OERTransport: Fundamentals of Math, Physics, and Statistics for Future Transportation Professionals

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Fundamentals of Math, Physics, and Statistics
for Future Transportation Professionals

FUNDAMENTALS OF MATH, PHYSICS, AND STATISTICS FOR FUTURE TRANSPORTATION PROFESSIONALS

ANURAG PANDE; PEYTON RATTO; AND AHMED FARID

Mavs Open Press

Arlington

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This open textbook is part of the ***OERT*ransport** collection of six transportation planning OER textbooks and their respective graduate course implementation at each of the three collaborating universities. The creation of each textbook benefitted from an independent industry advisory board's review and the savvy contributions of each institution's OER librarians. For more information on OERTransport: https://oertransportlab.uta.edu/

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PREFACE

This textbook and OER material cover tools and basic knowledge required to master the prerequisite essentials of physics, mathematics, and statistics applied in transportation engineering. To our knowledge, no such textbook currently exists to build the KSTs (Knowledge, Skills, Tools) for college freshmen' remedial courses required by most transportation engineering graduate programs that admit non-engineers (e.g., PHYS 141, MATH 142, STAT 321 at California Polytechnic State University, San Luis Obispo). The textbook modules are intended for students with undergraduate degrees in planning and other less technical fields who are interested in pursuing transportation careers where background engineering knowledge is required.

ABOUT THE AUTHORS

Anurag Pande, Ph.D.

Dr. Anurag Pande is a Professor of Civil Engineering at Cal Poly. His research interests include traffic simulation, engineering education, community engagement, data mining applications, and observational data analysis, including in the areas of traffic safety and crashes, driver behavior, transportation resilience, and emergency evacuation.

Peyton Ratto

Ms. Peyton Ratto graduated from Cal Poly with a dual MS degree in City and Regional Planning and Civil Engineering with a specialization in Transportation.

Ahmed Farid, Ph.D.

Dr. Ahmed Farid is a Postdoctoral Fellow in the department of Civil Engineering at Cal Poly. His research interests include traffic safety, data mining applications, and observational data analysis, including in the areas of traffic safety and crashes.

ABOUT THE INSTITUTION

California Polytechnic State University (Cal Poly) is located in San Luis Obispo on California's Central Coast, about halfway between Los Angeles and San Francisco. Cal Poly was founded in 1901 and has been consistently named the best public, master's-level university in the West by U.S. News & World Report since the late 1990s. It is part of the 23-campus California State University system and is well-known for its learn-by-doing pedagogy.

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- Stacey A. Slijepcevic, Ph.D., FIPSE Senior Program Manager/HIS STEM Senior Program Manager, U.S. Department of Education.
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- Brian Guenzel, MCRP, Senior Planner, City of Midlothian, TX.
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- John Liu, M.S., Deputy District Director, Maintenance and Operations, at Caltrans. Clovis, CA.
- David Weinreich, Ph.D., Postdoctoral Fellow in Urban Governance and Sustainable Transformation, University of Bergen, Norway.
- James Wood, Ph.D., Project Manager at the Dallas-Fort Worth International Airport.

PROJECT DIRECTION

- Ivonne Audirac, Ph.D., Professor, City and Regional Planning Program and Associate Director for Education, Center for Transportation, Equity Decisions, and Dollars CTEDD, University of Texas at Arlington (UTA). OERTransport Project PI.
- Jay M. Rosenberger, Ph.D., Professor, Industrial Engineering, University of Texas at Arlington (UTA). OERTransport Project Co-PI.
- Amber Raley, M.A., Urban Planning and Public Policy Ph.D. Student, University of Texas at Arlington (UTA). OERTransport Project Co-PI.

PROOFREADERS AND EDITORS

- Dr. Anurag Pande, Ph.D in Civil Engineering (Transportation).
- Dr. Ahmed Farid, Ph.D in Civil Engineering.
- Peyton Ratto, M.S./MCRP, Civil Engineering and City and Regional Planning with a specialization in Transportation Planning.
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- Anne M. Regan, M.A., English, Creative Commons (CC) License Certified.

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ABOUT THE COVER

CHAPTER 1: TRIGONOMETRY FUNCTIONS AND GEOMETRIC MEASUREMENTS

This chapter will discuss the basic geometric and **trigonometric functions** and measurements because they are important in transportation engineering for the functional design of facilities, including roadways, railroads, and intersection sight distance.

Learning Objectives

At the end of the chapter, the reader should be able to do the following:

- Recognize, rearrange, and simplify trigonometric functions.
- Apply properties of triangles to solve for sides and angles.
- Identify transportation engineering topics where the properties of triangles and trigonometric functions are used.
- Identify topics in the introductory transportation engineering courses that build on the concepts discussed in this chapter.

RECOGNIZE, REARRANGE, AND SIMPLIFY TRIGONOMETRIC FUNCTIONS

This section will explain basic manipulation of trigonometric functions with videos to help your comprehension. Also, short problems to check your understanding are included.

Radians & Degrees

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://uta.pressbooks.pub/oert-mpsfundamentals/?p=485#oembed-1

Check Your Understanding: Radians & Degrees

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Solving Similar Triangles

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Check Your Understanding: Solving Similar Triangles

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Introduction to the Trigonometric Ratios

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Check Your Understanding: Introduction to the Trigonometric Ratios

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The Trig Functions & Right Triangle Trig Ratios

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Check Your Understanding: The Trig Functions & Right Triangle Trig Ratios

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Solving for a Side in a Right Triangle Using the Trigonometric Ratios

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Check Your Understanding: Solving for a Side in a Right Triangle Using the Trigonometric Ratios

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The Inverse Trigonometric Functions

Introduction to Arcsine

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Introduction to Arctangent

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Introduction to Arccosine

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Check Your Understanding: The Inverse Trigonometric Functions

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Law of Sines

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Check Your Understanding: Law of Sines

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Law of Cosines

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Check Your Understanding: Law of Cosines

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Check Your Understanding: Recognize, Rearrange, and Simplify Trigonometric Functions Overall

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IDENTIFY PROPERTIES FOR CIRCULAR, PARABOLIC, AND SPIRAL CURVES

This section uses videos to understand the elements of simple circular curves, parabolic curves, and spiral curves.

Elements of Simple Circular Curves

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Elements of Parabolic Curves

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Elements of Spiral Curves

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RELEVANCE TO TRANSPORTATION ENGINEERING COURSEWORK

This section will explain horizontal straight road sections, curves, and spiral transition sections that connect them.

Horizontal Profile

> Transportation engineers apply the trigonometric functions discussed here in the process of designing roads, and in particular, their horizontal profiles. The horizontal profile describes the horizontal straight road sections, curves, and spiral transition sections (see Figure 1) that connect them. The curves' geometric design elements depend on principles discussed in the following sections of this chapter: "Radians & Degrees", "Solving Similar Triangles", "Introduction to the Trigonometric Ratios", "The Trig Functions & Right Triangle Trig Ratios", "Solving for a Side in a Right Triangle Using the Trigonometric Ratios", and "The Inverse Trigonometric Functions". For details of the design process (discussed in the Fundamentals of Transportation Engineering course) please review the above section titled "Elements of Simple Circular Curves".

Figure 1: Horizontal profile (top view) of a road with significant horizontal curvature. Landscape, Aerial View, Mountain Road by Freddy Dendoktoor under (CC0 1.0)

Stopping Sight Distance on Horizontal Curves

The stopping sight distance (SSD) is the minimum distance a driver needs to perceive an unusual situation, such as a deer crossing the road, and stop to avoid a collision. Along horizontal curves, the driver's line of sight may be obstructed by trees or other roadside objects (see Figure 2). The formula for computing the minimum required SSD on horizontal curves near sight obstructions is derived using the principles discussed in the following sections: "Radians & Degrees", "Solving Similar Triangles", "Introduction to the Trigonometric Ratios", "The Trig Functions & Right Triangle Trig Ratios", "Solving for a Side in a Right Triangle Using the Trigonometric Ratios", and "The Inverse Trigonometric Functions".

Figure 2: "Curvy road." No author. (CC0)

Sight Triangles at Intersections

When approaching an intersection, drivers should be able to see it from a distance to be able to stop in time before colliding with other approaching vehicles, pedestrians, and bicyclists. This distance is computed by first constructing "sight triangles" similar to the one shown in Figure 3. The sight triangle computations are made using the principle of similar triangles (see the section titled "Solving Similar Triangles" above).

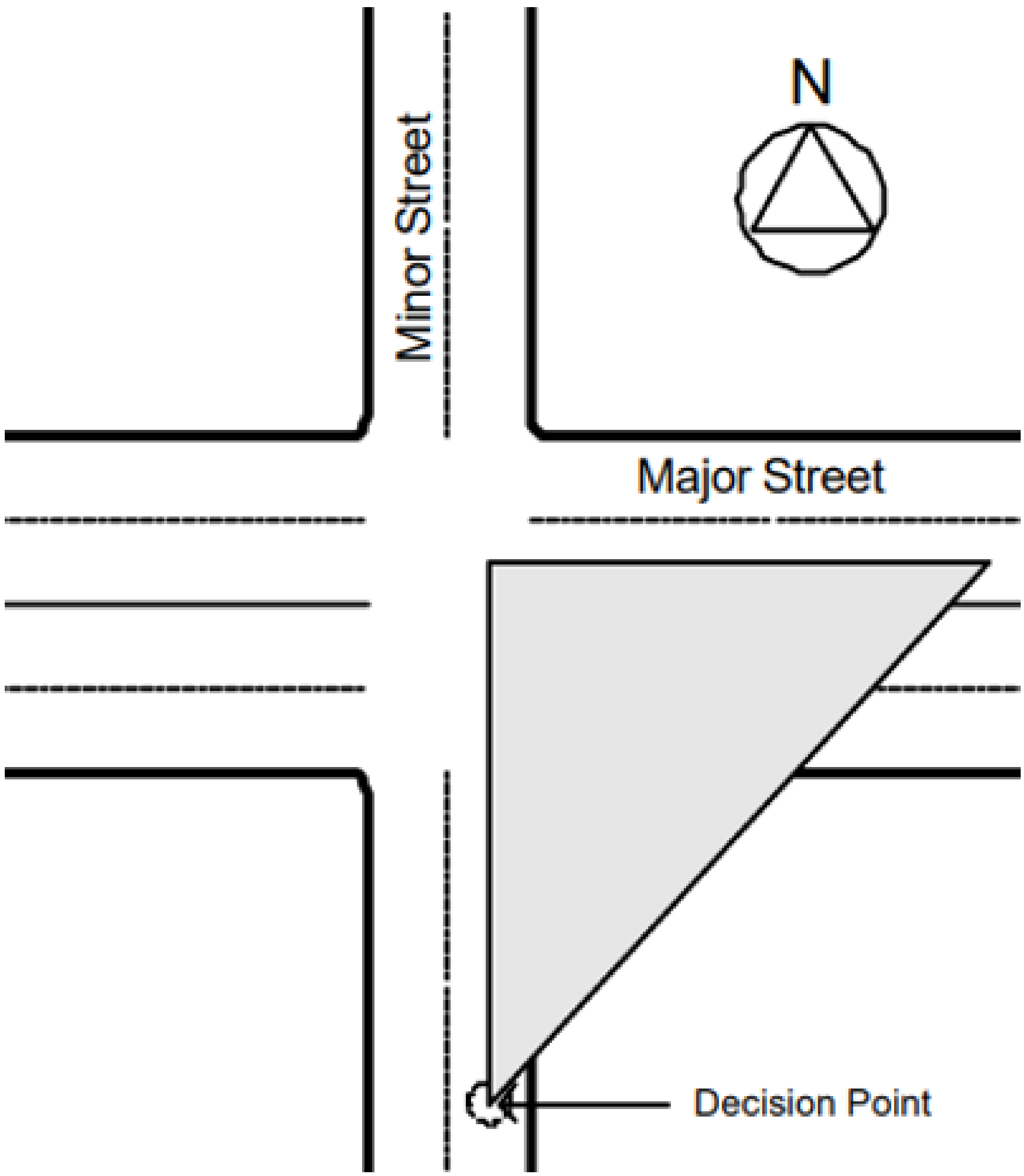

Figure 3: "Sample intersection sight triangle" by Lin et al. under public domain

Key Takeaways

- Transportation engineers apply trigonometric functions to design roads and railroad tracks, particularly their horizontal profiles. Knowledge of these functions is critical to ensuring that on segments with horizontal curvature (or bend around the corner), road users have adequate sight distance available to negotiate the curve on the travel way safely.
- The knowledge of trigonometry is also critical to solving intersection sight triangles and estimating Intersection Sight Distance (ISD) required to safely navigate the intersections.

GLOSSARY: KEY TERMS

Degree[1] – a unit of measure for angles equal to an angle with its vertex at the center of a circle and its sides cutting off $1/360$ of the circumference
Law of Cosines[1] – a law in trigonometry: the square of a side of a plane triangle equals the sum of the squares of the remaining sides minus twice the product of those sides and the cosine of the angle between them
Law of Sines[1] – a law in trigonometry: the ratio of each side of a plane triangle to the sine of the opposite angle is the same for all three sides and angles
Radian[1] – a unit of plane angular measurement that is equal to the angle at the center of a circle subtended by an arc whose length equals the radius or approximately 57.3 degrees
Trigonometric Function[1] – a function (such as the sine, cosine, tangent, cotangent, secant, or cosecant) of an arc or angle most simply expressed in terms of the ratios of pairs of sides of a right-angled triangle.

[1] https://www.merriam-webster.com/

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- Farid, A. (2022). Highway Design. Personal Collection of Ahmed Farid, California Polytechnic State University, San Luis Obispo, CA.
- Farid, A. (2022). Considerations for Intersections. Personal Collection of Ahmed Farid, California Polytechnic State University, San Luis Obispo, CA.

CHAPTER 2: POLYNOMIAL, EXPONENTIAL, AND LOGARITHMIC FUNCTIONS

Because polynomial, exponential, and **logarithmic functions** have several applications in transport engineering, this chapter will explain the functions. The exponential functions are used to model economic and population growth and to estimate the compound interest formula. Simplifying **polynomial** expressions and solving corresponding equations is helpful in the design of transportation facilities and understanding the relationship(s) between flow, speed, and density on roadway segments. Logarithmic functions are also sometimes used to model the relationship between traffic speed and density on a roadway segment, where the speed decreases as the density increases. Understanding the properties of these functions also provide a foundation to learn differential and integral calculus for these functions (See Chapter 4).

Learning Objectives

At the end of the chapter, the reader should be able to do the following:

- Generate graphs and charts from exponential and logarithmic functions.
- Solve the exponential and logarithmic equations.
- Visualize polynomial functions on a graph or chart.
- Set up and solve polynomial equations.
- Identify topics in the introductory transportation engineering courses that build on the concepts discussed in this chapter.

GENERATE GRAPHS AND CHARTS FROM EXPONENTIAL AND LOGARITHMIC FUNCTIONS

This section will explain **exponential growth** functions, exponential and logarithmic functions, exponential graphs, graphing exponential growth and decay, and introduce you to logarithms with videos to help your comprehension. Also, short problems to check your understanding are included.

Introduction to Exponential Functions

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Check Your Understanding: Exponential Functions

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Exponential Function Graph

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Graphing Exponential Functions

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Check Your Understanding: Graphing Exponential Functions

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Graphs of Exponential Growth

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Check Your Understanding: Graphs of Exponential Growth

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Graphing Exponential Growth and Decay

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Check Your Understanding: Graphing Exponential Growth and Decay

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Analyzing Graphs of Exponential Functions

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Check Your Understanding: Analyzing Graphs of Exponential Functions

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Analyzing Tables of Exponential Functions

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Check Your Understanding: Analyzing Tables of Exponential Functions

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Introduction to Logarithms

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Check Your Understanding: Logarithms

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Graphs of Logarithmic Functions

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Check Your Understanding: Graphs of Logarithmic Functions

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Relationship Between Exponentials and Logarithms: Graphs

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Relationship Between Exponentials and Logarithms: Tables

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Check Your Understanding: Relationship Between Exponentials and Logarithms

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SOLVE THE EXPONENTIAL AND LOGARITHMIC EQUATIONS

This section will explain how to solve exponential equations using **exponent** properties in both basic and advanced problems with videos to help your comprehension. Also, short problems to check your understanding are included.

Solving Exponential Equations Using Exponent Properties

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Check Your Understanding: Solving Exponential Equations Using Exponent Properties

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Solving Exponential Equations Using Exponent Properties (Advanced)

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Check Your Understanding: Solving Exponential Equations Using Exponent Properties (Advanced)

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Introduction to Logarithm Properties

Please read this link on Introduction to Logarithm Properties.

Check Your Understanding: Introduction to Logarithm Properties

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Justifying the Logarithm Properties

Please read this link on Justifying the Logarithm Properties.

Logarithm Change of Base Rule Introduction

Please read this link on Logarithm Change of Base Rule Introduction

Check Your Understanding: Justifying the Logarithm Properties & Logarithm Change of Base Rule

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Logarithmic Equations: Variable in the Argument

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Check Your Understanding: Logarithmic Equations: Variable in the Argument

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Logarithmic Equations: Variable in the Base

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Check Your Understanding: Logarithmic Equations: Variable in the Base

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VISUALIZE POLYNOMIAL FUNCTIONS ON A GRAPH OR CHART

This section will explain how to graph polynomial functions with videos to help your comprehension. Also, short problems to check your understanding are included.

Graphing Polynomial Functions

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Graphing Lab – Quadratics and Polynomials

If you'd like extra practice, please perform this lab.

Graphs of Polynomials Overview

Please read this link on Graphs of Polynomials Overview

Check Your Understanding: Graphs of Polynomials

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SET UP AND SOLVE POLYNOMIAL EQUATIONS

This section will explain how to set up and solve polynomials with variables, multiplying **binomials** with polynomials, and solving polynomial equations by factoring with videos to help your comprehension. Also, short problems to check your understanding are included.

Adding and Subtracting Polynomials

Please read this link on Adding and Subtracting Polynomials

Check Your Understanding: Adding and Subtracting Polynomials

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Adding and Subtracting Polynomials with Two Variables

Please read this link on Adding and Subtracting Polynomials with Two Variables

Check Your Understanding: Adding and Subtracting Polynomials with Two Variables

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Multiplying Binomials by Polynomials

Please read this link on Multiplying Binomials by Polynomials

Check Your Understanding: Multiplying Binomials by Polynomials

An interactive H5P element has been excluded from this version of the text. You can view it online here: https://uta.pressbooks.pub/oert-mpsfundamentals/?p=577#h5p-146

Solving Polynomial Equations by Factoring

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Factoring Polynomials by Taking a Common Factor

Please read this link on Factoring Polynomials by Taking a Common Factor

Check Your Understanding: Factoring Polynomials by Taking a Common Factor

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Dividing Polynomials: Long Division

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Check Your Understanding: Dividing Polynomials: Long Division

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Solving Polynomial Equations by using Synthetic Substitution

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Check Your Understanding: Solving Polynomial Equations by using Synthetic Substitution

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RELEVANCE TO TRANSPORTATION ENGINEERING COURSEWORK

This section explains the relevance of engineering economics, traffic assignment, vertical curve design, and traffic flow to transportation engineering coursework.

Engineering Economics

Engineering economics is the application of economic analysis methods to quantify benefits and costs to assist in the decision-making process. The decisions are related to transportation engineering alternative designs (e.g., constructing a stop-controlled intersection or a roundabout), prioritization of transportation projects, and maintenance budgets, among others. In engineering economics, the time value of money (present value of money and future value of money) and interest rate, also known as the discount rate, need to be considered to provide meaningful Benefit-Cost B/C analysis. The analysis includes the compound interest formula and the exponential growth of money. Exponential functions are also used to model the depreciation of assets. Relevant exponential functions are described in the section titled "Generate Graphs and Charts from Exponential and Logarithmic Functions" of this chapter.

Traffic Assignment

In the 4-step travel demand modeling process, traffic assignment is typically the last step, which is used to estimate the number of trips on each route of the roadway network. A critical part of traffic assignment is the Link Performance Function for each route, which defines travel time as a function of the number of users on a route. Travel time increases exponentially as traffic volume increases and is defined based on functional forms discussed in "Generate Graphs and Charts from Exponential and Logarithmic Functions".

Vertical Curve Design

In highway engineering, one of the fundamental design elements is the vertical profile (i.e., the upgrades, the downgrades, and the curves that connect consecutive roadway sections with different grades, also known as the vertical curves). The vertical curve's elevation is defined as a quadratic equation, a special type of polynomial function (see "Solve the Exponential and Logarithmic Equations" above).

Relationship among Traffic Flow, Speed, and Density

In traffic flow theory for uninterrupted flow facilities (such as freeways), the fundamental relationship relates speed and density on the segment with the flow rate past a point on the road segment. The decrease in speed as segment density increases may be modeled using linear, polynomial, or logarithmic functions. The nature of speed-density functions may be used to model relationships between the flow rate and density and/or the flow rate and speed. Polynomial and Logarithmic functions discussed in this chapter are critical to this analysis.

Key Takeaways

- Exponential functions are useful for transportation practitioners in applying the compound interest formula and modeling asset depreciation and population growth rate in engineering economics.
- Exponential functions are also used in defining link performance functions in traffic assignment to

model travel time as a function of road users on a network route

- Logarithmic functions are sometimes used to define speed as a function of roadway segment density.
- Polynomial functions are used in several applications, including the equation of parabolic curves connecting two consecutive grades on a roadway segment.
- Speed-flow and flow-density relationships on roadway segments are often also defined as polynomial functions.

GLOSSARY: KEY TERMS

Binomial[1] – a mathematical expression consisting of two terms connected by a plus sign or minus sign
Common Factor[1] – (also called common divisor) a number or expression that divides two or more numbers or expressions without remainder
Exponent[1] – a symbol written above and to the right of a mathematical expression to indicate the operation of raising to a power
Exponential Function[1] – a mathematical function in which an independent variable appears in one of the exponents
Exponential Growth[2] – growth whose rate becomes ever more rapid in proportion to the growing total number or size.
Logarithm[1] – the exponent that indicates the power to which a base number is raised to produce a given number
Logarithmic Function[1] – a function (such as y =loga x or y = ln x) that is the inverse of an exponential function (such as y = ax or y = ex) so that the independent variable appears in a logarithm
Polynomial[1] – a mathematical expression of one or more algebraic terms each of which consists of a constant multiplied by one or more variables raised to a nonnegative integral power (such as a + bx + cx2)

[1] https://www.merriam-webster.com/
[2] https://www.lexico.com/

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CHAPTER 3: SYSTEMS OF LINEAR EQUATIONS

The ability to solve systems of multiple equations is a critical component of an introductory transportation course. Systems of equations are helpful in several key transportation analyses, including travel demand estimation (e.g., traffic assignment), design (e.g., roadway alignment), and operations (e.g., speed-density relationships). This chapter highlights how to solve systems of equations with two variables, three variables, and when no solution occurs. It will also explain how to use Microsoft Excel to solve system equations.

Learning Objectives

At the end of the chapter, the reader should be able to do the following:

- Solve a system of equations of up to three variables.
- Implement solving equations up to five variables in Microsoft Excel.
- Identify topics in the introductory transportation engineering courses that build on the concepts discussed in this chapter.

SOLVING SYSTEMS OF EQUATIONS WITH TWO VARIABLES

In this section, you will learn how to setup a system of equations and solve linear equations by reading each description along with watching the videos. Also, short problems to check your understanding are included.

How to set up a system of equations

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Check Your Understanding: How to Set Up a System of Equations

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How to solve systems of equations graphically

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Check Your Understanding: How to Solve Systems of Equations Graphically

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Testing a solution to a system of equations

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Check Your Understanding: Testing a Solution to a System of Equations

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Solving System of Equations with Substitution

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Check Your Understanding: Solving System of Equations with Substitution

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Solving Systems of Equations with Elimination

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Check Your Understanding: Solving Systems of Equations with Elimination

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SOLVING SYSTEMS OF EQUATIONS WITH THREE VARIABLES

In this section, you will learn how to set up a system of equations with three variables and solve a system of equations when no solution occurs by watching the videos. Short problems to check your understanding are included.

Introduction to Linear Systems with Three Variables

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Solving Linear Systems with Three Variables

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Check Your Understanding: Solving Linear Systems with Three Variables

An interactive H5P element has been excluded from this version of the text. You can view it online here:
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Solving Linear Systems with Three Variables: No Solution

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://uta.pressbooks.pub/oert-mpsfundamentals/?p=477#oembed-8

Check Your Understanding: Solving Linear Systems with Three Variables: No Solution

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SYSTEMS OF EQUATIONS IN MICROSOFT EXCEL

In this section, you will learn how to set up a system of equations using Microsoft Excel and matrices and how to use the Microsoft Excel solver tool by watching the videos. Short problems to check your understanding are included.

Solving Systems of Equations Using Matrices

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://uta.pressbooks.pub/oert-mpsfundamentals/?p=477#oembed-9

Check Your Understanding: Solving Systems of Equations Using Matrices

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Solving System of Equations Using Excel

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Check Your Understanding: Solving System of Equations Using Excel

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How to Use the Solver Tool in Excel to Solve Systems of Linear Equations in Algebra

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Check Your Understanding: How to Use the Solver Tool in Excel to Solve Systems of Linear Equations in Algebra

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RELEVANCE TO TRANSPORTATION ENGINEERING COURSEWORK

This section explains the relevance of traffic assignment and operation along with highway design to transportation engineering coursework.

