write the effect of a change in x on profits as:

$$(10.2) \quad \Delta\pi = f(x + \Delta x) - f(x)$$

What we have described is a stand-alone investment that allows us to examine the effects on profits in response to a change in x without being required to consider other changes in the firm. To complicate the analysis, suppose that the firm's asset and liability configuration are describe by two interdependent variables x and y so that when we change x , to maximize profits we must also change y . Indeed, it is hard to imagine any change in x that doesn't require adjustments in other inputs to achieve the most beneficial outcomes. (If we purchase a sprinkler to reduce labor costs, we would surely expect labor costs to be reduced). To account for the simultaneity of x and y , we write the profit function as:

$$(10.3) \quad \pi = g(x, y)$$

Since x and y are interdependent, any change in x must also consider a change y to optimize its firm profits, so that to analyze a change in profits attributable to a change in x and y , we must calculate:

$$(10.4) \quad \Delta\pi = h(x + \Delta x, y + \Delta y) - h(x, y)$$

Equation (10.4) essentially describes incremental analysis. In accounts for changes in revenue and expenses associated with the incremental investments plus changes in the costs and revenue associated with other variables essential to the firm's operations.

A graphical approach. We now present the earlier results using graphical analysis familiar to students of microeconomics. Assume that a firm produces output using a combination of capital services and other inputs x and y . We graph the combinations of capital services and other inputs that produce the same output and call the result an isoquant (i.e. same quantity). In Figure 10.1 we represent two levels

of output by isoquants Q1, and Q2 drawn convex to the origin to reflect diminishing marginal productivity.

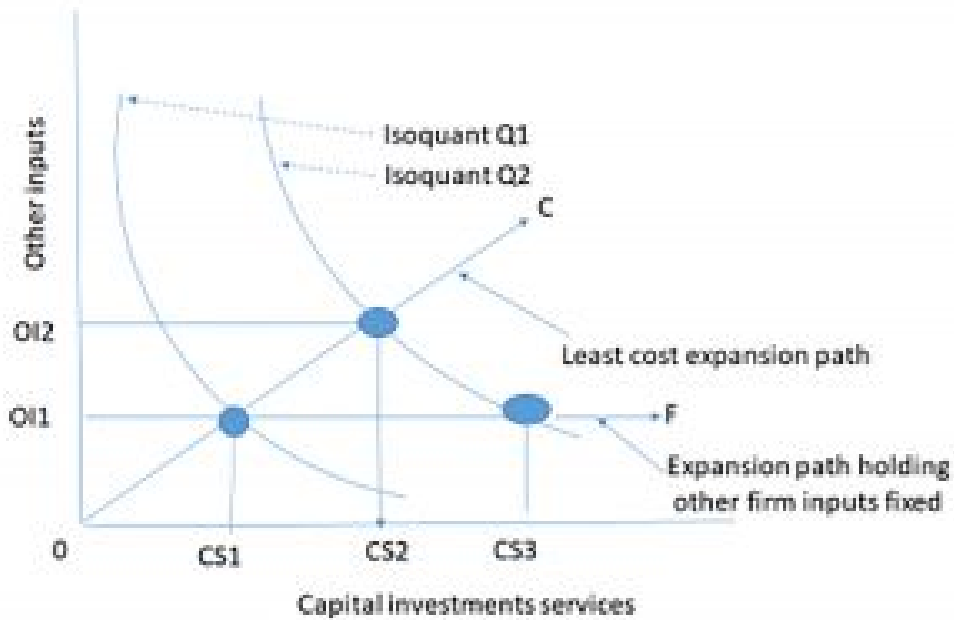


Figure 10.1. Isoquants

Now suppose that the firm considers an incremental investment, a capital investment that will allow the firm to increase its supply of capital services from CS1 to CS3 and increase output from Q1 to Q2. In this case, we hold constant the level of other inputs constant at OI1. However, economic production theory teaches that increasing production from Q1 to Q2 holding other inputs constant would be inefficient. Instead, the theory teaches that the firm should increase capital services and other inputs along some least cost line AC determined by the relative costs of increasing both the capital services from the new investment and from the other inputs.

Increasing production along the least cost line AC, the firm would increase capital service from CS1 to CS2 an amount less that was required to reach Q2 when we held other inputs constant. In other words, increasing the use of other inputs allows the firm to use less capital services from the new investment compared to the capital services required if other inputs were held constant $(CS2 - CS1) < (CS3 - CS1)$ while still producing the same level of output. Furthermore, expanding other inputs and services from the new durable along the least cost expansion path assures us that the cost of increasing other inputs from OI1 to OI2 is more than offset by reducing the cost of capital services from $(CS3 - CS1)$ to $(CS2 - CS1)$.

Which expansion path? We have demonstrated that whether to measure the cost of increasing production along the least cost line AC or the expansion path AF that holds other inputs constant at OI1 depends on the nature of the costs of increasing the use of other inputs depends mostly on the passage of time or use. If increases in other inputs makes the investment more productive, this productivity increase is credited to the investment. Finally, if we measure changes along the least cost path, we must recognize that we are answering a different kind of question than “what are the unique returns to a challenging investment?” When performing PV analysis on a stand-alone investment, we ask: does the investment earn a positive net present value (NPV)—greater than the PV sacrificed by liquidating the defender. In contrast, the incremental investment analysis asks: is this change in the capital structure of the firm and use of other inputs more efficient and profitable than before the change?

It is not a settled matter whether to measure the cost of increasing production along the least cost line AC or the expansion path AF that maintains other inputs constant at OI1. We come down on the side of measuring increases in production attributable to increases in durable services and other inputs because that is what the firm does. If the increased durable services make other inputs more productive, then why not include their contributions along with those of the increased durable services?

However, if we measure changes along the least cost path, we must recognize that we are answering a different kind of question than “what are the unique returns to a challenging investment?” When performing PV analysis on a stand-alone investment, we ask, does the investment earn a positive present value return—greater than the present value sacrificed by liquidating the defender. In contrast, the incremental approach asks: is this change in the capital structure of the firm and use of other inputs more efficient and profitable than before the change?

To emphasize the challenges of measuring the contributions of incremental investments, consider a scissors type of investment. With only one scissor blade, not much cutting occurs. Add a blade to an existing blade, and the scissor output returns to normal. The increase in the scissor production was because we changed the use of both blades. And the decision to add a second blade is made based on the change in output using one scissor blade to two and the change in output must be attributed to them both. If we analyzed the contribution of the second blade keeping the first blade idle, output would have remained at zero. Is it fair to attribute to the second blade the contributions of the first blade? Yes, if one treats the investment in the first blade as a sunk cost and not relevant for answering the question. On the other hand, the service extraction rate from the first blade increases when one adds a second blade. Part of the increased output must be attributed to increased services provided by the first blade. The scissor example is an extreme case. However, it illustrates the difficulty of applying the incremental approach if all other inputs are held constant.

Optimal Service Extractions Rates from Stand-alone and Incremental Investments

When analyzing incremental investment problems, we often assume that the investment being analyzed provides services at its optimal rate. Finding the optimal service extraction rate for capital investments is complicated and we often employ complex computer programs to find these optimal rates. Yet it is important to understand what determines optimal service extraction rates even though the focus of our analysis is on the investment's earnings and rates of return.

To be clear, when we discuss the cost of extracting services from our capital investment, we include the change in its salvage value, the cost of using other nondurable inputs, and the change in the value of other capital investments whose services support our investment. As it turns out, the costs of using a capital investment within a period conform to the categories already identified in our PV templates. The cost of using nondurable inputs are measures as cash costs of goods sold (COGS). The cost of extracting capital investment services that depend mostly on the passage of time, we measure as cash overhead expenses (OE). The cost equal to the change in the liquidation value of the capital investment we measure as depreciation (Dep) and most of the time we assume that this rate of depreciation is determined by the passage of time.

Service extraction rates when the cost of using the investment depends mostly on the passage of time. Consider, for example, an investment whose use cost depends mostly on the passage of time such as a storage facility. The roof, for example, has a finite time during which it can provide services. It simply wears out over time. Much the same is true for other parts of the building including its painted surfaces, floors, and other building features. While there may be some important costs associated with heating, ventilation, and air conditioning services, these costs depend mostly on the passage of time and weather conditions than on the number of items stored or the activity that occurs in the building. Finally, investments in buildings are usually fixed because their liquidation value is often low. To use a building requires that the new owner relocate to the building site. Thus, location fixity reduces the liquidity of the investment and contributes to its fixity in the firm's investment portfolio.

So, what is the optimal use of an investment whose costs depend mostly on the passage of time? Since its marginal use cost is usually low, their optimal use is their maximum capacity if the investment is making some positive contributions to the firm. For investments whose use costs depend mostly on the passage of time, we are not likely to change their use when adding an incremental investment to the firm unless we operate them at less than full capacity before the incremental investment.

Service extraction rates when the cost of using an investment depends mostly on its use. Consider another kind of investment whose use costs depends mostly on services provided by other capital investments, including maintenance costs, and nondurable inputs. Electric motors or gasoline powered equipment have their cost of service extraction dependent mostly on the cost of using nondurable inputs and services from other durables.

So, what determines the optimal service extraction rates from investments whose cost depends mostly on use and other inputs? The optimal service extraction rate depends mostly on the marginal use costs. Furthermore, we expect that service extraction rates change when incremental investments are added to the firm that alter these marginal use costs. In sum, investments whose costs depends mostly on their use will find that their optimal use is not fixed nor at their maximum capacity.

Opportunity costs. All investments that provide services for one period or more incur opportunity costs equal to the defender's IRR times the value of the investment at the beginning of the period. We incur the cost when we sacrifice a defending investment to employ a challenging one. We account for these costs when discounting future costs and returns to their present value.

So, what have we learned? We learned that the optimal service extraction rate from durables depends on the nature of costs (and returns). If costs vary with use, marginal cost and marginal returns play an important role in determining the optimal use of the durable in the short run. If costs are mostly time dependent, in the short run the durable's optimal service extraction rate is its maximum if each service unit provides value. However, in the long term, both time and use cost matter and determine whether the firm invests or disinvests in the durable.

Incremental Investments and Homogeneity of Size

We established earlier that comparing two challenging investments with different initial and periodic sizes could produce conflicting net present value (NPV) and internal rate of return (IRR) rankings. Incremental analysis violates the homogeneity of size condition and can produce conflicting NPV and IRR rankings. Consider the following.

Suppose a small firm has a profit function $\pi(x,y)$ where x and y represent the firm's existing assets (inputs). Next suppose that the firm is considering an incremental investment such that the new profit function will be $\pi(x + \Delta x, y + \Delta y)$. To decide whether to commit to the incremental investment or not, the firm has the following options: compare (1) incremental investment contributions $[\pi(x + \Delta x, y + \Delta y) - \pi(x,y)]$ versus (2) comparing whole firm profits before the incremental investment $\pi(x,y)$ and after the incremental investment: $\pi(x + \Delta x, y + \Delta y)$. An example follows.

Consider a firm with initial investments of \$1,000 expected to produce cash flow for the next periods of \$750 and \$500. The firm is considering an incremental investment of \$100 expected to produce an incremental change in cash flow for the next two periods of \$64 and \$51 respectively. The total cash

flow after the incremental investment for the next two periods are $\$750 + \$64 = \$814$ and $\$500 + \$51 = \$551$.

Assuming the defender's IRR is 8%, we compare IRR and NPV for the incremental investment with IRR and NPV for the firm's profits before and after the incremental investment. At first, one might conclude that the two approaches should produce identical rankings. They don't. Because they violate homogeneity of size condition, comparing the cash flow differences versus the cash flow of the entire firm before and after the investment, they need not produce consistent rankings and Table 10.1 demonstrates.

Table 10.1. Comparing NPV versus IRR Rankings for an incremental Investment

	Incremental Investment cash flow	Investment cash flow before the incremental investment	Investment cash flow after the incremental investment
Amount	\$100	\$1,000	\$1,000
R1	\$64	\$750	\$814
R2	\$51	\$500	\$551
NPV @ $r = 8\%$ (rankings)	\$2.98 (invest)	\$123.11 (2)	\$126.10 (1)
NPV @ $r = 12\%$ (rankings)	-\$2.20 (don't invest)	\$68.24 (1)	\$66.04 (2)
IRR (rankings)	10.26% (invest if $r=8\%$; don't invest if $r=12\%$)	17.54% (1)	16.86% (2)

The lesson from Table 10.1 is this. Incremental investment analysis recommends investing if the defender's IRR is 8% because it is greater than its IRR of 10.26% and produces a positive NPV of \$2.98. Incremental investment analysis recommends not investing if the defender's IRR is 12% because it is less than its IRR of 10.26% and produces a negative NPV of -\$2,20. In contrast, comparing whole firm NPVs and IRRs before and after the investment, NPV criterion recommends investing for a defender's IRR of 8% but not if the defender's IRR is 12%. Based on the IRR criterion, whole firm comparisons recommends against investing. In summary, NPV criterion produce consistent recommendation using incremental budgeting and whole firm comparisons. Incremental and whole firm IRR provide consistent recommendations if the defender's IRR is 8% but provide conflicting recommendation is the defender's IRR is 12%. Whole firm comparisons before and after the incremental investment based on their IRRs unambiguously recommend not investing: the IRR for not investing is 17.54% and greater than the IRR for investing of 16.88%.

We can produce consistent IRR and NPV results if we adjust for size—requiring us that we compare investments of the same size. This we achieve homogeneity of size by investing the differences in size between the incremental investment and the whole firm investments at the defender's IRR of 8% or 12%. When we do this, we create a disadvantage for incremental investments because it ignores returns from the firm greater than the defender's returns and instead assumes all the difference between the incremental investment and the whole firm earns the discount rate. To demonstrate, in Table 10.2 we

compare the incremental method and whole firm approach adjusted for size differences at the discount rates of 8% and 12%.

Table 10.2. Comparing NPV versus IRR Ranking for an incremental Investment after adjusting for size differences assuming the defender's IRR is 8%

	Incremental Investment cash flow	Investment cash flow before the incremental investment	Investment cash flow after the incremental investment
Amount	\$100	\$1,000	\$1,000
R1	\$0	\$750	\$814
R2	\$1,287	\$500	\$551
NPV @ r = 8% (rankings)	\$2.98 (invest)	\$123 (2)	\$126 (1)
IRR (rankings)	8.15% (invest)	13.88% (2)	14.02% (1)

For a discount rate of 8% and equal size investments, incremental and whole firm comparisons produce consistent results.

Table 10.3. Comparing NPV versus IRR Ranking for an incremental Investment after adjusting for size differences assuming the defender's IRR is 12%

	Incremental Investment cash flow	Investment cash flow before the incremental investment	Investment cash flow after the incremental investment
Amount	\$100	\$1,000	\$1,000
R1	\$0	\$750	\$814
R2	\$1,377.08	\$1,465.44	\$1,462.68
NPV @ r = 8% (rankings)	-\$2.20 (don't invest)	\$68.24 (1)	\$66.04 (2)
IRR (rankings)	11.89% (don't invest)	15.42% (1)	15.31% (2)

For a discount rate of 12% and equal size investments, incremental and whole firm comparisons produce consistent results.

So, what have we learned? We learned that without adjusting for size differences, IRR and NPV may produce inconsistent ranking results when comparing before and after an incremental investment analysis. Adjusting for size differences produce consistent rankings. However, it is important to note that whether to make the incremental investment or not depends on the discount rate. The discount rate of 8% leads to an invest decision. A discount rate of 12% leads to a don't invest decision.

Ranking two investments using differences. Incremental budgeting analysis leads to a more general question. In some cases, applied researchers compare two challenging investments by finding the NPV or IRR of their difference. The problem as we just demonstrated is this. If we violate homogeneity of size requirement, NPV and IRR rankings may be inconsistent when comparing the cash flow differences of two investments versus investment rankings of whole firms. In other words, IRR and NPV ranking of investments of A and B may be inconsistent with investment A-B without making appropriate size adjustment.

So why do some investors perform incremental investments? The answer is the same one we used to justify the payback method for ranking investments. It is perceived to be less data demanding than solving for an entire firm's cash flow before and after an incremental investment because independent effects need not be considered.

So, what have we learned? We learned that incremental analysis and whole firm comparisons may produce conflicting rankings. If performing incremental budgeting and comparing investment differences, we generally advise financial managers to use NPV because one will obtain the same rankings as one would obtain using NPV whole firm comparisons. A justification for this approach is that an incremental investment is not generally a scalable investment, otherwise it wouldn't be incremental.

Green and White Services: A Stand-Alone Investment

Lon is considering investing in a lawn care and snow removal business that will require he purchase lawnmowers, snow blowers, and other equipment. Lon intends to name his business Green and White Services (GWS). Assume that Lon has hired you to advise him on whether he should invest in the business. To solve Lon's investment problem, you intend to solve equation (A8.1) repeated here as equation (10.5) below:

(10.5)

$$\begin{aligned}
 NPV = & -(A_0 - D_0) + \frac{(CR_1 - CE_1 - iD_0)(1 - T) + TDep_1}{[1 + IRR^E(1 - T)]} \\
 & + \dots + \frac{(CR_n - CE_n - iD_{n-1})(1 - T) + TDep_n}{[1 + IRR^E(1 - T)]^n} \\
 & + \frac{Csh_0 - D_0 + TAccts_0 + (1 - T)Accts_n}{[1 + IRR^E(1 - T)]^n} \\
 & + \frac{(1 - T)V_n^{liquidation} + TV_n^{book} - [AP_n - AP_0 + AL_n - AL_0](1 - T)}{[1 + IRR^E(1 - T)]^n}
 \end{aligned}$$

Equation (10.5) is a generalized NPV equation that is used as the basis for the [PV template](#) that is used in this section. By setting debt to zero, using the tax rate for assets and the defender's IRR for return on assets—we can find NPV and IRR for assets. By account for debt, using the tax rate for equity and the defender's IRR for return on its equity—we can find NPV and IRR for equity. We can solve for IRRs by setting NPVs equal to zero. We illustrate how to operationalize equation (10.5) using the PV template described at the end of Chapter 8 by solving Lon's investment in GWS.

To complete the PV template corresponding to equation (10.5) Lon provides the information.

Acquisition and depreciation. Lon estimates the initial investment for GWS will cost \$40,000. We enter that number in cell B4. Since the focus is on return to assets, we ignore debt levels, leaving column C equal to zero. The equipment falls into the MACRS 3-year depreciation class (25%, 37.5%, 25%, and 12.50%). We calculate depreciation by multiplying MACR 3-year class rates (25%, 37.5%, 25% and 12.5%) times the value of the capital accounts equal to \$40,000. Depreciation is (\$40,000 x 25%) = \$10,000 in year one; \$40,000 x 37.5% = \$15,000 in year two, \$40,000 x 25% = \$10,000 in year three, and \$40,000 x 12.5% = \$5,000 in year four. Depreciation in each year is recorded in column F. Investment book value equals the initial purchase price less accumulated depreciation is recorded in column E. To simplify,

we set liquidated value of the investment equals to its book value and reported the results in column D.

Operating income and expenses. Lon recognizes that not all his customers will pay for his services when he provides them. As a result, he expects his accounts receivable (AR) to equal \$1,000 at the end of his first year of operation, to grow to \$1200 at the end of year two, to decline to \$800 at the end of year three, and to decline to \$400 at the end of year four. The outstanding \$400 balance at the end of year four, he expects to liquidate when he sells his business. Inventories (INV) are expected to equal zero. The sum of asset operating account balances (AR + INV) are reported in column G.

Lon intends to charge \$40 per service for both lawn care and snow removal. He estimates that his expenses (labor, fuel, main-tenance) will average \$18 per service. Lon projects the number of services for the next four years to be 500 in year one, 750 in year two, 900 in year three, and 1,000 in year four. To find sales we multiply the number of services per year by the price paid per service. We find that sales equal $\$40 \times 500 = \$20,000$ in year one; $\$40 \times 750 = \$30,000$ plus \$200 of miscellaneous income in year two (\$30,200); $\$40 \times 900 = \$36,000$ plus \$600 in miscellaneous income in year three (\$36,600); and $\$40 \times 1000 = \$40,000$ less \$400 of promotional discounts in year four (\$39,600). We report sales in column H.

We adjust sales to find cash receipts by accounting for changes in (AR+ INV) and report the results as cash receipts (CR) in column I. In year one AR increased by \$1,000. As a result, CR equal sales of \$20,000 less \$1,000 increase in AR or \$19,000. In year two AR increased by \$200 so that CR equal sales of \$30,000 less \$200 increase in AR or \$29,800. In year three, AR declined by \$400 so the CR was greater than sales of \$36,000 by \$400: $\$36,000 + \400 equals \$36,400. Finally, at the end of year four, the last year Lon intends to operate his business, he expects that AR will decline to \$400 increasing CR by \$400 to \$4,400. Finally, the remaining \$400 balance of CR Lon liquidates when he sells the business.

We assume that the sum of outstanding liability operating accounts, account payable (AL) and accrued liabilities (AL) equal zero and report the results in column J. We find COGS by multiplying the number of services provided times the cost to deliver a service unit. We find they equal $\$18 \times 500 = \$9,000$ in year one; $\$18 \times 750 = \$13,500$ in year two; $\$18 \times 900 = \$16,200$ in year three, and $\$18 \times 1000 = \$18,000$ in year four. We report expenses in column K. And because there are no changes in operating account, cash expenses equal expenses are reported in column L. Interest costs reported in column M equal debt at the beginning of the period times the average interest rate. Since our focus is on earnings and rates of return on equity, we set debt to zero and as a result, interest costs are zero.

Depreciation tax savings. Tax savings from depreciation recorded in column N equal depreciation recorded in column F times the average tax rate. Column O records annual cash flow equal to cash receipts (column I), less cash expenses (column L), minus interest costs (column M) whose sum is multiplied by one minus the average tax rate, plus depreciation tax savings (column N), plus change in outstanding debt (Δ Column C).

Liquidation values. We find the after-tax liquidated value of asset operating accounts (Column P), after-tax liquidated value of liability operating accounts (Column Q), and the after-tax difference between

the market and book value of capital assets (Column R). Finally, we sum cash flows from liquidations and cash flow in the last period (Column S).

Other exogenous variables. We report the defender's IRR on assets, average tax rate, and average interest rate in Table 10.4 below:

Table 10.4. Green and White Services (GWS) Asset Variables: Defender's IRR^A, Average Tax Rate, After-Tax Discount Rate, and Average Interest Rate on Debt.

	A	B
1	r	0.05
2	T	0.2
3	$r(1-T)$	0.04
4	Average interest rate	0.06

To simplify, we assume that Lon's marginal tax rate and capital gains tax rate both equal 20%. Lon also assumes that the before-tax IRR of the defending investment equals 5%. As a result, the after-tax IRR of the defending investment is 4%. The average interest rate (not relevant in this case because of the focus on assets) is 6%.

Rolling NPV, AE, and IRR calculations. Rolling NPVs, IRRs, and AEs equal their values calculated as if the term ended in each year. Lon assumes he will operate GCS for four years. He might have asked: what is the optimal life of GCS? To answer the optimal life question, we need to find annual or rolling NPV and AE values. In this case, Lon is correct, four years is preferred to one, two, or three years of operation. Rolling NPV reported in column N increased from a negative NPV of (\$769) if he operated the business for one year to an NPV of \$16,412 if he operated his business for four years as planned. Meanwhile rolling AEs increase from a negative (\$800) to a positive \$4,521 while rolling IRRs increase from 2% to 18.67%.

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[Open PV Template in Microsoft Excel.](#)

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Year Assets Debt capital Capital accounts liquidation value Capital accounts book value Depreciation
 (Dep= ΔE) Asset Operating accounts (AR + INV) Sales Cash Receipts (CR = H+ ΔG) Liability Operating
 Accounts (AP+AL) Expenses (COGS + OEs) Cash Expenses (CE=K+ ΔJ) Interest costs (Int= $i \cdot C$ (t-1))
 Depreciation tax savings (TxF) After-Tax Cash flow= (I-L-M)(1-tax rate) +N+ ΔC After-tax Liquidation
 asset operating accounts= $[G(t)-G(0)](1-\text{tax rate})$ Afer-tax Liquidation liability operating accounts: $[J(n)-$
 $J(0)](1-\text{tax rate})$ Sum liquidating Capital, operating, and debt accounts = (D-E)(1-tax rate)+P-Q+E-C
 Sum liquidations + last period cash flow = 0+R Rolling afer-tax NPV Rolling after-tax AE Rolling after-
 tax IRR

2

0

\$40,000
\$0
\$40,000
\$40,000

3

1

\$0
\$0
\$30,000
\$30,000
\$10,000
\$1,000
\$20,000

\$19,000
\$0
\$9,000
\$9,000
\$0
\$2,000
\$10,000
\$800
\$0
\$30,800
\$40,800
-\$769
-\$800
2.00%
\$0
\$0
\$15,000
\$15,000
\$15,000
\$1,200
\$30,200
\$30,000
\$0
\$13,500
\$13,500
\$0

\$3,000

\$16,200

\$960

\$0

\$15,960

\$32,160

-\$651

-\$345

3.03%

5

3

\$0

\$0

\$5,000

\$5,000

\$10,000

\$800

\$35,600

\$36,000

\$0

\$16,500

\$16,500

\$0

\$2,000

\$17,600

\$640

\$0

\$5,640
\$23,240
\$5,253
\$1,893
9.93%

6

4

\$0
\$0
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\$5,000
\$400
\$39,600
\$40,000
\$0
\$18,000
\$18,000
\$0
\$1,000
\$18,600
\$320
\$0
\$320
\$18,920
\$16,412
\$4,521

Cash flow used to Find Lon's Rolling NPV and IRRs. It may be helpful to make explicit the cash flow used to find the rolling NPVs, IRRs and AEs. Recall that in the last year, cash flow sums operating cash flow for that year and the liquidation of operating and capital accounts. The cash flow per year are summarized in Table 10.6.

Table 10.6. Cash flow used to find Lon's Rolling NPV^A, IRR^A, and AE^A on his assets. IRR^A are calculated at the end of each year.

	A	B	C	D	E
1	Economic Life	1 year	2 years	3 years	4 years
2		-\$40,000.00	-\$40,000.00	-\$40,000.00	-\$40,000.00
3		\$40,800.00	\$10,000.00	\$10,000.00	\$10,000.00
4		2.00%	\$32,160.00	\$16,200.00	\$16,200.00
5	...		3.03%	\$23,240.00	\$17,600.00
6	...			9.93%	\$18,920.00
7	...				18.67%

Earnings and rate of return on Equity. Once we have solved for Lon's earnings on his assets, (NPV^A) and rate of return on assets (IRR^A) we can easily find Lon's earnings and rate of return on his equity, NPV^E and IRR^E respectively. To do so we enter his defender's IRR^E equal to 9%, his average tax rate equal to 40%. As a result, his after-tax IRR^E on his defender is 5.4%. We assume that Lon borrows \$30,000 to finance his new business and retires the loan in equal installments of \$10,000 over the next three years.

Table 10.7. Green and White Services (GWS) Equity Variables: Defender's IRR^E, Average Tax Rate, After-Tax Discount Rate, and Average Interest Rate on Debt.

	A	B
1	r	0.09
2	T	0.4
3	r(1-T)	0.054
4	Average interest rate	0.06

The annual cash flow used to find Lon's rolling NPV^Es, IRR^Es, and AE^Es on his equity is reported in Table 10.8.

Table 10.8. Cash flow used to find Lon's Rolling NPV^E, IRR^E, and AE^E on his equity. IRR^E are calculated at the end of each year.

	A	B	C	D	E
1	Economic Life	1 year	2 years	3 years	4 years
2		-\$40,000.00	-\$40,000.00	-\$40,000.00	-\$40,000.00
3		\$40,800.00	\$10,000.00	\$10,000.00	\$10,000.00
4	IRR% (last entry)	1.50%	\$31,620.00	\$15,900.00	\$15,900.00
5	...		3.03%	\$21,180.00	\$15,700.00
6	...			7.63%	\$15,440.00
7	...				14.85%

After making the necessary changes, we resolve the PV model template and obtain the results in Table 10.9.

Table 10.9. PV template used to find

	A	B	C	D	E	F	G	H	I	J	K
1	Year	Assets	Debt capital	Capital accounts liquidation value	Capital accounts book value	Depreciation (Dep= ΔE)	Asset Operating accounts (AR + INV)	Sales	Cash Receipts (CR = H+ Δ G)	Liability Operating Accounts (AP+AL)	Expenses (COGS + OEs)
2	0	\$40,000	\$30,000	\$40,000	\$40,000						
3	1	\$0	\$20,000	\$30,000	\$30,000	\$10,000	\$1,000	\$20,000	\$19,000	\$0	\$9,000
4	2	\$0	\$10,000	\$15,000	\$15,000	\$15,000	\$1,200	\$30,200	\$30,000	\$0	\$13,500
5	3	\$0	\$0	\$5,000	\$5,000	\$10,000	\$800	\$35,600	\$36,000	\$0	\$16,500
6	4	\$0	\$0	\$0	\$0	\$5,000	\$400	\$39,600	\$40,000	\$0	\$18,000

An Incremental Investment Problem: Adding Landscaping Services to GWS

Partial budget analysis. We emphasize that we recommend incremental investment analysis compare before and after whole firm investments after making appropriate size adjustments. On the other hand, we recognize that sometimes an approximation result is adequate and that examining the changes in

before and after whole firm's NPV and IRR results may be enough, especially if the incremental investment is small and likely to have limited influence in the firm.

Examples of when partial budgeting has been employed include incremental investments that: expand an existing enterprise, purchase new equipment, change the commodity being produced when equipment requirements are similar, participating in a government program, and changing a production practice.

The most significant limitations are partial budgeting analysis as it is typically employed is that it ignores the time value of money and is limited to comparing two investments. Furthermore, it is often limited to analyzing benefits and costs in a single period, making it ineffective for long term investments¹. Were we to make an inter-temporal adaptation of partial budgeting analysis, we would find in each period the change in cash receipts less the change in cash expenses. Alternatively, Benefits = increased return from the new activity plus reduced costs of the replaced activity less Costs=reduced returns from old activity + increased costs associated with new activity Or, as suggested in partial budgeting analysis, the increase in cash costs from the new investment

GWS's *incremental investment*. Before finalizing his investment, Lon wonders about an adjustment to his financial plan. If he adds a trailer and a small tractor to his investment, he can perform not only yard care services but some basic landscaping services as well. The additional investment would equal \$30,000 that is also depreciable over the next four years. Comparing the four-year cash flow associated with GWS, Lon wonders what percent change would be required to break-even? In other words, Lon wants to find α in the following equation:

$$(10.6) \quad \$30,000 = \frac{\sigma x \$10,000}{1.04} + \frac{\sigma x \$16,200}{1.04^2} + \frac{\sigma x \$17,600}{1.04^3} + \frac{\sigma x \$18,920}{1.04^4}$$

1. A reference to partial budgeting analysis is: <https://sustainable-farming.rutgers.edu/wp-content/uploads/2014/09/Partial-Budgeting-Manual.pdf>

Table 10.10. Cash Flow Required to Break-Even with a \$30,000 Incremental Investment

	A	B	C
1	Year	Cash Flow	Cash Flow(α)
2	0	(\$30,000)	(\$30,000)
3	1	\$10,000	\$5,318
4	2	\$16,200	\$8,615
5	3	\$17,600	\$9,360
6	4	\$18,920	\$10,062
7	$r(1-T^*)$	0.04	0.04
8	NPV ^A	\$26,412.43	\$0.00
9	AE ^A	-\$7,276.36	\$0.00
10	IRR ^A	33.38%	4%
11	α		0.531798

The goal-seek results confirm that for a \$30,000 incremental investment would need to earn roughly 53% of the original cash flow to break-even, that is, earn the defender's IRR^A of 4%.

Brown and Round Doughnuts

An important kind of incremental investment is one that to adapt means replacing or modifying an existing investment. In this section we consider a doughnut machine replacement. Although less preferred, we can make incremental investment analyzes without using the PV model template. In this section we demonstrate how to perform incremental investment analysis without using the PV model template.

Brown and Round Doughnuts. We next consider two alternative machines for making doughnuts. One machine is the one already in operation. In this case the challengers are an older version of the original machine and a new machine. Should you continue with the used machine, buy a new machine, or get out of the doughnut business?

Continuing to make doughnuts with the old machine.

In this example the challenger is an older version of the original investment. To be specific, suppose you bought a doughnut machine 3 years ago for \$90,000. You are depreciating the machine over 5 years using the MACRS method, and the expected market value 5 years from today is \$10,000. The

machine generates \$30,000 in cash revenues and produces \$15,000 in cash expenses each year. If you sold the machine today, you could get \$30,000 from a local competitor.

Your after-tax IRR^A, $r(1 - T)$, is 10%, your marginal tax rate is 40%, and the capital gains tax is 20%. Assume book value depreciation can be used to offset ordinary income.

Let's begin with the old machine. Keeping the old machine is equivalent to reinvesting its \$30,000 current value (V_0) in the business and forfeiting the tax refund from capital losses. To determine the tax refund forfeited, recall that the used machine was purchased 3 years earlier for \$90,000, and has been depreciated using 5-year MACR, so the machine's book value is $\$90,000(100\% - 15\% - 25.5\% - 17.85\%) = \$37,485$.

Since book value exceeds liquidation value, a tax credit is owed to the seller equal to $T(V_0 - V_0^{\text{book}}) = .4(30,000 - 37,485) = \$2,994$. Now we can write the acquisition ATCF as $ATCF_0 = -\$30,000 - \$2,994 = -\$32,994$. Finally, 5 year MACR depreciation rates on the old machine equal: 16.66%, 16.66%, and 8.33%. We write the ATCF for the t^{th} period as:

$$\begin{aligned}
 ATCF_t &= (CR_t - CE_t - Dep_t)(1 - T) + Dep_{t-n} \\
 (10.7) \quad &= (CR_t - CE_T)(1 - T) + TDep_t - \Delta V_n
 \end{aligned}$$

However, because ATCF per period differs, we describe it using Table 10.11. below.

Table 10.11. Finding ATCF for Continuing to Operate with the Old Doughnut Machine

Year	CR _t	CE _t	Dep _t (MACR)	(CR _t -CE _t)(1-T) + TDep _t	ΔV _n	ATCF _t
1	30,000	15,000	14,994 (16.66%)	(15,000 x .6) + (14,994 x .4) = 14,998	0	\$14,998
2	30,000	15,000	14,994 (16.66%)	(15,000 x .6) + (14,994 x .4) = 14,998	0	\$14,998
3	30,000	15,000	7,470 (8.33%)	(15,000 x .6) + (7470 x .4) = 11,988	0	\$11,988
4	30,000	15,000	0	(15,000 x .6) = 9,000	0	\$9,000
5	30,000	15,000	0	(15,000 x .6) = 9,000	0	\$9,000

Finally, we write the salvage value for the old machine. Recalling that the salvage value of the old machine, V_5 , is \$10,000, that the old machine is completely depreciated, and the capital gains tax rate T_g is 0.2, we can write the salvage value as:

$$\begin{aligned}
 ATCF_5 &= V_5(\text{old}) - T[V_5(\text{old}) - V_5(\text{book of old})] \\
 (10.8) \quad &= \$10,000 - .2(\$10,000 - 0) = \$8,000
 \end{aligned}$$

Finally, we are prepared to combine the ATCF from the acquisition, operation, and liquidation of the old doughnut machine. We express these in Table 10.12.

Table 10.12. Old Doughnut Machine Operating ATCF

Year	CR _t	CE _t	Dep _t (MACR)	(CR _t -CE _t)(1-T) + TDep _t	ΔV _n	ATCF ₁
0	$-V_0 - T(V_0 - V_0^{\text{book}}) = -\$30,000 - (.4)(\$30,000 - \$37,485) = -\$30,000 - \$2,994 = -\$32,994$					-\$32,994 (Acquisition)
1	30,000	15,000	$\frac{14,994}{(16.66\%)}$	$(15,000 \times .6) + (14,994 \times .4) = 14,998$	0	\$14,998
2	30,000	15,000	$\frac{14,994}{(16.66\%)}$	$(15,000 \times .6) + (14,994 \times .4) = 14,998$	0	\$14,998
3	30,000	15,000	$\frac{7,470}{(8.33\%)}$	$(15,000 \times .6) + (7,470 \times .4) = 11,988$	0	\$11,988
4	30,000	15,000	0	$(15,000 \times .6) = 9,000$	0	\$9,000
5	30,000	15,000	0	$(15,000 \times .6) = 9,000$	0	\$9,000
6	$V_5 - T(V_5 - V_5^{\text{book}}) = \$10,000 - (.2)(\$10,000 - 0) = \$10,000 - \$2,000 = \$8,000$					\$8,000 (Liquidation)

The only other calculation left is to compute the NPV of the old doughnut machine which we complete using our Excel spreadsheet.

Table 10.13. Old Doughnut Machine
[Open Table 10.13. in Microsoft Excel](#)

	A	B	C
1	variables	data	formulas
2	rate	0.1	
3	Acquisition cost	-\$32,994	
4	ATCF1	\$14,998	
5	ATCF2	\$14,998	
6	ATCF3	\$11,988	
7	ATCF4	\$9,000	
8	ATCF5	\$9,000	
9	Salvage value	\$8,000	
10	NPV	\$18,745	$=-V_0 + NPV(\text{rate}, \text{ATCF1}:\text{ATCF5}) + V_n / (1 + \text{rate})^{\text{nper}}$
11	IRR	30%	$=IRR(\text{Acquisition cost}, \text{ATCF1}:\text{ATCF5}, \text{salvage value})$
12	AE	(\$4,944.92)	$=PMT(\text{rate}, \text{nper}, \text{NPV})$
13	nper	5	

A new doughnut machine.

Assume that, after completing the PV analysis of the old doughnut machine, a salesman for a “new and improved” doughnut machine stops by and wants to sell you a new machine. The new machine will

increase your revenues to \$45,000 each year, and decrease your expenses to only \$10,000 each year. The new machine costs \$90,000 and will require \$10,000 for delivery and installation costs. Thus, the acquisition $ATCF_0 = \$100,000$. The machine will fall into the MACRS five-year depreciation class (15%, 25.5%, 17.85%, 16.66%, 16.66%, and 8.33%). The machine is expected to have a \$40,000 salvage value after 5 years.

Next, we need to determine the operating ATCF for the new machine. Tax rates, the defender's after-tax IRR, and MACR depreciation rates are the same as before—equal to those used to evaluate the continued use of the old doughnut machine. Because of the level of sales, accounts receivable and accounts payable increase by \$5,000 and \$2,000 in the first year respectively. They are reduced by the same amount in the fifth year. The formula for operating ATCF is the same as before except that sales and expenses are no longer just cash items and must be adjusted by the term $\Delta V_n = \Delta CA - \Delta CL = \$5,000 - \$2,000 = \$3,000$

$$\begin{aligned}
 ATCF_t &= (\text{Sales}_t - \text{Expenses}_t - Dep_t)(1 - T) + Dep_t - \Delta V_n \\
 (10.9) \quad &= (\text{Sales}_t - \text{Expenses}_t)(1 - T) + TDep_t - \Delta V_n
 \end{aligned}$$

The new machine will be depreciated using the 5-year MACRS method. The depreciation for each machine, and the change in depreciation each year for the next 5 years is:

Table 10.14a. New Doughnut Machine Depreciation

Year	\$100,000 x MACR rate	=	Depreciation
1	\$100,000 x 15%	=	\$15,000
2	\$100,000 x 25.5%	=	\$25,500
3	\$100,000 x 17.85%	=	\$17,850
4	\$100,000 x 16.66%	=	\$16,660
5	\$100,000 x 16.66%	=	\$16,660

Remember that depreciation expense is a noncash expense, which by itself doesn't generate a cash flow. However, you can use the depreciation expense to reduce taxable income. As a result, the firm realizes an additional tax savings from depreciation. The amount of the tax savings are described below.

Table 10.14b. New Doughnut Machine Tax Savings

Year	Tax Savings = (Depreciation)(T = .4)
1	\$15,000 x (.4) = \$6,000
2	\$25,500 x (.4) = \$10,200
3	\$17,850 x (.4) = \$7,140
4	\$16,660 x (.4) = \$6,664
5	\$16,660 x (.4) = \$6,664

Finally, we find the salvage value as follows. We first recognize that the sale of the doughnut machine in 5 years will increase cash flow by \$40,000. Next, we find the asset's book value in year five as the difference between its acquisition value less its accumulated depreciation: \$100,000 - (\$15,000 + \$25,500 + \$17,850 + \$16,600 + \$16,660) = \$8,330. Since the asset's book value is less than its salvage value, there are capital gains taxes to be paid. Therefore, the salvage value ATCF can be written as the salvage value less the capital gains tax:

$$\begin{aligned}
 ATCF &= V_5 - T[V_5 - V_5(\text{book})] \\
 (10.10) \quad &= \$40,000 - .2(\$40,000 - \$8,330) = \$33,666
 \end{aligned}$$

Finally, we combine acquisition, operation, and liquidation ATCF associated with the new doughnut machine in Table 10.15.

Table 10.15. New Doughnut Machine Operating ATCF

Year	Sales	Expense	Dep _t (MACR)	(Sales - Expenses) x (1 - T) + T Dep.	ΔV _n	ATCF _t
0 ATCF ₀	(Acquisition) = -V ₀ =					-\$100,000
1 ATCF ₁	45,000	10,000	15,000 (15%)	(35,000 x .6) + (15,000 x .4) = 27,000	3,000	\$24,000
2 ATCF ₂	45,000	10,000	25,500 (25.5%)	(35,000 x .6) + (25,500 x .4) = 31,200	0	\$31,200
3 ATCF ₃	45,000	10,000	17,850 (17.85%)	(35,000 x .6) + (17,850 x .4) = 28,140	0	\$28,140
4 ATCF ₄	45,000	10,000	16,660 (16.66%)	(35,000 x .6) + (16,660 x .4) = 27,664	0	\$27,664
5 ATCF ₅	45,000	10,000	16,660 (16.66%)	(35,000 x .6) + (16,660 x .4) = 27,664	-3,000	\$30,664
5 ATCF ₅	(liquidation) = V ₅ - T[V ₅ - V ₅ (book)] = \$40,000 - .2(\$40,000 - \$8,330) =					\$33,666

We find the NPV for the new doughnut machine in Table 10.16:

Table 10.16. New Doughnut Machine NPV

Year	$\frac{1}{(1+r(1-T))^2} = \frac{1}{1.10)^t}$	ATCF	$\frac{ATCF}{(1+r(1-T))^t}$
0	1	-\$100,000	-\$100,000
1	.91	\$24,000	\$21,840
2	0.83	\$31,200	\$25,896
3	0.75	\$28,140	\$21,105
4	0.68	\$27,664	\$18,812
5	0.62	\$30,664 + \$33,666 = \$64,330	\$39,885

$$NPV = \sum_{t=0}^5 \frac{ATCF_t}{1+r(1-T)^2} = \$27,538 \text{ (27,584 using CF worksheet)}$$

Finally, we find the NPV for the new doughnut machine using our Excel Spreadsheet.

Table 10.17. New Doughnut Machine
[Open Table 10.17. in Microsoft Excel](#)

	A	B	C
1	variables	data	formulas
2	rate	0.1	
3	Acquisition cost	-\$100,000	
4	ATCF1	\$24,000	
5	ATCF2	\$31,200	
6	ATCF3	\$28,140	
7	ATCF4	\$27,664	
8	ATCF5	\$30,664	
9	Salvage value	\$33,666	
10	NPV	\$27,584	=-V0+NPV(rate,ATCF1:ATCF5)+Vn/(1+rate)^nper
11	IRR	18%	=IRR(Acquisition cost, ATCF1:ATCF5,salvage value)
12	AE	(\$7,276.60)	=PMT(rate,nper,NPV)
13	nper	5	

So, what have we learned? NPV will be positive for both options, continuing to use the old machine or investing in the new one. However, the firm's NPV will increase the most if it adopts the new machine—an increase of \$8,839 (\$27,584 - \$18,745) over continuing to use

the old machine. However, the IRR of the old machine is greater, and we have a conflict between IRR and NPV rankings. The ranking is resolved by deciding on the reinvestment rate for the differences in funding required by the two investments. If the difference in funding levels is invested at the discount rate, the NPV rankings are appropriate. If the investments can be scaled, then the IRR rankings are appropriate. It is assumed here that the reinvestment rate is the after-tax IRR of the defender, 10%, so that any differences in size would not change their respective NPVs, and NPV rankings are appropriate.

Summary and Conclusions

This chapter has reviewed two stand-alone and incremental investments. Of the two types of investments, incremental one may be the most common and therefore merit a separate chapter calling attention to their special nature.

Of importance, incremental investments require that its optimal use requires that the use of existing services and nondurable inputs be changed to accommodate the incremental investment. As a result, it has become a somewhat common approach to analyze incremental investments by measuring changes in all the variables used in the PV template and calculating NPVs, IRRs, and AEs based on the changes in cash flow.

We wondered how the rankings before and after the incremental investment would compare to the NPV, IRR, and AE of the difference? We note that the rankings might be inconsistent because the homogeneity of size condition was violated. NPV rankings generally were consistent.

So, we ask: why perform incremental investment at all? What not just compare the firm's financial rates of return and earnings before and after the investment? While we recommend this approach, we also recommend that PV problems are solved by practical people solving practical and immediate problems and may lack the resources required to solve a complete firm investment analysis. In these cases, reverting to incremental investments with perhaps less data requirements is the preferred approach.

Questions

1. Please describe the difference in investment focus between an AIS and a PV model?
2. What is the main difference between a stand-alone versus an incremental investment? Explain why some sources describe incremental investment analysis as incremental budgeting analysis.

3. This chapter presented two main PV model templates. One focused on analyzing investments. The second one focused on analyzing equity committed to the investment. Describe the differences between the two templates. Then explain how these differences permit us to calculate returns associated with investments versus returns on equity committed to investments.
4. Describe the nature of a fixed investment.
4. A capital investment provides services over several periods. A nondurable provides services once. Explain how this difference in capital goods and nondurable inputs are recognized in PV models.
5. A capital investment provides services for more than one period without losing its identity. The cost of providing these services is the change in the liquidation of the capital asset and the use of nondurable goods. The change in the liquidation value of a capital asset may mostly depend on the passage of time and the intensity of its use. Please explain how these two costs that result from the passage of time and use influence the optimal service extraction rate from the capital asset.
6. Operating accounts include AR, INV, AP, and AL. Describe how decreases (increases) in capital accounts influence operating and liquidation ATCF.
7. What is the difference between exogenous and endogenous variables used to solve for NPV, AE, and IRR in PV models?
8. Solving PV models requires that we forecast the values of exogenous and endogenous variables for the economic life of the investment. Describe how these forecasts might vary in their sophistication.

II. Forecasting and Present Value Models

LINDON ROBISON

Learning goals. At the end of this chapter, you should be able to: (1) distinguish between deterministic and statistical forecasts; (2) recognize the difference between endogenous and exogenous variable forecasts; (3) recognize the difference between dependent and independent variables in statistical forecasts; (4) recognize the value of quantitative forecasts; (5) understand how to apply economic theory to guide forecasts, and (6) be able to complete present value (PV) models by forecasting endogenous and exogenous variables.

Learning objectives. To achieve your learning goals, you should complete the following objectives:

- Understand the difference between forecasts and predictions.
- Be able to recognize different kinds of deterministic and statistical forecasts.
- Recognize the difference between exogenous and endogenous variables and forecasts.
- Distinguish between independent and dependent variables in regression equations.
- Learn the difference between quantitative forecasts and qualitative predictions.
- Understand why present value (PV) models require cash receipts and cash expense forecasts.
- Learn how to find forecast confidence intervals.
- Utilize information about time and use costs to forecast cash costs of goods sold (COGS) and overhead expenses (OEs).
- Understand how to use correlations among variables to make forecasts.
- Practice forecasting techniques for simple investment problems.

Introduction

1

Forecasts and predictions. This chapter is about forecasting. Forecasts use observations and perceptions about the past to predict the future. The origin of the word forecast comes from the word “fore” which means before plus the word “casten” which means to scheme, plan, or prepare. Thus, when we forecast, we predict something before it happens so that we can plan or prepare for it.

1. Much of the material in this chapter was developed in an article under review by the Agricultural Finance Review titled: “Coordinated Financial Statements: What-is, What-if, and How-much Questions” by Lindon J. Robison and Peter J. Barry submitted January 2021.

Related to forecasts are predictions. A prediction is also a statement about what someone believes will happen in the future. We sometimes distinguish between predictions and forecasts based on how we arrived at our estimate of the future. A prediction is a guess, less scientific, and more subjective than a forecast. In this view, predictions are more eclectic in the information they collect and use to describe the future—including past observations, informed opinions, conditional relationships, signs, impressions, and other foreshadowings of the future.

Everyday forecasts. We all make forecasts. We depend on them. Indeed, nearly every choice we make is preceded by a forecast. We are so dependent on our almost “every moment” forecasts that we may give them little notice. For example, we may give little thought to our forecast that if we turn on the water faucet, then water will pour out. Still our forecast precedes and motivates our act of turning on the faucet.

The relative importance of forecasts. What makes some forecasts more important than others are when they lead to significant resource commitments and are costly if not impossible to reverse. Buying a house, a new car, or choosing one’s companion may fall into the category of important forecasts. When making important forecasts, we may need to make several related forecasts. In the case of buying a car, we may want to forecast its frequency of repairs, its fuel efficiency, its riding comfort, and rather it impresses our partner. To improve our car-related forecasts, we may visit with friends who have made similar purchases. We may read professional car buying guides that evaluate the performance and cost of alternative cars of interest. We may even rent the car of interest to experience the car upfront and personal. But at the end of the day, we forecast our car purchase experience and make our choice—and live with the results.

Forecast accuracy. The accuracy of our forecasts varies depending on the quality of information and analysis behind them and the time distance to the future forecast. Consider the following. Because we have a long history of turning on the water faucet and experiencing the flow of water, we may be nearly certain of the relationship between turning on the faucet and experiencing the flow of water. In this case, the accuracy of our forecast is high. On the other hand, we may have heard that an eatery serves a tasty Rueben sandwich and based on the “expert” opinions of our friends, we may forecast that if we eat the Rueben at the eatery, we will be pleased with the experience. In this case the accuracy of our forecast is moderate. (Who is to say that our friends have the same tastes in sandwiches that we do?) Some forecasts are embedded in advertisements—that by consuming certain products, you will become popular and rich and your skin blemishes will disappear. These forecasts should be questioned unless supported by double-blind studies—because they are not motivated to report accurate results.

Forecasting, a two-step procedure. Forecasting is a two-step procedure. In the first step, we collect and analyze data to discover variables are related to each other over time. Suppose we observe an exogenous variable at time t , x_t , and find that it is significantly correlated with the same variable in the previous periods. Then we could write: $x_t = f(x_{t-1}, x_{t-2}, \dots) + \varepsilon_t$ where ε_t is an error term that represents what is not explained by the exogenous variable in the current and adjacent time periods.

Suppose that in the first step, we forecast a value for x_t that we designate as x_t^f since it is an estimate because we don’t know what the actual value is. Then to estimate two periods in the future we solve

using coefficient we found in step one to find: $x_{t+1} = f(x_t, x_{t-1}, \dots)$. In the case that $x_{t+1} - x_t = c$ then our forecast k periods in the future is: $x_{t+k} = x_t + kc$.

What follows. In what follows, we distinguish between different forecast equations based on the nearly certain information that provides the basis for our forecasts. While all forecast equations provide estimates of the future, some of them are derived deterministically and some statistically. In the case of statistically estimated forecast equations, it will be important to distinguish between dependent and independent variables. When describing the kinds of variables forecast, we will use the earlier language of Chapters 5 and 7, to distinguish between exogenous and endogenous variables. The purpose of this chapter is, at the end of the day, to find the forecasted endogenous and exogenous variables required to complete PV models. Questions at the end of the chapter can be used to practice and gain some forecasting experience.

Forecast Equations

Exogenous and endogenous variables. Exogenous variables are defined outside of a system. Endogenous variables are defined within the system. When we constructed coordinated financial statements (CFS) and PV templates, we distinguished between variables whose values were determined within the CFS and PV template, endogenous variables, and those variables whose values were determined outside of the CFS and PV template, exogenous variables. Within both the CFS system and PV templates, endogenous and exogenous variables are related to each other using deterministic equations.

Dependent and independent variables. When conducting experiments, it is common to use the language of dependent and independent variables. Independent variables are controlled by the experimenter and varied to determine their influence on dependent variables related to independent variables by statistical models. However, sometimes the relationship between dependent and independent variables is not determined experimentally but inferred and observed. This is the sense that we use dependent and independent variables in forecast equation.

Independent variables and what do we know for sure? We require independent variables to forecast dependent variables. Our independent variables are those used as exogenous variables in the CFS and PV templates used to determine endogenous variables. So, what independent variables do we know for sure? There are only two things that we know for sure that can be used as independent variables in our forecast equation: the time distance to a future date and the values of variables already realized, including lagged values of the dependent variable. If we were to include a contemporaneous variable to our independent variables in our forecast equation, we would need to forecast it as well as the dependent variable. This sometimes occurs when large forecast models forecast several dependent variables using multiple equations. In our discussion, we limit our forecasts to single equations. Finally, our confidence in the correlation between our dependent variables and time and lagged values increases with the quality and quantity of information used to infer the correlations.

Still, human behavior can often be predictable—we often act the same way in similar circumstances, and we develop habits that seem ingrained. We often use these tendencies to aid us in our forecasts. One important category of events that we use to guide our forecasts are commitments and covenants to perform in a predictable way in the future. For example, most have borrowed and agreed to repay on loans following specified terms. These performance commitments can be treated as nearly time dependent exogenous variables.

Forecasting endogenous variables. While forecasts mostly focus on exogenous variables when used within CFS and PV templates, we may find it useful to forecast endogenous variables. Forecasting endogenous variables is usually not necessary because they are determined by relationships established within the CFS and PV template. However, we recognize that our exogenous variables are estimated with error so that endogenous variable forecasts derived from the exogenous variable forecasts also contain errors. As a result, we may find it useful to over-identify some endogenous variables and a check on accuracy of our exogenous variables.

So, what have we learned? We learned that system variables may be either exogenous variables determined outside of the system or endogenous variables determined within the system by exogenous variables. Exogenous variables used to determine endogenous variables are dependent variables forecast by statistical or other models using independent variables that have been observed or that are highly correlated with the passage of time. In sum, we are mostly interested in forecasting exogenous variables (for our purposes) to use in CFS or PV templates. While we could technically, forecast endogenous variables, and sometimes we do, they are not our primary concern because they are determined within the CFS or PV templates.

In what follows, we will organize our forecast equation around what we know for sure. As a result, some forecast equations depend only on the passage of time. Others depend on the lagged values of a dependent variables. And still others depend on the lagged values of independent variables. Finally, we may create hybrid forecast equations that combine time and lagged values of dependent and independent variables. Forecasts that combine contemporaneous independent variables require accompanying equations that forecast their values—forecast equations in which they are the dependent variables.

Forecast Equations and What we Know

Trend lines. Consider forecasting exogenous variables that are highly correlated with or are dependent on the passage of time. Time may enter the forecast equation in several different forms, but to forecast future periods only requires time as the independent variable. An example may be an equation that

forecasts the condition of a building's roof k periods in the future. The condition of the roof and its remaining life might both be expressed as a function of the roof's age. Consider such an expression:

A roof's remaining useful life equals its useful life when constructed – its age.

Forecast equations that depend mostly on the passage of time are often referred to as trend lines. In this case, the endogenous variable y depends only on a constant and the independent variable t . This forecast equation can be expressed as:

$$(11.1) \quad y_t = \theta_0 + \theta_1 t + \epsilon_t$$

The trend line forecast k periods in the can simply be written as:

$$(11.2) \quad \hat{y}_{t+k} = \theta_0 + \theta_1(t + k)$$

In equation (11.2), the hat over the endogenous variable signifies that we have an estimate, not an actual value.

Lagged dependent variable. Suppose that Lon wants to forecast the demand for his services next season. His forecast might begin with the demand for his services he experienced during the previous period. One reason to emphasize the demand for Lon's services in the previous period is because so many conditions that existed last year will exist this year. To reflect the connection between this year's forecast and what happened last year, we specify:

Services in period t = services provided in period $(t - 1)$ + an error term

To complicate our forecasts, we recognize that the change in demand for Lon's services in period t will depend on the number of new lawn care providers (Lon's competition) plus change in the number of potential customers measured by the number of new households in Lon's service area with a yard. Call these changes in the demand less changes increases in services provided by others equal to change in services Δs_t and write the result as:

Services in period t = services provided in period $t - 1 + \Delta s_t + \text{error}$.

Distributed lags of independent variables. Closely related to forecasts that depend on the passage of time and lagged values of depend variables are lagged values of independent variable on which the dependent variable depends. This forecast equation is relevant when the dependent variable depends on the outcome of other independent variables. Some variables such as costs of goods sold (COGS), depend on production activities in the previous period. So, if we intend to forecast COGS, we may want to relate them to some function of production activities in the previous period.

COGS this period = some function of production activity levels in the previous period.

A popular forecast equation that combines both the lagged dependent variable and independent variables is the infinite geometric distributed lag (IGL) forecast equation. We derive it next.

Suppose we want to forecast a dependent variable at the end of period t , y_t , whose value we assume depends on the infinite lagged values of an independent variable z , $z_{t-1}, z_{t-2}, z_{t-3}, \dots$ weighted by the geometric factor $|\rho| < 1$. In this case we can write our forecast equation as:

$$(11.3) \quad y_t = \alpha + \beta(z_t + \rho z_{t-1} + \rho^2 z_{t-2} + \dots) + \epsilon_t$$

where α , β , and ρ are estimated coefficients. Without simplifying (11.3), it would be nearly impossible to estimate. However, using the geometric series sum technique, we can simplify (11.3) to an equation that we can estimate. To simplify, we lag (11.3) to obtain

$$(11.4) \quad y_{t-1} = \alpha + \beta(z_{t-1} + \rho z_{t-2} + \rho^2 z_{t-3} + \dots) + \epsilon_{t-1}$$

Then if we multiply (11.4) with ρ and subtract the result from (11.3), we obtain the geometric sum of the lagged values of z and an equation that we can estimate:

$$(11.5) \quad y_t - \rho y_{t-1} = \alpha - \rho\alpha + \beta z_t + \epsilon_t - \rho\epsilon_{t-1}$$

Solving for the current value of the endogenous variable and letting the error term equal $v_t = \epsilon_t - \rho\epsilon_{t-1}$ we can write:

$$(11.6a) \quad y_t = \alpha(1 - \rho) + \rho y_{t-1} + \beta z_t + v_t$$

And

$$(11.6b) \quad y_t = \gamma_0 + \rho y_{t-1} + \beta z_t + v_t$$

and where $\alpha = \gamma_0 / (1 - \rho)$. In addition, the proper estimation of equation (11.6b) and other linear regression models requires that the error term, in this case v_t , is distributed normally with a zero mean and a variance of σ^2 , a condition we express as $N(0, \sigma^2)$.

The geometric lag. The geometric lag structure that produced equation (11.6b) is specific. More general lag structures can be estimated. Indeed, another geometric lag structure is $(z_{t-1} + \rho z_{t-2} + \rho^2 z_{t-3} + \dots)$ that produces the model in equation (11.7) that doesn't require a forecasted value of z to estimate y and can be expressed as:

$$(11.7) \quad y_t = \gamma_0 + \gamma_1 y_{t-1} + \gamma^2 z_{t-1} + v_t$$

Forecasting exogenous variables using time and lagged values as independent variables. We have already described forecasting as a two-step procedure. The first step requires that we find the relationship between dependent variables and lagged dependent variables, time and/or lagged value of other relevant variables calculated using past observations. The second step requires that we project the forecast dependent or exogenous variable into the future. If the independent variable is time, then to find a forecast requires only that we update our time variables. On the other hand, if our independent variable includes lagged values, then we must forecast using a stepwise procedure. In the first step we find our next period forecast. In the second step our forecast becomes the independent variable in our

two-period future forecast—and so on to obtain future in the future forecasts. To illustrate, suppose that in the first-step we estimated that:

$$(11.8) \quad \hat{x}_t = a + bx_{t-1}$$

Then in the second step, our first period forecast becomes the independent variable is our two- period forecast allowing us to write:

$$(11.9) \quad \hat{x}_{t+1} = \alpha + \beta\hat{x}_t$$

Repeating the substitution process, we obtain a k period in future forecast equal to:

$$(11.10) \quad \hat{x}_{t+k} = \alpha + \beta\hat{x}_{t+k-1}$$

It should be noted that in equation (11.8), the forecast was only one period away from something we knew for sure, the lagged value of dependent variable. In equation (11.9) we were two periods away and in equation (11.10) we were k periods away from something we knew for sure. Therefore, we might conclude that the further away from lagged values we know for sure, the less confidence we should have in the accuracy of our forecast.

Deterministic versus linear regression forecasting equations. There are two ways to process data that describes the relationships between dependent and independent variables. The first method assumes that our relationship that describes the connection between dependent and independent variables is measured without error and the forecast is deterministic. The second method assumes that the relationship between dependent and independent variables approximates the true relationship which we don't know. So, in this second case, the best we can do is to estimate statistically the relationship between the dependent and independent variable(s) and recognize that the actual forecast is a sophisticated guess. We will refer to these methods here that explicitly recognize we are not completely accurate in our estimation of the dependent and independent variable relationship as a statistical regression model.

Several factors determine whether we should forecast using a deterministic or a regression model. These factors may include our confidence in our data, the importance of our forecasts, and difficulty and cost of gathering and processing the data. In what follows, we describe and give examples of both deterministic and linear regression models.

Deterministic Forecasts

A naïve forecast. Assume that the relationship between a dependent variable y is a linear function of itself lagged equal to:

$$(11.11) \quad y_t = a + by_{t-1}$$

If we are unable or unwilling to estimate coefficient a and b in equation (11.11), we may assume their values to equal: $a = 0$ and $b = 1$. This assumption implies that $y_t = y_{t-1}$. We call this a naïve forecast.

Even though the deterministic naïve forecast is our simplest forecast, it has its place. Suppose our question is: what would be the consequences on the investment's value if everything continued as it is right now? The answer to that question may indeed be a naïve (constant value) forecast. We demonstrate a naïve forecast using Lon's services provided during the previous four years. These are represented in Table 11.1. Our forecast description includes the actual value x_t , the forecast $x_t^{\hat{}}$, and the error of the forecast ϵ_t .

Table 11.1. Deterministic naïve forecasts of Lon's service calls

Time period	Actual value x_t	Forecast vales $x_t^{\hat{}}$	Forecast error: $\epsilon_t = x_t - x_t^{\hat{}}$
0	525	n.a.	
1	755	525	230
2	890	755	135
3	990	890	100
Forecast	n.a.	990	n.a.

One test of the accuracy of one's forecast is the pattern of the forecast error. We expect it to be randomly distributed about zero—in this case the sum of errors should be close to zero. In this case the sum of errors equal 365. In this case, our forecast model fails the test, and so we try a different forecast model.

Moving average forecasts. A step up in forecast sophistication that does not rely on statistical methods is the moving average. Suppose that we are interested in predicting the amount of services Lon could provide if the business continued another year. If we truly believed there was no trend in the data and the variation was due to random events, we might reduce the random effects on the forecast by averaging the previous four period $(990 + 890 + 725 + 525)/4 = 782.5$ or possible the last three $(990 + 890 + 755)/3 = 878.3$ or perhaps the most recent two observations $(990 + 890)/2 = 940$. However, because there is a positive trend, a weighted average forecast is even less accurate than the naïve forecast of 990 over the sample period, and we have no way of knowing its accuracy of the forecast beyond the sample period.

Geometric forecasts. PV models by their nature are geometrically designed created by compounding discount rates of the form $ab, ab^1, ab^2, \dots, ab^n$ where a and b are constants and superscripts represent the geometric factor. As a result, we may assume that our forecasts should also follow some form of a geometric series.

To make geometric forecasts, we begin by observing previous values of exogenous variable values reported in current and past Coordinated Financial Statements (CFS). To illustrate, suppose that over the past 10 years, corn yields g_1, g_2, \dots, g_{10} have increased by 13%. Then letting \bar{g} equal the average percentage increase in corn yields we find the average yield in equation (11.12) equal to:

$$(11.12) \quad (1 + g_1)(1 + g_2) \cdots (1 + g_n) = (1 + \bar{g})^n$$

Where g_t is the actual growth rate in the exogenous variable for periods $t = 1, \dots, n$ and the average geometric growth rate is \bar{g} . To illustrate suppose we had growth rates of corn yields for 10 year that we represent as:

$$(11.13) \quad (1 + g_1)(1 + g_2) \cdots (1 + g_{10}) = (1.13) = (1 + \bar{g})^n$$

Then solving deterministically for the average geometric mean over the past ten years of yields, we find:

$$(11.14) \quad (1 + \bar{g}) = (1.13)^{0.1} = 1.0123$$

Or an average, for the past 10 years, yields in period t that we represent with the variable y_t increased by 1.23%. Assuming the past trends in yield continue and if the yield last year were 140 bushels per acre, in 5 years we would expect yields to have increased to:

$$(11.15) \quad y_5 = 140(1.0123)^5 = 148.82$$

In other words, corn yields in period 5, assuming the average yield trend for the past ten years continues will equal 148.82 bushels per acre. In the case of Lon's services, over four years services have increased by $990/525 = 200\%$ or $1.89^{0.3} = 1.26$ of that services have increased on average by 26 percent per year so that next year's forecast for Lon's business is $(990)(1.26) = 1,221$.

Linear Regression Forecasts

What is a linear regression forecast? The second forecasting method is the statistical procedure referred to as multiple linear regression or in the case of a single explanatory variable, linear regression. Regression analysis finds a coefficient for a line (plane) through points representing values of a dependent variable such that the squared differences between the line and the values of the regression line are minimized. Hence, the regression method is sometimes referred to as least squares regression analysis.

Regression models can also be used to measure the percentage of the total variation around the mean of the data explained by regression equation using what is referred to as an R^2 that is between zero and 100 percent. Obviously, as we add more independent variables to our regression equation, we can increase the value of R^2 . However, as we add independent variables to our regression equation, the significant of each individual independent variable will likely decrease, a significance measured by t-tests. Eventually, we can add enough independent variables that our relationships is deterministic. When conducting regression analysis, we assume that independent variables are measured without error and not influenced by each other. Dependent variables are related to the independent variables and we expect them to be forecast with some error.

Finally, the ability to estimate regression equations are often complicated by missing variables, non-normal distributions of ϵ_t , autocorrelations between the error terms and correlations between the error terms and the exogenous variables and other properties keep econometricians occupied. These complications and others encountered when estimating regression equations, we ignore in this chapter in order to emphasize the fundamental concepts of forecasting.

A Lagged Regression Model and Forecast. Suppose we return to equation (11.11) and instead of assuming coefficient values of $a = 0$ and $b = 1$ we estimate coefficients a and b in regression equation (11.16).

$$(11.16) \quad y_t = a + by_{t-1} + \epsilon_t \text{ where } \epsilon_t \sim N(0, \sigma^2)$$

Fortunately, Excel has a statistical function that can estimate a and b so that the sum of ϵ_t^2 is minimized and the sum of ϵ_t 's is zero. To access regression analysis for estimating regression coefficient a and b in Excel we select Data and regression from the drop-down menu. If the Analysis tab does not appear, it may be necessary to access it from your options. Instructions to enable this option are in the [appendix](#) at the end of this chapter.

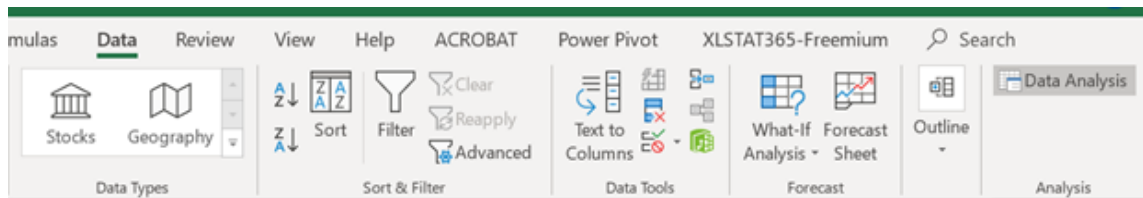


Figure 11.1. Data Ribbon for Excel Spreadsheet

After selecting the Data tab, select regression from the drop-down menu.

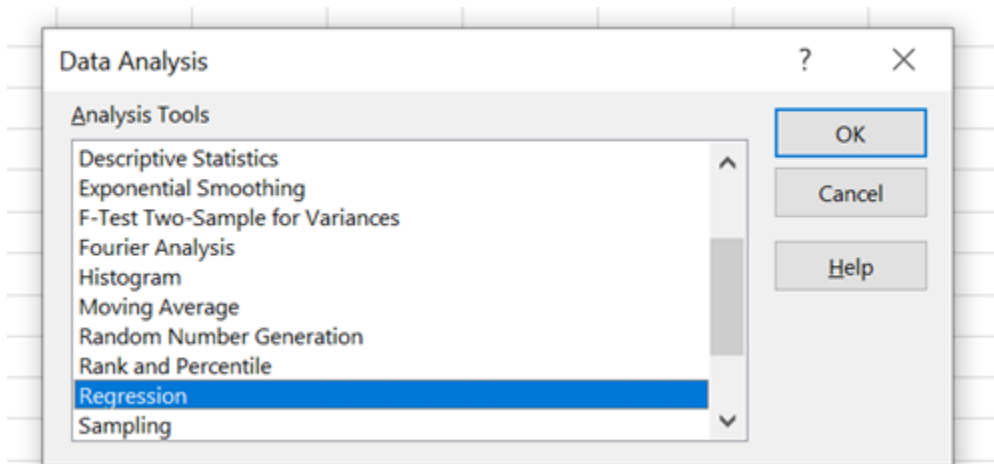


Figure 11.2. Data Analysis Pop-up Menu

After selecting regression from the drop-down menu, we are asked for regression analysis data. The lagged independent values for x_{t-1} (A2:A4) are entered in the Input X Range. The current or dependent values x_t (B2:B4) are then entered as the Input Y Range:

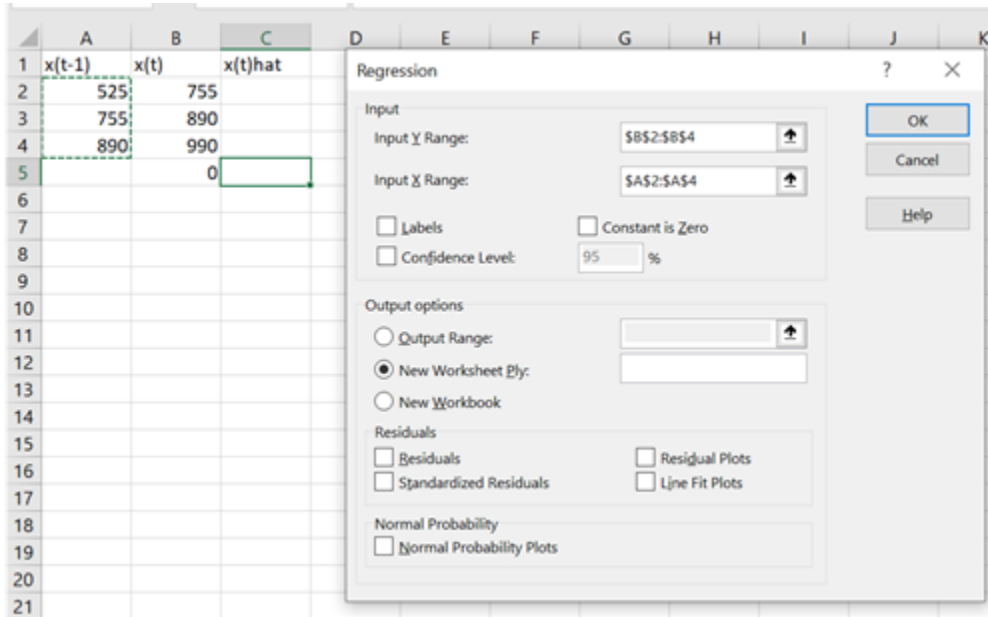


Figure 11.3. Data input page for the regression analysis.

We include a request that the regression equation find the constant value a . Clicking OK returns the regression estimates for our forecast equation equal to:

Table 11.2. The results from a linear regression analysis for Lon's sales.
[Open Table 11.2 in Microsoft Excel.](#)

	A	B	C	D	E	F	G
1	Summary Output						
2							
3	Regression Statistics						
4	Multiple R	0.997992					
5	R Square	0.995989					
6	Adjusted R	0.991978					
7	Standard	10.56297					
8	Observation	3					
9							
10	ANOVA						
11		df	SS	MS	F	Significance F	
12	Regression	1	27705.09	27705.09	248.3061	0.040346	
13	Residual	1	111.5763	111.5763			
14	Total	2	27816.67				
15							
16		Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
17	Intercept	417.0247	29.90354	13.94567	0.045572	37.06427	796.9852
18	X Variable (t)	0.637754	0.040472	15.75773	0.040346	0.123503	1.152005

Interpreting all the output described above is beyond the scope of this discussion and the proper topic for a beginning econometrics class. However, note that R^2 accounts for most of the variation in the data and the t-tests for the coefficients a and b are greater than 1.9 which is often reported as the minimum required for the independent variables to be retained.

Having found the forecast equation, we use the results to obtain our forecasts. To do so, we focus on the estimates of coefficient a reported in cell B17 equal to 417.02 and coefficient b reported in cell B18 equal to .638.

$$(11.18) \quad \hat{x}_t = a + bx_{t-1} = 417.2 + .638x_{t-1}$$

Next we create our table of lagged values, estimated values, errors, and a forecast value.

Table 11.3. Naïve forecasts of Lon's service calls

Time period	Actual value x_t	Estimated and Forecast vales \hat{x}_t	Forecast error: $\epsilon_t = x_t - \hat{x}_t$
0	525	n.a.	
1	755	752	3
2	890	899	-9
3	990	985	5
Forecast	n.a.	1049	n.a.

Notice that we obtain our forecast for x_{t-1} by substituting our last observed value of 990 into our forecast equation:

$$(11.19) \quad \hat{x}_t = a + bx_{t-1} = 417.2 + .638(990) = 1049.$$

Notice that accounting for a trend line significantly improved our estimated values over the sample range compared to previous forecast equations. Furthermore, the error term appears more nearly random and sums to a number close to zero: $3 - 9 + 5 = -1$. However, before we become too impressed with our estimates compared to our actual values, we must acknowledge that we have only three estimated values to estimate the two unknown parameters “a” and “b”. In other words, we almost have a perfectly identified systems so there is really no estimating challenge at all leaving us with a lack of confidence that our three-year trend will really continue.

Nonlinear trend lines. It is common for most new business to make impressive gains early on and then see gains level off. Indeed, it appears that for many processes, growth appears to follow a concave down pattern as might be described in the graph below.

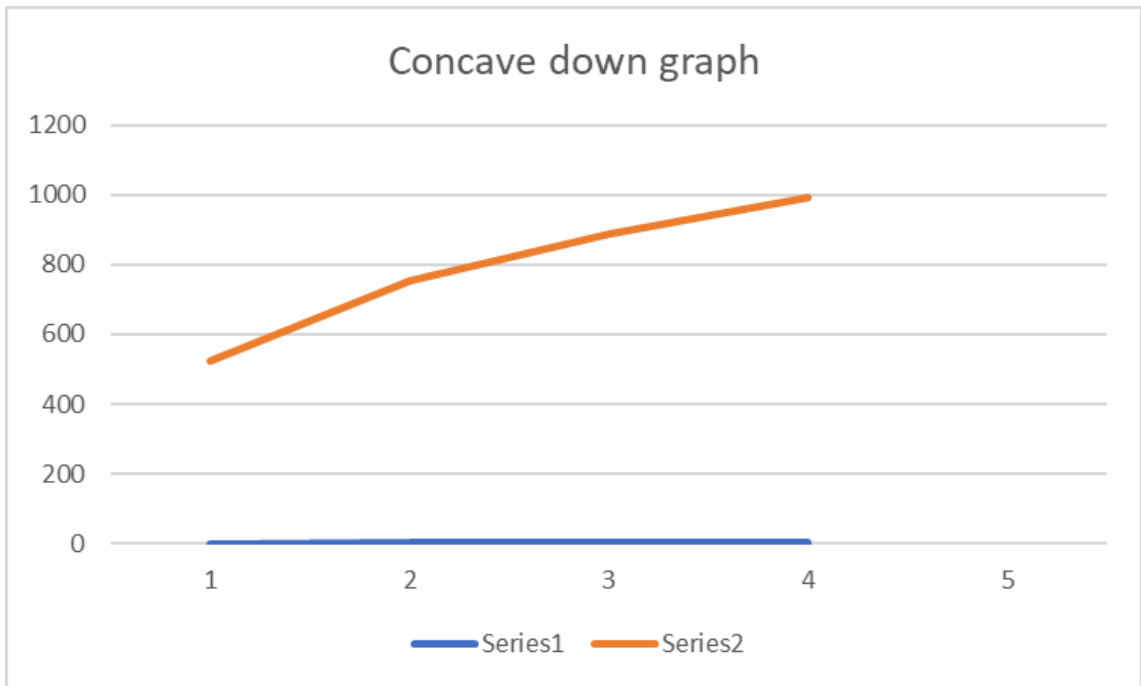


Figure 11.4. Quadratic Trend Line.

Such a function can be represented with a more complicated mathematical representation like the one below:

$$(11.20) \quad y_t = \beta_0 + \beta_1 t + \beta_2 t^2$$

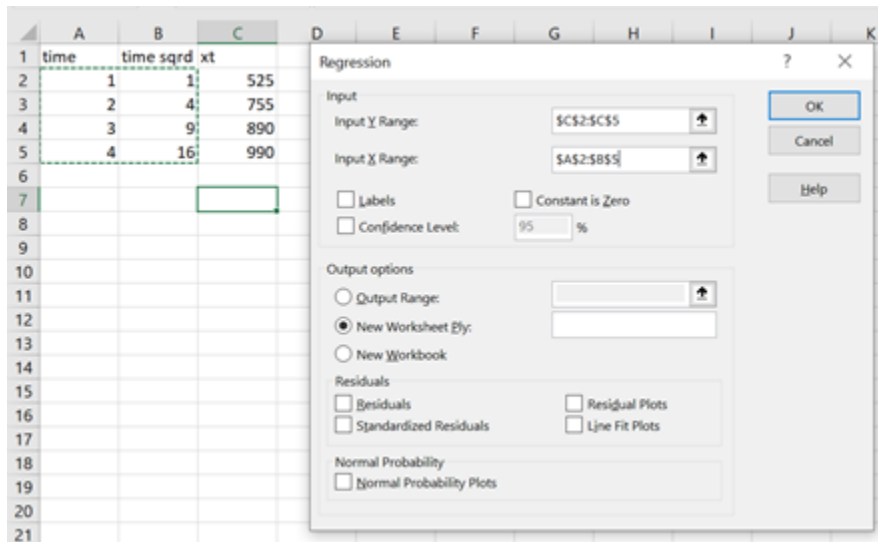


Figure 11.5. Data input page for a quadratic trend line.

Notice for the exogenous variable time and time squared, we entered the data matrix. Next we click OK to obtain the result:

Table 11.4. Regression results for forecasting Lon's services using a quadratic time function.
[Open Table 11.4 in Microsoft Excel.](#)

	A	B	C	D	E	F	G
1	Summary Output						
2							
3	Regression Statistics						
4	Multiple R	0.999259					
5	R Square	0.998518					
6	Adjusted R	0.995554					
7	Standard	13.41641					
8	Observation	4					
9							
10	ANOVA						
11		df	SS	MS	F	Significance F	
12	Regression	2	121270	60635	336.8611	0.038498	
13	Residual	1	180	180			
14	Total	3	121450				
15							
16		Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
17	Intercept	245	37.3497	6.559625	0.09631	-229.573	719.5729
18	X Variable (t)	315.5	34.07345	9.259409	0.068488	-117.444	748.4442
19	X Variable (t ²)	-32.5	6.708204	-4.84481	0.129583	-117.736	5273581

We are particularly interested in the coefficient for our estimated equation (11.21). We write the results as:

$$(11.21) \hat{x}_t = a + bt + ct^2 = 245 + 315.5t = 32.5t^2.$$

We create our table of lagged values, estimated values, errors, and a forecast value.

Table 11.5. Forecasts of Lon’s service calls based on a quadratic function of time.

Time period	Actual value x_t	Estimated and Forecast vales $x_t^{\hat{}}$	Forecast error: $\epsilon_t = x_t - x_t^{\hat{}}$
1	525	528	-3
2	755	746	9
3	890	899	-9
4	990	987	3
Forecast	n.a.	1010	n.a.

Note that the forecast based on a quadratic function of time is 1010 compared to 1049 based on a linear function of time. Since both equations report significant R^2 and t-values—we must choose between them based on economic logic—which model best fits the reality on the ground?

Confidence intervals. Assume the new potential owner is unsure about the reliability of the forecasts and asks what are the high and low forecast values? Using the forecast.confidence function, we can find statistical bounds around the projected trend line using the standard deviation of values around the trend link. Then we add and subtract the standard deviation multiplied by a constant such that given data and assuming the trend continues, we can be 95% sure that the “true” projection is included in the bounded confidence interval. Below we include the forecast values for Lon’s services during the next four year. Then we graph the original values, their forecasted values and the confidence intervals.

yr	value	Forecast(value)	Lower Confidence Bound(value)	Upper Confidence Bound(value)
1	575			
2	755			
3	890			
4	990	990	990.00	990.00
5		1136.801056	1082.95	1190.65
6		1273.468975	1217.94	1328.99
7		1410.136894	1352.98	1467.30
8		1546.804813	1488.04	1605.57

Table 11.6. Forecast values for the demand for Lon’s Green & White Services for the next four years, and 95% confidence intervals around the forecasts.

Forecasts and Shocks

Exogenous variable forecasts and shocks. Now suppose that in time $t + k$ we forecast the value of our dependent variable but something we call a shock, s_{t+k} , occurs that hasn't occurred in the past. Because the shock hasn't occurred in the past, we don't account for it in our forecast. Or suppose something that has occurred in the past that helped us forecast y_{t+k} doesn't occur? This is another kind of a shock.

To illustrate the first kind of shock, consider Christmas tree sales. Based on our past experiences, we forecast a significant rise in Christmas tree sales in December. But suppose this year, the world experiences a pandemic and tree-harvesters are quarantined and unable to supply the usual number of Christmas trees. Then we may find that our Christmas tree sales forecast to be widely inaccurate.

Shocks may have two kinds of consequences. And the kinds of forecasting consequences shocks may produce depend on what we are forecasting. If we are forecasting a quantitative value—e.g. Christmas tree sales, our forecast may be inaccurate such as we observe a reduction in sales of 50% below our forecast. Or, if we are forecasting an event with only two possible outcomes, e.g. win or lose, then our forecast is either right or wrong, correct or incorrect, true or false. Such would be the case if we forecast the winner of the Super Bowl and the wrong team wins. In this case, our forecast was not merely inaccurate, it was wrong.

Getting real. Before proceeding, we need to get real. Relationships between endogenous and exogenous variables are hardly ever stable or easily identified. Remember shocks? To illustrate, assume that we intended to forecast future corn prices. We begin by identifying the connection between past corn prices and other past exogenous variables. But then, we might find the relationship that we thought we had estimated was not stable and susceptible to forces not included in our model including shocks. Corn prices fell by two and one-half times in the 1970s leading to a fall in related prices and the financial crisis of the 1980s. Land values fell as much as 50 percent during the same time period. Later, in the 2006–2013 period corn prices nearly doubled over the previous twenty-year period only to fall again in 2013 with input prices substantially higher than in the previous period. In real terms, current profitability was lower than it was in the post 1980's period. Historically, it has taken grain and oilseed markets six to twelve years to find their “new normal” following large, sustained shocks. History again provides a guide to the future, but the internal confidence around these future projections should be wide.

Recognizing the difficulty of what we're about, in what follows we employ equation (11.1) beginning with its most simple form and then consider increasing complicated forecast equations. Our goal is to find first the relationship between endogenous and exogenous variables to forecast their future values.

So, what have we learned? We have learned that forecasts depend on their values being highly correlated with the passage of time where we determined the correlation between t

and independent variables by past observations. However, things in the future may not be just like they were in the past. Things that happened in the past that we assumed will happen in the future, don't. Or things that didn't happen in the past, do happen in the future. Both unplanned events we call shocks. So, in time period $t+k$, instead of observing y_{t+k} , we observe $y_{t+k} + s_{t+k}$. What is the influence of a shock on our forecast depends on if we are forecasting discrete outcomes (win versus lose) or continuous values (the level of Christmas tree sales).

Forecasting Exogenous Variables and the Accuracy of Endogenous Variables

Earlier we distinguished between dependent and independent variables in forecast equations and exogenous and endogenous variables in CFS and PV templates. When the exogenous variables are known in CFS and PV templates, endogenous variables are found using deterministic equations. We now ask, what about exogenous and endogenous variables in CFS and PV models when exogenous variables are forecast with error. Now we are less confident that the deterministic relationships from which we find endogenous variables as some deterministic function of exogenous variables can still be relied on?

We propose that we establish some over-identified checks on the values of our endogenous variables as means of validating our exogenous variable forecasts. In practice this proposal means that we use economic reasoning and other established relationships between endogenous and exogenous variables to obtain independent estimates of our exogenous variables besides the deterministic one obtains from CFS and PV templates. In other words, once we have obtained forecasts for our exogenous variable(s) x_t , we are prepared to use these forecasts to find the future values of our endogenous variables y_t .

These conditional forecasts of our endogenous variables take the form: if $x(t)$ then $y(t)$. To illustrate, we may forecast umbrella sales to increase if it rains—if it rains then umbrella sales will increase. We may not be able to predict the weather, but we can predict how consumer purchases will likely respond to the occurrence of rain. In the statement, if $x(t)$ then $\hat{y}(t)$, $x(t)$ is the exogenous variable forecast and $\hat{y}(t)$ is the endogenous variable forecast.

Another illustration of a conditional forecast, the forecast of an endogenous variable, may be the firm's profit forecasts. An agricultural firm's profit forecast may depend on several factors including weather, crop yields, trade policies, inventory levels, input costs, revenue and expense margins, and national economic variables such as stock market indices, consumer price indices, consumers' disposable incomes, the interest rate, and the level of consumer confidence to name a few.

Once we have obtained our exogenous variable forecasts from deterministic or regression equations, we now intend to use these to forecast future values of endogenous variables. The connection between exogenous and endogenous variables may be deterministic as was their calculations in CFS and PV templates. In other cases, they simply may be related to each other statistically even though economic logic may lead to the equation that forecasts endogenous variables from the base of exogenous variables.

Sometime correlations with exogenous variables can be useful even if the exogenous variable is not one with which we are directly interested. This is because several exogenous variables may be influenced in similar ways by time. Prices are a case in point. Some prices seem to move together through time and this correlation may be particularly helpful in our endogenous variable forecasts. In these cases, the relationship between exogenous and endogenous variables are not deterministic and must be estimated.

Price forecasts. In our original study of Lon's service, we predicted prices to be constant over the four years of Lon's operations. This forecast is naïve. Instead, suppose that we believed that Lon's prices for services would follow national price trends. One national price trend is the consumer price index (CPI). The CPI measures changes in the price level of a weighted average market basket of representative consumer goods and services purchased by households. There are, of course, several different CPI indexes. To calculate the CPI, we designate a base year for prices and assign it an index of 100. Then the updated value of the index is found by dividing the index of prices in the updated period by the index of prices in the base year:

$$(11.22) \quad \frac{CPI_t}{CPI_{base}} = \frac{\text{Average Prices}_t}{\text{Average Prices}_{base}} \approx \frac{\text{Lon's prices}_t}{\text{Average Lon's Prices}_{base}}$$

Suppose that we wish to know the average percentage increase from the base year to year t . If the current CPI were 123% and the base year were 10 years earlier, we could find the average geometric increase in prices to be those we found earlier:

$$(11.23) \quad (1 + \bar{g}) = (1.23)^{(1/10)} = 1.02$$

In other words, prices (including the price of lawn care services) have increased by 2% per year. If we applied this estimated average of prices to yard care services, we can forecast Lon's future price for yard care services:

Table 11.7. Forecast values for the price of Lon's Green & White Services Assuming a 2.1% increase

Year	Forecast Service Prices	Service times forecast price = sales	COGS = .45 x sales
Base	\$40		
1	\$40.84	$\$40.84 \times 525 = \$21,441$	\$965
2	\$41.69	$\$41.69 \times 755 = \$31,476$	\$14,164
3	\$42.67	$\$42.67 \times 890 = \$37,976$	\$17,089
4	\$43.46	$\$43.46 \times 990 = \$43,025$	\$19,361

Percentage forecasts and margins. One of the main ways we can forecast endogenous variables outside of the CFS and PV template is to observe the relationships between them and exogenous variables in the past—and the more observations, the better. One observation that is important using past data is the margin between revenue and costs. We can be confident that this difference is bounded, otherwise, competitors would flock to join our industry and begin replicating our operations. If the difference is negative for some time, we will become insolvent and forced to leave the industry. Sometimes, if we can discover the connection between margins and time—we may forecast margin as a function of time. For example, suppose we are interested in the price of hard red winter wheat but can only observe past prices of corn and soybeans. Using linear regression, a statistical technique that allows us to relate an endogenous variable to one or more exogenous variable—we can use the future estimates of corn and soybean prices to predict hard red winter prices.

Sales data. In column three of Table 11.1 and beginning in year one, we estimate cash sales or cash cost reductions. We can employ several assumptions about future cash flow associated with sales or reduced costs. The first one is that there is no change from data entered in year one. This assumption is consistent with capitalization formula that assume zero change in future cash flow. It may also be relevant if the model assumes that data are in real numbers so that numbers do not change with inflation. Furthermore, we may assume the future is represented by the present and these relationships between variables in the present should be preserved.

Economic logic and margins. Economic logic and margins may guide our endogenous variable forecasts. Recall that common size income statements provide us percentage relationships between endogenous and exogenous variables. Consider the common size income statement for HQN. Such a statement could, of course, be acquired for other industries more nearly like the industry in which we are focused.

Table 11.8. (Table 7.9) HQN's Accrued Common-Size Income Statements

Year	2017	2018	Industry Average
Total Revenue	100.00%	100.00%	100.00%
Cost of Goods Sold (COGS)	67.37%	70.00%	71.40%
Operating Expenses	29.82%	27.50%	22.50%
Depreciation	1.24%	0.88%	1.75%
EARNING BEFORE INTEREST AND TAXES (EBIT)	1.58%	1.62%	4.35%
Interest	1.22%	1.20%	0.55%
EARNINGS BEFORE TAXES (EBT)	0.36%	0.42%	3.80%
Taxes	0.17%	0.17%	1.52%
NET INCOME AFTER TAXES (NIAT)	0.19%	0.25%	2.28%

Economic logic suggests that some relationships between exogenous and endogenous variables may continue in the future. For example, taxes are likely to be function of the difference between revenue and sales. Another class of relationships likely to persist over time are margins, 100% minus the percent of costs as a function of revenue. Economic logic tells us that margins can't be zero or negative or the firm wouldn't survive. Nor can they be excessively high without inviting competitors and innovators to join the industry. Of course, changing competitive advantages and innovations may temporarily affect margins, in the short run and will likely encourage changes in the industry as firms expand to take advantage of the innovations. And as we increase in sophistication, we might worry about local, national, and international politics that may influence tariffs and the challenges of competing in a global economy—ultimately affecting our margins.

COGS and Revenue. The foundation for our forecasts is the economic relationship between the variables in our industry. Once we have price and/or yield forecasts, we can add other forecasts by treating them as dependent variables. Two variables of economic necessity must be highly correlated are costs and revenue. For example, we may expect to find that COGS has a predictable relationship to sales since the more we produce and sale the more expenses we are likely to occur. Therefore, once we have sales forecasts, we can forecast COGS using established relationships observed from past data.

One other relationship is the difference between revenue and costs. We can be confident that this difference is bounded. Otherwise, if the difference were strongly positive, competitors would flock to our industry and begin replicating our operations. If the difference is negative for a significant period of time, we will become insolvent and leave the industry. Taxes will be some function of the difference between revenue and sales. Changes in comparative advantages and preferences may also be included in the description of how the variables are related. Future preferences will influence demand for what we produce.

Changes in comparative advantage—who can produce least expensively and where—will also influence revenue and expenses. Favorable increases in comparative advantage will reduce our costs and put us in a position to expand our operation. And as we increase in sophistication, we might worry about

local, national, and international politics that may influence tariffs and the challenges of competing in a global economy.

Consider COGS in Table 11.1. In this example, we estimated COGS to be on average 45% of sales. Sales have been forecast to increase by 3 percent from base sales we observed in the CFS that described the firm. As a result, COGS will also increase by 3% maintaining a constant relationship of 45% between sales and COGS.

Still another method for estimating future cash revenue and expense cash flow is to employ someone else's guesses about the future. This is difficult because our investment and earning potential is often unique to our own firm. Still there are guesses about the future that are related to our own investment's future. And there are sophisticated forecasting firms and experts who make predictions about the future relevant to our own investment. For example, a dairy firm might be interested in forecasted milk prices or the expected cost of feed. Firms related to home sales would find future interest rate forecasts to be relevant since the demand for homes can be expected to be related to the cost of home financing. Indeed, most investments can be tied to some future forecast that is relevant.

COGS and OE. Associated with the business include what economists refer to as fixed and variable costs. We treat COGS as variable costs that follow the investment's cash revenue. In addition, we impose on the relationship between CR and COGS the restriction that long-term, Sales > COGS. OE are costs difficult to assign to any one production activity and vary over time. In our simplified analysis, we did not assume any OE. This is an unreasonable assumption since such fixed costs such as insurance, rental of office and storage space, and licenses all produce fixed costs. In the previous four years, fixed costs were \$350, \$390, \$450, \$515. In addition, in the most recent two years, Lon was required to rent additional space that cost \$100 in each period. We use the OE observations and estimate the following equation using our observed data.

$$(11.24) \quad OE = \alpha_0 + \alpha_1 \text{time} + \alpha_2 \text{ additional storage}$$

The way our forecasts work in practice is that we find some historical basis for the equation and from those relationships estimate our coefficients.

Now we illustrate the various forecasting methods using the data in our example template. To begin, suppose that we are confident in the cash flow data appearing in year two since year one may be a transition year. Dividing cash COGS by cash sales we find:

$$(11.25) \quad 0 < \gamma < \frac{COGS}{\text{Cash Sales}} = \frac{\$106,913}{\$137,068} = 78\%$$

As a result, in the future, we will assume that COGS equal 78% of whatever we estimate sales to equal. Whatever we estimate sales to equal, we will assume that COGS equal

Other forecasts. Sometimes, failing to find forecasts of direct interest to the calculation of our investment's NPV, AE, and IRR—we can often find forecasts highly correlated with our forecasts of interest. For example, suppose we are interested in the price of hard red winter wheat but can only observe past

prices of corn and soybeans. Finding past relationships between the price of hard red winter wheat and corn and soybean prices, we can use forecasts of corn and soybean prices to forecast future prices of hard red winter wheat.

Real versus nominal data values. In our PV templates, we project CR, cash COGS and cash OE. We can employ several assumptions about future cash flow associated with these values. The first one is that there is no change from data entered in year one. This assumption is consistent with capitalization formula that assume zero change in future cash flow. It may also be relevant if the model assumes that data are in real numbers so that numbers do not change with inflation. Furthermore, we may assume that the future is represented by the present and these relationships between variables in the present should be preserved.

Testing for robustness. That we populate our PV templates with estimates suggests that our templates will not be completely accurate. They will be consistent because the templates relate variables to each other consistently, but they are not likely to be completely accurate. We simply are not likely to estimate all of the future values of our variables correctly. As a result, we should calculate NPV, AE, and IRR under a variety of assumptions about future values so that we have some impression about the robustness of our estimates.

So, what have we learned? We have learned that forecasting recognizes two different kinds of statements. The first of these is of the form: if t then y_t . From this relationship between t and y_t , we make forecasts that include linear trend lines and other more complicated functions of time. Other kinds of forecast depend on lagged values of the dependent and independent variables.

We have been careful to distinguish between dependent and independent variables in forecast equations and exogenous and endogenous variables in CFS and PV templates. In the latter case, the relationship between exogenous and endogenous variables are deterministic that depend on accounting identities. The problem occurs that when we forecast exogenous variables, these are estimated with error that implies that our endogenous variables in CFS and PV templates are also estimated with error. Then it may be useful to find overidentified estimates of our endogenous variables to validate our exogenous variable forecasts—using the structure of our CFS and PV templates to check for accuracy. These require that we forecast our endogenous variables as well as our exogenous variables using economic and other logic to guide our estimation of endogenous variables. This type of estimation, like other forecasts will be of the form if x_t then y_t .

Qualitative Forecasts

Not everything we know can be reduced to a number. If all the lessons learned from the past could be translated into a quantitative forecast function, we could leave much of forecasting to computers (they already carry a significant forecasting role). But such is not the case. We all know much more that can be represented by a numerical forecast. When historical data is not available, we generally employ qualitative forecasting techniques that most often depend on the judgment of experts—that may be aided by quantitative forecasts.

Experts and all the rest of us observe, absorb, and process informative through our multiple sources and senses. So how do we do it? We may not be able to describe the process because our intuitive sense are complicated and profound. Book like “Thinking Fast and Slow” try to describe some aspects of our capacity to analyze and predict using information much more sophisticated that has been described in this chapter. This is not to disrespect the amazing information captured by digital coding, especially in the sciences, but when it comes to the “animal spirits” that operate in interpersonal interaction, it pays to leave room for qualitative judgments and forecasts.

So, we employ and appreciate professional forecasters to share their qualitative forecasts, sometimes augmented by quantitative forecasts. In may be the case that we will increasing translate qualitative forecasts into quantitative ones—and multitude are the examples of qualitative forecasts being improved using quantitative information.

Professional forecasts are usually experts in their specialty such a yields and prices of agricultural and forestry products as well as margins between revenue and cost. An example of expert forecast services are those offered by the [Food and Agricultural Policy Research Institute \(FAPRI\)](#) at the University of Missouri. FAPRI makes annual 10-year baseline projections for grain, oilseed, livestock markets, and a variety of other indicators. Even here, it is important to moderate forecasts using microeconomic principles and experience in making projections including a range of scenarios. Understanding intermediate and longer-term industry supply curves including regional and international dimensions is very important for most projects. Extension specialists, such as Dr. Jim Hilker, who have distinguished themselves with accurate forecasts can also be an invaluable resource.

One advantage of professional forecasts is that they account for so many influences that uncomplicated forecasters may ignore. The other advantage is that when we present our forecasts and responses to our forecasts are important—it is always helpful to have a reliable source for support.

PV Models and Forecasts

Forecasts and present value models. While this chapter is about forecasting, we are most interested in forecasting PV model exogenous variables. Solving PV models requires that we forecast the future

value of cash flow generated by an investment. PV models include exogenous and endogenous variables.

PV models discount future cash flows to find their equivalent worth in the present—since it is in the present where we live and make decisions. To repeat for emphasis, we cannot solve PV models without forecasting future exogenous variable cash flow on which the present value of an investment depends. When we invest in the present, all we know for sure is the cost of the investment, the time distance to different futures, lagged values of endogenous and exogenous variables, and future commitments like loan terms. Every other piece of data requires us to solve the PV problem and exogenous variable forecasts.

To illustrate, the template below is identified as: PV Template: Green & White Services. It will be helpful in our forecasting discussions in this chapter to continue using Lon’s yard care investment example. We will embellish it somewhat to focus on certain aspects of forecasting while maintaining the main characteristics of the problem.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
2	Assets	Debt capital	Capital	Capital	Deprcat	Asset	Sales	Cash	Liability	Expenses	Cash	Interest	depreciation	Cash flow	After-tax	After-tax	Sum	sum	Rolling	Rolling	Rolling	
		liquidation	accounts	ion	ion	Operating	Receipts	Receipts	Operating	(COGS +	Expenses	costs	tax	(CR-CE-De-	Liquidating	Liquidating	Sum	liquidations	NPV	AE	IRR	
		value	book	value	value	accounts	(CR)+ sales-	Δ (AR+INV)	Accounts	OE)	(CE)+	(I+O-)	savings=	-(I-T)+	asset	liability	Capital,	last period	cash flow			
			value	value		(AR + INV)			(AP+AL)	Expenses-	(AP+AL)		TiDeprca	Tdep. + (D-	operating	operating	and debt	operating				
													tion (T%)	De-)	accounts: (f-	accounts: (f-	accounts: (f-	cash flow				
3	Yr																					
4	0	\$40,000	\$20,000	\$40,000	\$40,000																	
5	1	\$0	\$20,000	\$30,000	\$30,000	\$1,000	\$21,000	\$20,000	\$0	\$9,000	\$9,000	\$1,200	\$2,000	\$9,800	\$800	\$0	\$10,800	\$20,640	-\$305	-\$325	3.20%	
6	2	\$0	\$20,000	\$35,000	\$35,000	\$1,200	\$30,200	\$30,000	\$0	\$13,500	\$13,500	\$1,200	\$3,000	\$15,240	\$960	\$0	-\$4,040	\$11,200	-\$413	-\$227	3.37%	
7	3	\$0	\$20,000	\$5,000	\$5,000	\$10,000	\$800	\$35,800	\$36,000	\$0	\$26,500	\$16,500	\$1,200	\$2,000	\$26,640	\$640	\$0	-\$14,360	\$2,280	\$5,246	\$1,977	20.37%
8	4	\$0	\$20,000	\$0	\$0	\$5,000	\$400	\$39,600	\$40,000	\$0	\$28,000	\$18,000	\$1,200	\$1,000	\$27,640	\$320	\$0	-\$18,680	-\$2,040	\$26,031	\$4,669	41.24%

Table 11.9. PV Template: Green & White Services [Open PV Template in Microsoft Excel.](#)

Data for Green & White Services. Looking to the future, Lon is considering the purchase of a lawn care/snow removal business. He will purchase the good will (customer base) of the previous owner of the business and equipment for \$40,000. His expected service calls, service prices and variable cost per service are reported below:

Table 11.10. Endogenous and exogenous variable forecasts for Green & White Services.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	Year	Number of Services	Price/Service	Cost/Service	Sales	AR+INV	CR	Expenses	AP+AL	CE	Depreciation Rates	V0	Depreciat
2	1	525	40	18	21000	1000	20000	9450	100	9350	0.25	40000	100
3	2	755	40	18	30200	1200	30000	13590	200	13490	0.375	40000	150
4	3	890	40	18	35600	800	36000	16220	100	16120	0.25	40000	100
5	4	990	40	18	39600	400	40000	17820	100	17820	0.125	40000	50

Forecast number of services are described Table 11.10 column B. Prices and cost per service forecasts are described in columns C and D and when multiplied by projected services produce sales revenue in column E and expenses in column H. Note that these are accumulated sales and costs, not cash. The sum of accounts receivable (AR) and inventory (INV) at the end of the period are reported in column F while the sum of accounts payable (AP) and accrued liabilities (AL) are reported in column I. Cash receipts (CR) and cash expenses (CE) are found by adjusting sales and expenses for changes in asset and liability accounts and the results are recorded in columns G and J respectively. Allowable depreciation rates are recorded in column K and when multiplied by the capital asset purchase of \$40,000 produce depreciation amounts recorded in M. Finally, debt outstanding is recorded in Column N.

Besides the annual data forecasts, we also record certain rate forecasts for Green & White Services. Some of these as well as some of the annual forecasts may be either exogenous or endogenous variables depending on how the endogenous defining equations are define. We include the current exogenous rates as:

Table 11.11. IRRs, average tax rate, average interest rate, and growth rates is applicable for Green & White Services.

	A	B
1	ROA	0.08
2	T	0.2
3	ROE(1-T)	0.064
4	Average Interest Rate	0.06
5	Debt Growth Rate	0
6	Liquid Growth Rate	0
7	Book Value Growth Rate	0
8	Account Balance Growth Rate	0
9	CR Growth Rate	0
10	COGS Growth Rate	0
11	OE Growth Rate	0
12	Beginning Cash	0
13	Liability op. Account Rate	0

We now ask, where can we find future values for template variables that will permit us to solve multi-period PV problems? The answer is that we forecast or estimate (guess) the value of exogenous variables in the template and based on these exogenous variable forecasts, we solve for the endogenous forecast variables.

Populating the PV template with exogenous and endogenous variables. Like Coordinated Financial Statements (CFS), we populate PV templates with exogenous and endogenous variables. The columns of data in Table 11.9 are organized into exogenous and endogenous variables. The exogenous variables are highlighted. Endogenous variables are not. The main distinction between exogenous and endoge-

nous variables within the template is the following. The values of exogenous variables are determined outside of the template. The values of endogenous variables are determined within the main template. Excel makes it easy to distinguish between exogenous and endogenous variables. We can identify endogenous variables within the template because they require an operations such as +, -, *, ^, or /. They may also require an operation embedded in an Excel function such as PV, PMT, NPER, RATE, IRR, or NPV.

Otherwise, they are exogenous. If the cell requires an operation, it will be signaled by an “=” sign that references other cells included in an operation. Exogenous variables may reference cells outside of the PV model with an equal sign when they contain values on which an operation will be performed. Alternatively, exogenous data cells in the template may include a number. The exception to this means for identifying endogenous variables is when they are calculated in separate pages and then copied into the main template.

We now describe in more detail the variables and data needed to populate our PV templates that enable us to find rolling NPV, AE, and IRR estimates for investments and invested equity. We begin with the only information we know for sure, future dates: 2019, 2020, 2021, etc. Therefore, we begin populating our PV templates with the one thing we know about the future: future dates recorded in column 1.

The investment acquisition amount. We record the investment acquisition amount in cell B4. Its value is determined outside of the template making it an exogenous variable. We treat the investment’s acquisition value as though it were a cash expenditure unless funded by debt, an amount recorded in cell C4. When we employ debt to acquire the investment, the focus is on the equity invested—equal to the investment amount less the debt used to acquire the investment.

If we purchased the investment(s) on credit, we enter the loan principal and interest payments in the years they are paid. We don’t enter the loan amount because money came from a lender and was received by the seller of the investment—Lon is only an intermediary in the exchange

The investment amount is also a well-documented number. Durable sellers post their offer prices. Markets establish and record the prices at which durables are exchanged. References for used durable prices like Kelly’s Blue Book for cars are generally available. Salvage values that occur sometime in the future are much harder to estimate. The IRS provides some economic life estimates and depreciation data equal to percent changes in the value of the original durable. Theses may help us establish book values and may provide some guide for determining the durables’ liquidation value.

Complicating our investment data are projects that incur investments costs over several time-periods. Indeed, one might consider repair and maintenance expenditures designed to improve the life and durable performance to represent additional investments. In other cases, the investment includes several durable that we replace at different intervals. For example, feeder calf operations routinely replace one cohort of calves with another but the feeding equipment and physical facilities we replace less often than the feed calves.

Liquidation values. To find rolling NPV, AE, and IRR estimates, we require not only an investment acquisition amount, but investment liquidation and market values over time. An investment’s acquisition

value, V_0 , is the amount of money exchanged to acquire the investment. An investment's book value in period t , V_t^{book} , equals its acquisition value minus its accumulated depreciation. An investment's liquidation value in period t , $V_t^{\text{liquidation}}$, is the amount of money a buyer is willing to exchange for the investment when it is sold. If an investment's liquidation value is greater than its acquisition value, we refer to the difference as capital gains. If an investment's liquidation value is less than its acquisition value but greater than its book value, we refer to the difference as depreciation recovery. Finally, if the liquidation value is less than its book value, we refer to the difference as capital losses. We describe capital gains, depreciation recovery, and capital losses in the following equations:

for $V_t^{\text{liquidation}} > V_0 > V_t^{\text{book}}$, capital gains = $V_t^{\text{liquidation}} - V_0 > 0$,

for $V_0 > V_t^{\text{liquidation}} > V_t^{\text{book}}$, depreciation recapture = $V_t^{\text{liquidation}} - V_t^{\text{book}} > 0$, and

for $V_0 > V_t^{\text{book}} > V_t^{\text{liquidation}}$, capital losses = $V_t^{\text{book}} - V_t^{\text{liquidation}} > 0$.

We determine accounting book values by applying tax regulations that specify depreciation amounts depending on the type and age of the investment. Since depreciation forecasts depend on the age and amount of the investment, we generally consider them endogenous variables. Furthermore, we recognize that liquidation values of investments are also generally determined by projecting market values and cost data and may be endogenously or exogenously variables. When we do not expect to liquidate the investments, but want to estimate their NPV, AE and IRR, we may equate market liquidation value with book values. That we populate our PV templates with forecasts, sophisticated guesses, suggests that our template is not completely accurate. As a result, we should calculate NPVs, AEs, and IRRs under a variety of guesses so that we have some impression about the robustness of our estimates.

Summary and Conclusions

We conclude this chapter with the observations that we are all forecasters—all the time. We are all today making decisions and plans that will influence our conditions in the future—and these require forecasts. As we review the nature of forecasts, we should be humbled by the process and two factors. The first one is that forecasts have been often wrong. As evidence, review the forecasts that predicted the great recession of 2008. And the second factor that should humble us is that the only thing we know for certain about the future is the time it will take us to get there.

We emphasize here that there exist many potential forecast equations. These should be evaluated based on their accuracy, their implementation difficulty, availability of data, and the importance of the forecast. Still, we need to accept that what should determine our confidence in the accuracy and correctness of our forecasts is finding exogenous variables highly correlated with the passage of time. Fortunately, there are several exogenous variables that satisfy this requirement. Most natural phenomenon have some predictable life cycle patterns of growth, production, and decay. We all make commitments that commit us to some action in the future that we often complete. As a result, our own

actions in the future can often be predicted. Still shocks often leave disturbs our confident in our forecasts.

Still, we must make decisions in the present that commit us to outcomes in the future—and there is no alternative. We will all continue to forecast and hope to become better in the effort. Hopefully, this chapter has provided some ideas on how we can all make better forecasts.

Questions

1. Describe the differences between forecasts and predictions.
2. How would you characterize the main business of forecasters?
3. List five forecasts and five predictions that you have made during the last week. Describe your confidence in the accuracy of your forecasts and predictions and what determined your confidence.
4. Forecasts can be accurate or inaccurate or wrong and right. Explain.
5. Evaluate the following forecast. A turkey woke up on Thanksgiving Day and forecast that it would be like every other day?
6. List two kinds of shocks. Provide an example from the nation news of the shocks.
7. Given the likelihood that not all of our forecasts will be accurate or correct, how can we plan these eventualities?
8. What is the difference between dependent and independent variables in forecast equations?
9. Explain the difference between exogenous and endogenous variables in the context of CFS and PV templates.
10. We may forecast exogenous and endogenous variables. Explain the difference in these forecasts and how they may be used.
11. Forecasting can be described as a two-step process.
12. The commodity price index is today 123 percent of its base 10 years ago.
 1. Using a deterministic method, forecast the price index next year. If corn prices today are \$3.18 forecast next year's corn prices using the following methods:
 2. Using a naïve forecast.
 3. Using a geometric average forecast based on the commodity price index.
 4. Assume that corn prices during the past four years were: \$2.56, \$3.90, \$2.89, \$3.25, and \$2.89. Use a linear regression model to forecast corn prices next year and the year after next.
 5. Using a regression equation with independent variables t and t^2 to forecast corn prices next year and the year after that.
 6. Compare and comment of the R^2 for the linear and quadratic function and the t statistics for coefficients of the independent variables in the equations.
13. Consider the PV template described in Table 11.1. Then resolve the template assuming that expenses (COGS + OEs) were an endogenous variable. Find the value of expenses using the exogenous variable sales data. Determine the relationship between COGS that vary with sales and OEs that mostly vary with the passage of time.

14. Assume the following endogenous and exogenous variables have been observed in the past. Forecast $Y(7)$ using the data below. Determine the accuracy of your forecast and the confidence interval around your forecast.

Time Period	$Y(t)$	$Y(t-1)$	$X(t-1)$
1	45	35	44
2	52	45	39
3	67	52	33
4	68	67	34
5	80	68	26
6	78	80	21
7			

Appendix

If the Data menu ribbon does not show the Data Analysis button at the far right, you can enable it in the Excel Options.

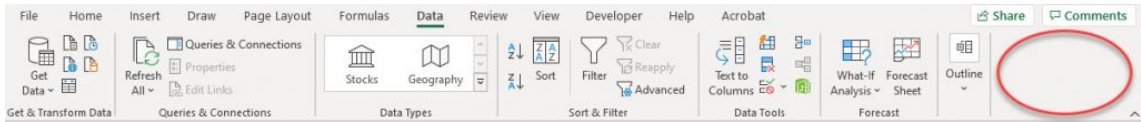


Figure 11.A1. Excel Data Menu Ribbon without the Data Analysis option.

Select File and then Options.

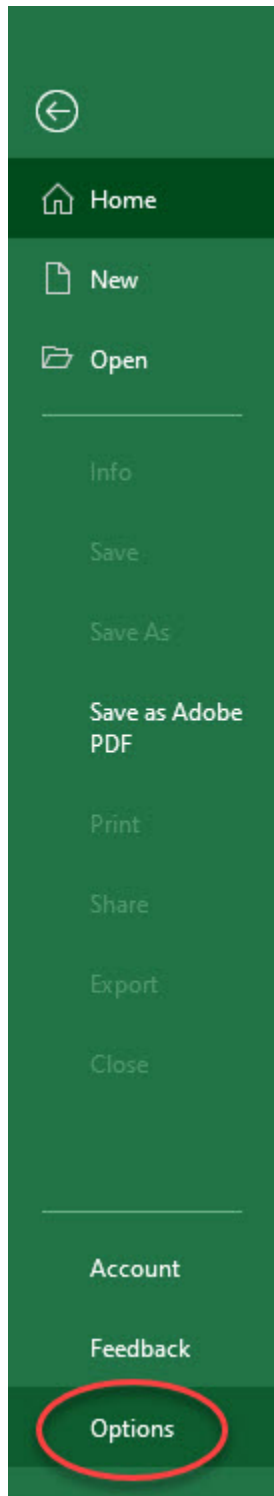


Figure 11.A2. Excel File Menu Options Selection

In the Excel Options popup window, first select the Add-ins menu, and then select the Go... button to Manage Excel Add-ins.

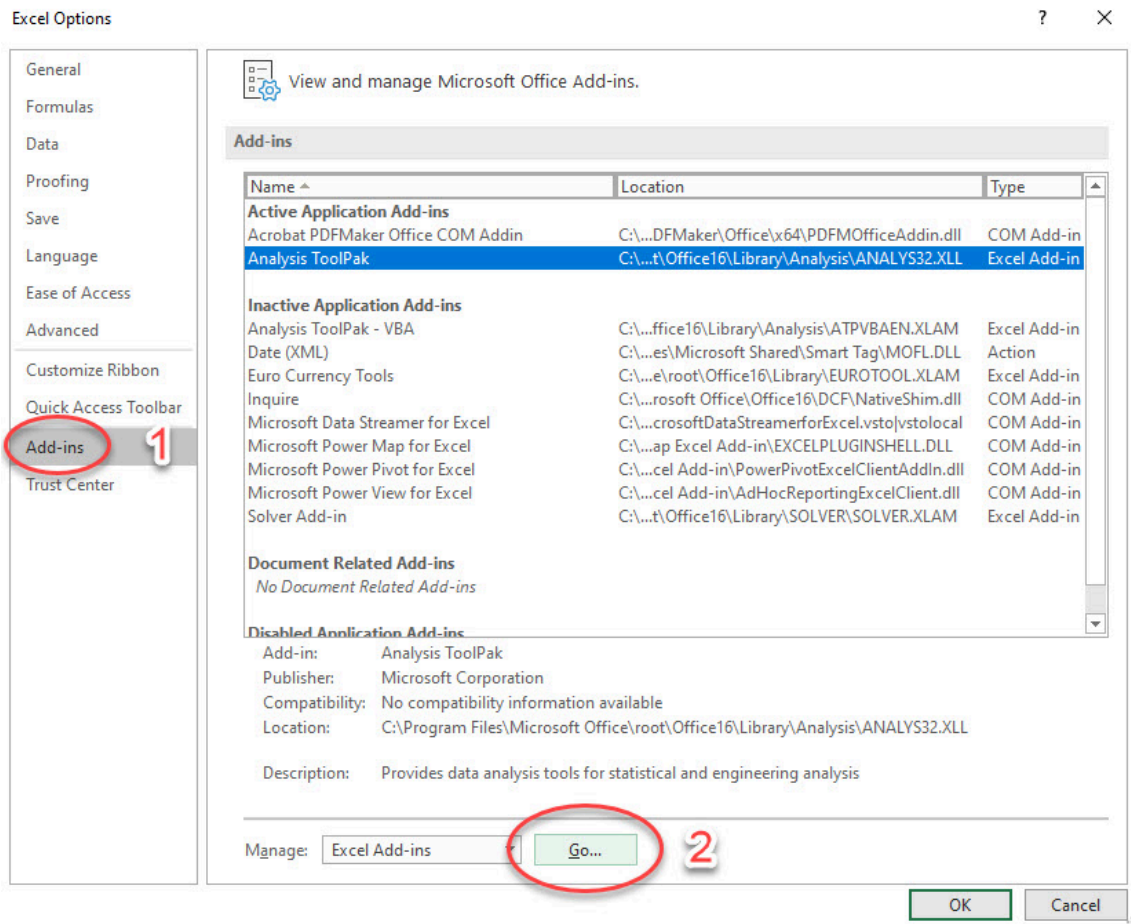


Figure 11.A3. Excel Options to Manage Add-ins.

Finally, in the Add-ins popup window, check the Analysis ToolPak option and select OK.

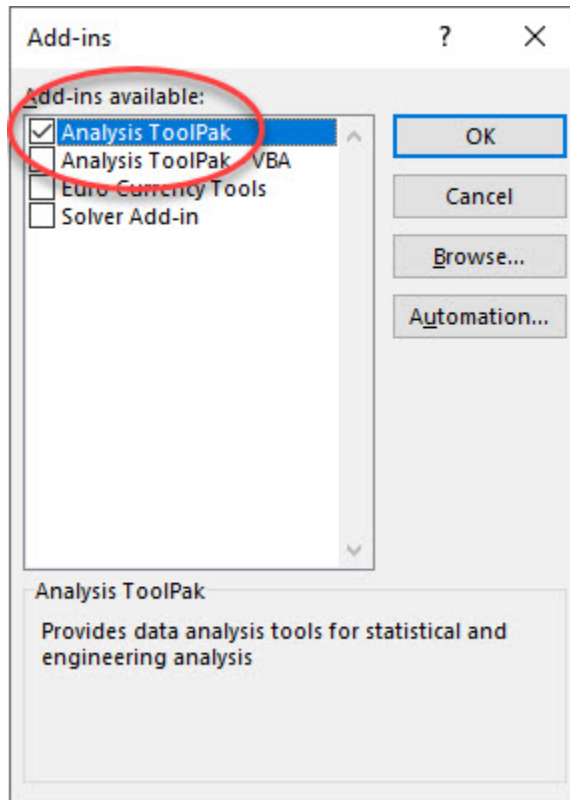


Figure 11.A4. Enable Analysis Toolpak.

When you return to the Excel spreadsheet Data menu ribbon, the Data Analysis button is now available.

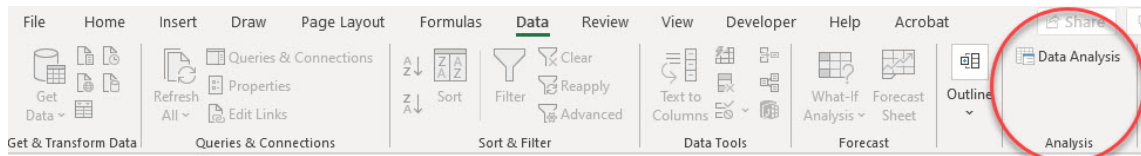


Figure 11.A5. Data Ribbon with Data Analysis option enabled.

PART IV

PART IV: HOMOGENEOUS MEASURES

12. Ranking Investments and Homogeneous Measures

LINDON ROBISON

Learning goals. At the end of this chapter, you should be able to: (1) explain why unambiguous rankings require one-dimensional measures; (2) understand why consistent rankings require homogeneous measures; and (3) describe present value (PV) model dimensions that require homogeneous measures.

Learning objectives. To achieve your learning goals, you should complete the following objectives:

- Understand the need for one dimensional ranking criterion.
- Evaluate and distinguish between one-dimensional net present value (NPV) and internal rate of return (IRR) ranking measures.
- Learn about PV model homogeneous measures that are required for consistent NPV and IRR rankings that include:
 - homogeneous funding sources;
 - homogeneous initial and periodic sizes;
 - homogeneous terms;
 - homogeneous tax rates; and
 - homogeneous liquidity and currency.

Introduction

Financial managers must decide whether to continue their commitment to an existing investment called a defender or disinvest in the defender and commit resources to a new investment called a challenger. More complicated investment problems may require that we choose between a defender and several challengers—mutually exclusive challengers, challengers that can be acquired in combination with other challengers, and time sequenced challengers. To describe the nature of defenders and challengers and the need for homogeneous measures, we introduce a simple one-period present value (PV) model.

Suppose we have already committed V_0^d to a defending investment in exchange for its future cash flow equal to R_1^d . Now suppose that the rate $(1 + r^d)$ at which the defender exchanges present and future dollars is:

$$(12.1) \quad (1 + r^d) = \frac{R_1^d}{V_0^d}$$

Now consider an alternative challenging investment. The rate at which the challenger's investment funds V_0^c are exchanged for future cash flow R_1^c is:

$$(12.2) \quad (1 + r^c) = \frac{R_1^c}{V_0^c}$$

In equations (12.1) and (12.2), r^c and r^d equal the defender's and challenger's internal rates of return. We cannot really say much about the relative internal rates of returns of the defender and challenger unless we require that the two investment amounts V_0^d and V_0^c are equal. If the investment amounts are equal and if ($R_1^c > R_1^d$), then we can make a definitive statement about their relative IRRs, ($r^c > r^d$).

Now suppose that we equate $r^c = r^d + \alpha$ and rewrite equation (12.2) as:

$$(12.3) \quad (1 + r^d + \alpha)V_0 = R_1^c$$

Finally, we divide both sides of equation (12.3) by $(1 + r^d)$ and write:

$$(12.4) \quad \frac{\alpha V_0}{1 + r^d} = -V_0 + \frac{R_1^c}{1 + r^d} = NPV$$

where the present value of earnings, $\alpha V_0 / (1 + r^d)$, will be referred to net present value, NPV.

So, what have we learned? By means of this example, we introduce some of the main themes of this chapter. Comparisons between defenders and challengers require homogeneous measures. Assuming homogeneous measures, challengers and defenders can be ranked for investment preference by their individual IRRs. Or, we can combine a defender and a challenging investment into one ranking equation we refer to as the NPV equation. NPV and IRR equations are the fundamental PV equations from which most others can be derived and extended for several periods. To ensure that they provide consistent rankings, we measure the characteristics of defenders and challengers in homogeneous units.

Opportunity Costs

An opportunity cost is a benefit, profit, or value of something that must be given up in order to acquire or achieve something else. An economic couplet declares that there is no free lunch. If you want lunch, you must pay for it—sacrifice something to obtain it. The opportunity cost, the cost of lunch, of invest-

ing in a challenger is the IRR of the defender. NPVs measure the difference in the present value of future earnings if the investor sacrificed the defender to invest in the challenger.

Prices are not opportunity costs. The price of a good is the amount of money, or money equivalent, paid to obtain it. The amount of money or money equivalent paid to obtain the good represents the direct cost of what is given up to acquiring something desired or to avoid something disliked. However, there may be other costs incurred to acquire a good—or to avoid a bad—besides the price paid. For example, one might consider the cost of attending a movie to be the price of the ticket. However, transportation costs to and from the theater could add to the actual value of what must be exchanged to attend the movie. And there may be other costs of attending the movie such as lost earnings as a result of missing work. Or, suppose the movie was playing while the moviegoer ate his or her prepaid dinner meal. Now the cost of the movie includes not only the price of the ticket plus transportation costs and lost wages but also the value of the skipped meal. The cost of the ticket, transportation costs, lost wages, and the skipped meal together represent the opportunity cost of attending the movie, which is different from the price of the movie ticket.

Opportunity costs and perfect capital markets. The capital (or financial) market is where people trade today's present dollars for future dollars and vice versa. In a perfect capital market, dollars trade between adjacent time-periods at the (same) market rate of interest. For a perfect capital market to exist, the following conditions are required: no barriers to entry; no participant can influence the price; transactions are costless to complete; relevant information about the market is widely and freely available; products and services are homogeneous; no distorting taxes exist; and investment possibilities are continuously divisible. Finally, the firm's opportunity cost of capital is the same regardless of the size or purpose of the amount being borrowed or lent. Finally, in perfect capital markets, investment NPVs are zero.

Though markets for some financial investments are considered highly efficient, they are not perfect. Rates of return on savings rarely equal the rate paid to borrow funds. Moreover, rates of return on investments typically depend on the size and economic lifetime of the investments. Hence, in the real world, investors face imperfect capital markets. We allow for imperfections in capital markets in our PV models by allowing the rate of return on an investment being considered for adoption, a challenger, to differ from the rate of return on an investment that must be sacrificed to adopt the challenger, a defender, to vary. In imperfect markets, the opportunity cost of the defender and the market rates of interest on loans are rarely equal.

So, what have we learned? We have learned that adopting a challenging investment requires that we sacrifice a defending one and what the defending investment earns. That's right, there is no free lunch. As a result, we recognize that opportunity costs are much more (and sometimes less) than just prices.

One-dimensional Ranking Measures

An analogy. To uniquely rank defending and challenging investments requires a one-dimensional ranking measure. To make this point, that unique investment rankings requires a one-dimensional measure, assume that we determined the winner and loser of a sporting event by several different measures.

To illustrate, suppose that the winner of the Super Bowl football game depended on the following measures: points earned, yards gained, yards earned on the ground divided by the yards earning passing, yards penalized, injuries sustained, and the number of persons viewing the contest. Most football sports fans would agree that the success measures just described matter—but they would also agree that we would be unable to determine who wins and who loses with such multi-dimensional measures unless in some rare event one team dominated in all dimensions. So, we must decide which measure matters most and in the case of football—the measure that matters most is points earned. Hopefully, all other measures are subsumed in the one defining measure.

We rank most investments using one of two one-dimensional measures, NPVs and IRRs. What does it mean that NPVs and IRRs provide one-dimensional measures? It means that they can be represented by a location along a straight line. Consider other details about NPVs and IRRs as ranking measures. NPVs measure the difference between a challenger and a defender described in present value dollars. Positive (negative) NPVs imply that the investors will increase (decrease) their earning by exchanging the defender for the challenger. Percentage rates of return measure the internal rate of return (IRR) of a single investment. Any comparison between defenders and challengers using IRRs will require an IRR be calculated for each investment.

NPV rankings. To rank investments uniquely using NPV measures requires that we reduce their current and future earning to a single number in the same period. Alternatively, we could compare two investments' future cash flow in each period of their economic lives. Then, we would likely be unable to rank them because one investment may earn more in one period and less in another than a challenging investment. To avoid the indecisiveness of multi-dimension metrics, we convert future cash flow to their present value creating a one-dimensional number that allows us to uniquely rank defender and challenging investments. To reduce future cash flow to equivalent in the present we assume that a challenger's earnings are reinvested in the defender and earn the defender's rate of return.

IRR rankings. Alternatively, investments can be ranked by their IRRs, a percent. But to find a percentage to represent the PV model, we treat each periodic cash flow as equally important by converting them to their present value equivalent. We use the word internal to describe the second ranking method because the earnings from the investment are reinvested in the challenger rather than being reinvested in the defending investment. IRRs have two possible interpretations. One, it is the (internal) rate of return earned by the challenger. Or, it is the defender's IRR that produces an NPV of zero. Of course, if the defender's IRR produces an NPV of zero, it means that the defender and challenger's IRR are equal. If the NPV is positive (negative), then the IRR of the challenger is greater than (less than) that of the defender although there may be some strange exceptions.

The idea that future cash flow can be ranked by their NPVs or IRRs may be one of the most important and pervasive concepts in financial management. It is the basis for ranking nearly every investment with future cash flow including physical investments, bonds, stocks, insurance, pension funds, housing, land, cars—and almost anything that has more than one period of economic life generating cash flow.

The problem of IRR and NPV rankings. There is a problem with having IRR and NPV ranking measures. We don't have just one—one-dimensional ranking number (points earned in a football game). We have two and possibly more. And when we have more than one, one-dimensional ranking measure, we have the possibility for inconsistent rankings. And we do.

IRR and NPV ranking criteria routinely produce inconsistent rankings and provide unlimited opportunities for academics to argue about their relative virtues. And here is the main point of this chapter, if we compare challengers and defenders using homogeneous measures, we can guarantee that IRR and NPV rankings will provide consistent rankings.

So, what have we learned? Unambiguous investment rankings require that we agree on one, one-dimensional ranking measure. And failing that, having more than one one-dimensional ranking measure, we must describe challengers and defenders using homogeneous measures to guarantee that we will have consistent IRR and NPV measures.

Homogeneous Measures

What adds complexity to choosing between a defender and a challenger and among challengers is that they often have different characteristics that make comparisons between and among them difficult. Moreover, these differences between challenging and defending investments, unless addressed, may result in inconsistent rankings. As a result, we may be unsure which is the preferred investment decision. This chapter intends to describe how we can avoid inconsistent investment rankings by requiring homogeneous (same) values for some investment characteristics. We do allow for some investment characteristics to be unequal—the ones that lead us to prefer one investment over another.

What is homogeneity? Homogeneity is the quality or state of being equal or of the same kind. Homogeneity synonyms include uniformity, equality, consistency, and harmony. Employed in our investment ranking discussion, homogeneity means that measures not relevant to the rankings of investments should be equal and not allowed to influence ranking outcomes. Homogeneous measures can be of two kind. We are measuring the same thing, inches, pounds, days of the year, time in the mountain time zone, etc. The second meaning of a homogeneous measure is that the value of the same characteristic being compared is the same across investments.

On the other hand, not all measures used to describe need or should be homogeneous, at least in the second meaning of the word. If all the measures were used to describe defending and challenging investments were homogeneous, they would be the same value and we would not be able to distinguish between investments. To emphasize the need for homogeneous measures, we introduce a horse racing analogy.

An analogy. To make the point that a lack of homogeneous (same) measures may create ranking conflicts, even when the comparisons involve one-dimensional measures, we compare ranking investments to a horse race. We organize horse races so that other factors besides the horses' speed do not influence the horse race outcome. For example, we expect that only horses enter the race. We require that the horses all begin the race at the same time and place. We expect that all horses will run the same distance and at the same time—that implies that not only has the starting point and time has been determined but the finish line as well. We require the jockey's weight not be a factor. Finally, we expect that everyone agrees that the criterion for ranking horses is the time interval between when each horse starts and finishes the race.

Ranking investments is, of course, not a horse race. Yet, the process of ranking investments according to their rates of return or earnings and horses according to their speed have many elements in common. First, we assume that we are comparing investments of the same size (only horses run the race). We also assume that we are comparing investments over the same time-period or term, (the length of the race is the same for all horses). Finally, we assume that the criterion used to rank investments (NPVs or IRRs) would provide the same rankings for each investment just as we would agree that timekeepers would agree on which horse crossed the finish line first. For the horse race to be fair, i.e. that the outcome is not influenced by irrelevant factors unrelated to the horses' speed, we control for as many differences as possible and meaningful. Only when controlling for irrelevant factors can we make the horse's speed the most important factor, the one nonhomogeneous factor that determines the winner.

To continue our analogy, suppose we wanted to compare the average speed on horses in different races. That may be difficult because so many conditions may not be controlled or made the same between races. Different locations, different conditions and length of the track, different weather conditions and time of day and time of year—all these and others may all influence the speed of the horses in that race. And if those difference between races were not enough to complicate the comparisons, consider the differences between horses: their age, weight, gender, genealogy, races run previously during a set period, their diets, and overall health, how it was trained—to name a few. All these differences may lead us to ask—what can we infer from race outcomes? What we want to know is what variables should not be measured in homogeneous measures? It may be that what we don't want to make homogeneous is something intangible—the horse's desire and will to win. Like we meant to say, it can be complicated.

Homogeneous Cash Flow Measures

We have emphasized that for NPV and IRR to provide consistent rankings, defender and challenger investments must be measured in homogeneous units. Perhaps the two most important homogeneous characteristics is that the economic activity of the defender and challenger must be both represented by cash flow. And secondly, the cash flow used to represent the economic activity of the defender and challenger must be measured in common present value units.

The cash flow principle. One investment dimension that both the defender and challenger must adopt to represent their economic activity, is the cash flow it generates and when. We refer to this requirement as the cash flow principle. When building a PV model, we represent economic activities of the firm by their cash flow. In our horse race analogy, the cash flow principle is equivalent to letting only horses run the race. Some activities of the firm produce what we characterize as noncash outcomes such as appreciation (depreciation) of assets, increases in inventories of unsold goods, and increases in accounts receivable and payable. But these events are not included in PV models because they do not produce cash flow.

One justification for the cash flow principle is that, at some point, we expect all economic activities to generate cash flow. At some point, we expect inventories to be liquidated and generate cash. At some point, we expect accounts receivable and accounts payable to be settled for cash. At some point we expect long-term assets that have appreciated (depreciated) to be sold and the difference between their acquisition and sale price to capture the noncash flow of depreciation (appreciation). Thus, in effect, we do count all economic activities of the firm by recognizing only cash flow; however, we count them only when they create cash flow.

The cash flow principle requires that careful distinction be made between a cash transaction and a noncash transaction. Sometimes the distinction is not always clear. For example, an asset's book value depreciation does not itself generate a cash flow. Depreciation expenses of an investment do, however, generate a tax shield that creates a cash flow in the form of reduced tax payment. Thus, we include the cash flow associated with tax savings resulting from depreciation of an asset but not the depreciation.

Present value equivalent cash flow. We already determined the need for homogenous measures to unambiguously rank investments using either NPVs or IRRs. Furthermore, we earlier accepted that a dollar in the present is not the same as a future dollar. But how do we reconcile that our NPV and IRR models consist of time-dated cash flow that are inherently differ from each other? The answer is that we exchange them with their present value equivalent by discounting them either with at the defender's or the challenger's IRR.

Consider a percent that describes the exchange between future dollars (A) and present dollars (B). We exchange future and present dollars every time we lend or borrow money, make or liquidate an investment, or invest in or withdraw money from a retirement account.

To illustrate, suppose I borrow present dollars V_0 now from a lender and in one year from now, I repay the lender future dollars R_1 where the subscript indicates the end of the time-period in which the cash

flow occurs. Since the ratio contains numbers of the same units, the rate of exchange between me and a lender is (R_1/V_0) equal to the decimal $(1 + r)$ or as sometime reported, as r percent. We summarize the ratio of future to present dollars expressed as a decimal equal to:

$$(12.5) \quad \frac{R_1}{V_0} = (1 + r)$$

To illustrate, suppose V_0 equals \$100 and R_1 equals \$110. In this example the ratio of future to present dollars is: $\$110/\$100 = 1.10$ or 110% and we could say that one present dollar can be exchanged for 1.10 future dollars.

Now suppose I want to receive R_1 dollars one period in the future in exchange for V_0 now. The percentage describing the exchange is (V_0/R_1) equal to $1/(1 + r)$ or if multiplied by 100%, I could say that I can exchange future dollars for present dollars at $(1/1 + r)$ percent.

$$(12.6) \quad \frac{V_0}{R_1} = \frac{1}{(1 + r)}$$

To illustrate, suppose V_0 equals \$100 and R_1 equals \$110. In this example the percentage of present to future dollars is $\$100/\$110 = 0.91$ and we could say that one future dollar can be exchanged for present dollars at .91 percent. Sometimes we say that one dollar in the future can be discounted to 0.91 present dollars, expressing our ratio as a decimal.

The variable r appears frequently in PV models that describe money exchanged across time. It is important to recognize that r is a decimal. However, when it is used in a ratio or equation with dollars—it is often referred to as a rate. For example, consider equation (12.7).

$$(12.7) \quad V_0(1 + r) = R_1$$

In equation (12.7) r is a decimal or percent that multiplies present dollars to find future dollars and we refer to it as the rate of return earned on initial investment V_0 .

Exchanging present dollars for a future dollar over more than one period. If we know the exchange rate r and are willing to assume it is constant over time, it is a small step to convert present dollars to their equivalent in the future or to convert future dollars to their equivalent in the present. We simply multiply or divide by the appropriate rate. To convert present dollars to their one period in the future value we multiply by $(1 + r)$. To convert future dollars to their present worth, we divide or discount by $1/(1 + r)$.

To illustrate the importance of calculating the present value of future cash flow, suppose that a “down-on-her-luck friend” approaches you. She explains that her wealthy aunt has promised her \$100 in one year, but she needs the money now. She asks you what you would offer in present dollars in return for her future \$100 dollars. You quickly calculate (assuming an exchange rate of 110% or 1.10), and report

that the present value of \$100 future dollars (received in one period from now) is \$90.91. Or, we might say that the discounted present value of \$100 future dollars is \$90.91.

Just as we found the present value of one period in future dollars, we can also find the present value of future cash flows received in two or more periods in the future. Returning to our earlier example, suppose this same down-on-her luck friend offers you R_1 dollars in one period and R_2 dollars two periods in the future and asks how many present dollars would you offer for the exchange if the exchange rate is $(1 + r)$.

We can find the present value of R_2 in two steps. First, convert R_2 to its equivalent value in period one dollars by discounting at the rate of $1/(1 + r)$ or $[R_2/(1 + r)]$. Then we discount $[R_2/(1 + r)]$ by dividing again by $(1 + r)$ to convert period one dollars to present period dollars. Important to note is that the exchange rate is assumed to be $(1 + r)$ between any two periods, including between periods one and two. We summarize our two period future cash flow discounted to their present value as:

$$(12.8) \quad V_0 = \frac{\left[R_1 + \frac{R_2}{(1 + r)} \right]}{(1 + r)} = \frac{R_1}{(1 + r)} + \frac{R_2}{(1 + r)^2}$$

We would follow a similar procedure to find the present value of dollars received in three periods from the present. Suppose you were offered \$100 for the next three periods. What is the present value of these future cash flows if the exchange rate were 110 percent between any two periods? The answer is:

$$(12.9) \quad V_0 = \frac{\$100}{1.1} + \frac{\$100}{1.1^2} + \frac{\$100}{1.1^3} = 90.91 + \$82.64 + \$75.13 = \$248.68$$

So, what have we learned? We learned that we can find the present value of future dollars by discounting them using one plus the appropriate discount rate r . Furthermore, we can find the value of present dollars in the future by compounding or multiplying then using one plus the appropriate reinvestment rate r . All this assumes, of course, that the exchange rate of dollars between time periods is a constant $(1 + r)$. By means of the exchange rate, we convert cash flow either in NPV or IRR models to their equivalent (homogeneous value) in the present.

Homogeneous measures and winning horses. In sum, when comparing horse races and investments, we try and make homogeneous any many irrelevant factors as possible. If we do not, the results will be influenced, biased, by factors we don't want to determine the outcome. And there is more. Not only

do irrelevant non-homogeneous investment dimensions bias rankings, that may also produce inconsistent IRR and NPV rankings. In other words, if we allow too many factors to vary, we may get a decisive race outcome but not be able to determine if it was meaningful. In the remainder of this chapter and in the following chapters, we focus on several investment dimensions that must be measured the same, homogeneously, so that our investment rankings will be unbiased and that NPV and IRR rankings will be consistent.

So, what have we learned? When ranking investments, we must decide which factors influencing the investment's performance should be made equal and which factors should be allowed to vary between defenders and challengers. Choosing which factors, we should hold constant or not, is somewhat of an art. There is no book of investment principles that we know of that can guide our choices. The best we can offer is a reasoned approach to help us determine what factors should be allowed to vary and to determine investment rankings and which ones should not be allowed to vary. The criteria we adopt for requiring homogeneous measures is this: we require homogeneous measures for those factors that allow IRR and NPV rankings to be inconsistent. We describe next those factors that if allowed to vary may produce inconsistent rankings.

Other Homogeneous Measures

In what follows, we briefly introduced those investment dimensions that should be made homogeneous to produce consistent NPV and IRR rankings. Later, each dimension will be emphasized in a separate chapter. We have already noted that PV models should measure economic activity using only cash flow and convert them to their present value equivalent. We also described in Chapter 8 the need to measure earnings and rates of return from either assets or equity. We should also compare investments of the same initial and periodic sizes, terms, tax codes, and liquidity and currency.

PV models are designed to measure time-dated cash flow in homogeneous units by converting them to their equivalent present value. This is key and fortunately is inherent in PV models. But there are other investment dimensions besides their time dated cash flow that may not only be measured in different units but may be measured in the same weight but given different values. As a result, investments not measured in homogeneous unit may produce inconsistent IRR and NPV rankings. We describe these next and they are so important that in what follows we devote a separate chapter to each measure.

Homogeneous funding sources. An important characteristic of an investment that should be homogeneous across investments being compared is the investment's funding source. This topic is covered in Chapter 8, but homogeneous funding sources is not the only focus of chapter 8. Homogeneity of investment funding requires that both defenders and challengers must attribute their earning and

rates of return to either its equity or assets. To be specific, if two investments are being ranked using their IRRs, then cash flow for the investments must be either attributed to cash flow from equity or assets. If two or more investments are being ranked using NPVs, then cash flow for challengers and the defender's discount rate must be attributed to either equity or assets. To avoid confusion, we will describe the rate of return on equity as ROE or IRR^E . And we will describe the return to the firm's assets or investment as ROA or IRR^A . If we wish to compare earnings on the investment or assets, we will compare NPV^A. If we wish to compare earnings on equity, we will compare NPV^E for the different investments.

So how do we build PV models that measure earnings and rates of return on assets or equity? When finding the ROA for the defender, we ignore cash flow that includes borrowing or lending activities. The advantage of this approach is that it measures the rate of return on assets independent of the returns from the loan used to finance the investment. We write the defender's cash flow and its corresponding IRR that measures ROA as:

$$(12.10) \quad V_0 = \frac{R_1}{(1 + ROA)}$$

where V_0 is the initial investment, R_1 is net cash flow earned in the first period, and S_1 is the liquidation value of the investment after one period. Then we write the ROA as:

$$(12.11) \quad ROA = \frac{R_1 - V_0}{V_0}$$

Interpreted, equation (12.11) equates ROA to cash returns R_1 including cash value of capital gains (losses) equal to the difference between the liquidation value and the beginning value of the asset all divided by the beginning value of the assets. It measures the rate of return on the firm's assets including returns generated by human capital, manufactured capital, social capital, natural capital, and financial capital.

In this approach, we do not account for the source of funding for the investment—equity or assets. Unless otherwise specified, it includes both debt and equity funding. In this calculation, we do not charge for interest costs charged on debt capital because the entire investment is charged at the beginning of the investment cycle and to charge for interest costs would double count the cost of using debt capital.

To measure the rate of return on the firm's equity, we only charge for equity initially invested and charge principal and interest payment in the time period they are repaid and the loan balance and the end of the analysis. In this formulation we net out the cost of using debt capital and attribute the remaining earnings to invested equity.

To calculate returns to earning by subtracting out the use of debt, assume that an asset is acquired using debt D_0 plus equity E_0 whose sum is equal to V_0 . At the beginning of the project, the firm supplies equity capital to purchase the investment creating a negative cash flow of E_0 dollars in the beginning period.

Then it receives D_0 from the lender and pays D_0 to the seller to complete the purchase of the investment. Since the investor receives D_0 from the lender and pays debt capital D_0 to the seller of the asset at the beginning of the period, they cancel out making net cash flow from debt capital at the beginning of the investment zero. At the end of the period, the investment repays debt capital D_0 plus interest iD_0 to the lender. In a one period model, the investor pays back to the lender, debt capital D_0 . Thus, we can express these cash flow as:

$$(12.12) \quad E_0 = \frac{R_1 - (1 + i)D_0}{(1 + ROE)}$$

And we can write the ROE as follows:

$$(12.13) \quad ROE = \frac{R_1 - (1 + i)D_0 - E_0}{E_0}$$

We are interested in the relationship between ROA and ROE. To determine that relationship, we first solve for R_1 in equation (12.11) and find that it equals:

$$(12.14) \quad R_1 = (1 + ROA)V_0$$

Then we make the substitution for R_1 in equation (12.13) using equation (12.14) and replacing V_0 with $(D_0 + E_0)$ we again the rates of return identity (RORI) result:

$$(12.15) \quad ROE = \frac{(ROA - i)D_0 + ROA}{E_0}$$

What equation (12.15) reveals is that $ROE > ROA$ as long as the firm earns a positive return on its debt capital ($ROA > i$).