Traffic Assignment

In travel demand modeling process, the last step is the traffic assignment for routes, which involves estimating the number of trips made on each route on the roadway network. Two commonly applied traffic assignment approaches, which involve solving a system of equations, are the user equilibrium (UE) and system optimal (SO) assignments. For simple UE problems helpful in conceptual understanding of these approaches, the system of equations is typically linear. The number of variables to solve for depends on the number of routes being considered for traffic assignment. The reader is referred to the above section titled "Solving Systems of Equations with Two Variables" for solving a system of linear equations with two variables and the section titled "Systems of Equations in Microsoft Excel" for solving a system of linear equations with three variables. For solving a system with more than three variables one requires use of computer-based tools which are discussed in "Systems of Equations in Microsoft Excel".

Highway Design

Analysis and design of roadway (or railway) elements, particularly horizontal alignment and vertical profiles, often require solving a system of equations. These equations are set up based on the geometric constraints of the design. For example, the roadway elevation at a certain location may be fixed due to an intersecting element. These constraints are used to set up system of simultaneous equations that require the use of methods discussed in the three previous sections in this chapter for solving them.

Traffic Operations

The relationships between highway speed, lane density (a measure of congestion), and traffic flow (number of vehicles past a point) are governed by a universal relationship of uninterrupted flow, i.e., flow rate is a multiple of speed and density. In addition, different segments are characterized by certain relationships between any of the variable pairs (i.e., speed-density; speed-flow; or flow-density). These relationships help set up equations for key parameters for roadway segments such as capacity flow rate, jam density, and free flow speed.

Key Takeaways

- There are two main approaches to traffic assignment in the travel demand modeling process: user equilibrium (UE) and system optimal (SO). Both of these approaches involve solving a system of equations to determine the number of trips that will be made on each route.
- In highway design equations that help ensure that the final design meets all necessary requirements and constraints need to be solved using the methods described in the chapter.
- In traffic operations, different segments of roadways with uninterrupted flow may have different relationships between speed, flow, and density, and these relationships can be used to set up equations for important parameters such as capacity flow rate, jam density, and free flow speed.

GLOSSARY: KEY TERMS

Augmented Matrix[1] – a matrix whose elements are the coefficients of a set of simultaneous linear equations with the constant terms of the equations entered in an added column
Matrix[1] – a rectangular array of mathematical elements of simultaneous linear equations that can be combined to form sums and products with similar arrays having an appropriate number of rows and columns
[1] https://www.merriam-webster.com/

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CHAPTER 4: CALCULUS – INTERPRETATION AND METHODS FOR INTEGRATION AND DIFFERENTIATION

This chapter discusses the interpretation of derivatives that helps to understand the origin of formulas for estimating values for roadway elevation along the vertical curve. Understanding **derivatives** as a rate of change can also help with traffic operations (e.g., capacity as a maximum flow point) and transportation economics (e.g., marginal costs and congestion pricing). Integrals are used to estimate the area under a given curve. This property of integrals is used in estimating earthwork volumes and quantifying aggregate delays at intersections or roadway segments with incidents.

Learning Objectives

At the end of the chapter, the reader should be able to do the following:

- Interpret derivatives as the rate of change.
- Differentiate various functions with single variables.
- Familiarize oneself with the rules and processes (e.g., chain rule, multiplication rule) of derivation.
- Use the integration method for estimating the area under a curve.
- Describe the process of integration and differentiation in correspondence with each other.
- Identify topics in the introductory transportation engineering courses that build on the concepts discussed in this chapter.

INTERPRET DERIVATIVES AS THE **RATE OF CHANGE**

In this section, you will learn about derivatives, **tangent slopes**, tangent lines, quotient rule, chain rule and power rule by watching the videos. Also, short problems to check your understanding are included.

Interpreting Derivatives

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://uta.pressbooks.pub/oert-mpsfundamentals/?p=497#oembed-1

Derivative as Slope of Curve

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://uta.pressbooks.pub/oert-mpsfundamentals/?p=497#oembed-2

Check Your Understanding: Interpreting Derivatives & Derivative as Slope of Curve

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The Derivative as Slope of Tangent Line

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Check Your Understanding: The Derivative as Slope of Tangent Line

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Tangent Slope as Instantaneous Rate of Change

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Check Your Understanding: Tangent Slope as Instantaneous Rate of Change

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Approximating Instantaneous Rate of Change with Average Rate of Change

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FAMILIARIZE ONESELF WITH THE RULES AND PROCESSES (E.G., CHAIN RULE, MULTIPLICATION RULE) OF DERIVATION

Basic Derivative Rules

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://uta.pressbooks.pub/oert-mpsfundamentals/?p=497#oembed-6

Basic Derivative Rules (Part 2)

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Check Your Understanding: Basic Derivative Rules

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Product Rule

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Check Your Understanding: Product Rule

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Quotient Rule

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Check Your Understanding: Quotient Rule

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Chain Rule

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Check Your Understanding: Chain Rule

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The Power Rule

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Check Your Understanding: The Power Rule

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Derivatives of Trigonometric Functions

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Check Your Understanding: Derivatives of Trigonometric Functions

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DIFFERENTIATE VARIOUS FUNCTIONS WITH SINGLE VARIABLES

In this section, you will learn how to differentiate various functions, products, quotients, **rational functions**, trigonometric functions, and logarithmic by watching the videos. Also, short problems to check your understanding are included.

Differentiability at a Point: Algebraic

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://uta.pressbooks.pub/oert-mpsfundamentals/?p=497#oembed-13

Check Your Understanding: Differentiability at a Point; Algebraic

An interactive H5P element has been excluded from this version of the text. You can view it online here: https://uta.pressbooks.pub/oert-mpsfundamentals/?p=497#h5p-92

Differentiating Polynomials

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://uta.pressbooks.pub/oert-mpsfundamentals/?p=497#oembed-14

Check Your Understanding: Differentiating Polynomials

An interactive H5P element has been excluded from this version of the text. You can view it online here: https://uta.pressbooks.pub/oert-mpsfundamentals/?p=497#h5p-93

Fractional Powers Differentiation

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Check Your Understanding: Fractional Powers Differentiation

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Radical Functions Differentiation Introduction

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Worked Example

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Check Your Understanding: Radical Functions Differentiation

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Differentiating Products

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Check Your Understanding: Differentiating Products

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Differentiate Quotients

Please read this link on differentiate quotients.

Check Your Understanding: Differentiate Quotients

An interactive H5P element has been excluded from this version of the text. You can view it online here:
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Differentiating Rational Functions

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Check Your Understanding: Differentiating Rational Functions

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Differentiate Trigonometric Functions

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Check Your Understanding: Differentiate Trigonometric Functions

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Differentiate Exponential Functions

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Check Your Understanding: Differentiate Exponential Functions

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Differentiate Logarithmic Functions

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Check Your Understanding: Differentiate Logarithmic Functions

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Differentiating Using Multiple Rules

Common Derivatives Review

Please read this link on common derivatives.

Derivative Rules Review

Please read this link on derivative rules.

Check Your Understanding: Differentiating Using Multiple Rules

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INTEGRALS OF COMMON FUNCTIONS

In this section, you will learn about **indefinite** integrals of sin(x), cos(x), and e^x and **definite** integrals functions, and U-substitution exponential function by watching the videos. Also, short problems to check your understanding are included.

Indefinite Integrals of Common Functions

Indefinite Integral of 1/x

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Check Your Understanding: Indefinite Integral of 1/x

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Indefinite Integrals of sin(x), cos(x), and e^x

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Check Your Understanding: Indefinite Integrals of sin(x), cos(x), and e^x

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Definite Integrals of Common Functions

Reverse Power Rule

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Check Your Understanding: Reverse Power Rule

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Rational Functions

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Check Your Understanding: Rational Functions

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Radical Functions

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Check Your Understanding: Radical Functions

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Trig Functions

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Check Your Understanding: Trig Functions

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Natural Logs

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Check Your Understanding: Natural Logs

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***Absolute Value** Functions*

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Check Your Understanding: Absolute Value Functions

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***Piecewise** Functions*

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Check Your Understanding: Piecewise Functions

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Integrating with U-Substitution

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U-Substitution: Definite Integral of Exponential Function

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Check Your Understanding: Integrating with U-Substitution

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USE THE **INTEGRATION** METHOD FOR ESTIMATING THE AREA UNDER A CURVE

In this section, you will learn about integral calculus, definite integrals, and the **Riemann** calculus technique by watching the videos. Also, short problems to check your understanding are included.

Introduction to Integral Calculus

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Introduction to Definite Integrals

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Introduction to Riemann Approximation

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Riemann Sums Review

Go to this link to read information on Riemann Sums.

Check Your Understanding: Riemann Approximation

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Definite Integral as the Limit of a Riemann Sum

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Check Your Understanding: Definite Integral as the Limit of a Riemann Sum

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Finding Area Under a Curve Using Integration

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Check Your Understanding: Finding Area Under a Curve Using Integration

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DESCRIBE THE PROCESS OF INTEGRATION AND DIFFERENTIATION IN CORRESPONDENCE WITH EACH OTHER

The Fundamental Theorem of Calculus

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Check Your Understanding: The Fundamental Theorem of Calculus

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RELEVANCE TO TRANSPORTATION ENGINEERING COURSEWORK

This section explains the relevance of designing vertical curves, estimating congestion pricing schedules, traffic operations and roadway capacity, and estimating aggregate delays in transportation engineering coursework.

Design of Vertical Curves

In roadway design, one grade or slope is connected with the next grade (For example, an upgrade of 3% and a downgrade of 2%) using a parabolic curve defined by a quadratic equation. First-order derivatives of these equations are used to define the highest or lowest point on these curves. If the road is being transitioned from an uphill to a downhill (or vice versa), then the highest point (or lowest point) also represents the point at which the slope is zero. Furthermore, the second derivative of the quadratic equation is a constant, i.e., independent of one's location on the curve. It provides for a smooth transition at a constant rate between the two grades. In this chapter, the sections titled "Interpret Derivatives as the Rate of Change" and "Differentiate Various Functions with Single Variables" provided the details on materials relevant to differentiating polynomials.

Estimating Congestion Pricing Schedules

The idea of congestion pricing is to direct roadway users to use transportation network elements (i.e., routes) that would minimize costs (in the form of travel times) for all road users. Developing a fundamental understanding of this idea requires the first-order differential of the function for the costs incurred by all road users to be set to zero. In this chapter, the sections titled “Interpret Derivatives as the Rate of Change” and “Differentiate Various Functions with Single Variables” provide the details on the relevant materials.

Traffic Operations and Roadway Capacity

Roadway capacity is defined as the maximum flow rate across a point on the roadway segment. It is defined based on the maxima of speed-flow or speed-density relationships. The maxima is found by setting the first derivative of a speed-density (or speed-flow) relationship to zero. In this chapter, the sections titled “Interpret Derivatives as the Rate of Change” and “Differentiate Various Functions with Single Variables” provide the relevant background for this application.

Earthwork Volumes

Roadway construction often requires earthwork, i.e., the addition or removal of material from the sites. The design process yields the roadway elevation and cross-section along a segment. These point locations are then used to estimate the aggregate earthwork volume using the principles described in the section titled “Use the Integration Method for Estimating the Area Under a Curve” of this chapter.

Estimating Aggregate Delays

The estimation of aggregate delays requires the estimation of the area under the curve for the input-output diagrams. These diagrams have time on the x-axis and the number of road users entering/exiting a roadway location on the y-axis. The time of entrance for a road user defies the input curve, while the time of exit defines the output curve. The area between the two curves defines the aggregate delay for the roadway entity under consideration (i.e., an intersection or roadway segment). The above section titled “Use the Integration Method for Estimating the Area Under a Curve” provides the details needed for this chapter.

KEY TAKEAWAYS

Key Takeaways

- The interpretation of first-order derivative as a rate of change and definition of maxima or minima of function as points where this derivative is equal to zero is critical to several transport applications. It is used in roadway design for a smooth transition between grades, minimization of total user costs via congestion pricing, and estimating roadway capacity based on speed-flow or flow-density relationships.
- Integration of a function over a defined interval may be interpreted as the area under the curve. This interpretation is used to estimate the volume of earthwork required, which is a common aspect of roadway construction. This information is important for anticipating the resources and equipment needed for planning and budgeting purposes. The same interpretation of integral as area under the curve is used to estimate the area under the input-output diagrams in traffic operations to estimate aggregate delay on intersections and roadway locations as well

GLOSSARY: KEY TERMS

Derivative[1] – the limit of the ratio of the change in a function to the corresponding change in its independent variable as the latter change approaches zero
Rate of Change[1] – a value that results from dividing the change in a function of a variable by the change in the variable
Slope[1] – the slope of the line tangent to a plane curve at a point
Tangent[1] – meeting a curve or surface in a single point if a sufficiently small interval is considered
Rational Function[1] – a function that is the quotient of two polynomials
Integration[1] – the operation of finding whose differential is known\
Indefinite[1] – having no exact limits
Definite[1] – having distinct or certain limits
Natural Logarithm[1] – a logarithm with e as a base
Absolute Value[1] – a nonnegative number equal in numerical value to a given real number
Piecewise[1] – with respect to a number of discrete intervals, sets, or pieces
Riemann Integral[1] – a definite integral defined as the limit of sums found by partitioning the interval comprising the domain of definition into subintervals, by finding the sum of products each of which consists of the width of a subinterval multiplied by the value of the function at some point in it, and by letting the maximum width of the subintervals approach zero
Fundamental Theorem of Calculus[2] – the theorem, central to the entire development of calculus, that establishes the relationship between differentiation and integration
Fundamental Theorem of Calculus, Part 1[2] – uses a definite integral to define an antiderivative of a function
Fundamental Theorem of Calculus, Part 2[2] – (also, evaluation theorem) we can evaluate a definite integral by evaluating the antiderivative of the integrand at the endpoints of the interval and subtracting

[1] https://www.merriam-webster.com/
[2] "Calculus Volume 1" by Gilbert Strang, Edwin "Jed" Herman on OpenStax, Chapter 5.3: The Fundamental Theorem of Calculus: https://openstax.org/books/calculus-volume-1/pages/5-3-the-fundamental-theorem-of-calculus

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CHAPTER 5: MOTION AND FORCES

This chapter discusses the basic elements of accelerated motion and motion at constant speed to help estimate braking distance and safe travel speeds. Concepts of motion along a straight line are also helpful in estimating headways and flow rates on the facilities. Accelerated motion along a circular path provides a conceptual understanding of roadway superelevation (or banking; also see Chapter 6).

This chapter also describes forces acting on static objects. An understanding of friction forces and normal reactions is critical to transportation infrastructure design. The gravity model is useful as a prerequisite to understanding the dynamics of travel demand (trip distribution between traffic analysis zones (TAZs)).

Learning Objectives

At the end of the chapter, the reader should be able to do the following:

- Describe the gravity model.
- Estimate the friction force and Normal Reaction on a static object.
- Describe the motion of an object in the graphic form through the time-space diagram.
- Use kinematic equations to solve for displacement, time, velocity, and acceleration of an object.
- Describe the motion of an object along a circular path.
- Identify topics in the introductory transportation engineering courses that build on the concepts discussed in this chapter.

UNITS AND MEASUREMENTS

Physical Quantities and Units

We define a **physical quantity** either by specifying how it is measured or by stating how it is calculated from other measurements. For example, we define distance and **time** by specifying methods for measuring them, whereas we define **average speed** by stating that it is calculated as distance traveled divided by time of travel.

Measurements of physical quantities are expressed in terms of **units**, which are standardized values. For example, the length of a race, which is a physical quantity, can be expressed in units of meters (for sprinters) or kilometers (for distance runners). Without standardized units, it would be extremely difficult for scientists to express and compare measured values in a meaningful way.

There are two major systems of units used in the world: SI units (also known as the **metric system**) and **English units** (also known as the customary or imperial system). English units were historically used in nations once ruled by the British Empire and are still widely used in the United States. Virtually every other country in the world now uses SI units as the standard; the metric system is also the standard system agreed upon by scientists and mathematicians. The acronym "SI" is derived from the French Système International.

SI Units of Time, Length, and Mass

SI units are part of the metric system. Metric systems have the advantage that conversions of units involve base-10 number system. There are 100 centimeters in a meter, 1000 meters in a kilometer, and so on. In nonmetric systems, such as the system of U.S. customary units, the relationships are not as simple—there are 12 inches in a foot, 5280 feet in a mile, and so on.

Length	**Mass**	**Time**
Meter (m)	Kilogram (kg)	Second (s)

It is often necessary to convert from one type of unit to another. Let us consider a simple example of how to convert units. Let us say that we want to convert 80 meters (m) to kilometers (km). The first thing to do is to list the units that you have and the units that you want to convert to. In this case, we have units in meters, and we want to convert to kilometers.

Next, we need to determine a **conversion factor** relating meters to kilometers. A conversion factor is a ratio expressing how many of one unit are equal to another unit. For example, there are 12 inches in 1 foot, 100 centimeters in 1 meter, 60 seconds in 1 minute, and so on. In this case, we know that there are 1,000 meters in 1 kilometer.

Now we can set up our unit conversion. We will write the units that we have and then multiply them by the conversion factor so that the units cancel out, as shown:

$$80\,\cancel{\text{m}} \times \frac{1\text{ km}}{1000\,\cancel{\text{m}}} = 0.080\text{ km}.$$

Note that the "m" unit cancels, leaving only the desired "km" unit. You can use this method to convert between any types of units.

Displacement

Position:

To describe the motion of an object, you must first be able to describe its **position**—where it is at any particular time. More precisely, you need to specify its position relative to a convenient reference frame. Earth is often used as a reference frame, and we often describe the position of an object as it relates to stationary objects in that reference frame. For example, a rocket launch would be described in terms of the position of the rocket with respect to the Earth as a whole, while a professor's position could be described in terms of where she is in relation to the nearby white board. (See Figure 1). In other cases, we use reference frames that are not stationary but are in motion relative to the Earth. To describe the position of a person in an airplane, for example, we use the airplane, not the Earth, as the reference frame. (See Figure 2).

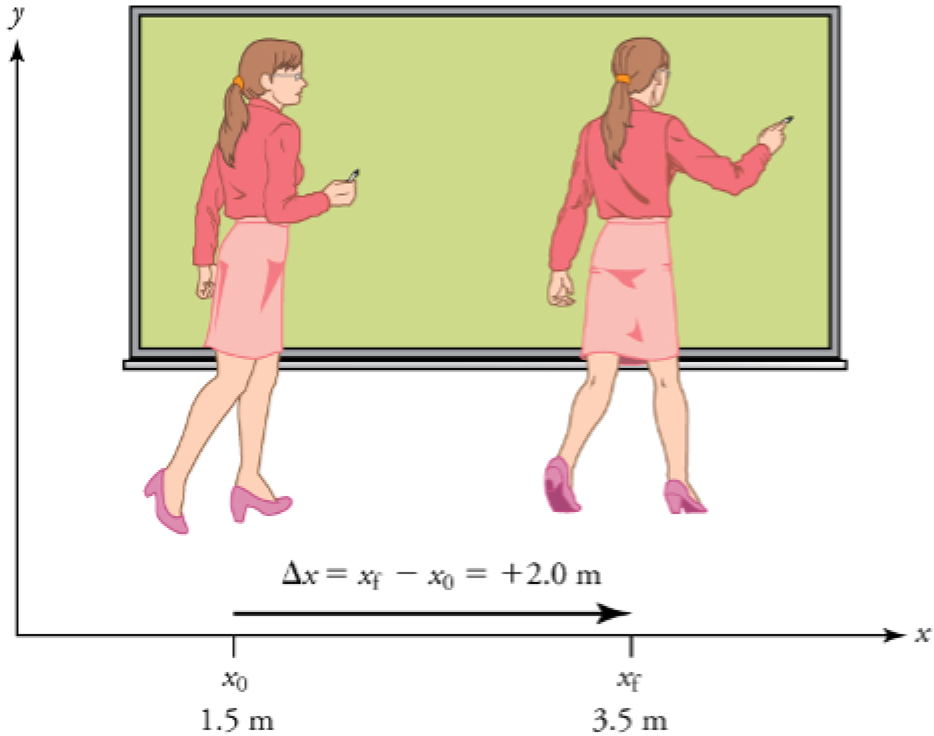

Figure 1: A professor paces left and right while lecturing. Her position relative to the blackboard is given by x. The $+2.0m$ *displacement of the professor relative to the blackboard is represented by an arrow pointing to the right.*

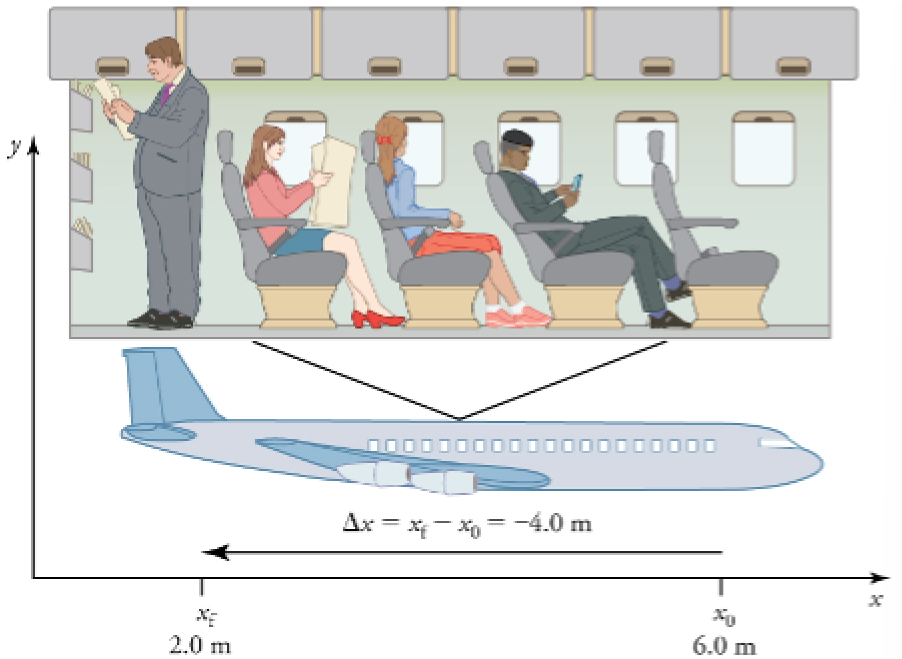

Figure 2: A passenger moves from his seat to the back of the plane. His location relative to the airplane is given by x. The −4 m displacement of the passenger relative to the plane is represented by an arrow toward the rear of the plane. Notice that the arrow representing his displacement is twice as long as the arrow representing the displacement of the professor (he moves twice as far) in Figure 1.

Displacement:

If an object moves relative to a reference frame (for example, if a professor moves to the right relative to a white board or a passenger moves toward the rear of an airplane), then the object's position changes. This change in position is known as **displacement.** The word "displacement" implies that an object has moved or has been displaced.

Displacement is the *change in position* of an object:

$$\Delta x = \genfrac{}{}{0pt}{}{\Delta}{x_f - x_0}$$

Where Δx is displacement, x_f is the final position, and x_0 is the initial position.

The Greek letter Δ (delta) means "change in" whatever quantity follows it; thus, Δx means *change in position.* Always solve for displacement by subtracting initial position x_0 from final position x_f.

Note that the SI unit for displacement is the **meter** (m), but sometimes kilometers, miles, feet, and other units of length are used. Keep in mind that when units other than the meter are used in a problem, you may need to convert them into meters to complete the calculation.

Note that displacement has a direction as well as a magnitude. The professor's displacement is 2.0 m to the right, and the airline passenger's displacement is 4.0 m toward the rear. In one-dimensional motion, direction can be specified with a plus or minus sign. When you begin a problem, you should select which direction is positive (usually that will be to the right or up, but you are free to select positive as being any direction). The professor's initial position is $x_0 = 1.5\ \mathrm{m}$ and her final position is $x_f = 3.5\ \mathrm{m}$. Thus, her displacement is

$$\Delta x = x_f - x_0 = 3.5m - 1.5m = +2.0m$$

In this coordinate system, motion to the right is positive, whereas motion to the left is negative. Similarly, the airplane passenger's initial position is $x_0 = 6.0\ \mathrm{m}$ and his final position is $x_f = 2.0\ \mathrm{m}$, so his displacement is

$$\Delta x = x_f - x_0 = 2.0\ \mathrm{m} - 6.0\ \mathrm{m} = -4.0\ \mathrm{m}$$

His displacement is negative because his motion is toward the rear of the plane, or in the negative *x* direction in our coordinate system.

Distance

Although displacement is described in terms of direction, distance is not. **Distance** is defined to be the magnitude or size of displacement between two positions. Note that the distance between two positions is different from the distance traveled between them. **Distance traveled** is the total length of the path traveled between two positions. Distance has no direction and, thus, no sign. For example, the distance the professor walks is 2.0 m. The distance the airplane passenger walks is 4.0 m.

Check Your Understanding: Displacement

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Vectors, Scalars, and Coordinate Systems

What is the difference between distance and displacement? Whereas displacement is defined by both direction and magnitude, distance is defined only by magnitude. Displacement is an example of a vector quantity. Distance is an example of a scalar quantity. A **vector** is any quantity with *both magnitude and direction*. Other examples of vectors include a velocity of 90 km/h east and a **force** of 500 newtons straight down.

The direction of a vector in one-dimensional motion is given simply by a plus (+) or minus (–) sign. Vectors are represented graphically by arrows. An arrow used to represent a vector has a length proportional to the vector's magnitude (e.g., the larger the magnitude, the longer the length of the vector) and points in the same direction as the vector.

Some physical quantities, like distance, either have no direction or none is specified. A **scalar**
20°C

is any quantity that has a magnitude, but no direction. For example, a 20°C temperature, the 250 kilocalories (250 Calories) of energy in a candy bar, a 90 km/h speed limit, a person's 1.8 m height, and a distance of 2.0 m are all scalars – quantities with no specified direction. Note, however, that a **scalar** can be negative, such as -20°C temperature. In this case, the minus sign indicates a point on a scale rather than a direction. Scalars are never represented by arrows.