Homogeneous initial and periodic investment sizes. Homogeneous size measures (Chapter 13) require that initial and periodic investment sizes for challengers and defenders must be equal. This seems like an obvious requirement. Nevertheless, it is routinely ignored and often produces conflicting NPV and IRR rankings. To apply the homogeneity of size requirements, initial investment differences must be resolved. We resolve initial size difference by accounting for their earnings especially that remain invested in a defender. Periodic cash flow in PV models represent investment withdrawal that alter the size of the investment. To maintain periodic size homogeneity, cash flow is compounded until a common ending period.

Our goal is to obtain unbiased investment and cash flow measures. This requires not only homogeneous measures but that all cash costs and cash returns associated with an investment should be included when determining an investment's present value. Consider how the total costs and returns principle is applied in several practical situations. Whenever low-interest loans or preferential tax treatments are tied to the ownership of a durable, these concessions will influence the present value of

the investment. To ignore these benefits (costs) would lead to an under or over evaluation of the worth of the investment.

Sometimes an investment such as land has more than one source of returns. Mineral deposits, potential recreational use, and urbanization pressures may create expected returns over and above those associated with agricultural use. Pollution standards may impose costs in addition to those normally experienced. All these expected cash costs and returns that influence the value of the durable should be included in the PV model. And cash flow associated with decreased accounts and inventories should also be included.

Requiring initial and periodic size equality is analogous requiring horses in the same race to be of the same size.

Homogeneous terms. Just as we would expect horses in the same race to run the distance, we would expect that investments being compared would be measured over the same term. Chapter 14 describes how we implement the homogeneity of term requirement. Similar to the method used when satisfying the homogeneity of size condition, when comparing investments of two different terms, we must account for the investments' earning between the term of their investments.

Complicating the application of the homogeneity of term requirement are sequenced investments, one s that cannot occur without a similar disinvestment. For example, we cannot invest in a new orchard or cohort of animals without deciding the term of the existing investments.

Homogeneous tax rates. Taxes represent withdrawal from the firm's earning by government. Homogeneous tax-rate measures (Chapter 15) require that we account for differences in effective tax rates for defenders and challengers that depend on cash flow patterns and capital gains (loses). The main point here, is that tax considerations should be equally considered when evaluating the challenger as well as the defender. On the other hand, some taxes may not be identically applied to both defenders and challengers and require a careful consideration of the effects of taxes.

Homogeneous liquidity and currency measures. Homogeneous liquidity measures (Chapter 16) recognize that firms and investments can be differentiated by their liquidity. This is a difficult concept to apply when ranking investments. And perhaps the best we can offer is to provide liquidity measures that differ by investment types.

A different homogeneity measure applies why engaged in global trade across countries and currencies. Now the rate of currency exchange becomes an important consideration and even more so when it varies over time.

So, what have we learned? To compare the value of a potentially new investment, a challenging investment, with an investment to be sacrificed, a defending investment, we need to apply homogeneous measures to both. Included in the list of homogeneous measures

required for meaningful and consistent rankings include: using only cash to represent economic activity, using homogeneous funding sources, using homogeneous initial (and periodic) investment sizes; using homogeneity investment terms; homogeneous currency and liquidity measures; homogeneous risk measures; homogeneous tax rate measures; and homogeneous measures of funding sources (assets versus equity).

Summary and Conclusions

The discount rate in a PV model represents the defender's IRR. It is the rate of return sacrificed by disinvesting in the defender to acquire the challenger. The defender's rate of return measures either its ROA or ROE. Choosing the appropriate discount rate for NPV models describing a challenger is perhaps the most difficult task of investment analysis. The choice essentially involves the identification of challengers and opportunity costs associated with defenders.

When ranking investments, we have available one of two possible one-dimensional ranking criterion, NPVs or IRRs. Ideally, these should provide consistent rankings regardless of whatever method we use. Unfortunately, this is not always the case and to be true in general requires that we adopt homogeneous measures for describing challenging and defending investments. These homogeneous measures are so important that they occupy much our effort in chapters 13, 14, 15 and 16 that follow. Our goal is that by using homogeneous measures to construct PV models we can obtain consistent, stable, and accurate investment ranking. Knowing that one's investment rankings are stable, accurate, and consistent is essential for the success of a financial manager whose goal is to maximize the present value of the firm's investments.

Questions

1. The fundamental concept of PV analysis is that a dollar today is not valued the same as a dollar received in the future. List several reasons that might explain why a dollar today may not be valued the same as a dollar in the future.
2. The idea that the value of a good today is not the same as the value of the good in the future is a universal concept. Suppose you were offered a 2020 Ford F-150 pickup truck for delivery today or the same truck in the same condition delivered in 2023. If the price today for the truck were \$40,000, what would you offer today for the truck to be delivered in 2023? If the prices you would offer are different, please explain why?
3. Explain the importance of valuing future cash flow in one single period, e.g., the present period.

4. Explain the difference between ratios, percentages, and rates.
5. To test your understanding of exchange rate between present and future dollars, please answer the following. Suppose that you could exchange one apple for 1.1 oranges. What is the exchange rate between apples and oranges, between oranges and apples? Compare the exchange rates between apples and oranges and oranges to apple to the exchange between present and future dollars is $r=.1$ or 10%.
6. Two important economic concepts associated with PV models and investments in general are price and opportunity costs. Opportunity costs are the sacrifices required to obtain a good. The market price of the good may be part of the opportunity cost, but likely not the entire opportunity cost. Consider the opportunity cost of attending the university. These may include income sacrificed. Then compare the opportunity cost of attending the university with the tuition costs, a direct cost. Describe the differences between opportunity costs versus direct costs in other personal activities.
7. Suppose that you invest \$100 today. Then assume that one year later your investment returns \$110 in cash. Assume that your opportunity cost of capital is 12%.
 1. What is the NPV of the investment?
 2. What is the IRR of the investment?
 3. What is the ratio that converts dollars received one period in the future to their present value?
 4. What is the ratio that converts present dollars to their value one period in the future?
8. A financial manager believes that a defending investment can earn \$500 for the next six years. He also believes that his defending investment can be sold today for \$2,500. What is the opportunity cost of sacrificing the defender to invest in a challenger? In other words, find the IRR of the defending investment.
9. Briefly distinguish between a defending and a challenging investment. Suppose you are considering replacing an outdated piece of equipment with two different pieces of equipment of different sizes, efficiency, and useful life. In addition, one company offers financing while the other one assumes buyers will provide their own financing. Describe the challenging you might face comparing and deciding between the two investments.
10. In practice, financial managers face more complicated investment problems than deciding between a single defender and a single challenger. Describe how you might solve an investment problem with more than one challenger? One challenger but more than one defender? When choosing between multiple defenders, what would be the appropriate criterion for the preferred defender? How would you analyze a problem consisting of multiple challengers and defenders? What principle from micro economic theory might guide you in preparing your answer?
11. Describe the fundamental different between rates of return and earnings on assets versus equity. Can you think of different considers when earnings and rates of return on assets would be the preferred measures versus conditions when earnings and rates of return on equity would be the preferred measure?
12. Explain how violating homogeneity of size may lead to inconsistent NPV and IRR rankings. (Hint: how do rankings based on a percent (IRRs) versus rankings based on dollar amounts (NPV) permit inconsistent rankings when initial investment sizes are different.)

13. Compare investments of different terms to two horses running a race of different lengths.
14. Since only death and taxes remain a constant, explain why it is important to measure a defender's and challenger's cash flow in both before and after-taxes. Suppose that the before and after-tax rankings were inconsistent, what would you conclude. Would you base your investment decision on before or after-tax results?
15. Suppose that you make a significant investment in U.S. currency in a foreign country factory that produces products that are sold and denominated in the currency of the host country? Homogeneous measures require that the investment be evaluated in the same currency. How would you convert a foreign currency to U.S. dollars?

13. Homogeneous Sizes

LINDON ROBISON

Learning goals. At the end of this chapter, you should be able to: (1) understand how internal rate of return (IRR) and net present value (NPV) models can produce inconsistent rankings; (2) produce consistent IRR and NPV rankings by adjusting for investment size differences and by adopting common reinvestment rate assumptions; (3) understand why some NPV and IRR rankings are unstable; (4) recognize how different initial size adjustment methods (addition, scaling, or some combination) produce consistent but sometimes different investment rankings; and (5) recognize how investment type differences can be used to identify the proper size adjustment method.

Learning objectives. To achieve your learning goals, you should complete the following objectives:

- Recognize that IRR and NPV rankings may be inconsistent and that NPV rankings may be unstable.
- Understand the difference between periodic investment sizes and initial investment sizes.
- Describe how different reinvestment rate assumptions can produce inconsistent IRR and NPV rankings.
- Show how resolving periodic and initial size differences and a common reinvestment rate produce consistent challenging investment rankings.
- Illustrate the different methods available for resolving periodic size investment differences.
- Demonstrate how scaling and addition can be used to resolve initial size differences in investments.
- Demonstrate that while different methods for resolving periodic and initial size differences can each produce consistent rankings, the consistent rankings may not be the same.
- Understand the conditions under which IRR and NPV provide the same rankings as the size adjusted IRR and NPV models.
- Recognize the four basic investment models
- Recommend the appropriate investment model based on challenger characteristics.

Introduction

1

1. Some of the material in this chapter was adapted from Robison, Myers, and Barry (2015).

In the single equation NPV model, we assumed that cash flow was exchanged between time periods at the defender's IRR. In the single equation IRR model, we assumed that cash flow was exchanged between time periods at the challenger's IRR. In the NPV model, the challenger (defender) was preferred to the defender (challenger) if the NPV were positive (negative). Furthermore, the NPV ranking of the defender and challenger was the same ranking obtained by comparing their respective IRRs. As a result, both IRR and NPV criteria produce the same ranking.

This chapter recognizes that when we rank multiple challengers funded by one defender, their IRR and NPV rankings may be different. Furthermore, we observe that changes in the defender's IRR may produce unstable NPV rankings. We demonstrate these results in Table 13.1 that ranks three challengers.

Table 13.1. Inconsistent IRR and NPV rankings and unstable NPV rankings among three mutually exclusive investments of different initial and periodic cash flows.

initial investments V_0^i where $i = 1,2,3$	period one cash flows R_t^i where $t = 1$ generated by initial investments V_0^i where $i = 1,2,3$	period two cash flows R_t^i where $t = 2$ generated by initial investments V_0^i where $i = 1,2,3$	NPV ⁱ earned by initial investments V_0^i where $i = 1,2,3$ assuming $r = 5\%$ (ranking)	NPV ⁱ earned by initial investments V_0^i where $i = 1,2,3$ assuming $r = 10\%$ (ranking)	IRR ⁱ earned by initial investments V_0^i where $i = 1,2,3$ (ranking)
\$1,000	\$800	\$400	\$124.72 (3)	\$57.85 (2)	14.8% (1)
\$2,000	\$1,560	\$800	\$211.34 (2)	\$79.34 (1)	13.3% (2)
\$3,000	\$2,250	\$1,200	\$231.29 (1)	\$37.10 (3)	11.0% (3)

We describe the challenging investments' NPV rankings in columns 4 and 5 for investments of \$1,000, \$2,000, and \$3,000. The investments' IRR rankings appear in column 6. We indicate investment rankings by parenthesized numbers in columns 4, 5, and 6.

If the defender's IRR is 5% for challenging investments 1, 2, and 3, their NPV rankings are 3, 2, and 1 respectively. If the defender's IRR is 10% for challenging investments 1, 2, and 3, their NPV rankings are 2, 1, and 3 respectively. Meanwhile, for challenging investments 1, 2, and 3 their IRR rankings are different still: 1, 2, and 3 respectively. These results demonstrate that IRR and NPV rankings may provide inconsistent rankings and NPV rankings may vary with the defender's IRR.

The inconsistent IRR and NPV rankings require decision makers to choose between conflicting recommendations. They must decide: which ranking method should they believe. Among practitioners, the NPV approach is considered more reliable for wealth maximization. However, the ease in which decision makers can interpret IRRs has made it the more popular of the two ranking methods. Both ranking methods, however, have their drawbacks. Investments may have multiple IRRs and NPV rankings may be unstable—varying with changes in the defender's IRR what is often referred to as the discount rate.

The problem produced by inconsistent IRR and NPV rankings is obvious—it leaves decision makers without a clear recommendation. On the other hand, NPV rankings that vary with the discount rate

(see columns four and five) can be just as problematic. The reason is that if we are not sure of the discount rate, then we cannot be sure of the NPV rankings either.

The purpose of this chapter is to demonstrate that by resolving periodic and initial investment size differences between challenging investments and by adopting a common reinvestment rate, we can guarantee consistent IRR and NPV rankings. We will also explain why some NPV rankings vary with changes in the defender's IRR. Finally, we show that depending on how we adjust for size differences and the reinvestment rate we adopt, we can produce four different present value (PV) models. We also discuss the conditions under which each investment model is appropriate for ranking challenging investments—rankings that will be consistent but possibly different across models.

Periodic and Initial Investment Sizes

Sometimes when IRR and NPV methods produce inconsistent rankings, it is because the challenging investments being ranked lack investment size homogeneity. Challenging investments may lack investment size homogeneity in two ways. First, their initial investment sizes may be different. In Table 13.1 note that challenging investments 1, 2, and 3 invested initial amounts of \$1,000, \$2,000, and \$3,000 respectively.

The second way challenging investment sizes may differ is their periodic cash flows. These differences change periodic investment amounts between challenging investments. Notice that in Table 13.1 cash flows in period one are different: \$800, \$1,560, and \$2,250 for investments 1, 2, and 3 respectively. Differences in initial and periodic investment sizes contribute directly to inconsistent and unstable rankings.

Notation. This chapter employs more than the usual amount of mathematical notation. It is required to tell the story about consistent and stable IRR and NPV rankings. The good news is that there are no complicated mathematical operations besides adding, subtracting, canceling like terms, and factoring. The notation that will be used throughout this chapter includes variables with subscripts and superscripts. The superscript on a variable associates it with an investment, $i = 1, 2, \dots, m$. The subscript on a variable associates it with a time period $t = 1, \dots, n$. If a superscripted variable is raised to a power, the variable is placed in parentheses and the power to which the variable is raised is placed outside of the parentheses. To illustrate, the variable $(R_5^2)^3$ describes the value for the variable R associated with investment $i = 2$ in the $t = 5$ time period raised to the third power. Finally, when we write an expression like $S(r, R_n^i)$ we are saying that function S depends on values assumed by variables r and R_n^i without specifying the exact nature of the function or the values of the variables represented by r and R_n^i . When we want to represent a vector of values for investment i we bold \mathbf{R} and write $\mathbf{R} = R_1^i, R_2^i, \dots, R_n^i$.

How differences in periodic cash flow can create periodic investment size differences. Applying the notation described earlier, we now demonstrate how differences in cash flow create differences in investments sizes over time. Assume that initial investments of V_0^i for investments $i = 1$ and $i = 2$ equal V_0 . Also assume that cash flows in period one earned by investments one and two are equal to R_1^1 and

R_1^2 respectively. These positive (negative) cash flows represent investments (disinvestments) in the underlying investment. If the initial investments earned a rate of return equal to the defender's IRR equal to r , then beginning in period two, investment one equals $V_0(1+r) - R_1^1$ and investment two equals $V_0(1+r) - R_1^2$ which are not equal if $R_1^1 \neq R_1^2$. As a result, unequal periodic cash flows create unequal periodic investments, even if their initial investments are equal, a cause of inconsistent rankings.

To illustrate numerically, suppose that $V_0 = \$100$ and $R_1^1 = \$15$ and $R_1^2 = \$20$. If r is 10% then after one period the amount invested in investment one is $\$100(1.1) - \$15 = \$95$ and the amount invested after one period in investment two is $\$100(1.1) - \$20 = \$90$. After one period the investment amounts are unequal and violate the homogeneity of measures requirement.

The implication of these results is that even though two investments begin with equal initial investments, IRR and NPV rankings may still be inconsistent and unstable if periodic cash flows are unequal.

So what have we learned? We've learned that the two homogeneous investments size conditions required for consistent NPV and IRR rankings are 1) equal initial challenging investments sizes, and 2) equal periodic cash flows except in the last period. We require equal periodic cash flows except in the last period to ensure that the challenging investments will satisfy equal periodic investment sizes. Besides these two size conditions, one of two reinvestment rate assumptions must be adopted for both NPV and IRR rankings. These two reinvestment rate assumptions are that cash flows are either reinvested at the defender's IRR or at the challengers' respective IRRs.

Consistent IRR and NPV Rankings When the Reinvestment Rate is the Defender's IRR

To be clear, reinvestment rates indicate where periodic cash flows will be reinvested. If the reinvestment rate is the defender's IRR, then we assume that earnings from the challenging investments will be reinvested in the defender. If the reinvestment rates are the challenging investments' respective IRRs, then we assume that earnings from the investments will be reinvested in the challenging investments. If there is some other reinvestment rate, it must correspond to a separate investment and be evaluated as another challenger. Hence, the only reinvestment rates we consider are the defender's IRR and the challenging investments' IRRs.

NPV models implicitly assume that the reinvestment rate is the defender's IRR and earnings from the investment will be reinvested in the defender. IRR models assume that the reinvestment rate is the investment's own IRR and that the funds will be reinvested in itself. However, to remove ranking incon-

sistencies, we impose the same reinvestment rate across IRR and NPV models and produce what others have referred to as modified IRR (MIRR) and modified NPV (MNPV) models.

Next we demonstrate that if the two sufficient size conditions are satisfied and challenger earnings are reinvested in the defender, we are guaranteed consistent NPV and IRR rankings. NPV rankings will be consistent and under some conditions stable.

NPV rankings. Consider ranking two mutually exclusive n period investments in a similar risk class by their respective NPVs. The NPV model for investment one is described as:

$$(13.1a) \quad NPV^1 = -V_0^1 + \frac{R_1^1}{(1+r)} + \dots + \frac{R_n^1}{(1+r)^n}$$

The NPV model for investment two is described as:

$$(13.1b) \quad NPV^2 = -V_0^2 + \frac{R_1^2}{(1+r)} + \dots + \frac{R_n^2}{(1+r)^n}$$

In the equations above, r is the defender's IRR, V_0^1 and V_0^2 are initial investments one and two respectively, and R_t^1 and R_t^2 are periodic cash flows in periods $t = 1, \dots, n$ generated by mutually exclusive investments one and two. We assume that the terms of the two investments are equal.

To resolve size differences caused by differences in periodic cash flows, we set them equal to zero in every period except the last one by reinvesting them at rate r until the last period. Generally, the discount rate and the reinvestment rate in NPV models are equal. To distinguish the original NPV model with the NPV model with funds reinvested until the last period, many finance scholars refer to the later model as the modified NPV model or the MNPV model. We write the MNPV model as:

$$(13.2a) \quad \begin{aligned} MNPV^1 &= -V_0^1 + \frac{(1+r)^{n-1}R_1^1 + (1+r)^{n-2}R_2^1 + \dots + R_n^1}{(1+r)^n} \\ &= -V_0^1 + \frac{S(r, \mathbf{R}^1)}{(1+r)^n} \end{aligned}$$

where $S(r, \mathbf{R}^1) = (1+r)^{n-1}R_1^1 + (1+r)^{n-2}R_2^1 + \dots + R_n^1$. Similarly the MNPV model for investment two can be rewritten as:

$$\begin{aligned}
MNPV^2 &= -V_0^2 + \frac{(1+r)^{n-1}R_1^2 + (1+r)^{n-2}R_2^2 + \dots + R_n^2}{(1+r)^n} \\
&= -V_0^2 + \frac{S(r, R^2)}{(1+r)^n}
\end{aligned}
\tag{13.2b}$$

where $S(r, \mathbf{R}^2) = (1+r)^{n-1}R_1^2 + (1+r)^{n-2}R_2^2 + \dots + R_n^2$. Finally, we impose the additional condition that $S(r, \mathbf{R}^1), S(r, \mathbf{R}^2) > 0$, otherwise the investment decision is no longer interesting.

The difference between $MNPV^1$ and $MNPV^2$ which ranks the two investments can be expressed as:

$$MNPV^2 - MNPV^1 = [V_0^1 - V_0^2] + \frac{[S(r, \mathbf{R}^2) - S(r, \mathbf{R}^1)]}{(1+r)^n}
\tag{13.3}$$

If the initial investment sizes are the same as required by the homogeneity of measures principle, then the difference in the first bracketed expression in equation (13.3) is zero and the rankings will depend only on the difference between the sums of reinvested cash flows:

$$MNPV^2 - MNPV^1 = \frac{[S(r, \mathbf{R}^2) - S(r, \mathbf{R}^1)]}{(1+r)^n}
\tag{13.4}$$

Equation (13.4) provides an interesting result. We obtained these results by reinvesting and discounting cash flow at the same rate so NPVs of the two investments are unaffected allowing us to write:

$$\frac{[S(r, \mathbf{R}^2) - S(r, \mathbf{R}^1)]}{(1+r)^n} = NPV^2 - NPV^1.
\tag{13.5}$$

We now illustrate the results of equation (13.5). Let the initial investment equal \$100 and cash flows in periods one and two equal \$50 and \$70 respectively. Finally let the discount rate and the implied reinvestment rate equal 10%. Then we write:

$$NPV = -\$100 + \frac{\$50}{(1+10\%)} + \frac{\$70}{(1+10\%)^2} = \$3.31
\tag{13.6a}$$

Then we write the equivalent MNPV model as:

$$MNPV = -\$100 + \frac{(\$50)(1+10\%) + \$70}{(1+10\%)^2} = \$3.31
\tag{13.6b}$$

The results illustrate NPV and MNPV models are equal and provide the same rankings when the reinvestment rate is the same as the discount rate.

Finally, it should be noted that the difference between the NPVs in equation (13.5) may be unstable and change as the defender's IRR changes as we demonstrated in Table 13.1.

IRR rankings. Consider ranking two mutually exclusive n period investments in a similar risk class by their respective IRRs. If the IRR rankings are to be consistent with the NPV rankings, they must satisfy the sufficient conditions described earlier, including assuming the same reinvestment rate r . The IRR model for investment one is described as:

$$(13.7a) \quad V_0^1 = \frac{R_1^1}{(1 + IRR^1)} + \dots + \frac{R_n^1}{(1 + IRR^1)^n}$$

The IRR model for investment two is described as:

$$(13.7b) \quad V_0^2 = \frac{R_1^2}{(1 + IRR^2)} + \dots + \frac{R_n^2}{(1 + IRR^2)^n}$$

In the equations above, V_0^1 and V_0^2 are initial investments one and two respectively, and R_t^1 and R_t^2 are periodic cash flows in periods $t = 1, \dots, n$ generated by mutually exclusive investments one and two.

To resolve differences in periodic cash flows, we set them equal to zero in every period except the last period by reinvesting them at rate r until the last period. Generally, the discount rate and the reinvestment rate in IRR models are equal. Here, the reinvestment rate is r rather than their respective IRRs. To distinguish the original IRR model from the IRR model with funds reinvested until the last period, many finance scholars refer to the later model as the modified IRR model, the MIRR model. We write the MIRR model as:

$$(13.8) \quad V_0^1 = \frac{S(r, \mathbf{R}^1)}{(1 + MIRR^1)^n}$$

Solving for $MIRR^1$ in equation (13.8) we obtain:

$$(13.9) \quad MIRR^1 = \left[\frac{S(r, R^1)}{V_0^1} \right]^{1/n} - 1$$

which has one positive root.

Investment two is described as:

$$(13.10) \quad V_0^2 = \frac{S(r, \mathbf{R}^2)}{(1 + MIRR^2)^n}$$

Solving for $MIRR^2$ in equation (13.9) we obtain:

$$(13.11) \quad MIRR^2 = \left[\frac{S(r, R^2)}{V_0^2} \right]^{1/n} - 1$$

which has one positive root.

Finally, we write the difference between $MIRR^1$ and $MIRR^2$ assuming initial investments V_0^2 and V_0^1 are equal to V_0 as:

$$(13.12) \quad \begin{aligned} MIRR^1 - MIRR^2 &= \left[\frac{S(r, R^1)}{V_0^1} \right]^{1/n} - \left[\frac{S(r, R^2)}{V_0^2} \right]^{1/n} \\ &= \left\{ [S(r, R^2)]^{1/n} - [S(r, R^1)]^{1/n} \right\} \left(\frac{1}{V_0} \right)^{1/n} \end{aligned}$$

If the initial sizes and terms of investments one and two are equal, then their MIRR ranking will depend on the same criterion that ranked the investments' MNPVs—the difference in the sum of their reinvested cash flows $S(r, \mathbf{R}^1)$ and $S(r, \mathbf{R}^2)$ and the MIRR rankings and MNPV rankings will be consistent. However, we cannot be certain that the MIRR rankings will equal the IRR rankings because we changed the reinvestment rate assumption in the IRR models.

Consistent IRR and NPV Rankings Assuming Reinvestment Rates are the Challenging Investments' IRRs

In this section, we demonstrate that we can also guarantee consistent NPV and IRR rankings by imposing the second reinvestment rate assumption—that periodic cash flows are reinvested in the challenging investments' IRRs.

Suppose the reinvestment rates were the IRRs of the challengers being evaluated. This reinvestment rate is appropriate under the condition that the cash flows are reinvested in the challengers as opposed to being reinvested in the defending investment. To demonstrate the consequences of assuming that the reinvestment rates equal the challengers' IRRs, consider the following.

Assume that the initial sizes of the challenging investments are equal and that their cash flows are reinvested in themselves earning their respective IRRs and the discount rate applied to the NPVs is the defender's IRR. Then we could write the difference between MNPVs as:

$$(13.13a) \quad \begin{aligned} &MNPV^2 - MNPV^1 = \\ &\frac{[S(MIRR^2, \mathbf{R}^2) - S(MIRR^1, \mathbf{R}^1)]}{(1 + r)^n} \end{aligned}$$

Meanwhile, we can write the difference between MIRRs as:

$$(13.13b) \quad \begin{aligned} &MIRR^2 - MIRR^1 = \\ &\left\{ [S(MIRR^2, \mathbf{R}^2)]^{1/n} - [S(MIRR^1, \mathbf{R}^1)]^{1/n} \right\} \left(\frac{1}{V_0} \right)^{1/n} \end{aligned}$$

Notice that the MNPV and MIRR rankings depend on the same sums making their rankings consistent. However, in contrast to the NPV model, the MNPV rankings are stable and likely different than the original NPV rankings. But there is one thing more. Look at the original IRR equations. They are equivalent to the MIRR equations. Reinvesting and discounting by the challengers' IRRs are offsetting operations and don't change the original IRRs. Therefore, if the initial sizes are equal, and the assumption is that cash flows are reinvested in the challenging investments, then IRR and MIRR rankings are equivalent! This is such an important point, we illustrate it with an example.

Assume net cash flows of \$50 and \$70 in periods one and two respectively, an initial investment of \$100, and a reinvestment rate of the investment's IRR or MIRR. Then we can write the IRR as:

$$(13.14a) \quad \$100 = \frac{\$50}{(1 + IRR)} + \frac{\$70}{(1 + IRR)^2} \quad \text{and} \quad IRR = 12\%$$

Now we can write the MIRR model as:

$$(13.14b) \quad \$100 = \frac{(\$50)(1 + MIRR) + \$70}{(1 + MIRR)^2} \quad \text{and} \quad MIRR = 12\%$$

Because the IRR and MIRR values are the same, they will return identical rankings. To emphasize the main point, if the periodic cash flows are reinvested in the challengers, then ranking investments using the investments' IRR is the same as ranking them with their MIRRs.

So what have we learned? We learned that when we satisfy the two sufficient size conditions required for consistent rankings, equal initial and period investment sizes (except in the last period), and satisfy the second reinvestment rate assumption that we reinvest in the challenging investments—then MNPV and MIRR rankings of the challengers will be consistent. And there is one thing more, the MIRR rankings will be the same as the original IRR rankings.

We have also learned that inconsistent challenging investments IRR and NPV rankings are often the results of the two ranking methods adopting different reinvestment rate assumptions. NPV models assume the reinvestment rate is the defender's IRR. IRR models assume the reinvestment rate is the challenger's IRR for the same investment. Consistency requires that the reinvestment rate for each investment be the same regardless of whether the ranking tool is IRR or NPV. Once we have satisfied initial and periodic investment size differences and adopted a consistent reinvestment rate assumption, investments can be ranked consistently using IRR or NPV methods.

Reinvestment rates and inconsistent rankings. To make clear that consistent IRR and NPV rankings require that the same reinvestment rate for each investment regardless of the ranking methods, IRR or NPV, we describe three investments with the same initial sizes and assume the defender's IRR is 5%. We allow NPV models to reinvest at the defender's IRR rate of 5%. We allow the challenger's cash flow to be reinvested at their respective IRRs. The inconsistent rankings produced by allowing investments' reinvestment rate to be different when using IRRs versus NPVs, we describe in Table 13.2. The results demonstrate that even if we adjust for initial and periodic size differences but fail to require consistent reinvestment rates, IRR and NPV rankings could still be inconsistent.

Table 13.2. Rankings Allowing NPV models (IRR models) to reinvest periodic cash flows at the defender's IRR (challengers' IRRs). Investment rankings indicated in parentheses below NPVs and IRRs values.

Period	Variables	Challenger 1 cash flow	Challenger 2 cash flow	Challenger 3 cash flow
0	V_0	– \$3,000	– \$3,000	– \$3,000
1	R_1	– \$100	\$2,010	\$2,250
2	R_2	\$3,800	\$1,560	\$1,200
	NPVs assuming $r=5\%$	\$351.47 (1)	\$329.25 (2)	\$231.29 (3)
	IRRs	10.89% (3)	13.01% (1)	11.03% (2)

Now assume that investment three was the defender and we used its IRR equal to 11.03% to reinvest period one cash flow to period two. Then because we employed the same reinvestment rate for each

model regardless of whether we used IRR or NPV ranking models, consistency is restored. We describe these results in Table 13.3.

Table 13.3. Allowing NPV and IRR models to reinvest periodic cash flows at same reinvestment rate equal to challenger 3's IRR. Investment rankings indicated in parentheses below NPVs and IRRs values.

Period	Variables	Challenger 1 cash flow	Challenger 2 cash flow	Challenger 3 cash flow
0	V_0	– \$3,000	– \$3,000	– \$3,000
1	R_1	\$0	\$0	\$2,250
2	R_2	$\$3,800 - \$100 \times 1.1103 =$ \$3688.97	$\$1,560 + \$2010 \times 1.1103 =$ \$3791.70	\$1,200
	NPVs assuming $r=11.03\%$	– \$7.57 (3)	\$83.82 (1)	\$0 (2)
	IRRs assuming $r=11.03\%$	10.89% (3)	12.57% (1)	11.03% (2)

So what have we learned? We learned that when we rank challengers using their NPVs and IRRs even when we adjust for initial and periodic size differences—we can produce conflicting rankings because IRR and NPV models assume different reinvestment rates. For IRRs and NPVs to produce consistent rankings, each investment must assume the same reinvestment rates. Different investments may assume different reinvestment rates; however, if IRR and NPV ranking are to be consistent, they must be the same reinvestment rate for the same investment regardless of whether the ranking models are IRRs or NPVs.

Resolving Initial Size Differences by Scaling

Scaling and a reinvestment rate equal to the defender's IRR

Consider the possibility of eliminating initial investment size differences between mutually exclusive investments by scaling the challengers to a common initial size. To scale challengers requires that they are available in continuous sizes. To scale an investment requires that it leaves the relative size of the investment and its cash flows constant over the continuous sizes. For example, scaling an investment by two assumes that there exists an investment twice the size that produces twice the cash flow of the original investment in each period. To illustrate, assume net cash flows of \$50 and \$70 in periods one and two respectively, an initial investment of \$100, and assume the reinvestment rate equals the defender's IRR of $r = 10\%$. Then we can write the MIRR model as:

$$(13.15a) \quad MIRR = \left[\frac{(\$50)(1 + 10\%) + \$70}{\$100} \right]^{1/2} - 1 = 12\%$$

Suppose now that we scale the MIRR model by 2. Then we can rewrite the model as:

$$(13.15b) \quad \text{scaled } MIRR = \left\{ \frac{2[(\$50)(1 + 10\%) + \$70]}{(2)\$100} \right\}^{1/2} - 1 = 12\%$$

Notice that the scaling factor “2” cancels so we are left with the original MIRR equation in (13.15a). And even if the reinvestment rate is the IRR, the scaling factor still cancels. Thus, the MIRR and the scaled MIRR are equal. (Can you demonstrate that the scaling factor cancels regardless of the reinvestment rate?)

Scaling and a reinvestment rate equal to the challengers’ IRR

What if the reinvestment rate is the challenger’s own IRRs, which assumes that the funds are reinvested in the challengers, then MIRRs, scaled MIRRs and the IRR equations are the same.

To make the point absolutely clear, that scaling will not change MIRRs or IRRs when the reinvestment rate is the investment’s IRR, we illustrate these results by comparing the original IRR calculations from Table 13.1 with MIRRs, and scaled MIRRs calculated in Table 13.4.

Table 13.4. Consistent MIRR, scaled MIRR, and IRR rankings obtained by scaling initial investment sizes and periodic cash flows assuming the reinvestment rates are equal to the investments’ IRRⁱ where i = 1,2,3.

initial investment V_0^i and scaling factor γ_i where $i = 1,2,3$	period one cash flows $\gamma_i R_t^1$ where $t = 1$	period two cash flows $R_2^i + (1 + IRR^i)R_1^i$ where $i = 1,2,3$	period two cash flows $\gamma_i R_2^i + \gamma_i(1 + IRR^i)R_1^i$ where $i = 1,2,3$	MIRR ⁱ where $i = 1,2,3$ assuming $r = IRR^i$ (ranking)	Scaled MIRR ⁱ where $i = 1,2,3$ assuming $r = IRR^i$ (ranking)	IRR ⁱ where $i = 1,2,3$ assuming $r = IRR^i$ (ranking)
$V_0^1 = \$1,000$ $\gamma_1 V_0^1 = \$3,000$ ($\gamma_1 = 3$)	0	\$400 + (1.148)\$800 = \$1,318.40	(3)\$400 + (3)(1.148)\$800 = \$3,955.20	14.8% (1)	14.8% (1)	14.8% (1)
$V_0^2 = \$2,000$ $\gamma_2 V_0^2 = \$3,000$ ($\gamma_2 = 3/2$)	0	\$800 + (1.133)\$1,560 = \$2,567.48	(3/2)\$800 + (3/2)(1.133)\$1,560 = \$3,851.22	13.3% (2)	13.3% (2)	13.3% (2)
$V_0^3 = \$3,000$	0	\$1,200 + (1.110)\$2,250 = \$3,652.50	\$1,200 + (1.110)\$2,250 = \$3,652.50	11.0% (3)	11.0% (3)	11.0% (3)

We next demonstrate how scaling achieves ranking consistency between MNPVs and MIRRs and scaled MNPVs when the reinvestment rate is the defender's IRR. The MIRR, scaled MIRR and MNPV rankings in Table 13.5 are consistent because the MNPV rankings have been scaled to conform to the MIRR, and scaled MIRR.

Table 13.5. Consistent size adjusted MIRR and MNPV rankings obtained by scaling initial investment sizes and periodic cash flows assuming the reinvestment rate equal to the defender's IRR of $r = 5\%$.

initial investment V_0^i and scaling factor γ_i where $i = 1,2,3$	period one cash flows γ_i R_t^i where $t = 1$	period two cash flows $\gamma_i R_2^i + \gamma_i(1 + r)R_1^i$ where $i = 1,2,3$	Scaled MNPV ⁱ where $i = 1,2,3$ assuming $r = 5\%$ (ranking)	MIRRs and Scaled MIRR ⁱ where $i = 1,2,3$ assuming $r = 5\%$ (ranking)
$\gamma_1 V_0^1 = \$3,000$ ($\gamma_1 = 3$)	0	$(3)\$400 + (3)(1.05)\$800 = \$3,720$	\$374.15 (1)	11.4% (1)
$\gamma_2 V_0^2 = \$3,000$ ($\gamma_2 = 3/2$)	0	$(3/2)\$800 + (3/2)(1.05)\$1,560 = \$3,657$	\$317.01 (2)	10.4% (2)
$V_0^3 = \$3,000$	0	$\$1,200 + (1.05)\$2,250 = \$3,562.50$	\$231.29 (3)	9.0% (3)

While MIRRs are not affected by scaling, even when the reinvestment rate is the defender's IRR, scaling does have an effect on MNPVs. We observe the effect of scaling on MNPVs by rewriting equation (13.4) as:

$$\begin{aligned}
 NPV^2 - NPV^1 &= [\gamma V_0^1 - V_0^2] + \frac{[S(r, \mathbf{R}^2) - \gamma S(r, \mathbf{R}^1)]}{(1+r)^n} \\
 &= \frac{[S(r, \mathbf{R}^2) - \gamma S(r, \mathbf{R}^1)]}{(1+r)^n}
 \end{aligned}
 \tag{13.16}$$

Note that the scaling factor does not cancel out in equation (13.19) and may affect the values and rankings of scaled and unscaled MNPVs. To summarize, scaled and unscaled MIRRs are equal in rank and amount. However, scaled and unscaled MNPVs are not necessarily equal in rank or amount.

To illustrate, consider the earlier example reconstructed as an MNPV model. Assume an MNPV model with an initial investment of \$100, and cash flows in periods one and two of \$50 and \$70 respectively and a reinvestment rate $r = 10\%$ and a discount rate of $r = 8\%$

$$MNPV = -\$100 + \frac{(\$50)(1 + 10\%) + \$70}{(1 + 8\%)^2} = \$7.17
 \tag{13.17a}$$

Suppose now that we scale the MNPV model by 2. Then, we can rewrite the model as:

$$(13.17b) \quad \text{scaled } MNPV = -(2)\$100 + \frac{(2)[(\$50)(1 + 10\%) + \$70]}{(1 + 8\%)^2} = \$14.34$$

Notice that the scaling factor “2” no longer cancels so we are left with a different MNPV equations before and after scaling. And even if the reinvestment rate is the same as the discount rate, the scaling factor still does not cancel. (How can you demonstrate that the scaling factor does changes the MNPV regardless of the reinvestment rate?) Notice something else that is very important: the scaled MNPV is equal to the unscaled MNPV times the scaling factor which in this case is two: $\$14.34 = (2) \7.17

So what does this mean? If the reinvestment rate is the investment’s IRR which assumes that cash flows are reinvested in the challenger, then we should rank investments using IRRs. Even if the investments are of unequal size which can be resolved by scaling, IRR rankings will still be consistent with scaled MNPV rankings which can be obtained by multiplying the unscaled MNPVs by their respective scaling factors.

Resolving Initial Size Differences by Addition

Suppose that the defender is scalable but that the challenging investments are unique and cannot be scaled. Since initial investment size equality is one of the conditions required for consistent MIRR and MNPV rankings, we must adopt an alternative to the scaling method to achieve initial size equality. One approach is to add sufficient amounts of the scalable defender to the smaller challenging investment so that its size equals the size of the larger challenger. For example assume that $(V_2 - V_1) = \delta$ is added to V_1 where it earns the defender’s IRR equal to r until the n^{th} period. This allows us to rewrite equation (13.4) as:

$$(13.18a) \quad MNPV^2 - MNPV^1 = [V_0^1 + \delta - V_0^2] + \frac{[S(r, \mathbf{R}2) - S(r, \mathbf{R}1) - \delta(1 + r)^n]}{(1 + r)^n}$$

Because compounding and discounting δ are offsetting effects, we can rewrite equation (13.18a) as (13.18b):

$$\begin{aligned}
 MNPV^2 - MNPV^1 &= [V_0^1 - V_0^2] \\
 &+ \frac{S(r, \mathbf{R}^2) - S(r, \mathbf{R}^1)}{(1+r)^n}
 \end{aligned}
 \tag{13.18b}$$

But this ranking would have been obtained by comparing the MNPVs before adjusting for size differences by addition. Therefore, equation (13.18b) can be rewritten as:

$$\begin{aligned}
 MNPV^2 - MNPV^1 &= NPV^2 - NPV^1 \\
 &= [V_0^1 - V_0^2] + \frac{S(r, \mathbf{R}^2) - S(r, \mathbf{R}^1)}{(1+r)^n}
 \end{aligned}
 \tag{13.18c}$$

These results demonstrate that when the reinvestment rate is the defender's IRR then NPV, MNPV, and added MNPV rankings will all be equal in amount and rank.

We can write the added MIRR as:

$$\begin{aligned}
 &\text{added } MIRR^2 - \text{added } MIRR^1 \\
 &= \left[\frac{S(r, \mathbf{R}^2)}{V_0^2} \right]^{1/n} - \left[\frac{S(r, \mathbf{R}^1) + \delta(1+r)^n}{(V_0^1 + \delta)} \right]^{1/n}
 \end{aligned}
 \tag{13.19}$$

Furthermore, these added MNPV and added MIRR rankings will be consistent because the ranking criterion are the same (see Table 13.6). Note, however, that the consistent rankings obtained through addition are not the same as those obtained when initial size differences were resolved through scaling. Clearly, the final rankings depend on what methods we use to adjust for initial size differences and what reinvestment rate we assume.

Table 13.6. Added MIRRs and added MNPVs assuming the reinvestment rate is the defender's IRR of r = 5%.

initial investment V_0^i and addition amount δ_i where $i = 1,2,3$	period one cash flows R_{1i} where $t = 1$	period two cash flows $R_2^i + \delta_i(1+r)^2 + R_1^i(1+r)$ where $i = 1,2,3$	MNPV ⁱ where $i=1,2,3$ assuming $r=5\%$ (ranking)	MIRR ⁱ where $i=1,2,3$ assuming $r=5\%$ (ranking)
$(\delta_1 + V_0^1) = \$3,000$ $(\delta_1 = \$2,000)$	0	$\$400 + \$2,000(1.05)^2 + \$800(1.05) = \$3,445$	\$124.72 (3)	7.2% (3)
$(\delta_2 + V_0^2) = \$3,000$ $(\delta_2 = \$1,000)$	0	$\$800 + \$1,000(1.05)^2 + \$1,560(1.05) = \$3,540.50$	\$211.34 (2)	8.6% (2)
$V_0^3 = \$3,000$	0	$\$1,200 + \$2,250(1.05) = \$3,562.50$	\$231.29 (1)	9.0% (1)

Four Consistent Investment Ranking Models

We have demonstrated that differences in initial and periodic sizes can cause inconsistent ranking. We resolved differences in periodic sizes by reinvesting cash flow until the last period. However, we noted that the reinvestment rate for each investment could be the challenger's IRR or the defender's IRR.

Then we noted that consistency requires that their initial sizes are equal as well as their periodic sizes. Several methods can be employed to resolve initial size differences. We emphasized two: scaling and addition. Each of these methods will generate consistent initial size adjusted MIRR and MNPV rankings as long as cash flows are reinvested until the last period.

The methods employed to adjust for periodic and initial size differences produce four distinct ranking models that depend on which size adjustment method and reinvestment rate we adopt. We describe the four investment ranking models in Table 13.7 below.

Table: 13.7. Alternative methods for adjusting challenging investments to resolve periodic and initial size differences assuming equal term and risk.

	Assume the reinvestment rate is the defender's IRR equal to r	Assume the reinvestment rate is the challengers' IRR
Resolve initial size differences by scaling the smaller challenger with a factor equal to $\gamma > 1$	model 1	model 2
Resolve initial size differences by adding amount δ to the smaller challenger	model 3	model 4

We now list below the PV models consistent with models 1 through 4. We first write the unadjusted IRR and NPV models as:

$$(13.20) \quad V_0 = \frac{R_1}{(1 + IRR)} + \frac{R_2}{(1 + IRR)^2} + \cdots + \frac{R_n}{(1 + IRR)^n}$$

And

$$(13.21) \quad NPV = -V_0 + \frac{R_1}{(1 + r)} + \frac{R_2}{(1 + r)^2} + \cdots + \frac{R_n}{(1 + r)^n}$$

Model 1. We write the adjusted IRR* and NPV* model 1 as:

(adjusted IRR model 1)

$$(IRR 1) \quad \gamma V_0 = \frac{(1 + r)^{n-1} \gamma R_1 + (1 + r)^{n-2} \gamma R_2 + \cdots + \gamma R_n}{(1 + IRR^*)^n}$$

(adjusted NPV model 1)

$$NPV^* = -\gamma V_0 + \frac{(1+r)^{n-1}\gamma R_1 + (1+r)^{n-2}\gamma R_2 + \dots + \gamma R_n}{(1+r)^n}$$

$$(NPV\ 1) \quad = \gamma NPV$$

Model 2. We write the adjusted IRR and NPV as:

(adjusted IRR model 2)

$$(IRR\ 2) \quad \gamma V_0 = \frac{(1+IRR)^{n-1}\gamma R_1 + (1+IRR)^{n-2}\gamma R_2 + \dots + \gamma R_n}{(1+IRR)^n}$$

(adjusted NPV model 2)

$$(NPV\ 2) \quad NPV^* = -\gamma V_0 + \frac{(1+IRR)^{n-1}\gamma R_1 + (1+IRR)^{n-2}\gamma R_2 + \dots + \gamma R_n}{(1+r)^n}$$

Model 3. We write the adjusted IRR* and NPV* model 3 as:

(adjusted IRR model 3)

$$(IRR\ 3) \quad V_0 + \delta = \frac{(1+r)^{n-1}R_1 + (1+r)^{n-2}R_2 + \dots + R_n + \delta(1+r)^n}{(1+IRR^*)^n}$$

(adjusted NPV model 3)

$$(NPV\ 3) \quad NPV^* = -(V_0 + \delta) + \frac{(1+r)^{n-1}R_1 + (1+r)^{n-2}R_2 + \dots + R_n + \delta(1+r)^n}{(1+r)^n} = NPV$$

Model 4. We write the adjusted IRR* and NPV* model 4 as:

(adjusted IRR model 4)

$$\begin{aligned}
 & V_0 + \delta \\
 \text{(IRR 4)} \quad &= \frac{(1 + IRR)^{n-1}R_1 + (1 + IRR)^{n-2}R_2 + \cdots + R_n + \delta(1 + IRR)^n}{(1 + IRR)^n}
 \end{aligned}$$

(adjusted NPV model 4)

(NPV 4)

$$\begin{aligned}
 NPV^* &= -(V_0 + \delta) \\
 &+ \frac{(1 + IRR)^{n-1}R_1 + (1 + IRR)^{n-2}R_2 + \cdots + R_n + \delta(1 + IRR)^n}{(1 + r)^n} \\
 &= MNPV
 \end{aligned}$$

Comments on the Four Investment Models and Ranking Consistency

A comment about ranking consistency. Before commenting on models 1 through 4, we note the following. In some cases periodic and initial size-adjusted IRRs and NPVs produce the same rankings as the original IRRs and NPVs. In these cases, it simplifies our efforts to be able to use the original NPV or IRR rankings knowing they will provide the consistent rankings produced by the size adjusted IRR and NPV models.

Comments on model 1. Notice that we can factor out the scaling factor in NPV model 1 producing the original NPV multiplied by the scaling factor— γ NPV. Moreover, ranking investments by γ NPV provides the same ranking as size adjusted IRRs and NPVs. So, if ranking needs to occur under conditions consistent with model 1, our recommendation is to rank challengers using γ NPV.

Comments on model 2. Notice that the scaling factor in IRR model 1 cancels so that we are left with our original IRR model except that periodic cash flow are reinvested until the last period. But, compounding and discounting by the same term are offsetting actions. As a result, the IRR rankings provides the same ranking as the adjusted IRR model. So, if ranking needs to occur under conditions consistent with model 2, our recommendation is rank challengers using their original IRRs.

Comments on model 3. Notice that compounding and discounting by the factor $(1 + r)$ in model 3 are offsetting actions so that the adjustment produces the original NPV model except for the added factor δ that is added at the beginning and subtracted at the end of the period—eliminating its influence in the model. As a result, the original IRR rankings are the same as the size adjusted IRR and NPV models. So, if ranking needs to occur under conditions consistent with model three, our recommendation is rank challengers using their original IRRs.

Comments on model 4. Notice that compounding and discounting using the challenger's IRR are offsetting actions so that the adjustment produces the original IRR model except for the added factor δ . However, δ is added at the beginning and subtracted at the end of the period eliminating it from the model. As a result, the IRR rankings are the same rankings provided by the size adjusted IRR and NPV models. So, if ranking needs to occur under conditions consistent with model four, our recommendation is rank challengers using their original IRRs.

Other size adjustment models. For completeness, it is important to note that there are a number of other size adjustment methods we could employ. For example, we could combine challengers 1 and 2 in Table 13.1 to achieve initial size consistency with challenger 3. Or we could use some combination of scaling and addition to adjust for initial size differences. And the alternative reinvestment rates we could employ to resolve periodic size differences is large indeed. The only consideration when employing these nonstandard size adjustments is that the recommendations included in comments for models one through four no longer apply, requiring decision makers to find size adjusted NPVs or IRRs.

When to Rank Challengers Using Four Models

Adjusting initial investment sizes: scaling versus addition. The appropriate initial size adjustment method depends on the characteristics of the challengers in which investments are made. To compare challengers of different initial sizes, scaling to achieve initial size differences is appropriate when the challengers are available in continuous sizes or approximately so. We describe such investments as producing constant returns to scale. For constant returns to scale, the relationship between the investment and the cash flow it produces is constant. To illustrate, hours of services may often be purchased in divisible units, each unit producing the same output as earlier ones. In sum, models 1 and 2 apply when the investments are models available in continuous sizes.

On the other hand, some investments are available only in a few sizes. Machinery is available in a limited number of sizes. Sometimes, land may be purchased in fixed quantities depending on physical configurations. The same is true for buildings. So how do we compare investments available in a few sizes?

Suppose we are comparing two investments of different sizes. Assume we find the difference in their initial cost and ask: if the smaller challenger is adopted, where is the difference in their initial cost employed and what does it earn? The answer is it remains invested in the defender and these must

be included with the earnings of the smaller challenger. In this way we achieve initial size consistency while allowing the underlying investments to be different in their initial sizes.

Adjusting periodic size differences: reinvestment rate assumptions. We resolved periodic size differences by setting cash flow in all of the periods to zero except cash flow in the last period. We set all of the cash flow in all but the last period equal to zero by reinvesting them. The implicit reinvestment rate for NPV and IRR models is the defender's or the challengers' IRRs.

Ranking Investments when the Challengers are not Mutually Exclusive

So far we have treated our challengers as mutually exclusive investments—only one can be chosen. For firms with a fixed investment budget, treating investments as mutually exclusive may be justified. Suppose that we face a set of challenging investments that are not mutually exclusive and their adoption depends on their passing some IRR hurdle. How should the investment problem be analyzed? In other words, suppose that we have a portfolio of different investment challengers and of different sizes. How big or how many challengers should be adopted ?

Economic theory can help us answer this question. Usually we array our challengers according to their IRRs and invest in the most profitable ones first. Of course, the size of the defender must match the size of the challenger, but the criterion for arranging defenders to match the size of our challenger is somewhat different. We give up, sacrifice, the defender with the smallest IRR first. To assume that defenders are infinitely scalable is to believe that the firm is in a perfect market, an assumption we earlier rejected.

Of course, investments may not come in infinitely small amounts; one large investment may be more profitable than a small investment because it is larger. If challengers come in lumpy amounts, that is the size of the challenger we investigate.

So, we are back to the place we started. Begin with the highest rate of return challenger, and sacrifice the lowest rate of return defender. Then analyze the next highest rate of return challenger and sacrifice the next lowest rate of return defender, all the time maintaining size equality. Keep analyzing challengers and defenders sequentially, maintaining size homogeneity, until you fail to adopt the next highest rate of return challenger compared to the next lowest rate of return defender.

This rule is comparable to the marginal cost equals marginal return rule employed in comparative economic static analysis.

The logic behind this arrangement is that we begin by considering the most favorable challenger, and compare it with the least favorable defender. Then we continue to the next most favorable challenger and second least favorable defender. Once we have found a challenger unable to earn a higher IRR than the IRR of the defender being sacrificed, we end our investment analysis.

Common Practice, Common Problems, and Reinvestment Rates

Common practice is to rank challengers using IRR or NPV methods. We have emphasized that this common practice can frequently produce a common problem of inconsistent rankings. Then we made the point that when challengers were adjusted for initial and periodic size differences and employed a common reinvestment rate—either the defender or the challengers' IRRs—then the ranking were consistent. The question is, can consistent IRR and NPV rankings be achieved under fewer restrictions? For example, could we relax the common reinvestment rate assumption and still find consistent rankings—all the time? The answer is no!

Since we require only one counter-example to disprove the general claim, we provided one in Table 13.2. Consider three challengers of equal initial size and different periodic cash flow. Since reinvesting the cash flow to achieve periodic size equality at the defender's IRR doesn't change NPV rankings and reinvesting at the challenger's IRR doesn't change IRR rankings, we can rank the three challengers using NPV and IRR under initial and periodic size equality. We reported the results of such a comparison earlier in Table 13.2.

Note the conflicting NPV and IRR rankings. NPV ranks the three challengers 1, 2, and 3. IRR ranks the three challengers 3, 1, and 2. These results provide the one counter example needed to disprove the claim that different reinvestment rates are not required for consistent rankings. Furthermore, since we are describing the reinvestment rate for the same investment and are only changing the ranking method, consistency would require the same reinvestment rate assumption regardless of the ranking method employed.

Summary and Conclusions

This chapter has demonstrated that equal initial investments and periodic cash flows reinvested until the last period plus a common reinvestment rate across IRR and NPV models are sufficient conditions for consistent MIRR and MNPV investment rankings. We can make unequal initial investments equal by scaling or adding investments to the small challengers—and by some other methods, if required. Whether or not initial size differences are resolved by scaling or by addition depends on whether the challenger is available in continuous sizes in which case scaling is appropriate. Otherwise, if it is unique and cannot be scaled, size differences are resolved by additions. All other methods for resolving initial size differences ultimately create additional challengers.

Resolving periodic size differences required that we reinvest cash flow until the final period. We considered the defender's IRR and the challengers' IRRs as possible reinvestment rates. Of course, there are a number of other possible reinvestment rate choices. The important point is that if the discount rate and the reinvestment rate are different, then the NPV rankings will be stable and will not vary with the discount rate.

So what have we learned? As long as the homogeneity of size principle is observed, whatever common reinvestment rate is assumed or whatever initial size adjustment method is employed will generate consistent rankings. These possibilities we described as four basic models. In addition, while all size adjusted PV models will produce consistent rankings, the ranking may vary with the methods adopted. The main implication of these results is to shift the debate from which ranking method to use, IRR versus NPV, to what is the appropriate size adjustment model to adopt?

Questions

1. Discuss why some investment analysts prefer NPV methods to rank investments while others prefer IRR methods. Which do you prefer? Defend your answer.
2. Provide three reasons why NPV and IRR rankings may be inconsistent.
3. Refer to the investment problems described in Table Q13.1 below. Notice that initial investment sizes are unequal as are the periodic cash flows in period one which produce the inconsistent NPV and IRR ranking reported in the last two columns and rows of Table Q13.1. We will refer to the results in Table Q13.1 in several questions that follow.

Table Q13.1. NPVs and IRRs for two mutually exclusive investments assuming a discount rate equal to the defender's IRR of $r = 5\%$.

initial investments V_0^i where $i = 1,2$	period one cash flows R_t^i where $t = 1$ generated by initial investments V_0^i where $i = 1,2$	period two cash flows R_t^i where $t = 2$ generated by initial investments V_0^i where $i = 1,2$	NPV ⁱ earned by initial investments V_0^i where $i = 1,2$ assuming $r = 5\%$ (ranking)	IRR ⁱ earned by initial investments V_0^i where $i = 1,2$ (ranking)
$V_0^1 = \$3,000$	\$2,000	\$1,500	\$265.31 (1)	11.51% (2)
$V_0^2 = \$1,000$	\$800	\$400	\$124.72 (2)	14.83% (1)

- a. Complete Table Q13.2 by finding MNPVs and MIRRs for the two investments assuming a discount rate and reinvestment rate equal to the defender's IRR, of $r=5\%$. Include the investment rankings in the parentheses.

Table Q13.2. MNPVs and MIRR for two mutually exclusive investments assuming a discount and reinvestment rate of $r = 5\%$.

initial investments V_0^i where $i = 1,2$	period one cash flows R_t^i where $t = 1$ generated by initial investments V_0^i where $i = 1,2$	period two cash flows R_t^i where $t = 2$ generated by initial investments V_0^i where $i = 1,2$	MNPV ⁱ earned by initial investments V_0^i where $i = 1,2$ assuming $r = 5\%$ (ranking)	MIRR ⁱ earned by initial investments V_0^i where $i = 1,2$ assuming $r = 5\%$ (ranking)
$V_0^1 = \$3,000$		$\$1,500 + \$2,000(1.05) = \$3,600$	\$ ()	% ()
$V_0^2 = \$1,000$		$\$400 + \$800(1.05) = \$1,240$	\$ ()	% ()

- b. Complete Table Q13.3 by finding MNPVs and MIRR for the two investments assuming a discount rate and reinvestment rate equal to the defender's IRR, of $r = 5\%$ and a reinvestment rates equal to the investments' IRRs. Include the investment rankings in the parentheses.

Table Q13.3. MNPVs and MIRR for two mutually exclusive investments assuming a discount and reinvestment rate of $r = 5\%$, and reinvestment rates equal to the investments' IRRs

initial investments V_0^i where $i = 1,2$	period one cash flows R_t^i where $t = 1$ generated by initial investments V_0^i where $i = 1,2$	period two cash flows R_t^i where $t = 2$ generated by initial investments V_0^i where $i = 1,2$	MNPV ⁱ earned by initial investments V_0^i where $i = 1,2$ assuming the discount and reinvestment rate equal $r = 5\%$ (ranking)	MIRR ⁱ earned by initial investments V_0^i where $i = 1,2$ assuming the reinvestment rate equals $r = 5\%$ (ranking)
$V_0^1 = 3,000$		$\$1,500 + \$2,000(1.1151) = \$3,730.20$	\$ ()	% ()
$V_0^2 = \$1,000$		$\$400 + \$800(1.1483) = \$1,318.64$	\$ ()	% ()

- c. Complete Table Q13.4 by finding scaled MNPVs and scaled MIRR for the two investments assuming a discount rate and reinvestment rate equal to the defender's IRR, of $r=5\%$. What is the scaling factor you are using? Include the investment rankings in the parentheses.

Table Q13.4. Scaled MNPVs and MIRR for two mutually exclusive investments assuming a discount and reinvestment rate of $r = 5\%$.

initial investment $\gamma_i V_0^i$ where $i = 1,2$	period one cash flows $\gamma_i R_t^i$ where $t = 1$ generated by initial investments $\gamma_i V_0^i$ where $i = 1,2$	period two cash flows $\gamma_i R_t^i$ where $t = 2$ generated by initial investments $\gamma_i V_0^i$ where $i = 1,2$	Scaled MNPV ⁱ earned by scaled initial investments $\gamma_i V_0^i$ where $i = 1,2$ assuming the discount and reinvestment rate equal $r = 5\%$ (ranking)	Scaled MIRR earned by scaled initial investments $\gamma_i V_0^i$ where $i = 1,2$ assuming the discount and reinvestment rate equal $r = 5\%$ (ranking)
$V_0^1 = 3,000$		$\$1,500 + [\$2,000(1.05)] = \$3,600$	\$ ()	% ()
$\gamma_2 V_0^2 = \$3,000$ $\gamma_2 = 3$		$[(3)\$400] + [(3)\$800(1.05)] = \$3,720$	\$ ()	% ()

- d. Complete Table Q13.5 by finding scaled MNPVs and scaled MIRR for the two investments

assuming a discount rate equal to the defender's IRR, of $r = 5\%$ and reinvestment rates equal to the investments' IRRs. Include the investment rankings in the parentheses.

Table Q13.5. Scaled MNPVs and MIRRs for two mutually exclusive investments assuming a discount rate of $r = 5\%$ and reinvestment rates equal to the investments' IRRs

initial investment $\gamma_i V_0^i$ where $i = 1, 2$	period one cash flows $\gamma_i R_t^i$ where $t = 1$ generated by initial investments $\gamma_i V_0^i$ where $i = 1, 2$	period two cash flows $\gamma_i R_t^i$ where $t = 2$ generated by initial investments $\gamma_i V_0^i$ where $i = 1, 2$	Scaled MNPV ⁱ earned by scaled initial investments $\gamma_i V_0^i$ where $i = 1, 2$ assuming the discount rate $r = 5\%$ and reinvestment rates equal to the investments' IRRs (rankings)	Scaled MIRRs earned by scaled initial investments $\gamma_i V_0^i$ where $i = 1, 2$ assuming the reinvestment rate equals $r = 5\%$ (ranking)
$V_0^1 =$ \$3,000		\$1,500 + [\$2,000(1.1151)] = \$3,730.20	\$ ()	% ()
$\gamma_2 V_0^2 =$ \$3,000 where $\gamma_2 =$ 3		[(3)\$400] + [(3)\$800(1.1483)] = \$3,955.92	\$ ()	% ()

- e. Complete Table Q13.6 by finding added MNPVs and MIRRs for the two investments assuming a discount and reinvestment rate equal to the defender's IRR, of $r = 5\%$. What is the additional investment amount γ_i added to the smaller investment? Include the investment rankings in the parentheses.

Table Q13.6. Added MNPVs and MIRRs for two mutually exclusive investments assuming a discount rate of $r = 5\%$ and reinvestment rates equal to the defender's IRR of $r = 5\%$.

Initial plus added investments $(V_0^i + \delta_i)$ where $i = 1, 2$	period one cash flows R_t^i where $t = 1$ generated by initial plus added investments $(V_0^i + \delta_i)$ where $i = 1, 2$	period two cash flows $R_t^i + \delta_i(1+r)^2$ where $t = 2$ generated by initial plus added investment $(V_0^i + \delta_i)$ where $i = 1, 2$	Added MNPV ⁱ earned by initial plus added investments $(V_0^i + \delta_i)$ where $i = 1, 2$ assuming the discount rate and reinvestment rate $r = 5\%$ (rankings)	Added MIRR ⁱ where $i = 1, 2$ earned by initial plus added investments $(V_0^i + \delta_i)$ where $i = 1, 2$ assuming the reinvestment rate $r = 5\%$ (rankings)
$V_0^1 =$ \$3,000		\$1,500 + \$2,000(1.05) = \$3,600	\$ ()	% ()
$(V_0^2 + \delta_2) =$ \$3,000 where $\delta_2 =$ \$2,000		\$400 + \$800(1.05) + $\$2,000(1.05)^2 =$ \$3,445	\$ ()	% ()

- f. Complete Table Q13.7 by finding added MNPVs and MIRRs for the two investments assuming a discount rate equal to the defender's IRR, of $r = 5\%$ and reinvestment rates equal to the investments' IRRs. Include the investment rankings in the parentheses.

Table Q13.7. Added MNPVs and MIRRs for two mutually exclusive investments assuming a discount rate of $r = 5\%$ and reinvestment rates equal to the investments' IRRs

Initial plus added investments ($V_0^1 + \delta_i$) where $i = 1,2$	period one cash flows R_t^1 where $t = 1$ generated by initial plus added investments ($V_0^1 + \delta_i$) where $i=1,2$	period two cash flows R_t^1 where $t=2$ generated by initial plus added investment ($V_0^1 + \delta_i$) where $i = 1,2$	Added MNPV ⁱ earned by initial plus added investments ($V_0^1 + \delta_i$) where $i = 1,2$ assuming the discount rate $r = 5\%$ and reinvestment rates equal to the investments' IRRs (rankings)	Added MIRR ⁱ earned by initial plus added investments ($V_0^1 + \delta_i$) where $i = 1,2$ assuming the reinvestment rate $r = 5\%$ (rankings)
$V_0^1 =$ \$3,000		\$1,500 + [\$2,000(1.1151)] = \$3,730.20	\$ ()	% ()
$(V_0^2 + \delta_2) =$ \$3,000 where $\delta_2 =$ \$2,000		\$400 + [\$800(1.1483)] + \$2,000(1.1483) ² = \$3,955.83	\$ ()	% ()

4. To answer the questions that follow, first complete Table Q13.8 using data from the completed Tables in question 3. Cells indicated with an “n.a.” do not require an answer.

Table Q13.8. Summary of rankings described in Tables Q13.1 through Q13.7

Table provide the data	Type of ranking	Assuming reinvestment rate is defender's IRR		Assuming reinvestment rate is investments IRR	
		Investment 1	Investment 2	Investment 1	Investment 2
Q13.1	NPV			n.a.	n.a.
	IRR	n.a.	n.a.		
Q13.2	MNPV			n.a.	n.a.
	MIRR			n.a.	n.a.
Q13.3	MNPV	n.a.	n.a.		
	MIRR	n.a.	n.a.		
Q13.4	Scaled MNPV			n.a.	n.a.
	Scaled MIRR			n.a.	n.a.
Q13.5	Scaled MNPV	n.a.	n.a.		
	Scaled MIRR	n.a.	n.a.		
Q13.6	Added MNPV			n.a.	n.a.
	Added MIRR			n.a.	n.a.
Q13.7	Added MNPV	n.a.	n.a.		
	Added MIRR	n.a.	n.a.		

- a. Referring to your completed Table Q13.8, which ranking methods produce consistent results? Please explain your answer.
 - b. Referring to your completed Table Q13.8, which consistent ranking method(s) are consistent with NPV rankings? Please explain your answer.
 - c. Referring to your completed Table Q13.8, which consistent ranking method(s) are consistent with scaled NPVs? Please explain your answers.
 - d. Referring to your completed Table Q13.8, which consistent ranking method(s) are consistent with IRRs? Please explain your answers.
5. Explain why investments of unequal term can be examined using the ranking methods described in this chapter.

14. Homogeneous Terms

LINDON ROBISON

Learning goals. Assuming reinvestment rate is investments IRR After completing this chapter, you should be able to: (1) consistently rank one-time investments using internal rate of return (IRR) or net present value (NPV) methods; (2) find time adjusted cash flow averages called annuity equivalents (AE); (3) use AE to find the optimal life of a repeatable investment; and (4) use capitalization rates to find the present value of long-lived investments and repeatable investments.

Learning objectives. To achieve your learning goals, you should complete the following objectives:

- Learn appropriate methods for ranking one-time investments with unequal terms using IRR or NPV ranking criteria.
- Learn how to represent the time adjusted average of an investment's cash flow using AE.
- Learn how IRR and NPV methods may provide conflicting optimal replacement ages for repeatable investments.
- Learn how to use an investment's cash flow patterns and its AE to determine the optimal life of a repeatable investment.
- Learn how to find the present value of future earnings from repeatable investments using the capitalization formula.

Introduction

The word term refers to the length of time an object is expected to last. An investment refers to a commitment of resources from which investors expect to earn a return. The term of an investment refers to the length of time investor commit resources to a project. In this chapter we explore the connections between investments and their terms. We begin by considering different kinds of investments.

One-time investments. In the previous chapter, we developed methods for ranking one-time mutually exclusive investments with unequal initial and periodic sizes. In that effort, we employed a simplifying assumption: that the terms of the competing challengers and the defender were equal. This assumption allowed us to rank investments consistently using IRR and NPV methods. This assumption is frequently violated. Not all challengers and defenders have equal terms. Therefore, we develop methods for ranking one-time mutually exclusive challengers of unequal terms which is the first goal of this chapter. An important characteristic of one-time investments is that owning a one-time investment does not require replacement in order to invest in a similar investment.

Repeatable investments. Some investments provide essential services. As a result, when they wear out, they need to be replaced (think of a light bulb). What replaces the existing investment may be an identi-

cal, improved, or remodeled version of the original investment. We call these investments that require replacement, repeatable investments because they are owned in sequence rather than simultaneously. Examples of repeatable investments include orchards, breeding livestock, roofs on houses, and equipment. Since repeatable investments are owned in sequence, we must determine what is the optimal time to replace a repeatable investment.

Finding the optimal life of a repeatable investment requires that we review the concept of a time-adjusted average cash flow, an AE. AE, of course, change as new periods of cash flow are included in its calculation. Using AE for different time periods and patterns of cash flow, we can determine the optimal life of the investment.

Long-lived investments. While the term of an investment refers to the length of time investors are committed to the project, the economic life of an investment refers to the length of time an investment has the potential to generate economic services. As a result, the economic life of an investment can exceed the term of an investment and in fact does when it is sold to a subsequent owner. Sometimes we account for the difference between the economic life and term of an investment by including salvage value when finding the NPV or IRR of an investment for a particular owner. In this chapter we explicitly account for salvage value by finding the present value of an investment's cash flow over its economic life. This approach requires that we discuss the concept of capitalization and capitalized values.

In what follows we first discuss how to value long-lived assets and capitalization of an investment's earning. We begin with discussing how to value long-lived assets because in that discussion we review capitalization and AEs, concepts important for finding optimal terms and ranking repeatable investments later on.

Present Values and Capitalization Rates

The economic life and term of an investment. The term of an investment is the number of periods the financial manager expects to manage an investment. The economic life of an investment is the number of periods the investment is expected to generate cash flow. The PV of an investment for an individual depends on its cash flow during the term of the investment plus its liquidation value. The liquidation value of the investment depends on its economic life. Therefore, to estimate the present value for an investment, we must find the discounted present value of all future cash flows. In practice, we often estimate the present value of an investment using the capitalization formula.

The capitalization formula and capitalization rate. Consider a challenging investment that earns R constant cash flow dollars for n period and is liquidated at the end of the n th period for V_n . We assume that the defender funding the investment has an IRR of r . We write the maximum bid (minimum sell) price PV model that equates V_0 to its discounted future earnings over a term of n periods discounted by its defender's IRR as:

$$(14.1) \quad V_0 = \frac{R}{(1+r)} + \cdots + \frac{R + V_n}{(1+r)^n}$$

Now assume that the second owner of the investment has the same earnings expectations as the first owner of the durable so that we can write:

$$(14.2) \quad V_n = \frac{R}{(1+r)} + \cdots + \frac{R + V_{2n}}{(1+r)^{2n}}$$

Finally, substituting for V_n in equation (14.1) the right hand side of equation (14.2) we obtain:

$$(14.3) \quad V_0 = \frac{R}{(1+r)} + \cdots + \frac{R + V_{2n}}{(1+r)^{2n}}$$

And if the investment's salvage value were continually exchanged with its expected cash flows we could write:

$$(14.4) \quad V_0 = \frac{R}{(1+r)} + \frac{R}{(1+r)^2} + \cdots$$

Clearly, the farther away from the present is the constant cash flow R , the less it contributes to the present value of the investment. To demonstrate the diminishing contributions of future cash flow, note how the value of discount rate that multiplies R decreases with n . To illustrate we let $r = 10\%$, $R=100$, and alternative values of n .

$$\frac{\$100}{(1.1)^{20}} = \$14.86$$

$$\frac{\$100}{1.1)^{40}} = \$2.21$$

$$\frac{\$100}{(1.1)^{60}} = \$0.33$$

$$\frac{\$100}{(1.1)^{80}} = \$0.05$$

And

$$\frac{\$100}{(1.1)^{100}} = \$0.01$$

So what is the sum of an infinite stream of constant payments R discounted by r percent? To find that sum, multiply both sides of equation (14.1) by $(1 + r)$ and subtract from the result the original equation:

$$(14.5) \quad V_0(1 + r) - V_0 = R - \frac{R}{(1 + r)} + \frac{R}{(1 + r)} - \frac{R}{(1 + r)^2} + \frac{R}{(1 + r)^2} - \dots - \frac{R}{(1 + r)^{mn}} + \frac{V_{mn}(1 + r) + V_{mn}}{(1 + r)^{mn}}$$

After subtracting and simplifying and letting m get very large, we find the sum of the infinite series to equal:

$$(14.6) \quad V_0 = \lim_{m \rightarrow \infty} \frac{R}{r} \left[1 - \frac{1}{(1 + r)^{mn}} \right] + \frac{V_{mn}}{(1 + r)^{mn}} = \frac{R}{r}$$

The far right-hand side of equation (14.6) we refer to as the capitalization formula where the investment's constant earnings R is divided by the defender's IRR equal to r , the capitalization rate. The capitalization formula allows us to relate the present value of all future cash flow to the value of the investment.

$$(14.7) \quad V_0 = \frac{R}{r}$$

To illustrate the capitalization formula, suppose that you purchase an annuity for \$100 that pays you and your heirs \$5 a year forever. The capitalization formula for this investment is equal to:

$$(14.8) \quad V_0 = \frac{R}{r} = \frac{\$5}{r}$$

If we knew the investment's initial value and its infinite stream of earnings, we could always estimate the capitalization rate as:

$$(14.9) \quad r = \frac{R}{V_0} = \frac{\$5}{\$100} = 5\%$$

Comments about the capitalization formula and capitalization rate. In textbooks and references to the capitalization rate, it is often expressed as:

$$(14.10) \quad r = \frac{\text{annual cash income}}{\text{purchase price}}$$

Then when practitioners implement the capitalization formula, equation (14.9), they estimate R as the first period's net cash flow and an industry capitalization rate for r . For example, if the industry standard were 5%, we would write the capitalization formula that estimates the maximum bid price for the investment as:

$$(14.11) \quad V_0 = \frac{R}{r} = \frac{\$5}{.05} = \$100$$

PV models and capitalization formulas. The capitalization formula is a PV model. The unknown variable in the capitalization formula identifies the kind of PV model represented. If the unknown variable is the discount rate or the capitalization rate r , the capitalization formula is an IRR model. Most often, capitalization formulas solve for V_0 making them maximum bid (minimum sell) type models.

So what have we learned? We learned that we can best understand the capitalization formula and the capitalization rate in the context of a PV model. Despite its various descriptions in applied publications, the capitalization formula is the constant income or AI of the investment earned over its economic life divided by defender's IRR.

Present Value (PV) Models and Averages

In the previous section, we described how the capitalization formula depended on a constant R . However, this is an unrealistic assumption since an investment's cash flow is hardly ever (never) equal to a constant. So instead, we assume that R is a time adjusted cash flow measure, an AE that we describe next.

Arithmetic means, expected values, geometric means and AEs. There are several measures of central tendency in a numeric series that include arithmetic means, expected values, and geometric means. An example of an arithmetic mean or average follows. Consider three numbers 3, 5, and 7. The average of these numbers can be calculated by dividing their sum by 3 since there are three numbers: $(3 + 5 + 7) / 3 = 5$. This is the average of this series. Now suppose we wanted to find the mean of the three numbers weighted by their probability of occurring. If the probability of 3 occurring were 25%, if the probability of 5 occurring were 25%, and the probability of 7 occurring were 50%, then the expected value of the series would be: $[(.25)3] + [(.25)5] + [(5)7] = 5.50$. This is the weighted average or expected value of the series.

Next consider an example of a geometric mean. Consider three rates of return: 105%, 110% and 115%. The geometric mean is that number which, when multiplied together three times, equals the product of 105%, 110%, and 115%: $[(1.05)(1.10)(1.15)] = (1.0991)^3$. Alternatively, the geometric mean is

$[(1.05)(1.10)(1.15)]^{1/n} = 1.33^{1/n} = 1.0992$ where $n = 3$ because there are three numbers in the series. Note that the geometric mean is not equal to the arithmetic mean: $[(1.10 + 1.11 + 1.12)] / 3 = 1.11$. It is also not necessarily equal to the expected value.

To illustrate AE, the constant R in equation (14.9) is an AE whose present value sum equals the present value of the sum of discounted cash flow R_1, R_2, \dots, R_n .

(14.12)

$$\frac{R_1}{(1+r)} + \frac{R_2}{(1+r)^2} + \dots + \frac{R_n}{(1+r)^n} = \frac{R}{(1+r)} + \frac{R}{(1+r)^2} + \dots + \frac{R}{(1+r)^n}$$

For example, consider the AE in the following problem:

$$(14.13) \quad \frac{15}{(1.1)} + \frac{20}{(1.1)^2} = \frac{17.38}{(1.1)} + \frac{17.38}{(1.1)^2}$$

On the left hand side of equation (14.12) is a stream of unequal periodic cash flow. On the right hand side of equation (14.12) is a stream of equal periodic cash flow each of which is an AE. The important fact, however, is that the discounted AE on the right-hand side of equation (14.12) equals the discounted periodic cash flow on the left-hand side of equation (14.12). The AE for the series on the right-hand side of equation (14.12) is 17.38. We demonstrate how to find an AE using Excel as follows.

Calculating AE from an irregular stream of cash flow is a two-step procedure. The first step is to find the net present value (NPV) of the irregular cash flow stream. We illustrate this step using Excel's NPV function. In our example, the NPV of \$15 received at the end of period one and \$20 received at the end of period two is \$30.17.

Table 14.1a. How to Find an AE for an Irregular Stream of Cash Flow

[Open Table 14.1a. in Microsoft Excel](#)

	B5		f_x	=NPV(B1,B2:B3)
	A	B		C
1	rate	0.1		
2	R1	15		
3	R2	20		
4	nper	2		
5	NPV	\$30.17		=NPV(rate,R1:R2)

The next step is to find the AE, a constant payment, for the NPV equal to \$30.17. Using Excel's PMT function we find the AE for an NPV of \$30.17 equal to \$17.38. We display the Excel solution below.

Table 14.1b. How to Find an AE for an Irregular Stream of Cash Flow

[Open Table 14.1b. in Microsoft Excel](#)

	B6		f_x	=PMT(B1,B4,B5,,0)
	A	B	C	
1	rate	0.1		
2	R1	15		
3	R2	20		
4	nper	2		
5	NPV	\$30.17	=NPV(rate,R1:R2)	
6	AE	(\$17.38)	=PMT(rate,nper,NPV,,0)	

So, what have we learned about AE? An AE is a constant cash flow stream whose discounted value is to equal the present value of another stream of unequal cash flow. Each value in the stream of equal cash flow is an AE. If one calculated the arithmetic mean of the AE in the stream of cash flow, the arithmetic average would also equal the value of any other AE in the series. If the discount rate were an interest rate and V_0 were an amount borrowed, then AE would be the constant loan payment whose present value would equal the original loan amount.

Some observations on NPV and AE rankings. Consider an NPV model and its value expressed as the present value of a series of AE in equation (14.14). Since the present value of the AE is equal to the NPV of the investment, then the two sums must provide equal NPV rank. Important to note, however, is that it is the present value of the series of AE payments that is equal to the NPV, while a number of unequal payments could equal the same NPV. Therefore, there is a direct relationship (i.e. they both go up or down together) between NPVs and AEs. Furthermore,

$$\begin{aligned}
 NPV &= -V_0 + \frac{R_1}{(1+r)} + \frac{R_2}{(1+r)^2} + \dots + \frac{R_n}{(1+r)^n} \\
 &= \frac{R}{(1+r)} + \frac{R}{(1+r)^2} + \dots + \frac{R}{(1+r)^n}
 \end{aligned}
 \tag{14.14}$$

any change in NPV must be matched by a corresponding change in the AE and in the same direction. For any two investments of equal size and term, and where investment one has a larger NPV than investment two, then investment two must increase its AE in order for it to equal the higher NPV amount. We illustrate this point in more detail.

Consider again the expression:

$$(14.15) \quad NPV = \frac{R}{(1+r)} + \frac{R}{(1+r)^2} + \cdots + \frac{R}{(1+r)^n}$$

Then, suppose the term of the model in equation (14.15) is increased by one period. Then the equality in equation (14.15) no longer holds:

$$(14.16) \quad NPV < \frac{R}{(1+r)} + \frac{R}{(1+r)^2} + \cdots + \frac{R}{(1+r)^n} + \frac{R}{(1+r)^{n+1}}$$

To preserve the equality in equation (14.15), the AE equal to R must be decreased by some amount δ to reestablish the equality allowing us to rewrite the earlier equality:

$$(14.17) \quad \begin{aligned} NPV &= \frac{R}{(1+r)} + \frac{R}{(1+r)^2} + \cdots + \frac{R}{(1+r)^n} \\ &= \frac{R-\delta}{(1+r)} + \frac{R-\delta}{(1+r)^2} + \cdots + \frac{R-\delta}{(1+r)^{n+1}} \end{aligned}$$

So, what have we learned? We learned that NPVs and the present value of their AE provide equal rankings. Furthermore, any sum or vector of cash flow can be converted to its present value equivalent annuity. To maintain the equality between a present value and a stream of annuities requires the following. If the present value sum is increased, then either the term or the amount of the annuity must increase.

More Complicated Capitalization Formulas

In the discussion to this point, we implemented the capitalization formula by assuming that the future cash flow were constants, AEs. This of course, is rarely the case. We now ask: how can we find the capitalization formula if future cash flow are expected to increase (decrease) at $g\%$ over time as you might expect on rental property? To answer this question we return to our PV model.

Suppose that we wanted to find the capitalization formula for an investment whose cash flow are expected to increase (decrease) at an average rate of g percent ($g < 0$ percent). Then the capitalization formula that accounts for R is increasing (decreasing) at rate g ($g < 0$) as:

$$(14.18) \quad V_0 = \frac{R(1+g)}{(1+r)} + \frac{R(1+g)^2}{(1+r)^2} + \dots = \frac{R(1+g)}{(r-g)}$$

Now the capitalization rate, the defender's IRR is equal to: $(r - g)/(1 + g)$. We write the capitalization rates depending on alternative values of g and $r = 10$ in Table 14.2

Table 14.2a. Alternative percentage capitalization rates $(r - g)/(1 + g)$ depending on alternative values of $g > 0$ and r

	$g = 0\%$	$g = 1\%$	$g = 2\%$	$g = 3\%$	$g = 4\%$	$g = 5\%$
$r = 10\%$	0.1	0.089	0.078	0.068	.058	0.048
$r = 9\%$	0.09	0.079	0.069	0.058	0.048	0.038
$r = 8\%$	0.08	0.069	0.059	0.049	0.038	0.029
$r = 7\%$	0.07	0.059	0.049	0.039	0.029	0.019
$r = 6\%$	0.06	0.050	0.039	0.029	0.019	0.01

Table 14.2b. Alternative percentage capitalization rates $(r - g)/(1 + g)$ depending on alternative values of $g < 0$ and r

	$g = 0\%$	$g = -1\%$	$g = -2\%$	$g = -3\%$	$g = -4\%$	$g = -5\%$
$r = 10\%$	0.10	0.111	0.122	0.134	0.146	0.158
$r = 9\%$	0.09	0.101	0.112	0.124	0.135	0.147
$r = 8\%$	0.08	0.091	0.102	0.113	0.125	0.137
$r = 7\%$	0.07	0.081	0.092	0.103	0.115	0.126
$r = 6\%$	0.06	0.071	0.082	0.093	0.104	0.116

So, what have we learned? We have learned that expected increases in cash flow decrease capitalization rates and increase the value of capitalized income. For example if $R = \$100$, $r = 10\%$ and $g = 0\%$ so that the capitalization rate is 10%, then $V_0 = \$1000$. If $R = \$100$, $r = 10\%$, and $g = 5\%$ then the capitalization rate is 4.8%, and $V_0 = \$2,083.33$. We have also learned that expected decreases in cash flow increase capitalization rates and decrease the value of capitalized income. For example, if $R = \$100$, $r = 10\%$, and $g = -5\%$ then capitalization rate is 15.8%, then $V_0 = \$632.91$.

Ranking One-time Investments with Unequal Terms using NPV Models

Notation. Before proceeding to the first focus of this chapter, ranking one-time investments, we confirm the notation used earlier and which will be used again in this chapter. The mathematical notation will describe two challenging investments and a defender. We assume that the initial investment sizes are equal to V_0 . The defender's IRR for the two investments is r . The term of investments one and two are n_1 and n_2 respectively. Periodic cash flows for investments $i = 1, 2$ in period $t = 1, \dots, n_i$ can be expressed as R_t^i . And the vector of cash flows is represented as: $\mathbf{R}^i = R_1^i, R_2^i, \dots, R_{n_i}^i$ for $i = 1, 2$.

Finally, we define the sum S of compounded periodic cash flows at rate r as: $S(r, \mathbf{R}^i, n_{in}) = (1+r)^{(n_i-1)} R_1^i + (1+r)^{(n_i-2)} R_2^i + \dots + R_{n_i}^i$. We also define the sum of periodic cash flows compounded at the investment's IRR as: $S(\text{IRR}^i, \mathbf{R}^i, n_i) = (1+\text{IRR}^i)^{(n_i-1)} R_1^i + (1+\text{IRR}^i)^{(n_i-2)} R_2^i + \dots + R_{n_i}^i$. Note that the value of the function S depends on three variables defined in the equation: the reinvestment rate r or IRR^i , the vector of periodic cash flows \mathbf{R}^i , and the term of the investment n_i .

Sufficient conditions for consistently ranking mutually exclusive one-time investments using IRRs and NPVs. We discovered in Chapter 13 that there are two sufficient conditions for consistently ranking investments using NPVs and IRRs—assuming their terms were equal. These two sufficient conditions are equal initial investment sizes and equal periodic cash flows except in their common last period.

In this chapter we will assume that initial sizes of investments are equal. However, we will allow for differences in periodic cash flows because investment terms differ. As a result, we can no longer be sure that the NPVs and IRRs rank investments the same. To solve the problem of unequal periodic cash flows, we need to rationalize investment term differences. Our focus in the first part of this chapter is on how to create equal terms and periodic cash flows except for the last period for mutually exclusive challengers.

Compounding and discounting by the same rate are offsetting operations. Essential to rationalizing term differences is the obvious fact that NPVs for investment one and two compounded and discounted at rate r have the same value as their original NPV function. We demonstrate this point using the following equations. For investment one, the result is:

$$\begin{aligned}
NPV^1 &= -V_0 + \frac{R_1^1}{(1+r)} + \cdots + \frac{R_{n_1}^1}{(1+r)^{n_1}} \\
&= -V_0 + \frac{(1+r)^{(n_1-1)}R_1^1 + (1+r)^{(n_1-2)}R_2^1 + \cdots + R_{n_1}^1}{(1+r)^{n_1}} \\
&= -V_0 + \frac{S(r, \mathbf{R}^1, n_1)}{(1+r)^{n_1}}
\end{aligned}$$

(14.19a)

Similarly, for investment two:

$$\begin{aligned}
NPV^2 &= -V_0 + \frac{R_1^2}{(1+r)} + \cdots + \frac{R_{n_2}^2}{(1+r)^{n_2}} \\
&= -V_0 + \frac{(1+r)^{(n_2-1)}R_1^2 + (1+r)^{(n_2-2)}R_2^2 + \cdots + R_{n_2}^2}{(1+r)^{n_2}} \\
&= -V_0 + \frac{S(r, \mathbf{R}^2, n_2)}{(1+r)^{n_2}}
\end{aligned}$$

(14.19b)

What equations 14.19a and 14.19b illustrate is the obvious: multiplying by one, the compound rate divided by the identical discount rate, cannot change the value of what is being multiplied.

In Chapter 13, we created equal periodic cash flows by reinvesting the periodic cash flows until the last period, creating MNPV and MIRR models. Then we learned that if the reinvestment rate was equal to the discount rate that NPV and MNPV and IRR and MIRR models produced identical results because reinvesting and discounting by the same rate are offsetting operations. What we learn in this chapter confirms this principle, that reinvesting and discounting by the same rate are offsetting operations. Furthermore, this principle can be used to resolve differences in investment terms.

Resolving term differences. We can convert challengers to the same term by reinvesting and discounting their cash flow to a common term. The compound factor that converts the periodic cash flows and the discount rate from period n_2 to n_1 is $(1+r)^{(n_1-n_2)}$. This is applied to the extreme right-hand side of equation (14.19b) resulting in the expression:

$$\begin{aligned}
 NPV^2 &= -V_0 + \frac{(1+r)^{(n_1-n_2)}S(r, \mathbf{R}^2, n_2)}{(1+r)^{(n_1-n_2)}(1+r)^{n_2}} \\
 &= -V_0 + \frac{(1+r)^{(n_1-n_2)}S(r, \mathbf{R}^2, n_2)}{(1+r)^{n_1}} = -V_0 + \frac{S(r, \mathbf{R}^2, n_1)}{(1+r)^{n_1}}
 \end{aligned}
 \tag{14.20}$$

Equation (14.20) confirms once again that compounding and discounting by the same rate are offsetting operations even when used to extend the term of investments. Nevertheless, by compounding and discounting by the same rate (multiplying by one), we convert the term of investment two to the term of investment one without changing the value of the function. It is still equal to the original NPV equation.

The rankings of investments one and two—assuming their reinvestment rates and discount rate are the defender's IRR, r —can be expressed as:

$$NPV^1 - NPV^2 = \frac{S(r, \mathbf{R}^1, n_1)}{(1+r)^{n_1}} - \frac{S(r, \mathbf{R}^2, n_1)}{(1+r)^{n_1}}
 \tag{14.21}$$

Clearly, the only difference between the two NPVs are their vector of cash flow since they have equal initial investments, discount (reinvestment) rates, and terms.

So, what have we learned? We learned that it is okay to compute and rank mutually exclusive investments using their original NPV equations even though their terms are not the same as long as the reinvestment rate is the defender's IRR. In other words, the compounding and discounting are offsetting operations so that NPV models that convert future streams of cash flow to the present are unaffected.

IRR and NPV Models for Analyzing Repeatable Investments

In our earlier discussions, we showed that IRR and NPV rankings were consistent as long as certain size conditions were satisfied and the investments were equal terms. We can no longer claim IRR and NPV provide consistent ranking when their terms are different.

The two preceding sections resolve term differences by assuming the reinvestment rates were either the defender's or the challengers' IRRs. The problem is that we can no longer be certain that the NPV

and IRR methods rank the challengers consistently since we violated the common reinvestment rate assumption—the reinvestment rates were the challengers' own IRRs. If we allow each investment to reinvest in itself, we lose our consistency guarantee.

Another problem using IRR methods besides losing consistency with NPV rankings is that methods for finding the optimal term involve finding the term with the greatest IRR instead of finding the term with the largest AE. This complicates an already complicated subject.

For these and several other reasons, in the remainder of this chapter we will analyze repeatable investments using NPV model assumptions, the most notable of which is that the reinvestment rate is the defender's IRR common to both challengers. To be clear, we could assume a common defender's IRR or each challenger's IRR but having more than one challengers' IRR may produce inconsistent rankings and asymmetry in exchanges. Were we to assume a still different reinvestment rate besides these, the defender's or the challengers' IRR would imply there exists another challenger besides those being considered and, if so, should be considered as a separate challenger.

So in what follows, we assume that earnings from the challengers are reinvested in the defender and asymmetry between exchanges of dollars between time periods. In the next section we begin building the AE tool using NPV assumptions required to analyze repeatable investments.

Ranking One-Time Investments

Term difference and inconsistent rankings. Consider Table 14.3. Note that the two challengers are not periodic size consistent because they withdraw funds at different rates. Challenger one withdraws all of its earnings after one period. Meanwhile, challenger two withdraws some of its earnings in period one and the remainder in period two. As a result, the two investments have unequal terms.

In Table 14.3 panel a, we rank the two challengers using their NPV, IRR, and AE assuming that the discount rate and the reinvestment rate is 10%. In Table 14.3 panel b, we make the two challengers periodic size-consistent by reinvesting period one earnings for one period at the defender's IRR. This operation also resolves term differences between the two investments.

Table 14.3. The Influence of periodic Size and Term Differences Created by Differential Withdrawals.

Panel a. NPV, IRR and AE rankings assuming different periodic cash flows and terms for challengers one and two and a discount rate equal to the defender's IRR of 10%.						
Challengers	Initial Outlay	Cash Flows in period one	Cash Flows in period two	NPV (rankings)	IRR (rankings)	AE (rankings)
C1	\$1,000	\$1,180	\$0	\$72.73 (2)	18% (1)	\$72.73 (1)
C2	\$1,000	\$160	\$1,160	\$104.13 (1)	16% (2)	\$60.00 (2)
Panel b. NPV, IRR, and AE rankings assuming equal periodic cash flows and terms for challengers one and two where equal periodic cash flows and terms are achieved by reinvesting period one cash flows at the defender's IRR rate of 10% to a common ending period. NPV and IRR rankings after adjusting for term differences assuming a reinvestment rate of r.						
Challengers	Initial Outlay	Cash Flows in Period one	Cash Flows in Period two	NPVs and MNPVs (rankings)	IRR (rankings)	AE (rankings)
C1	\$1,000	\$0	\$1,180 (1.10) = \$1,298	\$72.73 (2)	6.7% (2)	\$49.91 (2)
C2	\$1,000	\$0	\$160 (1.10) + \$1,160 = \$1,336	\$104.13 (1)	15.6% (1)	\$60.00 (1)
Panel c. NPV, IRR, and AE rankings assuming equal periodic cash flows and terms for challengers one and two where equal periodic cash flows and terms are achieved by reinvesting cash flows at the investments' IRRs. The discount rate is assumed to equal the defender's IRR of 10%						
Challengers	Initial Outlay	Cash Flows in Period one	Cash flows in Period two	NPV (MNPV) (rankings)	IRR (rankings)	AE (rankings)
C1	\$1,000	\$0	\$1,180 (1.18) = \$1,392.40	\$150.74 (1)	18% (1)	\$86.66 (1)
C2	\$1,000	\$0	\$160 (1.16) + \$1,160 = \$1,345.60	\$112.07 (2)	16% (2)	\$64.57 (2)

Term and periodic cash flow differences in Table 14.3 panel a produced inconsistent rankings using NPV versus IRR and AE methods. However, in Table 14.3 panel b, when term and periodic cash flow differences were eliminated except for the common last period through reinvesting at the defender's IRR of 10%, NPV, IRR, and AE rankings were consistent. In Table 14.3 panel c, when term and periodic cash flow differences were eliminated through reinvesting at the challengers' respective IRRs of 18% and 16%, NPV, IRR, and AE rankings were again consistent but changed from the rankings produced when the reinvestment rate was the defender's IRR.

There are two things to be emphasized about Table 14.3. First, NPVs in panel a and panel b are the same even after adjusting for differences in periodic cash flows and terms. This is because the reinvestment rate was the defender's IRR, and reinvesting and discounting cash flows are offsetting operations. Second, when we adjusted for periodic size inconsistencies using the investments' IRRs as the reinvest-

ment rate, NPVs changed but the investments' IRRs were equal to their MIRRs in panels a and c. This is again the result of reinvesting and discounting by the same rate—the investments' IRRs.

So what have we learned? We learned that if investments have term differences and the reinvestment rate is the defender's IRR then NPV rankings are appropriate. If investments have term differences and the reinvestment rates are the investments' IRRs then IRR rankings are appropriate.

Alternative reinvestment rate assumptions. In Table 14.3, we made the first challenger into a two-period model by reinvesting its earnings at the defender's IRR. Suppose the one-period challenger was available for investment in each period. In other words, suppose that challenger one could be repeated. Since challenger one is an investment of size \$1,000, then only \$1,000 of period one earnings could be reinvested in the one-period challenger. The difference between the challenger's first-period earnings and \$1,000 we assume will be invested at the defender's IRR. The new investment problem is summarized in Table 14.4.

Table 14.4. Resolving Term Differences Between Two Challengers by Reinvesting \$1,000 of Period One Earnings at its One-period IRR of 18% and Reinvesting other Funds at the Defender's IRR of 10%.

Challengers	Initial Outlay	Cash Flow in period one	Cash Flow in period two	NPV assuming defender's IRR is 10% (rankings)	IRR(rankings)	AE (rankings)
C1	\$1,000	\$0	\$1,000 (1.18) + \$180 (1.1) = \$1,378	\$138.84 (1)	17.4% (1)	\$80.00 (1)
C2	\$1,000	\$0	\$1,160 + \$160 (1.1) = \$1,336	\$104.13 (2)	15.6% (2)	\$60.00 (2)

The interesting result of Table 14.4 provides an example of a blended reinvestment rate not equal to either the defender's IRR of 10% or the investment's own IRR but a weighted average of them both equal to 17.4% for investment one and 15.6% for investment two. However, viewing investment opportunities as combinations of investments in the defender and the challenger must be considered to be a new challenger with a unique reinvestment rate.

Using Annuity Equivalent (AE) to Rank Repeatable Investments

The difficulty of finding finite number of replacements to resolve term differences. In our previous example, we resolved term differences between a one-period investment and a two-period investment by repeating the first investment. Now suppose we have a more complicated term inconsistency problem. For example, assume challenger one's term is 7 periods while challenger two's term is 8 periods. Now repeating an investment one or several times won't resolve term differences. Indeed, to resolve term

differences in this problem would require that challenger one be repeated 8 times and challenger two be repeated 7 times. Now we have a 58-period model—which requires a lot of work.

Using Annuity Equivalents (AE) to rank repeatable investments. We can resolve term differences by calculating and comparing the AE of the investments, even though they have different terms. The reason we can use the AE to rank investments of different terms is because the annuity equivalent doesn't change when you increase the term by repeating investments. Thus, the AE from one investment repeated 2, 3, 4, 7, 8, m , or an infinite number of times is the same. This is an important fact because the AE calculated over the lives of multiple (infinite) replacements can be compared to the AE of another repeatable investment and the two investments can be ranked by their difference.

We now support the claim that we can rank repeatable investments by their AE. We write the one-period model as:

$$(14.22a) \quad NPV = -V_0 + \frac{R_1}{(1+r)} + \frac{R_2}{(1+r)^2} + \cdots + \frac{R_n}{(1+r)^n}$$

We write the NPV model with one replacement as:

$$(14.22b) \quad NPV + \frac{NPV}{(1+r)^n} = -V_0 + \frac{R_1}{(1+r)} + \frac{R_2}{(1+r)^2} + \cdots + \frac{R_n}{(1+r)^n} + \frac{1}{(1+r)^n} \left[-V_0 + \frac{R_1}{(1+r)} + \frac{R_2}{(1+r)^2} + \cdots + \frac{R_n}{(1+r)^n} \right]$$

And we could write the NPV model with enough replacements to equalize their term as:

(14.22c)

$$NPV + \frac{NPV}{(1+r)^n} + \cdots = -V_0 + \frac{R_1}{(1+r)} + \frac{R_2}{(1+r)^2} + \cdots + \frac{R_n}{(1+r)^n} + \frac{1}{(1+r)^n} \left[-V_0 + \frac{R_1}{(1+r)} + \frac{R_2}{(1+r)^2} + \cdots + \frac{R_n}{(1+r)^n} \right] + \cdots$$

Next, factoring, we obtain:

$$(14.22d) \quad NPV \left[1 + \frac{1}{(1+r)^n} + \dots \right] = \left[-V_0 + \frac{R_1}{(1+r)} + \frac{R_2}{(1+r)^2} + \dots + \frac{R_n}{(1+r)^n} \right] \left[1 + \frac{1}{(1+r)^n} + \dots \right]$$

Then, canceling the two bracketed terms at the end of each equation, we regain our original one-investment problem:

$$(14.22e) \quad NPV = -V_0 + \frac{R_1}{(1+r)} + \frac{R_2}{(1+r)^2} + \dots + \frac{R_n}{(1+r)^n} \\ = \frac{R}{(1+r)} + \frac{R}{(1+r)^2} + \dots + \frac{R}{(1+r)^n}$$

A numerical example that the AE for a single investment is equal to the AE calculated over two investments. Note that the same AE that solved the one-investment problem solves the multiple-replacement problem.

We demonstrate this result in Table 14.5 which calculates AE for one investment and then recalculates the AE for an investment and one replacement.

Table 14.5. Resolving Term Inconsistencies by Calculating AE for an Investment and the Investment and a Replacement Assuming Defender's IRR is 10%

Challenger	V ₀	R ₁	R ₂	R ₃	R ₄	NPV (rankings)	IRR (rankings)	AE (rankings)
C1	\$2,000	\$1,200	\$1,200			\$82.64 (2)	13.1% (1)	\$47.65 (1)
C1 plus replacement	\$2,000	\$1,200	\$1,200 - \$2,000 = -\$800	\$1,200	\$1,200	\$150.95 (1)	13.1% (1)	\$47.65 (1)

Table 14.5 illustrates the importance of AE rankings to resolve term (and size) inconsistencies for repeatable investments. Note first that term inconsistencies produce different NPVs. The NPVs are positive because the challenger earns a higher rate of return than the defender. Furthermore, collecting these returns for two challengers, the challenger and its replacement, earns more than just one investment. Hence, the NPV for the challenger and its replacement is greater than the NPV for just one

challenger. On the other hand, both the IRR and the AE rank the investments the same, because their calculations are adjusted for the term of the investment. These results are also helpful because they confirm that the IRR of a single defender can be used to discount the challenger and still maintain consistency.

So what is our best advice? Resolve term inconsistencies for repeatable investments by calculating AE for the first investment or the IRR for the first investment.

So, what have we learned about ranking repeatable investments? When asked to rank two challengers that are repeatable but have unequal term and their reinvestment rate is the defender's IRR, rank them by their respective first investment's AE.

Finding AE that account for technologically improved replacements. Suppose that one of the challengers will be replaced by a technologically improved replacement that perhaps costs more, but also produces higher returns. For $\gamma > 1$ the investment problem takes the following form:

$$(14.23) \quad NPV + \frac{(1 + \gamma)NPV}{(1 + r)^n} + \dots$$

Including enough replacement to equalize terms, we find the sum of the discounted NPVs. Call this sum S which is equal to:

$$(14.24) \quad \begin{aligned} S &= NPV \left[1 + \frac{1 + \gamma}{(1 + r)^n} + \frac{(1 + \gamma)^2}{(1 + r)^{2n}} + \dots \right] \\ &= NPV \left[\frac{1}{1 - \frac{1 + \gamma}{(1 + r)^n}} \right] \end{aligned}$$

The details of the derivation are not included here, but require nothing more than the summation of geometric series. The interpretation of equation (14.24) is that technological improvements result in NPV increases in the first replacement by γ percent. The second replacement's NPV increase by $(1 + \gamma)(1 + \gamma)$ —over the first one. Thus, the NPVs increased from NPV to $(1 + \gamma)NPV$ $(1 + \gamma)^2 NPV$ etc. We demonstrate the effect of technologically improved replacement on the ranking of investments in Table 14.6. To simplify our calculations, we assume that we have already found the NPV of the two challengers equal to \$100 for the first investment and \$150 for the second investment. The term differences of the

two investments are $n_1 = 10$ for the first investment and $n_2 = 20$ for the second investment. The rate of technological improvements are 5% for investment one and 3% for investment two. To demonstrate the importance of accounting for technological improvements, we find the AE unadjusted for technological improvement and the AE accounting for technological improvement.

Table 14.6. Ranking Adjusted for Term Differences and Technologically Improved Replacements. Rankings Assuming Defender's IRR is 10%

Challenger	NPV (rankings)	Terms	Technological change coefficient γ	Adjustment coefficient	AE not adjusted for technological change (rankings)	NPVs adjusted for technological change (rankings)
				$\left[\frac{1}{1 - \frac{1 + \gamma}{(1 + r)^n}} \right]$		
C1	NPV ₁ = \$100 (2)	n ₁ = 10	$\gamma_1 = 5\%$	1.68	\$16.27 (2)	\$100(1.68) = \$168 (1)
C2	NPV ₂ = \$150 (2)	n ₂ = 20	$\gamma_2 = 3\%$	1.18	\$17.62 (1)	\$150(1.18) = \$177.12 (2)

It is useful to note in Table 14.6 that NPV and AE unadjusted for technological change are consistent as our theory implies. However, once we account for technological change, the investment rankings based on the investments AE are reversed. Investment one is preferred even though its NPV is less than that of challenger's two NPV.

Inconsistent rankings were not caused by failing to adjust for differences in size and terms. As we have already demonstrated, these can be rationalized using AE. What produced the inconsistencies was comparing the rankings without technologically improved replacements (unadjusted AE) versus including the assumption of technologically improved replacements (adjusted for technology rankings).

Capitalizing AE to find the present value of a stream of repeatable investments. If we are comparing repeatable investments with different terms, then the comparisons are not between individual investments but with the present value sum of all the investments in each cash flow stream. In this regard we could compare the AE since the AE ordering of investment is the same as the NPV orderings. Or, we could capitalize the AE to find the present value sum of all of the investments. To understand how to capitalize the AE, that is, to find its value over an infinite number of repeatable investments, we write equation (14.25):

$$\begin{aligned}
NPV \left[1 + \frac{1}{(1+r)^n} + \dots \right] &= \\
\left[\frac{R}{(1+r)} + \frac{R}{(1+r)^2} + \dots + \frac{R}{(1+r)^n} \right] \left[1 + \frac{1}{(1+r)^n} + \dots \right] & \\
= \frac{R}{(1+r)} + \frac{R}{(1+r)^2} + \dots + \frac{R}{(1+r)^{mn}} & \\
= \lim_{m \rightarrow \infty} \frac{R}{r} \left[1 - \frac{1}{(1+r)^{mn}} \right] = \frac{R}{r} &
\end{aligned}$$

(14.25)

To get some idea of how fast convergent to the capitalization formula R/r occurs in equation (14.25), if $n = 10$ and $r = 10\%$, then for $m = 2$, then $[1 - 1/(1+r)^{mn}] = .85$; for $m = 3$, then $[1 - 1/(1+r)^{mn}] = .94$, and for $m = 4$, then $[1 - 1/(1+r)^{mn}] = .98$. Finally, for $m = 5$, then $[1 - 1/(1+r)^{mn}] = .991$.

Finding the Optimal Replacement Age for Different Repeatable Investments

The condition that identifies the optimal replacement age for a repeatable investment. The optimal age for each repeatable investment in a stream of repeatable investments is that age that maximizes the NPV for the entire stream of repeatable investments. Finding the optimal age of a repeatable investment is a ranking problem. Only in this case, each challenger is defined by its replacement age, and each age-differentiated investment is considered to be a different challenger. Our goal is to find optimal replacement age.

The key to understanding when to replace repeatable investments (without employing a lot of calculus) is to think about averages—or in our case, AE. We want to maximize the present value sum of NPVs for the entire stream of repeatable investments—not just the NPV for an individual investment.

If the investments in a series of repeatable investments have identical cash flow patterns, then the rule for maximizing the NPV of the present value sum for all repeatable investment is to find the term that maximizes the AE for a single investment. Thus, for a repeatable investment, if holding the investment for an additional period increases the AE for the challenger, then the investment should be held for at least another period—until holding the investment another period decreases the AE. Of course, the periodic cash flows could still be positive and NPV increasing even though AE are decreasing. Thus

holding the investment for the term that maximizes its NPV is definitely not the same rule as holding the investment for the term that maximizes its AE.

The pattern of cash flows is the ultimate determinant of an investment’s optimal life. Since the calculus requires a smooth inverted cup-like shape for maximization, we typically assume investment cash flows have corresponding patterns. However, the pattern for an investment’s periodic cash flows are not the same as the pattern of AE for an investment at different ages. Consider some different kinds of investments and cash flow patterns and their corresponding optimal lives.

Finding the optimal replacement age for a growth and decay type investment. The growth and decay type of investment, after the initial investment, is identified by increasing periodic cash flows followed by decreasing periodic cash flows. Specifically, suppose that we have a repeatable investment with cash flows reported for 6 periods. Assuming the reinvestment rate and the discount rate are 10%, we find the NPV for each investment assuming it has an economic life of one period, two periods, three periods, and up to six periods. Then we find the AE for the investments at their alternative economic lives. Finally, we capitalized the AE by 10% (divide the AE by .1) to find the lifetime present value of the multiple investments at their alternative ages. The results are reported in Table 14.7.

Table 14.7. An Example of a Growth and Decay Type Investment.

Time period	Cash flow per period	NPV per investment for alternative investment lives discounted at 10%	AE per investment for alternative investment lives discounted at 10%	Present value sum of an infinite number of repeatable investments
0	(\$300.00)	(\$300.00)		
1	\$150.00	(\$163.64)	(\$180.00)	(\$18,000.00)
2	\$275.00	\$63.64	\$36.67	\$3,667.00
3	\$130.00	\$161.31	\$64.87	\$6,487.00
4	\$70.00	\$209.12	\$65.97	\$6,597.00
5	\$30.00	\$227.75	\$60.08	\$6,008.00
6	(\$10.00)	\$222.10	\$51.00	\$5,100.00

If our goal were to maximize the NPV for one of the repeatable investments, we would hold the investment until it no longer produced positive cash flows—in our example until period 5 with a cash flow of \$30 and lifetime present earnings of \$6,008.00. But if our goal is to optimize our lifetime earnings from a large number of repeatable investments, then we would hold each investment until they reach age 4 with a cash flow of \$70 and lifetime earnings of \$6,597.00. Another way to report the results of Table 14.7 is to note that as long as the periodic cash flow exceeds the AE in a period, adding that period to the life of the investment will increase the AE and the lifetime earnings of the repeatable investments.

Finding the optimal replacement age for a light bulb type investment. Recall that the “light bulb” type of investment describes a category of investments which, after the initial investment, produce a nearly constant level of services that are virtually undiminished over its economic life. Then, at some point, the investment stops providing services and the investment dies a sudden death. An example cash flow

pattern for this investment assumes that the investment dies in period 5. The investment is described in Table 14.8.

Table 14.8. An Example of a Light Bulb Type Investment.

Time period	Cash flow per period	NPV per investment for alternative investment lives discounted at 10%	AE per investment for alternative investment lives discounted at 10%	Present value sum of an infinite number of repeatable investments
0	(\$300.00)	(\$300.00)		
1	\$100.00	(\$209.09)	(\$230.00)	(\$23,000.00)
2	\$100.00	(\$126.45)	(\$72.86)	(\$7,286.00)
3	\$100.00	(\$51.31)	(\$20.63)	(\$2,063.00)
4	\$100.00	\$16.99	\$5.36	\$536.00
5	\$0	\$16.99	\$4.48	\$448.00
6	\$0	\$16.99	\$3.90	\$390.00

The light bulb type investment has an important pattern that is easily recognized. It is that as long as its constant cash flows are positive, its NPV and AE are increasing. However, once the investment dies and its positive cash flows end, its NPV is constant but its AE is continually decreasing. Thus, the optimal life of a light bulb is to keep it until it dies. Of course, this recommendation could be modified if there were serious costs associated with an interruption of services and that the exact period in which the investment died was not known with certainty.

Finding the optimal replacement age for a continuous decay type investment. The continuous decay type investment is one in which the investment performs best when new and then, with use and time, its service capacity decreases and its maintenance requirements increase so that its periodic cash flows exhibit a continuous decay. An example of such a periodic cash flow pattern is described in Table 14.9.

Table 14.9. An Example of the Continual Decay Type Investment.

Time period	Cash flow per period	NPV per investment for alternative investment lives discounted at 10%	AE per investment for alternative investment lives discounted at 10%	Present value sum of an infinite number of repeatable investments
0	(\$4300.00)	(\$300.00)		
1	\$190.00	(\$127.27)	(\$139.96)	(\$13,996.00)
2	\$152.00	(\$1.65)	(\$.95)	(\$95.00)
3	\$129.00	\$95.27	\$38.31	\$3,831.00
4	\$85.00	\$153.32	\$48.37	\$4,837.00
5	\$38.00	\$175.92	\$46.41	\$4,641.00
6	\$8.00	\$181.43	\$41.66	\$4,166.00

In the continual decay model, after the cost of the initial investment is paid, the periodic cash flows of the investment continually decrease. Still they increase the NPV of the investment as long as they are positive. At some point, the value of earning high returns during the early life of the investment

swamps the cost of acquiring a new investment and the AE decrease indicating the optimal age of the investment. In this example, the optimal life of the investment is 4 periods with a corresponding present value sum of earnings over an infinite number of repeatable investments equal to \$4,837.00.

Finding the optimal replacement age for an investment with irregular periodic cash flows. The last category of investments considered are those whose cash flow patterns are unique. That is, the cash flow pattern for an investment owned one year is different than the same investment owned two years, three years, and so on. To illustrate this type of investment, consider a machinery owner who custom hires using his machine to perform services for customers. The cash flow pattern for the machine begins with a capital purchase followed by three years of nearly constant cash flows which then decrease by 25% per year—mostly because of repairs but also because the demand for custom hires using older machines decreases. In the year the machine is replaced, the old machine earns a liquidation value that depends on the age of the machine. In this problem, the machine at each age is considered a unique challenger even though it is the same machine differentiated by age. A description of the cash flows for this problem follows.

Table 14.10. Investments With Irregular Cash Flow Characterized by Constant and then Declining Cash Flow with an Income Spike in the last Period of the Investment's Economic Life. The Discount and Reinvestment Rates are Assumed to Equal 10%.

Period	3 year old challenger	4 year old challenger	5 year old challenger	6 year old challenger	7 year old challenger	8 year old challenger
0	(\$100)	(\$100)	(\$100)	(\$100)	(\$100)	(\$100)
1	\$40	\$40	\$40	\$40	\$40	\$40
2	\$40	\$40	\$40	\$40	\$40	\$40
3	liquidation = \$64	\$32	\$32	\$32	\$32	\$32
4	0	liquidation = \$51.20	\$25.60	\$25.60	\$25.60	\$25.60
5	0	0	liquidation = \$40.96	\$20.48	\$20.48	\$20.48
6	0	0	0	liquidation = \$32.77	16.38	\$16.38
7	0	0	0	0	liquidation = \$26.21	\$13.11
8	0	0	0	0	0	liquidation = \$20.97
Summary Measures						
NPVs	\$17.51	\$28.43	\$36.38	\$42.16	\$46.36	\$49.42
AE	\$7.04	\$8.97	\$9.60	\$9.68	\$9.52	\$9.26
Capitalized AE	\$70.40	\$89.70	\$96.00	\$96.80	\$95.20	\$92.60

Note that NPVs are increasing with the age of the challengers. The maximum AE is earned in the sixth period and declines for each of the older challengers. Thus the optimal age for the challengers is age 6.

Summary and Conclusions

In Chapter 13, we found two sufficient conditions for consistently ranking mutually exclusive investments. The two conditions are that initial investment sizes and periodic cash flows are equal except for their last common period. In this chapter we extended the results of Chapter 13 by developing methods to rank investments of unequal terms. Ranking unequal term investments are problematic because unequal terms create unequal periodic cash flows, violating the second of two sufficient conditions for consistently ranking investments using IRRs and NPVs.

In our efforts to find methods for ranking investments of unequal terms, we found that MNPV and MIRR models would rank investments equally. But we also found that under some conditions, IRR and NPV models could produce equal rankings. In this chapter, we emphasized that adjusting for term differences by reinvesting and discounting by the defender's IRR that NPV and MNPV and MIRR models would produce consistent rankings. These results would not hold when some other reinvestment rate applies. These findings led to some important practical results: when the reinvestment rate and the discount rate are the same, rank investments using NPVs. When the reinvestment rates are the investments' IRRs, rank using investments' IRRs.

In the second part of this chapter, we considered repeatable investments. If by repeating investments for a required number of times the investments had a common ending date, then individual investment term differences could be ignored. In effect, the entire stream of repeatable investments could be considered to be a single investment. Of course, if the number of repeatable investments was considered to be infinite then the term problem is resolved.

Assuming equal initial investment sizes, we found that the repeatable investment's optimal age was the age that maximized AE for a single investment. Thus, we can find the optimal replacement age for repeatable investments in a stream of replacements by finding the age that maximizes AE for any one investment. Capitalizing the AE provides us with an estimate of the present value of the earnings from the stream of repeatable investments.

Questions

1. Describe the three sufficient conditions required for consistent IRR and NPV rankings for one-time investments? Explain how investment term differences violate one of the two sufficient conditions.
2. Some investments are one-time investments. Others are repeatable. Describe what conditions produce one-time investments. Then describe what conditions produce repeatable investments. Give examples of one-time and repeatable investments.
3. One way to resolve term differences is to reinvest the periodic cash flows of both investments to a common ending period. Explain the implications of assuming that the reinvestment rates are

the defender's IRR, the investments' IRR, or some other rate.

4. Explain the effects on an investment's NPV if term differences are resolved by reinvesting its periodic cash flows to some common period using the defender's IRR, while discounting the reinvested funds over the changed terms by the same rate, the defender's IRR.
5. Explain the difference, if any, between an investment's IRR and its MIRR if the reinvestment rate is the investment's IRR. Depending on your answer, what practical recommendation would you offer to financial managers wanting to rank investments whose earnings would be reinvested in themselves?
6. In the table below, term differences are resolved by reinvesting periodic cash flows to the common ending period assuming the reinvestment rate and the discount rate is the defender's IRR of 10%. Produce a similar table assuming the same initial investment sizes and cash flows, only assume the defender's IRR is 5% not 10%. Then associate your results with the ranking possibilities described in Table 14.1 by declaring which of the four models correspond to your table.

Table Q14.1. The influence of periodic size and term differences created by differential withdrawals assuming a reinvestment rate equal to the defender's IRR of 10%. The discount rate for the NPV and MNPV models equal the defender's IRR. The discount rate for the IRR and MIRR models are the IRRs and MIRR

Challengers	Initial Outlay	Cash Flow in period one	Cash Flow in period two	NPV (rankings)	IRR (rankings)	MNPV (rankings)	MIRR (rankings)
C1	\$900	\$1090	0	\$90.91 (2)	\$21.11% (1)	\$90.91 (2)	15.42% (2)
C2	\$900	\$160	\$1050	\$113.22 (1)	17.27% (2)	\$113.22 (1)	16.71% (1)

7. In the table below, term differences are resolved by reinvesting periodic cash flows to the common ending period assuming the reinvestment rates are the investments' IRRs and the discount rate is 10%. Produce a similar table assuming the same initial investment sizes and cash flows, only assume the defender's IRR is 5% not 10%. Then associate your results with the ranking possibilities described in Table 14.1 by declaring which of the four models correspond to your table.

Table Q14.2. The influence of periodic size and term differences created by differential withdrawals assuming a reinvestment rate equal to the challenger's IRRs. The discount rate for the NPV and MNPV models equal the defender's IRR. The discount rate for the IRR and MIRR models are the IRRs and MIRR

Challengers	Initial Outlay	Cash Flow in period one	Cash Flow in period two	NPV (rankings)	IRR (rankings)	MNPV (rankings)	MIRR (rankings)
C1	\$900	\$1090	\$0	\$90.91 (2)	\$21.11% (1)	\$190.91 (1)	21.11% (1)
C2	\$900	\$160	\$1050	\$113.22 (1)	17.27% (2)	\$122.83 (2)	17.27% (2)

8. Referring to the completed tables in Questions 6 and 7, please answer the following. Why are NPV and MNPV ranking consistent and equal in amounts in the Question 6 table but inconsistent and different in amounts in the Question 7 table? And why are IRR and MIRR rankings inconsistent and unequal in the Question 6 table but consistent and equal in amount in the Question 7 table? What are the practical implications of these results?
9. Annuity equivalents are elements in a stream of constant periodic cash flows whose present value

equals the present value of some fixed amount or the present value of a non-constant cash flow stream. What is the arithmetic mean of a series of AE? If the discount rate is 8% and the term is 10 periods, find the AE for the periodic cash flows 21, 34, 5, and 13. Then find the AE for a fixed present value amount of \$199 assuming the same discount rate and term. Finally, recalculate the AE if the term is decreased from 10 periods to 5 periods.

10. A potential Uber driver can purchase a new car for \$18,000. Then the car is expected to earn constant periodic cash flows for the next three years of \$6,000. After that, mostly because of decreased demand for rides in older cars and higher repair costs, periodic cash flows decrease by 25% per year. The liquidation value of the new car after three years is \$9,000 and then declines each year thereafter by 25%. Find the optimal age at which the Uber driver should replace cars. Then find the capitalized value of an investment in one car assuming each car is owned until its optimal age.

15. Homogeneous Tax Rates

LINDON ROBISON

Learning goals. After completing this chapter, you should be able to: (1) construct effective tax rates on defending and challenging investments; (2) describe how effective tax rates depend on investment types; and (3) find effective tax rates on defending and challenging investments by calculating tax adjust coefficients.

Learning objectives. To achieve your learning goals, you should complete the following objectives:

- Learn how to compute a defender's after-tax IRR by finding its tax adjustment coefficient.
- Learn how capital gains alter a defending investment's tax adjustment coefficient.
- Learn how to compute the tax adjustment coefficient for a single period.
- Learn how to compare the average effective tax rate for an investment.
- Learn how to find the tax adjustment coefficient for loans and investments with increasing (decreasing) cash flow.
- Learn how taxes and tax adjustment coefficients influence investment rankings.

Introduction

According to Benjamin Franklin, taxes are one of the two constants in life (the other one is death). Taxes need to be included in present value (PV) models because only what the firm earns and keeps after paying taxes really matters. Therefore, proper construction of PV models requires that taxes be consistently applied to defending and challenging investments.

A single-equation PV model compares two investments: a challenger, described by time-dated cash flow, and a defender, usually described by its internal rate of return (IRR) of its time-dated cash flow. Homogeneous tax rates require that the same units are used when describing the cash flow of the challenger as the cash flow used when finding the IRR of the defender. And in particular, homogeneous tax rates require that when the cash flow of the challenger are adjusted for taxes, the IRR of the defender must also be adjusted for taxes. This chapter shows how to find homogeneous tax rates in PV models.

It is popular to adjust a defender's before-tax IRR to an after-tax IRR by multiplying it by $(1 - T)$, where T is the average tax rate applied to the investment. This chapter points out that this method for adjusting the defender's IRR for taxes is only appropriate in a few special cases. Finally, this chapter shows how to find appropriate tax rates for adjusting the defender's IRR for taxes in a variety of tax environments.

After-tax PV Models for Defenders and Challengers

The goal is to find a defender's after-tax IRR. The first step is to find the defender's before-tax IRR. Recall that the defender's before-tax IRR is that rate of return such that the NPV of the defender's before-tax net cash flow is zero. The defender's after-tax IRR discounts the defender's after-tax cash flow so that its NPV remains equal to zero.

There is an easy test to determine if taxes have been properly introduced into a PV model. First consider the defender's cash flow stream and its IRR. Then introduce taxes into the defender's cash flow stream and its IRR. If taxes have been properly introduced, then the NPV of the defender's after-tax cash flow stream discounted by its after-tax IRR is still zero.

To illustrate, consider a defending investment that earns constant net cash flow of R dollars per period in perpetuity. The maximum bid (minimum sell) price V_0 for this defending investment whose before-tax IRR is r . We describe this investment in equation (15.1).

$$(15.1) \quad V_0 = \frac{R}{(1+r)} + \frac{R}{(1+r)^2} + \dots = \frac{R}{r}$$

where $r = R/V_0$ is the before-tax IRR. Now suppose that we describe the defender's cash flow and the IRR for this same investment on an after-tax basis. We do so by introducing the constant marginal income tax rate T into the model:

$$(15.2) \quad V_0 = \frac{R(1-T)}{[1+r(1-T)]} + \frac{R(1-T)}{[1+r(1-T)]^2} + \dots = \frac{R}{r}$$

Income R and the before-tax IRR of the defender are both adjusted for taxes by multiplying by $(1-T)$. We can be sure that $r(1-T)$ is the after-tax IRR, since V_0 is the same in the before and after-tax models. However, we need to be aware that only in the special case of constant infinite income can we multiply r by $(1-T)$ to obtain the after-tax IRR.

A General Approach for Finding the Defender's After-tax IRR

A general approach for finding the defender's after-tax IRR follows. Consider a defender that has a market-determined value of V_0 and earns a before-tax cash flow stream of R_1, R_2, \dots , and whose defender's before-tax IRR is r . A before-tax PV model for this defender can be written as:

$$\begin{aligned}
 V_0 &= \frac{R_1}{(1+r)} + \frac{R_2}{(1+r)^2} + \dots \\
 (15.3) \quad &= \frac{R_1}{(1+r)} + \frac{1}{(1+r)} \left[\frac{R_2}{(1+r)} + \frac{R_3}{(1+r)^2} + \dots \right]
 \end{aligned}$$

Now we express the PV model for the defender one period later as:

$$(15.4) \quad V_1 = \frac{R_2}{(1+r)} + \frac{R_3}{(1+r)^2} + \dots$$

Note that the bracketed expression in equation (15.3) equals the right-hand side of equation (15.4). This equality allows us to substitute V_1 for the bracketed expression in equation (15.3). After making the substitution in equation (15.3) we solve for capital gains equal to:

$$(15.5) \quad V_1 - V_0 = rV_0 - R_1$$

Using the expression in equation (15.5) we can solve for the defender's before-tax IRR equal to:

$$(15.6) \quad r = \frac{(V_1 - V_0) + R_1}{V_0}$$

We now want to find the after-tax IRR for the defender described in equation (15.6). To do so, the IRR and the cash flow for the defender must be adjusted for taxes in such a way that V_0 in equation (15.3) is not changed (and the defender's NPV is still zero). The defender's cash flow is adjusted for taxes by multiplying them by $(1 - T)$. The defender's before-tax IRR is adjusted for taxes by multiplying it by $(1 - \theta T)$ where θ , a tax adjustment coefficient, adjusts r to its after-tax equivalent such that V_0 is the same whether calculated on a before-tax or after-tax basis. Besides these tax adjustments, let T_j equal other taxes applied to the defender's cash flow that may include property taxes in the case of land, depreciation tax credits in the case of depreciable investments, and capital gains taxes when appropriate for assets earning capital gains. What all these taxes have in common is their functional dependence on the previous period's asset value.

The after-tax PV model corresponding to equation (15.3) that leaves V_0 unchanged can be written as:

$$\begin{aligned}
 V_0 &= \frac{R_1(1 - T) + T_j f(V_0)}{[1 + r(1 - \theta T)]} \\
 (15.7) \quad &+ \frac{1}{[1 + r(1 - \theta T)]} \left[\frac{R_2(1 - T) + T_j f(V_1)}{[1 + r(1 - \theta T)]} + \frac{R_3 + T_j f(V_2)}{[1 + r(1 - \theta T)]^2} + \dots \right]
 \end{aligned}$$

Similarly, V_1 can be expressed as:

$$(15.8) \quad V_1 = \frac{R_2(1 - T) + T_j f(V_1)}{[1 + r(1 - \theta T)]} + \frac{R_3 + T_j f(V_2)}{[1 + r(1 - \theta T)]^2} + \dots$$

Replacing the bracketed expression in equation (15.7) with V_1 , we find the after-tax IRR for the defender as:

$$(15.9) \quad r(1 - \theta T) = \frac{(V_1 - V_0) + R_1(1 - T) + T_j f(V_0)}{V_0}$$

Finally, substituting for r in equation (15.9) from the right-hand side of equation (15.6) and solving for θ , we obtain:

$$(15.10) \quad \theta = \frac{R_1 - \left(\frac{T_j}{T}\right) f(V_0)}{R_1 + (V_1 - V_0)}$$

The value for θ in equation (15.10) adjusts the defender's before-tax IRR to obtain homogeneity of measures between the defender's after-tax IRR and the defender's after-tax cash flow in equation (15.7). When homogeneity of measures is maintained, the defender's NPV is still zero whether calculated on a before-tax or after-tax basis. We now find some specific after-tax IRRs for various types of defenders.

Case 15.1. $(V_1 - V_0) = 0$ and $T_j f(V_0) = 0$. In this case, the defender earns neither capital gains nor suffers capital losses, in which case $T_j = 0$. This case is illustrated by an infinite constant cash flow series, described in equations 15.1 and 15.2. Because the cash flow is constant, capital gains are zero. Furthermore, the defender's return in each period is equal to its cash receipts, which is fully taxed at income tax rate T . Therefore, the entire before-tax rate of return must be adjusted by $(1 - T)$. In equation (15.10), substituting zero for capital gains and depreciation or capital gains tax results in:

$$(15.11) \quad \theta = \frac{R_1 - \left(\frac{T_j}{T}\right) f(V_0)}{R_1 + (V_1 - V_0)} = \frac{R_1}{R_1} = 1$$

Case 15.2. $(V_1 - V_0) > 0$ and $T_j f(V_0) = 0$. In this case the investment earns capital gains that are not taxed, which lowers the defender's effective tax rate. Thus, $\theta < 1$ and $(1 - \theta T) < (1 - T)$. The greater the capital gains, the lower the effective tax rate in period t . We illustrate this type of model next.

Consider a defending investment whose before-tax cash flow grows geometrically at rate g . Then before-tax cash flow in period t , R_t , equals: $R_0(1 + g)^t$, and we write the investment's IRR model as:

$$\begin{aligned}
 V_0 &= \frac{R_0(1+g)}{(1+r)} + \frac{1}{(1+r)} \left[\frac{R_0(1+g)^2}{(1+r)} + \frac{R_0(1+g)^3}{(1+r)^2} + \dots \right] \\
 &= \frac{R_0(1+g)}{(r-g)}
 \end{aligned}
 \tag{15.12}$$

One period later, we can write:

$$V_1 = \frac{R_0(1+g)^2}{(1+r)} + \frac{R_0(1+g)^3}{(1+r)^2} + \dots = V_0(1+g)
 \tag{15.13}$$

and capital gains equal:

$$V_1 - V_0 = V_0(1+g) - V_0 = gV_0
 \tag{15.14}$$

Then, substituting the right-hand side of equation (15.12) for V_0 in equation (15.14), we obtain:

$$gV_0 = \frac{gR_0(1+g)}{r-g}
 \tag{15.15}$$

Now we are ready to find the tax-adjustment coefficient in this model. Substituting into equation (15.10) for capital gains and first period cash flow, we obtain:

$$\begin{aligned}
 \theta &= \frac{R_1 - \left(\frac{T_j}{T}\right) f(V_0)}{R_1 + (V_1 - V_0)} = \frac{R_0(1+g)}{R_0(1+g) + gV_0} \\
 &= \frac{R_0(1+g)}{R_0(1+g) + \frac{gR_0(1+g)}{r-g}} = \frac{r-g}{r}
 \end{aligned}
 \tag{15.16}$$

Suppose that we naively introduced taxes into the growth model described in equation (15.12) and obtained:

$$\begin{aligned}
 V_0 &= \frac{R_0(1+g)(1-T)}{[1+r(1-T)]} + \frac{R_0(1+g)^2(1-T)}{[1+r(1-T)]^2} + \dots \\
 &= \frac{R_0(1+g)(1-T)}{r(1-T) - g}
 \end{aligned}
 \tag{15.17}$$

In this case, an increase in T increases V_0 :

$$\frac{dV_0}{dT} = \frac{R_0(1+g)g}{[r(1-T) - g]^2} > 0.
 \tag{15.18}$$

Since increasing T increases V_0 , we know that $r(1-T)$ cannot equal the after-tax IRR for the defending investment.

So what have we learned? We learned how to interpret θ in the geometric growth model. Earlier we demonstrated that in the geometric growth model, capital gains are earned at rate g . But in this model, capital gains are not converted to cash, and so they are not taxed. Therefore, not all earnings on the investment are taxed, only the portion earned as cash. As a result, the effective tax rate is not T but θT , where θ is the percentage of returns earned as cash.

If, for example, $r = 8\%$, $g = 3\%$, then

$$\theta = \frac{.08 - .03}{.08} = .63.
 \tag{15.19}$$

Furthermore, for an investor in the 35% tax bracket whose cash flow is increasing at 3%, the effective tax rate would be $\theta T = (.63)(.35) = 22\%$.

Case 15.3. $(V_1 - V_0) < 0$ and $T_{jf}(V_0) = 0$. In this case the investment suffers capital losses for which no tax savings are allowed, which increases the defender's effective tax rate. Thus, $\theta > 1$ and $(1 - \theta T) > (1 - T)$. The greater the capital losses, the higher is the effective tax rate in period t . We illustrate this type of model next.

Consider a defending investment whose before-tax cash flow declines geometrically at rate $d > 0$. Then before-tax cash flow in period t , R_t , equals: $R_0(1-d)^t$, and we write the investment's maximum bid price as:

(15.20)

$$V_0 = \frac{R_0(1-d)}{(1+r)} + \frac{1}{(1+r)} \left[\frac{R_0(1-d)^2}{(1+r)} + \frac{R_0(1+d)^3}{(1+r)^2} + \dots \right] = \frac{R_0(1-d)}{(r+d)}$$

One period later, we can write:

$$(15.21) \quad V_1 = \frac{R_0(1-d)^2}{(1+r)} + \frac{R_0(1+d)^3}{(1+r)^2} + \dots = V_0(1-d)$$

and capital losses equal:

$$(15.22) \quad V_1 - V_0 = V_0(1-d) - V_0 = -dV_0$$

Then, substituting the right-hand side of equation (15.20) for V_0 in equation (15.22) we obtain:

$$(15.23) \quad dV_0 = \frac{dR_0(1-d)}{r+d}$$

Now we are ready to find the tax-adjustment coefficient in this model. Substituting into equation (15.10) for capital losses and first period cash flow, we obtain:

$$(15.24) \quad \begin{aligned} \theta &= \frac{R_1 - \left(\frac{T_j}{T}\right) f(V_0)}{R_1 + (V_1 - V_0)} = \frac{R_0(1-d)}{R_0(1-d) - dV_0} \\ &= \frac{R_0(1-d)}{R_0(1-d) + \frac{dR_0(1-d)}{r+d}} = \frac{r+d}{r} \end{aligned}$$

So what have we learned? We learned that in the geometric decay model, capital losses are suffered at rate d . But since capital losses are not converted to cash, they do not create a tax shield. As a result, the effective tax rate is not T but θT , where θ is $(r+d)/r$ which is greater than one and depends on the decay rate d . It should be obvious that the two previous cases are equivalent since d could be replaced by $g < 0$. On the other hand, it may be

helpful later on to distinguish between growth in cash receipts $g\%$ and book value depreciation $d\%$.

If for example, $r = 8\%$, $d = 3\%$, then,

$$(15.25) \quad \theta = \frac{.08 + .03}{.08} = 1.375.$$

Furthermore, for an investor in the 35% tax bracket, the effective tax rate on cash flow described by a geometric growth pattern would be $\theta T = (1.375)(.35) = 48\%$. The conclusion of Case 15.3 is that suffering losses on investments which are not used to shield income from taxes increases the investor's effective tax rate.

Case 15.4. $(V_1 - V_0) > 0$ and $T_{jf}(V_0) = -T_p(1 - T)V_0$. In this model, the defending investment earns capital gains but pays property tax on its previous period's value times the property tax rate T_p . However, property taxes are tax deductible, so we can write the after-tax IRR model for the defending land investment as:

$$(15.26) \quad \begin{aligned} V_0 &= \frac{R_0(1 + g)(1 - T) - T_p(1 - T)V_0}{[1 + r(1 - \theta T)]} \\ &+ \frac{R_0(1 + g)^2(1 - T) - T_p(1 - T)V_0(1 + g)}{[1 + r(1 - \theta T)]^2} + \dots \\ &= \frac{R_0(1 + g)(1 - T)}{r(1 - \theta T) - g} - \frac{V_0(1 + g)T_p(1 - T)}{r(1 - \theta T) - g} \end{aligned}$$

We can sum the IRR model above by recognizing that it consists of two geometric sums. However, after summing, V_0 appears on both sides of the equation. Therefore, we solve for V_0 using equation (15.26) and obtain:

$$(15.27) \quad V_0 = \frac{R_0(1 + g)(1 - T)}{r(1 - \theta T) - g + T_p(1 - T)}$$

Capital gains and V_0 in this model are the same as in Case 15.2, which allows us to write:

$$\begin{aligned}
\theta &= \frac{R_1 - \left(\frac{T_f}{T}\right) f(V_0)}{R_1 + (V_1 - V_0)} = \frac{R_0(1 + g) + \left(\frac{T_p(1 - T)}{T}\right) V_0}{R_0(1 + g) + gV_0} \\
&= \frac{R_0(1 + g) + \left(\frac{T_p(1 - T)}{T}\right) \frac{R_0(1 + g)}{r - g}}{R_0(1 + g) + \frac{gR_0(1 + g)}{r - g}} = \frac{r - g + \frac{T_p(1 - T)}{T}}{r}
\end{aligned}$$

(15.28)

Consider the effect of property taxes compared to the geometric model without property taxes. In Case 15.2, we assumed that $r = 8\%$ and $g = 3\%$, so that $\theta = (.08 - .05) / .08 = .63$. We continue with the assumption that the decision maker is in the 35% tax bracket and add the assumption that the property tax rate is 2%. We now substitute into equation (15.28) and find:

$$\begin{aligned}
\theta &= \frac{r - g + \frac{T_p(1 - T)}{T}}{r} \\
&= \frac{.08 - .03 + \frac{.02(1 - .35)}{.35}}{.08} = 1.09
\end{aligned}$$

(15.29)

So what have we learned? We learned that property taxes increase the effective tax rate. To illustrate, for an investor in the 35% tax bracket, the effective tax rate on cash flow described by a geometric growth pattern with property taxes would be $\theta T = (1.09)(.35) = 38\%$. Obviously, property taxes increase an investor's effective tax rate.

Case 15.5. $(V_1 - V_0) > 0$ and $T_f(V_0) = TdV_0$. A most fortuitous tax environment is an investment that appreciates but is considered to depreciate at rate d for tax purposes. In this case the investor is not required to pay taxes on the capital gains and is allowed a tax credit for depreciation that occurs only on the books. An example of such a model is the following:

(15.30)

$$\begin{aligned} V_0 &= \frac{R_0(1+g)(1-T) - TdV_0}{[1+r(1-\theta T)]} + \frac{R_0(1+g)^2(1-T) + TdV_0(1-d)}{[1+r(1-\theta T)]^2} + \dots \\ &= \frac{R_0(1+g)(1-T)}{r(1-\theta T) - g} + \frac{V_0Td}{r(1-\theta T) + d} \end{aligned}$$

Because the before-tax model is the geometric growth model, we can write:

$$\begin{aligned} \theta &= \frac{R_1 - \left(\frac{T_f}{T}\right) f(V_0)}{R_1 + (V_1 - V_0)} = \frac{R_0(1+g) - dV_0}{R_0(1+g) + gV_0} \\ &= \frac{R_0(1+g) - d + \frac{R_0(1+g)}{r-g}}{R_0(1+g) + \frac{gR_0(1+g)}{r-g}} = \frac{r-g-d}{r} \end{aligned}$$

(15.31)

So what have we learned? We learned that the effective tax rate for depreciable assets whose value is actually increasing while the investor is able to claim a tax deduction for book value depreciation results in a significant reduction in the investor's effective tax rate.

To illustrate, suppose that $d = 5\%$. Then, using numbers from our previous example, we substitute into equation (15.31) and find:

$$(15.32) \quad \theta = \frac{r-g-d}{r} = \frac{.08 - .03 - .05}{.08} = 0\%$$

In other words, the tax rate for the investment is effectively zero.

Tax Adjustment Coefficients in Finite Models

So far we have obtained tax adjustment coefficient models for infinite time examples. As a result, the tax adjustment coefficients have been the same for each period. This approach has been convenient for exposition purposes. However, this result is not generally applicable. In practice, the tax adjustment coefficients vary by time period. We demonstrate using a finite time horizon model, a loan model in which the interest paid is tax deductible.

In many applications, the defender is a loan. When a loan is the defender, we immediately want to know if the interest paid is tax deductible. To keep our analysis simple, assume a constant-payment loan made for n periods at an interest rate of i percent. In which case, loan amount L_0 at interest rate i with annuity payments A can be written as:

$$(15.33) \quad L_0 = \frac{A}{(1+i)} + \frac{A}{(1+i)^2} + \cdots + \frac{A}{(1+i)^n}$$

And the firm's before-tax IRR is i . The loan balance after one period can be written as:

$$(15.34) \quad L_1 = \frac{A}{(1+i)} + \frac{A}{(1+i)^2} + \cdots + \frac{A}{(1+i)^{n-1}}$$

Furthermore, capital loss for loans can be expressed as:

$$(15.35) \quad L_1 - L_0 = -\frac{A}{(1+i)^n}$$

The after-tax IRR model can be expressed as:

$$\begin{aligned}
L_0 &= \frac{A - AT \left[1 - \frac{1}{(1+i)^n} \right]}{[1 + i(1 - \theta T)]} + \frac{A - AT \left[1 - \frac{1}{(1+i)^{n-1}} \right]}{[1 + i(1 - \theta T)]^2} \\
&+ \dots + \frac{A - AT \left[1 - \frac{1}{(1+i)} \right]}{[1 + i(1 - \theta T)]^n} \\
&= \frac{A(1 - T) + \left[\frac{AT}{(1+i)^n} \right]}{[1 + i(1 - \theta T)]} + \frac{A(1 - T) + \left[\frac{AT}{(1+i)^{n-1}} \right]}{[1 + i(1 - \theta T)]^2} \\
&+ \dots + \frac{A(1 - T) + \left[\frac{AT}{(1+i)} \right]}{[1 + i(1 - \theta T)]^n}
\end{aligned}
\tag{15.36}$$

With equation (15.36) in a familiar form, we can write θ as:

$$\theta = \frac{R_1 - \left(\frac{T_f}{T} \right) f(V_0)}{R_1 + (V_1 - V_0)} = \frac{A - \frac{A}{(1+i)^n}}{A - \frac{A}{(1+i)^n}} = 1
\tag{15.37}$$

In words, the effective after-tax IRR for a loan whose interest is tax-deductible is $(1 - T)$.

Ranking Investments and Taxes

In this section, we acknowledge that the effective tax rates vary by the different types of investments being considered. Furthermore, if the effective tax rate for defenders and challengers differ, then before and after-tax investment rankings may be changed. To make this point, consider the following example. Assume the defender's before-tax IRR is 8%, the income tax rate is 35%, the rate of growth in

net cash flow is 3%, and the last year's before-tax net cash flow was \$100. The maximum bid price for this investment with geometric growing income is:

$$(15.38) \quad V_0^b = \frac{R_0(1+g)}{r-g} = \frac{\$100(1.03)}{(.08-.03)} = \$2,060.$$

If the minimum sell price is \$3,000, then the NPV for the investment is a negative \$940 and the challenger is rejected.

Now suppose we introduce taxes into the model and recalculate the investment's NPV. To simplify we assume an infinite life for the investment and assume $\theta = 1$ for the defender and find:

$$(15.39) \quad \begin{aligned} NPV &= -V_0 + \frac{R_0(1+g)(1-T)}{[1+r(1-T)]} + \frac{R_0(1+g)^2(1-T)}{[1+r(1-T)]^2} \\ &\quad + \frac{R_0(1+g)^3(1-T)}{[1+r(1-T)]^3} + \dots \\ &= -V_0 + \frac{R_0(1+g)(1-T)}{r(1-T)-g} \\ &= -\$3,000 + \frac{\$100(1.03)(1-.35)}{.08(1-.35)-.03} = \$43.18 \end{aligned}$$

Now the challenger's NPV is positive. Introducing taxes into the NPV model has reversed the ranking: the challenger is now preferred to the defender. These results should alert financial managers to the importance of after-tax rankings.

Summary and Conclusions

Taxes are important when valuing and ranking investments because what really matters is the amount of your earnings you get to keep after paying taxes. It's not the before-tax IRR that matters but the after-tax IRR that counts because that's the rate you really earned on your defending investment.

In this chapter, we have introduced a method for finding a defender's after-tax IRR. The method required that when a defender's cash flow adjusted for taxes and was discounted by its after-tax IRR its

change in NPV was zero. Then we used the equality to find the after-tax discount rate IRR. The implied tax rate in the after-tax IRR was compared to the firm's average marginal income tax-rate.

While the usual practice is to multiply the defender's before-tax IRR by $(1 - T)$ to obtain the defender's after-tax IRR, this chapter demonstrated that this cannot be generally relied on to obtain the defender's properly adjusted after-tax IRR. Indeed, we demonstrated that capital gains that are not taxed can lower the effective tax rate. Capital losses that do not create tax shields can increase the effective tax rate. Property tax paid on land and other real property increases the effective tax rate paid. Allowing investors to write off an investment's depreciation lowers the effective tax rate.

So what have we learned? We learned that when comparing defenders and challengers, it is important to find the correct after-tax IRR of the defender to discount the challenger's after-tax cash flow. Other things being equal, we prefer challengers whose effective tax rate is less than the effective tax rate of the defender.

Finally, this chapter has employed simplifying assumptions, such as large values for n and average depreciation and growth rates, to find effective after-tax discount rates that have nice, closed-form solutions. Usually, this is not the case. It is often more difficult to find effective after-tax discount rates, and often these are not closed-form solutions.