Coordinate Systems for One-Dimensional Motion

To describe the direction of a vector quantity, you must designate a coordinate system within the reference frame. For one-dimensional motion, this is a simple coordinate system consisting of a one-dimensional coordinate line. In general, when describing horizontal motion, motion to the right is usually considered positive, and motion to the left is considered negative. With vertical motion, motion up is usually positive, and motion down is negative. In some cases, however, as with the jet in Figure 3 below, it can be more convenient to switch the positive and negative directions. For example, if you are analyzing the motion of falling objects, it can be useful to define downwards as the positive direction. If people in a race are running to the left, it is useful to define left as the positive direction. It does not matter as long as the system is clear and consistent. Once you assign a positive direction and start solving a problem, you cannot change it.

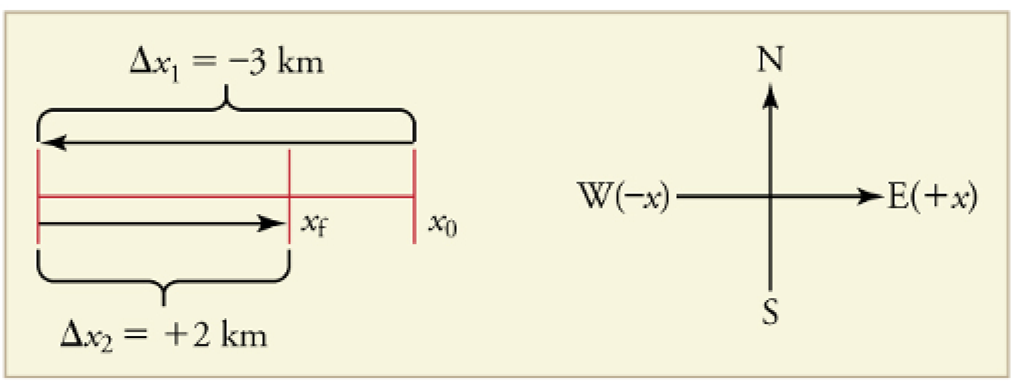

Figure 3: It is usually convenient to consider motion upward or to the right as positive (+) and motion downward or to the left as negative (-)

Check Your Understanding: Vectors, Scalars, and Coordinate Systems

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TIME, VELOCITY, SPEED, AND ACCELERATION

In this section, you will learn about time, velocity, speed and **acceleration** along with motion graphs and motion diagrams by reading each description. Also, short problems to check your understanding are included.

Describe the Motion of an Object in the Graphic Form through the Time-Space Diagram

Time, Velocity, and Speed, Acceleration

In **physics**, the definition of time is simple — time is change, or the interval over which change occurs. It is impossible to know that time has passed unless something changes. The amount of time or change is calibrated by comparison with a standard. The SI unit for time is the **second**, abbreviated s. We might, for example, observe that a certain pendulum makes one full swing every 0.75 s. We could then use the pendulum to measure time by counting its swings or, of course, by connecting the pendulum to a clock mechanism that registers time on a dial. This allows us to not only measure the amount of time, but also to determine a sequence of events.

How does time relate to motion? We are usually interested in **elapsed time** for a particular motion, such as how long it takes an airplane passenger to get from his seat to the back of the plane. To find elapsed time, we note the time at the beginning and end of the motion and subtract the two. For example, a lecture may start at 11:00 A.M. and end at 11:50 A.M., so that the elapsed time would be 50 min. Elapsed time Δt is the difference between the ending time and beginning time,

$$\Delta t = t_f - t_0$$

Where Δt is the change in time or elapsed time, t_f is the time at the end of the motion, and t_0 is the time at the beginning of the motion. (As usual, the delta symbol, Δ, means the change in the quantity that follows it.)

Velocity:

Your notion of velocity is probably the same as its scientific definition. You know that if you have a large displacement in a small amount of time you have a large velocity, and that velocity has units of distance divided by time, such as miles per hour or kilometers per hour.

Average velocity is *displacement (change in position divided by the time of travel,*

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_0}{t_f - t_0}$$

Where $\bar{v}$ is the *average* (indicated by the bar over the *v*) velocity, Δx is the change in position (or displacement), and x_f and x_0 are the final and beginning positions at times t_f and t_0, respectively. If the starting time $t_0$is taken to be zero, the average velocity is simply

$$\bar{v} = \frac{\Delta x}{t}$$

Notice that this definition indicates that velocity is a vector because displacement is a vector. It has both magnitude and direction. The SI unit for velocity is meters per second or m/s, but many other units, such as km/h, mi/h (also written as mph), and cm/s, are in common use. Suppose, for example, an airplane passenger took 5 seconds to move -4 m (the minus sign indicates that displacement is toward the back of the plane). His average velocity would be

$$\bar{v} = \frac{\Delta x}{t} = \frac{-4 \text{ m}}{5 \text{ s}} = -0.8 \text{ m/s}$$

The minus sign indicates the average velocity is also toward the rear of the plane.

The average velocity of an object does not tell us anything about what happens to it between the starting point and ending point, however. For example, we cannot tell from average velocity whether the airplane passenger stops momentarily or backs up before he goes to the back of the plane. To get more details, we must consider smaller segments of the trip over smaller time intervals.

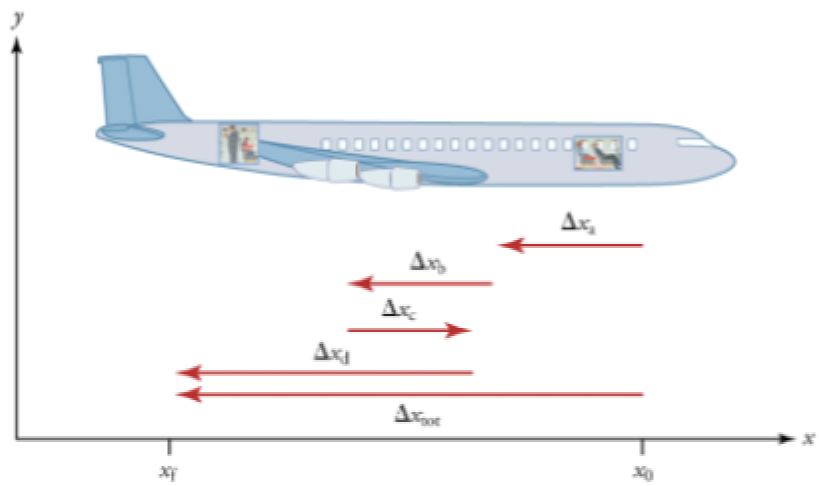

Figure 4: A more detailed record of an airplane passenger heading toward the back of the plane, showing smaller segments of his trip.

The smaller the time intervals considered in a motion, the more detailed the information. When we carry this process to its logical conclusion, we are left with an infinitesimally small interval. Over such an interval, the average velocity becomes the **instantaneous velocity** or the velocity at a specific instant. A car's speedometer, for example, shows the magnitude (but not the direction) of the instantaneous velocity of the car. (Police give tickets based on instantaneous velocity, but when calculating how long it will take to get from one place to another on a road trip, you need to use average velocity.) Instantaneous velocity v is the average velocity at a specific instant in time (or over an infinitesimally small-time interval).

Speed:

In everyday language, most people use the terms "speed" and "velocity" interchangeably. In physics, however, they do not have the same meaning and they are distinct concepts. One major difference is that speed has no direction. Thus, speed is a scalar. Just as we need to distinguish between instantaneous velocity and average velocity, we also need to distinguish between instantaneous speed and average speed.

Instantaneous speed is the magnitude of instantaneous velocity. For example, suppose the airplane passenger at one instant had an instantaneous velocity of –3.0 m/s (the minus meaning toward the rear of the plane). At that same time, his instantaneous speed was 3.0 m/s. Or suppose that at one time during a shopping trip your instantaneous velocity is 40 km/h due north. Your instantaneous speed at that instant would be 40 km/h—the same magnitude but without a direction. Average speed, however, is vastly different from average velocity. Average speed is the distance traveled divided by elapsed time.

We have noted that distance traveled can be greater than displacement. So average speed can be greater than average velocity, which is displacement divided by time. For example, if you drive to a store and return home in half an hour, and your car's odometer shows the total distance traveled was 6 km, then your average speed was 12 km/h. Your average velocity, however, was zero, because your displacement for the round trip is zero. (Displacement is change in position and, thus, is zero for a round trip.) Thus, average speed is not simply the magnitude of average velocity.

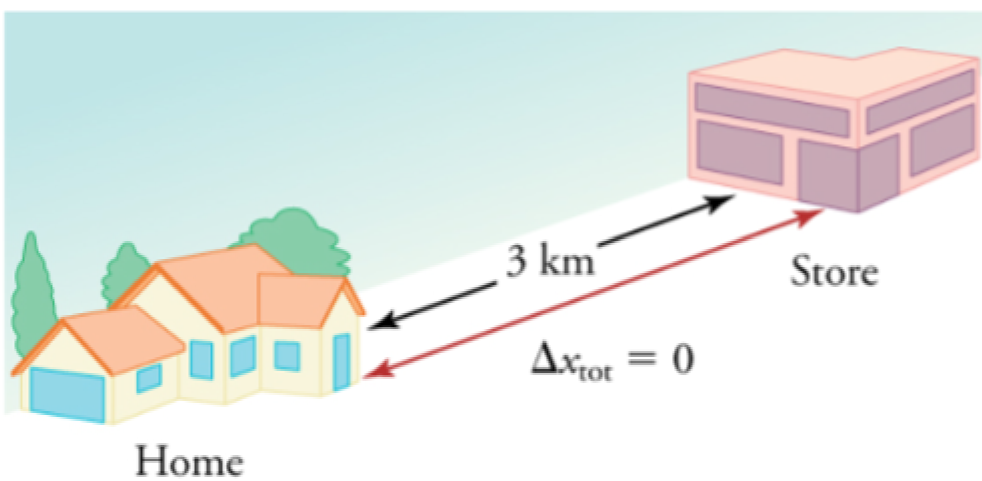

Figure 5: During a 30-minute round trip to the store, the total distance traveled is 6 km. The average speed is 12 km/h. The displacement for the round trip is zero since there was no net change in position. Thus, the average velocity is zero.

Another way of visualizing the motion of an object is to use a graph. A plot of position or of velocity as a function of time can be extremely useful. For example, for this trip to the store, the position, velocity, and speed-vs.-time graphs are displayed in Figures 6-8. (Note that these graphs depict a very simplified model of the trip. We are assuming that speed is constant during the trip, which is unrealistic given that we will probably stop at the store. But for simplicity's sake, we will model it with no stops or changes in speed. We are also assuming that the route between the store and the house is a perfectly straight line.)

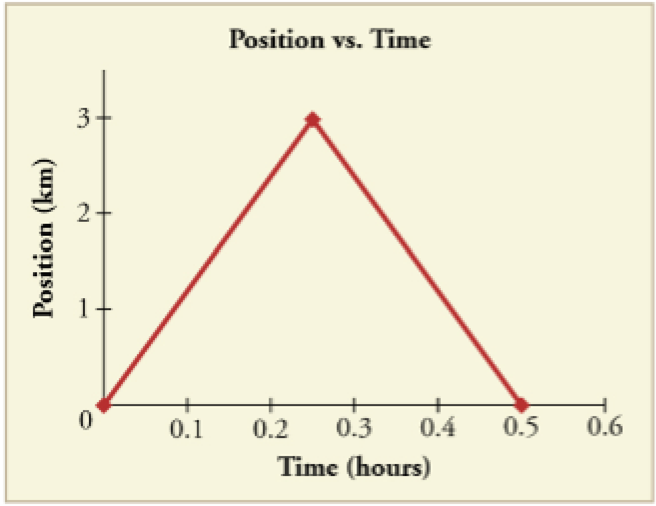

Figure 6: Position vs. time on a trip. Note that the velocity for the return trip is negative.

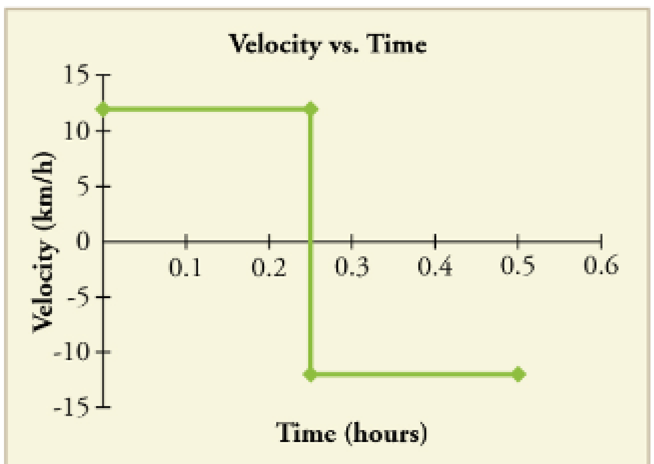

Figure 7: Velocity vs. time on a trip. Note that the velocity for the return trip is negative.

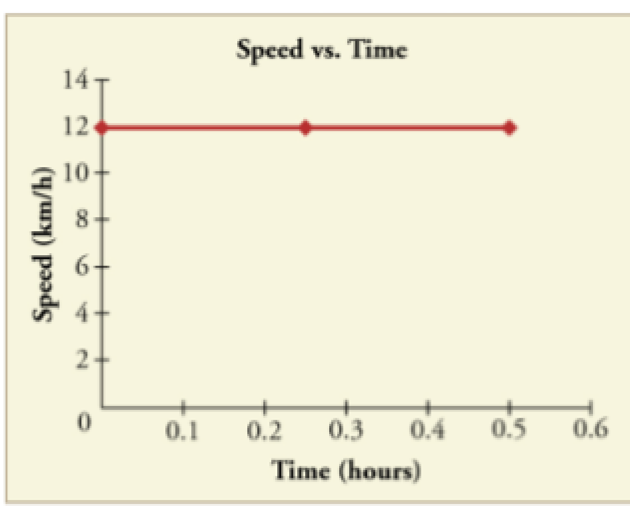

Figure 8: Speed vs. time on a trip. Note that the velocity for the return trip is negative.

Acceleration:

In everyday conversation, to accelerate means to speed up. The accelerator in a car can in fact cause it to speed up. The greater the acceleration, the greater the change in velocity over a given time. The formal definition of acceleration is consistent with these notions, but more inclusive.

Average Acceleration is *the rate at which velocity changes,*

$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0}$

Where $\bar{a}$ is average acceleration, *v* is velocity, and *t* is time.

Because acceleration is velocity in m/s divided by time in s, the SI units for acceleration are $\mathrm{m/s^2}$ meters per second squared or meters per second per second, which means by how many meters per second the velocity changes every second.

Recall that velocity is a vector—it has both magnitude and direction. This means that a change in velocity can be a change in magnitude (or speed), but it can also be a change in direction. For example, if a car turns a corner at constant speed, it is accelerating because its direction is changing. The quicker you turn, the greater the acceleration. So, there is an acceleration when velocity changes either in magnitude (an increase or decrease in speed) or in direction, or both.

Keep in mind that although acceleration is in the direction of the change in velocity, it is not always in the direction of motion. When an object's acceleration is in the same direction of its motion, the object will speed up. However, when an object's acceleration is opposite to the direction of its motion, the object will slow down. Speeding up and slowing down should not be confused with a positive and negative acceleration.

Instantaneous Acceleration

Instantaneous acceleration *a*, or the acceleration at a specific instant in time, is obtained by the same process as discussed for instantaneous velocity, that is by considering an infinitesimally small interval of time. How do we find instantaneous acceleration using only algebra? The answer is that we choose an average acceleration that is representative of the motion. Figure 9 shows graphs of instantaneous acceleration versus time for two quite different motions. In (a) the acceleration varies slightly and the average over the entire interval is nearly the same as the instantaneous acceleration at any time. In this case, we should treat this motion as if it had a constant acceleration equal to the average (in this case about $1.8\ \mathrm{m/s^2}$. In (b) the acceleration varies drastically over time. In such situations it is best to consider smaller time intervals and choose an average acceleration for each. For example, we could consider motion over the time intervals from 0 to 1.0 s and from 1.0 to 3.0 s as separate motions with accelerations of $+3.0\ \mathrm{m/s^2}$ and $-2.0\ \mathrm{m/s^2}$, respectively.

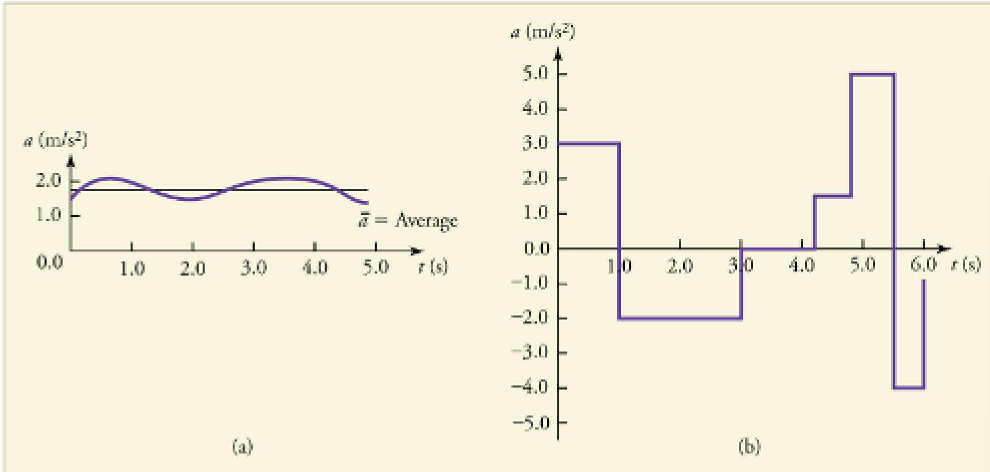

Figure 9: Graphs of instantaneous acceleration versus time for two different one-dimensional motions. (a) Here acceleration varies only slightly and is always in the same direction since it is positive. The average over the interval is nearly the same as the acceleration at any given time. (b) Here the acceleration varies, perhaps representing a package on a post office conveyor belt that is accelerated forward and backward as it bumps along. It is necessary to consider small time intervals (such as from 0 to 1.0 s) with constant or nearly constant acceleration in such a situation.

Check Your Understanding: The Motion of an Object in the Graphic Form through the Time-Space Diagram

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Interpreting Motion Graphs

The Moving Man

Please complete this simulation.

Check Your Understanding: Interpreting Motion Graphs

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Motion Diagram

Comparing Motion Diagrams (Simulation)

Please complete this simulation.

USE KINEMATIC EQUATIONS TO SOLVE FOR DISPLACEMENT, TIME, VELOCITY AND ACCELERATION OF AN OBJECT

In this section, you will learn how to use kinematic equations to solve for displacement, time, velocity and acceleration of an object by reading each description. Also, short problems to check your understanding are included.

Motion Equations for Constant Acceleration in One Dimension

We might know that the greater the acceleration of, say, a car moving away from a stop sign, the greater the displacement in a given time. But we have not developed a specific equation that relates acceleration and displacement. In this section, we develop some convenient equations for kinematic relationships.

Notation: *t, x, v, a*

First, let us make some simplifications in notation. Taking the initial time to be zero, as if time is measured with a stopwatch, is a great simplification. Since elapsed time is $\Delta t = t_f - t_0$, taking $t_0 = 0$ means that $\Delta t = t_f$, the final time on the stopwatch. When initial time is taken to be zero, we use the subscript 0 to denote initial values of position and velocity. That is, x_0 *is the initial position and* v_0, *is the initial velocity.* We put no subscripts on the final values. The is, *t is the final time, x is the final position, and v is final velocity.* This gives a simpler expression for elapsed time – now, $\Delta t = t$. It also simplifies the expression for displacement, which is now $\Delta x = x - x_0$. Also, it simplifies the expression for change in velocity, which is now $\Delta v = v - v_0$. To summarize, using the simplified notation, with the initial time taken to be zero,

$$\Delta t = t$$

$$\Delta x = x - x_0$$

$$\Delta v = v - v_0$$

Where *the subscript 0 denotes an initial value and the absence of a subscript denotes a final value* in whatever motion is under consideration.

We now make the important assumption that acceleration is constant. This assumption allows us to avoid using calculus to find instantaneous acceleration. Since acceleration is constant, the average and instantaneous accelerations are equal. That is,

$$\bar{a} = a = \text{ constant}$$

So, we use the symbol *a* for acceleration at all times. Assuming acceleration to be constant does not seriously limit the situations we can study nor degrade the **accuracy** of our treatment. For one thing, acceleration is constant in a considerable number of situations. Furthermore, in many other situations we can accurately describe motion by assuming a constant acceleration equal to the average acceleration for that motion. Finally, in motions where acceleration changes drastically, such as a car accelerating to top speed and then braking to a stop, the motion can be considered in separate parts, each of which has its own constant acceleration.

Solving for Displacement (Δx) and the Final Position (*x*) from Average Velocity when Acceleration (*a*) is Constant.

To get our first two new equations, we start with the definition of average velocity:

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

Substituting the simplified notation for $\Delta x \text{ and } \Delta t$ yields

$$\bar{v} = \frac{x - x_0}{t}$$

Solving for *x* yields

$$x = x_0 + \bar{v}t$$

Where the average velocity is, with constant a

$$\bar{v} = \frac{v_0 + v}{2}$$

The equation $\bar{v} = \frac{v_0 + v}{2}$ reflects the fact that, when acceleration is constant, v is just the simple average of the initial and final velocities. For example, if you steadily increase your velocity (that is, with constant acceleration) from 30 to 60 km/h, then your average velocity during this steady increase is 45 km/h. Using the equation $\bar{v} = \frac{v_0 + v}{2}$, we see that $\bar{v} = \frac{v_0 + v}{2} = \frac{30 \text{ km/h} + 60 \text{ km/h}}{2} = 45 \text{ km/h}.$

The equation $x = x_0 + \bar{v}t$ gives insight into the relationship between displacement, average velocity, and time. It shows, for example, that displacement is a linear function of average velocity. (By linear function, we mean that displacement depends on $\bar{v}$ rather than on $\bar{v}$raised to some other power, such as $\overline{v^2}$. When graphed, linear functions look like straight lines with a constant **slope**.) On a car trip, for example, we will get twice as far in a given time if we average 90 km/h than if we average 45 km/h.

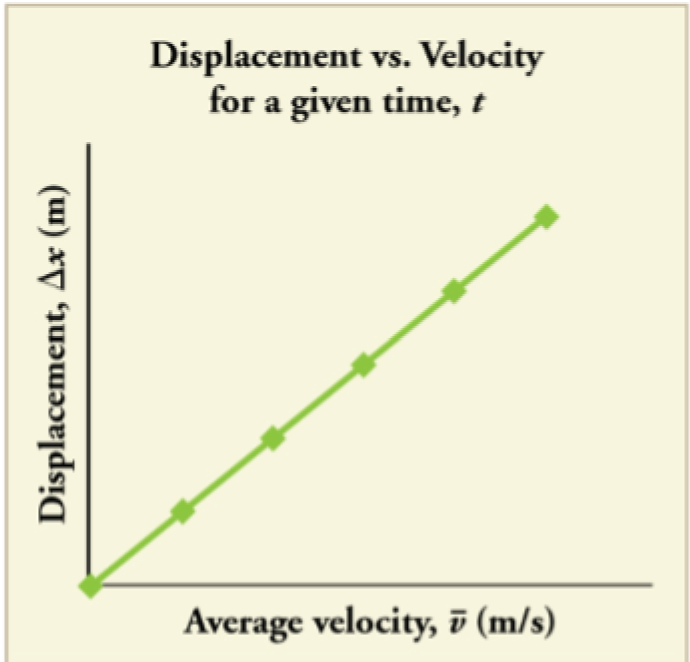

Figure 10: There is a linear relationship between displacement and average velocity. For a given time t, an object moving twice as fast as another object will move twice as far as the other object.

Solving for Final Velocity:

We can derive another useful equation by manipulating the definition of acceleration.

$a = \frac{\Delta v}{\Delta t}$

Substituting the simplified notation for Δv and Δt gives us (constant a)

$a = \frac{v - v_0}{t}$

Solving for *v* yields (constant a)

$v = v_0 + at$

In addition to being useful in problem solving, the equation $v = v_0 + at$ gives us insight into the relationships among velocity, acceleration, and time. From it we can see, for example that

- Final velocity depends on how large the acceleration is and how long it lasts
- If the acceleration is zero, then the final velocity equals the initial velocity $(v = v_0)$, as expected (i.e., velocity is constant)
- If *a* is negative, then the final velocity is less than the initial velocity.

Solving for Final Position when Velocity is Not Constant $(a \neq 0)$

We can combine the equations above to find a third equation that allows us to calculate the final position of an object experiencing constant acceleration. We start with

$v = v_0 + at$

Adding v_0 to each side of this equation and dividing by 2 gives

$\frac{v_0+v}{2} = v_0 + \frac{1}{2}at$

Since $\frac{v_0+v}{2} = \bar{v}$ for constant acceleration, then

$\bar{v} = v_0 + \frac{1}{2}at$

Now we substitute this expression for $\bar{v}$ into the equation for displacement, $x = x_0 + \bar{v}t$, yielding (constant a)

$x = x_0 + v_0t + \frac{1}{2}at^2$

What else can we learn by examining the equation $x = x_0 + v_0t + \frac{1}{2}at^2$? We see that:

- Displacement depends on the square of the elapsed time when acceleration is not zero.
- If acceleration is zero, then the initial velocity equals average velocity $(v_0 = \bar{v})$ and $x = x_0 + v_0t + \frac{1}{2}at^2$ becomes $x = x_0 + v_0t$

Solving for Final Velocity when Velocity is not Constant $(a \neq \mathbf{0})$

A fourth useful equation can be obtained from another algebraic manipulation of previous equations.