Questions

1. Describe the appropriate test for determining whether or not taxes have been properly introduced into the defender's IRR.
2. When finding after-tax IRRs for defenders, we solve for θ and claim that θT is the defender's effective tax rate. Interpret the meaning of θ .
3. Under what condition is the defender's effective tax rate equal to its income tax rate T or that $\theta = 1$?
4. Assume that a defending investment's price is V_0 , that d is the defender's book value depreciation rate, that R is the constant stream of income earned by the defender, that T is the income tax rate, and that r is the defender's before-tax IRR. Also assume that the defender is allowed to write off book value depreciation even though its income stream is constant. We write such a model as:

$$(Q15.1) \quad V_0 = \frac{R(1 - T) + TdV_0}{[1 + r(1 - \theta T)]} + \frac{R_0(1 - T) + TdV_0}{[1 + r(1 - \theta T)]^2} + \dots$$

In this model, the depreciation is constant for $100/d$ periods beyond which the discounted cash flow is small enough to be ignored. The before-tax IRR model can be written as:

$$(Q15.2) \quad V_0 = \frac{R}{(1+r)} + \frac{R}{(1+r)^2} + \dots = \frac{R}{r}$$

Solve for θ in the after-tax IRR model by substituting for V_0 the right hand side of the second model R/r . Next, solve for a numerical value for θ assuming that $d = 3\%$ and $r = 8\%$.

5. Suppose that you have found the NPV for a challenger. What impact will an increase in the defender's effective tax rate have on the challenger's IRR? Please explain your answers.
6. Calculate the effective tax rate θT for a depreciating challenger where d is the depreciation rate, T is the income tax rate, r is the defender's before-tax IRR, and V_0 is the price of defender. The IRR model is written below. (Hint: find θ by setting the right hand side of the equation below to the equivalent model without taxes ($T = 0$) and solve for θ .)

$$(Q15.3) \quad \begin{aligned} V_0 &= \frac{R_0(1-d)(1-T) + TdV_0}{[1+r(1-\theta T)]} \\ &+ \frac{R_0(1-d)^2(1-T) + TdV_0(1-d)}{[1+r(1-\theta T)]^2} + \dots \\ &= \frac{R_0(1-d)(1-T) + TdV_0}{r(1-\theta T) + d} + \dots \end{aligned}$$

7. Discuss the following. Suppose that cash flow was constant ($R_t = R_0$), yet tax laws allowed the owner of the durable to claim depreciation at rate d . Without solving for θ , can you deduce whether it would be equal to, less than, or greater than one? Defend your answer.
8. Assume you are investing in land and that you borrow money to finance its purchase. Also assume that you pay property taxes on the land. In other words, land is the challenger and the loan is the defender. Which investment has the higher effective tax rate under three scenarios: $g = 0$, $g > 0$, and $g < 0$? What do you need to know about the property tax rate T_p to answer this question?
9. In this chapter, five different investment tax scenarios were described. Please provide an example of each type of investment and how the tax scenario might change its rankings compared to before-tax rankings.

16. Homogeneous Liquidity and Currency

LINDON ROBISON

Learning goals. 1. Learn how to measure an investment's liquidity. 2. Learn how to compare liquidity for a firm versus liquidity for an investment. 3. Learn how to rank investments according to their liquidity using cash flow measures.

Learning objectives. To achieve your learning goals, you should complete the following objectives:

- Learn to distinguish between two types of returns earned by an investment: time dated cash flow and capital gains (losses).
- Learn how to distinguish between a firm's liquidity and an investment's liquidity.
- Learn how to describe an investment's liquidity at a point in time by using its current-to-total returns (CTR) ratio.
- Learn how to describe an investment's liquidity over time by using its inter temporal CTR ratio.
- Learn how to connect CTR ratios to price-to-earnings (PE) ratios.
- Learn how PE ratios can be used to infer the liquidity of an investment.
- Learn how coverage (C) ratios can be used to infer the liquidity of an investment.
- Learn how to evaluate investments whose cash flow are in foreign currencies.

Introduction

An investment may earn two types of returns (losses) for investors: (1) time-dated cash flow called current returns, and (2) capital gains (losses). This chapter demonstrates that capital gains earned on an investment depend on the investment's pattern of future cash flow.

This chapter also demonstrates that the combination of current returns versus capital gains (losses) has important liquidity implications for investors, especially when an investment is financed with debt capital. Debt-financed investments whose earnings are expected to grow over time may experience a cash shortfall called a financing gap, in which the cash returns are less than the scheduled payments of principal plus interest. This gap is most likely to occur early in the investment's life and is exacerbated by inflation. As time passes and the cash returns grow in size, they will eventually exceed the repayment obligation, and the liquidity problem is solved.

The term liquidity is used to describe "near-cash investments" because of the similarity between liquids and liquid investments. A liquid such as water can fill the shape of its container and is easily transferred from one container to another. Similarly, liquid funds easily meet the immediate financial needs of their owners. Illiquid investments such as land and buildings, like solids, are not easily accessed to meet financial needs because their ownership and control are not easily transferred.

A firm's liquidity reflects its capacity to generate sufficient cash to meet its financial commitments as they come due. A firm's failure to meet its financial commitments results in bankruptcy, even though the firm might be profitable and have positive equity. Consequently, firms must account for an investment's liquidity in addition to earning positive net present value (NPV).

There are several measures that reflect a firm's or an investment's liquidity. One measure used to reflect the liquidity of the firm is the current ratio, the ratio of current assets divided by current liabilities. Because current assets can be quickly converted to cash, they are an important source of liquidity for the firm.

While the current ratio measures the liquidity of the firm, we want a liquidity measure specific to an investment, and one that describes the liquidity of an investment over time—what we will refer to as a periodic liquidity measure. The liquidity measure that provides investment-specific liquidity information over time is the current-to-total returns (CTR) ratio. The CTR ratio is derived from market equilibrium conditions where NPV is zero and represents the ratio of current (cash) returns to total returns. The price-to-earnings (PE) ratio is a commonly reported ratio reflecting the price of an investment reflected by its current earnings. A liquidity measure related to the CTR ratio is the coverage (C) ratio. A C ratio is the ratio of cash returns to loan payments or other debt obligations in any particular period.

Current-to-Total>Returns (CTR) Ratio

The current-to-total returns (CTR) ratio is a periodic liquidity measure derived from the relationship in a given period between an investment's current (cash) returns, its capital gains (losses), and its total return, or the sum of current returns and capital gains (losses). To derive the CTR measure, we begin with an expression of an investment's total returns:

Total investment returns = current returns + capital gains (losses)

The above relationship is expressed symbolically as follows. Let the investment's total return equal the investment's internal rate of return r times the value of the investment at the beginning of the period V_0 or rV_0 . Meanwhile, let R_1 be defined as the asset's net cash, or current returns earned at the end of the first period. In addition, let V_0 and V_1 equal the investment's beginning and end-of-period values respectively, while their difference ($V_1 - V_0$) is equal to the investment's capital gains (losses) in the first period. Summarizing, we write the investment's total returns as:

$$(16.1) \quad rV_0 = R_1 + (V_1 - V_0)$$

Next we solve for the investment's internal rate of return earned in each period, r . We solve for this measure by dividing the investment's total returns by the investment's value at the beginning of the period V_0 :

$$(16.2) \quad r = \frac{R_1}{V_0} + \frac{(V_1 - V_0)}{V_0}$$

A useful measure obtained from equation (16.2) is the ratio of current returns R_1 divided by total returns rV_0 . This ratio indicates the percentage of total returns the firm receives as cash. The higher this ratio, the greater the likelihood that the firm has the liquidity required to meet its debt obligations in the current period. Rearranging the previous equation yields the ratio of current to total returns in the first period, equal to CTR_1 :

$$(16.3) \quad CTR_1 = \frac{R_1}{r_1 V_0} = 1 - \frac{\left(\frac{V_1 - V_0}{V_0}\right)}{r_1} = 1 - \frac{\eta_1}{r_1}$$

where $\eta_1 = (V_1 - V_0) / V_0$ is the capital gains (loss) rate in period one.

To illustrate the CTR ratio, we now derive the CTR ratio for the geometric series of cash flow. Consider an investment that is expected to generate a perpetual series of cash flow that grow at average geometric rate of g percent per period. If R_0 is the initial net cash flow, and r is the IRR of the investment, then the PV of the growth model can be expressed as:

$$(16.4a) \quad V_0 = \frac{R_0(1+g)}{(1+r)} + \frac{R_0(1+g)^2}{(1+r)^2} + \frac{R_0(1+g)^3}{(1+r)^3} + \dots$$

One year later:

$$(16.4b) \quad V_1 = \frac{R_0(1+g)^2}{(1+r)} + \frac{R_0(1+g)^3}{(1+r)^2} + \frac{R_0(1+g)^4}{(1+r)^3} + \dots = V_0(1+g)$$

and in general

$$(16.4c) \quad V_t = V_0(1+g)^t$$

and the capital gains rate for the geometric growth model is found to equal:

$$(16.4d) \quad \eta = \frac{V_t - V_{t-1}}{V_{t-1}} = \frac{V_{t-1}(1+g) - V_{t-1}}{V_0} = g$$

Using our results from equation (16.3) we can find our CTR ratio as:

$$(16.5) \quad CTR_1 = \frac{R_1}{r_1 V_0} = 1 - \frac{\eta_1}{r_1} = 1 - \frac{g}{r} = \frac{r - g}{r}$$

To illustrate, if an investment earns cash flow in perpetuity described by R_0 , $r = 10\%$ and $g = 5\%$, the investment's CTR ratio can be found equal to:

$$(16.6) \quad CTR_1 = \frac{.1 - .05}{.1} = 50\%$$

It is left as an exercise to demonstrate that if the CTR ratio is constant over time, the CTR ratio is 1. Namely, that:

$$(16.7a) \quad CTR_1 = \frac{r - g}{r} \text{ for } t = 1, 2, \dots, n.$$

To illustrate equation (16.7a), if in period one, $r = 10\%$ and $g = 5\%$, then 50% of the investment's returns will be earned in the form of capital gains. On the other hand, if the geometric mean of changes in future cash flow was negative, ($g < 0$), the current returns to total returns would exceed total returns. If $g = -5\%$, the CTR1 ratio would equal:

$$(16.7b) \quad CTR = \frac{r - g}{r} = \frac{.15}{.1} = 150\%$$

In words, equation (16.7b) informs us that 150% of the investment's return in period one will be received as current returns in order to compensate for the 50% capital losses experienced by the investment.

So, what have we learned? It means that the investments with increasing cash flow are less liquid than those with decreasing cash flow. A less (more) liquid investment means that a part of its return in each period is earned in capital gains (losses) that are not converted to cash until the investment is liquidated.

Liquidity, Inflation, and Real Interest Rates

Prices reflect the ratio of money exchanged for a unit of a good. For example, assume that the price for gasoline is \$3.00 per gallon. If you purchased 10 gallons of gasoline, you would pay \$30.00 ($\3.00×10). Conversely, the ratio of \$30.00 expended for gas divided by the gallons of gas purchased is the price of gas: $\$30.00/10$ equal to \$3.00.

Now suppose that last year you purchased roughly \$50,000 worth of goods. For convenience, let p_j be the price of good j where $j = 1, \dots, n$. Furthermore, let q_j represent the quantity of good j purchased. Multiplying the price times the quantity of each good purchased equals your total expenditures of \$50,000. To summarize, $\$50,000 = \sum_{j=1}^n p_j q_j$ equals the sum of price times quantity of all of the goods you purchased.

In case you had not noticed, prices for goods and services change over time. Consider the price of a dozen large eggs. The price of a dozen of large eggs over the past 38 years is graphed in Figure 16.1 ranging from less than \$1 per dozen in 1980 to almost \$3 per dozen in 2016.

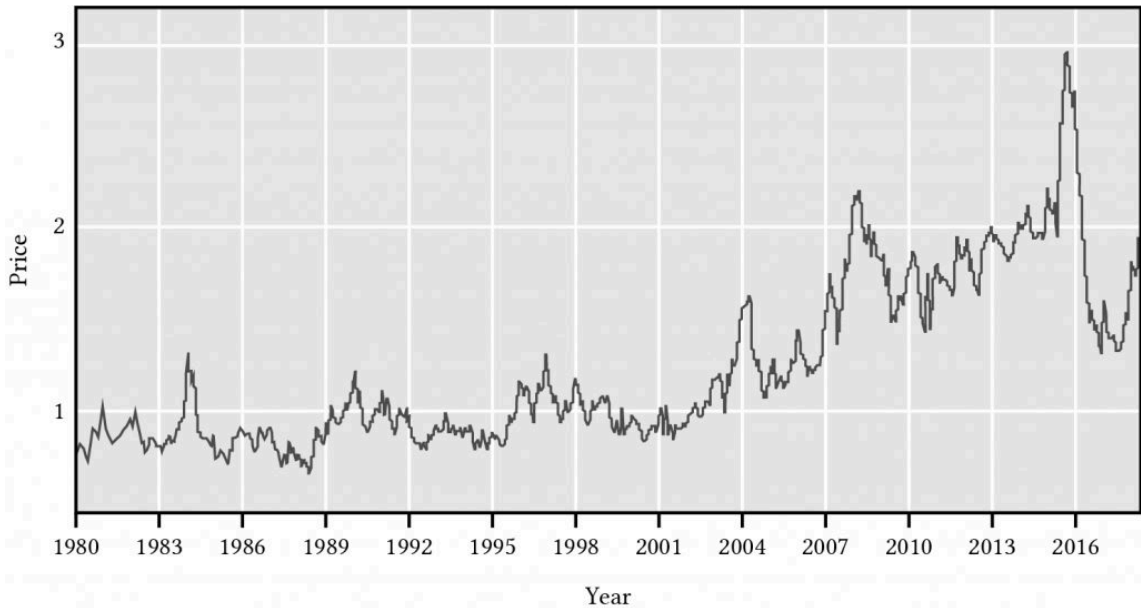


Figure 16.1. The price of a dozen large eggs over the period 1980 to 2016. (Bureau of Labor Statistics, 2016)

Prices over time for an individual good can change for several reasons. The demand for eggs can rise—more people wanting to consume more eggs will likely cause egg prices to rise. Or the cost of supplying eggs may change—chicken have become more efficient at producing eggs making them cheaper or chicken feed may become more expensive making eggs more expensive to produce.

The quantity theory of money

There is another reason why prices may change over time—there is more money in circulation. This explanation for why prices rise we call the quantity theory of money (QTM). The quantity theory of money states that there is a direct relationship between the quantity of money in an economy and the

level of prices of goods and services sold. What constitutes the money supply is a complicated subject that we can avoid here and still make our point.

Returning to our earlier discussion, let the total of goods and services sold times their prices equal the amount of money M in the economy times its velocity (V) or the number of times the same unit of money is used during a time period. We write this relationship as

$$(16.8) \quad \sum_{j=1}^n p_j q_j = MV$$

According to the QTM, if people purchased the same goods in the same quantity and if V were held constant and M increased by i percent, then the price of each individual good must also increase by i percent.

$$(16.9) \quad \sum_{j=1}^n (1 + i) p_j q_j = (1 + i) MV$$

Finally, if we measured the prices times the same goods in each year and calculated the ratio of the current year by the previous year we could obtain an estimate of the how prices in general have changed or the rate of inflation. Adding a superscript t to the price variable, we find the inflation rate plus one in year t is equal to:

$$(16.10) \quad \frac{\sum_{j=1}^n p_j^t q_j}{\sum_{j=1}^n p_j^{t-1} q_j} = (1 + i)$$

An index $(1 + i)$ can be computed for all different bundles of purchased goods. One popular inflation index is called the Consumer Price Index (CPI) that measures the rate of increase in prices holding the bundle of goods constant or at least as far as it is possible. Consider how inflation has affected consumer purchases over time. The vertical axis measures the CPI. The horizontal axis measures time. All indices select one year at their base in which the index is one. In Figure 16.2, the base year is somewhere between 1982-1984.

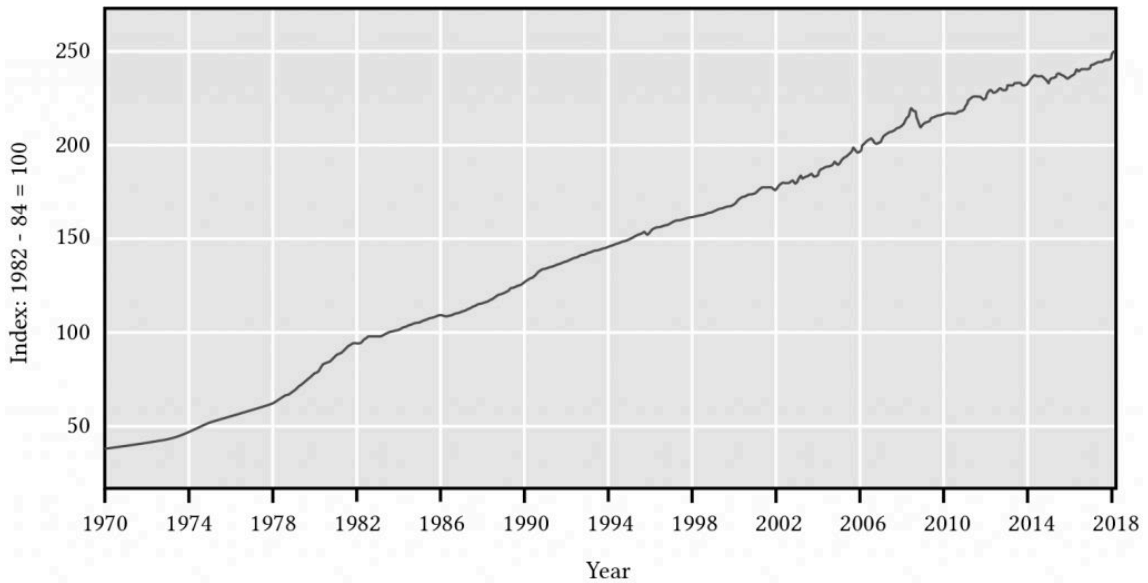


Figure 16.2. The consumer price index for all the U.S. letting average prices between 1982-84 equal 100.

According to Figure 16.2, in 2018 you paid on average two and one-half times as much money to purchase the same good as you did in 1982-1984.

So why is inflation important to our discussion of PV models and profit measures? If your income increases by i percent and the cost of everything you purchase increases by the same percent i , then in real terms what you can actually purchase is the same as before since your income and costs increased by the same rate.

Real interest rates. Let say that your after-tax cash flows increased by rate $g\%$ but that g is influenced by the rate of inflation. Then after accounting for inflation, what is left is attributed to other factors besides an increase in the money supply. Let this real growth in after-tax cash flow (ATCF) be g^* . Then we can write:

$$(16.11) \quad (1 + g) = (1 + g^*)(1 + i)$$

Interest rates, discount rates, and inflation. Inflation also influences interest rates paid to borrow money and the discount rate that reflects the opportunity cost for one's resources. To motivate this discussion intuitively, assume you lent a friend \$1000 for one year. Also assume that inflation is 5% so that when your friends repays the loan, to make the same purchases at the time the loan was made, they would have to pay back $\$1,000 * (1.05) = \$1,050$. But they would have to pay more than 5% for being able to rent your money for a year—let's call this rate the real interest rate and denote it as r^* . To account for both the reduced purchase price of your money because an increase in prices by inflation i and the cost of postponing the use of one's money, the interest rate (and opportunity cost of resources invested in a defender) is r equal to:

$$(16.12) \quad (1 + r) = (1 + r^*)(1 + i)$$

We graph real interest rates r^* over time in Figure 16.3.

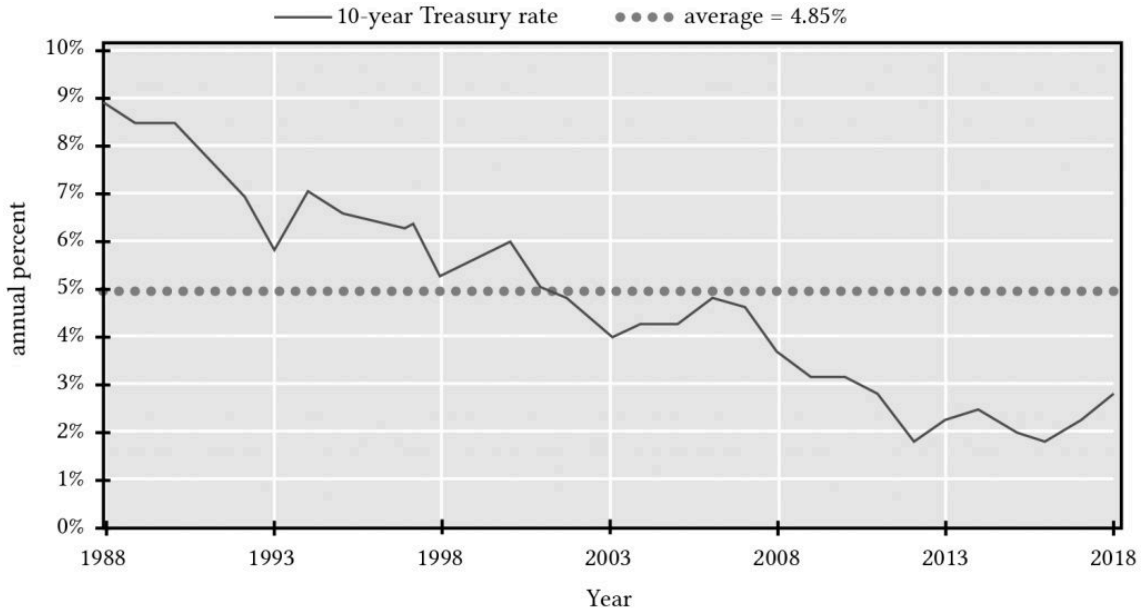


Figure 16.3. 10-year Treasury constant maturity interest rate, U.S., 1988 - 2018

So, what have we learned? It means that real interest rates have been falling since 1988 and have recently started to increase. Increasing real interest rates will make long-term investments less liquid because it increases interest rate changes on loans and also increases the rate of growth g that we already demonstrated increases capital gains relative to current earnings.

Liquidity Implications

The mix of capital gains (losses) versus current returns depends directly on the average rate of growth ($g > 0$) or decline ($g < 0$) in future cash flow. As inflation increases the average rate of growth g , then inflation also reduces an investment's liquidity, especially during the investment's initial years.

For geometric decay in earnings ($g < 0$), current returns are more than 100% of total earnings—and current returns have increased in importance relative to capital loss or depreciation. These results clearly show the paradoxical conditions that growth in projected earnings can weaken the liquidity of the investment due to the increasing relative importance of capital gains. In contrast, a decline in projected earnings can strengthen the liquidity due to the increasing relative importance of current returns (See Table 16.1).

Also note in Table 16.1 that increases in the capital gains rate g holding r constant always leads to a reduction in CTR ratio. This is easily explained. Since the total rate of return r is fixed and composed of current rate of returns and capital gains rate of returns, increases in the capital gains rate of return must necessarily decrease the current rate of returns.

Table 16.1. Values of CTR ratios where $CTR_t = (r - g) / r$

	r%				
g%	10	8	6	4	2
-5	150%	163%	183%	225%	350%
-3	130%	138%	150%	175%	250%
0	100%	100%	100%	100%	100%
3	70%	63%	50%	25%	
5	50%	38%	17%		

Price-to-earnings (PE) Ratios

Related to the CTR ratio is the price-to-earnings (PE) ratio equal to:

$$(16.13) \quad PE = \frac{V_0}{R_1}$$

If we describe the investment using the PV expression in equation (16.4a), then we can write:

$$(16.14) \quad V_0 = \frac{R_0(1+g)}{(1+r)} + \frac{R_0(1+g)^2}{(1+r)^2} + \frac{R_0(1+g)^3}{(1+r)^3} + \dots + \frac{R_0(1+g)}{r-g}$$

Next we substitute $R_0(1+g)$ for R_1 , and the right-hand side of equation (16.9) for V_0 into the PE expression in equation (16.8), and obtain:

$$(16.15) \quad PE = \frac{V_0}{R_1} = \frac{1}{r-g}$$

But from equation (16.7a) we know that $rCTR = (r - g)$, which allows us to connect CTR and PE ratios, namely:

$$(16.16) \quad PE = \frac{1}{r - g} = \frac{1}{rCTR}$$

In words, what we learn from equation (16.16) is that, in general, higher CTR ratios imply lower PE ratios and lower CTR ratios imply higher PE ratios. These results may at first appear to be somewhat counter intuitive—that, for liquidity purposes, we prefer investments with lower PE. Upon reflection, it makes sense—assets whose returns are mostly capital gains will be earning a lower percentage of current returns compared to their values than assets that earn little or no capital gains.

To make the point that CTR and PE ratios are inversely related, we construct Table 16.2 for PE ratios. Compare the results with those found in Table 16.1 to confirm in your minds the implications of equation (16.16). Note that PE ratios uniformly increase with increases in g and the rate of return earned in the form of capital gains.

Table 16.2. Values of PE ratios where $PE_t = 1/(r - g) = 1 / (rCTR_t)$

	r%				
g%	10	8	6	4	2
-5	6.67%	7.67%	9.11%	11.11%	14.29%
-3	7.69%	9.06%	11.11%	14.29%	20.00%
0	10%	13.50%	16.67%	25.00%	50.00%
3	14.29%	20.00%	33.33%	100.00%	
5	20.00%	33.33%	100.00%		

PE ratios are frequently used to describe the financial condition of an asset. Generally speaking, high PE ratios reflect an expectation of income growth for an asset. Unfortunately, high levels of PE ratios may reflect bubbles or unrealistic expectations for income growth for a particular investment—and high PE ratios are often followed with a market adjustment in which the earnings expectation of an investment are adjusted downward, which reduces the investment's PE ratio. For example, in 1929 for stocks (see Figure 16.4), and 2010 for housing (see Figure 16.5), expected income growth was unrealistic, and in both cases, PE ratios fell when earnings expectations were adjusted downward.

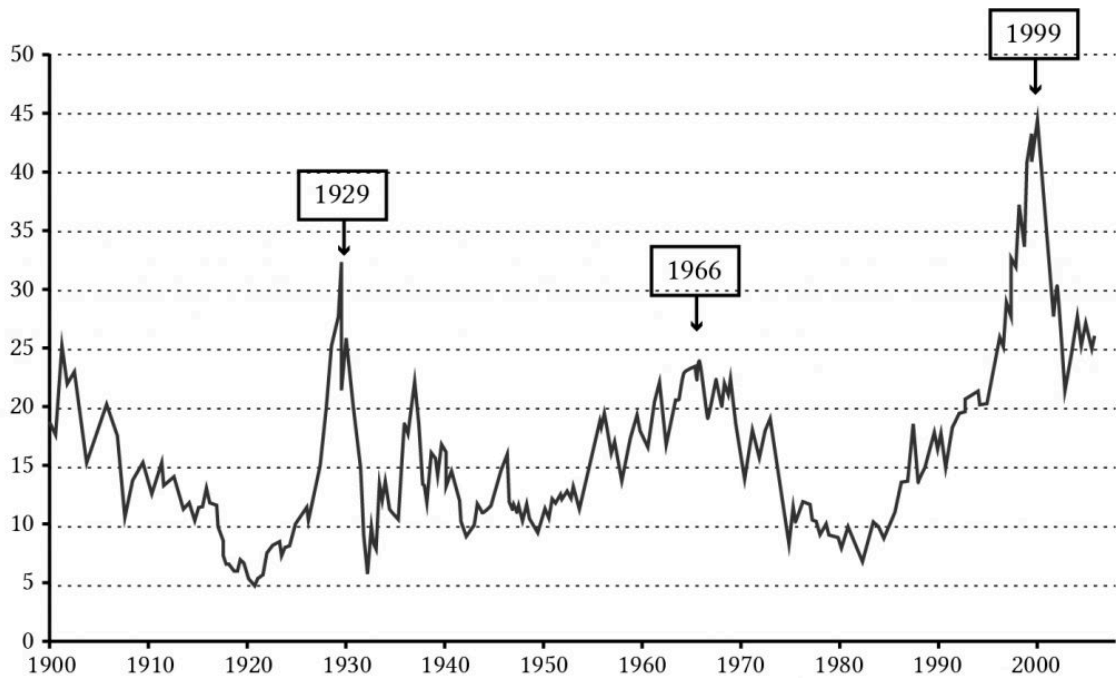
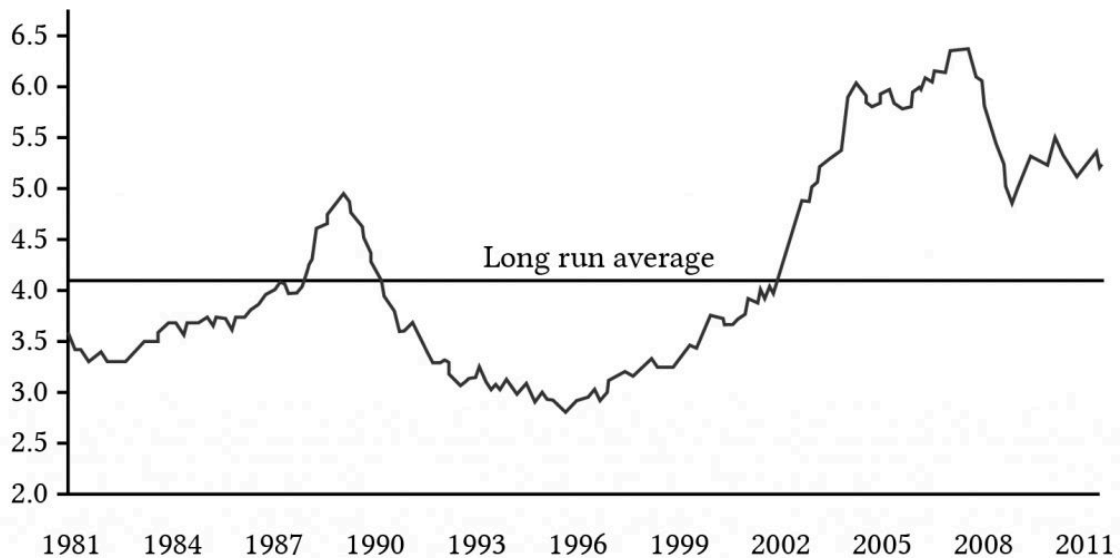


Figure 16.4. Historic PE ratios for stocks described on the Standard and Poor's Stock Exchange. (Earnings are estimated based on a lagged 10-year average.)



Source: Nationwide, ONS

Figure 16.5. Historic PE ratios for UK housing stock.

Compare the value of the PE ratios in Figures 16.4 and 16.5. PE values for stocks that reflect depreciable assets ($g < 0$) have average values around 20. PE values for housing stock that reflect non-depreciable assets ($g > 0$) have recently averaged around 5. (See Table 16.2.)

Coverage (C) Ratios

Investments can be evaluated according to both profitability and periodic liquidity criteria. Profitability analysis focuses on whether the acceptance of an investment will increase the investor's present wealth. Periodic liquidity measure analysis considers whether an investment will generate cash flow consistent with the terms of financial capital, especially debt capital, that are used to finance the investment. Is an investment capable of generating sufficient net cash flow in each period to satisfy the requirements for repayment of the loan's principal plus interest? If not, then the investment is illiquid because liquid funds drawn from other sources are required to meet cash flow requirements associated with the investment. Thus, a simple test of investment liquidity is to compare the net cash flow generated by the investment in a given period with the debt-servicing requirement.

The ratio of net cash flow in a period t , R_t , divided by the loan plus principal payment in the same period, A_t , is called the coverage ratio, C_t :

$$(16.17) \quad C_t = \frac{R_t}{A_t}$$

If the coverage ratio is equal to or exceeds 1.0, the investment is liquid in that period. If coverage is less than one, the investment is illiquid in that period. In most cases, the loan payment that includes principal plus interest is a constant "A," that depends on the price of the purchased asset, V_0 ; the financed proportion of the purchase price γ ; the term of the loan n ; and the interest rate charged on the loan r^f . Using our previous notation, we can express the relationship between the amount financed, the fixed loan payment, the interest rate on the loan, and the term of the loan as:

$$(16.18) \quad \gamma V_0 = A U S_0(r^f, n)$$

We solve equation (16.18) for A and substitute the result for A_t in equation (16.19). In the same equation, we also substitute $R_0(1 + g)^t$ for R_t , and obtain:

$$(16.19) \quad C_t = \frac{R_0(1 + g)^t}{\gamma V_0} U S_0(r^f, n)$$

Replacing V_0 with $[R_0(1 + g)]/(r - g)$, we can rewrite equation (16.19) as:

$$(16.20) \quad C_t = \frac{(r - g)(1 + g)^{t-1}}{\gamma} US_0(r^f, n)$$

We illustrate equation (16.20) as follows. Assume $r = 8\%$, $g = 2\%$, $n = 20$, and $r^f = 7\%$. Also assume the investment is 80% financed. Using an Excel spreadsheet, we find that $US_0(7\%, 20) = 10.59$.

Table 16.3. Calculating $US_0(r^f=7\%, n=20)$
[Open Table 16.3 in Microsoft Excel](#)

B4	fx		=PV(B1,B2,B3,,0)
	A	B	C
1	rate	7%	
2	nper	20	
3	pmt	-1	
4	NPV	\$10.59	=PV(rate,nper,pmt,,0)

Then we find the coverage value for the first year of the investment to equal:

$$(16.21a) \quad C_1 = \frac{(.08 - .02)(1.02)^0 10.59}{.8} = .79$$

Interpreted, in year one, the investment described above will only pay for 79 percent of the loan payment due. However, 10 years later and half way into the term of the loan, the coverage ratio has increased to:

$$(16.21b) \quad C_{10} = \frac{(.08 - .02)(1.02)^9 10.59}{.8} = .95$$

Finally, in the last year of the loan, the coverage ratio is:

$$(16.21c) \quad C_{20} = \frac{(.08 - .02)(1.02)^{19} 10.59}{.8} = 1.16$$

In other words, in year 20, the last year of the loan, the investment cash flow not only repays the loan but 16 percent more than the required loan payment. Of course, after the loan is repaid, the periodic cash flow is available to the firm to meet its cash flow requirements.

One word about the term of the investment, it is assumed to be infinite. However, for any one investor, the term may be finite, but since the terminal value for any one owner is simply the present value of the continued cash flow, we can express the investment as having an infinite life. Therefore, we are justified in ignoring the term of the infinite-life investment.

When is the coverage equal to 100 percent? There is another way to approach the issue of coverage. It is to ask in what year is the coverage equal to 1? To answer this question for the geometric growth model with an infinite life and using the already established notation, we set:

$$(16.22) \quad A = R_0(1 + g)^t$$

and solving for t we find:

$$(16.23a) \quad t = \frac{\ln A}{\ln R_0 + \ln(1 + g)}$$

And after making substitutions for A and R_0 and simplifying, we can write:

$$(16.23b) \quad t = \frac{\ln \gamma - \ln(r - g) - \ln US_0(r^f, n)}{\ln(1 + g)}$$

We illustrate equation using numbers from the previous example where: $\gamma = .8$, $r = .08$, $g = .02$, $r^f = .07$, and $n = 20$. We find:

$$(16.24) \quad t = \frac{-.22 + 2.81 - 2.36}{.02} = 11.50$$

In words, what we have found that the 12th payment will be the first period that more than pays for its financing.

So, what have we learned? As the previous example demonstrates, it means that for investments with increasing cash flow, its liquidity increases over time.

Exchange Rates and PV Models

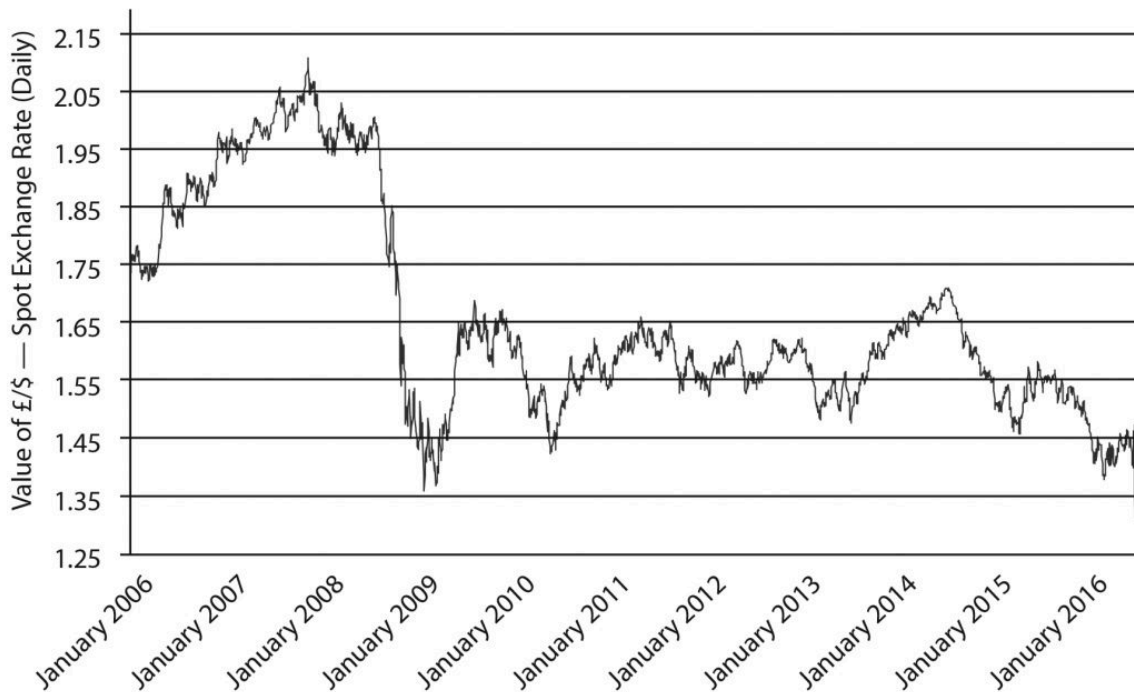
In today's globalized economy, commodity exchanges are likely to occur between countries and involve exchanging dollars for the currency of another country such as Euros, pounds peso, yen, and yuan to name a few. And if we are engaged in some form of international business or travel, we will likely be required to exchange dollars for the currency of the country in which we are doing business or which we are visiting. What follows is intended to describe how to convert measures of foreign currency into our own. We begin with some definitions.

Exchange rate. The exchange rate is the rate at which one currency trades against another currency in the foreign exchange market. To be more precise, an exchange rate is a ratio of the prices of two currencies: the price of the exchange currency being acquired divided by the base currency being sacrificed. During the summer of 2020, the exchange rate for dollars and euros was .89 euros per 1 dollar and, alternatively, 1.12 dollars per 1 euro. Table 16.4 provides a table of several exchange rates in which the dollar is the base currency.

Table 16.4. US Dollar Exchange Rates

US Dollar	1.00 USD
Euro	0.890730
British Pound	0.805050
Indian Rupee	75.521451
Australian Dollar	1.444682
Canadian Dollar	1.357791
Singapore Dollar	1.394211
Swiss Franc	0.948127
Malaysian Ringgit	4.286344
Japanese Yen	107.568396
Chinese Yuan Renminbi	7.067112

Foreign exchange markets. A foreign exchange market is a market in which one currency trades against another. Currencies are continuously traded and currency exchange rates are continuously changing in the foreign exchange market as currency dealers adjust to changing demand and supply for a country's currency. For example, consider the exchange rate between the English pound and the U.S. dollar. Note the drop in the exchange rate (how many dollars an English pound could purchase around the time England's economy experienced difficult times).



Source: Bank of England

Figure 16.6. Exchange Rate: Pound Sterling to US Dollar

Floating versus fixed exchange rates. The exchange rate influences a country's economy. A high exchange rate makes it more economical for people in the base country to make purchases from or visit the exchange country. A low exchange rate means that it is more expensive for people in the base country to make purchases from or visit the exchange country. As a result, countries often take actions to influence the exchange rate through their monetary policies.

Depending on how countries manage their exchange rates, we describe them as floating or fixed. A floating exchange rate is determined by the supply and demand for currencies in the global currency markets. Therefore, if the demand for the currency is high, its exchange rate will increase. In some extreme cases, some governments may seek to keep the value of their currency fixed at a certain rate compared to other currencies.

What forces influence exchange rates? As Figure 16.6 demonstrates, exchange rates are continually changing. There are several theories that explain the price level of goods and services in a economy and in turn influence their exchange rates with other countries. One such theory already referred to in this chapter is the quantity theory of money. This theory assumes that the quantity of money in an economy is an important determinant on the country's economic activity and that a change in a country's money supply will either increase prices or the supply of its goods and services or both. The

quantity theory of money is only one of many theories attempting to explain prices in an economy. We invite students to explore others.

To review the quantity theory of money and apply it to exchange rates, suppose that an economy produces a basket of goods and services G during a period. Moreover, assume that during the period, the money supply is M and used in exchanges for goods G on average V times. The theory adopts the reasonable hypothesis that goods time their prices must equal the money time the number of times it is used in an exchange or $PG = MV$ or that the price of the currency is:

$$(16.25) \quad P = \frac{MV}{G}$$

Think of the price of money P as the purchasing power of a unit of the country's currency. When P goes down, it requires less of the country's currency to buy the same quantity of the country's goods and services. Then if the country become more productive and G increases, holding everything else constant, people in the country can buy the same amount of goods and services at a lower price P . If the amount of money M in the economy increases, then the currency required to buy the same amount of goods and services increases. (We'll not worry about velocity for now.)

Suppose we have two countries A and B and that people in those countries exchange currency for the purpose of international purchases or to pay expenses when visiting the country or for speculation assuming that one can guess relative changes in the price of each country's currency. Then subscripting the variables in equation (16.25), we can write the exchange rate between the two countries as:

$$(16.26) \quad \frac{P_A}{P_B} = \frac{G_B}{G_A} \frac{M_A}{M_B} \frac{V_A}{V_B}$$

To apply the quantity theory of money, think of country B as the U.S. and country A as Canada. The exchange rate describes how many Canadian dollars one U.S. dollar can purchase. Assume to begin, that the exchange rate is one Canadian dollar for one U.S. dollar. Then assume that production in the U.S. increases G_B . Then holding all other things constant, the exchange rate must increase. One U.S. dollar can now buy more Canadian dollars because those wanting to buy U.S. dollars realize that they can now purchase more than before and are willing to pay more for a U.S. dollar. For an exercise, describe the consequences of an increase in the U.S. money supply.

Now imagine that as a U.S. businessperson, you invest V_0^B in a Canadian business and expect to receive Canadian cash flow for the next n periods of R_1^A, \dots, R_n^A . Assuming your defender's discount rate is r percent and the predicted exchange rate for Canadian and U.S. dollars is $X_1^{A/B}, \dots, X_n^{A/B}$.

$$(16.27) \quad NPV = -V_0^B + \frac{R_1^A X_1^{A/B}}{(1+r)} + \dots + \frac{R_n^A X_n^{A/B}}{(1+r)^n}$$

An example:

Suppose that Lon decides to franchise his business and open Lon's Yard Care Business in Liverpool, England. He wonders what his NPV would equal if an investment in US dollars of \$1000 earns three years of £300 British pounds and the exchange rate of pounds for dollars because of the impact of BREXIT is expected to equal: 1.5, 1.4 and 1.3. Also assume that the defender's IRR is 10%.

$$(16.28) \quad NPV = -\$1,000 + \frac{(300)(1.5)}{1.1} + \frac{(300)(1.4)}{1.1^2} + \frac{(300)(1.3)}{1.1^3} = \$49.21$$

After adjusting for changing exchange rates between the British pound and the U.S. dollar, Lon finds that a three-year investment in a Liverpool franchise would earn \$49.21 NPV valued in U.S. dollars.

So, what have we learned? Suppose that I can exchange 1 dollar for .8 British pound. Then the exchange rate for pounds/dollars is .8. On the other hand if I can exchange 1 British pound for 1.25 dollars, then the exchange rate for dollars/pounds is 1.25. If I am receiving a payment in pounds and I want to know the dollars equivalent of what I am receiving, I multiply the pounds by the dollars/pound exchange rate. So if I were receiving a payment of 300 British pounds, I would multiply by the dollar/pounds exchange rate of 1.25 and compute that I would be receiving the equivalent of \$375.

Summary and Conclusions

An investment may earn two types of returns: time dated cash flow and capital gains (losses). Moreover, these returns can be earned over the lifetime of the investment. The relative importance of the two forms of returns, cash flow versus capital gains (losses), will determine the inter temporal liquidity of the asset. The greater the portion of the return is earned as cash, the more liquid will be the investment.

This chapter demonstrated whether an investment earns capital gains (losses) is determined in PV models by its pattern of expected future cash flow. Expected increases (decreases) in the number, frequency, or amount of the expected future cash flow will produce capital gains (losses). Inflation causes long-term investments to become less liquid because they increase the rate of growth in the investment's cash flow.

Investments that earn capital gains are called appreciating investments. Investments that experience capital losses or depreciation are referred to as depreciating investments.

Perhaps paradoxically, investments that earn capital gains are less liquid than investments that experience capital losses. Assuming investments earn similar total rates of return, the greater the cash returns are compared to total returns, the greater are the liquid funds available to meet liquidity needs including repayment of funds and interest required to maintain control of the asset.

In this chapter we developed current-to-total returns (CTR) measures to indicate an investment's liquidity over time. Furthermore, CTR measures were shown to be related to price-to-earnings (PE) ratios which also depended on an investment's liquidity. Finally, one feature of an investment, is its coverage, its ability in any one period to repay principal and interest payments due in any one period. If funds borrowed to purchase an appreciating investment are repaid with a constant loan payment, then initially, it is likely that cash flow generated by the investment will not cover the interest and principal payments. This pattern is very likely during periods of inflation. Coverage ratios will also indicate the liquidity of investments over their productive lives.

This chapter acknowledged that we do business in an increasingly globalized economy. As a result, many investments involve exchanges of foreign currencies. This chapter explored exchange rates and how to perform PV analysis when more than one currency is involved.

In summary, the periodic liquidity of an investment is an important characteristic that should influence the capital decisions of financial managers along with the impact of an investment on the firm's solvency, profitability, efficiency, and leverage.

Questions

1. Please describe three possible appreciating and three depreciating investments that are available for most farm financial managers.
2. Describe the connections between capital gains (losses) with an investment's periodic liquidity.
3. Describe the connection between the inflation rate i and an investment's CTR ratio.
4. How is an investment's periodic liquidity related to its CTR (current-to-total returns) ratio and its C (coverage) ratio? How are the two periodic liquidity measures related to each other?
5. Compare the firm's current (CT) ratio and its debt-to-service (DS) ratio with an investment's CTR ratio and C ratio. In what ways are they similar? How do they differ?
6. Consider the CTR ratio described in equation (16.3). Find the CTR ratio for the investment whose anticipated cash flow is a constant R in perpetuity described below:

$$(Q16.1) \quad V_0 = \frac{R}{(1+r)} + \frac{R}{(1+r)^2} + \frac{R}{(1+r)^3} + \dots$$

(Hint: Observe the CTR ratio described in equation (16.3). Then derive capital gains (losses) measures for the PV model described in this question.)

7. Find the CTR ratio for an investment whose defender's IRR is 8% and its anticipated growth rate is 2%. Then find the PE ratio for the same investment.
8. Assume that PE ratios can be approximated using equation (16.10). What would account for the large swings in PE ratios described in housing stock described in Figure 16.2—changes in expected r or g ? Or due to some other influence on the value of housing stock?
9. Find the coverage (C) ratio for year 5 for the investment described in the text where: $\gamma = .7$, $r = .08$, $g = .02$, $r^f = .07$, and $n = 20$. Call the result, the result for the base model.
 - a. Recalculate the base model C ratio where $n = 20$ is changed to $n = 15$. Describe how reducing n changed the coverage compared to the base model. Then provide an explanation for the change in coverage.
 - b. Recalculate the base model C ratio where $g = .02$ is changed to $g = -.02$. Describe how reducing g changed the coverage compared to the base model. Then provide an explanation for the change in coverage.
 - c. Recalculate the base model C ratio when $\gamma = .8$ is changed to $\gamma = .6$. Describe how increasing γ changed the coverage compared to the base model. Then provide an explanation for the change in coverage.
10. In this chapter we derived equation (16.18) which solved for the time period in which the coverage was 100%. Find the time period in which coverage is 100% when $\gamma = .7$, $r = .08$, $g = .02$, $r^f = .06$, and $n = 20$. Call the result, the base model result. Then use equation (16.18) to find how t associated with 100 percent coverage changes for the following cases:
 - a. Recalculate the base model and find a new t value associated with 100% coverage where $n = 20$ is changed to $n = 15$. Describe how reducing n changed t and then provide an explanation for the change in t .
 - b. Recalculate the base model and find a new t value associated with 100% coverage where $r^f = .06$ is changed to $r^f = .08$. Describe how increasing the interest rate r^f changed t associated with 100% coverage. Then provide an explanation for the change in t .
 - c. Recalculate the base model and find a new t value associated with 100% coverage where $r = .08$ is changed to $r = .06$. Describe how decreasing the discount rate r changed t associated with 100% coverage. Then provide an explanation for the change in t .
11. Suppose the firm you managed was facing financial stress due to falling prices. Moreover, the stress is reflected in low liquidity and solvency measures for the firm. It appears that there is no way the firm can survive its current crises without liquidating not only short-term assets, but also long-term investments. Based on what you have learned about investment liquidity, propose a plan for the firm to meet its liquidity crisis.

PART V

PART V: PRESENT VALUE MODEL
APPLICATIONS

17. Loan Analysis

LINDON ROBISON

Learning goals. After completing this chapter, you should be able to: (1) understand the different ways interest can be calculated on a loan; (2) recognize the different kinds of interest rates that are used to calculate interest costs on loans; (3) compute comparable effective interest rates; (4) use Excel worksheets to calculate the loan payment, interest rate, term, elasticity of term, and amortization schedule for a particular loan; and (5) find the PV of refinanced and concessionary interest rate loans..

Learning objectives. To achieve your learning goals, you should complete the following objectives:

- Learn the differences between and how to compute the following interest rates: the annual percentage rate (APR), the actuarial rate, and the effective rate.
- Learn how to use the loan equality equation to find a loan's effective interest rate, its constant payment, its term, and its original loan amount.
- Learn how to find the elasticity relationship between interest rates, loan payments, and the term of the constant payment loan.
- Learn how to create an amortization table for constant payment loans.
- Learn how to find break-even points when refinancing loans.
- Learn how to find the effective rate for several disguised interest rate loans including discount loans and points added loans.

Introduction

1

Loan formulas and PV models have many similarities. Interest rates on loans are like opportunity costs, the loan amount is like an investment, and loan payments are like an investment's cash flow. These and other similarities between PV models and loan formulas allow us to use PV tools to analyze different types of loans and interest rates. The next section focuses on alternative interest rate definitions. This chapter will also identify the relationship between the term of a loan and the size of the loan payment. Finally, this chapter finds break-even points when refinancing loans and effective interest rates for a variety of disguised interest rate loans.

1. Some of the material in this chapter was adapted from Robison, L.J. and P.J. Barry (1996); Present Value Models and Investment Analysis. "Chapter 12: Loan Analysis". MSU Press, East Lansing, Michigan.

Comparing the Actuarial Rate, Annual Percentage Rate (APR), and Effective Interest Rate

Loans charge interest rates of which there are at least three closely related kinds. These rates and their commonly used synonyms are listed below. The interest rate name used in this chapter is italicized. They are:

- (1) actuarial rate, compound rate, true rate, or periodic rate (r^f);
- (2) Annual Percentage Rate (APR), annual rate, or nominal rate (r/m); and
- (3) effective interest rate or effective annual rate (r^e).

Actuarial Rate. In financial transactions, interest may be computed and charged more than once a year. For example, interest on savings deposits is usually calculated on a daily basis while many corporate bonds pay interest on a semiannual basis. The interest rate used in computations for periods of less than one year is called an actuarial interest rate. The actuarial rate is defined as the interest rate per compounding period or the interest rate per period of conversions. It is the actuarial rate used to charge interest on the principal sum during each successive conversion period. For example, consider a 1% actuarial rate charged monthly on \$1,000. In this case, in the first month of the loan, 1% of \$1,000 or \$10 of interest is charged. In the second month, interest is charged on \$1,010 equal to \$10.10, etc.

Annual Percentage Rate (APR). Let r^f represent the APR. Let m stand for the number of times during the year the interest is calculated or charged. Thus, m equals the number of compounding periods per year. The ratio of r^f/m is the actuarial rate, the compound rate, the true rate, or the periodic rate.

We find the APR from actuarial rates by expressing the actuarial rate on an annual basis. To convert the actuarial rate to an APR, we multiply the actuarial rate by m . In the previous example, we multiply the actuarial rate of 1% per month by 12 to yield an APR of 12%. When the compound period or conversion period is one year in length, then the actuarial rate and the APR are equal.

Consider two savings institutions, both offering the same APR. The only difference is that institution A offers monthly compounding of interest, while institution B offers annual compounding. Which one should the saver prefer? Obviously, monthly compounding is preferred because the saver earns interest on the interest earned during the same year. With institution B, interest is earned during the year only on the principal saved and on interest earned in previous years.

Effective interest rates. Effective interest rates, designated r^e , are the actual interest charged measured on an annual basis. When APRs have different numbers of compound periods per year, the different actuarial rates should be converted to their effective interest rates for comparison. The effective rate is obtained by compounding the actuarial rate for a period of one year. As the number of compounding periods per year increases, the difference between the APR and the effective rate increases.

Relationship between interest rates. The relationships between an actuarial rate, an APR, and an effective rate can be easily summarized. Let m be the number of compounding periods per year, let r^f be the APR, let r^f/m be the actuarial rate, and let r^e be the effective rate. The relationship between the effective rate r^e , the APR rate r^f , and the actuarial rate r^f/m can be expressed as:

$$(17.1) \quad r^e = \left[\left(1 + \frac{r^f}{m} \right)^m - 1 \right]$$

Note that when $m = 1$, the effective interest rate, the APR rate, and the actuarial rate are equal. However, when m is not 1, the rates are no longer equal. For example, suppose we wish to find the r^e assuming r^f were compounded quarterly. To solve this problem let $m = 4$, and $r^f = 12$. Substituting .12 for r^f and 4 for m in equation (17.1), we obtain the results described in Table 17.1 using Microsoft Excel. In cell B3 we enter the function $f_x = ((1 + (B1 / B2))^B2) - 1$ which returns 12.55%.

Table 17.1. Finding an effective interest rate compounded quarterly.
[Open Table 17.1 in Microsoft Excel.](#)

B3			f_x	$= (1 + B1/B2)^{B2} - 1$
	A	B	C	
1	APR	0.12		
2	m	4		
3	effective rate	12.55%	$= (1 + APR/m)^m - 1$	

If m is increased to 12, or monthly compounding periods, the effective rate is found as before. Let $m = 12$, and $r^f = 12$ in equation (17.1). Then

$$(17.2) \quad \left[\left(1 + \frac{.12}{12} \right)^{12} - 1 \right]$$

To solve this equation in Excel, we change cell B2 to 12 and find the effective interest rate to equal 12.68% as shown in Table 17.2.

Table 17.2. Finding an effective interest rate compounded monthly.
[Open Table 17.2 in Microsoft Excel.](#)

B3			f_x	$= (1 + B2/B3)^{B3} - 1$
	A	B	C	
1	APR	0.12		
2	m	12		
3	effective rate	12.68%	$= (1 + APR/m)^m - 1$	

A special compounding formula is obtained by allowing the number of periods compounded to be very large. This idea is expressed as:

$$(17.3) \quad \lim_{m \rightarrow \infty} \left(1 + \frac{r^f}{m} \right)^m$$

which means that as m approaches infinity the effective rate r^e equals:

$$(17.4) \quad r^e = \lim_{m \rightarrow \infty} \left(1 + \frac{r^f}{m} \right)^m - 1 = e^{r^f} - 1$$

To solve this problem using Excel, we use the EXP function. In Excel, Euler's number "e" is found as the function EXP(1) and 12% compounded continuously can be calculated as $f_x = \text{EXP}(.12) - 1 = 12.75$. Using the previous cell designation, to find the value of continuously compounding of 12% we would enter the formula in B3 as: $f_x = \text{EXP}(B1) - 1 = 12.75$.

Table 17.3. Finding an effective interest rate compounded continuously.
[Open Table 17.3 in Microsoft Excel](#)

B3			f_x =EXP(B1) - 1
	A	B	C
1	APR	0.12	
2	m	Infinite	
3	effective rate	12.75%	=EXP(APR) - 1

Constant Payment Loans

Having defined interest rates in financial models, we now use PV models to analyze the most common type of loan, the constant payment loan. Constant payment loans are repaid with a series of equal payments A at equal time intervals. These payments may occur m times during a year over n years, yielding a total of mn payments.

The fundamental equality is that the sum of the loan payments discounted at the actuarial interest rate must equal the amount loaned. The relationship between loan amount, L_0 , received in time-period zero, with payment A , made for mn periods, at actuarial interest rate (r^f/m) beginning in period one, is:

$$(17.5) \quad L_0 = \frac{A}{1 + \frac{r^f}{m}} + \frac{A}{\left(1 + \frac{r^f}{m} \right)^2} + \cdots + \frac{A}{\left(1 + \frac{r^f}{m} \right)^{mn}}$$

In this formula, the actuarial rate (r^f/m) is the IRR for the PV model, and since loan payments are constant, it is unique.

Fortunately, the calculations described in equation (17.5) can be easily performed using Excel. We calculate L_0 in the following example.

Example 17.1. Loan amount supported by a constant payment loan. Suppose a borrower can make constant payments of \$150 per month for 48 months (four years). The borrower wants to know what size loan can be repaid if the APR interest rate is 5% and the actuarial rate is $.05/12 = .42\%$. Using equation (17.5) we can solve for the loan amount supported by the constant loan payment of \$150 using the Excel formula in Table 17.4 below:

Table 17.4. Finding the Loan Amount in a Constant Payment Loan
[Open Table 17.4 in Microsoft Excel](#)

B5			f_x	=PV(B1/B2,B2*B3,B4,,0)
	A	B		C
1	rate		.05	
2	m		12	
3	n		4	
4	pmt		-150	
5	PV		\$6,513.44	=PV(rate/m,mn,pmt,,0)

The answer displayed is: \$6,513.44. In other words, 48 monthly payments of \$150 on a loan charging a 5% APR interest rate and a monthly actuarial interest rate of .42% will repay a loan in the amount of \$6,513.44.

Fortunately, Excel can also solve for the constant payment if the loan amount, term, and interest rate are known. A constant payment loan problem is described in Example 17.2.

Example 17.2. Constant Payment Loan Annuities. Suppose \$5,000 is borrowed from a lending institution for five years at an APR of 12% or a monthly actuarial rate of 1%. The loan is to be repaid with 60 equal monthly installments. What is the payment or annuity necessary to retire the loan?

We solve for the loan payment using Excel's PMT function in Table 17.5. The answer displayed is: -\$111.22 which means that 60 payments of \$111.22 on a loan charging 12% APR interest and a monthly actuarial interest rate of 1% will repay a loan in the amount of \$5,000.

Table 17.5. Finding the Constant Loan Payment
[Open Table 17.5 in Microsoft Excel](#)

B4			f_x =PV(B1/B2,B2*B3,B5,,0)
	A	B	C
1	rate	0.12	
2	m	12	
3	n	5	
4	pmt	(\$111.22)	=PMT(rate/m,mn,PV,,0)
5	PV	\$5,000.00	

Finally, consider a loan problem where the unknown variable is the term of the loan. We introduce such a problem in Example 17.3 and the Excel solution in Table 17.6.

Example 17.3. Loan term required to retire a constant payment loan. Suppose a borrower can make constant payments of \$150 per month. The borrower wants to know how many monthly payments will be required to retire a loan of \$8,000 if the APR interest rate is 6% and the monthly actuarial rate is .5%.

We solve the equation using Excel's NPER function shown in Table 17.6. The answer displayed is 62.19 which means that 62 regular payments and one partial payment will be required to retire a loan of \$8,000 if the APR interest rate charged on the loan is 6%.

Table 17.6. Finding the Number of Payments required to Retire a Loan
[Open Table 17.6 in Microsoft Excel](#)

B6			f_x =NPER(B1/B2,B4,B5,,0)
	A	B	C
1	rate	0.06	
2	m	12	
3	n		
4	pmt	-150	
5	PV	\$8,000.00	
6	nper	62.18593	=NPER(rate/m,pmt,PV,,0)

Comparing Interest Paid, Loan Term, and Payment Amounts for Constant Payment Loans

An important relationship exists between the loan's term and total interest paid. To illustrate, consider a \$30,000 loan at 15% APR to be repaid in monthly payments over 30 years. The monthly payment for

this loan is \$379.33. Total interest TI paid on a constant payment loan is found by multiplying the constant loan payment A times the term of the loan mn , minus the amount of the loan L_0 :

$$(17.6) \quad TI = (A)(mn) - L_0 = (\$379.33)(360) - \$30,000 = \$106,560$$

In this case, the total interest paid is \$106,560.

Increasing the payment amount by 10% to \$417.27 reduces the term of the loan by 48% to just over 15.36 years (verify the results above using Excel). Meanwhile, total interest paid is reduced by 56% to \$46,961.

The term reduction in response to an increased loan payment is not always so significant. For example, if the above loan had an 8% APR, the monthly payment would equal \$220.13 instead of \$379.22. Increasing the payment by 10% would decrease the term of the loan by only 27% from 30 years to 21.93 years, and the total interest paid would be decreased by 32% from \$49,247 to \$33,722.

It would be useful to know how changing the term of the loan affects payment size and total interest paid. It can be shown that as mn becomes large, the payment A approaches the interest cost per period, i.e., the smallest payment possible equals the interest charged on the outstanding loan balance. If the borrower wished to minimize his or her payment, the appropriate term is the one that permits the borrower to repay only interest. The shortest repayment period, on the other hand, is one. Obviously, there is a trade-off between the size of the loan payment and the length of the loan.

The point elasticity of term, measured in years n with respect to the payment A, measures the percentage change in the term n in response to a 1% change in the loan payment. The term elasticity, $E_{(n,A)}$, has been calculated as:

$$(17.7) \quad E_{n,A} = \frac{1 - e^{rm}}{rn} < 0$$

For example, a 30-year loan and an APR of 15% would have an elasticity of term equal to:

$$(17.8) \quad E_{n,A} = \frac{1 - e^{(.15)(30)}}{(.15)(30)} = -19.78\% < 0$$

In other words, increasing the payment by 1% would decrease the term by approximately 19.78%. In contrast, the arc elasticity, rather than the point elasticity, compares the percentage change in the loan term to a 10% increase in the loan payment and finds the percentage change in the term to equal 48%, or an arc elasticity of term equal to 4.8%. Note that a point elasticity of 19.78% versus an arc elasticity of 48% is the result of comparing large changes in loan payments of 10% versus comparing tiny changes in loan payments (e.g. .00001%). See Table 17.7 of point elasticities below.

Table 17.7. Point elasticity measures for loans of alternative terms and interest rates.

$n / r \%$	1%	5%	7.5%	10%	15%	20%
1	1.01	1.03	1.04	1.05	1.08	1.11
5	1.08	1.14	1.21	1.30	1.49	1.72
10	1.17	1.30	1.49	1.72	2.32	3.19
15	1.26	1.49	1.85	2.32	3.77	6.36
20	1.37	1.72	2.32	3.19	6.36	13.40
25	1.49	1.99	2.94	4.47	11.07	29.48
30	1.49	2.32	3.77	6.36	19.78	67.07
60	2.99	6.36	19.78	67.07	900.23	13,563.00

Example 17.4. Term and Loan Payment Trade-Offs. Lucy Landlord is financing the renovation of a property. She needs a loan for \$28,000. Her lender offers her a loan for 20 years at the current interest rate of 15%. She calculates her annual payment to be \$4,473.32. If she increases her payment by 1% to \$4,518.05, her term is reduced to 19 years, or a reduction of 5%. This percentage reduction is nearly equal to the tabled value of 6.36 in the Table 17.1, found at the intersection of the row labeled 20 and the column labeled 0.15. Large percentage increases in A, such as 10%, may not be accurately reflected in the table of point elasticities. This is because the percentage changes in n with respect to A are large compared to the very small changes in n with respect to A used to calculate the table.

Creating an Amortization Table for Constant Payment Loans

The word “amortize” originally meant “to kill.” Thus when we amortize a loan, we kill or extinguish it by making regular payments—killing the loan if you will. One feature of the constant payment loan is that, while the loan payment is constant, the amount of the payment devoted to paying off the loan—the principal portion of the payment—and the amount of the payment devoted to paying the interest on the loan are constantly changing. As the loan principal is reduced or killed off, the amount of the payment devoted to interest charges is reduced and the amount of the payment devoted to reducing the loan is increased. Lending institutions, when asked, will provide amortization tables that detail the amount of interest and principal paid on each payment during the life of the loan. Fortunately, Excel provides us the tools needed to create our own amortization tables.

Finding principal portion of the t^{th} loan payment using Excel. The Excel PPMT function can be used to find the principal portion of the t^{th} loan payment. The function is expressed as:

$$(17.9) \quad PPMT(\text{rate}, \text{per}, \text{nper}, PV, , 0)$$

To illustrate the PPMT function, consider a \$25,000 loan to be repaid in monthly installments for four years. The APR for the loan is 5%. We want to know what portion of the 5th payment (period) will be applied to the loan's principal. The Excel solution is represented in Table 17.8.

Table 17.8. Finding the Loan Principal Paid on the tth Payment
[Open Table 17.8 in Microsoft Excel](#)

B6			f_x =PPMT(B1/B2,B4,B2*B3,B5,,0)
	A	B	C
1	rate	0.05	
2	m	12	
3	n	4	
4	period	5	
5	PV	\$25,000	
6	Principal paid	(\$479.47)	=PPMT(rate/m,period,m*n,PV,,0)

It turns out that \$479.47 of the 5th payment is applied to the outstanding loan principal. In addition, students may verify that the principal portion of the 25th payment is \$521.05. This is because the outstanding principal on which interest is charged decreases over the life of the loan. As interest decreases, more of the loan payment can be applied to the outstanding loan principal.

Finding interest payment $IP(t)$ on the t^{th} loan payment using Excel. The Excel IPMT function can be used to find the interest portion of a loan payment in the t^{th} period and is expressed as:

$$(17.10) \quad IPMT(\text{rate}, \text{per}, \text{nper}, PV, , 0)$$

To illustrate the IPMT function, we return to the earlier example: a loan amount of \$25,000, a term of 48 monthly payments, at 5% APR, and a desire to find the interest paid on the 5th loan payment. We illustrate the Excel solution for finding interest paid on the t^{th} period. The solution is entered in cell B8 as \$54.68. Table 17.9 describes the solution in more detail.

Table 17.9. Finding the Loan Interest Paid on the tth Payment
[Open Table 17.9 in Microsoft Excel](#)

B6			f_x =IPMT(B1/B2,B4,B2*B3,B5,,0)
	A	B	C
1	rate	0.05	
2	m	12	
3	n	4	
4	period	5	
5	PV	\$25,000	
6	Interest paid	(\$96.26)	=IPMT(rate/m,period,m*n,PV,,0)

Together, the interest payment of \$96.26 and the principal payment of \$479.47 equals the constant loan payment of \$575.73.

PVs of Special Loans

Loans and credit (one's borrowing capacity) make possible a modern economy and successful firms. Sometimes sellers offer special loan arrangements to encourage the potential buyers to purchase their products. These may include concessionary interest rate loans, skip payment loans, skip principal payment loans, variable interest rate loans, and balloon payment loans. Other times firms may want to expand their investment base and need to refinance their loans, or decreases in interest rates may provide them an incentive to refinance. Fortunately, all of these special loans can be analyzed using PV models developed earlier. We consider the benefits and costs of several special loans in what follows.

Concessionary Interest Rate Loans

A concessionary interest rate loan is one whose interest rate is below the market interest rate. Were the issue to exchange an old loan for a new one at a lower interest rate, there wouldn't really be much of a decision to be made. However, concessionary interest rate loans are often involved in trade-offs. For example, a concessionary interest rate loan may be offered as an incentive to make a long-term purchase—and the borrower wants to determine if the NPV of the loan is recaptured in the investment's higher cost? Or, suppose that the economy has exerted downward pressure on interest rates. But in order to refinance, borrowers must pay a percentage of the loan to refinance. In this instance, borrowers may want to know, what percentage of the loan, points, can they afford to pay and still break-even?

Three steps for evaluating concessionary interest rate loans. To evaluate a concessionary interest rate loan, we follow three steps. First, we find the loan payment of the concessionary interest rate loan. Second, we find the PV of the concessionary loan payments using either the market interest rate or the interest rate on the original loan as the defender's IRR. And third, we find the NPV of the concessionary interest rate loan by subtracting from the loan the PV of the concessionary interest rate loan. We now illustrate how to evaluate concessionary interest rate loans.

Example 17.5. Finding the NPV of a concessionary interest rate loan. Suppose that you want to buy a used car whose price is listed at \$8,000. To encourage you to make the purchase, the car salesperson offers you a concessionary interest rate loan of $i = 1\%$ while the opportunity costs of capital, your IRR, is $r = 8\%$. If the loan is to be repaid over five years of monthly payments, you ask: what is the NPV of the concessionary rate loan? You might also ask: accounting for the PV of the used car loan, what is the actual price of the used car?

To find the value of the seller's concessionary interest rate loan, we follow the three steps outlined earlier. First, using Excel financial spreadsheet, we calculate the loan payment (PMT) of the concessionary interest rate loan and report our results in Table 17.10 below.

Table 17.10 Finding the loan payment for a concessionary interest rate loan for \$8,000 at a concessionary rate of 1%, repaid with five years of monthly payments.
[Open Table 17.10 in Microsoft Excel.](#)

B4	fx		=PMT(B1/B2,B2*B3,B5,,0)
	A	B	C
1	rate	0.01	
2	m	12	
3	n	5	
4	pmt	(\$136.75)	=PMT(rate/m,m*n,PV,,0)
5	PV	\$8,000.00	

Our calculations find that a loan payment of \$136.75 (pmt) is sufficient to repay an \$8,000 loan (PV) at an APR of 1% (rate), with monthly payments (m) over five years (n).

Step two discounts the concessionary interest rate loan payments using the borrower's IRR of 8%/12. The results are summarized in Table 17.11 below.

Table 17.11. Finding the present value of an \$8,000 loan at a concessionary interest rate 8%, for monthly payment of \$136.75 for five year.
[Open Table 17.11 in Microsoft Excel.](#)

B5	fx		=PV(B1/B2,B2*B3,B4,,0)
	A	B	C
1	rate	0.08	
2	m	12	
3	n	5	
4	pmt	(\$136.75)	
5	PV	\$6,744.30	=PV(rate/m,m*n,pmt,,0)

The calculations reported in Table 17.11 find that the present value of the concessionary interest rate loan of \$8,000 is equal to \$6,744.30.

Step three calculates the NPV for the concessionary interest rate loan by subtracting from the original loan of \$8,000 the discounted present value of the concession interest loan equal to \$6,744 and finds:

$$(17.11) \quad NPV = \$8,000 - \$6,744.30 = \$1,255.70$$

Once you know the NPV of the concessionary rate loan, you can negotiate—will the seller lower the price by more than the NPV of the concessionary rate loan if you provide your own financing or pay with cash?

The Refinance Problem

A common financial transaction is the refinancing of a constant payment loan. If the current interest rate is less than the interest rate on one's existing loan, a reasonable borrower would refinance. But what if refinancing requires a fee, a percentage of the new loan, to be paid at the loan closing? We want to know what the borrower can afford to pay as a refinancing cost, or points of the loan, to break even. Finding break-even points is what we do next. We will use as our starting point the numbers introduced into the previous example.

The refinance problem is clearly an NPV problem. Its solution requires that we identify the defending and challenging investments. The defender in this case is the borrower current loan. The challenger is the new loan with its reduced interest rate and points charged to close the loan. What the borrower will earn on the loan is not relevant here because we assume that those earnings will be the same regardless of whether they are financed with the new loan or continue to be financed with the old loan.

Example 17.6. Finding break-even points for a refinanced constant payment loan. To solve a refinance problem with points, we recognize that the breakeven points (p) equate the cost of refinancing the loan with the NPV of the refinanced loan. Let the loan (L_0) times points charged to refinance the loan equal the NPV of the new loan. Setting refinancing costs equal to the NPV of the new loan, using data from Example 17.5 we write:

$$(17.12) \quad pL_0 = NPV(\text{new loan}) = \$1,255.70$$

Then solving for points p as a percentage of the refinanced loan, we divide both sides of equation 17.12 by the loan amount L_0 and find:

$$(17.13) \quad p = \frac{NPV(\text{new loan})}{L_0} = \frac{\$1,255.70}{\$8,000} = 15.7\%$$

In words, equation (17.13) computes the points we can pay to refinance an existing loan at a concessionary interest rate and still break even. In this example, break-even points equal 15.7%.