If we solve $v = v_0 + at$ for t, we get

$t = \frac{v-v_0}{a}$

Substituting this and $\bar{v} = \frac{v_0+v}{2}$ into $x = x_0 + \bar{v}t$, we get (constant a)

$v^2 = v_0^2 + 2a\left(x - x_0\right)$

An examination of the equation $v^2 = v_0^2 + 2a\left(x - x_0\right)$ can produce further insights into the general relationships among physical quantities:

- The final velocity depends on how large the acceleration is and the distance over which it acts
- For a fixed **deceleration**, a car that is going twice as fast does not simply stop in twice the distance – it takes much further to stop. (This is why we have reduced speed zones near schools).

Summary of Kinematic Equations (Constant *a*)

$x = x_0 + \bar{v}t$

$\bar{v} = \frac{v_0 + v}{2}$

$v = v_0 + at$

$x = x_0 + v_0 t + \frac{1}{2}at^2$

$v^2 = v_0^2 + 2a\left(x - x_0\right)$

What are the Kinematic Formulas?

Please read "What Are Kinematic Formulas?"

Choosing Kinematic Equations

FREE-BODY DIAGRAMS

In this section, you will learn about free-body diagrams and how to make a free-body diagram by reading each description. Also, short problems to check your understanding are included.

Introduction to Forces and Free-body Diagrams

Key Terms

Term	Meaning
Force	A push or pull on an object, usually has symbol *F*. Has SI units of Newtons $\left(\mathrm{N}\right)$ or $\frac{\mathrm{kg}m}{\mathrm{s}^2}$
Contact force	A force that requires contact between objects. Examples are tension, normal force, and friction.
Long range force	A force that does not need contact between objects to exist.
Free body diagram	A diagram showing the forces acting on the object. The object is represented by a dot with forces drawn as arrows pointing away from the dot. Sometimes called force diagrams.

Types of Forces

Force (symbol)	Force type	Description
Weight (F_g or *W*)	Long range	Force from gravity acting on an object with mass. Sometime called force of gravity. Pulls towards the Earth (down) always.
Tension (F_T or *T*)	Contact	Force of something pulling on an object. Can be caused by a string, rope, chain, cord, cable, or wire. Pulls along the direction of the rope on the object.
Normal force (F_N or *N*)	Contact	Force between two objects when they touch. Pushes perpendicularly to the object's surface.
Friction (F_f or *f*)	Contact	Force resisting sliding between surfaces. Pushes parallel to the contact surface and in the opposite direction of sliding.

How to make a free body diagram:

1. Start by identifying the contact forces. Let us look for what the object is touching by outlining the object (see Figure 11). Draw a dot where something touches the outline; where there is a dot, there must be at least one contact force. Draw the force vectors at the contact points to represent how they push or pull on the object (including correct direction).

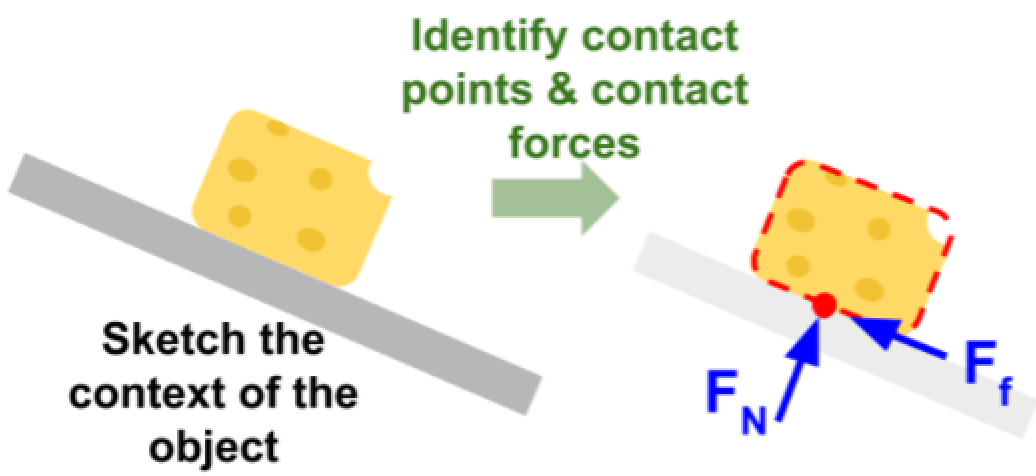

Figure 11: Contact force identification for a block of cheese resting on a ramp.

2. After we have identified the contact forces, draw a dot to represent the object we are interested in (see Figure 12 in example). We only want to find the forces acting on our object and not forces the object exerts on other objects.
3. Draw a coordinate system and label the positive directions. If the object is on an incline, then align the axes with the incline.
4. Draw the contact forces on the dot with an arrow pointing away from the dot. Make sure the arrow lengths are relatively proportional to each other. Label all forces.
5. Draw and label our long-range forces. This will usually be weight unless there is electric charge or magnetism involved.
6. Draw and label your acceleration vector off to the side of the dot – not touching the

dot. If there is no acceleration, then write $a = 0$.

Here is an example of a free body diagram for a block of cheese resting on a table (See Figure 12). Gravity pulls down on the cheese's mass with **weight** *(W)* and the table pushes up on the cheese with a normal force *(N)*. Since there are no ropes and the cheese is not trying to slide, there is no **tension** or friction.

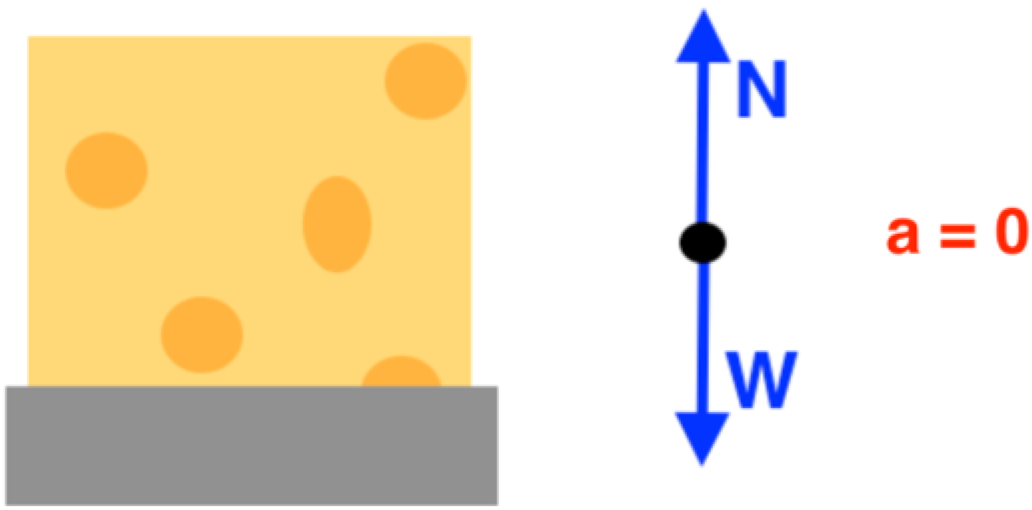

Figure 12: The free body diagram for a block of cheese resting on a table has normal force N up, weight W down, and no acceleration.

Common mistakes and misconceptions

1. Sometimes people draw the forces of the object acting on other things. We only want to draw the forces pushing or pulling on our object. Only focus on what is happening to the object of interest.
2. Sometimes people forget the directions of the different types of forces. Weight is always down, friction is always parallel to the contact surface, normal force is always perpendicular to the contact surface, and tension only pulls.

Check Your Understanding: Forces and Free-body Diagrams

An interactive H5P element has been excluded from this version of the text. You can view it online here:
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ESTIMATE THE FRICTION FORCE AND NORMAL REACTION ON A STATIC OBJECT

In this section, you will learn how estimate the friction force and normal reaction on a static object and learn about kinetic and **static friction** forces by reading each description along with watching the videos. Also, short problems to check your understanding are included.

Normal Force

If the force supporting a load is perpendicular to the surface of contact between the load and its support, this force is defined to be a **normal force** and is given the symbol N (different from newton). The word normal means perpendicular to a surface.

Friction

Friction is a force that opposes relative motion between systems in contact. One of the simpler characteristics of friction is that it is parallel to the contact surface between systems and always in a direction that opposes motion or attempted motion of the systems relative to each other. If two systems are in contact and moving relative to one another, then the friction between them is called **kinetic friction**. For example, friction slows a hockey puck sliding on ice. But when objects are stationary, static friction can act between them; the static friction is usually greater than the kinetic friction between the objects.

As seen in Table 1 below, the coefficients of kinetic friction are less than their static counterparts. The equations explained below include the dependence of friction on materials and the normal force.

Table 1: Coefficients of Static and Kinetic Friction

System	μ_s Static Function	μ_k Kinetic Function
Rubber on dry concrete	1.0	0.7
Rubber on wet concrete	0.7	0.5
Wood on wood	0.5	0.3
Waxed wood on wet snow	0.14	0.1
Metal on wood	0.5	0.3
Steel on steel (dry)	0.6	0.3
Steel on steel (oiled)	0.05	0.03
Teflon on steel	0.04	0.04
Bone lubricated by synovial fluid	0.016	0.015
Shoes on wood	0.9	0.7
Shoes on ice	0.1	0.05
Ice on ice	0.1	0.03
Steel on ice	0.04	0.02

Kinetic and Static Friction Forces

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Static Friction Equation

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Where there is no motion between the objects, the **magnitude of static friction** is $f_S \leq \mu_S N,$ where μ_S is the coefficient of static friction and *N* is the magnitude of the normal force.

The symbol $\leq$ means less than or equal to, implying that static friction can have a minimum and a maximum value of $\mu_s N$. Static friction is a responsive force that increases to be equal and opposite to whatever force is exerted, up to its maximum limit. Once the applied force exceeds $f_{s(\max)}$, the object will move. Thus $f_{s(\max)} = \mu_S N$.

Kinetic Friction Equation

Once an object is moving, the **magnitude of kinetic friction** f_k is given by $f_k = \mu_k N,$ where μ_k is the coefficient of kinetic friction.

Check Your Understanding: Friction

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Static and Kinetic Friction Example

Check Your Understanding: Static and Kinetic Friction

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FORCES AND NEWTON'S LAWS

In this section, you will learn about Newton's three laws of motion along with laws of gravitation by reading each description along with watching the videos. Also, short problems to check your understanding are included.

Newton's 3 Laws of Motion

Newton's First Law of Motion

There exists an **inertial frame of reference** such that a body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net **external force**.

The first **law** of motion postulates the existence of at least one frame of reference which we call an inertial reference frame, relative to which the motion of an object not subject to forces is a straight line at a constant speed. An inertial reference frame is any reference frame that is not itself accelerating. A car traveling at constant velocity is an inertial reference frame. A car slowing down for a stoplight, or speeding up after the light turns green, will be accelerating and is not an inertial reference frame. Finally, when the car goes around a turn, which is due to an acceleration changing the direction of the velocity vector, it is not an inertial reference frame. Note that Newton's laws of motion are only valid for inertial reference frames.

Rather than contradicting our experience, **Newton's first law of motion** states that there must be a cause (which is a **net external force**) for there to be any change in velocity (either a change in magnitude or direction) in an inertial reference frame. An object sliding across a table or floor slows down due to the net force of friction acting on the object.

The idea of cause and effect is crucial in accurately describing what happens in various situations. For example, consider what happens to an object sliding along a rough horizontal surface. The object quickly grinds to a halt. If we spray the surface with talcum powder to make the surface smoother, the object slides farther. If we make the surface even smoother by rubbing lubricating oil on it, the object slides farther yet. Extrapolating to a frictionless surface, we can imagine the object sliding in a straight line indefinitely. Friction is thus the cause of the slowing (consistent with Newton's first law). The object would not slow down at all if friction were completely eliminated. Consider an air hockey table. When the air is turned off, the puck slides only a short distance before friction slows it to a stop. However, when the air is turned on, it creates a nearly frictionless surface, and the puck glides long distances without slowing down. Additionally, if we know enough about the friction, we can accurately predict how quickly the object will slow down. Friction is an external force.

Mass: The property of a body to remain at rest or to remain in motion with constant velocity is called inertia. Newton's first law is often called the **law of inertia**. As we know from experience, some objects have more **inertia** than others. It is obviously more difficult to change the motion of a large boulder than that of a basketball, for example. The inertia of an object is measured by its mass.

An object with a small mass will exhibit less inertia and be more affected by other objects. An object with a large mass will exhibit greater inertia and be less affected by other objects. This inertial mass of an object is a measure of how difficult it is to alter the uniform motion of the object by an external force.

Roughly speaking, mass is a measure of the amount of "stuff" (or matter) in something. The quantity or amount of matter in an object is determined by the numbers of atoms and molecules of several types it contains. Unlike weight, mass does not vary with location. The mass of an object is the same on Earth, in orbit, or on the surface of the Moon. In practice, it is exceedingly difficult to count and identify all of the atoms and molecules in an object, so masses are not often determined in this manner. Operationally, the masses of objects are determined by comparison with the standard **kilogram**.

Check Your Understanding: Newton's First Law of Motion

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Newton's Second Law of Motion

Newton's second law of motion is closely related to Newton's first law of motion. It mathematically states the cause-and-effect relationship between force and changes in motion. Newton's second law of motion is more quantitative and is used extensively to calculate what happens in situations involving a force. Before we can write down Newton's second law as a simple equation giving the exact relationship of force, mass, and acceleration, we need to sharpen some ideas that have already been mentioned.

First, what do we mean by a change in motion? The answer is that a change in motion is equivalent to a change in velocity. A change in velocity means, by definition, that there is an acceleration. Newton's first law says that a net external force causes a change in motion; thus, we see that a net external force causes acceleration.

Another question immediately arises. What do we mean by an external force? An intuitive notion of external is correct—an external force acts from outside the **system** of interest. An internal force acts between elements of the system. Only external forces affect the motion of a system, according to Newton's first law.

To obtain an equation for Newton's Second Law, we first write the relationship of acceleration and net external force as the proportionality

$$a \propto F_{\text{net}}$$

Where the symbol $\propto$ means "proportional to," and F_{net} is the net external force. This proportionality states acceleration is directly proportional to the net external force. Now, it also seems reasonable that acceleration should be inversely proportional to the mass of the system. In other words, the larger the mass (the inertia), the smaller the acceleration produced by a given force. This proportionality is written as $a \propto \frac{1}{m}$, where *m* is the mass of the system. Combining the two proportionalities just given yield's Newton's Second Law of Motion.

Newton's Second Law of Motion: The acceleration of a system is directly proportional to and in the same direction as the net external force acting on the system, and inversely proportional to its mass. In equation form, Newton's Second Law of Motion is:

$a = \frac{F_{net}}{m}$

Often written as:

$F_{\text{net}} = ma$

Units of Force: $F_{\text{net}} = ma$ is used to define the units of force in terms of the three basic units for mass, length, and time. The SI unit of force is called the newton (abbreviated N) and is the force needed to accelerate a 1-kg system at the rate of 1 m/s^2. That is, since $F_{\text{net}} = ma, 1 \text{ N} = 1 \text{ kg} \cdot \text{m/s}^2$.

Weight and the Gravitational Force: When an object is dropped, it accelerates toward the center of Earth. Newton's second law states that a net force on an object is responsible for its acceleration. If air resistance is negligible, the net force on a falling object is the gravitational force, commonly called its **weight w.**

Since the object experiences only the downward force of gravity, $F_{\text{net}} = w$. We know that the acceleration of an object due to gravity is $g, \text{ or } a = g$. Substituting these into Newton's Second Law gives $w = mg$. Since $g = 9.81 \text{ m/s}^2$ on Earth, the weight of a 1.0 kg object on Earth is 9.81N, as we see: $w = mg = (1.0kg)\left(9.81\frac{m}{s^2}\right) = 9.81N$.

When the net external force on an object is its weight, we say that it is in **free-fall**. That is, the only force acting on the object is the force of gravity. In the real world, when objects fall downward toward Earth, they are never truly in free-fall because there is always some upward force from the air acting on the object.

Common misconceptions: Mass vs. Weight: It is important to be aware that weight and mass are very different physical quantities, although they are closely related. Mass is the quantity of matter (how much "stuff") and does not vary in **classical physics**, whereas weight is the gravitational force and does vary depending on gravity. It is tempting to equate the two, since most of our examples take place on Earth, where the weight of an object only varies a little with the location of the object.

Check Your Understanding: Newton's Second Law of Motion

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Newton's Third Law of Motion

Whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that it exerts.

This law represents a certain symmetry in nature: Forces always occur in pairs, and one body cannot exert a force on another without experiencing a force itself. We sometimes refer to this law loosely as "action-reaction," where the force exerted is the action and the force experienced as a consequence is the reaction. Newton's third law has practical uses in analyzing the origin of forces and understanding which forces are external to a system. You might think that two equal and opposite forces would cancel, but they do not because they act on different systems.

Other examples of Newton's third law are easy to find. As a professor paces in front of a whiteboard, she exerts a force backward on the floor. The floor exerts a reaction force forward on the professor that causes her to accelerate forward. Similarly, a car accelerates because the ground pushes forward on the drive wheels in reaction to the drive wheels pushing backward on the ground. You can see evidence of the wheels pushing backward when tires spin on a gravel road and throw rocks backward.

In another example, rockets move forward by expelling gas backward at high velocity. This means the rocket exerts a large backward force on the gas in the rocket combustion chamber, and the gas therefore exerts a large reaction force forward on the rocket. This reaction force is called **thrust**. It is a common misconception that rockets propel themselves by pushing on the ground or on the air behind them. They actually work better in a vacuum, where they can more readily expel the exhaust gases. Helicopters similarly create lift by pushing air down, thereby experiencing an upward reaction force. Birds and airplanes also fly by exerting force on air in a direction opposite to that of whatever force they need.

Check Your Understanding: Newton's Third Law of Motion

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Gravity Force Lab

Laws of Gravitation

Newton's Universal Law of Gravitation

The gravitational force is relatively simple. It is always attractive, and it depends only on the masses involved and the distance between them. Stated in modern language, **Newton's universal law of gravitation** states that every particle in the universe attracts every other particle with a force along a line joining them. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

Misconception alert: The magnitude of the force on each object (one has larger mass than the other) is the same, consistent with Newton's Third Law.

For two bodies having masses *m* and *M* with a distance *r* between their centers of mass, the equation for Newton's Universal Law of Gravitation is

$$F = G\frac{mM}{r^2}$$

Where *F* is the magnitude of the gravitation force and *G* is a proportionality factor called the **gravitational constant**. *G* is a universal gravitational constant—that is, it is thought to be the same everywhere in the universe. It has been measured experimentally to be in SI units,

$$G = 6.673 \times 10^{-11}\frac{N{\cdot}m^2}{kg^2}$$

Introduction to Spatial Interaction Modelling

Check Your Understanding: Spatial Interaction Modelling

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DESCRIBE THE MOTION OF AN OBJECT ALONG A CIRCULAR PATH

In this section, you will learn about the motion of an object along a circular path by reading each description along with watching the videos. Also, short problems to check your understanding are included.

Angular Rotation

Angle Units

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Angular Position and Displacement

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://uta.pressbooks.pub/oert-mpsfundamentals/?p=544#oembed-4

Angular Velocity *and Acceleration*

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://uta.pressbooks.pub/oert-mpsfundamentals/?p=544#oembed-5

Uniform Circular Motion

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Rotation Angle and Angular Velocity

We begin the study of **uniform circular motion** by defining two angular quantities needed to describe rotational motion.

Rotation Angle

When objects rotate about some axis – for example, when the CD in Figure 13 below rotates about its center – each point in the object follows a circular arc. Consider a line from the center of the CD to its edge. Each **pit** used to record sound along this line moves through the same angle in the same amount of time. The **rotation angle** is the amount of rotation and is analogous to linear distance. We define the rotation angle $\Delta\theta$ to be the ratio of the **arc length** to the radius of curvature:

$$\Delta\theta = \frac{\Delta s}{r}$$

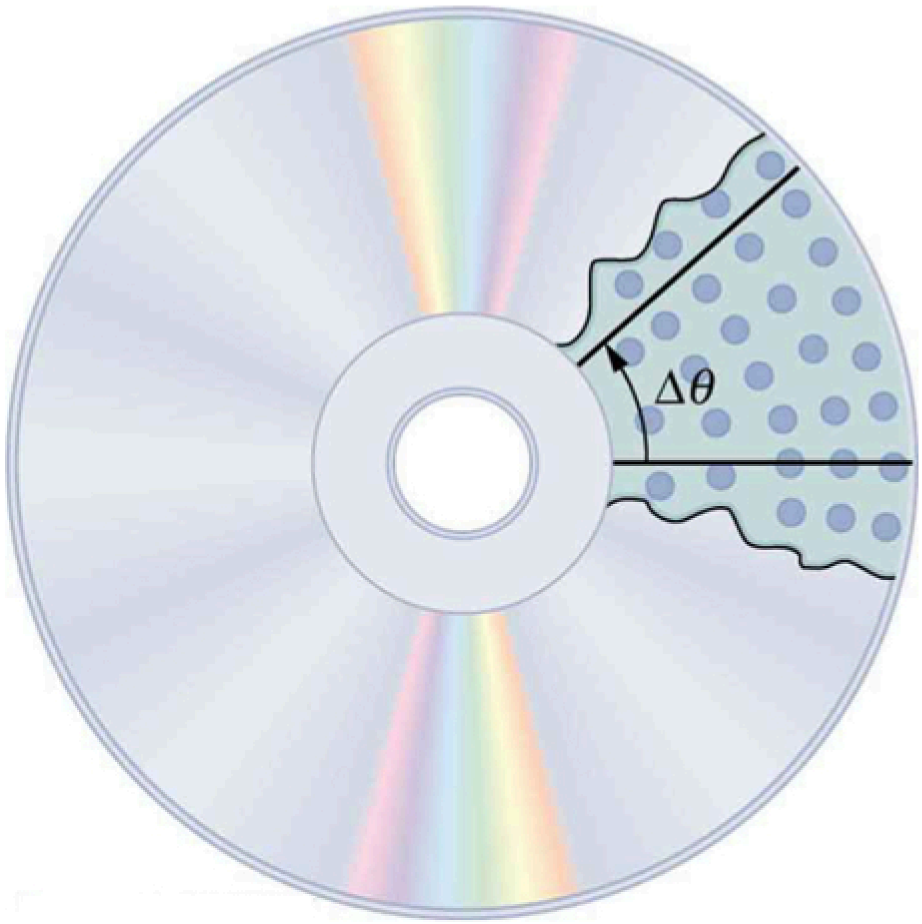

Figure 13: All points on a CD travel in circular arcs. The pits along a line from the center to the edge all move through the same angle $\Delta\theta$ in a time Δt

The arc length ΔS is the distance traveled along a circular path as shown in Figure 14 below. Note that *r* is the **radius of curvature** of the circular path.

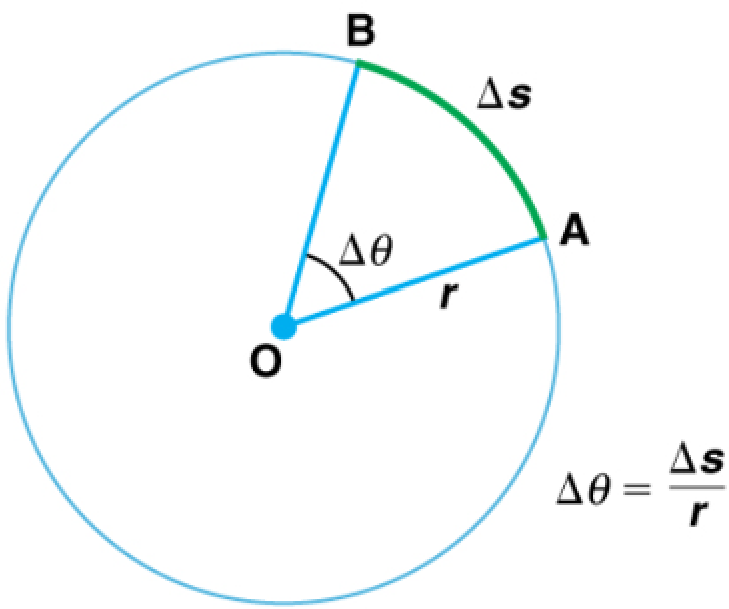

Figure 14: The radius of a circle is rotated through an angle $\Delta\theta$. The arc length ΔS is described on the circumference.

We know that for one complete revolution, the arc length is the circumference of a circle of radius r. The circumference of a circle is $2\pi r$. Thus, for one complete revolution the rotation angle is

$$\Delta\theta = \frac{2\pi r}{r} = 2\pi$$

This result is the basis for defining the units used to measure rotation angles, $\Delta\theta$ to be **radians** (rad), defined so that

$$2\pi \text{ rad } = 1 \text{ revolution}$$

A comparison of some useful angles expressed in both degrees and radians is shown in Table 2 below.

Table 2: Comparison of Angular Units

Degree Measures	Radian Measure
30°	$\frac{\pi}{6}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
120°	$\frac{2\pi}{3}$
135°	$\frac{3\pi}{4}$
180°	π

Angular Velocity

How fast is an object rotating? We define **angular velocity** $\boldsymbol{\omega}$ as the rate of change of an angle. In symbols, this is

$$\omega = \frac{\Delta\theta}{\Delta t}$$

Where an angular rotation $\Delta\theta$ takes place in a time Δt. The greater the rotation angle in a given amount of time, the greater the angular velocity. The units for angular velocity are radians per second (rad/s).

Angular velocity ω is analogous to linear velocity *v*. To get the precise relationship between angular and linear velocity, we again consider a pit on the rotating CD. This pit moves an arc length ΔS in a time Δt, and so it has a linear velocity

$$v = \frac{\Delta s}{\Delta t}$$

From $\Delta\theta = \frac{\Delta s}{r}$ we see that $\Delta s = r\Delta\theta$. Substituting this into the expression for *v* gives

$$v = \frac{r\Delta\theta}{\Delta t} = r\omega$$

We write this relationship in two different ways and gain two different insights:

$$v = r\omega \text{ or } \omega = \frac{v}{r}$$

The first relationship in $v = r\omega \text{ or } \omega = \frac{v}{r}$ states that the linear velocity *v* is proportional to the distance from the center of rotation, thus, it is the largest for a point on the rim (larges *r*), as you might expect. We can also call this linear speed *v* of a point on the rim the *tangential speed.* The second relationship in $v = r\omega \text{ or } \omega = \frac{v}{r}$ can be illustrated by considering the tire of a moving car. Note that the speed of a point on the rim of the tire is the same as the speed *v* of the car. See Figure 15 below. So, the faster the car moves, the faster the tire spins – large *v* means a large ω, because $v = r\omega$. Similarly, a larger-radius tire rotating at the same angular velocity (ω) will produce a greater linear speed (*v*) for the car.

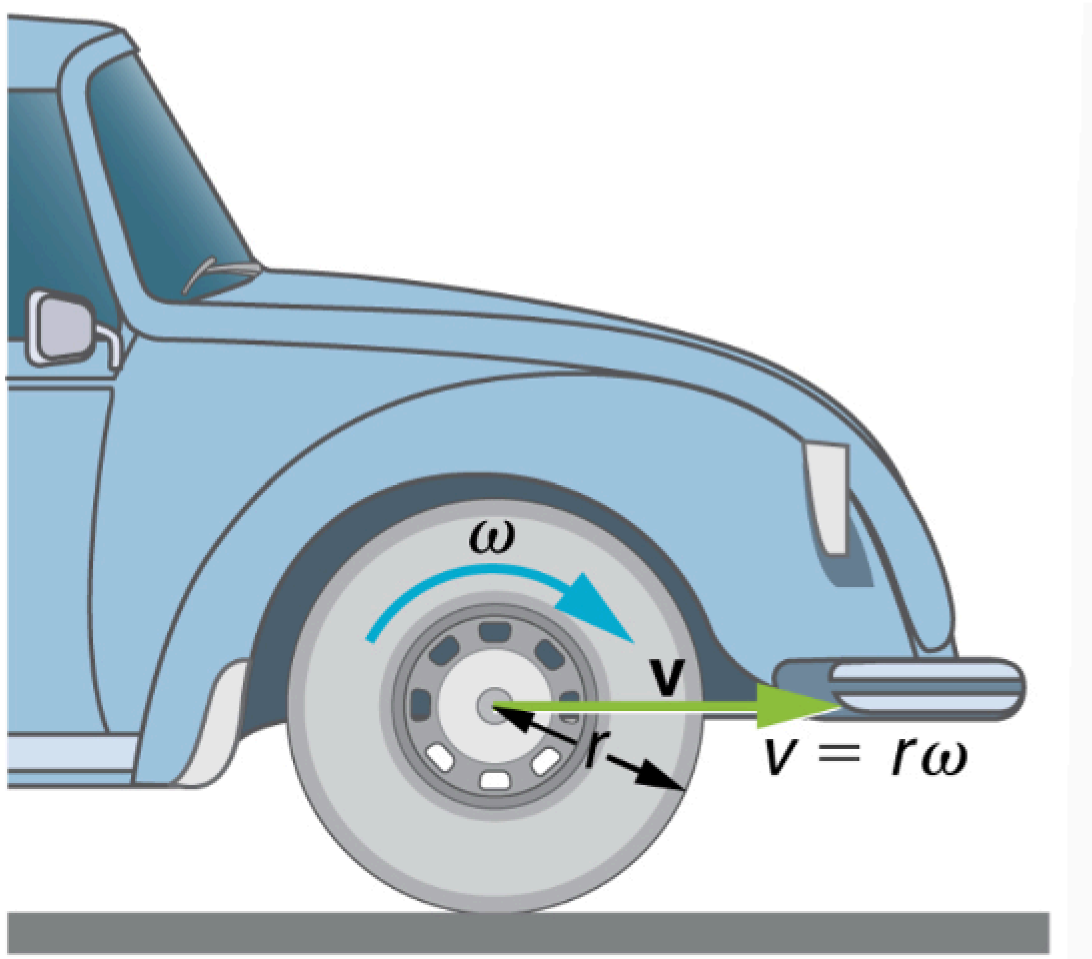

Figure 15: A car moving at a velocity v to the right has a tire rotating with an angular velocity ω*. The speed of the tread of the tire relative to the axle is v, the same as if the car were jacked up. Thus, the car moves forward at linear velocity* $v = r\omega$*, where r is the tire radius. A larger angular velocity for the tire means a greater velocity for the car.*

Check Your Understanding: Angular Rotation

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Circular Motion and Centripetal Acceleration

***Centripetal Force** and Acceleration Intuition*

Visual Understanding of Centripetal Acceleration Formula

Centripetal Acceleration

We know from **kinematics** that acceleration is a change in velocity, either in its magnitude or in its direction, or both. In uniform circular motion, the direction of the velocity changes constantly, so there is always an associated acceleration, even though the magnitude of the velocity might be constant. You experience this acceleration yourself when you turn a corner in your car. (If you hold the wheel steady during a turn and move at constant speed, you are in uniform circular motion.) What you notice is a sideways acceleration because you and the car are changing direction. The sharper the curve and the greater your speed, the more noticeable this acceleration will become. In this section we examine the direction and magnitude of that acceleration.

Figure 16 below shows an object moving in a circular path at constant speed. The direction of the instantaneous velocity is shown at two points along the path. Acceleration is in the direction of the change in velocity, which points directly toward the center of rotation (the center of the circular path). This pointing is shown with the vector diagram in the figure. We call the acceleration of an object moving in uniform circular motion (resulting from a net external force) the centripetal acceleration (a_c); centripetal means “toward the center” or “center seeking.”

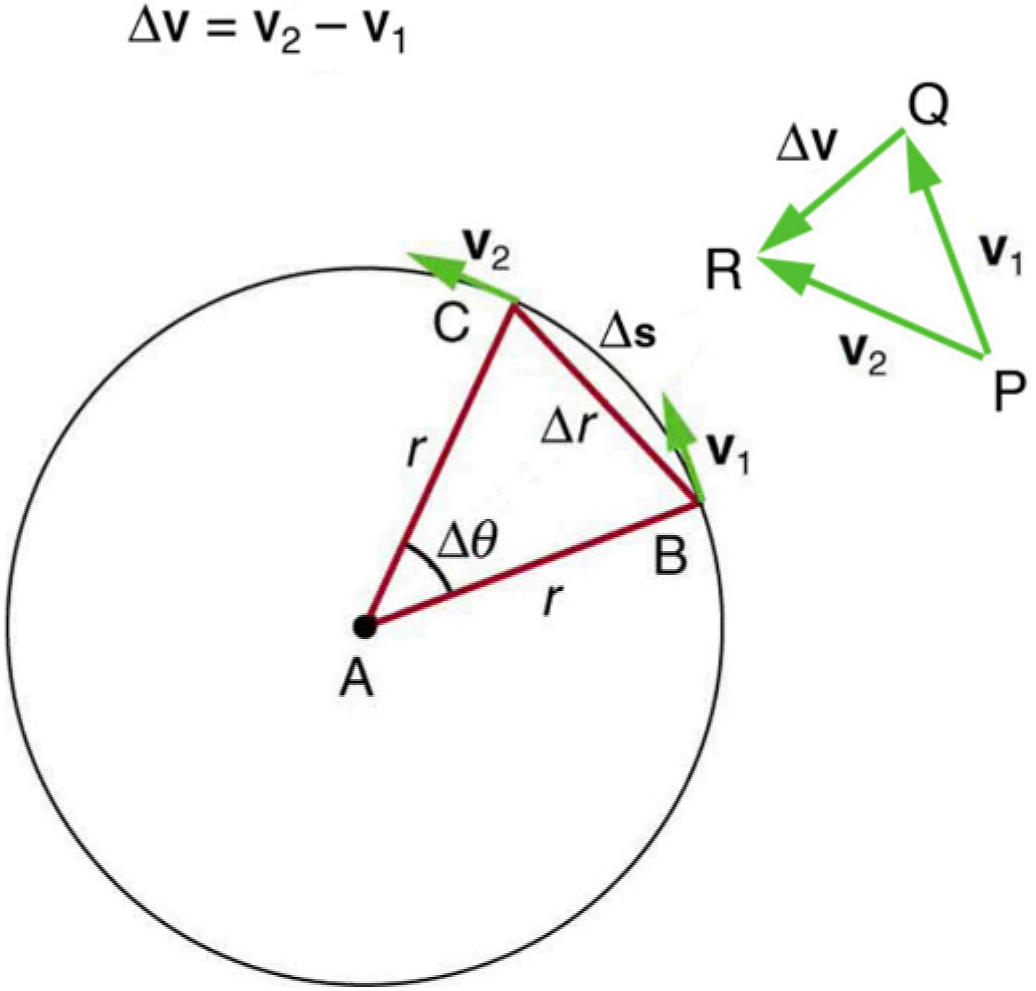

Figure 16: The directions of the velocity of an object at two different points are shown, and the change in velocity Δv is seen to point directly toward the center of curvature. (See small inset). Because $a_c = \frac{\Delta v}{\Delta t}$, the acceleration is also toward the center; a_c is called centripetal acceleration. (Because $\Delta\theta$ is exceedingly small, the arc length Δs is equal to the chord length Δr for small time differences).

The direction of centripetal acceleration is toward the center of curvature, but what is its magnitude? Note that the triangle formed by the velocity vectors and the one formed by the radii *r* and Δs are similar. Both the triangles ABC and PQR are isosceles triangles (two equal sides). The two equal sides of the velocity vector triangle are the speeds $v_1 = v_2 = v$. Using the properties of two similar triangles, we obtain

$$\frac{\Delta v}{v} = \frac{\Delta s}{r}$$

Acceleration is $\frac{\Delta v}{\Delta t}$, and so we first solve this expression for Δv:

$$\Delta v = \frac{v}{r}\Delta s$$

Then we divide this by Δt, yielding

$$\frac{\Delta v}{\Delta t} = \frac{v}{r} \times \frac{\Delta s}{\Delta t}$$

Finally, noting that $\frac{\Delta v}{\Delta t} = a_c$, and that $\frac{\Delta s}{\Delta t} = v$, the linear or tangential speed, we see that the magnitude of the centripetal acceleration is

$$a_c = \frac{v^2}{r}$$

Which is the acceleration of an object in a circle of radius *r* at a speed *v*. So, centripetal acceleration is greater at high speeds and in sharp curves (smaller radius), as you have noticed when driving a car. But it is a bit surprising that a_c is proportional to speed squared, implying, for example, that it is four times as hard to take a curve at 100 km/h than at 50 km/h. A sharp corner has a small radius, so that a_c is greater for tighter turns, as you have probably noticed.

It is also useful to express a_c in terms of angular velocity. Substituting $v = r\omega$ into the above expression, we find $a_c = \frac{(r\omega)^2}{r} = r\omega^2$. We can express the magnitude of centripetal acceleration using either of the two equations:

$$a_c = \frac{v^2}{r}; a_c = r\omega^2$$

Recall that the direction of a_c is toward the center. You may use whichever expression is more convenient.

Check Your Understanding: Circular Motion and Centripetal Acceleration

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Centripetal Forces

Centripetal Force Problem Solving

Centripetal Force

Any force or combination of forces can cause a centripetal or radial acceleration. Just a few examples are the tension in the rope on a tether ball, the force of Earth's gravity on the Moon, friction between roller skates and a rink floor, a banked roadway's force on a car, and forces on the tube of a spinning centrifuge.

Any net force causing circular motion is called a centripetal force. The direction of a centripetal force is toward the center of curvature, the same as the direction of centripetal acceleration. According to Newton's Second Law of Motion, the net force is mass times acceleration: net $F = ma$. For uniform circular motion, the acceleration is the centripetal acceleration – $a = a_c$. Thus, the magnitude of centripetal force F_c is

$$F_c = ma_c$$

By using the expressions for centripetal acceleration a_c from $a_c = \frac{v^2}{r}$; $a_c = r\omega^2$, we get two expressions for the centripetal force F_c in terms of mass, velocity, angular velocity, and radius of curvature:

$$F_c = m\frac{v^2}{r}; F_c = mr\omega^2$$

You may use whichever expression for centripetal force is more convenient. Centripetal force F_c is always perpendicular to the path and pointing to the center of curvature, because a_c is perpendicular to the velocity and pointing to the center of curvature.

Note that if you solve the first expression for *r*, you get

$$r = \frac{mv^2}{F_c}$$

This implies that for a given mass and velocity, a large centripetal force causes a small radius of curvature – that is, a tight curve.

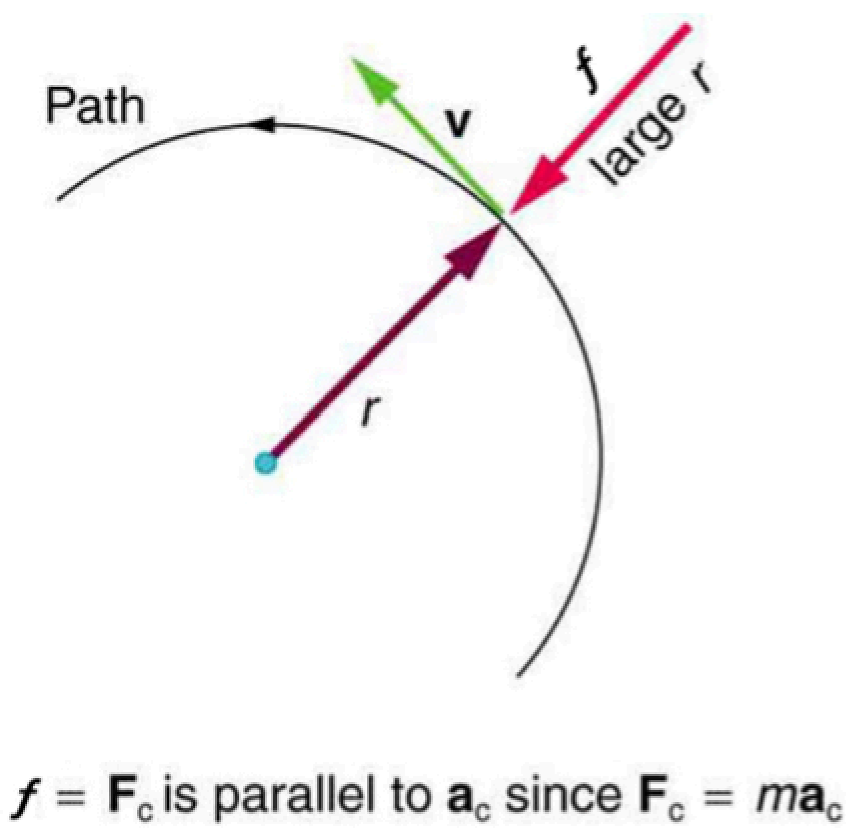

Figure 17: The frictional force supplies the centripetal force and is numerically equal to it. Centripetal force is perpendicular to velocity and causes uniform circular motion. The larger the F_c, the smaller the radius of curvature r and the sharper the curve.

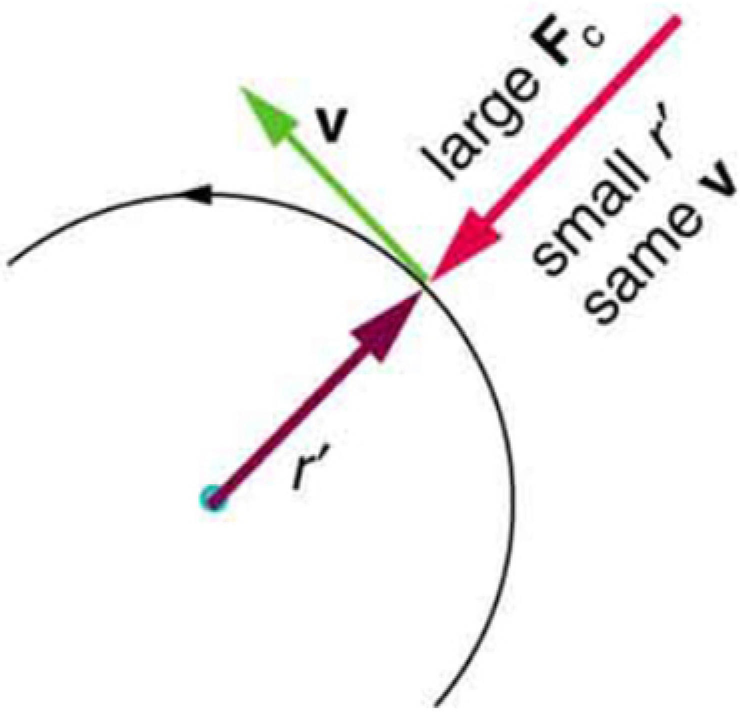

Figure 18: The frictional force supplies the centripetal force and is numerically equal to it. Centripetal force is perpendicular to velocity and causes uniform circular motion. The larger the F_c, the smaller the radius of curvature r and the sharper the curve. This curve has the same v, but a larger F_c produces a smaller r'.

Example 1

1. What coefficient of friction do car tires need on a flat curve?
   (a) Calculate the centripetal force exerted on a 900 kg car that negotiates a 500 m radius curve as 25.0 m/s.
   (b) Assuming an unbanked curve, find the minimum statis coefficient of friction, between the tires and the road, static friction being the reason that keeps the car from slipping (See Figure 19 below).

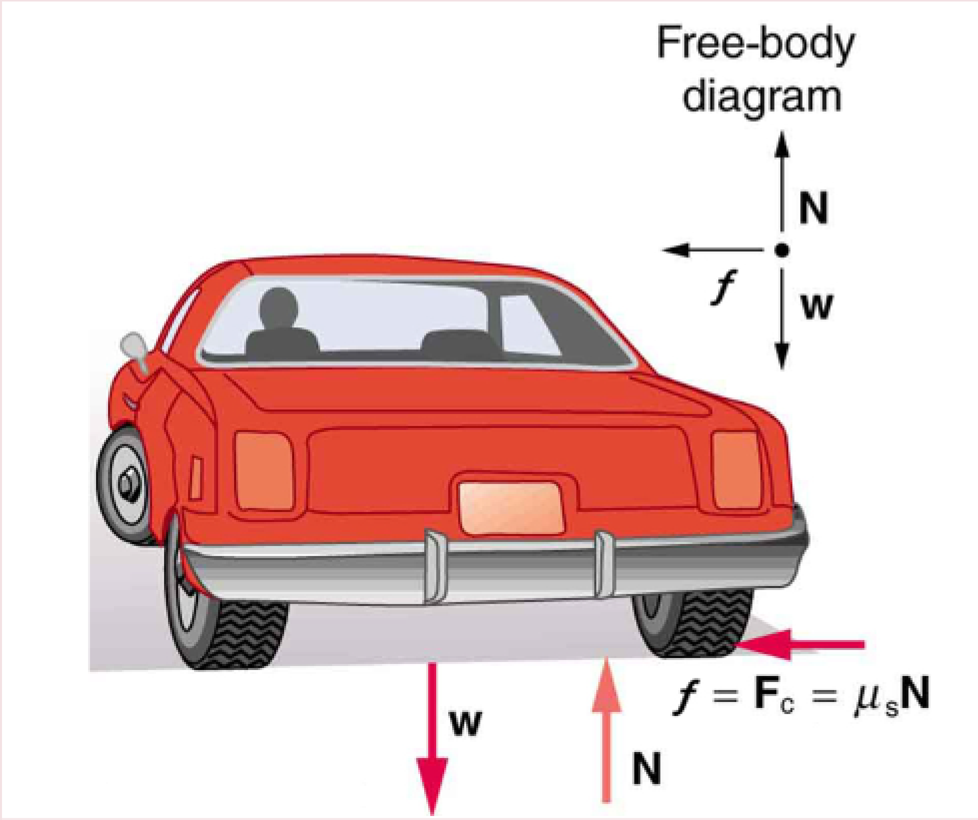

Figure 19

Answer: (a) $F_c = \frac{mv^2}{r} = \frac{(900\ \text{kg})(25.0\ \text{m/s})^2}{(500\ \text{m})} = 1125\ \text{N}$

(b) Figure 19 shows the forces acting on the car on an unbanked (level ground) curve. Friction is to the left, keeping the car from slipping, and because it is the only horizontal force acting on the car, the friction is the centripetal force in this case. We know that the maximum static friction (at which the tires roll but do not slip) is $\mu_s N, \text{ where } \mu_s$ *is the static coefficient of friction and N is the normal force. The normal force equals the car's weight on level ground, so that* $N = mg$*. Thus the centripetal force in this situation is*

$$F_c = f = \mu_s N = \mu_s mg$$

Now we have a relationship between centripetal force and the coefficient of friction. Using the first expression for F_c *from the equation*

$$\left.\begin{array}{l} F_c = m\frac{v^2}{r} \\ F_c = mr\omega^2 \end{array}\right\}$$

$$m\frac{v^2}{r} = \mu_s mg$$

We solve this for μ_S*, noting that mass cancels, and obtain*

$$u_s = \frac{v^2}{rg}$$

Substituting the knowns,

$$\mu_s = \frac{(25.0 \text{ m/s})^2}{(500 \text{ m})(9.80 \text{ m/s}^2)} = 0.13$$

Let us now consider **banked curves**, where the slope of the road helps you negotiate the curve. See Figure 20 below. The greater the angle θ, the faster you can take the curve. Racetracks for bikes as well as cars, for example, often have steeply banked curves. In an "ideally banked curve," the angle θ is such that you can negotiate the curve at a certain speed without the aid of friction between the tires and the road. We will derive an expression for θ for an ideally banked curve and consider an example related to it.

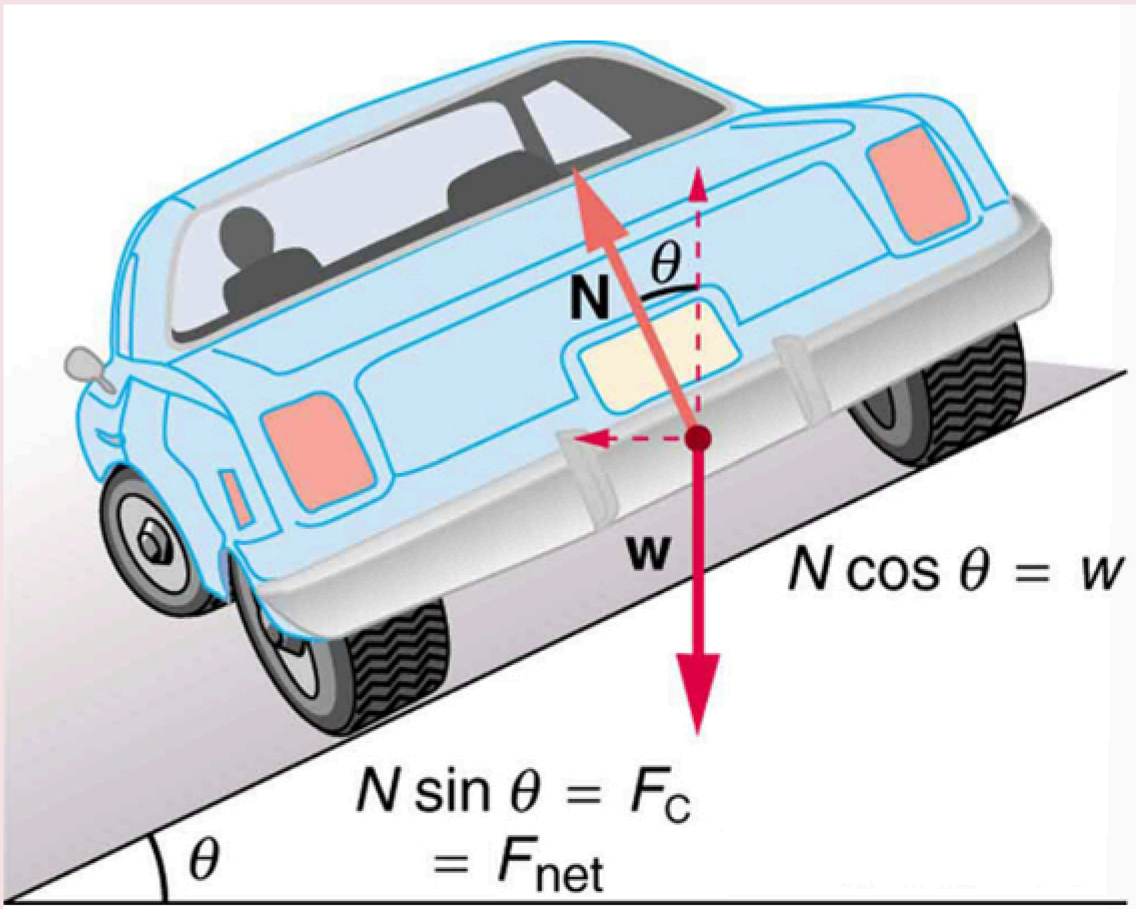

Figure 20: The car on this banked curve is moving away and turning to the left.

For **ideal banking**, the net external force equals the horizontal centripetal force in the absence of friction. The components of the normal force N in the horizontal and vertical directions must equal the centripetal force and the weight of the car, respectively. In cases in which forces are not parallel, it is most convenient to consider components along perpendicular axes – in this case, the vertical and horizontal directions.