Moderated Payment Loans

Sometimes, we borrow for increased liquidity. Small to medium size firms often face variability of cash receipts that do not coincide with expenses. This means that a firm's liquidity measures such as the current ratio (CT) or the times-interest-earned (TIE) ratio may also vary significantly over time, creating liquidity needs—that are often addressed with increased borrowings. And sometimes, borrowers may apply for loans because the original loan was written for a much shorter period than the investment's productive life creating a mismatch between the term of financing and producing activities of

the firm. Whatever the case, liquidity concerns may require the firm to renegotiate the terms of their loans with their lenders to solve liquidity needs.

Increasing a loan's term. Consider a moderated payment loan designed to reduce immediate liquidity needs. We use as our example, the loan described in Example 17.2. In this case the required payment was found to equal \$111.22. To improve liquidity, suppose the borrower asks the lender to increase the term of the loan by 12 monthly payments. What would be the new loan payment? Resolving the Excel PMT function returns:

$$(17.14) \quad PMT(rate, nper, PV, , 0) = PMT(.01, 72, 5000, , 0) = \$97.75.$$

Now a loan payment of \$97.75 will retire a loan of \$5,000 with 72 payments rather than 60 payments at an actuarial rate of 1%. The loan payment is reduced from \$111.22 to \$97.75. In this example, the term increased by 12, from 60 months to 72 months or 20% while the payment decreased by $[(\$111.22 - \$97.75) / \$111.22] = 12\%$ and the elasticity of term is $20\% / 12\%$ or 1.67%. Compare this result with those in Table 17.7 at the intersection of column headed by 1% and row labeled 60. Can you explain why there is a difference from the results above and those in Table 17.7? (Hint: can you recall the difference between elasticity measured at a point versus elasticity measure over some interval?)

Skip payments and student loans. Sometimes, we borrow funds to accommodate immediate liquidity needs, expecting that sometime in the future our liquidity will improve and allow us to repay our loans. Such an incentive exists for many students whose incentive for borrowing is to delay making interest and loan payments until after graduating from school.

Loans that delay interest and principal payments, we refer to as skip payment loans. The attractive feature of a skip payment loan is that you don't have to make payments until after graduation. The not so nice feature of this loan is that interest is accumulating over those periods that you are not making payments. The effect of accumulating interest cost is that the principal loan balance is increasing—you are in effect borrowing to pay the interest costs on your loan.

Example 17.7. Borrowing to learn. Suppose you obtain a student loan of \$12,000 to cover your last year of schooling expenses. No payments are required during your last year in school. After graduation you are required to repay the loan plus accumulated interest over 3 years of monthly payments. Assume the market interest rate is 7.5% which is the interest rate charged on your student loan. What is the NPV of this loan. What is the effective rate on the loan? The timeline for this loan is described in Figure 17.1.



Figure 17.1. The payment timeline for a 12-month skip payment loan with three years of payments.

Step 1 for analyzing the skip payment loan is to find the payment A to be made over three years beginning 13 months after the loan was obtained. The amount of the loan to be repaid beginning in month $t = 13$ is $\$12,000(1.075)$, the original amount plus interest charges. This amount must be repaid with 36 payments at an interest rate of $7.5\%/12 = 0.625\%$. What makes the solution to this loan interesting is that we are finding the solution as though we were standing at time period $t = 13$ where the amount to be repaid equals the discounted value of payments at the beginning of of loan month 13 or the end of loan month 12. We describe the solution using an Excel spreadsheet:

Table 17.12. Finding the loan payment on a skip payment loan.
[Open Table 17.12 in Microsoft Excel.](#)

B4			f_x	=PMT(B1/B2,B2*B3,B5,,0)
	A	B	C	
1	rate	0.075		
2	m	12		
3	n	3		
4	pmt	(\$401.27)		=PMT(rate/m,m*n,PV,,0)
5	PV	\$12,900.00		\$12,900 = 1.075 * \$12,000

Our calculations find that to repay a one-year skip payment loan to be repaid over three years of monthly payments at an interest rate of 7.5% is equal to \$401.27. We may be interested in what would have been the loan payment if repaid over four years—without one year of skipped payments? Resolving our Excel payment problem by replacing PV of \$12,000 with \$12,900 and $n = 3$ with $n = 4$, we find:

Table 17.13. Finding the loan payment on the loan without skipping payments.
[Open Table 17.13 in Microsoft Excel.](#)

B4	f _x		=PMT(B1/B2,B2*B3,B5,,0)
	A	B	C
1	rate	0.075	
2	m	12	
3	n	4	
4	pmt	(\$290.15)	=PMT(rate/m,m*n,PV,,0)
5	PV	\$12,000.00	

Skipping one year of payments led to an increase payment from \$290.15 to \$401.27. Was the immediate increase in liquidity equal to the decreased liquidity over the three years of repayment? We can't answer that question, the best financial managers can do is describe the consequences of one's financial choices.

Finally, we ask: what is the difference between the effective interest rate on the old and new loan? There is no difference, None! Can you explain why? For the same reason that we don't change NPVs when we reinvestment and discount at the same interest rate. Acquiring a skip payment loan at the market interest rate may increase future cash flow obligations, but it doesn't change the NPV of the loan.

Disguised Interest Rate Loans

One of the challenges financial managers face when considering borrowing decisions is knowing the actual cost of borrowing—or, stated another way, knowing the effective APR interest rate. Sometimes lenders offer loans that disguise the real cost of their loans. We call loans with disguised interest rates, disguised interest rate loans.

Disguised interest rate loans have effective interest rates than are presented in the loan description. For example, interest costs can be subtracted in the initial period, reducing the actual loan amount received by the borrower (a discount loan). Interest can be charged as though the original loan balance was outstanding throughout the life of the loan (an add-on loan). Alternatively, the lender can charge a loan closing fee, reducing the actual loan balance received by the borrower. Additionally, the interest can compound more frequently than loan payments occur. Each of these methods will increase the effective interest rate above the stated interest rate. Consider several types of disguised interest rate loans.

Discount loan. A borrower approaches his lender for a loan of L_0 for mn periods. The borrower learns that the stated interest rate or disguised interest rate is r^d . When the borrower picks up the check for his loan, the amount he receives equals only:

$$(17.15) \quad L^d = L(1 - r^d n)$$

the amount of the loan requested less the stated interest rate times the term of the loan. Meanwhile, the constant loan payment is calculated as:

$$(17.16) \quad A = \frac{L_0}{mn}$$

To calculate the APR associated with this loan, treat payments of amount A as if they were associated with a constant payment loan that retires a principal of L_0 .

Example 17.8. The effective interest rate for a discount loan. To illustrate the discount loan, assume a consumer obtains an installment loan for \$10,000, from which \$2,500 is deducted for interest costs. The loan is to be repaid over 2 years, with monthly payments equal to \$416.67 = (\$10,000/24). To solve this problem and find the effective interest rate, we key into our Excel RATE function:

$$(17.17) \quad \text{RATE}(\text{nper}, \text{pmt}, \text{pv}, , 0) = \text{RATE}(24, \$416.67, -\$7,5000) = 2.44\%.$$

Displayed is the actuarial monthly rate of 2.44% or, after multiplying by 12, we find the corresponding APR rate of 29.3%. The effective rate r^e is: $r^e = (1.0244)^{12} - 1 = 33.55\%$.

This is quite a difference compared to the stated interest rate of 12.5%. Hence, the discount loan effectively disguises its true APR.

Discounts for on-time payments. Sometimes disguised interest rates can provide opportunities for significant savings. One such opportunity is discounts offered for on-time savings. You may make a purchase on account and be offered a discount for early payment. For example, suppose you are offered a 2% discount if you pay your bill by the 10th day of the month. After 30 days, interest is charged on your account at the rate of 12% per year so by paying early you give up 20 days of free credit. In other words, you are asked to forfeit 20 days of free credit for a discount of 2%.

What is the effective interest rate you would earn by paying your bill by the 10th of the month? The solution requires that we convert the 2% discount to an annual rate by equating 2% is to 20 days as $x\%$ is to 365 days.

$$(17.18) \quad \frac{2\%}{20 \text{ days}} = \frac{x\%}{365 \text{ days}}$$

Then solving for x , we find:

$$(17.19) \quad x = (2\%) \frac{365}{20} = 26.5\%$$

And an interest rate of 36.5% for funds paid on time is really an impressive investment.

Points added loans. Earlier we discussed points added to close a loan when refinancing. Now we discuss points added loans in a different view. We ask: what do points added to a loan do to the effective interest rate? In this case, suppose you are quoted an interest rate of $r\%$ for a loan amount of L_0 to be repaid over mn periods—and then just before you sign, you are informed that there are p points charged to close the loan adding a transaction charge of pL_0 to obtain the loan. To find the effective rate we simply compare the rate without the refinancing cost with the rate associated with the rate.

For this example, we continue with the loan described in Table 17.8 of \$8,000 for monthly payments over 5 years at 1%. The payment for this loan has already been calculated as \$136.75. Now suppose that the lender requires an 8% closing fee to close the loan of $8\% * \$8,000 = \640 . In effect what points required to close a loan means that the actual loan amount available to the borrow is reduced by the closing fee, in this case \$640 so that the actual loan is \$7,460 rather than \$8,000. To find the rate associated with the point added loan we employ the Excel Rate function and find:

$$(17.20) \text{RATE}(nper, pmt, PV) = \text{RATE}(60, \$136.75, \$7,460) = .3175\%$$

Then multiplying the actuarial rate by 12 to find the new APR: $12 * .3175 = 3.8\%$ so that adding points to a loan more than doubled its effective interest rate from 1% to 3.8%.

Future Values

Up until now, we have emphasized the importance of converting future cash flow to their equivalent worth in the present. Hence, the title to this book includes the words, present value. On the other hand, there are times when values in the future deserve our focus. One future focus may be an anticipated event such a future purchase that requires an accumulation of funds and interest. For example, suppose that one anticipates a large capital purchase such as a house or machinery. Or one may be interested in having sufficient funds to generate an annuity at the time of retirement. We represent a future value at the beginning of period n as FV_n .

Example 17.9. Saving for a down payment. Suppose you want to buy a house for \$200,000, but to obtain a loan, you must pay a down payment of 10% or \$20,000. If you intend to save a monthly amount for 3 years which you reinvest at 5%—what is the amount you must save? To answer this question, we need to distinguish between two different kinds of annuities. Annuities due are annuities paid at the beginning of the period (type = 1 in Excel's PMT function). Ordinary annuities are paid at the end of the period (type = 0 in Excel's PMT function). Knowing when payments (annuities) are paid, at the beginning or end of the period, is important when making payments to achieve some future financial goal.

So, we want to find what amount must we save and invest for 36 monthly payments that earn 5% APR such that their compounded accumulations at the beginning of period $n + 1$ will equal the \$20,000 required for a down payment for a future capital purchase? Suppose you make payments at the beginning of n period so that at the beginning of the $n + 1$ period or the end of the n^{th} period you have available a future value of FV_{n+1} equal to:

$$(17.21) \quad FV_{n+1} = A(1+r)^n + A(1+r)^{n-1} + \dots + A(1+r)$$

Table 17.14. Finding annuity due required to save \$20,000 in 36 monthly payments.
[Open Table 17.14 in Microsoft Excel.](#)

B6	fx		=PMT(B1/B2,B2*B3,,20000,1)
	A	B	C
1	rate	0.05	
2	m	12	
3	n	3	
4	nper	36	=(m*n) if type = 1
5	rate/m	0.41%	
6	pmt	(\$513.94)	=PMT(rate/m,m*n,PV=0,FV,type=1)
7	FV	\$20,000.00	

Alternatively, we could find the 36 monthly payments required if the payments were made at the end of each period that compound at an APR of 5% APR so that at the beginning of the n^{th} period the accumulated savings will equal \$20,000.

$$(17.22) \quad FV_n = A(1+r)^{n-1} + A(1+r)^{n-2} + \dots + A(1+r) + A$$

Table 17.15. Finding the annuity with payments made at the end of each period.
[Open Table 17.15 in Microsoft Excel.](#)

B6	fx		=PMT(B1/B2,B2*B3,,20000,0)
	A	B	C
1	rate	0.05	
2	m	12	
3	n	3	
4	nper	35	=(m*n)-1 if type = 0
5	rate/m	0.41%	
6	pmt	(\$516.08)	=PMT(rate/m,m*n,PV=0,FV,type=0)
7	FV	\$20,000.00	

Summary and Conclusions

In this chapter we demonstrated the versatility of PV models by using them to analyze loans. For constant payment loans, we used PV models to solve for constant loan payments, terms, loan amounts, and interest rates—remembering that one PV equation can solve for at most one unknown.

Using PV models to analyze loans required that we identify the various kinds of interest rates. This was an important exercise because we discovered the difference between stated interest rates and effective interest rates—the interest rate actually paid on the loan funds made available.

Another important exercise was discovering the sensitivity of the relationship between the size of the loan payment and the term of the loan. In most cases, the relationship is not one-to-one. In other words, a 1% increase in the size of the loan payment rarely leads to a 1% drop in the term. The corresponding percent decline in the term of the loan is usually much, much more. Hence, we discovered that, when applying for loans, it pays to explore various terms and sizes of loan payments and find the best match—the one with the optimal trade-off between term and liquidity.

A common problem is that existing loans often need to be refinanced. Such may be the case when interest rates drop or a project currently financed is expanded and additional funds are required. In the text we considered refinancing existing loans and found break-even loan closing points. In the supplemental materials at the end of this chapter, we will find the more general formula for refinancing when the term, interest rate, and size of the new loan may be different than on the existing loan.

Lastly, we demonstrated how some loans may disguise the true interest rate. While we only illustrated the solution for the discount loan and the points-added loan, there are several other kinds of loans that disguise the true interest rate.

Questions

1. Which would you prefer to earn on your savings? An APR rate of 12.5% or a 1% actuarial rate compounded monthly? Given an APR of r percent, what is the most that the effective rate can earn above the APR rate if it is compounded continuously?
2. Consider a loan of \$80,000 at an APR of 13%. What is the loan payment that would retire the loan if repaid in monthly payments for 10 years? If repaid in monthly installments for 9 years? Compare the percentage change in the term versus the percentage change in the loan payment (the arc elasticity). Finally, find the point elasticity $E_{(n,A)}$ on the original loan.
3. Assume a loan of \$54,000 with a remaining term of 21 years. The existing loan requires monthly payments at an APR of 11.25%. For a 3% closing fee, the borrowers could refinance their loan at an APR rate of 10% for the same term. What is the effective interest rate on the new loan? What are the break-even points for refinancing the loan? What is the total interest paid on the two loans?
4. A consumer obtains an installment loan of \$12,000 from which \$2,700 is deducted for interest costs. The loan is to be repaid over two years with monthly payments equal to \$500 ($\$12,000/24$). Please determine the effective interest rate r_e on this loan.
5. A farm supply store offers its customers 30 days same-as-cash arrangements. That is, for bills paid within 30 days after purchases are made, no interest is charged. On the other hand, to encourage early payments, the supply store offers a 2% discount on bills paid with 10 days. Please calculate the effective interest rate the store offers its buyers for giving up 20 days of free credit.

6. Suppose you borrowed \$5,000 for 3 years at an APR rate of 8%. Create an amortization table for this loan.
7. Home Depot mailed to some of its customers a coupon entitling them to either a 10% discount on their next purchase or two years of free credit. Under what conditions would you be indifferent between the two options? (Hint: the answer does not depend on the amount purchased.)
8. A farm firm has a mortgage loan for \$150,000 at an APR rate of 5%. The term of the loan is 15 years and the payments equal \$14,451. Cash flow problems from reduced farm income leaves the firm only able to pay \$10,000 on this loan. What would the new term equal if the lender allowed the borrower to repay over a longer term?

18. Land Investments

LINDON ROBISON

Learning goals. After completing this chapter, you should be able to: (1) describe land's unique investment characteristics; (2) understand how land's enduring nature affects its price variability; (3) recognize how transaction costs reduce the liquidity of land investments; and (4) evaluate land investments using present value (PV) models developed earlier.

Learning objectives. To achieve your learning goals, you should complete the following objectives:

- Learn what makes land distinct from other types of investments.
- Learn why land prices are so volatile compared to other investments.
- Learn how to distinguish between real and nominal discount rates.
- Learn how to distinguish between inflationary and real growth in earning rates.
- Learn how expected growth rates in earnings from land are capitalized into land values.
- Learn how to find the real growth rate for land.
- Learn how to calculate the maximum bid (minimum sell) price for land.
- Learn how transaction costs associated with buying and selling land influence land's liquidity.
- Learn how taxes influence the maximum bid (minimum sell) price for land.
- Learn how to find the tax adjustment coefficient for investments in land.
- Learn how to use land price-to-earnings ratios to predict adjustments in the price of land.

Introduction

1

Land's immobility and durability make it unique among investments and deserving of special attention in PV analysis. Land's immobility means that it cannot be moved and that its services must be extracted by those physically on site. Durability means that land has the capacity to provide services over time without significant change in its service provision capacity.

Earlier, an asset's liquidity was defined as its nearness to cash. One dimension of an asset's liquidity depends on the form of its earnings—cash versus capital gains (see Chapter 16). In this chapter we discuss a different dimension of liquidity—the cost of converting an asset to cash through its sale. Land's

1. Some of the material in this chapter was adapted from Robison, L.J. and P.J. Barry (1996); *Present Value Models and Investment Analysis*. "Chapter 14: Transaction Costs and Land Purchase/Sale Decision". MSU Press, East Lansing, Michigan.

immobility makes it less liquid than assets that can be moved because it cannot be moved to meet the convenience of the buyer. Land's immobility also limits the potential buyers to those near enough to extract the land's services. Another reason that land is illiquid is because buyers and sellers pay fees to complete its purchase and sale, including Realtor fees, legal costs of changing and recording its title, and other related fees—but not to each other. Evidence of farmland's low liquidity is its infrequent transfer. On average, only 2% to 3% of the privately owned farmland in the United States is sold each year.

On the other hand, lenders prefer land as collateral for loans for the same reason that makes land illiquid—its immobility. Land's immobility reduces the riskiness of it being stolen, hidden, or moved. Lenders also prefer land as collateral for loans because of its durability, which reduces the riskiness of it losing its value as security for loans. To offset some of land's illiquidity, special institutions and programs have developed for financing residential and farm real estate.

The immobility and durability of land also make it a popular object on which to assess taxes. Taxing agencies have easy and indisputable records of the amount of land subject to tax and who owns the land. Thus, most land-owners pay property taxes on land and buildings but pay no similar tax on more mobile and less durable investments. Consequently, land is one of the few investments for which taxes are based on the market value of the investment at the beginning of the period as well as on the earnings of the investment during the period.

This chapter develops PV models to estimate maximum bid (minimum sell) prices for land investments, to describe how transaction costs contribute to land's illiquidity, to understand how land's durability increases its price volatility, and to demonstrate the influence of taxes on maximum bid (minimum sell) prices. We begin by describing the connection between land's durability and its price volatility and why land and other durables are subject to inflationary bubbles and crashes.

Why are land prices so volatile?

Durable price volatility. One interesting feature of farmland and other durables prices such as housing stock, is their historically high swings in values. Figure 18.1 describes year-to-year changes in Michigan farmland values over the 2000 to 2010 period. Figure 18.2 describes changes year-to-year changes in housing values over the same period. Part of these changes can be attributed to capital gains (losses). But there is another explanation for changes in the value of land and other durables, and it has to do with changes in real interest rates and real growth rates.

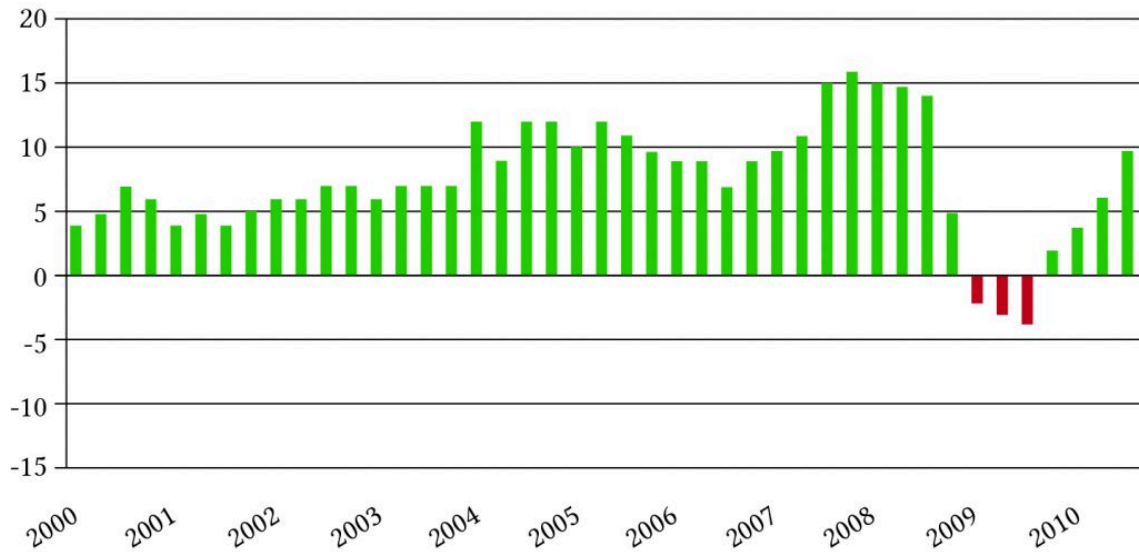


Figure 18.1. Year-to-year changes in farmland values.



Figure 18.2. Year-to-year changes in housing prices.

Inflationary, nominal, and real interest rates. To understand price volatility of durables, it is necessary to describe inflationary, nominal, and real interest rates. Recall that the inflation rate i is equal to the rate of change in average prices, changes often linked to monetary or fiscal policies of governments. The nominal interest rate r depends on the rate of inflation and a real component that is dependent on

factors other than the rate of inflation such as changing market conditions or changes in productivity. To describe the effects of inflation on the nominal interest, let one plus the nominal interest rate r equal one plus the real rate r^* times one plus the inflation rate i so that:

$$(18.1) \quad (1 + r) = (1 + r^*)(1 + i)$$

We solve for the real interest rate in equation (18.1) as:

$$(18.2) \quad r^* = \frac{r - i}{(1 + i)}$$

Historically, real interest rates vary by sectors but are highly correlated. Real interest rates on home loans are described graphically in Figure 18.3:

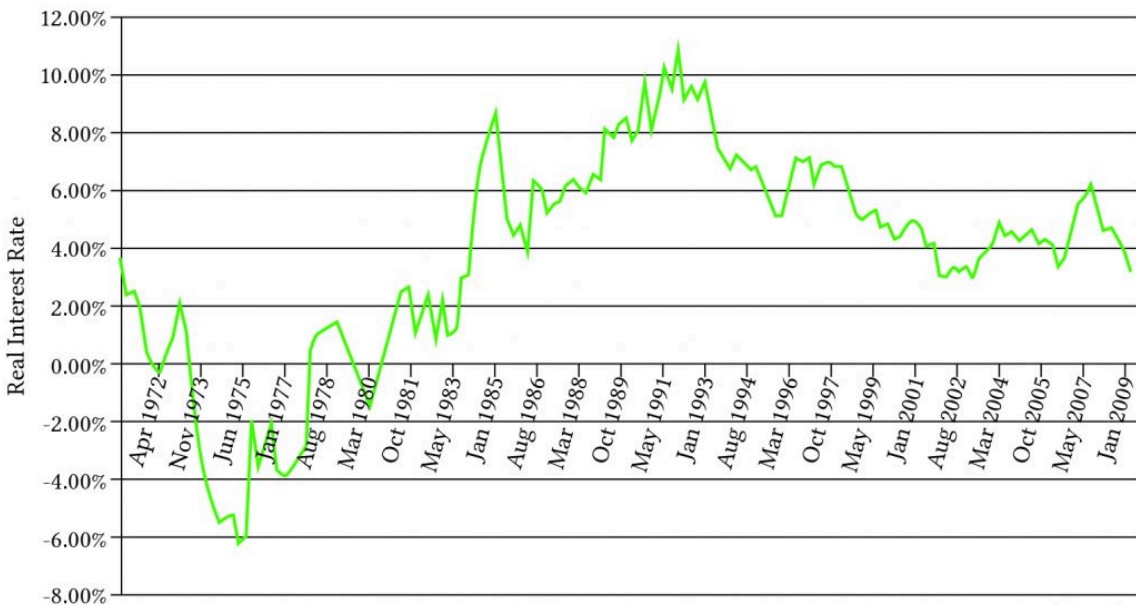


Figure 18.3. Real home loan interest rates.

We can describe the effects of changes in the real growth rate on durable asset prices by letting the nominal growth rate g equal one plus the real growth rate g^* times one plus the inflation rate i , so that:

$$(18.3) \quad (1 + g) = (1 + g^*)(1 + i)$$

We solve for the real growth rate in equation (18.3) as:

$$(18.4) \quad g^* = \frac{g - i}{(1 + i)}$$

To understand the volatility of farmland prices, we build a general PV model where V_0 is land's present value, g is the average nominal growth (decay) rate of net cash flow, r is the nominal interest rate, and R_0 represents initial cash flow. For the moment we ignore taxes and write the relationship between the variables just described in a growth model as:

$$(18.5) \quad V_0 = \frac{R_0(1+g)}{(1+r)} + \frac{R_0(1+g)^2}{(1+r)^2} + \frac{R_0(1+g)^3}{(1+r)^3} + \dots = \frac{R_0(1+g)}{(r-g)}$$

We call the result the value in use model because there are no sales or purchases implied. Suppose we replace $(1+r)$ with the right-hand side of equation (18.1) and $(1+g)$ with the right-hand side of equation (18.3). After the substitution, we obtain the result:

$$(18.6) \quad V_0 = \frac{R_0(1+g^*)(1+i)}{(1+r^*)(1+i)} + \frac{R_0[(1+g^*)(1+i)]^2}{[(1+r^*)(1+i)]^2} + \frac{R_0[(1+g^*)(1+i)]^3}{[(1+r^*)(1+i)]^3} + \dots$$

Notice that the inflationary effects on the discount rate and the growth (decay) rate cancel, so that we can write equation (18.6) as a function of only real rates and initial cash flow:

$$(18.7) \quad V_0 = \frac{R_0(1+g^*)}{(1+r^*)} + \frac{R_0(1+g^*)^2}{(1+r^*)^2} + \frac{R_0(1+g^*)^3}{(1+r^*)^3} + \dots = \frac{R_0(1+g^*)}{(r^*-g^*)}$$

Suppose that in equation (18.7), r^* changes. To measure the impact of a change in r^* , we differentiate V_0 with respect to r^* and obtain:

$$(18.8) \quad \frac{dV_0}{dr^*} = \frac{-R_0(1+g^*)}{(r^*-g^*)^2}$$

The percentage change in V_0 , with respect to a change in r^* , equals the change in the asset's value in response to a change in the real interest rate, equation (18.8), divided by the asset's initial value V_0 , the right-hand side of equation (18.7):

$$(18.9) \quad \frac{dV_0/dr^*}{V_0} = \frac{-1}{(r^*-g^*)}$$

Now suppose that g^* changes. To measure the impact of a change in g^* we differentiate V_0 with respect to g^* in equation (18.7) and obtain:

$$(18.10) \quad \frac{dV_0}{dg^*} = \frac{R_0(1 + g^*)}{(r^* - g^*)^2}$$

The percentage change in V_0 with respect to a change in g^* equals the change in the asset's value in response to a change in the real interest rate, equation (18.10) divided by the asset's initial value V_0 described in equation (18.7).

The percentage change in V_0 with respect to a change in g^* equals:

$$(18.11) \quad \frac{dV_0/dg^*}{V_0} = \frac{1}{(r^* - g^*)}$$

We now describe the percentage changes in asset values in response to changes in the real discount rate and the real growth rate for alternative real interest rates and real capital gains (capital loss) rates. In the previous chapter we described changes in asset values over time. This analysis is different (and static or timeless). We consider what happens to an asset's value at a point in time if one of the underlying variables that determines its value is changed.

In Table 18.1 calculations are based on equation (18.11). In the calculations we set the percentage change in asset values for alternative values of r^* and g^* . We set r^* equal to 2%, 4%, 6%, 8%, and 10%. We set g^* equal to -20%, -10%, -5%, 0%, 2% and 4%. Negative values for g^* correspond to depreciable durable assets such as buildings and machinery. Positive values of g^* are harder to justify and may reflect speculative bubbles in which asset earnings are expected to grow in real terms in perpetuity—an assumption which is hard to justify.

Table 18.1. Percentage changes in asset values corresponding to small changes in real interest rates for alternative values of r^* and for assets with various degrees of durability characterized by the variable g^* .

	$g^* = -20\%$	$g^* = -10\%$	$g^* = -5\%$	$g^* = 0\%$	$g^* = 2\%$	$g^* = 4\%$
$r^* = 2\%$	-4.5%	-8.3%	-14.3%	-50%	n.a.	n.a.
$r^* = 4\%$	-4.1%	-7.1%	-11.1%	-25%	-50%	n.a.
$r^* = 6\%$	-3.8%	-6.3%	-9.1%	-16.7%	-25%	-50%
$r^* = 8\%$	-3.6%	-5.6%	-7.7%	-12.5%	-16.7%	-25%
$r^* = 10\%$	-3.3%	-5%	-6.7%	-10%	-12.5%	-16.7%

Consider Table 18.1. Note that an asset's durability is reflected by its real growth (decay) rate—an asset's durability decreases with its g^* value. Also note that the percentage change in asset values in response to increases in real interest rates decreases with an asset's durability. For example, if the real interest rate were 4%, a small increase in the real interest rate would cause a 4.1% decrease in the value of a depreciable asset with a negative real growth rate of -20%. Meanwhile, a small increase in the real interest rate for a durable asset with a 4% real growth rate would result in a 50% decrease in the value of an asset.

Now consider Table 18.2, which examines the effect of small increases in the real growth rate for the same scenarios described in Table 18.1 and also based on equation (18.11). The difference between Table 18.1 and Table 18.2 is that the signs within the body of the tables are reversed. An increase in the real interest rate reduces asset values because future dollars are worth less with a higher discount rate. In contrast, an increase in the real growth rates means there are more future cash flows to discount—making the change in the value of the assets positive in response to a small increase in the real growth rate.

Table 18.2. Percentage changes in asset values corresponding to 1% changes in real growth rate for alternative values of r^* and for assets with various degrees of durability characterized by the variable g^* .

	$g^* = -20\%$	$g^* = -10\%$	$g^* = -5\%$	$g^* = 0\%$	$g^* = 2\%$	$g^* = 4\%$
$r^* = 2\%$	4.5%	8.3%	14.3%	50%	n.a.	n.a.
$r^* = 4\%$	4.1%	7.1%	11.1%	25%	50%	n.a.
$r^* = 6\%$	3.8%	6.3%	9.1%	16.7%	25%	50%
$r^* = 8\%$	3.6%	5.6%	7.7%	12.5%	16.7%	25%
$r^* = 10\%$	3.3%	5%	6.7	10%	12.5%	16.7%

One interesting feature of Table 18.2 is that the percentage change in asset values in response to small increases in the real growth rate are dampened by higher real interest rates. For that reason, when the economy is “overheated” and prices are rising, one response is to reduce the money supply, increasing the real interest rate and slowing price increases. For example, suppose the real growth rate were zero% and the real interest rate was 2%. If the real growth rate increased slightly, the asset value would increase by 50%. If, instead, the real interest rate were 10% and the real growth rate increased slightly, the asset value would increase by only 10%.

So, what have we learned? We learned that durable assets such as land and houses are subject to significant percentage changes in value in response to changes in real interest rates and real growth rates—much more so than depreciable assets. In addition, changes in the asset’s real growth rate in earnings have less effect when real interest rates are high.

To test your understanding of the material just covered, demonstrate that a 1% increase in the real growth rate produces the same percentage changes in asset values as those described in Table 18.1, except opposite in sign.

Maximum Bid (Minimum Sell) Price Models and Transaction Costs

Maximum bid (minimum sell) price models. We now deduce maximum bid price models for buyers and minimum sell price models for sellers. At least two costs differentiate maximum bid price models for buyers and minimum sell price models for sellers: taxes and transactions costs. Let c equal the percentage of land's sale price paid by the buyer to purchase the land—but not paid to the seller. These may include loan closing fees, title examination and registration fees, attorney's costs, and closing points paid by the buyer. Let s equal the percentage of the land's sale price paid by the seller to sell the land—but not paid to the buyer. This may include loan closing fees, Realtor fees, advertising, and other fees.

Denote V_0^b as the buyer's maximum bid price and V_0^s as the seller's minimum sell price. Ignoring taxes, and assuming that the land will never be sold again, the expression for V_0^b can be written as:

(18.12)

$$(1 + c)V_0^b = \frac{R_0(1 + g)}{(1 + r)} + \frac{R_0(1 + g)^2}{(1 + r)^2} + \frac{R_0(1 + g)^3}{(1 + r)^3} + \dots = \frac{R_0(1 + g)}{(r - g)}$$

Expression $(1 + c)V_0^b$ represents the maximum bid price, including the effect of closing costs. To isolate the maximum bid price, we solve for V_0^b .

$$(18.13) \quad V_0^b = \frac{R_0(1 + g)}{(r - g)(1 + c)}$$

In the maximum bid price model, the discounted stream of future cash flow represents the challenging investment—the present value of earnings the buyer will receive if the land is purchased.

Example 18.1: Finding the Maximum Bid Price for Land. What is the maximum bid price of a parcel of land if the buyer pays 6% of the acquisition price as a closing fee? To solve the problem, assume last year's net cash flow was \$22,000, which is expected to grow in perpetuity at 4.5%. Assume that the buyer's opportunity cost of capital is 12%. The maximum bid price is calculated using equation (18.13):

$$(18.14) \quad V_0^b = \frac{R_0(1 + g)}{(r - g)(1 + c)} = \frac{\$22,000(1.045)}{(.12 - .045)(1.06)} = \$289,182$$

If closing costs, as a percentage of the price, increase from 6% to 9%, V_0^b decreases to \$281,223.

Minimum sell price models. Realtor fees and other closing costs paid by the seller are now introduced into the minimum sell price model. To build the minimum sell price model, let sV_0^s be subtracted from the seller's proceeds. Ignoring taxes and assuming that the land will only be sold one time, we can write the minimum sell price model as:

(18.15)

$$(1 - s)V_0^s = \frac{R_0(1 + g)}{(1 + r)} + \frac{R_0(1 + g)^2}{(1 + r)^2} + \frac{R_0(1 + g)^3}{(1 + r)^3} + \dots = \frac{R_0(1 + g)}{(r - g)}$$

Solving for V_0^s , we find the minimum sell price equal to:

$$(18.16) \quad V_0^s = \frac{R_0(1 + g)}{(r - g)(1 - s)}$$

In the minimum sell price model, the discounted stream of future cash flow represents the defending investment—it represents the earnings the seller would not earn if the land were sold.

Example 18.2: Finding the Minimum Sell Price. Letting $s = 2\%$ and using the data from Example 18.1, we find the minimum sell price using equation (18.13):

$$(18.17) \quad V_0^s = \frac{R_0(1 + g)}{(r - g)(1 - s)} = \frac{\$22,000(1.045)}{(.12 - .045)(.98)} = \$312,789$$

Now, suppose s increases from 2% to 3%. In turn, V_0^s increases to \$316,014. In contrast to the effects of an increase in c that reduced the maximum bid price, an increase in s increases the minimum sell price:

Land Values, Transaction Costs, and Liquidity

Suppose a buyer and a seller had the same earnings expectations and opportunity costs for a durable such as farmland. It must follow that their present-value for the durable in use would be the same. But if the buyer had to pay closing fees to acquire the durable, his maximum bid price would be reduced. And if the seller received only a portion of the present value after the sale, her minimum sell price would increase. Therefore, transaction costs that increase the seller's minimum sell price and reduce the buyer's maximum bid price reduce the liquidity of the durable, even when the buyer and seller have the same earnings expectations and opportunity costs.

In other words, an asset is illiquid if its value in use (V_0) is bounded below by a buyer's maximum bid price (V_0^b) and bounded above by the seller's minimum sell price (V_0^s). We can easily show that assets for individuals with identical expectations of land's value in use are illiquid when transaction costs are imposed on its sale and purchase.

Substitute the value in use from equation (18.5) into the maximum bid price model in equation (18.13), and the result is:

$$(18.18) \quad V_0^b = \frac{R_0(1+g)}{(r-g)(1+c)} = \frac{V_0}{(1+c)} < V_0$$

It should be clear that for $c > 0$, $V_0^b < V_0$.

Next, substitute the value in use from equation (18.5) into the minimum sell price model in equation (18.16) and the result is:

$$(18.19) \quad V_0^s = \frac{R_0(1+g)}{(r-g)(1-s)} = \frac{V_0}{(1-s)} > V_0$$

It should be clear that for $s > 0$, $V_0 < V_0^s$. Finally combining these results allows us to express the effects of transaction costs on the maximum bid, minimum sell, and value-in-use models:

$$(18.20) \quad V_0^b < V_0 < V_0^s$$

Despite equal earning expectations, as a result of transaction costs, the maximum bid price is less than the minimum sell price making land illiquid. To make this point another way, suppose that we formed the ratio of the maximum bid price and the minimum sell price for land and assumed that $c = 5\%$ and $s = 6\%$. Then the buyer in this example would only be willing to offer 89.2% of the seller's minimum sell price, and a sale would not occur.

$$(18.21) \quad \frac{V_0^b}{V_0^s} = \frac{(1-s)}{(1+c)} = \frac{1-.06}{1+.05} = 89.2\%$$

Other Factors Contributing to Asset Liquidity

Differences in financing costs for buyers and sellers may also contribute to an asset's illiquidity. For example, a loan tied to the ownership of durables at below-market interest rates may create a wedge between the maximum bid price and the minimum sell price. If the loan has a due-on-sale clause, the

seller gives up something of value that is not received by the buyer when the investment is sold. Thus, concessionary interest rate loans with due-on-sale clauses may create illiquidity.

Differences in information between the buyer and seller may also contribute to an asset's illiquidity. When two trusted friends exchange a durable, the information costs are insignificant—they freely exchange information. However, when two strangers exchange a durable, the buyer must commit resources to discover the quality of the information provided by the seller. If an investment's service capacity is easily observed, as with certified seed or fertilizer, then the information cost is small and the investment will be more liquid. Durables like used cars or tractors, whose remaining service capacity is not easily observed, will be less liquid. The issuance of a warranty is a signal of higher quality, and thus a warranty is an attempt to reduce information costs for the prospective buyer and to increase the liquidity of the asset.

The cost of gathering information about the asset's service capacity may also contribute to its liquidity. To illustrate, a new asset's service capacity may be well known. Once the durable is used, however, the uncertainty associated with the intensity of use and care of the durable creates an immediate decline in the value of the durable. On the other hand, a used durable whose service extraction is nearly depleted is more liquid, because there is nearly perfect information about the durable's remaining capacity.

The liquidity of durables is also affected by disposal costs. Disposal cost is the cost of removing durables from service paid by its current owner. Most durables are mobile. Buildings, however, must be disassembled at considerable costs. In some cases, the investment is so valuable that assembly and disassembly are warranted. An example is the London Bridge. Most durables with high disposal costs become completely illiquid and remain in service to the first owner until destroyed. Often it is easier to transfer the owner rather than the investment itself. Examples of durables with high disposal costs are land and houses.

Durables whose value increases with time and use are called appreciating durables. Durables whose values decrease with time and use are called depreciating durables. A durable whose value increases with time and use will be more liquid than one whose value declines with time and use, with all other liquidity attributes held constant. Appreciating durables provide greater security for lenders than depreciating durables because their value and the lender's security are increasing. Finally, liquidity is affected by whether or not time or use determines the durables' service capacity. Durables whose service capacity is determined by time have a fixed pattern of changes in service capacity and cannot adjust to changes in the market. Durables whose change in service capacity is determined by use can adjust to changes in the marketplace for the value of its services. Thus, durables whose service capacity is tied to use are more liquid than those whose service capacity is linked to time.

The liquidity of a durable is also influenced by the number of services the durable can provide. Durables become more liquid if the buyer has opportunities for the durables' services that are not available to the seller. A durable is more likely to have multiple uses when it is not fixed geographically. Multiple use capability is also tied to the acquisition characteristics of the durable. If the durable is lumpy in acquisition, potential buyers must take it or leave it. Durables divisible in acquisition, on the

other hand, provide options for prospective buyers. They can buy a gallon or a tankful of gas, a sack of seeds or a truckload.

Finally, durables whose service extraction rate is fixed, and hence irreversible, are less liquid than durables whose service extraction rate is variable. Variations in service capacity provide greater adjustment potentials, and thus reduce the differences in perceptions between buyers and sellers. Moreover, durables whose service extraction rates vary tend to have multiple uses, while those with fixed service extraction rates tend to be single-use durables.

Maximum Bid (Minimum Sell) Land Price Models with Taxes

Taxes are now introduced into our maximum bid (minimum sell) price for land models. As mentioned earlier, land is subject to a wide variety of taxes, not all of which are imposed on other assets. Four kinds of taxes must be considered in our maximum bid (minimum sell) price for land models: First, the cash flow stream is taxed at the average income tax rate T . Second, the opportunity cost of capital r is expressed on an after-tax basis. Since the discount rate may reflect the opportunity cost of capital associated with investments that earn both capital gains and cash, the tax rate is θT . Third, property taxes (T_p) are another tax charged against the market price of land in each period. Thus, land's maximum bid price in future periods must be included in each period. The study of capital gains indicated that land prices should increase at the same rate as land earnings. Finally, one tax not included in the land price models is tax-savings rate T_g , resulting from durable depreciation. The reason is simple—land is not considered a depreciable investment.

The Maximum Bid Price Model with Taxes. The expanded maximum bid price model with taxes is written as:

(18.22)

$$(1 + c)V_0^b = \frac{R_0(1 + g)(1 - T) - V_0^b T_p(1 - T)}{[1 + r(1 - \theta T)]} + \frac{R_0(1 + g)^2(1 - T) - V_0^b(1 + g)T_p(1 - T)}{[1 + r(1 - \theta T)]^2} + \dots = S_1 + S_2$$

The cash flow stream contains two geometric series. In both series, the geometric factor is $(1 + g)/[1 + r(1 - \theta T)]$ and the sums of the two series are equal to:

$$\begin{aligned}
 S_1 &= \frac{R_0(1+g)(1-T)}{[1+r(1-\theta T)]} + \frac{R_0(1+g)^2(1-T)}{[1+r(1-\theta T)]^2} + \dots \\
 &= \frac{R_0(1+g)(1-T)}{r(1-\theta T) - g}
 \end{aligned}
 \tag{18.23}$$

while

$$\begin{aligned}
 S_2 &= -\frac{V_0^b T_p (1-T)}{[1+r(1-\theta T)]} - \frac{V_0^b (1+g) T_p (1-T)}{[1+r(1-\theta T)]^2} - \dots \\
 &= -\frac{V_0^b T_p (1-T)}{r(1-\theta T) - g}
 \end{aligned}
 \tag{18.24}$$

Finally, setting $(1+c)V_0^b$ equal to the sum of S_1 and S_2 and solving for V_0^b , the maximum bid price with taxes of land can be written as:

$$V_0^b = \frac{R_0(1+g)(1-T)}{[r(1-\theta T) - g](1+c) + T_p(1-T)}
 \tag{18.25}$$

It is still the case that the challenging investment in equation (18.16) is represented by the discounted after-tax cash flow stream while the defender is the maximum bid price plus closing fees, which earns the discount rate r given up to acquire the income earning durable.

Example 18.3: Maximum Bid Price with Taxes. An investor is interested in buying a crop farm. According to the financial statements, last year the farm generated a net cash receipts of \$48,000. Past data indicated that net cash returns have been growing at 6% per year. The investor's opportunity cost of capital is 11%, and closing fees for the transaction are 3%. The investor's income tax bracket is 30%, the property tax is 2%, and we assume that $\theta = .75$. Using equation (18.25), the maximum bid price for the farm is

$$\begin{aligned}
 V_0^b &= \frac{\$48,000(1.06)(1-.3)}{\{.11[1-(.75)(.3)] - .06\}(1+.03) + .02(1-.3)} = \frac{\$35,616}{.026 + .014} \\
 &= \$890,400
 \end{aligned}
 \tag{18.26}$$

The Value in Use model. It will be convenient later on to find the value of what the seller will sacrifice instead of selling his or her land and continuing to extract services from it. In this case there are no transaction costs, and property taxes are associated with its use value as opposed to its maximum bid

price. We can derive the value in use with taxes model, V_0^u , by setting $c = 0$ in equation (18.25). The result is:

$$(18.27) \quad V_0^u = \frac{R_0(1+g)(1-T)}{[r(1-\theta T) - g] + T_p(1-T)}$$

To illustrate, the value in use model for the previous example is:

$$(18.28) \quad V_0^u = \frac{\$48,000(1.06)(1-.3)}{\{.11[1 - (.75)(.3)] - .06\} + .02(1-.3)} = \$913,230$$

One way to interpret the value in use with taxes model is that it represents the defending asset for the seller of land—the present value of future earnings it would receive, provided the land isn't sold.

The Minimum Sell Price Model with Taxes. The minimum sell price model with taxes is similar to the maximum bid price model with taxes, with a few exceptions: First, assume the seller originally bought his or her land t periods earlier at a price of V_{-t}^b . Then, if V_{-t}^b is less than (more than) the current selling price of V_0^s , the seller pays (earns) capital gains taxes (credits) at a rate of α times the average income tax rate of $\alpha T\%$ or αT .

Another difference is the transaction costs. These costs affect taxable capital gains because the capital gains tax is only paid on the net gain from the sale of land. Like the maximum bid price model, however, the property tax rate is assessed against the inflating value in use since there is no benchmark sell price to calibrate its tax value.

The value in use with taxes is the challenging investment for the seller. We equate the after-tax sale price to the value in use with taxes in equation (18.29). We summarize these results in equation (18.29):

$$(18.29) \quad \begin{aligned} (1-s)V_0^s - [(1-s)V_0^s - (1+c)V_{-t}^b]\alpha T \\ = V_0^u = \frac{R_0(1+g)(1-T)}{[r(1-\theta T) - g] + T_p(1-T)} \end{aligned}$$

Solving for V_0^s the minimum sell price with taxes we find:

$$(18.30) \quad V_0^s = \frac{V_0^u - (1+c)V_{-t}^b\alpha T}{(1-s)(1-\alpha T)}$$

Example 18.4: Minimum Sell Price with Taxes. ABC Corporation wants to sell a large farm. They want to know their minimum sell price to accept—or the price a buyer would need to offer in order for them to

be as well off selling as they would be continuing to farm the land. ABC has already calculated the farm-land's value in use is equal to \$913,230 (see equation 18.27). This is the present value of their defender.

Recognizing its minimum sell price adjusted for transaction costs and capital gains taxes (credits) as its challenger, ABC equates the two, and solves for its minimum sell price with taxes. The result is expressed in equation (18.30). Summarizing the information needed to find ABC's minimum sell price: $V_0^u = \$913,230$, $c = 3\%$, $V_{-t}^b = \$600,000$, $\alpha = .4$, $T = 30\%$, and $s = 5\%$.

The result is:

$$(18.31) \quad V_0^s = \frac{V_0^u - (1 + c)V_{-t}^b\alpha T}{(1 - s)(1 - \alpha T)} = \frac{\$913,230 - (1.03)(\$600,000)(.4)(.3)}{(.95)[1 - (.4)(.3)]} = \$1,003,672$$

Clearly, the buyer and seller in Examples 18.3 and 18.4 would never exchange land—the minimum sell price of \$1,003,672 exceeds the buyer's maximum bid price of \$890,400. What would have to change for land to become liquid—for the buyer to offer the seller at least his or her minimum sell price? Well, for one thing, the buyer would have to expect initial earnings greater than \$48,000. Or the buyer could find means of reducing its property tax rate. Or the buyer could reduce its transaction costs. Or the buyer may have a lower opportunity cost than the seller. Other changes, such as changes in g and T , have ambiguous effects on the maximum bid price V_0^b .

Finding the Tax Adjustment Coefficient θ

The homogeneity of measures principle requires that after-tax cash flow be discounted by an after-tax discount rate—in most cases the IRR of the defender. The question is, how do we find the after-tax IRR that corresponds to the after-tax cash flow of the defender? First, we specify the defender's cash flow stream and PV and solve for the discount rate or the IRR of the investment. We are guaranteed that we have found the IRR of the defender in a maximum bid (minimum sell) price model and value in use model because the NPV in these models is always zero. We demonstrate how to find the defender's IRR first for the simple geometric growth model and then for more complicated models that include taxes.

Now we introduce taxes into equation (18.5), holding V_0 constant, and express the after-tax geometric growth model as:

(18.32)

$$\begin{aligned}
 V_0 &= \frac{R_0(1+g)(1-T)}{[1+r(1-\theta T)]} + \frac{R_0(1+g)^2(1-T)}{[1+r(1-\theta T)]^2} + \frac{R_0(1+g)^3(1-T)}{[1+r(1-\theta T)]^3} + \dots \\
 &= \frac{R_0(1+g)(1-T)}{r(1-\theta T) - g}
 \end{aligned}$$

To summarize, because equation (18.32) describes a defender's after-tax cash flow stream, then $r(1 - \theta T)$ is the defender's after-tax IRR—because NPV in equation (18.32) is zero. Another way to summarize the results above is to note that taxes in equation (18.32) are neutral—meaning that they are introduced so that NPV remains zero. Since V_0 is the same in both equations (18.5) and (18.32), we can equate their right-hand sides and write:

$$(18.33) \quad V_0 = \frac{R_0(1+g)(1-T)}{r(1-\theta T) - g} = \frac{R_0(1+g)}{(r-g)}$$

Up to this point, we have not specified the value of θ . However, we do require that its value be chosen in such a way that $r(1 - \theta T)$ is the after-tax IRR of the defender's after tax cash described using a geometric growth model. We find such a θ for the geometric cash flow model using equation (18.32), which equals:

$$(18.34) \quad \theta = \frac{r-g}{r}$$

The above results are intuitive. For $g \neq 0$ in equation (18.34), capital gains (losses) are earned (lost) without creating any tax consequences. If the before-tax IRR is r , an after-tax rate of $r(1 - T)$ implies cash and capital gains (losses) are taxed at the average income tax rate T , which is not true because only cash receipts are taxed. Since capital gains (losses) are earned (lost) at rate g , subtracting from $r\%$, $g\%$, makes taxes neutral in the geometric growth model described above. To confirm that θ coefficient makes taxes neutral, the right-hand side of equation (18.34) is substituted into equation (18.33) to obtain the result:

$$(18.35) \quad V_0 = \frac{R_0(1+g)(1-T)}{\left\{ r \left[1 - \left(\frac{r-g}{r} \right) T \right] - g \right\}} = \frac{R_0(1+g)}{(r-g)}$$

Following a similar procedure, we can find the value for θ in the value in use model for land. In this case, we equate the right-hand side of the value in use land model to the right hand side of the geometric growth model. This new equality can be written as:

$$(18.36) \quad V_0^u = \frac{R_0(1+g)(1-T)}{[r(1-\theta T) - g] + T_p(1-T)} = \frac{R_0(1+g)}{r-g}$$

As before, θ in equation (18.36) is the coefficient that makes taxes neutral in the land value in use model because it ensures that the effects of taxes on the defender's cash flow are equally offset by changes in the discount rate. Therefore, the discount rate continues to equal the defender's IRR. We can find such a θ for a value in use land model by solving for θ in equation (18.36):

$$(18.37) \quad \theta = \frac{r-g + (T_p/T)(1-T)}{r}$$

Clearly, property taxes increase the tax adjustment coefficient and the effective tax rate.

Non-neutral taxes. When we calculated the tax-adjustment coefficient, we assumed tax-neutrality. This was an appropriate assumption for finding the after-tax IRR of the defender. In other PV models, there is no reason to assume that the after-tax IRR of the defender used to calculate the maximum bid (minimum sell) price or NPV will be tax-neutral. For example, assume the challenger's cash flow is represented by a geometric growth equation and assume that the tax adjustment coefficient of the defender is one—an assumption consistent with a defender's whose returns are only cash. In this case, we find the value in use as:

$$(18.38) \quad \begin{aligned} V_0 &= \frac{R_0(1+g)(1-T)}{[1+r(1-T)]} + \frac{R_0(1+g)^2(1-T)}{[1+r(1-T)]^2} + \dots \\ &= \frac{R_0(1+g)(1-T)}{r(1-T) - g} \end{aligned}$$

To test for tax-neutrality, we differentiate V_0 with respect to T and find:

$$(18.39) \quad \frac{dV_0}{dT} = \frac{gR_0(1+g)}{[r(1-T) - g]^2} > 0$$

In this particular case, taxes are not neutral, and increasing the tax rate makes the defender less attractive relative to the challenger because the challenger's capital gains are shielded from income taxes.

Price-to-earnings Ratios for Land

An interesting ratio can be derived from the value in use with taxes model, equation (18.32). It is the ratio of land's value in use, divided by the previous period's cash flow, approximated by cash rents. We write the ratio as:

$$(18.40) \quad \frac{V_0}{R_0} = \frac{(1 + g)(1 - T)}{[r(1 - T) - g] + T_p(1 - T)}$$

Note that the value-to-earnings ratio for land depends on specific tax rates, opportunity costs, and expected growth rates, and these vary from location to location. Figure 18.4 below describes land values to rent ratios for land in Iowa, Illinois, and Indiana since 1967. Suppose that in equation (18.40) we were to ignore taxes and assume a real growth rate of zero. Then equation (18.40) reduces to the traditional capitalization ratio of:

$$(18.41) \quad \frac{V_0}{R_0} = \frac{1}{r^*}$$

This is how the traditional capitalization ratio is derived. As an earlier graph demonstrated, however, the ratio has not been constant over time.

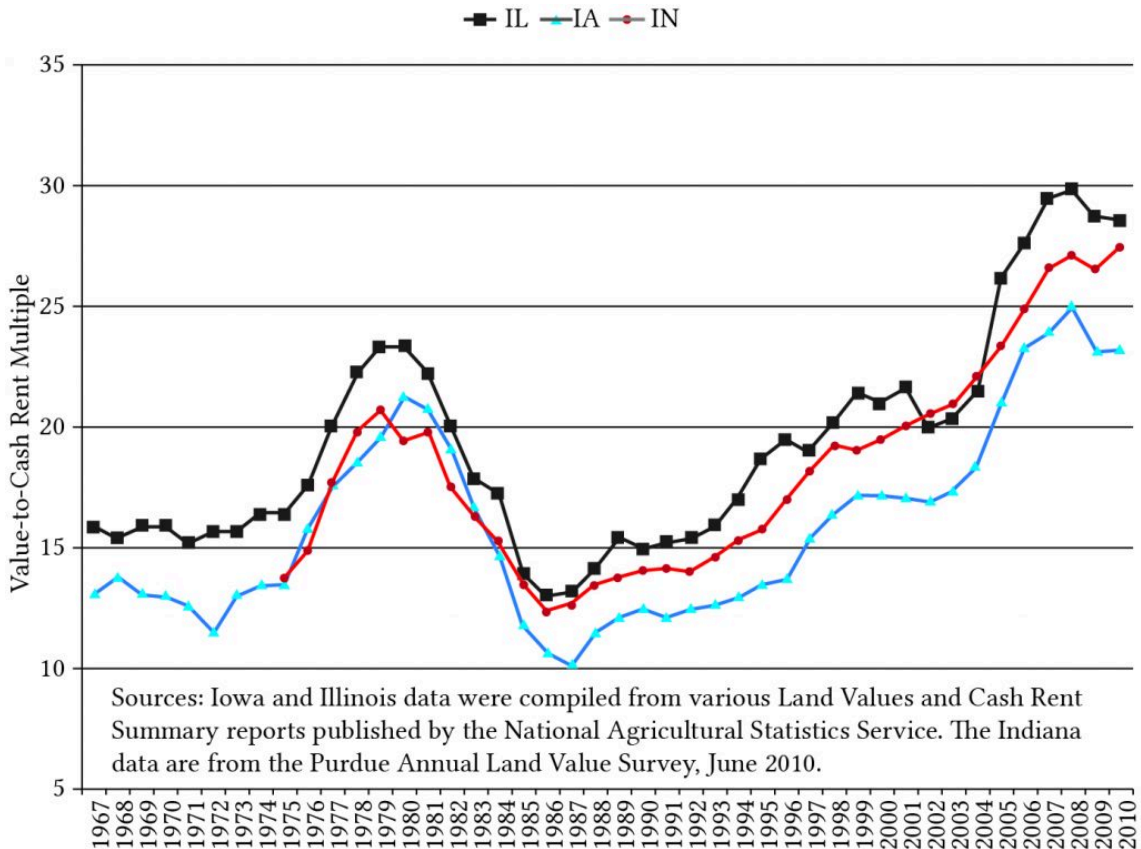


Figure 18.4. Land values to rent ratios for land in Iowa, Illinois, and Indiana since 1967.

An interesting exercise would be to approximate the discount rate in equation (18.41) with the real interest rate described in an earlier figure and then compare the results with the ratios described above.

Summary and Conclusions

Land's immobility and durability make it an important asset for securing loans. Its immobility means that lenders know where to find it, and its durability means that it will have the capacity to supply services and earn a return almost undiminished into the future. Both of these characteristics provide assurances to lenders that if they loan money to buy land, they will be able to recover their loans even if the borrowers fail to meet their obligations. On the other hand, land's immobility and durability also mean that its owners' can be easily identified and taxed. Land sales are carefully recorded to establish its value. Thus, land is a favorite source of tax revenue not only for its market value but also at the time of sale if capital gains (losses) are incurred.

Land has another feature. Land is infrequently traded, and transfers of ownership are costly for both buyers and sellers—often requiring the help of outside agents who must be paid for their services. As a result, both buyers and sellers pay funds they do not receive when buying and selling land. These transaction costs, including tax payments, mean that land exchange will likely not occur unless the buyer expects to earn more from land's services than currently earned by sellers. As a result, land is illiquid and its purchase and sale is a very infrequent occurrence.

In this chapter, PV models were used to model three types of prices: maximum bid prices, the most a buyer could offer to purchase the land; minimum sell prices, the least a seller would accept to sell the land; and value in use prices, the value of the land assuming continued use. These models accounted for three different types of taxes (property tax, income tax, and capital gains tax) as well as cash flow. These models provide helpful tools for assisting buyers and sellers in determining investment strategies.

Questions

1. Describe the features that make land an illiquid investment.
2. Why do lenders seem to prefer land to secure their loans? Compare the loan security provided by land compared to that offered by used equipment or breeding livestock?
3. Figures 18.1 and 18.2 compare year-to-year changes in farmland and housing prices. What is similar between those forces contributing to the variability in the prices of land and housing stock?
4. Commodities are nondurable goods and are likely used up in a single period. Meanwhile, land and houses are durables and last for many years. Can you describe how these differences between durable and nondurable goods may contribute to differences in the variability of their prices? Does the fact that the supply of land is relatively fixed versus the supply of commodities that can vary from year to year influence their variability?
5. Equation (18.9) describes the percentage change in the price of a long term asset (in this case land) in response to a change in the real interest rate r^* . Assume that the real growth rate is 3% and the real interest rate is 4%. Find the percentage increase in V_0 if r^* increases by 1%. Find the percentage increase in V_0 if g^* decreases by 1%.
6. Consider the maximum bid price and minimum sell price models. Consider the following data: the closing fee for the buyer is 3%, the closing fee for the seller is 4%, and in the last period land rented for \$150 per period. Land is liquid if the maximum bid price is greater than the minimum sell price. Using this data, please answer the following questions. Calculate the ratio of the maximum bid price to the minimum sell price. Is land liquid or illiquid? What must occur for the land to become liquid?
7. An investor is interested in buying a crop farm. According to the financial statements last year, the farm generated a net cash flow of \$52,000. Past data indicates that net cash returns have been growing at 3% per year. The investor's opportunity cost of capital, the IRR on its defender, is 8%, and closing fees for the transaction are 4%. The property is taxed at 1.5%, and the investor is in the 35% tax bracket. The tax adjustment coefficient θ is assumed to equal 75%. Find the maxi-

mum bid price for the parcel of land using maximum bid price with taxes model.

8. Farmland owners Rob and Ruthie Bell want to retire and sell their farm. They purchased their farm over 30 years ago for \$325,000. They recognize that their minimum sell price must leave them as well off as they would be if they continued to operate their farm. They estimate that closing fees will equal 3%, they are in the 30% tax bracket, and their capital gain tax rate α_T is 12%. Finally, they realize that their closing fees will be 5%. Find the minimum sell price for Rob and Ruthie Bell. (Hint: first find the value in use with taxes for the farmland. Then using the value in use with taxes and data described in this question, calculate the minimum sell price using the minimum sell price with taxes model.)
9. In previous problems, the tax adjustment coefficient was assumed to be 75%. Using the data supplied for calculating the maximum bid price for land with taxes, find the tax adjustment coefficient using equation (18.37).
10. Use equation (18.41) and land value to cash rent data described in Figure 18.4 to approximate real interest rates over time for Iowa, Indiana, or Illinois. If you believe that real interest rates are constant, what other explanations can you offer for time varying land value to rent ratios?

19. Leases

LINDON ROBISON

Learning goals. After completing this chapter, you should be able to: (1) understand how leasing can provide an alternative method for gaining control of and access to capital resources; and (2) evaluate leases using present value (PV) models developed earlier.

Learning objectives. To achieve your learning goals, you should complete the following objectives:

- Learn how leasing arrangements offer a different method for gaining control of and access to capital resources.
- Distinguish between the different forms of leases including the sale and leaseback lease, the operating lease, and the financial or capital lease.
- Learn the differences between leases and rental agreements.
- Learn how lease agreements are taxed.
- Learn the advantages and disadvantages of entering a lease agreement versus owning a capital resource.
- Learn how to find the break-even purchase price when compared to a leasing option.

Introduction

1

Control of a durable can be transferred between parties in several ways. A lease agreement is one important way. A lease is a contract through which control over the right to use a durable is transferred from one party, the lessor, to another party, the lessee who acquires control of the durable. In exchange for the right to use or control a durable for a specified time period, the lessee pays the lessor a rental payment or share in the output produced by the leased durable. The lease payment or shared output must cover the opportunity cost of funds invested in the leased item, depreciation of the durable, and other incidental ownership costs incurred by the lessor. Thus, leasing is a method of financing the control of a durable that separates its use from its ownership.

There are several kinds of lease arrangements and several kinds of durables that are leased. Farmland is frequently leased as are machines, houses, cars, computers, coping equipment, buildings, breeding

1. Some of the material in this chapter was adapted from Robison, L.J. and P.J. Barry (1996). *Present Value Models and Investment Analysis*. "Chapter 18, Leasing Decision." Michigan State University Press.

livestock, and many kinds of management services. In this chapter, we describe various types of leases, evaluate their advantages and disadvantages, and apply present value (PV) tools to analyze leasing benefits for both the lessor and the lessee.

Types of Lease Agreements

Sale and leaseback. Under the sale-and-leaseback agreement, a firm owning a durable to be leased sells the durable to a financial institution and simultaneously executes an agreement to lease the durable back. Thus, the lease becomes the alternative to a purchase with a specific advantage to the lessee: namely, he or she is allowed to write off the entire lease payment as an expense instead of just interest costs and depreciation. The lessor will not enter into this type of lease agreement unless the lease payments are sufficient to return the full purchase price plus a return on the investment to the lessor.

Operating leases. Operating leases, sometimes called service leases, provide for both financing and maintenance. These leases ordinarily call for the lessor to maintain the leased equipment. The cost of the maintenance is built into the lease payment. Computers and office copiers, together with cars and trucks, are the primary types of equipment involved in operating leases.

In the case of operating leases, lease payments are often insufficient to recover the full cost of the equipment. To offset this feature, the lease is written for a time period considerably less than the expected life of the leased equipment, leaving the lessor to recover his or her investment in renewal payments of the lease or through disposal of the leased equipment.

One of the main advantages of the operating lease for lessees is the cancellation of the short lease period which allows them to adopt and bring into use more advanced equipment. Thus, for durables subject to rapid changes in technology, an operating lease is often a preferred method for gaining control of a durable.

Financial or capital leases. A financial lease is a fully amortized lease whose PV of the lease payments equals the full price of the leased equipment. It does not provide for maintenance service nor may it be canceled. The financial lease begins with a lessee selecting the specific items it requires and agreeing with the lessor about the price and the delivery of the item. The lessor arranges with a bank or another financial institution to purchase the equipment and simultaneously executes a financial lease with the lessee intending to use the equipment. The purchase of the equipment builds into the lease payments a rate of return equivalent to what would be charged on a loan, and the lease is canceled when the purchase price of the durable plus a return for the lessor is paid.

Under financial leases, lessees generally pay property taxes and insurance and, in many cases, can acquire ownership of the durable at the end of the lease. The significant difference between the sale-and-leaseback lease and the financial lease is that the lessee purchases the durable directly from the manufacturer rather than from the lessor under the terms of the financial lease.

Lease Agreements and Taxes

One of the major effects of the lease is to alter the tax obligations of the lessee and lessor. Therefore, special attention is required to make sure that lease agreements are acceptable under current tax codes as interpreted in the United States by the Internal Revenue Service. There is an important distinction between a lease agreement and a loan for tax purposes. If the lease agreement cannot be distinguished from an ordinary loan agreement, then any special tax provisions associated with the lease agreement are lost.

To distinguish a lease from a sale agreement, the term over which the investment durable is leased must be less than 75 percent of the economic life of the durable. Nor should the lessee be granted any special repurchase option not available to others not involved in the lease. There are other conditions as well; tax codes are evolving documents that are constantly being updated.

Lease Liquidity and Risk

Normally, the decision to acquire the durable is not at issue in the typical lease analysis. The issue at hand is whether to acquire control of the durable through a lease or by purchasing it—often by borrowing funds to finance the purchase. The decision requires a careful examination of the advantages of leasing. Leasing offers several possible advantages relative to owning a durable that we describe next.

Release of cash and credit. When a firm leases or sells a durable on a leaseback arrangement, the lessee avoids the cash down payment required to purchase the investment. More generally, the liquidity of a lease versus the liquidity of a purchase depends on which option requires the most accelerated payments.

Some texts discuss in detail the effect on the firm's credit reserve as a result of leasing. If leasing uses up credit at a slower rate than borrowing, there may be credit incentives for leasing rather than borrowing. However, lenders are likely to recognize that long-term lease agreements place the same requirements on the firm's future cash flow as loans do.

Obsolescence risk. Investments may experience significant obsolescence risks. In addition, there is some risk that the need for durables' services will change before the durables' service capacity is exhausted. Part of this obsolescence and use risk may be reduced through a lease arrangement for a short time period, especially if the lessor is less subject to obsolescence risk than the lessee. This is likely the case where the leased equipment has alternative uses in other firms or industries and where the risk can be spread over many lessees.

Idle capacity risk. Another risk that can often be reduced through lease arrangement is the risk of holding idle equipment. If the demand for services from a durable is not sufficient to employ the durable full time, the lessee can reduce idle capacity risk by leasing rather than owning the durable. From

the lessor's point of view, the durable can be completely employed because many lessees will use the equipment.

Foreclosure risk. In many respects, leasing is similar to borrowing because it represents an obligation of the firm to a series of future cash payments. There is one significant difference between agreeing to lease and borrowing to purchase the durable. In the case of financial difficulties, the lessor simply takes back the equipment because he or she holds the legal title. In the case of a loan, inability to meet loan payments may result in more complex foreclosure proceedings.

Tax advantages. A tax advantage may be gained when the term of the lease is shorter than the allowable tax depreciation period for ownership. However, tax incentives for the lessee must be a result of a lower total tax burden for the lessee and lessor. This situation implies that the lease arrangement has legitimately allowed for a reduction in the total amount of taxes paid. Needless to say, the Internal Revenue Service imposes conditions on what does and does not constitute a legal lease.

Net Present Cost of Leasing versus Purchasing

The decision to lease or to buy a durable depends on the net present cost (NPC) of leasing versus NPC of purchasing the durable. Since the size and timing of the cash flow, influence of taxes, and risks associated with buying and leasing are different, they represent two different investment opportunities which are amenable to analysis using PV models. In the following discussion, the after-tax cash flow is considered risk-adjusted, and the problem is treated as riskless.

To compare the lease with the purchase option, the firm considers itself in both roles—that of a lessee and lessor—and finds the maximum lease payment it could pay and be indifferent between purchasing or leasing the durable. We begin by describing the NPC of the lease (NPC^L). Because we are describing the NPC of the lease, cash costs are treated as positive flow, and income that reduces the cost of leasing is treated as negative cash flow. For simplicity, we will assume a maximum constant lease payment equal to C . Then we assume the lease payment at age t is C_t , the lessee's constant marginal tax rate is T , the lessee's before-tax opportunity cost of capital is r , and the lessee's tax adjustment coefficient is θ as described in Chapter 11. With these definitions in place, the net present cost of a lease for n periods, NPC^L , can be expressed as:

$$(19.1) \quad NPC^L = C(1 - T) + \frac{C(1 - T)}{[1 - r(1 - \theta T)]} + \dots + \frac{C(1 - T)}{[1 - r(1 - \theta T)]^{n-1}}$$

Notice in equation (19.1) that the first lease payment is made at the beginning of the lease payment rather than at the end of the lease period, which would be the case of a loan. Therefore, leases can be viewed as an annuity due type of financial arrangement. Multiplying both sides of equation (19.1) by $[1 + r(1 - \theta T)]$ and summing the resulting geometric series, we write the result as:

$$(19.2) \quad NPC^L = C(1 - T) \{1 + US_0[r(1 - \theta T), n - 1]\}$$

where:

$$(19.3) \quad US_0[r(1 - \theta T), n - 1] = \frac{C(1 - T)}{[1 - r(1 - \theta T)]} + \dots + \frac{C(1 - T)}{[1 - r(1 - \theta T)]^{n-1}}$$

We now consider the net present cost of a cash purchase, NPC^P , for the same financial manager who previously considered the lease option. We begin by assuming that the market value of the durable leased or purchased is equal to V_0 . We also assume that V_t^B is the book value of the depreciable durable at age t . We continue to assume that the purchaser's opportunity cost of capital is r , the average tax rate is T , and γ_t is the percentage of the investment's value allowed to be deducted in year t where $t = 1, \dots, n$, and the tax adjustment coefficient is θ . Of course, the sum of the amount deducted cannot exceed the value of the investment so that $\gamma_1 + \dots + \gamma_n \leq 1$. All these assumptions allow us to write the NPC^P as:

$$(19.4) \quad NPC^P = V_0 - \frac{T\gamma_1 V_0}{[1 + r(1 - \theta T)]} - \dots - \frac{T\gamma_n V_0}{[1 + r(1 - \theta T)]^n} - \frac{V_n - T\gamma_n V_n}{[1 + r(1 - \theta T)]^n}$$

The purchaser's cost on the right-hand side of equation (19.4) is the purchase price V_0 , less n periods of tax savings from depreciation equal to $T\gamma_t V_0$ in the period discounted to the present period. The liquidation value V_n minus a tax adjustment term $T(V_n - V_n^B)$ reduces the purchaser's cost. The adjustment term accounts for the tax consequences of a difference between the liquidation value and the book value of the investment. If the investment is completely depreciated by the n^{th} period, and the liquidation value is zero, then the NPC^P can be written as:

$$(19.5) \quad NPC^P = V_0 - \frac{T\gamma_1 V_0}{[1 + r(1 - \theta T)]} - \dots - \frac{T\gamma_n V_0}{[1 + r(1 - \theta T)]^n}$$

Finally, if the NPC^L , expressed as the right-hand side of equation (19.2), is equated to NPC^P , expressed as the left-hand side of equation (19.5), we can solve for the lease payment C , a maximum bid lease, which equates the two alternatives for controlling the services from the durable.

$$(19.6) \quad C = \frac{NPC^P}{(1 - T) \{1 + US_0[r(1 - \theta T), n - 1]\}}$$

Consider an example. Go Green is a lawn service that requires a new truck to service its customers. It wants to know the largest lease payment it could afford and still be as well off as it would be if it

purchased the truck. The truck in question has a new sticker price of \$30,000 and can be depreciated using a straight-line method over five years. At the end of the lease, the truck has a liquidation value of \$5,000. To find the maximum bid payment, the following assumptions are used: $T = 32\%$, $\gamma_t = 10\%$ for $t = 1, \dots, 5$, $r = 14\%$, $\theta = 1$, $n = 5$, $V_n = \$5,000$, and $V_n^B = 0$. Finally, the after-tax discount rate is $r(1 - T) = .14(1 - .32) = .095$

To find the maximum bid payment, we first find NPC^P by substituting numerical values into equation (19.6):

$$(19.7) \quad NPC^P = \$30,000 - (.32)(\$6,000)US_0(0.095, 5) - \frac{\$5,000(1 - .32)}{(1.0952)^5}$$

We record the solution to this problem using Excel in Table 19.1 below.

Table 19.1. Finding the NPC of the Purchase.
[Open Table 19.1 in Microsoft Excel](#)

B9	fx		=NPV(B2,B3:B8)
	A	B	C
1	Purchase Price	\$30,000	
2	rate	.095	
3	Depreciation tax savings period 1	(\$1,920)	
4	Depreciation tax savings period 2	(\$1,920)	
5	Depreciation tax savings period 3	(\$1,920)	
6	Depreciation tax savings period 4	(\$1,920)	
7	Depreciation tax savings period 5	(\$1,920)	
8	After-tax salvage value period 6	(\$3,100)	
9	NPV of depreciation tax savings + after-tax salvage	(\$9,170.60)	=NPV(rate, value1:value6)
10	NPC: purchase price - PV of tax-savings + salvage value	\$20,829.40	=NPV of depreciation tax savings + salvage value + purchase price

In cell B9 we enter the function: = NPV(rate,values) = NPV(B2,B3:B8) which returns the value (\$9,170.60). In cell B10 we add the purchase price of \$30,000 to the NPV of after-tax depreciation savings plus the liquidation value which returns the NPC of the purchase equal to \$20,829.36.

Having found NPC^P equal to \$20,829.40, we can find C using equation (19.6).

We solve this problem using Excel in Table 19.2 by finding the present value of Annuity Due with 5 payments of -1. We enter the arguments of the PV function and label the terms in adjacent cells. Excel's PV function has the following arguments listed in column A: rate (rate), number of payments (nper), annuity (pmt), future value (FV), annuity type (type). For this problem, the discount rate is 9.5% = .095, the number of payments equals 5, the payments are -1, the future value FV is zero, and the type of annuity is an annuity due which requires we enter 1.

Table 19.2. Finding the PV of the Annuity Due
[Open Table 19.2 in Microsoft Excel](#)

B6			f_x =PV(B1,B2,B3,B4,B5)
	A	B	C
1	rate		.095
2	nper		5
3	pmt		-1
4	FV		0
5	type		1
6	PV		\$4.20 =PV(rate,nper,pmt,FV,type)

In cell B6 we type in the PV function: = PV(B1,B2,B3,B4,B5). The result returned is PV = \$4.20. Then we multiply by (1 - T) as required in equation (19.6). Finally we find C:

$$(19.8) \quad C = \frac{NPC^P}{(1 - T) \{1 + US_0[r(1 - \theta T), n - 1]\}} = \frac{\$20,829.40}{.68(4.2)} = \$7,283.01$$

Now suppose that we wanted to know the number of periods required to reach a future value of \$2,000. With the data already entered, we turn to the “goal seek” option in Excel. To access goal seek from the main menu, press the [Data] tab, then on the ribbon menu press the [What-if Analysis] button, and on the drop down menu press [Goal Seek...]. The goal seek menu allow us to identify a target value for a particular cell and a variable to change. In our case the target cell is B6 and the target amount is \$2,000. The cell we allow to vary is the number of periods cell, B2. Then entering these into Excel's solver, the solution returned is 19.57. Thus, by continuing to invest for 8.57 periods, we obtain a future value of \$2,000 compared to the future value of \$1,005.18 reached after 10 years of investing.

Alternatively, we could have asked what is the payment amount required to reach a future value of \$2,000 during the original term. In this case we vary the annuity in cell is B3, and the solver returns the value of \$182.65. Thus, by increasing our annuity due from \$90 to \$182.65, we reach a future value amount of \$2,000 in the same term required to reach a future value of \$1,005.18 with \$90 annuity due payments.

What we have found in this example is that we would be indifferent between purchasing the truck for \$30,000 and making five lease payments of \$7,283.01. Obviously, the high discount rate of 14% and making the payment for the truck up front contributes to the high lease payment. In the questions at

the end of this chapter, students will be asked to repeat the problem assuming the same values except that the opportunity cost of capital is 5%.

Comparative Advantages for Lessee and Lessor

So far, we have described the technical properties of leases and how a financial manager may decide between gaining control of a durable through a lease or purchase arrangement. In this section, we allow the lessee and the lessor to be two different persons whose comparative advantages lead them into an exchange arrangement in which one party purchases an asset and then leases it to another party. The solution in this case considers the NPC from the lessee's (lessor's) position, assuming that the lessor (lessee) pays their maximum bid purchase price (lease payment). What makes this an interesting problem is that differences in the lessee's and lessor's tax rates, as well as opportunity costs of capital, can create incentives to exchange.

Comparative advantage. A fundamental principle of economics is that firms should specialize in tasks for which nature, institutions, or luck has granted them an advantage. To illustrate, suppose two farmers, A and B, can both produce dry edible beans and carrots. Also assume A can grow dry edible beans and carrots more profitably than B, and B can grow carrots more profitably than it can grow dry beans. If both beans and carrots must be produced, then A should produce beans, and B should grow carrots. They could then trade to obtain what they did not produce, and both would be better off.

Even though A produced both beans and carrots more profitably than B, it would still be to A's and B's advantage to specialize. They should decide in which product A had the greatest advantage in production, or B had the least comparative disadvantage, and then specialize and trade as before.

An application of the law of comparative advantage explains why two firms may lease. Suppose firm A is able to purchase a durable at a lower price than firm B. But firm B, not firm A, has need of the services of the durable. In this case, firm A's advantage is in purchasing while B's advantage is in using the durable. Thus, A might purchase the durable and either sell or lease to B.

Another reason why A and B might both agree to lease has to do with relative tax rates. Suppose A, the purchaser of a durable, pays taxes at an average rate T^P , which is higher than T^L , the average rate at which B, the lessee, pays taxes. As the durable is depreciated, the depreciation creates a tax shield of greater value to the purchaser than to the lessee. Thus, we might say that A, the purchaser, has a comparative tax advantage over B, the lessee, in claiming tax depreciation. The lease allows the purchaser, A, and the lessee, B, to benefit from their comparative advantages associated with the tax shields created by depreciation.

Still further reasons why A and B might purchase and lease a durable are that they face different opportunity costs of capital, have different opportunities for using the durable's services, or face different marginal costs of credit.

Comparative advantages for lessee and lessor created by different tax rates and opportunity costs of capital. To demonstrate the idea of comparative advantage with a lease agreement, consider that person B, the lessee, will not lease for more than it would cost to control the durable through a purchase agreement. We want to find B's NPC if the durable were purchased. To simplify, assume the purchase price of the durable is V_0 , the book value depreciation rate in year t is $\gamma_t = \gamma(1 - \gamma)^t$ where $t = 0, \dots, n - 1$, r^B is B's opportunity cost of capital, T^B is B's average tax rate, and assume the lessee tax adjustment coefficient is 1. The NPC^P of the purchase for B is:

$$(19.9) \quad NPC^L(\text{purchase}) = V_0 - \frac{T^B \gamma V_0}{[1 + r^B(1 - T^B)]} - \frac{T^B \gamma V_0(1 - \gamma)}{[1 + r^B(1 - T^B)]^2} - \frac{T^B \gamma V_0(1 - \gamma)^2}{[1 + r^B(1 - T^B)]^3} - \dots$$

Equation (19.9) is equivalent to equation (19.4) except that it is expressed in a form that makes it analytically tractable. Obviously, more detailed expressions with finite time horizons could easily be calculated. What makes equation (19.9) tractable is that it is expressed as an infinite series whose solution can be written as:

$$(19.10) \quad NPC^L(\text{purchase}) = V_0 - \frac{T^B \gamma V_0}{r^B(1 - T^B) + \gamma}$$

Now consider what happens to B's NPC resulting from the purchase if B's tax rate increases. We find the change in B's NPC associated with the purchase of the durable by differentiating NPC^P with respect to T^B . (This operation in essence examines the change in NPC^P with respect to a small change in T^A . The results is that NPC^P decreases with an increase in T^B . Thus, a firm in a higher tax bracket can buy a durable at a lower after-tax cost than someone in a lower tax bracket.

Next, we find B's NPC^L associated with leasing the durable which, after summing geometrically, can be expressed as:

$$\begin{aligned}
 NPC^L (\text{lease}) &= \frac{C(1 - T^B)}{[1 + r^L(1 - T^B)]} + \frac{C(1 - T^B)}{[1 + r^L(1 - T^B)]^2} \\
 &+ \frac{C(1 - T^B)}{[1 + r^L(1 - T^B)]^3} + \dots \\
 &= \frac{C[1 + r^L(1 - T^B)]}{r^L}
 \end{aligned}
 \tag{19.11}$$

To illustrate equation (19.10) consider the following example: Suppose Affordable Assets (AA) buys durables and leases them to other firms. As a well-to-do established firm, AA's tax rate, T^A is high—45 percent. On the other hand, the Unendowed User (UU) lacks capital and prefers to lease rather than own in order to protect its limited credit. Because of its lower earnings, UU's average tax rate T^U is 21 percent. In addition, assume that the leased durable wears out at the rate of $\gamma = 10$ percent, and assume the market rate of return for both AA and UU is 14 percent. Finally, assume that the purchase price of the durable is \$4,000.

Making the appropriate substitutions into equation (19.10), we find the NPC of purchasing the durable for UU is:

$$NPC^{UU} = \$4,000 - \frac{[(.21)(.1)(\$4,000)]}{[(.14)(1 - .21) + .1]} = \$3,600
 \tag{19.12}$$

In other words, the after-tax purchase price of the durable for UU is \$3,600.

Now consider the after-tax NPC of the purchase for firm AA. Making the appropriate substitutions into equation (19.10), we find the NPC of purchasing the durable for AA is:

$$NPC^{AA} = \$4,000 - \frac{[(.45)(.1)(\$4,000)]}{[(.14)(1 - .45) + .1]} = \$3,000
 \tag{19.13}$$

In other words, the after-tax cost of the durable purchase for AA was only \$3,000 compared to \$3,600 for UU.

Having calculated NPC for purchasing the durable for firms UU and AA, we can find their respective maximum bid lease prices. Making the appropriate substitutions into equation (19.11), we find $NPC^{UU}(\text{leasing}) = C^{UU}[1 + (.14)(1 - .21)] / .14 = \$3,600$. Solving for C^{UU} we find $C^{UU} = \$453.81$.

Similarly we find $NPC^{AA}(\text{leasing}) = C^{AA}[1 + (.14)(1 - .45)] / .14 = \$3,000$. Solving for C^{AA} we find $C^{AA} = \$389.97$.

Reflections on comparative advantages for firms UU and AA. Because of their differences in tax rates, firms UU and AA have different effective after-tax cost of the durable even though the market price of the durable is the same for each firm. In this case, the effective cost for AA is less than for firm UU—\$3,000 versus \$3,600. On the other hand and as a result of the differences in the effective after tax cost of the durable, the maximum bid lease price for firm UU is greater than for firm AA: \$453.81 versus \$389.97.

These results provide opportunities for firms AA and UU to take advantage of their comparative advantages: firm AA purchases the durable at a lower effective cost than is available to UU. UU, on the other hand, is quite willing to pay a higher lease price than AA's maximum bid lease price. Under these arrangements, both firms AA and UU are made better off.

Leases and NPV. Assume that firm AA buys the durable and UU leases the durable from AA. Now the lease price is an expense for firm UU but an income for AA. We may want to ask: what is the minimum lease payment AA could accept from UU and still break even? To find this amount we set AA's NPV as:

$$(19.14) \quad NPV^{AA} = \frac{C[1 + r(1 - T^{AA})]}{r - NPC^{AA}(\text{purchase})}$$

After making the appropriate substitutions we find

$$(19.15) \quad C = \frac{NPC^{AA}(\text{purchase})(r^{AA})}{[1 + r^{AA}(1 - T^{AA})]} = \frac{\$3,000(.14)}{[1 + (.14)(1 - .45)]} = \$389.97$$

Thus, any arrangement with UU which generates a lease payment more than \$389.97 will result in a positive NPV for AA.

Summary and Conclusions

Trades occur when each party to the exchange gives up something of value in return for something of greater value. In most trades, a physical object or service is exchanged for an agreed-on amount of cash. Moreover, in most cases, ownership is transferred along with the good or service.

This chapter has considered leasing, a different kind of trade, in which the control over the use of a durable is transferred, but ownership of the durable is not. Leasing exists because it provides benefits to the lessee and lessor that might not be realized if control of a durable also required ownership.

Different types of leases described in this chapter include sale-and-leaseback agreements, operating leases, and financial or capital leases. Leases offer particular advantages for lessees including the avoidance of capital requirements associated with a purchase. As a result, lessees may use their limited

credit for other purchases or as a credit reserve. Obsolescence risk of ownership is also reduced for lessees. Additionally, leasing offers the chance to better match service requirements to the delivery of services. Ownership of durables may require holding idle capacity—less likely when services of a durable are leased.

Tax considerations are critical in the decision to lease or to purchase. Ownership allows for tax depreciation shields; leasing allows the entire lease payment to be claimed as an expense.

An important principle of comparative advantage is involved in the lease decision. If the tax depreciation from ownership is greater for one firm than for another, leasing permits the firm which can benefit the most from the depreciation shield to claim it by purchasing the asset and leasing it to another firm. Not only can comparative tax advantages be optimally used through leasing, there can also be comparative advantages in acquiring financing. Comparative advantage and the incentive to lease may also result from differences in opportunity costs, access to credit, and use for the durables' services. In essence, leasing is a critical tool that allows firms to benefit from their comparative advantages.

Questions

1. Describe three different lease types and their essential differences.
2. For every lease agreement, there is a lessor and a lessee who both believe they will be better off by executing the lease. Describe how a sale and leaseback agreement might make both the lessor and lessee better off.
3. Assume a lease agreement that requires \$100 payments at the beginning of each period for 15 years. The lessee's marginal tax bracket is 25% and opportunity cost of capital is 8%. The tax adjustment coefficient is assumed to equal 80%. Find the NPC of this lease. Now assume that the lessor agreed to accept payments at the end of each period instead of the beginning. What would be the NPC of the lease agreement under the new arrangement?
4. Resolve the Go Green lease problem assuming that the firm's opportunity cost of capital is 5% rather than 14%. Compare your solution with the previous one. Did the lease payment go up or down? Can you explain why?
5. Resolve the Go Green lease problem by assuming that the lessee will lease the durable for an extra period. The lessee will make 6 lease payments instead of 5, but the depreciation schedule will not change. In other words, find the maximum lease payment and explain your results.
6. Explain in your own words the concept of comparative advantage. Provide one example of how the concept of comparative advantage has influenced your choice or the choice of others.
7. Explain how the principle of comparative advantage might explain why two firms may agree to a lease arrangement.
8. In the example of a lessee UU and a lessor AA, the lessee and lessor are in different average tax brackets (45 versus 21 percent) but were assumed to face the same opportunity cost of capital. Resolve the example by assuming UU's opportunity cost of capital is 7% instead of 14%. Find the effective after-tax purchase price of the durable for the two firms. Then find the maximum bid

lease price for the two firms. Do the results allow for the two firms to take advantage of the principle of comparative advantage? Please explain.

9. Consider the case of firms UU and AA described in the text. Since AA has the lowest NPC for purchasing the durable, it should do so. Meanwhile the maximum bid lease price for UU is higher than for AA, so UU should lease rather than purchase the durable. Suppose that UU agrees to pay AA a lease price of \$410.00. What is AA's NPV from buying the durable and leasing it back to UU? (Hint: In this arrangement, the lease cost for UU is an income for AA.)

20. Financial Investments

LINDON ROBISON

Learning goals. After completing this chapter, you should be able to: (1) distinguish between financial investments and capital investments; (2) recognize the different kinds of financial investments; and (3) evaluate the different kinds of financial investments using present value (PV) models developed in earlier chapters.

Learning objectives. To achieve your learning goals, you should complete the following objectives:

- Distinguish between financial investments and nonfinancial investments.
- Learn how PV tools can be used to analyze financial investments using methods similar to those used when comparing nonfinancial investments.
- Learn to distinguish between financial and nonfinancial objectives.
- Learn the role of brokers, dealers, and financial intermediaries in the securities markets.
- Learn how to value riskless securities such as time deposits.
- Learn why Albert Einstein called compounding interest “the greatest mathematical discovery of all time”.
- Learn the nature of bonds and the variables that determine their value.
- Learn how tax rates affect the value of financial investments.

Introduction

In its broadest meaning, to invest means to give up something in the present for something of greater value in the future. Investments, what we invest in, can differ greatly. They may range from lottery tickets and burial plots to municipal bonds and corporate stocks. It is helpful to organize these investments into two general categories: real and financial. Capital, or real, investments involve the exchange of money for nonfinancial investments that produce services such as storage services from buildings, pulling services from tractors, and growing services from land. Financial investments involve the exchange of money in the present for a future money payment. Financial investments include time deposits, bonds, and stocks.

This chapter applies PV models developed earlier to financial investments. Large corporations and other business organizations require financial investments from a large number of small investors to provide funds for operation and growth. The collection of these funds would be impossible if each investor were required to exercise a managerial role over them. Moreover, the collection of investment funds from a large number of small investors would be impossible unless the investors could be assured of investment safety and limited liability. A number of financial investments have been designed to overcome these and other investment obstacles.

The market in which financial investments are traded is called the securities market or financial market. Activities in the financial markets are facilitated by brokers, dealers, and financial intermediaries. A broker acts as an agent for investors in the securities markets. The securities broker brings two parties together to obtain the best possible terms for his or her client and is compensated by a commission. A dealer, in contrast to a broker, buys and sells securities for his or her own account. Thus, a dealer also becomes an investor. Similarly, financial intermediaries (e.g. banks, investment firms, and insurance companies) play important roles in the flow of funds from savers to ultimate investors. The intermediary acquires ownership of funds loaned or invested by savers, modifies the risk and liquidity of these funds, and then either loans the funds to individual borrowers or invests in various types of financial assets.

Numerous books discuss the institutional arrangements associated with securities trading. In this chapter, the focus is not on the details of trading in the securities market but on how to value financial investments or securities.

Valuation of Riskless Securities

The prices of financial investments, like other prices, are determined in markets—financial markets. The characteristics of financial markets, however, may differ significantly depending on the type of financial investment traded. Common to all financial markets, though, is that prices are established by matching the desires of buyers and sellers. Moreover, in equilibrium there is neither excess demand nor excess supply. However, equilibrium does not mean that all prospective buyers and sellers agree that an investment's price is equal to its value. It only means that a price has been found that, in a sense, balances the different goals of buyers and sellers.

Time deposits. Time deposits are money deposited at a banking institution for a specified period of time. The main difference between different kinds of time deposits is their liquidity. Generally speaking, the less liquid the time deposits are, the greater their yield is.

The most liquid form of time deposits are sometimes known as “on call” deposits. On call deposits can be withdrawn at any time and without any notice or penalty. Examples of on call deposits include money deposited in a savings or checking account in a bank. Another kind of time deposit is a certificate of deposit (CDs). When CDs mature, they can be withdrawn or they can be held for another term. CDs are generally considered to be liquid because they are negotiable and can be re-discounted when the holder needs some liquidity.

Some time deposits must be held until maturity or suffer penalties for early withdrawal. The rate of return for these time deposits is higher than on call deposits because the requirement that the deposits be held for a designated term gives the bank the ability to invest in financial products that earn greater returns. In addition, the return on time deposits is generally lower than the average return on long-term investments in riskier products like stocks and some bonds. Some banks offer market-linked time deposit accounts which offer potentially higher returns while guaranteeing principal.

Investments in time deposits may consist of a one-time investment or a series of deposits. What all time deposits have in common is that the payment(s) is compounded until some ending date at which time they are partially or totally withdrawn. Thus, our interest is not in the present value of the payments but in their future value. The market determined price in the case of financial investments is the compound rate at which time deposits earn interest. Typically, the compound rate of interest earned on time deposits is low—partly because they provide nearly perfect liquidity—especially for time deposits for which there is no penalty nor transaction cost associated with withdrawal of funds.

Ordinary annuities and annuities due. Previously we described an annuity as a time series of constant payments made or received at the end of each period. Annuities paid and received at the end of a period are referred to as ordinary annuities. Annuities paid (or received) at the beginning of each period are referred to as annuity due payments.

To illustrate an annuity due, consider a series of deposits R made every period, the first one occurring in the present. We write the future value V_n of the time series of deposits R compounded at rate \hat{r} for withdrawal in period n as:

$$(20.1) \quad R(1 + \hat{r})^n + R(1 + \hat{r})^{n-1} + \dots + R(1 + \hat{r})^2 + R(1 + \hat{r}) = V_n$$

Note that the first time deposit R is compounded for n periods—and is equal in value to its compounded value at the beginning of period n . Similarly, the second payment is compounded for $n - 1$ period and is equal to its compounded value at the beginning of period n . Finally, the last payment is made at the beginning of period $n - 1$ and is compounded once to find its compounded value at the beginning of period n .

To illustrate, suppose a saver deposits a \$90 payment (PMT) at the beginning of each period for $n = 10$ periods. If the deposits are compounded at a rate of $r = 2\%$, what amount is available for withdrawal at the beginning of period 10? We solve this problem using Excel in Table 20.1. Note that to indicate annuity due payments at the beginning of the period, “type” in the Excel future value FV function is changed from its default of zero to 1. In place of PV in the equation, we enter a blank closed with a comma.

Table 20.1. The Future Value of 10 Annuity Due Payments of \$90 Compounded at 2%
[Open Table 20.1 in Microsoft Excel](#)

B4	f_x		=FV(B1,B2,B3,,B6)
	A	B	C
1	rate	0.02	
2	nper	10	
3	pmt	(\$90.00)	
4	FV	\$1,005.18	=FV(rate,nper,pmt,[pv],type)
5	PV	0	
6	type	1	

What our calculations determine is that at the beginning of the 10th period, we have \$1,005.18 available. However, there are several other interesting questions that we might answer using our Excel formulas. For example, we might want to know how many periods would be required to have \$2,000 available at some future date. Continuing with the entries already made in Excel, we add the following:

Table 20.2. Finding the Number of \$90 Annuity Due Payments Compounded at 2% Required to Earn a Future Value of \$2,000

[Open Table 20.2 in Microsoft Excel](#)

B2	f_x		=NPER(B1,B3,B4,B6)
	A	B	C
1	rate	0.02	
2	nper	29.693477	=NPER(rate,pmt,FV,type)
3	pmt	(\$90.00)	
4	FV	\$2,000.00	
5	PV	0	
6	type	1	

We might be interested in knowing the periodic investment required to have \$2,000 available in 10 years if the investments were compounded at 2%. The required Excel formula is represented in Table 20.3.

Table 20.3. Finding the Annuity Due Payment Compounded at 2% Required to Earn a Future Value of \$2,000 in 10 Years

[Open Table 20.3 in Microsoft Excel](#)

B3	f_x		=PMT(B1,B2,B5,B4,B6)
	A	B	C
1	rate	0.02	
2	nper	10	
3	pmt	\$179.07	=PMT(rate,nper,PV,FV,type)
4	FV	\$2,000.00	
5	PV	0	
6	type	1	

To reach one's savings goal of \$2,000 in 10 years at a compound rate of 2% would require a periodic payment into a time deposit at the beginning of each period equal to \$179.07.

Compound Interest

The magic of compounding. It is claimed that Albert Einstein called compound interest “the greatest mathematical discovery of all time”. It is probable that no other concept in finance has more importance for investors than what is sometimes called the “wonder of compound interest.” Compounding interest creates earnings not only on the original amount saved or invested but also creates earning on the interest on earnings, and it does so period after period. Another way to describe this process is to say that compounding interest generates earnings on the investment and reinvested earnings from the investment. To realize the power of compound interest requires two things: the re-investment of earnings and time.

Our previous calculations confirm the importance of compound interest. In 10 payments of \$90 we accumulated a future value of \$1,005.18. By making an additional 8.57 payments, we doubled our future value. We found a similar effect when we increased loan payments and observed significant decreases in term. When finding the effect of increased loan payments on loan terms, we calculated the elasticity of the term of the loan with respect to the size of the loan payment. Similarly, we can find the impact of an increase in the term of periodic savings on future values (FV). We refer to this elasticity as the FV elasticity with respect to term. The derivation of the FV elasticity with respect to term is found in Derivation 20.1 at the end of this chapter. The discrete approximation to the FV elasticity with respect to n is:

$$(20.2) \quad E_{FV,n} = \frac{dFV}{dn} \frac{n}{FV} \approx \frac{n}{US_0(r,n)}$$

Table 20.4 provides elasticity estimates for alternative reinvestment rates and terms.

Table 20.4. Tabled Values of elasticity $E_{FV,n} = n/US_0(r,n)$

n	$r = .01$	$r = .03$	$r = .05$	$r = .08$	$r = .10$	$r = .15$
1	1.00%	1.03%	1.05%	1.08%	1.10%	1.15%
3	1.02%	1.06%	1.10%	1.16%	1.21%	1.31%
5	1.03%	1.09%	1.15%	1.25%	1.32%	1.49%
10	1.05%	1.17%	1.30%	1.49%	1.63%	1.99%
20	1.08%	1.34%	1.60%	2.04%	2.35%	3.20%
30	1.16%	1.53%	1.95%	2.66%	3.18%	4.57%
60	1.34%	2.17%	3.17%	4.85%	6.02%	9.00%

A related question with a more intuitive answer is how long does it take to double, triple, or quadruple one’s investment? The derivation for the equation that answers this question can be found in Derivation 20.2 at the end of this chapter.

Let m be the number of times one wants to multiply his or her FV obtained at period n . Let n_m equal the period in which the last deposit is made. Then we can find the number of periods ($n_m - n$) required to increase the original FV by m times. The formula found in Derivation 20.2 is provided below:

$$(20.3) \quad n_m - n = \frac{\ln [m + (1 - m)e^{-rn}]}{r}$$

To illustrate, let $r = .1$, $n = 5$, and $m = 2$. The number of periods required to double the FV of one's original investment equals:

$$(20.4) \quad n_2 - n = \frac{\ln [2 - e^{-(.1)(5)}]}{.1} = \frac{\ln(1.39)}{.1} = 3.3$$

In this example, an investment consisting of 5 equal payment compounded at 10% could be doubled in additional 3.3 payments. And the investment could be tripled in:

$$(20.5) \quad n_3 - n = \frac{\ln [3 - e^{-(.1)(5)}]}{.1} = \frac{\ln(1.79)}{.1} = 5.8$$

The second example illustrates the power of compounding interest. Starting after 5 periods of investing, in 3.3 more periods the FV of the investment doubled and in 5.8 additional periods the FV of the original investment tripled.

Finally, we conclude this section by constructing Table 20.5 demonstrating the power of compounding interest. The cells of the table indicate the time required to double, triple, and quadruple an initial investment for reinvestment rates of 1%, 3%, 5%, and 10%.

Table 20.5. Periods required to double, triple, or quadruple the FV of an investment consisting of 5 equal payments assuming alternative reinvestment rates.

Alternative re-investment rates	Number of time periods required to increase the FV of the original investment m times		
	$m = 2$ (double the original FV)	$m = 3$ (triple the original FV)	$m = 4$ (quadruple the original FV)
1%	4.8	9.3	13.7
3%	4.4	8.2	11.6
5%	4.0	7.3	10.2
10%	3.3	5.8	7.8

The rule of 72 approximates the amount of time it takes to double your investment at a given rate of return. To apply the rule, you divide the rate of return by 72. For example, assume you invest \$1000 at an interest rate of 5%. It would take 14.4 years ($72/5$) to double your investment to \$2000.

Bonds

A bond is a financial asset frequently traded in financial markets. Bonds represent debt claims on the assets and income of the entity issuing the bond. A bond's value is equal to the present value of its future cash flow (interest and principal) discounted at an appropriate interest rate. Bonds usually have a known maturity date at which time the bond holder receives the bond's face or par value. Bonds are unique because their redemption or liquidation value is fixed. Typical redemption amounts are \$1,000 or \$10,000.

Consider the following example. Suppose a bond can be purchased at a price of V_0 (an initial cash outflow) and redeemed n periods later at a cash value of V_n , a cash inflow to the bond holder. Moreover, suppose that the bond generates no cash return except when it is sold. Further, assume that the before-tax discount rate is r percent. Ignoring any tax consequences, the NPV of this bond is the sum of the cash outflow plus the present value of the cash inflow.

$$(20.6) \quad NPV = -V_0 + \frac{V_n}{(1+r)^n}$$

Those who purchase bonds may want to calculate the “yield” on a bond, which is the discount rate that equates the present value of the bond's cash flow to its present market value—the bond's internal rate of return (IRR). If, for example, the bond's market value is \$321.97 and its cash flow of \$1,000 occurs at the end of year 10, then the bond's yield, or its IRR, is 12 percent.

$$(20.7) \quad V_0 = \frac{\$1,000}{(1.12)^{10}} = \$321.97$$

Now consider the effect of taxes. First, assume capital gains taxes are paid by the bond purchaser at rate T_g and income taxes are paid at rate T . The after-tax NPV of the bond is calculated by adjusting the discount rate to its after-tax equivalent and by subtracting from the liquidation value of the bond, the capital gains tax:

$$(20.8) \quad NPV = -V_0 + \frac{V_n - T_g(V_n - V_0)}{[1 + r(1 - \theta T)]^n}$$

where θ is the tax adjustment coefficient defined in Chapter 11. If the NPV in equation (20.5) is zero then $r(1 - \theta T)$ is the bond's after-tax IRR. To find the effective tax rate we set NPV equal to zero in equations (20.8) and (20.6) so that before-tax cash flow discounted by the before-tax IRR in equation (20.6) is equal to the after-tax cash flow discounted by the after-tax IRR in equation (20.8). Then we solve for θ that makes the two equations equal. The result is:

$$(20.9) \quad \theta = \frac{(1+r)}{rT} \left\{ 1 - \left[1 - T_g + T_g \left(\frac{V_0}{V_n} \right) \right]^{1/n} \right\}$$

To illustrate using the numbers from our previous example, let the before-tax IRR equal 12%, let V_0 equal \$321.97, let V_n equal \$1,000, and let the income tax and the capital gains tax T and T_g both equal 15 percent. Next making the appropriate substitutions into equation (20.6) we find θ equal to:

$$(20.10) \quad \theta = \frac{1.12}{(.12)(.15)} \left\{ 1 - \left[1 - .15 + .15 \left(\frac{\$321.97}{\$1,000} \right) \right]^{1/10} \right\} = .66.$$

For $\theta = .66$ the effective tax rate is reduced from 0.15 to $(.66)(0.15) = 0.10$.

In the example, just completed, we were able to deduce a closed form solution for θ . In most cases, closed form solutions are very difficult to obtain. Nevertheless, in most practical cases involving numerical estimates, we can still find estimates for the effective tax rate and after-tax IRRs. We will demonstrate the empirical approach to finding effective tax rates in the next section.

Coupons and Bonds

Most bonds, in addition to capital gains (or losses), provide “coupon” (interest) payments. The number and amount of the coupon payments will alter the NPV as well as the price of the bond. Usually, the coupon rate r^c is a percentage of the redemption value of the bond.

The before-tax NPV of a bond with n coupon payments is:

$$(20.11) \quad \begin{aligned} NPV &= -V_0 + \frac{r^c V_n}{(1+r)} + \frac{r^c V_n}{(1+r)^2} + \dots + \frac{r^c V_n + V_n}{(1+r)^n} \\ &= -V_0 + (r^c V_n) US_0(r, n-1) + \frac{r^c V_n + V_n}{(1+r)^n} \end{aligned}$$

To illustrate, suppose an investor wants to find the maximum bid price V_0^b for a three-year \$1,000 bond if it offers coupon payments of $r^c V_n = (.05)(1,000) = \50 , and $r = 10\%$. Then using equation (20.11) and setting NPV equal to zero, find the maximum bid price V_0^b equal to:

$$\begin{aligned}
 V_0^b &= (r^c V_n)US_0(r, n) + \frac{V_n}{(1+r)^n} = \$50US_0(.10, 2) + \frac{\$1,050}{(1+.10)^3} \\
 (20.12) \quad &= \$875.65
 \end{aligned}$$

Taxes, of course, affect the NPV of bonds with coupon payments. Only now, taxes may or may not be paid on the coupon payments and capital gains. For example, coupon payments of many municipal bonds are not taxed but their capital gains are taxed. Tax exemptions, of course, raise the maximum bid price and NPVs of the bonds for all investors but especially for higher tax bracket investors. The after-tax present value of the bond with tax-exempt coupon payments is written as:

$$(20.13) \quad NPV = -V_0 + (r^c V_n)US_0[r(1 - \theta T), n - 1] + \frac{V_n - (V_n - V_0)T_g}{[1 + r(1 - \theta T)]^n}$$

To illustrate, suppose an investor wants to find the maximum bid price V_0^b for a three-year \$1,000 bond if it offers coupon payments of $r^c V_n = (.05)(1,000) = \50 and $r = 10\%$ and pays capital gains and income taxes at the rate of 10%. Then using equation (20.13) and setting NPV equal to zero, we find the maximum bid price V_0^b for the bond whose earlier maximum bid price was calculated equal to \$875.65:

$$\begin{aligned}
 V_0^b &= (r^c V_n)US_0[r(1 - \theta T), n] + \frac{V_n - (V_n - V_0)T_g}{[1 + r(1 - \theta T)]^n} \\
 (20.14) \quad &= \$50US_0[.10(1 - \theta.20), 3] + \frac{\$1,050 - (\$1,000 - \$875.65)(.10)}{[1 + .1(1 - \theta.2)]^3}
 \end{aligned}$$

In the application of equation (20.13), the discount rate was the IRR—because it was the discount rate that resulted in NPV equal to zero. We introduced taxes into the cash flow, but we don't know the effective after-tax discount rate that would set NPV equal to zero. We have to solve for θ that will adjust the discount rate for taxes in the same magnitude as were the cash flow adjusted for taxes. For equation (20.13), we find the after-tax IRR using Excel:

Table 20.6. Finding the After-tax IRR
[Open Table 20.6 in Microsoft Excel](#)

B5	f_x		=IRR(B1:B4)
	A	B	C
1	Bond's max bid price	-\$875.65	
2	First coupon payment	\$50.00	
3	Second coupon payment	\$50.00	
4	Salvage value (\$1,000) + Coupon	\$1,037.60	
5	IRR	9.59%	=IRR(B1:B4)

The IRR calculation of 9.59% equals the after-tax IRR of 9.59% compared to the before tax IRR of 10%. We can find θ and the effective tax rate by setting

$$(20.15) \quad r(1 - \theta T) = .1[1 - \theta(.1)] = .0959.$$

From the above equation we find $\theta = .41$. Thus the effective tax rate is $(.2)(.41) = 8\%$ —far less than the actual income tax rate of 20%.

To review the process, we began by solving for the maximum bid price of the bond. This required that the stated discount rate was the before-tax IRR for the bond. Then we reasoned as follows. If the maximum bid price is the same whether calculated on a before-tax or after-tax basis, then the effect of taxes on the cash flow—in this case the capital gains tax—must have the same effect on the discount rate. Thus, we required that the maximum bid price in the after-tax model be the same maximum bid determined in the before-tax model and solved for the after-tax IRR and the tax adjustment coefficient θ .

Common Stocks

In contrast to bonds, common stocks have neither a fixed return nor a fixed cost. The terminal value of bonds is usually fixed, but the terminal value of stocks depends on the market value of the stock at the time of sale. The equity capital generated by the sale of stock is an alternative to debt capital generated by the sale of bonds. It also is a means of sharing risk among numerous investors.

Stocks offer significant benefits for stockholders as well as the companies issuing the stocks. Stockholders have the opportunity for ownership in the major businesses of the world with the consequent share in profits while their liability is limited to their investments. Moreover, stock ownership frees them from decision-making responsibilities in the management of the company, although common stock allows its owners to vote for directors and sometimes other matters of significance facing the company.

Stock owners receive dividend payments on their stocks, usually on a quarterly basis. The amount of dividends paid on stocks is determined by a corporation's board of directors. The board of directors' dividend policies may influence the kinds of stock they issue and the kinds of investors they attract.

The relevant question for a potential stock purchaser is what is the maximum bid price for a particular stock? If r is the nominal discount rate, and R_1, R_2, \dots are projected dividends paid on the stock in periods 1, 2, \dots , the maximum bid price for the stock is:

$$(20.16) \quad V_0^b = \frac{R_1}{(1+r)} + \frac{R_2}{(1+r)^2} + \dots$$

The model above assumes an infinite life. This assumption is consistent with the "life of the investment" principle discussed in Chapter 8 because to know the terminal value of the stock, V_n , we must know the value of the dividends in periods $(n+1), (n+2), \dots$. Knowing all future dividends converts the problem to one in which the number of periods equals the life of the firm.

A simplified form of equation (20.16) is possible if expected dividends R_1, R_2, \dots are replaced by their expected annuity equivalent R . Then we can write:

$$(20.17) \quad V_0^b = RUS_0(r, \infty) = \frac{R}{r}$$

And r , the stock's IRR, is R/V_0 . Thus, with long-term constant dividends of \$100 and the stock valued at \$1000, the rate of return is 10%. Another important ratio derived from equation (20.16) is what is commonly referred to as the price-to-earnings ratio, $V_0/R = 1/r$ which is often viewed as the bellwether of financial irregularities in the financial market. Higher than usual price-to-earnings ratios may signal what has been called "irrational exuberance" for the investment. Lower than usual price-to-earnings ratios may signal that the investment is undervalued.

Finally, debt capital must be repaid regardless of the financial fortunes of the business. However, the return to stockholders depends critically on the performance of the company. This makes NPV analysis of stock investments subject to considerable uncertainty.

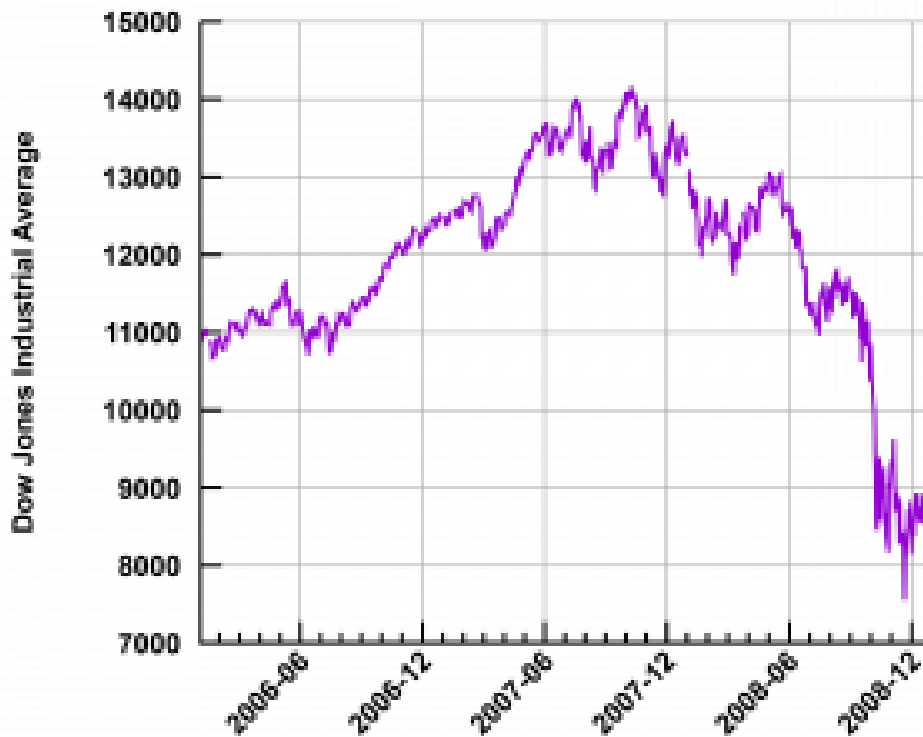


Figure 20.1. Dow Jones crash in 2008.

So, what have we learned? We learned to distinguish between real and financial investments. Earlier chapters in this book focused on real investments. Our focus in this chapter was on financial investments including time deposits, certificates of deposits, bonds, and stocks. The important point is that we can examine both real and financial investments using PV tools.

Summary and Conclusions

This chapter has introduced concepts related to financial investments which involve exchanges of money over time. In relation to financial investments, the focus is on the amount of one's investment available at some future period of time. We find the FV of a financial investment by compounding inter-

est. As we demonstrated in Table 20.1, the power of compounding is truly amazing. Therefore, the advice for most investors is to invest early and continuously, and then let one's investment grow.

Time deposits represent an important financial investment opportunity. These differ mostly in their liquidity—those that are least liquid are also the ones with the higher yields. Bonds are an important class of financial investment. In contrast to most other financial investments, their liquidation value is fixed. What is not fixed is their purchase price which is established in the market and depends on their yield—their expected IRR.

One of the interesting aspects of bonds is the way they are taxed. Often when bonds are issued by municipalities and some other institutions, they receive special tax considerations. In the extreme neither their coupon payments nor their capital gains are taxed. In other circumstances, capital gains are taxed and coupon payments are not. Regardless, an important question is how do special tax provisions provided with some bonds affect their after-tax IRRs. While the effective after-tax IRR can sometimes be found in a formulaic expression, often they are too complicated to be expressed in a closed form solution but can be found numerically by calculating the before and after-tax IRRs.

Finally, we only introduced the concept of common stocks. Investments in common stocks are most often discussed in the context of risk analysis and include discussions of many alternative of risk response strategies. A more complete discussion of investments in stocks is beyond the scope of this financial management text.

Questions

1. To invest means to give up something in the present for something of value in the future. Can you describe three of your most important personal investments? Describe what you sacrificed including money and time, and describe your expected future returns.
2. Describe the difference between real and financial investments. Also describe how the investment settings for real and financial investments may differ.
3. Describe the main characteristics of alternative types of time deposits? What are the rates of return currently offered by major banks or credit unions on alternative types of time deposits?
4. Suppose a saver invests \$65 in a time deposit at the beginning of each period for 10 periods. If the deposits are compounded at a rate of 4%, what amount is available for withdrawal at the beginning of period 11?
5. Suppose a young couple now renting an apartment wants to invest in their own home. The average price range of their desired home is approximately \$250,000. If they make monthly deposits in a time deposit account at the beginning of the period for 5 years and their deposits earn an APR compound rate of 3%, what would be the required size of their monthly deposit to pay for the 20% required down payment of the price of their home (\$50,000)? If they desired to save the required down payment in three years, what would be the required size of their monthly deposits?

6. It is claimed that Albert Einstein called compound interest “the greatest mathematical discovery of all time.” What is it about compounding that is so important that he would make such a claim? Do you agree? If so, why? If you disagree, what would be an alternative mathematical discovery of greater importance?
7. Describe in words the meaning of the elasticity of the future value of an investment with respect to its term. Then calculate the elasticity of the future value of an investment compounded at an APR rate of 4% for 12 periods. How would the elasticity measure change if the compound rate were increased to 7%? Can you explain the direction of the change?
8. Find the number of periods required to double the future value of an investment compounded for 5 periods at alternative reinvestment rates of 2%, 4%, and 8%.
9. An investor desires to save \$10,000. If the compound rate is 3% per month and the investor plans on saving \$200 at the beginning of each month, how many months will be required for the investor to reach her savings goal of \$10,000? Once the saving goal is reached, how many additional months will be required for the saver to double the amount saved if she saves at the same rate?
10. Consider an 8-year bond with a par value of \$10,000. If the purchase price of the bond is \$4,250, what is the yield on the bond? (Ignore the influence of taxes.)
11. What is the relationship between the bond’s yield (its IRR) and the bond’s purchase price? What would you expect to happen to the bond’s purchase price if the expected yield on the bond increased? Please explain.
12. In the example illustrating equation (20.8), the effective tax rate for a bond purchaser was found to equal $\theta T = .66(.15) = 10\%$. Please recalculate the effective tax rate in the example assuming the capital gains tax rate is only 50% of the income tax rate. In other words, if the capital gains tax rate is 7.5%, what is the bond purchaser’s effective tax rate? How does lowering the effective tax rate change the firm’s effective after-tax IRR?
13. Consider the equation:

$$(Q20.1) \quad V_0 = (r^c V_n)(1 - T)US_0[r(1 - \theta T), n] + \frac{V_n - (V_n - V_0)T_g}{[1 + r(1 - \theta T)]^n}$$

where $r(1 - \theta T)$ is the after-tax IRR of a bond.

Next consider the equation:

$$(Q20.2) \quad V_0 = (r^c V_n)US_0(r, n) + \frac{V_n}{(1 + r)^n}$$

where r is the before-tax IRR of the bond.

Then assume that $V_0 = \$500$, $V_n = \$1000$, $r^c V_n = .04(\$1,000) = \40 , $T = 20\%$, and $T_g = 10\%$. Find the before-tax and after-tax IRRs using the two equations described in this question and the values listed. Then use the before-tax and after-tax IRRs to find the value of the tax-adjustment coefficient θ .

14. Compare the rule of 72 with the actual time required to double your investment.

Derivation 20.1.

The derivation of the elasticity of future value (FV) with respect to term:

The elasticity of FV with respect to n is defined as:

$$(20.i) \quad E_{FV,n} = \frac{dFV}{dn} \frac{n}{FV}$$

The future value of n payments compounded at rate r is equal to:

$$(20.ii) \quad FV = \int_0^n Re^{rt} dt = \frac{R}{r}(e^{rn} - 1)$$

And the derivative of FV with respect to n is:

$$(20.iii) \quad \frac{dFV}{dn} = Re^{rn}$$

Substituting into the elasticity formula we find:

$$(20.iv) \quad E_{FV,n} = \frac{dFV}{dn} \frac{n}{FV} = \frac{nRe^{rn}}{\frac{R}{r}(e^{rn} - 1)} = \frac{rn}{1 - e^{-rn}}$$

The discrete approximation of the elasticity of FV with respect to n is:

$$(20.v) \quad E_{FV,n} = \frac{dFV}{dn} \frac{n}{FV} = \frac{nRe^{rn}}{\frac{R}{r}(e^{rn} - 1)} = \frac{rn}{1 - e^{-rn}} \approx \frac{n}{US_0(r, n)}$$

Derivation 20.2.

Deriving the formula for calculating the number of periods to double the FV of an n period investment reinvested at an interest rate of $r\%$.

To begin, the future value (FV) of an n period investment R compounded at rate r is equal to:

$$(20.vi) \quad FV = \int_0^n Re^{rt} dt = \frac{R}{r}(e^{rn} - 1)$$

Now m times the FV of the original investment is set equal to the same investment compounded for nm periods:

$$(20.vii) \quad m \int_0^n Re^{rt} dt = \frac{mR}{r}(e^{rn} - 1) = \int_0^{n_m} Re^{rt} dt = \frac{R}{r}(e^{rn_m} - 1)$$

and simplifying by equating the two integrated equations above we find:

$$(20.viii) \quad m(1 - e^{-rn}) = (e^{rn_m - rn} - e^{-rn})$$

Then solving for the difference between $(nm - n)$, we find the time it takes to increase the FV of the original investment at period n by m times.

$$(20.ix) \quad n_m - n = \frac{\ln[m + (1 - m)e^{-rn}]}{r}$$

21. Yield Curves

LINDON ROBISON

Learning goals. After completing this chapter, you should be able to: (1) understand how interest and earning rates on a variety of debt and investments instruments are related to yield curves and periodic interest rates; (2) describe how yield curves and periodic interest rates defined over future time periods can be used to predict market participants' expectation of future opportunities and threats; and (3) use yield curves and periodic interest rates to better understand and predict future financial opportunities and threats.

Learning objectives. To achieve your learning goals, you should complete the following objectives.

- Learn how to calculate single period discount rates from investments of varying terms.
- Understand the relationship between yield curves and single period discount rates.
- Learn how yield curves can be used to predict future financial opportunities and threats.

Introduction

In most present value (PV) models, the discount rate is a constant even when the term of the model has changed. In reality, each period's discount rate may be different because factors that influence the discount rate are not constant. These factors include the level of economic growth in the economy, the inflation rate, national and international events likely to influence our economy, activities in the stock and bond market, housing and land markets, the unemployment rate, and election outcomes.

One way to observe the difference in period discount rates is to observe the interest rates on loans with different terms. They are not constant. To demonstrate, note the changes in variable interest rate loans. They aren't constant either. Or, relevant to this discussion, notice the difference in average yields on bonds with different maturities even in the same risk class. Alternatively, notice the change in the internal rate of return (IRR) of a bond as its time to maturity changes over times.

In what follows, we pursue two objectives. The first objective is to demonstrate how to calculate the periodic discount rate for no-coupon bonds. The approach we describe can also be used to find periodic discount rates implied by bonds that are more complicated as well as other financial instruments. Our second objective is to connect the shapes and patterns of periodic interest rate curves and their corresponding yield curves patterns to predict future economic activity and opportunities and threats. Yield curves, as we will explain in more detail later, are the geometric means of periodic interest rates at a point in time with various maturities.

Geometric Means and Periodic Discount Rates

Geometric means and long-term discount rates.

Suppose we have information that allows us to predict n future periodic discount rates r_1, r_2, \dots, r_n . The geometric mean for the n periodic rates satisfies the following equation:

$$(21.1) \quad (1 + r)^n = (1 + r_1)(1 + r_2) \cdots (1 + r_n)$$

In words, one plus the geometric mean multiplied by itself n times would equal the product of 1 plus n periodic discount rates. We can solve for the geometric mean of the n periodic discount rates in equation (21.1) by finding the n th root of the products of the n periodic discount rates and subtracting one:

$$(21.2) \quad r = [(1 + r_1)(1 + r_2) \cdots (1 + r_n)]^{(1/n)} - 1$$

To demonstrate, suppose that periodic discount rates for period one through three were $r_1 = .07$, $r_2 = .03$, and $r_3 = .075$. We can solve for the geometric mean of the three periodic interest rates as:

$$(21.3) \quad r = [(1 + .07)(1 + .03)(1 + .075)]^{(1/3)} - 1 = .058 \text{ or } 5.8\%$$

How to calculate periodic discount rates.

Consider a zero coupon bond that can be purchased for V_0^1 at the beginning of the period and redeemed at the end of the period for its par value of \$1,000. The periodic rate of return r_1 for the one-period bond satisfies the PV model:

$$(21.4) \quad V_0^1 = \frac{\$1,000}{(1 + r_1)}$$

Solving for the one-period discount rate r_1 we find:

$$(21.5) \quad r_1 = \frac{\$1,000}{V_0^1} - 1$$

For example if $V_0^1 = \$960$ then

$$(21.6) \quad r_1 = \frac{\$1,000}{\$960} - 1 = .042 \text{ or } 4.2\%$$

In the case of a one-period bond, the one-period discount rate is also the geometric mean.

Now consider a zero-coupon bond that matures in two periods with similar risk and tax provisions as the one-period bond described in equation (21.4). Assume the zero-coupon two-period bond can be purchased for V_0^2 at the beginning of the period and redeemed at the end of the period for its par value of \$1,000. The periodic rate of return r_2 for the two-period bond satisfies the PV model:

$$(21.7) \quad V_0^2 = \frac{\$1,000}{(1 + r_1)(1 + r_2)}$$

Solving for the second period discount rate r_2 we find:

$$(21.8) \quad r_2 = \frac{\$1,000}{V_0^2(1 + r_1)} - 1$$

For example if $V_0^2 = \$910$ then

$$(21.9) \quad r_2 = \frac{\$1,000}{\$910(1.042)} - 1 = 0.055 \text{ or } 5.5\%$$

When we purchase a two period bond, we acquire a financial instrument with a single yield for two periods. The yield is the geometric mean of the product of the single period discount rates. In our example, the yield or geometric mean is equal to:

$$(21.10) \quad r = [(1 + 0.042)(1 + .055)]^{1/2} - 1 = .048 \text{ or } 4.8\%$$

Continuing our example, if $V_0^3 = \$855$ then

$$(21.11) \quad r_3 = \frac{\$1,000}{\$855(1.042)(1.055)} - 1 = .064 \text{ or } 6.4\%$$

Finally, consider a zero-coupon n period bond that can be purchased for V_0^n at the beginning of the period and redeemed at the end of the period for its par value of \$1,000. The periodic rate of return r_n for the n period bond satisfies the PV model equal to:

$$(21.12) \quad V_0^n = \frac{\$1,000}{(1 + r_1)(1 + r_2) \cdots (1 + r_n)}$$

Solving for the one-period discount rate r_n we find:

$$(21.13) \quad r_n = \frac{\$1,000}{V_0(1+r_1)(1+r_2)\cdots(1+r_{n-1})} - 1$$

Periodic rates of return and geometric means.

When we purchase a three-period bond, we acquire a single yield for three periods. Consistent with the PV model, the yield is the geometric mean of the product of the single-period discount rates. In our example, the yield or geometric mean for the three-period bond is equal to:

$$(21.14) \quad r = [(1 + .042)(1 + .055)(1 + .064)]^{1/3} - 1 = .054 \text{ or } 5.4\%$$

While one could expect to earn the bond's yield to maturity if the investor held the bond to maturity, a three-year bond becomes a two-year bond after one year, and if sold, the bond's price would reflect the yield on a two-year bond.

Graphing the geometric means of bonds against their varying time to maturity produces the bond's yield curve. In Figure 21.1, we graph the periodic discount rates and the corresponding yield curves for one, two, and three period bonds.

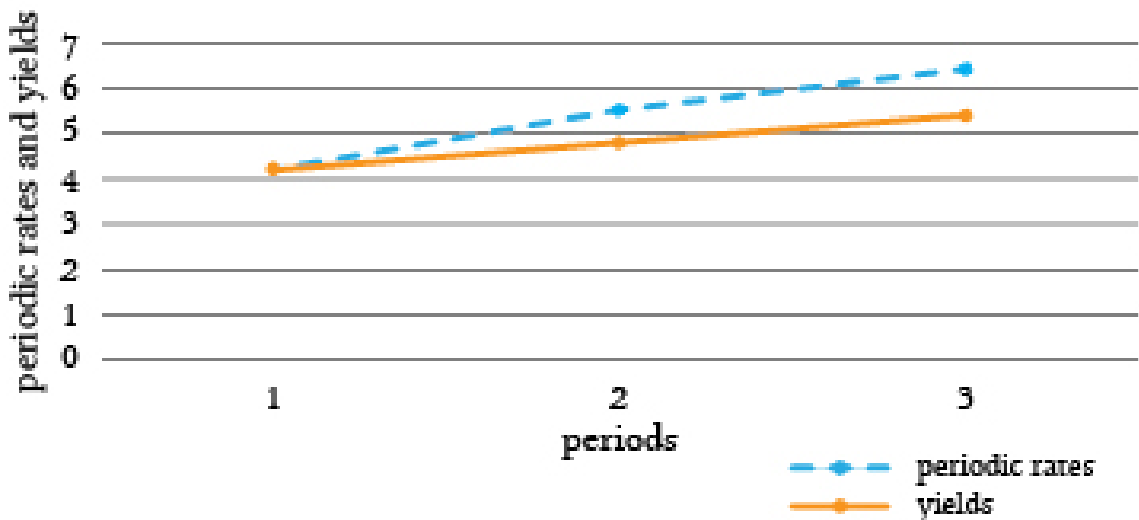


Figure 21.1. A comparison of periodic interest rates and the corresponding yield curves of bonds of varying maturities

Predicting Future Economic Activities and Opportunities and Threats

Marginal periodic rates and average geometric mean yield curves.

The periodic rate is a marginal rate while the yield rate is an average or mean—in this case a geometric mean. In our example, the periodic rate is increasing. As a result, the yield curve is also increasing but at a slower rate because previous and lower values of the periodic rate influence the yield. Therefore, if the yield curve is increasing, it is because the marginal periodic rates added to the series are greater than the geometric mean of the previous values included in the calculation. Moreover, if the yield curve were decreasing, it would require that periodic rates added were less than the geometric mean. To be complete, if the yield curve were constant, it would suggest that the marginal periodic rate were equal to the geometric mean.

Of course, there are many bond yield curves depending on the type of bond considered. When producing a yield curve, it is essential that bonds used to produce the yield curve belong to a similar risk class—even though this may be difficult because term differences produce different risks. To do the best we can to hold risk constant when producing yield curves, government backed debt is often considered. The most frequently reported yield curve compares three-month, two-year, five-year and thirty-year U.S. Treasury debt. We present graphs of U.S. Treasury yield curves at two points in time in Figure 21.2 below

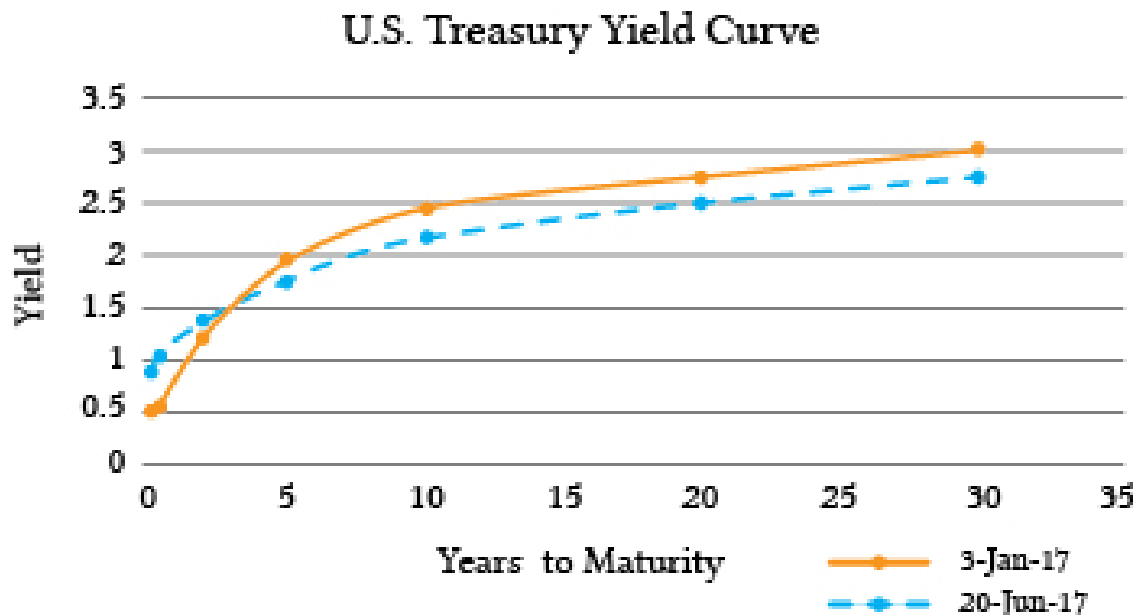


Figure 21.2. Yield curves using U.S. Treasury debt calculated on January 3, 2017 and June 20, 2017

Interpreting the shapes of yield curves.

In the previous section, we described the relationship between the period or marginal interest rates and the yield curve that, according to our PV model, is the geometric mean of the periodic rates. We now suggest some interpretations of the yield curves. Figures of periodic rate calculated from yield curves are not generally available.

- *Positively sloped yield curves.* To invest or lend one's financial capital to another person or entity requires a sacrifice on the part of the lender. Furthermore, the longer the funds are committed to another person or entity, the greater is the sacrifice. As a result, many economists claim that upward sloping yield curves predict a healthy economy in the future where borrowers have a bright view of future earnings and are willing to pay increased interest rates for the privilege of investing in the future. Another explanation is that borrowers expect periodic interest rates in the future to rise that in turn will produce an increasing yield curve. One reason for periodic rates to rise is expected increases in inflation and subsequent response to inflation by the Federal Reserve to increase interest rates on government debt instruments to offset inflationary pressures.
- *Flat or humped yield curves.* A flat yield curve is consistent with constant periodic interest rates so that all bond maturities have similar yields. A humped yield curve implies that periodic interest rates for a period lie above then fall below the yield curve and are constant before and after the hump. Economists generally view constant or humped yield curves as uncertain indicators of the future well-being of the economy.
- *Negatively sloped or inverted yield curves.* During periods when financial market participants expect periodic rates of return to decrease, yield curves have been downward sloping or inverted. Some financial economist connect inverted yield curves with pending downturns in the economy or recessions. In support of this connection, an inverted yield curve has indicated a worsening economic situation in the future 7 times since 1970 (Adrian, 2010). See Table 21.1.

Summary and Conclusions

Previous chapters have treated the multi-period discount rate in PV models (including IRR) as constants. This chapter has emphasized that these constant discount rates are composed of time varying periodic discount rates. Many time-varying factors would prevent periodic discount rates from being constant. Such forces acting on periodic interest rates include monetary and fiscal policies, inflation rates, unemployment rates, and national and international trade and treaties to name a few.

So, what have we learned? We learned that rate of return expectations built into periodic

discount rates and reflected in yield curves of bonds (and interest rates) of varying maturities reflect expected future economic conditions.

While there is not universal agreement on how to interpret yield curves, and indeed different yield curves may be subject to varying interpretations, there is support for interpreting downward sloping or inverted yield curves as foreshadowing a slow-down in future economic activities.

The goal of this chapter has been to acquaint students with another resource for predicting future opportunities and threats.

Table 21.1. The connection between inverted yield curves and future recessions (1970-2009).

Event	Date of Inversion Start	Date of the Recession Start	Time from Inversion to Recession Start	Duration of Inversion	Duration of Recession
			Months	Months	Months
1970 Recession	Dec-68	Jan-70	13	15	11
1974 Recession	Jun-73	Dec-73	6	18	16
1980 Recession	Nov-78	Feb-80	15	18	6
1981-1982 Recession	Oct-80	Aug-81	10	12	16
1990 Recession	Jun-89	Aug-90	14	7	8
2001 Recession	Jul-00	1-Apr	9	7	8

Questions

1. Compare geometric means and periodic discount rates.
2. Find the geometric mean of three periodic discount rates equal to $r_1 = 8\%$, $r_2 = 6.5\%$, and $r_3 = 2\%$.
3. Suppose the geometric mean of two periodic discount rates was 6%, and the periodic discount rate in the second period was 7%. Find the periodic discount rate in the first period.
4. Suppose a one-year zero-coupon bond with a par value of \$1,000 was selling for \$962. Find the bond's yield.
5. Suppose the yield curve was decreasing. Some economics would view this as a sign the economy

is slowing down. Do you agree or disagree? Defend your answer.

6. Find data on the yield curve in the economy and discuss what it portends.

22. Econs and Humans

LINDON ROBISON

Learning goals. After completing this chapter, you should be able to: (1) understand that, as humans, our needs include socio-emotional goods as well as those goods that can be purchased with money; and (2) that our needs motivate us to allocate resources to achieve both economic as well as socio-emotional goals.

Learning objectives. To achieve your learning goals, you should complete the following objectives:

- Learn the difference between “econs” and “humans”.
- Be reminded of the five distinct motives for pursuing a goal.
- Learn the difference between commodities and relational goods.
- Learn why humans exchange goods on terms and levels that econs view as irrational.
- Learn how including relational goods in our models explains exchanges that many econs would call irrational.

Introduction

The famous Beatles singing group titled one of their hit songs, “Can’t Buy Me Love.” While coming from an unlikely source of philosophical insight, their song title did proclaim an important truth: money can’t buy love nor most any other goods whose value depends on relationships. So, the purpose of this chapter is to remind us that relationships and the intangible goods produced in relationships may be our most important resources that need careful managing.

Econs versus humans. The behavioral economist Richard Thaler (2008) described two types of decision makers: “econs” and “humans.” Econs, according to Thaler, make decisions like persons described in graduate school economic textbooks. They are perfectly selfish, possess perfect information about the outcome of their choices, and their will power is absolute. Humans, on the other hand, fail to satisfy any of those assumptions. Their choices are influenced by relationships with other humans that lead them to unselfish choices. They lack complete information about the outcomes of their choices and about alternative choices. Finally, they sometimes overeat, drink too much, and also often don’t follow through on their choices.

Motives. A homework question at the end of Chapter 1 asked you what are your motives for wanting a college degree? The list of possible motives are reported below.

- I want a college degree so I can increase my lifetime earnings and get a better job.
- I want a college degree so important people in my life will be pleased with my achievements.

- I want a college degree to live up to the expectations I have for myself.
- I want a college degree so I will feel part of groups to which I want to belong.
- I want a college degree so that, in the future, I will be better able to help others.

An econ would have said that 100% of his or her motive was to increase lifetime earnings. Assuming your responses were not those of an econ, then you undoubtedly make decisions like a human. Your selection of motives can be used to infer that you care about and are motivated by other considerations than your desire to obtain a college degree to increase your potential income. The evidence from past surveys supports the conclusion that we are all human.

Anomalies

One outcome of managers behaving like humans instead of econs is that we observe behavior that is inconsistent with the prediction of profit and net present value (NPV) maximizing behavior associated with econs. One such example occurs regularly in the sale and purchase of farmland. Some of the anomalies we observe in the farmland market include terms and level of trade and selection of trading partners that depend on relationships, unselfish exchanges where sellers sacrifice higher prices for lower ones, and prefer to sell to friends and family rather than strangers and unfriendly neighbors.

To illustrate anomalies in the U.S. farmland markets, consider the minimum sell price of land. Compared to the arm's length market price, fifteen hundred farmland owner-operators in Illinois, Michigan, and Nebraska were surveyed. They reported discounting farmland prices to friends and family members by 5.57% and 6.78% respectively. These same owner-operators reported that they would require a minimum sell price premium of 18.4% to sell their land to their unfriendly neighbors (Siles, et. al., 2000).

Another farmland market study found that strangers entering the Linn County, Oregon, farmland market were at a decided disadvantage because they were forced to rely on public advertisements and Realtors to gain access to farmland sales information. Friendly neighbors and family of sellers accessed farmland purchase opportunities directly from the sellers. One consequence of this differential information access was that a stranger buying an 80-acre parcel of Class II non-irrigated farmland through a Realtor was projected to pay \$2,535 per acre while a neighbor of the seller buying the same land was projected to pay 20% less (Perry and Robison, 2001).

Finally, as a result of premiums and discounts and preferential access to farmland markets that depend on relationships, farmland sellers reported that less than 2% of their sales were to unfriendly neighbors while up to 70% of land sales were to friendly neighbors and family. Others reported similar observations in which relationships altered the terms and level of farmland trades. Indeed, terms and level of trade and selection of trading partners in farmland exchanges that are not influenced by relationships have a special name, "arm's length sales."

How do we explain the regular anomalies we observe in farmland markets? One explanation follows. Econs trade and consume only commodities. Humans trade and consume both commodities and relational goods. When we ignore relational goods in human exchanges and consumption, the results appear as anomalies. Consider next the nature of relational goods.

Commodities and Relational Goods

Expanding the decision maker's choice set. For the most part, economic theory focuses on physical goods and services that decision makers obtain for themselves and whose values do not depend on their connection to a particular person(s). We call these goods commodities. Describing this focus on commodities, Becchetti, Pelloni and Rossetti (2008) wrote: “in mainstream economics agents are mostly considered in isolation as they impersonally interact through markets, and consumption goods and leisure are assumed to be ‘sufficient statistics’ of their utility.”

Nothing in economic theory, however, prevents us from expanding the set of properties used to describe goods in decision makers' choice sets. For example, we could add the goods' relational properties to the description of those goods. Relational properties of goods include the identity of persons who produce, exchange, consume, and store goods in the choice set. Furthermore, this added description of goods could be justified if it were shown that decision makers' preference ordering depended on the relational properties of goods.

Social scientists in the past have connected a good's relational properties to its preference orderings (Bruni and Stanca, 2008). Indeed, Adam Smith may have foreshadowed the concept of relational goods when he described fellow-feeling or sympathy as essential for human happiness.

Emphasizing that the identity of exchange partners matters when defining relational goods, Uhlaner (1989) wrote, “goods which arise in exchanges where anyone could anonymously supply one or both sides of the [exchange] are not relational”. Luigino and Stanca (2008) concluded in their review of relational goods that “genuineness” is foundational, and the identity of the other person is essential for the value, and in some cases even for the existence, of the relational good. Gui and Sugden (2005) defined relational goods as “the affective components of interpersonal relations [that] are usually perceived as having value through their sincerity or genuineness”.

Defining relational goods. Relational goods are those goods whose value depends at least in part on their connections to people who produce, exchange, consume, and store them. Three concepts describe relational goods. The flow of relational goods is called socio-emotional goods (SEG). The stock or inventory of relational goods is called attachment value goods (AVG). SEG embedded in persons are said to create investments in social capital (Robison and Flora, 2003). Finally, social capital is required to produce SEG. We describe these, SEG, AVG, and social capital, in more detail in what follows.

SEG are intangible goods that satisfy socio-emotional needs. While there is no universally accepted list of socio-emotional needs that relational goods are expected to satisfy, generally accepted needs

include the need for internal validation or self-actualization, the need for external validation, the need for connectedness (belonging, love and friendship), and the need for knowing (Robison, Schmid, and Siles, 2002). SEG differ from other intangible goods and services because they are produced by social capital—sympathy (empathy), regard, or trust that one person has for another person or group.

SEG like other intangible goods can become attached to, associated with, or embedded in durable goods and change the meaning and value of the durable goods they act on. Durable goods embedded with SEG are called attachment value goods (AVG) and represent a stock of SEG. Since SEG and AVG “spring out of interpersonal relationships, and comprise the often intangible, interpersonal side of economic interactions”, they qualify as relational goods (Robison and Ritchie, 2010).

In mainstream economics, the production of commodities employs manufactured capital (tools and implements), natural capital, human capital, and financial capital. All of these contribute to the creation of a good or service valued for its mostly observable physical properties. In contrast, relational goods are produced in sympathetic (empathetic), trusting, and high regard relationships referred to here and by others as social capital. While there are other definitions of social capital, many of these do not satisfy the requirement of being capital or social. Instead they focus on where social capital lives (networks), what it can produce (cooperation), the rules that organize its use (institutions) and how to produce it (Robison, Schmid and Siles, 2002).

Human needs are satisfied by commodities and relational goods. The distinguishing properties of commodities and relational goods are described next.

Distinguishing properties of commodities. The properties that describe commodities have little or no connection to people or relationships among people and are described next. (1) Commodities are exchanged in impersonalized markets. (2) The terms and level at which commodities are exchanged are determined by the aggregate of market participants and apply generally. (3) Commodities are standardized goods of uniform quality which makes them perfect substitutes for each other so that little or no connection exists between their value and those who produce, exchange, consume, and store them. (4) The value of commodities can be inferred from their (mostly) observable properties. (5) Manufactured, natural, human, and financial capital may all play an important role in the production of commodities. (6) The value of commodities can be altered by changing their form, function, location, or other physical properties. (7) Commodities satisfy mostly physical needs and wants. (8) Commodities are mostly nondurable goods not likely to become embedded with SEG because of their short useful lives. (9) Commodities are most likely to have their quantity and quality certified by arm’s length agencies established for that purpose.

Distinguishing properties of relational goods. The properties that describe relational goods are wholly or partially dependent on the good’s connection to people who produce, exchange, consume, and store them and are described next. (1) Relational goods are exchanged in personalized settings in which either the buyer or the seller or both are known to each other. (2) The terms and level of relational goods exchanged are influenced by the social capital inherent in the relationships of those producing, exchanging, consuming, and storing them. (3) Relational goods are poor substitutes for each other because they are produced in unique relational settings. (4) The value of relational goods depend on their mostly unobservable intangible properties. (5) While other forms of capital may be used in their

production, relational goods cannot be produced without social capital. (6) The value of relational goods can be altered by changing their connections to people who produce, exchange, consume, or store them. (7) Relational goods satisfy mostly socio-emotional needs and wants. (8) Relational goods are mostly durable goods likely to become embedded with SEG because of their extended useful lives. (9) Relational goods are not likely to have their quantity or quality value certified by arm's length agencies established for that purpose.

We summarize the differences between commodities and relational goods in Table 22.1.

Table 22.1. Properties of commodities and relational goods.

Property	Commodities	Relational goods
1. Exchange setting.	Impersonal setting in which buyer and seller are not known to each other.	Personalized setting in which buyer and seller are connected through a social relationship.
2. How terms and level of exchange are determined.	Terms and level of goods exchanged are determined by the aggregate influence of market participants.	Terms and level of goods exchanged are uniquely determined by the social capital inherent in the persons engaged in the exchange.
3. Substitutability of goods.	Standardized goods with uniform quality which allows one commodity to substitute for another.	Unique good with few substitutes because its value is uniquely determined by the social capital involved in its exchange.
4. Value determining properties.	Mostly observable, physical properties.	Mostly unobservable intangible SEG exchanged directly or embedded in an AVG.
5. Capital used in their production.	Manufactured, natural, human, and financial capital may all be important in the production of commodities.	While other forms of capital may be used, social capital is required in the production of relational goods.
6. How the value of the good is changed.	Value is altered by changing the physical properties of the good including its form, function, location, taste, color, and other physical properties.	Value is altered by humanistic acts that produce SEG that may lead to increased investments in social capital or that may become embedded objects creating AVG.
7. Needs satisfied.	Mostly physical	Mostly socio-emotional including the need for validation, belonging, and knowing.
8. Durability.	Mostly nondurable or used infrequently.	Mostly durable or if nondurable, used frequently.
9. Certification.	Arm's length agencies empowered with regulatory and inspection duties.	Within the relationships associated with the good's production, consumption, or exchange.