Figure 20 above shows a free-body diagram for a car on a frictionless banked curve. If the angle θ is ideal for the speed and radius, then the net external force will equal the necessary centripetal force. The only two external forces acting on the car are its weight w and the normal force of the road N**.** (A frictionless surface can only exert a force perpendicular to the surface – that is, a normal force). These two forces must add to give a net external force that is horizontal toward the center of curvature and has magnitude mv^2/r. Because this is the crucial force and it is horizontal, we use a coordinate system with vertical and horizontal axes. Only the normal force has a horizontal component, and so this must equal the centripetal force – that is,

$$N \sin \theta = \frac{mv^2}{r}$$

Because the car does not leave the surface of the road, the net vertical force must be zero, meaning that the vertical components of the two external forces must be equal in magnitude and opposite in direction. From the figure, we see that the vertical component of the normal force is $N \cos \theta$, and the only other vertical force is the car's weight. These must be equal in magnitude; thus,

$$N \cos \theta = mg$$

Now we can combine the last two equations to eliminate N and get an expression for θ, as desired. Solving the second equation for $N = mg/(\cos \theta)$, and substituting this into the first yields

$$mg \frac{\sin \theta}{\cos \theta} = \frac{mv^2}{r}$$

m g \tan (\theta)=\frac{m v^2}{r}

$$\tan \theta = \frac{v^2}{rg}$$

Taking the inverse tangent gives

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right); (\text{ ideally banked curve, no friction })$$

This expression can be understood by considering how θ depends on v and r. A large θ will be obtained for a large v and a small r. That is, roads must be steeps banked for high speeds and sharp curves. Friction helps, because it allows you to take the curve at greater or lower speed than if the curve is frictionless. Note that θ does not depend on the mass of the vehicle

Example 2

1. What is the **ideal speed** to take a steeply banked tight curve? Curves on some test tracks and racecourses, such as the Daytona International Speedway in Florida, are very steeply banked. This banking, with the aid of tire friction and very stable car configurations, allows the curves to be taken at very high speed. To illustrate, calculate the speed at which a 100 m radius curve banked at 65.0°should be driven if the road is frictionless.

Answer: $\text{starting with } \tan \theta = \frac{V^2}{rg}, \text{ we get } v = (rg \tan \theta)^{\frac{1}{2}}$

Noting that $\tan 65.0^{\circ} = 2.14$*, we obtain*

$$v = \left[(100m)\left(\frac{9.80m}{s^2}\right)(2.14)\right]^{\frac{1}{2}} = 45.8 \text{ m/s}$$

Frames of Reference

Fictitious Forces

Fictitious Forces and Non-Inertial Frames: ***The Coriolis Force***

What do taking off in a jet airplane, turning a corner in a car, riding a merry-go-round, and the circular motion of a tropical cyclone have in common? Each exhibits fictitious forces—unreal forces that arise from motion and may seem real, because the observer's frame of reference is accelerating or rotating.

When taking off in a jet, most people would agree it feels as if you are being pushed back into the seat as the airplane accelerates down the runway. Yet a physicist would say that you tend to remain stationary while the seat pushes forward on you, and there is no real force backward on you. An even more common experience occurs when you make a tight curve in your car—say, to the right. You feel as if you are thrown (that is, forced) toward the left relative to the car. Again, a physicist would say that you are going in a straight line, but the car moves to the right, and there is no real force on you to the left. Recall Newton's first law.

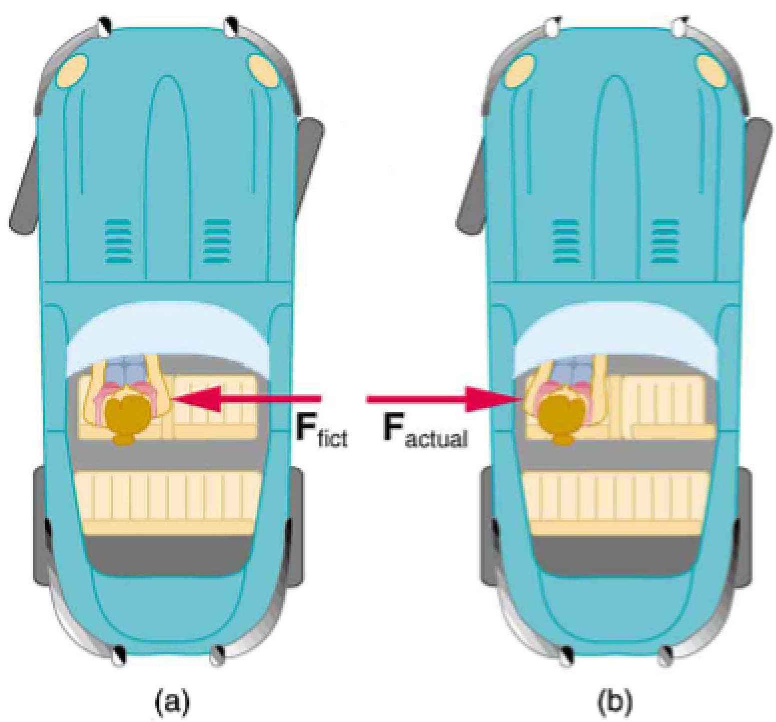

Figure 21: (a) The car driver feels herself forced to the left relative to the car when she makes a right turn. This is a fictitious force arising from the use of the car as a frame of reference. (b) In the Earth's frame of reference, the driver moves in a straight line, obeying Newton's first law, and the car moves to the right. There is no real force to the left on the driver relative to Earth. There is a real force to the right on the car to make it turn.

We can reconcile these points of view by examining the frames of reference used. Let us concentrate on people in a car. Passengers instinctively use the car as a frame of reference, while a physicist uses Earth. The physicist chooses Earth because it is very nearly an inertial frame of reference—one in which all forces are real (that is, in which all forces have an identifiable physical origin). The car is a **non-inertial frame of reference** because it is accelerated to the side. The force to the left sensed by car passengers is a fictitious force having no physical origin. There is nothing real pushing them left—the car, as well as the driver, is actually accelerating to the right.

Let us now take a mental ride on a merry-go-round—specifically, a rapidly rotating playground merry-go-round. You take the merry-go-round to be your frame of reference because you rotate together. In that non-inertial frame, you feel a fictitious force, named **centrifugal force** (not to be confused with centripetal force), trying to throw you off. You must hang on tightly to counteract the centrifugal force. In Earth's frame of reference, there is no force trying to throw you off. Rather you must hang on to make yourself go in a circle because otherwise you would go in a straight line, right off the merry-go-round.

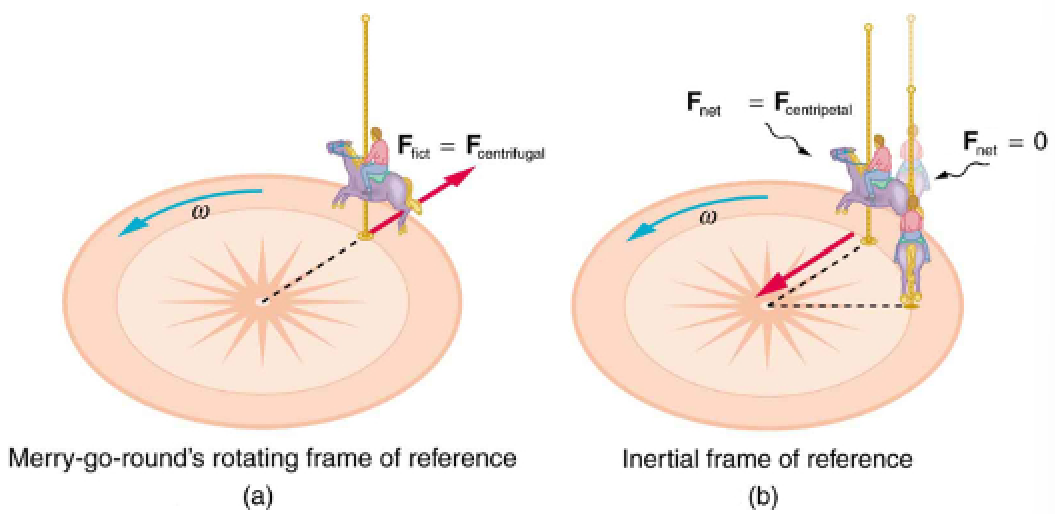

Figure 22: (a) A rider on a merry-go-round feels as if he is being thrown off. This fictitious force is called the centrifugal force—it explains the rider's motion in the rotating frame of reference. (b) In an inertial frame of reference and according to Newton's laws, it is his inertia that carries him off and not a real force (the unshaded rider has $F_{\text{net}} = 0$ *and heads in a straight line). A real force,* $F_{\text{centripetal}}$*, is needed to cause a circular path.*

This inertial effect, carrying you away from the center of rotation if there is no centripetal force to cause circular motion, is put to good use in centrifuges (See Figure 23). A centrifuge spins a sample very rapidly. Viewed from the rotating frame of reference, the fictitious centrifugal force throws particles outward, hastening their sedimentation. The greater the angular velocity, the greater the centrifugal force. But what really happens is that the inertia of the particles carries them along a line tangent to the circle while the test tube is forced in a circular path by a centripetal force.

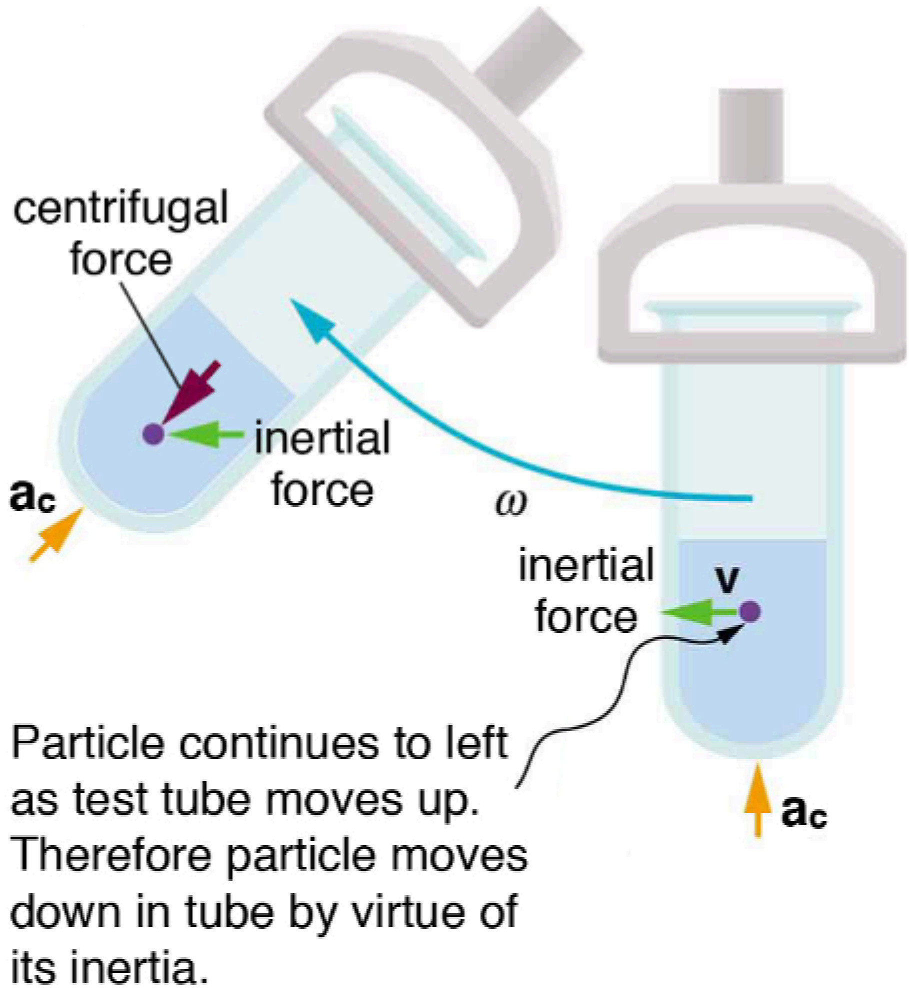

Figure 23: Centrifuges use inertia to perform their task. Particles in the fluid sediment come out because their inertia carries them away from the center of rotation. The large angular velocity of the centrifuge quickens the sedimentation. Ultimately, the particles will come into contact with the test tube walls, which will then supply the centripetal force needed to make them move in a circle of constant radius.

Let us now consider what happens if something moves in a frame of reference that rotates. For example, what if you slide a ball directly away from the center of the merry-go-round, as shown in Figure 24 below? The ball follows a straight path relative to Earth (assuming negligible friction) and a path curved to the right on the merry-go-round's surface. A person standing next to the merry-go-round sees the ball moving straight and the merry-go-round rotating underneath it. In the merry-go-round's frame of reference, we explain the apparent curve to the right by using a fictitious force, called the **Coriolis force**, which causes the ball to curve to the right. The fictitious Coriolis force can be used by anyone in that frame of reference to explain why objects follow curved paths and allows us to apply Newton's Laws in non-inertial frames of reference.

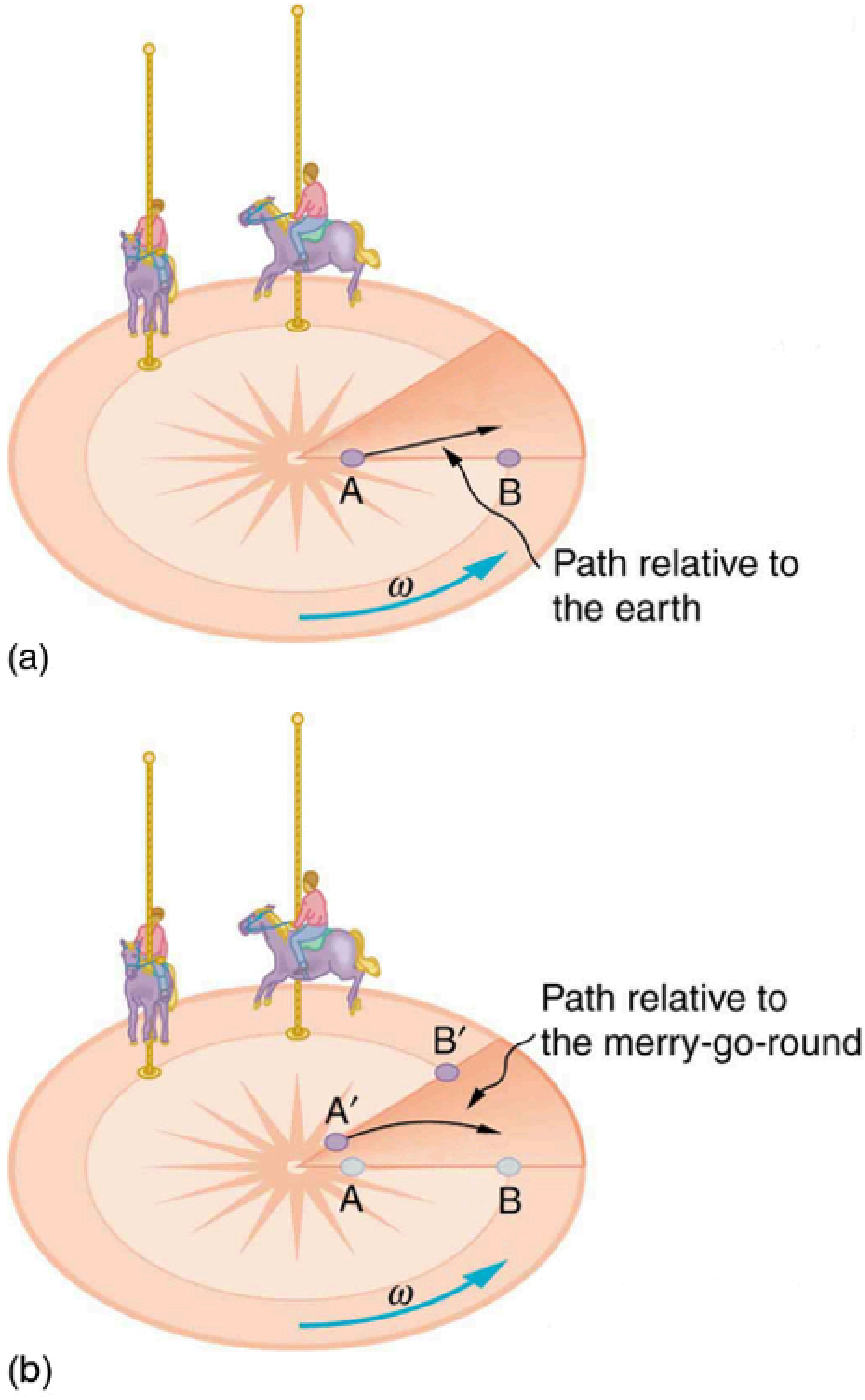

Figure 24: Looking down on the counterclockwise rotation of a merry-go-round, we see that a ball slid straight toward the edge follows a path curved to the right. The person slides the ball toward point B, starting at point A. Both points rotate to the shaded positions (A' and B') shown in the time that the ball follows the curved path in the rotating frame and a straight path in Earth's frame.

RELEVANCE TO TRANSPORTATION ENGINEERING COURSEWORK

This section explains the relevance of trip distribution, superelevation, stopping sight distance, time and space headways to transportation engineering coursework.

Trip Distribution

In travel demand modeling, one of the major steps (typically Step 2 of the 4-step process) involves estimating the number of trips made from one designated traffic analysis zone (TAZ) to another. This step is termed the trip distribution step. There are multiple trip distribution computational techniques, or "**models**," and the gravity model, derived from the physics gravity model (see the above section titled "Laws of Gravitation"), is one of the most useful models. In the physics gravity model, the greater the masses of two objects and the shorter the distance between them, the greater the gravitational pull. In the trip distribution step, the total number of trips produced at the origin zone and attracted at the destination zone are analogous to the object masses. Furthermore, just like the gravity model from Physics, the number of trips is also inversely proportional to an exponent of the distance between zones.

Superelevation

The above section titled "Circular Motion and Centripetal Acceleration" describes the basics of circular motion. This understanding and the discussion on the balancing of forces in Chapter 6 are critical to the cross-section design of the road segments with horizontal curves.

Stopping Sight Distance

The stopping sight distance, SSD, is the distance needed for a driver to perceive an unusual situation and stop to avoid crashing into an object or another road user. The SSD is estimated based on the time needed for the driver to detect and recognize the situation, also known as perception-reaction time (PRT), and the distance needed to brake to a complete stop, also known as the braking distance. These distances are estimated using the principles of kinematics for motion at constant speeds (see the above section titled " 1.1 Describe the Motion of an Object in the Graphic Form through the Time-Space Diagram"), with accelerated motion (see sections "Motion Equations for Constant Acceleration in One Dimension" and "What are the Kinematic Formulas?"), and traveling along a grade (see the section titled "Forces and Newton's Laws").

Time and Space Headways

The time headway is the time difference between the front bumper of one vehicle and that of another traveling behind it, passing the same point on a roadway segment. Space headway is the physical distance between the front bumper of a vehicle and that of another behind it. The relationship between time and space headways is based on kinematic relationships discussed in the sections titled "Time, Velocity, Speed, and Acceleration" and " Use Kinematic Equations to Solve for Displacement, Time, Velocity and Acceleration of an Object". The time headway may be used to estimate the traffic flow rate at points of the roadway segments. It is defined as the number of vehicles passing a point on the road per unit of time (in the units of vehicles/hour). The space headway may be used to estimate the density of the traffic stream on a roadway segment. It is the number of vehicles observed per unit distance on a roadway stretch (vehicles/mile).

Key Takeaways

- The gravity model from physics described in the chapter provides the foundational understanding of the trip distribution modeling between traffic analysis zones (TAZs). The gravity model is based on the idea that the number of trips between two zones is related to the size of the zones (i.e., total trips produced/attracted at a zone based on characteristics such as population or employment) and the distance between them.
- To ensure safe travel, the SSD (Stopping Sight Distance) is estimated based on the design speed appropriate for the roadway segments. Principles of kinematics for motion discussed in this chapter are used to derive formulas for estimating SSD appropriate for the context.
- The idea of motion at constant speed also relates to the time and space headways between vehicles and is useful for estimating flow rate past a point and density on roadway segments.

GLOSSARY: KEY TERMS

Acceleration[1] – the rate at which an object's velocity changes over a period of time
Acceleration due to Gravity[1] – acceleration of an object as result of gravity
Accuracy[1] – the degree to which a measured value agrees with correct value for that measurement
Angular Velocity[1] – �, the rate of change of the angle with which an object moves on a circular path
Arc Length[1] – Δ�, the distance traveled by an object along a circular path
Average Acceleration[1] – the change in velocity divided by the time over which it changes
Average Speed[1] – distance traveled divided by time during which motion occurs
Average Velocity[1] – displacement divided by the time over which displacement occurs
Banked Curve[1] – the curve in a road that is sloping in a manner that helps a vehicle negotiate the curve
Center of Mass[1] – the point where the entire mass of an object can be thought to be concentrated
Centrifugal Force[1] – a fictitious force that tends to throw an object off when the object is rotating in a non-inertial frame of reference
Centripetal Acceleration[1] – the acceleration of an object moving in a circle, directed toward the center
Centripetal Force[1] – any net force causing uniform circular motion
Classical Physics[1] – physics that was developed from the Renaissance to the end of the 19th century
Conversion Factor[1] – a ratio expressing how many of one unit are equal to another unit
Coriolis Force[1] – the fictitious force causing the apparent deflection of moving objects when viewed in a rotating frame of reference
Deceleration[1] – acceleration in the direction opposite to velocity; acceleration that results in a decrease in velocity
Displacement[1] – the change in position of an object

Distance[1] – the magnitude of displacement between two positions
Distance Traveled[1] – the total length of the path traveled between two positions
Elapsed Time[1] – the difference between the ending time and beginning time
English Units[1] – system of measurement used in the United States; includes units of measurement such as feet, gallons, and pounds
External Force[1] – a force acting on an object or system that originates outside of the object or system
Fictious Force[1] – a force having no physical origin
Force[1] – a push or pull on an object with a specific magnitude and direction; can be represented by vectors; can be expressed as a multiple of a standard force
Free-body Diagram[1] – a sketch showing all of the external forces acting on an object or system; the system is represented by a dot, and the forces are represented by vectors extending outward from the dot
Free-Fall[1] – a situation in which the only force acting on an object is the force due to gravity
Friction[1] – a force past each other of objects that are touching; examples include rough surfaces and air resistance
Gravitational Constant, G[1] – a proportionality factor used in the equation for Newton's universal law of gravitation; it is a universal constant—that is, it is thought to be the same everywhere in the universe
Ideal Banking[1] – the sloping of a curve in a road, where the angle of the slope allows the vehicle to negotiate the curve at a certain speed without the aid of friction between the tires and the road; the net external force on the vehicle equals the horizontal centripetal force in the absence of friction
Ideal Speed[1] – the maximum safe speed at which a vehicle can turn on a curve without the aid of friction between the tire and the road
Inertia[1] – the tendency of an object to remain at rest or remain in motion
Inertial Frame of Reference[1] – a coordinate system that is not accelerating; all forces acting in an inertial frame of reference are real forces, as opposed to fictitious forces that are observed due to an accelerating frame of reference
Instantaneous Acceleration[1] – acceleration at a specific point in time
Instantaneous Speed[1] – magnitude of the instantaneous velocity
Instantaneous Velocity[1] – velocity at a specific instant, or the average velocity over an infinitesimal time interval
Kilogram[1] – the SI unit for mass, abbreviated (kg)
Kinematics[1] – the study of motion without considering its causes
Kinetic Friction[1] – a force that opposes the motion of two systems that are in contact and moving relative to one another
Law[1] – a description, using concise language or a mathematical formula, a generalized pattern in nature that is supported by scientific evidence and repeated experiments
Law of Inertia[1] – see Newton's first law of motion
Magnitude of Kinetic Friction[1] – ��=���, where �� is the coefficient of kinetic friction
Magnitude of Static Friction[1] – ��≤���, where �� is the coefficient of static friction and � is the magnitude of the normal force
Mass[1] – the quantity of matter in a substance; measured in kilograms
Meter[1] – the SI unit for length, abbreviated (m)

Metric System[1] – a system in which values can be calculated in factors of 10
Model[1] – simplified description that contains only those elements necessary to describe the physics of a physical situation
Net External Force[1] – the vector sum of all external forces acting on an object or system; causes a mass to accelerate
Newton's First Law of Motion[1] – in an inertial frame of reference, a body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net external force; also known as the law of inertia
Newton's Second Law of Motion[1] – the net external force ���� on an object with mass � is proportional to and in the same direction as the acceleration of the object, �, and inversely proportional to the mass; defined mathematically as $a = \frac{F_{net}}{m}$
Newton's Third Law of Motion[1] – whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that the first body exerts
Newton's Universal Law of Gravitation[1] – every particle in the universe attracts every other particle with a force along a line joining them; the force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them
Non-Inertial Frame of Reference[1] – an accelerated frame of reference
Normal Force[1] – the force that a surface applies to an object to support the weight of the object; acts perpendicular to the surface on which the object rests
Physical Quantity[1] – a characteristic or property of an object that can be measured or calculated from other measurements
Physics[1] – the science concerned with describing the interactions of energy, matter, space, and time; it is especially interested in what fundamental mechanisms underlie every phenomenon
Pit[1] – a tiny indentation on the spiral track moulded into the top of the polycarbonate layer of CD
Position[1] – the location of an object at a particular time
Radians[1] – a unit of angle measurement
Radius of Curvature[1] – radius of a circular path
Rotation Angle[1] – the ratio of arc length to the radius of curvature on a circular path: $\Delta\theta = \frac{\Delta s}{r}$
Scalar[1] – a quantity that is described by magnitude, but not direction
Second[1] – the SI unit for time, abbreviated (s)
SI Units[1] – the international system of units that scientists in most countries have agreed to use; includes units such as meters, liters, and grams
Significant Figures[1] – express the precision of a measuring tool used to measure a value
Slope[1] – the difference in y-value (the rise) divided by the difference in x-value (the run) of two points on a straight line
Static Friction[1] – a force that opposes the motion of two systems that are in contact and are not moving relative to one another
System[1] – defined by the boundaries of an object or collection of objects being observed; all forces originating from outside of the system are considered external forces
Tension[1] – the pulling force that acts along a medium, especially a stretched flexible connector, such as a rope or cable; when a rope supports the weight of an object, the force on the object due to the rope is called a tension force

Thrust[1] – a reaction force that pushes a body forward in response to a backward force; rockets, airplanes, and cars are pushed forward by a thrust reaction force
Time[1] – change, or the interval over which change occurs
Ultracentrifuge[1] – a centrifuge optimized for spinning a rotor at very high speeds
Uniform Circular Motion[1] – the motion of an object in a circular path at constant speed
Units[1] – a standard used for expressing and comparing measurements
Vector[1] – a quantity that is described by both magnitude and direction
Weight[1] – the force � due to gravity acting on an object of mass �; defined mathematically as: �=��, where � is the magnitude and direction of the acceleration due to gravity
Y-Intercept[1] – the y-value when �=0, or when the graph crosses the y-axis

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- "Traffic Flow" by David Levinson https://www.youtube.com/watch?v=NjxIhgG4cNw

CHAPTER 6: BASIC DYNAMICS AND STATIC EQUILIBRIUM

This chapter explains momentum and collision concepts to support an understanding of how speed plays a critical role in the safety of road users, particularly Vulnerable Road Users (VRUs), i.e., pedestrians and bicyclists. Balancing forces on stationary objects is crucial for designing banking or superelevation (slope along the cross-section of the road) of road segments with horizontal curvature.