Two setting for exchanging relational goods

There are two types of relational good exchanges. In the first type of exchange, the focus is on the relationships, and goods exchanged are mostly SEG. In the second type of exchange, the focus is on the good exchanged that is almost always an AVG, a tangible thing embedded with SEG.

Exchanges focused on relationships. In relationship focused exchanges, SEG are exchanged directly between persons in social capital rich relationships and require no object besides the persons involved in the exchange to complete the transaction. For example, two persons with strong feelings of affection for each other may express those feelings, SEG, in any number of settings including meals, cultural events, conferences, religious gatherings, or work settings. And if there is an object exchanged, it is incidental to the exchange of SEG.

Exchanges focused on objects. In object focused exchanges, AVG are exchanged, things or objects embedded with SEG. AVG result from prior or anticipated connections between social capital rich persons in which SEG are produced and embedded in objects. AVG are most likely a durable. However, AVG may sometimes be non durables that are often exchanged repeatedly such as a meal prepared to celebrate special occasion or a song or dance performed to mark milestones.

Anomalies and Isoutilities

At the beginning of this chapter, we claimed that econs and humans made decisions differently. Sometimes these differences are called anomalies. Furthermore, because many human decisions differ from econ choices, human choices are sometimes called irrational. The claim here is that human choices can appear irrational because relational goods are excluded from the analysis.

To illustrate how including relational goods in exchanges can resolve important economic anomalies, consider the following example. Suppose a seller has the option of exchanging his farmland with a stranger for a commodity (the market price) or exchanging his farmland with a friend or family member for a combination of commodities and relational goods. If the seller prefers the combination of relational goods and commodities offered by friends and family members to the commodities only offered by a stranger even though the commodities offered by the stranger exceed those offered by friends and family members, we might consider that an economic exchange anomaly has occurred. The seller accepted a lower commodity price from a friend or family member when a higher commodity price was available from a stranger. This is only an anomaly if the relational goods included in the exchange are ignored.

To explain further how including relational goods in exchanges can resolve anomalies, we consider the concept of an isoutility line. Suppose that a decision maker is offered alternative combinations of two goods, a commodity and a relational good. Furthermore, allow that the amounts of the commodity and relational good can be exchanged at some rate that leaves the well-being of the decision maker unaffected. The combinations of relational goods and commodities that leave the decision maker's well-being unaltered are referred to isoutility combinations and are represented in Figure 22.1 as $P_{\text{buyer}}P_{\text{buyer}}$. Curve $P_{\text{seller}}P_{\text{seller}}$ represents the seller's isoutility combinations of relational goods and commodity prices.

Commodity Price P

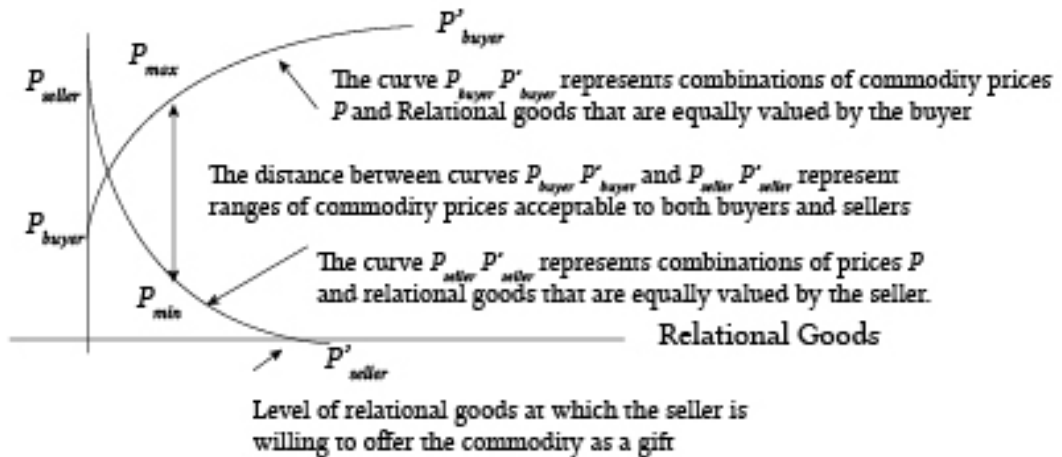


Figure 22.1. Combinations of Commodity Prices and Relational Goods that Leave Buyers' and Sellers' Well-Being Unchanged.

The implication of the graph in Figure 22.1 is that as more of a relational good is received, the seller (buyer) would be willing to accept (offer) a lower (higher) commodity price without suffering a loss in well-being. Furthermore, as relational goods are included in the transaction, the range of commodity prices acceptable to both buyers and sellers increases which also increases the likelihood that persons rich in social capital will exchange. For example, in Figure 22.1 persons without social capital would not trade since, with no relational goods exchanged, the minimum sell price is above the maximum bid price.

In one of the first studies designed to test the influence of relationships on terms and level of exchange, Robison and Schmid (1989) asked faculty and graduate students what would be their minimum sell price of a used car to people who offered them various levels of social capital. Since the Robison and Schmid article was published, the essence of the study has been repeated multiple times with similar results. A recent unpublished survey by Richard Winder found the results reported in Table 22.2. The mode of the distributions of responses by relationship are in bold.

Table 22.2. Average minimum sell price for a used car with a market value of \$3,000 reported by 600 survey respondents.

	Nasty neighbor	Stranger	Friend	Family
> \$3,500	65			
\$3,500	263	39	2	
\$3,250	21	33	5	1
\$3,000	236	476	122	29
\$2,750	11	30	135	24
\$2,500	4	22	298	199
< \$2,500			38	348

Notice that in the absence of social capital (exchanges with a stranger), the distribution of minimum sell prices centers around the commodity exchange price of \$3,000. However, when the exchange is conducted with a social capital rich partner such as a friend or family member, the minimum sell price is significantly below the market price with a mode of \$2,500 for a friend and a mode price below \$2,500 for a family member.

Summary and Conclusions

The material covered in this book is intended to provide instructions about how to behave like an econ. We hope you don't always follow our advice. There are times and places and circumstances when commodity considerations should be softened by social capital and the importance of relational goods. Learning how to make the appropriate trade-offs between commodities and investments in social capital and relational goods, when to behave like an econ versus a human, may be our most important management task. Good luck.

Questions

1. By 2013 Bill and Melinda Gates had donated 38 billion dollars to various charities and especially to fight hunger in Africa. They have donated billions more to these causes since. If Bill and Melinda Gates were to have taken the same survey you took at the beginning of the class where "I want a college degree" is replaced with "We donated billions of dollars...", how do you think they would have answered? To better speak for them, read a brief interview of Bill Gates by Neil Tweedie at: <http://www.telegraph.co.uk/technology/bill-gates/9812672/Bill-Gates-interview-I-have-no-use-for-money.-This-is-Gods-work.html>. Now answer the survey that follows as though you were channeling either Bill or Melinda Gates by writing the relative importance of each motive using percentages in the blank besides each motive. The sum of the motives must equal 100%.

- _____ We have donated billions of dollars so that we could increase our lifetime earnings and get a better job.
 - _____ We have donated billions of dollars so that important people in our lives would be pleased with our achievements.
 - _____ We have donated billions of dollars to live up to the expectations we have for ourselves.
 - _____ We have donated billions of dollars so we will feel part of different groups to which we want to belong.
 - _____ We have donated billions of dollars to help others.
2. Consider a hypothetical gas purchase.
 - a. Your gas tank holds 15 gallons and is nearly empty. You normally fill up your car with gas at a station on your way home. How many additional miles would you drive to fill up your car with gas if you could save 10 cents per gallon? I would drive an additional _____ miles to save 10 cents per gallon.
 - b. Your gas tank holds 15 gallons and is nearly empty. You normally fill up your car with gas at a station on your way home. How many additional miles would you drive to fill up your car with gas at a gas station owned and operated by your favorite uncle? I would drive an additional _____ miles to purchase gas at a gas station owned and operated by my favorite uncle.
 - c. If your answer to parts a) and b) were different from zero, can you explain why?
 3. Summarize the difference between relational goods and commodities as discussed in this chapter. Please list two commodities and two relational goods that you own.
 4. An isoutility line describes different combinations of two different goods that provide equal satisfaction. Use the concept of isoutility to explain why you might sell your used car at different prices to a friend, a stranger, a family member, and someone you disrespect.
 5. Commodities are sold in the market place, and their prices are determined by anonymous market forces. The terms of trade of relational goods depend on the relationship between persons exchanging them. Give an example in which you have exchanged relational goods in which relationships altered the terms and level of trade. Then give another example in which you have exchanged commodities in which relationships had no influence on the terms and level of trade.
 6. One could donate one's blood or blood plasma at a local Red Cross and receive in return a small amount of juice served in a paper cup and possibly a cookie. One could also sell one's blood at a number of places (the current price is \$25 to \$60 per bag). Since some people sell their blood for money and others donate it for free, explain the difference in the way these two groups of people dispose of their blood.
 7. Suppose you needed a medical procedure that required a skilled physician. Assume that a number of equally skilled physicians were available to perform the procedure. Would your choice of a physician to perform the procedure depend on your relationship to the physician? If he were a family friend? If he were a stranger? If you knew the physician only by reputation—that he performed volunteer work in developing countries? If he was rude to his/her patients, inconsiderate to his/her assistants, and lived a lavish life style? For each of the physicians, answer the questions that follow using the scale included in each question:

- a. If the physician were skilled and a family friend, the likelihood I would select this physician to perform my procedure is:
- Not likely 1 3 5 7 Very likely (circle one)
- b. If the physician were skilled and a stranger, the likelihood I would select this physician to perform my procedure is:
- Not likely 1 3 5 7 Very likely (circle one)
- c. If the physician were skilled and someone I knew only by reputation—that he performed volunteer work in developing countries, the likelihood I would select this physician to perform my procedure is:
- Not likely 1 3 5 7 Very likely (circle one)
- d. If the physician were skilled and someone I knew only by reputation—that he was rude to his/her patients, impatient with his assistants, and lived a lavish lifestyle, the likelihood I would select this physician to perform my procedure is:
- Not likely 1 3 5 7 Very likely (circle one)
8. If you were equally likely to select one of the physicians described in question 7, explain why. If you were not equally likely to select one of the physicians described above, explain why.

Appendix

LINDON ROBISON

Introduction

1

This appendix introduces Excel spreadsheets and describes how they can help solve financial management problems discussed in the text. The appendix describes how to input and manipulate data in Excel and illustrates both with a compound interest example. Then, the appendix describes how Excel's "goal seek" tool can test for solution sensitivity and answer "what if" kinds of questions. Finally, the appendix introduces Excel's financial functions and employs them to solve a variety of financial, accounting, and capital budgeting problems.

This appendix assumes readers know how to open, save, print, export, and import an Excel worksheet. In addition, we assume that readers are familiar with functions and number formatting options available in Excel. Having a knowledge of these basic Excel concepts, we are now ready to present Excel tools useful for solving financial management problems. Before doing so, however, we emphasize that what follows is an introduction to Excel. For more Excel details applied to financial management problems, readers should consult such texts as *Principles of Finance with Excel* by Simon Benninga (2006).

Opening an Excel spreadsheet. To open an Excel spreadsheet, find and press the Excel icon on your desktop. A blank worksheet should appear on your screen. Cells in Excel spreadsheets are identified with letter columns and row numbers. In Figure A.1, for example, the letter x appears in column D and row 5. When the cell with x is selected in the spreadsheet, the cell location is identified in the location tracker cell located in the top left of the spreadsheet as D5. (Try selecting different cells and observe the location tracker cell report the new location.)

1. The figures in the Appendix are screenshots of Microsoft Excel worksheets. The links to open Figures A.1, A.2, A.3, A.4, A.5, A.6, A.7, A.9, A.10, A.14, A.20, A.21, A.22, A.23, and A.24 all open the same Excel file. The worksheet tabs at the bottom of the Excel window can be selected for each figure.

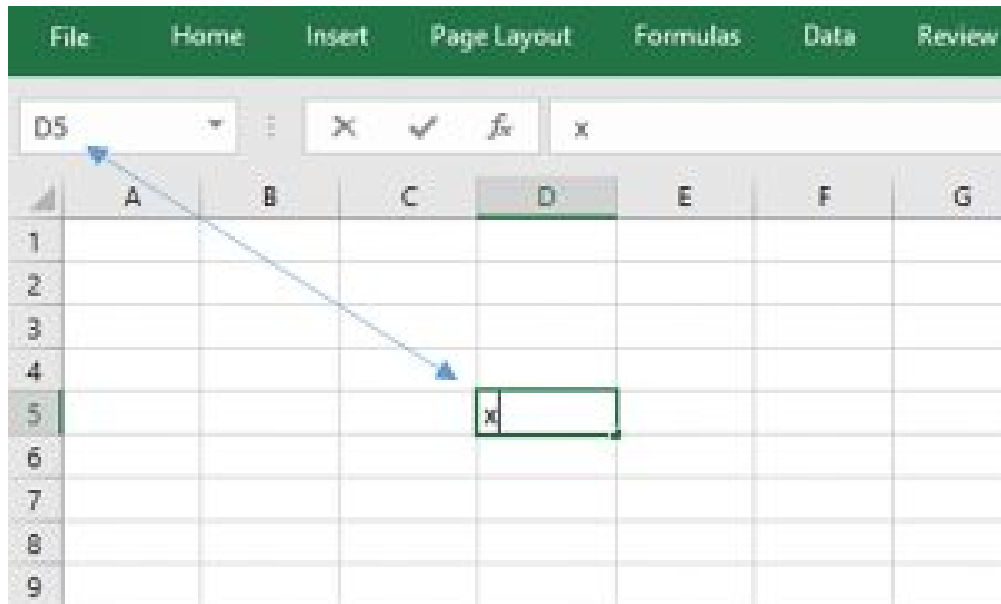


Figure A.1. Identifying cell locations in Excel
[Open Example in Microsoft Excel](#)

Users can instruct Excel to perform operations on data in cells by entering a function or calling up an Excel supplied function. Users enter a function to be executed by first pressing the “=” sign. The “=” sign instructs Excel that what follows is a function to be executed.

Entering user supplied functions in Excel. Excel has significant computing capacity that can execute user supplied functions. We illustrate a simple user supplied function in Figure A.2. We enter the numbers 5 and 4 into cells B1 and B2 respectively and in cell B3 instruct Excel to add the two numbers together. Note that the function appears to the right of the Excel function symbol *fx* on the blank space above the letters identifying the columns. After pressing enter, the function calculates the sum of 5 in cell B1 and 4 in cell B2 which is equal to the number 9 displayed in cell B3.

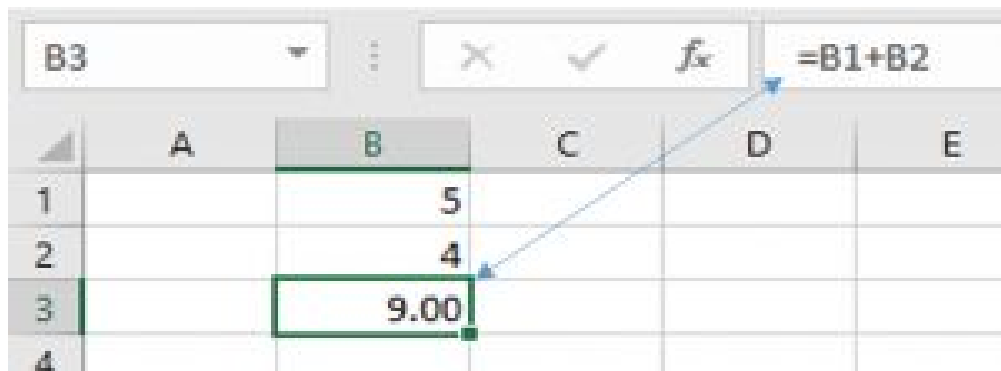


Figure A.2. A user supplied addition function.
[Open Example in Microsoft Excel](#)

We can always recover the equation embedded in cell B3 by pointing the cursor to cell B3. One of significant advantage of Excel is that we can replace the numbers in cells B1 and B2 and the function will automatically recalculate their sum. To illustrate, in Figure A.3 we change the number 5 in cell B1 to 7. Excel uses the same function entered earlier to add cells B1 and B2 to return the sum of 11.

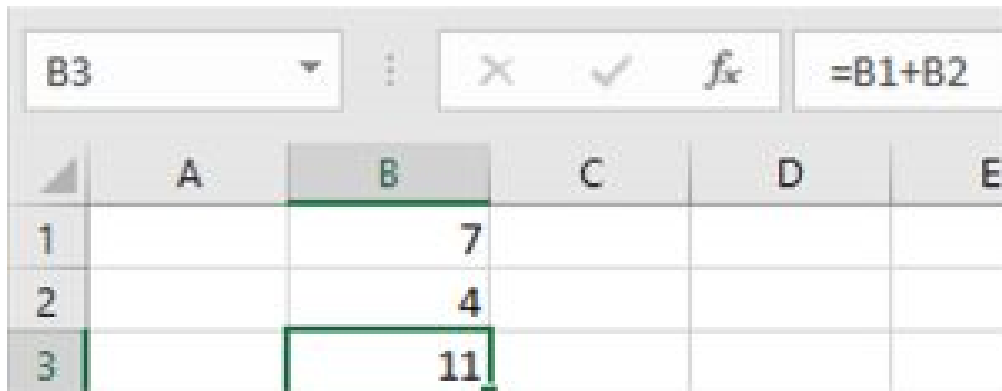


Figure A.3. An Externally supplied addition function with a changed entry.
[Open Example in Microsoft Excel](#)

Excel's library of functions. In addition to user supplied functions, Excel has a library of functions capable of operating on data identified in the spreadsheet. To illustrate, we employ Excel's SUM function for adding numbers. The SUM function lists the location of the beginning and ending cell in the string of numbers to be added, separated by a colon. We use the SUM function to solve the example described in Figure A.2 by entering into cell B3 the function `SUM(B1:B2)`.

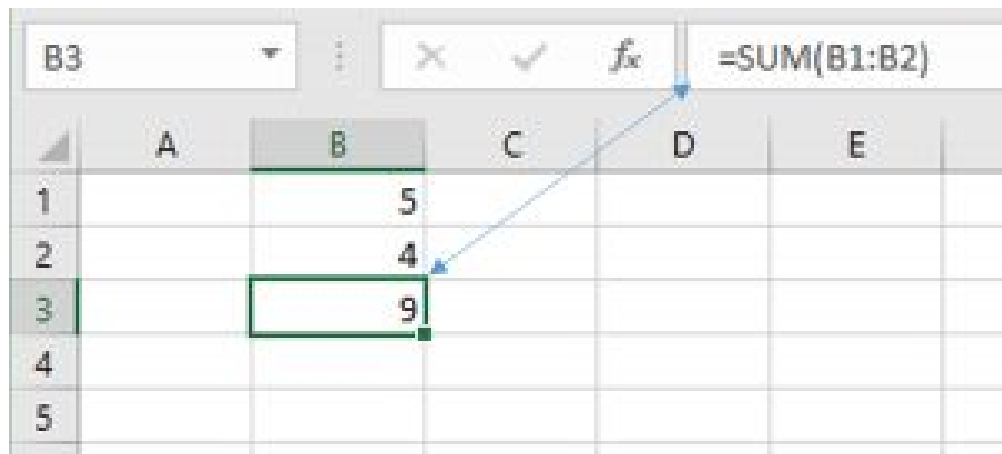


Figure A.4. Adding numbers using Excel SUM function.
[Open Example in Microsoft Excel](#)

Excel and handles. A “handle” is powerful Excel tool that enables Excel users to repeat operations or values in a cell by dragging the cell handle to a new location. In the new cell, the function in the handle cell is repeated and updated or the value in the handle cell is repeated. We describe handles next.

In Figure A.4, note the bold lines around cell B3 where the cursor is pointed. At the bottom right corner of the cell is a solid square called the handle. If we click on the handle and drag it to a new cell, B4, the function in cell B3 is repeated in cell B4, and the cell locations in the SUM function will be updated by one row value. In the other words, the function in the original cell SUM(B1:B2) is updated by one row in cell B4 to SUM(B2:B3). Just as the SUM function in cell B3 added the preceding two numbers, so does the SUM function in B4 add the preceding two numbers—4 and 9. We illustrate the use of handles in Figure A.5.

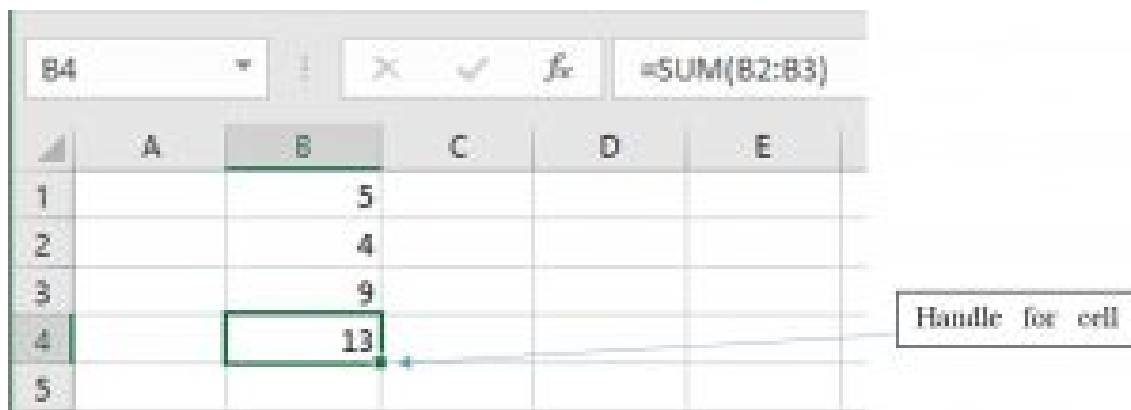


Figure A.5. Dragging the handle in cell B3 to enter the same updated function in cell B4.
[Open Example in Microsoft Excel](#)

Fixed cells. The function in B3 was SUM(B1:B2), and the updated function in B4 was SUM(B2:B3). However, there may be occasions when we want to fix a cell’s location in a function rather than have it automatically updated when dragged. In our example, suppose that we wanted to add the string of numbers that began in cell B1 rather than having the addition begin with the number in cell B2 when dragged one cell. To do so, we merely fix cell B1’s location in our function by preceding the column letter and row number with a “\$” sign as in \$B\$1. Other cells identified in the function that are not fixed are updated when the cell containing the function is dragged by its handle. In our example, the function in B4 updates the cell at the end of the string but keeps fixed the cell B1 at the beginning of the string. To illustrate, in Figure A.6 the SUM function contains a fixed cell B1 but updates the ending cell in the string when the function in B3 is dragged to location B4.

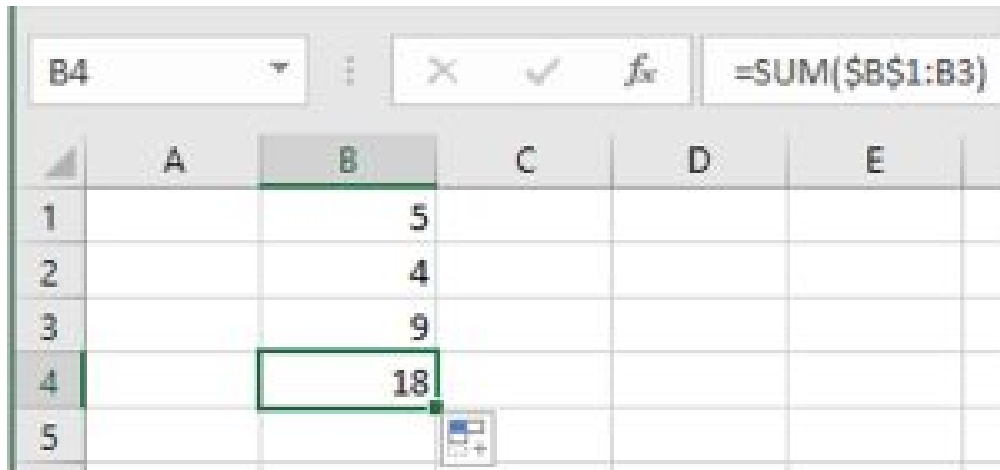


Figure A.6. Using Excel’s SUM function to add a string of numbers with the number in the first cell fixed.

[Open Example in Microsoft Excel](#)

Formatting cells. Suppose we wanted to indicate that the numbers in the string and function reflect dollars (or percentages, etc.). For the cells to reflect dollar amounts, we must format the cells in the Excel. To format Excel spreadsheet cells, we click “Home” on the main ribbon and then click “Format” on the drop-down menu. Notice that the last entry in the drop down Format menu is labeled “Format Cells” (See Figure A.7.).

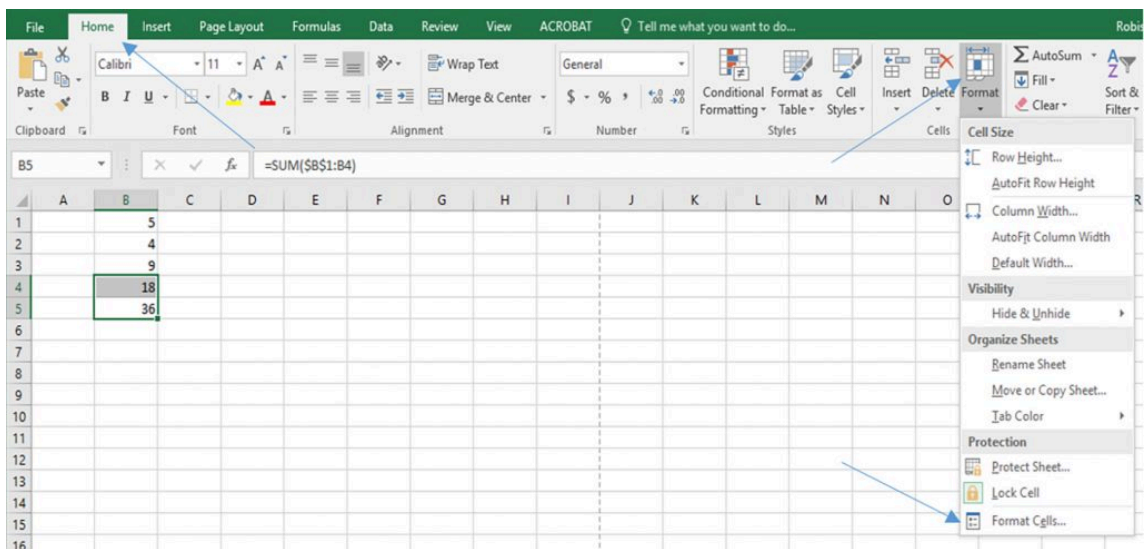


Figure A.7. Formatting cells in Excel by accessing the Format cell options.

[Open Example in Microsoft Excel](#)

Lastly, we click on “format cells” and note the several format options. After we highlight the cells we wish to reformat, we click the currency line with two decimal places.

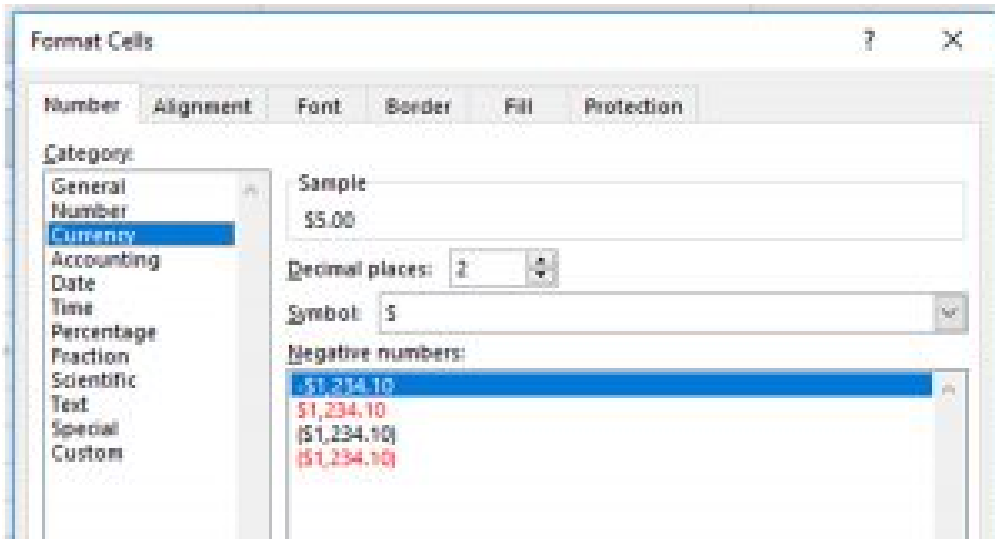


Figure A.8. Cell formatting options.

We display in Figure A.9 the numbers reformatted as currency.

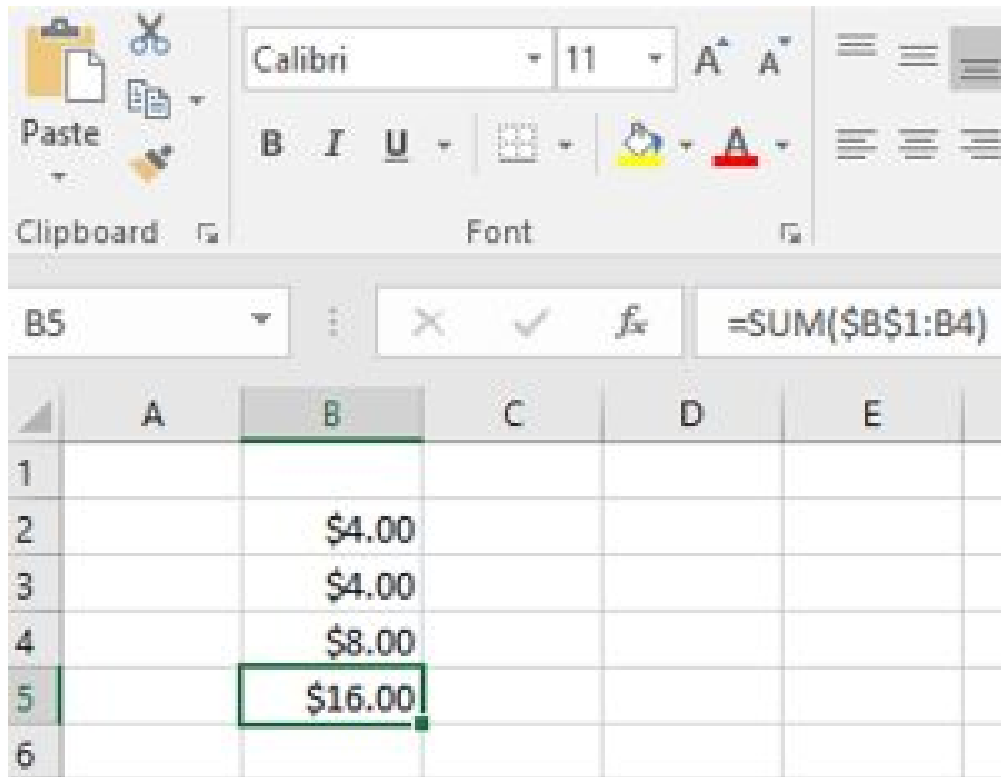


Figure A.9. Currency Formatted cells
[Open Example in Microsoft Excel](#)

Excel Financial Functions

Compound interest. We illustrate the important concepts of user supplied functions, handles, and fixed cell locations by calculating compound interest. Suppose we begin in the present, period zero, with an investment of \$10,000 that earns 5% per period. Therefore, at the beginning of period 1, our original investment has grown to $\$10,000(1.05) = \$10,500$. If we allow the original investment plus interest to grow another period, we have at the beginning of period two an investment of $\$10,000(1.05)(1.05) = \$11,025$. It might be interesting, although tedious, to see how our investment would grow over 10 periods following the same procedure. Excel can easily solve this problem using cell handles.

To begin, we label cell A3 “Years” and in cells A4 through A14, we list the number of years for which we want to compute the value of an investment that earns compound interest beginning in the present. We label cell B3, “Compound Values.” At the beginning of year 0 in cell B4, we list the original investment amount. At the beginning of year 1 in cell B5, we enter the function for compounding the value in the previous cell—multiplying the previous cell’s value by 1.05. Clicking on the handle in cell B5, and dragging it to year 10 in cell B14, we calculate the value of the investment in each of the 10 periods. To

make clear this operation, we display the function in each cell to the right of the cell in which the function is embedded. If we wish to explore the effects of compounding with alternative interest rates, we change the interest rate listed in cell C16. For example, compare the investment's value compounded at 5% percent rate compared to the investment's value compounded at 10%. The comparison is reported in Figure A.10.

	A	B	C	D	E		A	B	C	D	E
1	Compound Interest and Investment Values					1	Compound Interest and Investment Values				
2						2					
3	Year		Value	formula		3	Year		Value	formula	
4	0		\$10,000			4	0		\$10,000		
5	1		\$10,500.00	=C4*(1+\$C\$16)		5	1		\$11,000.00	=C4*(1+\$C\$16)	
6	2		\$11,025.00	=C5*(1+\$C\$16)		6	2		\$12,100.00	=C5*(1+\$C\$16)	
7	3		\$11,576.25	=C6*(1+\$C\$16)		7	3		\$13,310.00	=C6*(1+\$C\$16)	
8	4		\$12,155.06	=C7*(1+\$C\$16)		8	4		\$14,641.00	=C7*(1+\$C\$16)	
9	5		\$12,762.82	=C8*(1+\$C\$16)		9	5		\$16,105.10	=C8*(1+\$C\$16)	
10	6		\$13,400.96	=C9*(1+\$C\$16)		10	6		\$17,715.61	=C9*(1+\$C\$16)	
11	7		\$14,071.00	=C10*(1+\$C\$16)		11	7		\$19,487.17	=C10*(1+\$C\$16)	
12	8		\$14,774.55	=C11*(1+\$C\$16)		12	8		\$21,435.89	=C11*(1+\$C\$16)	
13	9		\$15,513.28	=C12*(1+\$C\$16)		13	9		\$23,579.48	=C12*(1+\$C\$16)	
14	10		\$16,288.95	=C13*(1+\$C\$16)		14	10		\$25,937.42	=C13*(1+\$C\$16)	
15						15					
16	interest rate		5%			16	interest rate		10%		
17						17					

Figure A.10. Analyzing the Effect of a 5% and a 10% Compound Interest Rate on Investment Values
[Open Example in Microsoft Excel](#)

Goal Seek. In the previous example, we found that \$10,000 compounded at 5% for 10 periods produced a value of \$16,288.95 and \$25,937.42 if \$10,000 were compounded at 10%. Suppose our goal was to have \$20,000 at the end of 10 years of compounding. We might ask: what initial investment, an exogenous variable, compounded for 10 years at 5% would produce a \$20,000, an endogenous variable whose value is determined within the system? We could, of course, answer this question by experimenting with alternative investment values in cell C4 where the initial investment amount is stored. However, this would be tedious and may be difficult to find the exact amount.

Excel offers an alternative approach to finding the desired value of an endogenous variable by changing an exogenous variable using an algorithm called “Goal Seek.” Excel’s Goal Seek algorithm answers “what if” kinds of questions. Specifically, it answers the question “what if” the value of the endogenous variable located in a particular cell has some specified value, then what must be the value of the exogenous variable located in another designated cell? Excel records the new values for endogenous and exogenous variables in the Excel spreadsheet. We illustrate the process using the compounding investment example described in Figure A.10.

To access Goal Seek, press the [Data] tab and the [What If Analysis] button. Finally, press “Goal Seek...” in the drop-down menu. The Data tab, the What If Analysis button, and the drop-down menu displaying the Goal Seek option are displayed in Figure A.11.

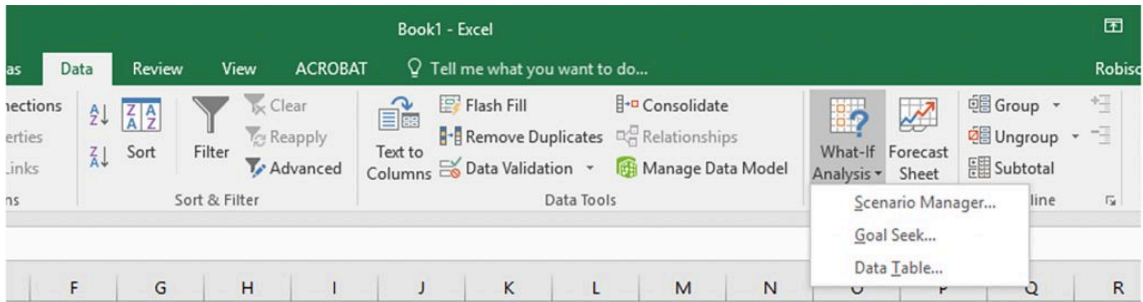


Figure A.11. Accessing Goal Seek in an Excel spreadsheet

The Goal Seek dialogue box asks that the endogenous variable’s cell location be identified in the “Set cell” entry field. We enter the desired value of the endogenous variable in the “To value” field. Finally, we identify the cell location of the exogenous variable that is to be changed in the “By changing cell” entry field. After pressing “OK”, Goal Seek finds the value of the exogenous variable required to obtain the desired value of the endogenous variable.

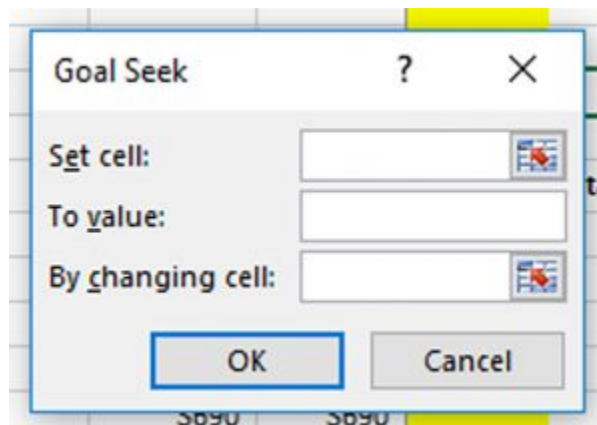


Figure A.12. The Goal Seek dialogue box asking for the endogenous variable location to be recorded in the “Set” field, the desired value of the endogenous variable to be recorded in the “To value” field, and the location of the exogenous variable to be recorded in the “By changing” field.

To illustrate, suppose we wanted to have an investment value of \$20,000 at the end of 10 years. Our “what if” question is: What if the value of our original investment compounded at 5% for 10 years were \$20,000? Then what would be our required investment amount? To solve this problem, in the

Goal Seek dialogue box we enter cell C14—the endogenous variable location—in the “Set cell” field; we enter \$20,000 in the “To value” field; and we enter cell location A4—the exogenous variable—in the “By changing cell” field. The results are recorded in Figure A.13.

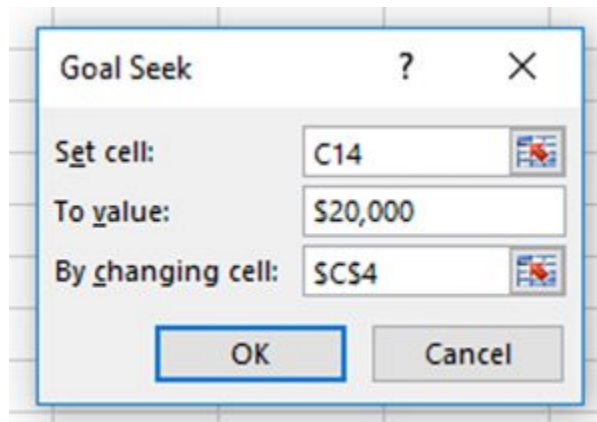


Figure A.13. A Completed Goal Seek dialogue box that instructs Excel to find the original investment compounded at 5% required to produce a \$20,000 investment value at the beginning of period 10.

Pressing the “OK” button returns the Goal Seek solution reported in Figure A.14.

	A	B	C	D	E
1	Compound Interest and Investment Values				
2					
3	Year		Value	formula	
4	0		\$12,278		
5	1		\$12,892.18	=C4*(1+\$C\$16)	
6	2		\$13,536.79	=C5*(1+\$C\$16)	
7	3		\$14,213.63	=C6*(1+\$C\$16)	
8	4		\$14,924.31	=C7*(1+\$C\$16)	
9	5		\$15,670.52	=C8*(1+\$C\$16)	
10	6		\$16,454.05	=C9*(1+\$C\$16)	
11	7		\$17,276.75	=C10*(1+\$C\$16)	
12	8		\$18,140.59	=C11*(1+\$C\$16)	
13	9		\$19,047.62	=C12*(1+\$C\$16)	
14	10		\$20,000.00	=C13*(1+\$C\$16)	
15					
16	interest rate		5%		
17					

Figure A.14. The Goal Seek solution, \$12,278, equals the initial Investment compounded at 5% required to produce a \$20,000 value at the beginning of period 10.

[Open Example in Microsoft Excel](#)

We have determined using Goal Seek that an investment value of \$20,000 at the beginning of period 10 requires that we invest \$12,278 in the present—an increase of \$2,278 over the original investment.

An Application of Goal Seek to HQN’s Coordinated Financial Statements (CFS)

In the text, we explored HQN’s financial condition reflected by its CFS and associated ratios, DuPont equation, and common size balance sheet and income statements. We now use Goal Seek to find the value of an exogenous variable required to produce the desired value of an endogenous variable. To begin, we copy HQN’s original CFS in Figure A.15.

	A	B	C	D	E	F	G	H	I
	BALANCE SHEET			ACCRUAL INCOME STATEMENT			STATEMENT OF CASH FLOW		
1	DATE	12/31/2017	12/31/2018	DATE	2018	DATE	2018		
3	Cash and Marketable Securities	\$930	\$600	+	Cash Receipts	\$38,990	+	Cash Receipts	\$38,990
4	Accounts Receivable	\$1,640	\$1,200	+	Change in Accounts Receivable	(\$440)	-	Cash Cost of Goods Sold	\$27,000
5	Inventory	\$3,750	\$5,200	+	Change in Inventories	\$1,450	-	Cash Overhead Expenses	\$11,078
6	Notes Receivable	\$0	\$0	+	Realized capital gains (losses)	\$0	-	Interest paid	\$480
7	Total Current Assets	\$6,320	\$7,000	=	Total Revenue	\$40,000	-	Taxes	\$68
8							=	Net Cash Flow from Operations	\$364
9	Depreciable Assets	\$2,990	\$2,710	+	Cash Cost of Goods Sold	\$27,000			
10	Non-depreciable Assets	\$690	\$690	+	Change in Accounts Payable	\$1,000	+	Realized Cap. Gains + Depr. Recapture	\$0
11	Total Long-Term Assets	\$3,680	\$3,400	+	Cash Overhead Expenses	\$11,078	+	Sales Non-depreciable Assets	\$0
12	TOTAL ASSETS	\$10,000	\$10,400	+	Change in Accrued Liabilities	(\$78)	-	Purchases Non-depreciable Assets	\$0
13				+	Depreciation	\$350	+	Sale Depreciable Assets	\$30
14	Notes Payable	\$1,500	\$1,270	=	Total Expenses	\$39,350	-	Purchases Depreciable Assets	\$100
15	Current Portion Long-Term Debt	\$500	\$450				=	Net Cash Flow from Investment	(\$70)
16	Accounts Payable	\$3,000	\$4,000		Earnings Before Interest and Taxes (E	\$650			
17	Accrued Liabilities	\$958	\$880	-	Less Interest Costs	\$480	+	Change in Non-Current Long Term Debt	(\$57)
18	Total Current Liabilities	\$5,958	\$6,600	-	Earnings Before Taxes (EBT)	\$170	+	Change in Current Portion of Long Term Debt	(\$50)
19	Non-Current Long Term Debt	\$2,042	\$1,985	-	Less Taxes	\$68	+	Change in Notes Payable	(\$230)
20	TOTAL LIABILITIES	\$8,000	\$8,585	=	Net Income After Taxes (NIAT)	\$102	-	Less Dividends and Owner Draw	\$287
21				-	Less Dividends and Owner Draw	\$287	=	Net Cash Flow from Financing	(\$624)
22	Contributed Capital	\$1,900	\$1,900	=	Addition to Retained Earnings	(\$185)			
23	Retained Earnings	\$100	(\$85)					CHANGE IN CASH POSITION (Q11+Q18+Q24)	(\$330)
24	Total Equity	\$2,000	\$1,815						
25	TOTAL LIABILITIES AND EQUITY	\$10,000	\$10,400						
27	BASE SPELL RATIOS FOR HQN								
29		12/31/2018	Industry						
30	SOLVENCY RATIOS								
31	Times Interest Earned (TIE)	1.35	2.5						
32	Debt Service Ratio (DSR)	1.02	1.40						
33	PROFITABILITY RATIOS								
34	Profit margin (m)	0.43%	1.03%						
35	Return on assets (ROA)	6.50%	3.30%						
36	After-tax ROA [ROA(1-T*)]	5.82%							
37	Return on equity (ROE)	8.50%	10.70%						
38	After-tax ROE [ROE(1-T)]	5.10%							
39	EFFICIENCY RATIOS								
40	Inventory Turnover (ITO)	10.67	7.7						
41	ITOT (365/ITO)	34.22	47.4						
42	Asset Turnover (ATO)	4.00	3.2						
43	ATOT (365/ATO)	91.25	114.1						
44	Receivable Turnover (RTO)	24.39	11.41						
45	RTOT (365/RTO)	14.97	32						
46	Payable Turnover (PTO)	9.33	12.59						
47	PTOT (365/PTO)	39.11	29.00						
48	LIQUIDITY RATIOS								
49	Current ratio (CR)	1.06	1.30						
50	Quick ratio (QR)	0.43	0.70						
51	LEVERAGE RATIOS								
52	Debt/Asset (D/A)	0.80	0.91						
53	Debt/Equity (D/E)	4.00	2.00						
54	Equity multiplier (A/E)	5.00	2.20						
57	CFS template	Exogenous Variables	SPELL Ratios	Common Size Balance Sheets	Common Size Income Statement				

Figure A.15. HQN's Coordinated Financial Statements and Embedded Functions and Associated Ratios. [Open CFS Example in Microsoft Excel](#)

Now suppose we are concerned about the low cash receipts efficiency ratio m and ask: what must our cash receipts equal to produce an m efficiency ratio of .8% or 8/10th of a penny on each dollar of sale? To answer this question, we enter the following into our Goal Seek dialogue box.

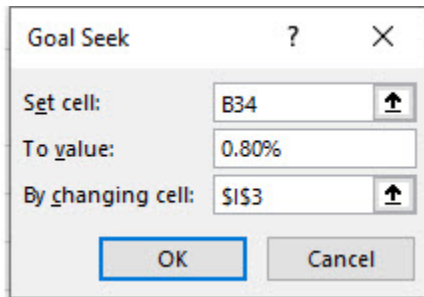


Figure A.16. Solving the “What if” question, how much does cash receipts need to increase to obtain an m ratio of .8% using Goal Seek

After pressing OK, we return the answer to our “What if” question in figure A.17.

BALANCE SHEET			ACCRUAL INCOME STATEMENT			STATEMENT OF CASH FLOW		
DATE	12/31/2017	12/31/2018	DATE	2018	DATE	2018		2018
Cash and Marketable Securities	\$930	\$752	+ Cash Receipts	\$39,142	+ Cash Receipts	\$39,142		\$39,142
Accounts Receivable	\$1,640	\$1,200	+ Change in Accounts Receivable	(\$440)	- Cash Cost of Goods Sold	\$27,000		\$27,000
Inventory	\$3,750	\$5,200	+ Change in Inventories	\$1,450	- Cash Overhead Expenses	\$11,078		\$11,078
Notes Receivable	\$0	\$0	+ Realized capital gains (losses)	\$0	- Interest paid	\$480		\$480
Total Current Assets	\$6,320	\$7,152	= Total Revenue	\$40,152	- Taxes	\$68		\$68
					= Net Cash Flow from Operations	\$516		\$516
Depreciable Assets	\$2,990	\$2,710	+ Cash Cost of Goods Sold	\$27,000				
Non-depreciable Assets	\$690	\$690	+ Change in Accounts Payable	\$1,000	+ Realized Cap. Gains + Depr. Recapture	\$0		\$0
Total Long-Term Assets	\$3,680	\$3,400	+ Cash Overhead Expenses	\$11,078	+ Sales Non-depreciable Assets	\$0		\$0
TOTAL ASSETS	\$10,000	\$10,552	+ Change in Accrued Liabilities	(\$78)	- Purchases Non-depreciable Assets	\$0		\$0
			+ Depreciation	\$350	+ Sale Depreciable Assets	\$30		\$30
Notes Payable	\$1,500	\$1,270	= Total Expenses	\$39,350	- Purchases Depreciable Assets	\$100		\$100
Current Portion Long-Term Debt	\$500	\$450			= Net Cash Flow from Investment	(\$70)		(\$70)
Accounts Payable	\$3,000	\$4,000	Earnings Before Interest and Taxes (E	\$802				
Accrued Liabilities	\$958	\$880	- Less Interest Costs	\$480	+ Change in Non-Current Long Term Debt	(\$57)		(\$57)
Total Current Liabilities	\$5,958	\$6,600	= Earnings Before Taxes (EBT)	\$322	+ Change in Current Portion of Long Term Debt	(\$50)		(\$50)
Non-Current Long Term Debt	\$2,042	\$1,985	- Less Taxes	\$68	+ Change in Notes Payable	(\$230)		(\$230)
TOTAL LIABILITIES	\$8,000	\$8,585	= Net Income After Taxes (NIAT)	\$254	- Less Dividends and Owner Draw	\$287		\$287
			- Less Dividends and Owner Draw	\$287	= Net Cash Flow from Financing	(\$624)		(\$624)
Contributed Capital	\$1,900	\$1,900	= Addition to Retained Earnings	(\$33)				
Retained Earnings	\$100	\$67			CHANGE IN CASH POSITION (Q11+Q18+Q24)	(\$178)		(\$178)
Total Equity	\$2,000	\$1,967						
TOTAL LIABILITIES AND EQUITY	\$10,000	\$10,552						

A	B	C
BASE SPELL RATIOS FOR HQN		
	12/31/2018	Industry
SOLVENCY RATIOS		
Times Interest Earned (TIE)	1.67	2.5
Debt Service Ratio (DSR)	1.18	1.40
PROFITABILITY RATIOS		
Profit margin (m)	0.80%	1.03%
Return on assets (ROA)	8.02%	3.30%
After-tax ROA [ROA(1-T*)]	7.34%	
Return on equity (ROE)	16.11%	10.70%
After-tax ROE [ROE(1-T)]	12.71%	
EFFICIENCY RATIOS		
Inventory Turnover (ITO)	10.71	7.7
ITOT (365/ITO)	34.09	47.4
Asset Turnover (ATO)	4.02	3.2
ATOT (365/ATO)	90.90	114.1
Receivable Turnover (RTO)	24.48	11.41
RTOT (365/RTO)	14.91	32
Payable Turnover (PTO)	9.33	12.59
PTOT (365/PTO)	39.11	29.00
LIQUIDITY RATIOS		
Current ratio (CR)	1.06	1.30
Quick ratio (QR)	0.43	0.70
LEVERAGE RATIOS		
Debt/Asset (D/A)	0.80	0.91
Debt/Equity (D/E)	4.00	2.00
Equity multiplier (A/E)	5.00	2.20

Figure A.17. The Goal Seek solution to the “What if” question, how much will cash receipts need to increase to produce an m value of .8%?

[Open “What if” CFS Example in Microsoft Excel](#)

The Goal Seek answer to the “what if” question described earlier is that cash sales must increase to \$39,142 in order for HQN to produce an m ratio equal to .8%. We may also observe that the new level of cash receipts would also increase both ROA and ROE values to well above industry standards. The ROE ratio increases from 8.5% to 16.11%. The ROA increases from 6.50% to 8.02%. Furthermore, increased cash receipts would improve the debt-to-service ratio from 1.02 to 1.18.

An overview of financial functions.

This section discusses how to use Excel financial functions, some of which were discussed in earlier book chapters—particularly Chapter 17 on loan analysis. Obviously, the financial functions discussed in this section are only a sampling of the many financial and other functions available to Excel users. We begin by writing the general present value (NPV) function in equation A.1. The various financial functions we will discuss in what follows are only special cases of equation A.1.

$$(A.1) \quad NPV = -V_0 + \frac{R_1}{(1 + r/m)} + \frac{R_2}{(1 + r/m)^2} + \cdots + \frac{R_{mn} + S_{mn}}{(1 + r/m)^{mn}}$$

We begin by noting a fundamental algebraic reality. A single equation can solve for at most one unknown variable—although the unknown variable may have more than one solution. Therefore, equation A.1 can produce a solution(s) for at most one unknown. As a result, all of the financial models described in this appendix assume that one of the variables in equation A.1 is unknown. Consider the following financial models that result by assuming one of the variables in A.1 is unknown.

The net present value (NPV) model. The NPV model solves for the net present value of an investment, the present value difference between the cost of the investment V_0 and future cash flows discounted by rate r/m plus the investment's liquidation value V_n .

The internal rate of return (IRR) model. The IRR model finds the value of the discount rate r that makes NPV equal to zero. The variable r is unknown, and because it enters the equation with an exponent, r may have more than one solution.

The maximum bid (minimum sell) price model. The maximum bid price model finds the value of V_0 such that NPV is zero. It represents the most an investor could pay for a future cash flow stream and still break-even. The value of V_0 that makes NPV zero may also represent the least-price an investor could receive in exchange for the investment's future cash flow stream and still break-even.

The annuity equivalent (AE) model. There are two versions of the annuity equivalent model. The first one finds the value of a constant cash flow R received for mn periods whose NPV equals the present value sum of cash flows R_i ($i = 1, \dots, mn$) plus S_{mn} , the equation's NPV.

$$(A.2) \quad \begin{aligned} NPV &= -V_0 + \frac{R_1}{(1 + r/m)} + \frac{R_2}{(1 + r/m)^2} + \cdots + \frac{R_{mn} + S_{mn}}{(1 + r/m)^{mn}} \\ &= \frac{R}{(1 + r/m)} + \frac{R}{(1 + r/m)^2} + \cdots + \frac{R}{(1 + r/m)^{mn}} \end{aligned}$$

In the second version of the annuity equivalent model, we set NPV equal to zero and find the annuity equivalent whose discounted value is equal to V_0 . Cash flows associated with a constant loan payment

fit this model where V_0 is the loan amount, R is the constant loan payment, r/m is the interest rate, and mn is the term of the loan.

$$\begin{aligned}
 V_0 &= \frac{R_1}{(1 + r/m)} + \frac{R_2}{(1 + r/m)^2} + \dots + \frac{R_{mn} + S_{mn}}{(1 + r/m)^{mn}} \\
 \text{(A.3)} \quad &= \frac{R}{(1 + r/m)} + \frac{R}{(1 + r/m)^2} + \dots + \frac{R}{(1 + r/m)^{mn}}
 \end{aligned}$$

Break-even models. In some cases, we might be interested in a particular value of R_i or the liquidation value S_{mn} that reduces NPV to zero. In these cases, we often employ Goal Seek to find that particular R_i or S_{mn} value.

The term model. There are two term models considered in this book. In the first, the unknown variable is the term, mn , and its solution depends on the other known variables in the equation. In the second term model, called the optimal term model, the solution for mn depends on a calculus-defined condition requiring that we solve the model for various values of mn until the optimal conditions are satisfied. This is a difficult model to solve because it requires that we know additional future values of variables. Forecasting the future values of variables is discussed in Chapter 11.

The future value model. The future value model solves for S_{mn} , the liquidation value of the investment, assuming NPV is equal to zero. The uniqueness of this model is that instead of discounting future cash flows to their present value, we compound cash flows to their future value in the mn^{th} period

Excel functions.

Corresponding to the general NPV equation variables described in A.1 are several Excel functions. Excel's PV function finds NPV and V_0 . Its PMT function finds R , an AE. Its NPER function finds mn . Its RATE function finds r/m , and its FV function finds the value of S_{mn} after setting NPV equal to zero.

Corresponding to the Excel functions PV, RATE, NPER, PMT, and FV are the arguments pv, rate, nper, pmt, and fv. Functions are capitalized. Arguments corresponding to functions are lowercased. When we solve for a particular function, the corresponding argument is eliminated from the function's list of arguments. We list the following Excel functions, their arguments, and the order in which the arguments appear. In addition to the list of arguments, functions call for "type" variables which tells the function whether cash flows occur at the beginning of the period in which case type=1 or at the end of the period in which case type=0. If values are not entered for "type" and "fv", Excel assumes they are both equal to zero. Excel financial functions and their arguments are listed next.

- The PV function is =PV(rate, nper, pmt, fv, type).
- The FV function is= FV(rate, nper, pmt, pv, type).

- The RATE function is =RATE(nper, pmt, pv, fv, type).
- The NPER function is =NPER(rate, pmt, pv, fv, type).
- The PMT function is =PMT(rate, nper, pv, fv, type).

To solve the functions listed above, we enter the cell locations of the argument values and enter them into the Excel function. For example, if the value for rate were listed in cell C4, nper were listed in cell C5, pmt were listed in cell C6, and fv and type were both zero, we would write the PV function as =PV(C4, C5, C6, 0, 0).

Calling up Excel Functions.

There are two ways to call up Excel functions. The first way is to write the function with its argument values in a designated location. For example, assume the value for rate were listed in cell C4, nper were listed in cell C5, pmt were listed in cell C6, and fv and type were both zero and listed in cells C8 and C9. Then, we would write the PV function in cell C7 as =PV(C4, C5, C6, C8, C9).

A second way to call up an Excel function, particularly if the name of the function or its arguments are not known, is to click on the Formulas tab on the main ribbon and the Financial button that produces a drop-down menu with a list of financial functions. The financial functions NPER, RATE, PV, FV, and PMT are included in the list. To illustrate, we find and click the PV function in Figure A.18 below.

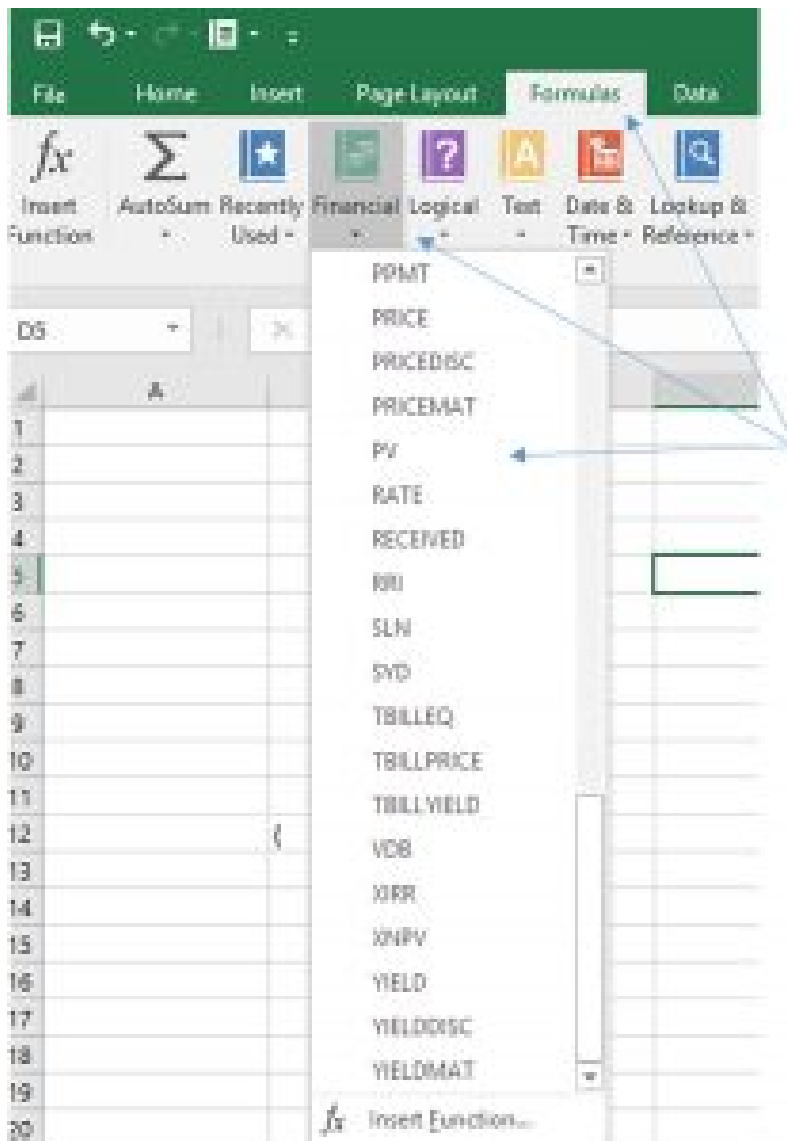


Figure A.18. Opening the Excel PV function.

After clicking on PV in the drop-down menu, the PV function dialogue box with its list of arguments is displayed in Figure A.19.

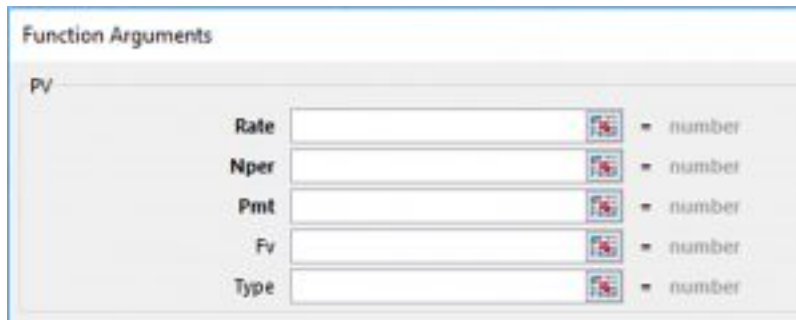


Figure A.19. The PV functions and its arguments.

To review, the PV function arguments correspond to variables in equation A.1. Rate corresponds to “ r/m ”, the discount rate in an NPV model or the interest rate in a constant loan model. Nper corresponds to the product of mn , the number of periods we compound interest in equation A.1. Pmt corresponds to the annuity equivalent R . FV corresponds to future value. One other argument not explicitly described in equation A.1, type, identifies if cash flows occur at the beginning of the period, in which case we enter the number 1, or if calculated at the end of the period, in which case we enter the number 0.

To illustrate, we solve for pv using the PV function. We begin by listing value for all of the PV arguments and in the cell that calls for a pv value, we enter the PV function. Our example in Figure A.21 corresponds to the example in Chapter 17. The annual percentage rate r equals 5%. The number of compound periods per year m is 12. The number of years n is 4. And the annuity equivalent R entered with a negative sign is $-R = -150$. We now transform the numbers corresponding to equation A.1 into PV function arguments. Rate equals $r/m = .05/12 = .004$; Nper equals $mn = 12 \times 4 = 48$. In cell C7 we enter the function PV with its arguments. For the first argument of the PV function we list the location of rate in cell C4. The second argument of the PV function is nper located in cell C5. The third argument is pmt located in cell C6. The fourth argument is pv in cell C7. Since pv is an endogenous variable in the PV function, we enter the PV function in cell C7. Finally, type is equal to zero in cell C8 and fv is also zero located in cell C9. The PV formula in cell C7 is expressed as $=PV(C4, C5, C6, C8, C9)$. The solution to the PV equation with argument values described in Chapter 17 and repeated above are illustrated in Figure A.20.

	A	B	C
1	Excel PV functions and arguments		
2			
3	Excel arguments	Eqn. A.1 variables	Excel argument values
4	rate	r/m	0.0040
5	nper	mn	48
6	pmt	R	(\$150.00)
7	pv	NPV+Vo	6539.14
8	type	beg = 0	0.00
9	fv	Vn=0	\$0.00
10			

Figure A.20. An Excel PV function solution for equation A.1 where A.1 variables are described in cells C4 through C9.

[Open Example in Microsoft Excel](#)

Notice that the argument values for the PV function are listed in cells C4, C5, C6, C8, and C9. The unknown (endogenous variable) pv is located in cell C7. So in cell C7 we list the PV function and its arguments described in the function box located at the top of the figure.

To illustrate another function, suppose that we knew that $pv = \$5,000$ but that nper was unknown. In which case we would enter \$5000 in cell C7 and enter the NPER function in cell C5: =NPER(C4, C6, C7, C8, C9). The solution is described in cell C5, and the NPER function is entered in the function box. These results are described in Figure A.21.

C5			
=NPER(C4,C6,C7,C8,C9)			
	A	B	C
1	Excel NPER function and arguments		
2			
3	Excel arguments	Eqn. A.1 variables	Values
4	rate	r/m	0.0040
5	nper	mn	35.85
6	pmt	R	(\$150.00)
7	pv	NPV+Vo	5000.00
8	type	beg = 0	0.00
9	fv	Vn=0	\$0.00

Figure A.21. Answering the “what if” the nper was unknown.
[Open Example in Microsoft Excel](#)

Of course, we can explore “what if” questions by changing arguments values of the function NPER and noting the consequences on nper. For example, what if we change the variable r in cell B4 to 8% so that $r/m = 8\% / 12 = .0067$. Then we ask, what is the new value of nper? The worksheet with the revised value for rate is described in Figure A.22.

C5				
=NPER(C4,C6,C7,C8,C9)				
	A	B	C	D
1	Excel NPER function and arguments			
2				
3	Excel arguments	Eqn. A.1 variables	Excel argument Values	
4	rate	r/m	0.0067	
5	nper	mn	37.85	
6	pmt	R	(\$150.00)	
7	pv	NPV+Vo	5000.00	
8	type	beg = 0	0.00	
9	fv	Vn=0	\$0.00	

Figure A.22. Answering the question: “what if” rate were 8%, then what would be the value of nper?
[Open Example in Microsoft Excel](#)

To this point, we have assumed time varying cash flows in equation A.1 have been replaced by their AE equal to R. But in some cases, it is important to consider the time varying cash flow rather than the AE cash flow. Excel's NPV function allows us to do just that—consider time varying cash flows. Excel's NPV and IRR function considers the complete A.1 equation including its time varying cash flow.

The NPV function. The NPV function has only two arguments: the rate argument and the stream of periodic cash flows that can be entered individually separated by commas or entered as the first and last value in the cash flow stream beginning in period 1 and ending in period *mn* separated by a constant. The original investment value is not included in the function but added separately to the function value. We illustrate the NPV function using data from the Green and White Services investment problem.

Green and White Services wants to know the NPV of an investment of \$40,000 in lawn care equipment with expected cash flow during its four years of operation equal to: \$9,800, \$16,200, \$17,840, and \$19,600. In period five, Green and White intend to liquidate their investment and receive an after-tax cash flow liquidation value of \$8,000. We enter these values in Figure A.23 and point the cursor at cell C12 where the NPV function is inserted. The first argument of the function is the discount rate of 8% recorded in cell C4. The second argument is the periodic cash flow in cells C7 through C11 separated by a colon. To the NPV function value we add the investment amount of (\$40,000) located in cell C6. After pressing enter, the NPV function returns an NPV value of \$17,061.91.

	A	B	C	D	E
1	Excel NPV function and arguments				
2					
3	Excel arguments	Eqn. A.1 variables	Excel argument values		
4	rate	r	0.08		
5	after-tax cash flow				
6		Vo	(40,000.00)		
7		R1	9,800.00		
8		R2	16,300.00		
9		R3	17,840.00		
10		R4	19,600.00		
11		R5	8,000.00		
12		NPV	\$17,061.91		
13		IRR	23%		
14					
15					

Figure A.23. The NPV function for Green and White Services' investment of \$40,000 that produces four periods of return and then is salvaged in year 5.

[Open Example in Microsoft Excel](#)

The IRR function with time varying cash flows. The IRR function finds the discount rate such that NPV is zero. The IRR function has only two arguments. The first argument is investment value. The second argument is the stream of periodic cash flows that can be entered individually separated by commas or entered as the first and last value in the cash flow stream beginning in period 1 and ending in period mn separated by a constant. We illustrate the IRR function using data from the Green and White investment problem. We enter the investment value in cell C6 and periodic cash flows in cells C7 through C11. We enter the IRR function in cell C13. Its first argument is the initial investment of (\$40,000). The second argument is the periodic cash flow in cells C7 through C11 separated by a colon. After pressing enter, the IRR function returns an IRR value of 23%. We summarize this IRR example in Figure A.24.

	A	B	C	D	E
1	Excel NPV function and arguments				
2					
3	Excel arguments	Eqn. A.1 variables	Excel argument values		
4	rate	r	0.08		
5	after-tax cash flow				
6		V0	(40,000.00)		
7		R1	9,800.00		
8		R2	16,300.00		
9		R3	17,840.00		
10		R4	19,600.00		
11		R5	8,000.00		
12		NPV	\$17,061.91		
13		IRR	23%		
14					
15					
16					

Figure A.24. The IRR function for Green and White Services' investment of \$40,000 that produces four periods of return and then is salvaged in year 5.
[Open Example in Microsoft Excel](#)

Excel spreadsheets provide a powerful means for computing what otherwise would be tedious calculations. However, their ability to provide solutions to complex equations rapidly and repeatedly allows for a potential problem. The problem is that we may not spend sufficient time and effort to carefully specify the calculations we need Excel to perform. Cars are valuable means of getting us from point A to point B, but we still need to know the location of point A and point B. Excel spreadsheets provide valuable solutions to complex equations, but we still need to carefully define the problem to be solved and carefully derive the appropriate equation that solves the problem. Then, we are ready to enjoy the benefits of powerful Excel spreadsheets and formulas.

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U

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- unequal terms
- unlimited liability
- $US_0(r^f/m, mn)$
- $US_0(r, n)$

V

- value in use model
- variance

W

- weaknesses
- weather forecast metaphor
- weighted cost of capital (WCC)
- what if analysis
 - scenarios

Y

- yield curve

Author's Note

LINDON ROBISON

This book is the culmination of many years' research, learning, teaching, writing, and revising with feedback from peers and students alike. When deciding whether or not to make this book open, we realized that by doing so we were giving every financial management student a scholarship and every small business financial manager free tools to grow a better business. We decided that using an open license would give this book a broader reach to help more people.

I welcome feedback from professors, students, and financial managers who have found and used this resource. You may contact me via email at robison@msu.edu. I look forward to hearing how this book has been used in or out of the classroom and appreciate constructive comments for improving it for future editions.

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LINDON ROBISON



Dr. Lindon J. Robison is Professor of Agricultural, Food, and Resource Economics (AFRE) at Michigan State University. He holds B.S., M.S., and Ph.D. degrees from Utah State University, University of Illinois, and Texas A&M University respectively. He has published numerous books and articles, including the text for ABM 435. He has consulted for governments, firms, and international organizations such as the World Bank, particularly in Latin America. He has worked for the US Government as an agricultural economist, has been a visiting faculty member at Brigham Young University, the University of Minnesota, and the Swedish University of Agricultural Sciences in Uppsala, Sweden. He has won many academic awards including Best Ph.D. thesis for his work on risk and portfolio management of rural banks. He recently completed his assignment as the interim chair of AFRE.



Dr. Steven D. Hanson is associate provost and dean of International Studies and Programs (ISP) at Michigan State University. He received his PhD in agricultural economics from Iowa State University. He joined MSU in 1988 in the Department of Agricultural, Food, and Resource Economics (AFRE) where he served as Chair of AFRE, providing leadership to one of the most productive and highly ranked departments in the country. Dr. Hanson's research focuses on risk management with financial contracts and the valuation of financial contracts and assets, such as land. His work has been published in leading journals such as the American Journal of Agricultural Economics and the Journal of Empirical Finance.



Dr. J. Roy Black was Professor of Agricultural, Food, and Resource Economics (AFRE) at Michigan State University. He was from a ranch in the Northern Great Plains on the Canadian border. He had joint undergraduate majors in Animal Science and Agricultural Economics at Montana State University and had a Ph.D. degree from the University of Minnesota. He had a three way appointment with responsibilities in extension, research, and teaching. His focus was in the area of agricultural production and marketing systems and markets for risk instruments used in the global food system. He was the recipient of many state and national awards for his outreach and research work. His research and outreach work had been supported by a wide array of sources including the Michigan AgBio and Michigan Cooperative Extension, USDA, EPA, NSF, and private companies.

Dr. Black passed in 2020.