Learning Objectives

At the end of the chapter, the reader should be able to do the following:

- Balance forces on a stationary object in equilibrium.
- Use of momentum preservation to describe the post-collision movement of objects.
- Identify topics in the introductory transportation engineering courses that build on the concepts discussed in this chapter.

USE OF MOMENTUM PRESERVATION TO DESCRIBE THE POST-COLLISION MOVEMENT OF OBJECTS

In this section, you will learn about the use off momentum preservation to describe the post-collision movement of objects by watching the videos. Also, examples are for your understanding are included.

Introduction to Momentum

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://uta.pressbooks.pub/oert-mpsfundamentals/?p=517#oembed-1

Impulse and Momentum Dodgeball Example

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://uta.pressbooks.pub/oert-mpsfundamentals/?p=517#oembed-2

Bouncing Fruit Collision Example

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://uta.pressbooks.pub/oert-mpsfundamentals/?p=517#oembed-3

Momentum: Ice Skater Throws a Ball

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2-Dimensional Momentum Problem (Trig Review!)

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Part 2

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Force vs. Time Graphs

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Check Your Understanding: Introduction to Momentum

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Momentum Notes – Overview

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Linear Momentum

The scientific definition of **linear momentum** is consistent with most people's intuitive understanding of momentum: a large, fast-moving object has greater momentum than a smaller, slower object. Linear momentum is defined as the product of a system's mass multiplied by its velocity. In symbols, linear momentum is expressed as $p = mv$

Momentum is directly proportional to the object's mass and also its velocity. Thus, the greater an object's mass or the greater its velocity, the greater its momentum. Momentum *p* is a vector having the same direction as the velocity *v*. The SI unit for momentum is $kg \cdot m/s$

Example 1

Calculating Momentum: A Football Player and a Football

(a) Calculate the momentum of a 110-kg football player running at 8.00 m/s.

(b) Compare the player's momentum with the momentum of a hard-thrown 0.410-kg football that has a speed of 25.0 m/s.

Answers:

(a) $p_{\text{player}} = (110 \text{ kg})(8.00 \text{ m/s}) = 880 \text{ kg} \bullet \text{m/s}$

(b) $p_{\text{ball}} = (0.410 \text{ kg})(25.0 \text{ m/s}) = 10.3 \text{ kg} \bullet \text{m/s}$

The ratio of the player's momentum to that of the ball is

$$\frac{p_{\text{player}}}{p_{\text{ball}}} = \frac{880}{10.3} = 85.9$$

Momentum and Newton's Second Law

The importance of momentum, unlike the importance of energy, was recognized early in the development of classical physics. Momentum was deemed so important that it was called the "quantity of motion." Newton actually stated his **second law of motion** in terms of momentum: The net external force equals the **change in momentum** of a system divided by the time over which it changes. Using symbols, this law is

$$F_{net} = \frac{\Delta p}{\Delta t}$$

Where F_{net} is the net external force, Δp is the change in momentum, and Δt is the change in time.

This statement of Newton's Second Law of Motion includes the more familiar $F_{\text{net}} = ma$ as a special case. We can derive this form as follows. First, note that the change in momentum Δp is given by

$$\Delta p = \Delta(mv)$$

If the mass of the system is constant, then

$$\Delta(mv) = m\Delta v$$

So that for constant mass, Newton's Second Law of Motion becomes

$$F_{\text{net}} = \frac{\Delta p}{\Delta t} = \frac{m\Delta v}{\Delta t}$$

Because $\frac{\Delta v}{\Delta t} = a$, we get the familiar equation $F_{\text{net}} = ma$ when the mass of the system is constant.

Example 2

During the 2007 French Open, Venus Williams hit the fastest recorded serve in a premier women's match, reaching a speed of 58 m/s (209 km/h). What is the average force exerted on the 0.057-kg tennis ball by Venus Williams' racquet, assuming that the ball's speed just after impact is 58 m/s, that the initial horizontal component of the velocity before impact is negligible, and that the ball remained in contact with the racquet for 5.0 ms (milliseconds)?

Answer:

$$\Delta p = m\left(v_f - v_i\right)$$

$$= (0.057 \text{ kg})(58 \text{ m/s} - 0 \text{ m/s})$$

$$= 3.306 \text{ kg} \cdot \text{m/s} \approx 3.3 \text{ kg} \cdot \text{m/s}$$

Now the magnitude of the net external force can be determined by using

$F_{net} = \frac{\Delta p}{\Delta t}$

$F_{\text{net}} = \frac{\Delta p}{\Delta t} = \frac{3.306 \text{ kg·m/s}}{5.0 \times 10^{-3} \text{ s}} = 661 \text{ N} \approx 660 \text{ N}$

Impulse

The effect of a force on an object depends on how long it acts, as well as how great the force is. In the Example 2 above, a large force acting for a brief time had a significant effect on the momentum of the tennis ball. A small force could cause the same **change in momentum**, but it would have to act for a much longer time. For example, if the ball were thrown upward, the gravitational force (which is much smaller than the tennis racquet's force) would eventually reverse the momentum of the ball. Quantitatively, the effect we are talking about is the change in momentum Δp.

By rearranging the equation $F_{net} = \frac{\Delta p}{\Delta t}$ to be

$\Delta p = F_{\text{net}} \ \Delta t$

We can see how the change in momentum equals the average net external force multiplied by the time this force acts. The quantity $F_{\text{net}} \ \Delta t$ is given the name **impulse.** Impulse if the same as the change in momentum.

Example 3

1. **Making connections – illustrations of force exerted:**

A 1.2-kg block slides across a horizontal, frictionless surface with a constant speed of 3.0 m/s before striking a fixed barrier and coming to a stop. In Figure 1, the force exerted by the barrier is assumed to be a constant 15 N during the 0.24-s collision. The impulse can be calculated using the area under the curve.

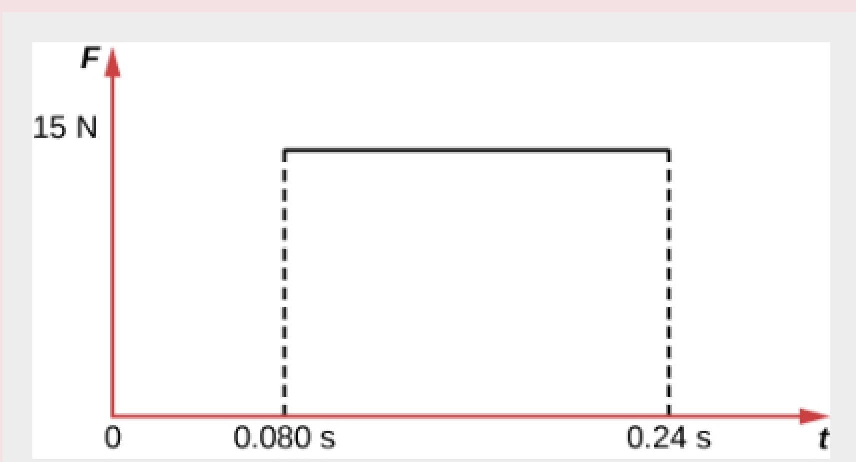

Figure 1: This is a graph showing the force exerted by a fixed barrier on a block versus time.

$$\Delta p = F\Delta t = (15N)(0.24 \text{ s}) = 3.6 \text{ kg} \cdot \text{m/s}$$

Note that the initial momentum of the block is

$$p_{\text{initial }} = mv_{\text{initial }} = (1.2 \text{ kg}) \left(-\frac{3.0 \text{ m}}{\text{s}}\right) = -3.6 \text{ kg} \cdot \text{m/s}$$

We are assuming that the initial velocity is –3.0 m/s. We have established that the force exerted by the barrier is in the positive direction, so the initial velocity of the block must be in the negative direction. Since the final momentum of the block is zero, the impulse is equal to the change in momentum of the block.

Suppose that, instead of striking a fixed barrier, the block is instead stopped by a spring. Consider the force exerted by the spring over the time interval from the beginning of the collision until the block comes to rest.

In this case, the impulse can be calculated again using the area under the curve (the area of a triangle):

$$\Delta p = \frac{1}{2}(\text{ base })(\text{ height }) = \frac{1}{2}(0.24 \text{ s})(30 \text{ N}) = 3.6 \text{ kg} \cdot \text{m/s}$$

Again, this is equal to the difference between the initial and final momentum of the block, so the impulse is equal to the change in momentum.

2. **Calculating Magnitudes of Impulses – Two Billiard Balls Striking a Rigid Wall:**

Two identical billiard balls strike a rigid wall with the same speed and are reflected without any change of speed. The first strikes perpendicular to the wall. The second ball strikes the wall at an angle of 30°">30° from the perpendicular and bounces off at an angle of 30°">30° from perpendicular to the wall.

(a) Determine the direction of force on the wall due to each ball.

(b) Calculate the ratio of the magnitudes of impulses on the two balls by the wall.

Answer:

Strategy for A: In order to determine the force on the wall, consider the force on the ball due to the wall using Newton's second law and then apply Newton's third law to determine the direction. Assume the x-axis to be normal to the wall and to be positive in the initial direction of motion. Choose the y-axis to be along the wall in the plane of the second ball's motion. The momentum direction and the velocity direction are the same.

Solution for A: The first ball bounces directly into the wall and exerts a force on it in the +x direction. Therefore, the wall exerts a force on the ball in the –x direction. The second ball continues with the same momentum component in the y direction, but reverses its x-component of momentum, as seen by sketching a diagram of the angles involved and keeping in mind the proportionality between velocity and momentum.

These changes mean the change in momentum for both balls are in the –x direction, so the force of the wall on each ball is along the –x direction.

Strategy for B: Calculate the change in momentum for each ball, which is equal to the impulse imparted to the ball.

Solution for B: Let � be the speed of each ball before and after collision with the wall, and � the mass of each ball. Choose the x-axis and y-axis as previously described and consider the change in momentum of the first ball which strikes perpendicular to the wall.

$p_{xi} = mu; p_{yi} = 0$

$p_{xf} = -mu; p_{yf} = 0$

Impulse is the change in momentum vector. Therefore, the x-component of impulse is equal to –2�� and the y-component of impulse is equal to zero.

Now consider the change in momentum of the second ball.

$p_{xi} = mu\cos 30°; p_{yi} = -mu\sin 30°$

$p_{xf} = -mu\cos 30°; p_{yf} = -mu\sin 30°$

It should be noted here that while p_x changes sign after the collision, p_y does not. Therefore, the x-component of impulse is equal to –2mucos 30°">–2��cos30° and the y-component of impulse is equal to zero. The ratio of the magnitudes of the impulse imparted to the balls is $\frac{2mu}{2mu\cos 30°} = \frac{2}{\sqrt{3}} = 1.155$

3. **Making connections – baseball:**

In most real-life collisions, the forces acting on an object are not constant. For example, when a bat strikes a baseball, the force is small at the beginning of the collision since only a small portion of the ball is initially in contact with the bat. As the collision continues, the ball deforms so that a greater fraction of the ball is in contact with the bat, resulting in a greater force. As the ball begins to leave the bat, the force drops to zero. Although the changing force is difficult to precisely calculate at each instant, the average force can be estimated very well in most cases.

Suppose that a 150-g baseball experiences an average force of 480 N in a direction opposite the initial 32 m/s speed of the baseball over a time interval of 0.017 s. What is the final velocity of the baseball after the collision?

$\Delta p = F\Delta t = (480)(0.017) = 8.16 \text{ kg} \cdot \text{m/s}$

$mv_f - mv_i = 8.16 \text{ kg} \cdot \text{m/s}$

$(0.150 \text{ kg})v_f - (0.150 \text{ kg})(-32 \text{ m/s}) = 8.16 \text{ kg} \cdot \text{m/s}$

$v_f = 22 \text{ m/s}$

Conservation of Momentum

Momentum is an important quantity because it is conserved. Yet it was not conserved in the examples in Impulse and Linear Momentum and Force, where large changes in momentum were produced by forces acting on the system of interest. Under what circumstances is momentum conserved?

The answer to this question entails considering a sufficiently large system. It is always possible to find a larger system in which total momentum is constant, even if momentum changes for components of the system. If a football player runs into the goalpost in the end zone, there will be a force on him that causes him to bounce backward. However, the Earth also recoils —conserving momentum—because of the force applied to it through the goalpost. Because Earth is many orders of magnitude more massive than the player, its recoil is immeasurably small and can be neglected in any practical sense, but it is real, nevertheless.

Consider what happens if the masses of two colliding objects are more similar than the masses of a football player and Earth – for example, one car bumping into another, as shown in Figure 2. Both cars are coating in the same direction when the lead car (labeled m_2) is bumped by the trailing car (labeled m_1). The only unbalanced force on each car is the force of the collision. (Assume that the effects due to friction are negligible). Car 1 slows down as a result of the collision, losing some momentum, while car 2 speeds up and gains some momentum. We shall now show that the total momentum of the two-car system remains constant.

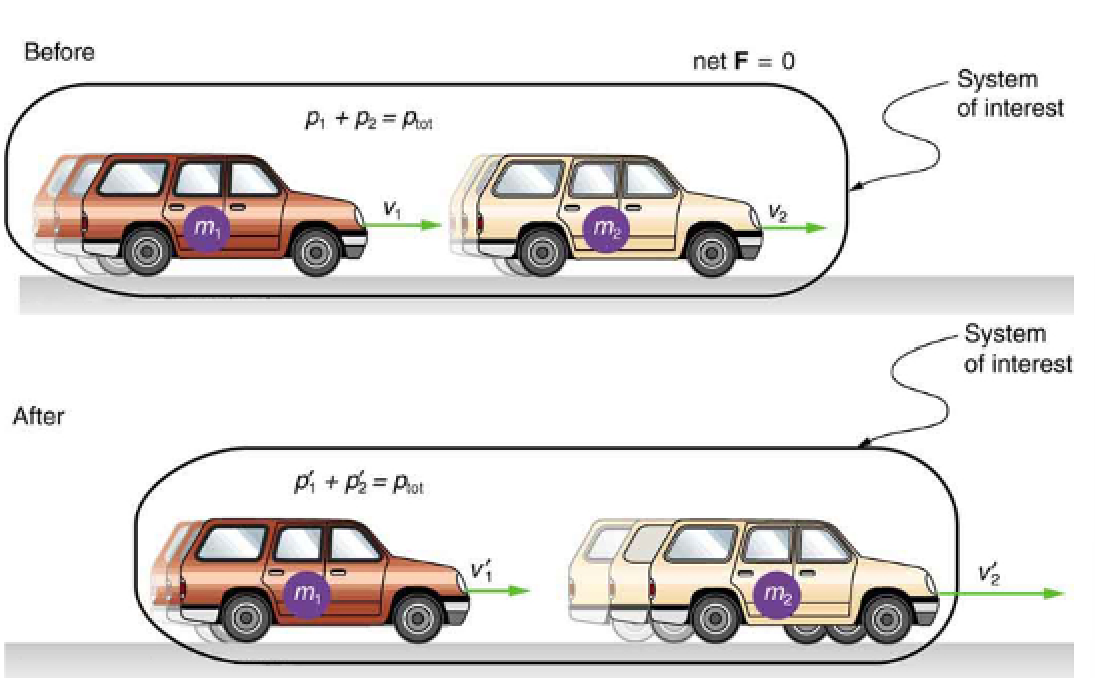

Figure 2: A car of mass m_1 moving with a velocity of $\boldsymbol{v}_1$ bumps into another car of mass m_2 and velocity $v_2$that it is following. As a result, the first car slows down to a velocity of v_1' and the second speeds up to a velocity of v_2'. The momentum of each car is changed, but the total momentum p_{tot} of the two cars is the same before and after the collision (if you assume friction to be negligible)

Using the definition of impulse, the change in momentum of car 1 is given by

$$\Delta p_1 = F_1 \Delta t,$$

where F_1 is the force on car 1 due to car 2, and Δt is the time the force acts (the duration of the collision). Intuitively, it seems obvious that the collision time is the same for both cars, but it is only true for objects traveling at ordinary speeds. This assumption must be modified for objects travelling near the speed of light, without affecting the result that momentum is conserved.

Similarly, the change in momentum of car 2 is

$$\Delta p_2 = F_2 \Delta t,$$

Where F_2 is the force on car 2 due to car 1, and we assume the duration of the collision Δt is the same for both cars. We know from Newton's third law that $F_2 = -F_1$, and so

$$\Delta p_2 = -F_1 \Delta t = -\Delta p_1$$

Thus, the changes in momentum are equal and opposite, and

$$\Delta p_1 + \Delta p_2 = 0$$

Because the changes in momentum add to zero, the total momentum of the two-car system is constant. That is,

$$p_1 + p_2 = \text{ constant,}$$

$$p_1 + p_2 = p_1' + p_2'$$

Where $p_1'1 \text{ and } p_2'$ are the momenta of cars 1 and 2 after the collision. (We often use primes to denote the final state.)

This result—that momentum is conserved—has validity far beyond the preceding one-dimensional case. It can be similarly shown that total momentum is conserved for any isolated system, with any number of objects in it. In equation form, the **conservation of momentum principle** for an isolated system is written

$$p_{tot} = \text{ constant}$$

Or

$$p_{tot} = p_{tot}'$$

Where p_{tot} is the total momentum (the sum of the momenta of the individual objects in the system) and p'_{tot} is the total momentum some time later. (The total momentum can be shown to be the momentum of the center of mass of the system.) An **isolated system** is defined to be one for which the net external force is zero $(F_{\text{net}} = 0)$

Elastic and Inelastic Collisions

Elastic Collisions in One Dimension

Let us consider several types of two-object collisions. These collisions are the easiest to analyze, and they illustrate many of the physical principles involved in collisions. The conservation of momentum principle is particularly useful here, and it can be used whenever the net external force on a system is zero.

We start with the elastic collision of two objects moving along the same line—a one-dimensional problem. An **elastic collision** is one that also conserves internal kinetic energy. **Internal kinetic energy** is the sum of the kinetic energies of the objects in the system. Figure 3 below illustrates an elastic collision in which internal kinetic energy and momentum are conserved.

Truly elastic collisions can only be achieved with subatomic particles, such as electrons striking nuclei. Macroscopic collisions can be very nearly, but not quite, elastic—some kinetic energy is always converted into other forms of energy such as heat transfer due to friction and sound. One macroscopic collision that is nearly elastic is that of two steel blocks on ice. Another nearly elastic collision is that between two carts with spring bumpers on an air track. Icy surfaces and air tracks are nearly frictionless, more readily allowing nearly elastic collisions on them.

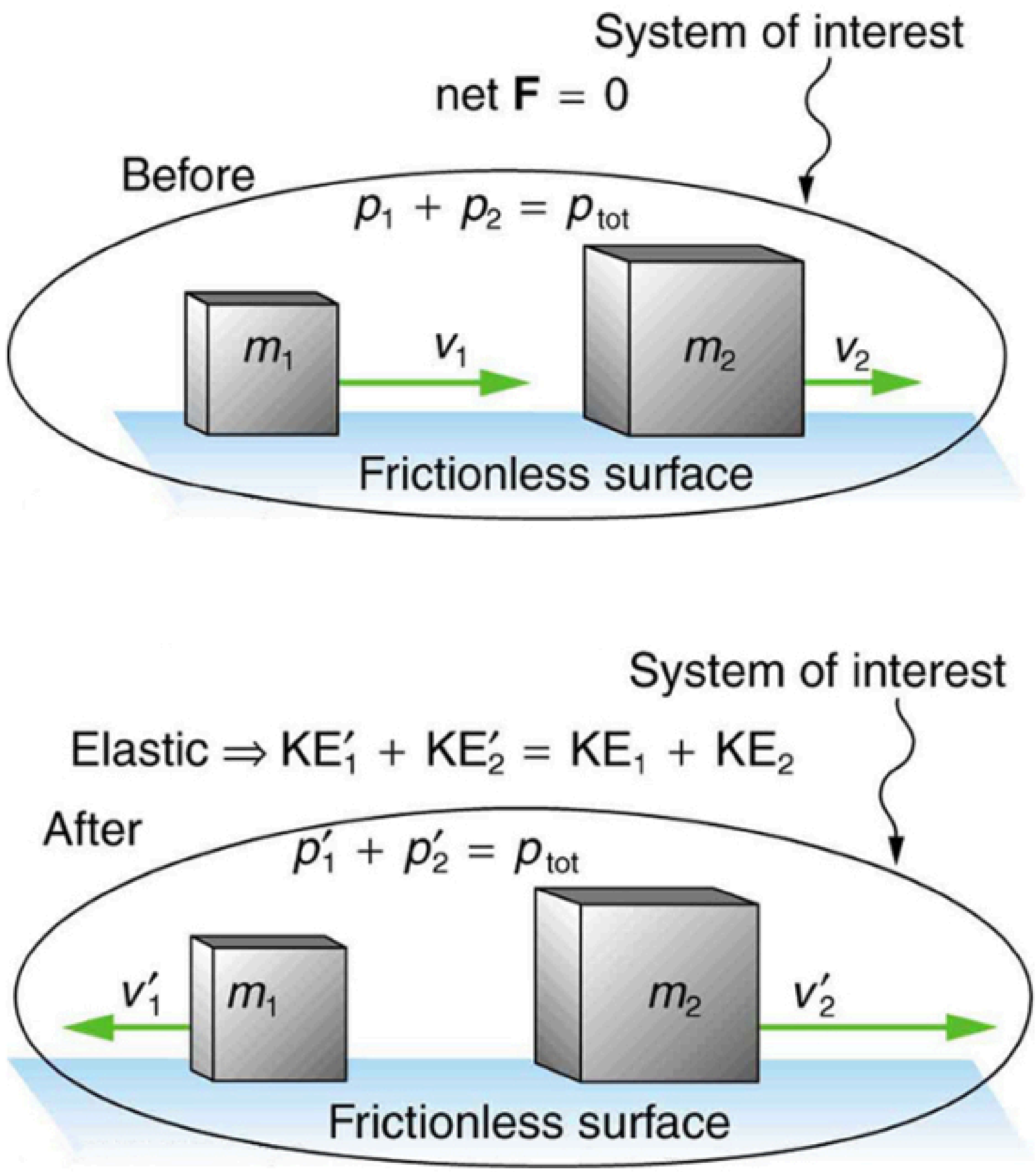

Figure 3: An elastic one-dimensional two-object collision. Momentum and internal kinetic energy are conserved.

Now, to solve problems involving one-dimensional elastic collisions between two objects we can use the equations for conservation of momentum and conservation of internal kinetic energy. First, the equation for conservation of momentum for two objects in a one-dimensional collision is

$$p_1 + p_2 = p'_1 + p'_2 \, (F_{\text{net}} \ = 0)$$

Or

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2 \, (F_{\text{net}} \ = 0)$$

Where the primes (‘) indicate values after the collision. By definition, an elastic collision conserves internal kinetic energy, and so the sum of kinetic energies before the collision equals the sum after the collision. Thus,

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 (\text{ two } - \text{ object elastic collision })$$

Expresses the equation for conservation of internal kinetic energy in a one-dimensional collision.

Example 4

Calculating velocities following an elastic collision:

Calculate the velocities of two objects following an elastic collision, given that

$$m_1 = 0.500 \text{ kg}, m_2 = 3.50 \text{ kg}, v_1 = 4.00 \frac{\text{m}}{\text{s}}, v_2 = 0.$$

Answer: For this problem, note that $v_2 = 0$ and use conservation of momentum.

Thus, $p_1 = p'_1 + p'_2$ or $m_1 v_1 = m_1 v'_1 + m_2 v'_2$

Using conservation of internal kinetic energy and that $v_2 = 0$,

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$$

Solving the first equation (momentum equation) for v'_2, we obtain

$$v'_2 = \frac{m_1}{m_2} (v_1 - v'_1).$$

Substituting this expression into the second equation (internal kinetic energy equation) eliminates the variable v′2">�′2,leaving only v'_1 as an unknown. There are two solutions to any quadratic equation; in this example, they are $v'_1 = 4.00 \text{ m/s}$ and $v'1 = -3.00 \text{ m/s}$. Both solutions may or may not be meaningful. In this case, the first solution is the same as the initial condition. The first solution thus represents the situation before the collision is discarded. The second solution $(v'_1 = -3.00 \text{ m/s})$ is negative, meaning that the first object bounces backward. When this negative value of v'_1 is used to find the velocity of the second object after the collision, we get

$$v'_2 = \frac{m_1}{m_2} (v_1 - v'_1) = \frac{0.500 \text{ kg}}{3.50 \text{ kg}} [4.00 - (-3.00)] \text{m/s}$$

$$v'_2 = 1.00 \text{ m/s}$$

Collision Lab – PhET Simulation

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Inelastic Collisions in One Dimension

We have seen that in an elastic collision, internal kinetic energy is conserved. An **inelastic collision** is one in which the internal kinetic energy changes (it is not conserved). This lack of conservation means that the forces between colliding objects may remove or add internal kinetic energy. Work done by internal forces may change the forms of energy within a system. For inelastic collisions, such as when colliding objects stick together, this internal work may transform some internal kinetic energy into heat transfer. Or it may convert stored energy into internal kinetic energy, such as when exploding bolts separate a satellite from its launch vehicle.

Figure 4 below shows an example of an inelastic collision. Two objects that have equal masses head toward one another at equal speeds and then stick together. Their total internal kinetic energy is initially $\frac{1}{2}mv^2 + \frac{1}{2}mv^2 = mv^2$. The two objects come to rest after sticking together, conserving momentum. But the internal kinetic energy is zero after the collision. A collision in which the objects stick together is something called a **perfectly inelastic collision** because it reduces internal kinetic energy more than does any other type of inelastic collision. In fact, such a collision reduces internal kinetic energy to the minimum it can have while still conserving momentum.

Perfectly inelastic collision – a collision in which the objects stick together is sometimes called "perfectly inelastic."

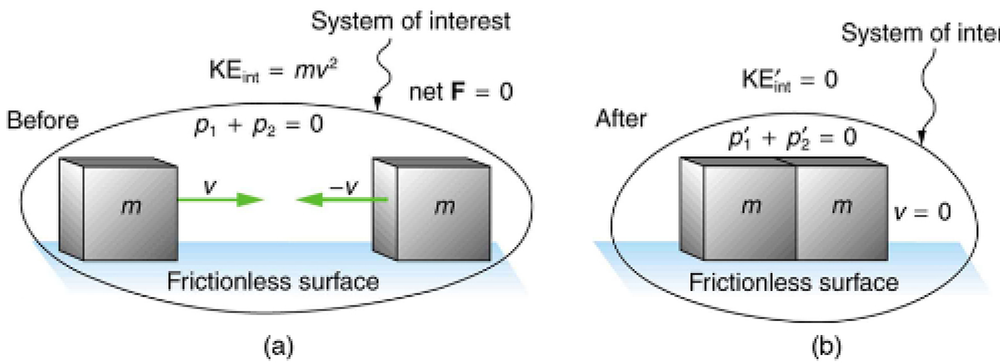

Figure 4: An inelastic one-dimensional two-object collision. Momentum is conserved, but internal kinetic energy is not conserved. (a) Two objects of equal mass initially head directly toward one another at the same speed. (b) The objects stick together (a perfectly inelastic collision), and so their final velocity is zero. The internal kinetic energy of the system changes in any inelastic collision and is reduced to zero in this example.

Example 5

Calculating final velocity and energy release – two carts collide:

In the collision pictured below in Figure 5, two carts collide inelastically. Cart 1 (denoted m_1 carries a spring which is initially compressed. During the collision, the spring releases its potential energy and converts it to internal kinetic energy. The mass of cart 1 and the spring is $0.350\ \text{kg}$, and the cart and the spring together have an initial velocity of $2.00\ \text{m/s}$. Cart 2 (denoted m_2) has a mass of 0.500 *kg* and an initial velocity of -0.500*m/s*. After the collision, cart 1 is observed to recoil with a velocity of -4.00*m/s*.

(a) What is the final velocity of cart 2?

(b) How much energy was released by the spring (assuming all of it was converted into internal kinetic energy)?

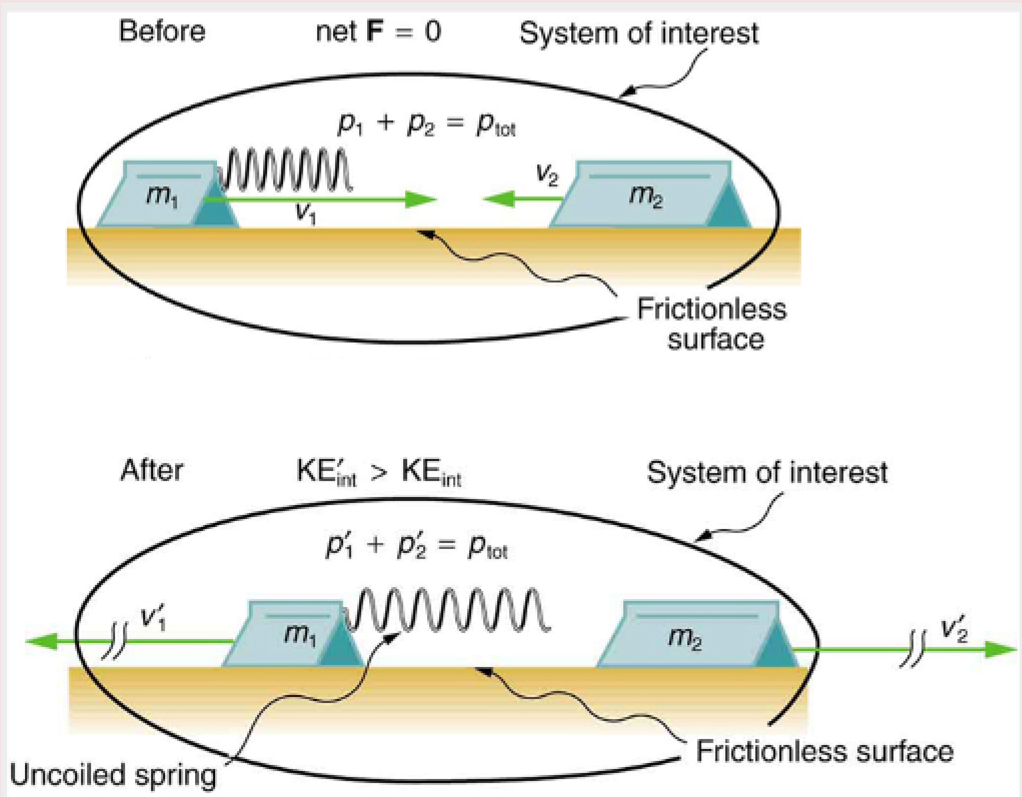

Figure 5: An air track is nearly frictionless, so that momentum is conserved. Motion is one-dimensional. In this collision, examined in the Example above, the potential energy of a compressed spring is released during the collision and is converted to internal kinetic energy.

Answer:

(a) the equation for conservation of momentum in a two-object system is
$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'$.

The only unknown in this equation is v_2'. Solving for v_2' and substituting known values into the previous equation yields.

$$v_2' = \frac{m_1v_1 + m_2v_2 - m_1v_1'}{m_2}$$

$$= \frac{(0.350\ \text{kg})(2.00\ \text{m/s}) + (0.500\ \text{kg})(-0.500\ \text{m/s})}{0.500\ \text{kg}} - \frac{(0.350\ \text{kg})(-4.00\ \text{m/s})}{0.500\ \text{kg}}$$

$$= 3.70\ \text{m/s}$$

(b) the internal kinetic energy before the collision is

$$KE_{int} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

$$= \frac{1}{2}(0.350\ \text{kg})(2.00\ \text{m/s})^2 + \frac{1}{2}(0.500\ \text{kg})(-0.500\ \text{m/s})^2$$

$$= 0.763\ \text{J}$$

After the collision, the internal kinetic energy is

$$KE'_{int} = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2_2$$

$$= \frac{1}{2}(0.350 \text{ kg})(-4.00 \text{ m/s})^2 + \frac{1}{2}(0.500 \text{ kg})(3.70 \text{ m/s})^2$$

$$= 6.22 \text{ J}$$

The change in internal kinetic energy is thus

$$KE'_{int} - KE_{int} = 6.22J - 0.763J = 5.46J$$

Collisions of Point Masses in Two Dimensions

In the previous two sections, we considered only one-dimensional collisions; during such collisions, the incoming and outgoing velocities are all along the same line. But what about collisions, such as those between billiard balls, in which objects scatter to the side? These are two-dimensional collisions, and we shall see that their study is an extension of the one-dimensional analysis already presented. The approach taken (similar to the approach in discussing two-dimensional kinematics and dynamics) is to choose a convenient coordinate system and resolve the motion into components along perpendicular axes. Resolving the motion yields a pair of one-dimensional problems to be solved simultaneously.

One complication arising in two-dimensional collisions is that the objects might rotate before or after their collision. For example, if two ice skaters hook arms as they pass by one another, they will spin in circles. To avoid rotation, we consider only the scattering of **point masses** – that is, structureless particles that cannot rotate or spin.

We start by assuming that $F_{\text{net}} = 0$, so that momentum *p* is conserved. The simplest collision is one in which one of the particles is initially at rest. See Figure 4 below. The best choice for a coordinate system is one with an axis parallel to the velocity of the incoming particle, as shown in Figure 6. Because momentum is conserved, the components of momentum along the x- and y- axes $(p_x \text{ and } p_y)$will also be conserved, but with the chosen coordinate system, p_y is initially zero and p_x is the momentum of the incoming particle. Both facts simplify the analysis. (Even with the simplifying assumptions of point masses, one particle initially at rest, and a convenient coordinate system, we still gain new insights into nature from the analysis of two-dimensional collisions.)

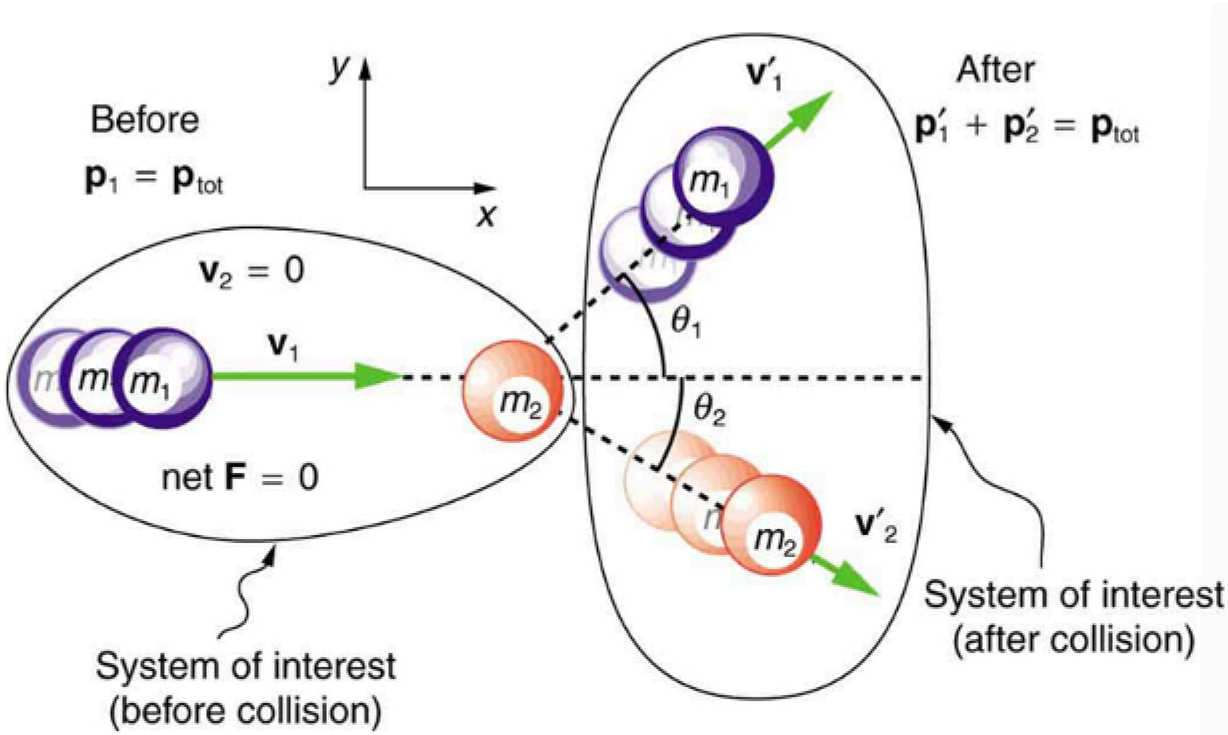

Figure 6: A two-dimensional collision with the coordinate system chosen so that m_2 is initially at rest and v_1 is parallel to the x -axis. This coordinate system is sometimes called the laboratory coordinate system, because many scattering experiments have a target that is stationary in the laboratory, while particles are scattered from it to determine the particles that make-up the target and how they are bound together. The particles may not be observed directly, but their initial and final velocities are.

Along the x-axis, the equation for conservation of momentum is

$$p_{1x} + p_{2x} = p'_{1x} + p'_{2x}$$

Where the subscripts denote the particles and axes, and the primes denote the situation after the collision. In terms of masses and velocities, this equation is

$$m_1 v_{1x} + m_2 v_{2x} = m_1 v'_{1x} + m_2 v'_{2x}$$

But because particle 2 is initially at rest, this equation becomes

$$m_1 v_{1x} = m_1 v'_{1x} + m_2 v'_{2x}$$

The components of the velocities along the x-axis have the form $v \cos\theta$. Because particle 1 initially moves along the x-axis, we find $v_{1x} = v_1$.

Conservation of momentum along the x-axis gives the following equation:

$$m_1 v_{1x} = m_1 v'_{1}1 \cos\theta_1 + m_2 v'_2 \cos\theta_2$$

Where θ_1 and θ_2 are shown in Figure 4.

Along the y-axis, the equation for conservation of momentum is

$$p_{1y} + p_{2y} = p_{1y} + p_{2y}$$

Or

$$m_1 v_{1y} + m_2 v_{2y} = m_1 v'_{1y} + m_2 v'_{2y}$$

But v_{1y} is zero, because particle 1 initially moves along the x-axis. Because particle 2 is initially at rest, v_{2y} is also zero. The equation for conservation of momentum along the y-axis becomes

$$0 = m_1 v'_{1y} + m_2 v'_{2y}$$

The components of the velocities along the y-axis have the form $v \sin \theta$. Thus, the conservation of momentum along the y-axis gives the following equation:

$$0 = m_1 v'_{1}{}_{1} \sin \theta_1 + m_2 v'_2 \sin \theta_2$$

Making Connections: Real World Connections

We have seen, in one-dimensional collisions when momentum is conserved, that the center-of-mass velocity of the system remains unchanged as a result of the collision. If you calculate the momentum and center-of-mass velocity before the collision, you will get the same answer as if you calculate both quantities after the collision. This logic also works for two-dimensional collisions.

For example, consider two cars of equal mass. Car A is driving east (+x-direction) with a speed of 40 m/s. Car B is driving north (+y-direction) with a speed of 80 m/s. What is the velocity of the center-of-mass of this system before and after an inelastic collision, in which the cars move together as one mass after the collision?

Since both cars have equal mass, the center-of-mass velocity components are just the average of the components of the individual velocities before the collision. The x-component of the center of mass velocity is 20 m/s, and the y-component is 40 m/s.

Using momentum conservation for the collision in both the x-component and y-component yields similar answers:

$$m(40) + m(0) = (2m) v_{\text{final}}\ (x)$$

$$v_{\text{final}}\ (x) = 20\ \text{m/s}$$

$$m(0) + m(80) = (2m) v_{\text{final}}\ (y)$$

$$v_{\text{final}}\ (y) = 40\ \text{m/s}$$

Since the two masses move together after the collision, the velocity of this combined object is equal to the center-of-mass velocity. Thus, the center-of-mass velocity before and after the collision is identical, even in two-dimensional collisions, when momentum is conserved.

Elastic and Inelastic Collisions

Elastic and Inelastic Collisions

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Deriving the Shortcut to Solve Elastic Collision Problems

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How to Use the Shortcut for Solving Elastic Collisions

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BALANCE FORCES ON A STATIONARY OBJECT IN EQUILIBRIUM

Equilibrium Forces

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Check Your Understanding: Equilibrium Forces

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Static Equilibrium

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Balancing Act PhET Simulation

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The First Condition for Equilibrium

The first condition necessary to achieve equilibrium is that the net external force on the system must be zero. Expressed as an equation, this is simply

$$\text{net } F = 0$$

Note that if net F is zero, then the net external force in *any* direction is zero. For example, the net external forces along the typical x- and y-axes are zero. This is written as $\text{net } F_x = 0$ and $F_y = 0$.

The Second Condition for Equilibrium

The second condition necessary to achieve equilibrium involves avoiding accelerated rotation (maintaining a constant angular velocity). A rotating body or system can be in equilibrium if its rate of rotation is constant and remains unchanged by the forces acting on it. The magnitude, direction, and point of application of the force are incorporated into the definition of the physical quantity called **torque**.

Torque is the rotational equivalent of a force. It is a measure of the effectiveness of a force in changing or accelerating a rotation (changing the angular velocity over a period of time). In equation form, the magnitude of torque is defined to be

$$\tau = rF\sin(\theta)$$

Where τ is the symbol for torque, r is the distance from the pivot point to the point where the force is applied, F is the magnitude of the force, and θ is the angle between the force and the vector directed from the point of application to the pivot point.

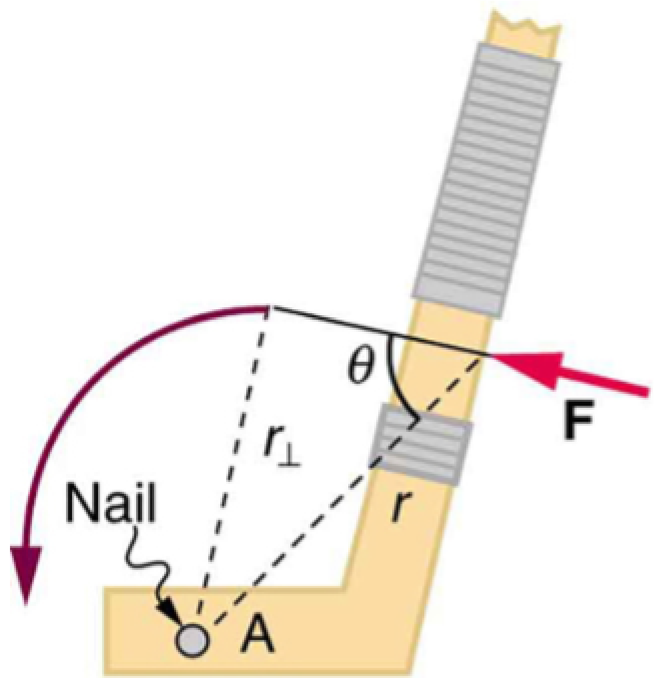

Figure 7: A force applied to an object can produce a torque, which depends on the location of the pivot point. The three factors $r, F,$ and θ *for pivot point A on a body are shown here – r is the distance from the chosen pivot point to the point where the force F is applied, and* θ *is the angle between F and the vector directed from the point of application to the pivot point. If the object can rotate around point A, it will rotate counterclockwise. This means that torque is counterclockwise relative to pivot A.*

The **perpendicular lever arm** $r_\perp$ is the shortest distance from the pivot point to the line along which F acts; it is shown as a dashed line in Figure 7. Note that the line segment that defines the distance $r_\perp$ is perpendicular to *F*, as its name implies. It is sometimes easier to find or visualize $r_\perp$ than to find both *r* and θ. In such cases, it may be more convenient to use $\tau = r_\perp F$ rather than $\tau = rF\sin(\theta)$ for torque, but both are equally valid.

The SI unit of torque is newtons time meters, usually written as $N \cdot m$. For example, if you push perpendicular to the door with a force of 40 N at a distance of 0.800 m from the hinges, you exert a torque of $32N \cdot m\,(0.800m \times 40N \times \sin{(90°)}$ relative to the hinges. If you reduce the force to 20 N, the torque is reduced to $16N \cdot m$, and so on.

Now, the second condition necessary to achieve equilibrium is that the net external torque on a system *must be zero.* An external torque is one that is created by an external force. You can choose the point around which the torque is calculated. The point can be the physical pivot point of a system or any other point in space—but it must be the same point for all torques. If the second condition (net external torque on a system is zero) is satisfied for one choice of pivot point, it will also hold true for any other choice of pivot point in or out of the system of interest. (This is true only in an inertial frame of reference.) The second condition necessary to achieve equilibrium is stated in equation form as $\text{net } \tau = 0$ where net means total. Torques, which are in opposite directions are assigned opposite signs. A common convention is to call counterclockwise (ccw) torques positive and clockwise (cw) torques negative.

When two children balance a seesaw as shown in Figure 8, they satisfy the two conditions for equilibrium. Most people have perfect intuition about seesaws, knowing that the lighter child must sit farther from the pivot and that a heavier child can keep a lighter one off the ground indefinitely.

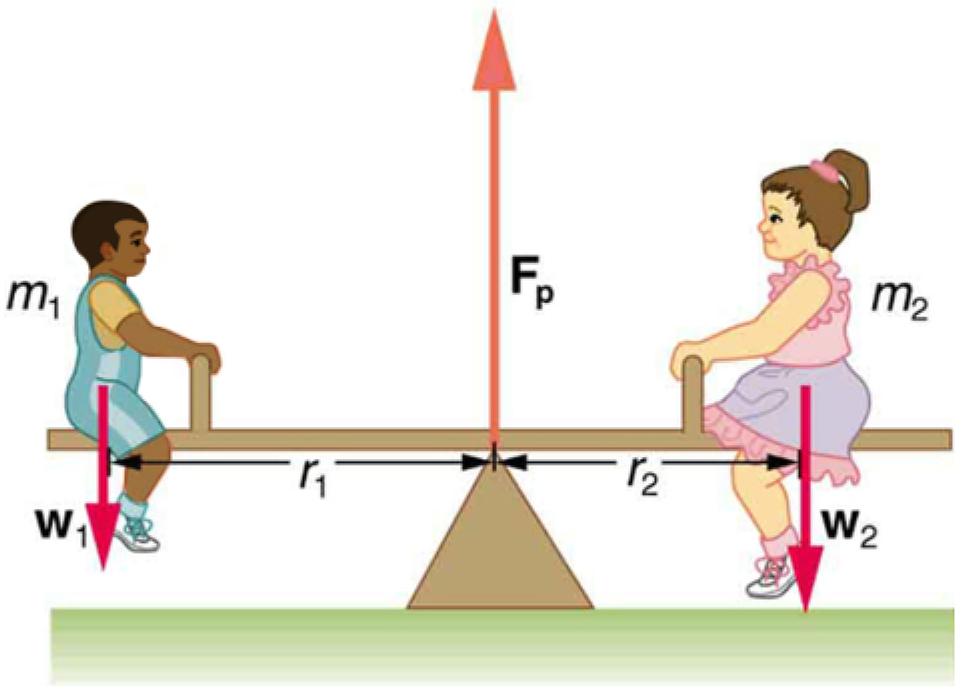

Figure 8: Two children balancing a seesaw satisfy both conditions for equilibrium. The lighter child sits farther from the pivot to create a torque equal in magnitude to that of the heavier child.

Example 6

The two children shown in Figure 8 above are balanced on a seesaw of negligible mass. (This assumption is made to keep the example simple). The first child has a mass of 26.0 kg and sits 1.60 m from the pivot. (a) If the second child has a mass of 32.0 kg, how far is she from the pivot? (b) What is F_p the supporting force exerted by the pivot?

Strategy:

Both conditions for equilibrium must be satisfied. In part (a), we are asked for a distance; thus, the second condition (regarding torques) must be used, since the first (regarding only forces) has no distances in it. To apply the second condition for equilibrium, we first identify the system of interest to be the seesaw plus the two children. We take the supporting pivot to be the point about which the torques are calculated. We then identify all external forces acting on the system.

Solution (a)

The three external forces acting on the system are the weights of the two children and the supporting force of the pivot. Let us examine the torque produced by each. Torque is defined to be

$$\tau = rF\sin(\theta)$$

Here $\theta = 90°$, so that $\sin(\theta) = 1$ for all three forces. That means $\tau_{\perp} = r$ for all three. The torques exerted by the three forces are first,

$$\tau_1 = r_1 w_1$$

Second,

$$\tau_2 = -r_2 w_2$$

And third,

$$\tau_p = r_p F_p$$

$$= 0 \cdot F_p$$

= 0

Note that a minus sign has been inserted into the second question because this torque is clockwise and is therefore negative by convention. Since F_p act directly on the pivot point, the distance r_p is zero. A force acting on the pivot cannot cause a rotation, just as pushing directly on the hinges of a door will not cause it to rotate. Now, the second condition for equilibrium is that the sum of the torques on both children is zero. Therefore

$$\tau_2 = -\tau_1$$

Or

$$r_2 w_2 = r_1 w_1$$

Weight is mass times the acceleration due to gravity. Entering *mg* for *w*, we get

$$r_2 m_2 g = r_1 m_1 g$$

Solve this for the unknown r_2:

$$r_2 = r_1 \frac{m_1}{m_2}$$

The quantities on the right side of the equation are known; thus, r_2 is

$$r_2 = (1.60\ \text{m})\frac{26.0\ \text{kg}}{32.0\ \text{kg}} = 1.30\ \text{m}$$

As expected, the heavier child must sit closer to the pivot (1.30 m versus 1.60 m) to balance the seesaw.

Solution (b)

This part asks for a force F_p. The easiest way to find it is to use the first condition for equilibrium, which is

$$\text{net}\ F = 0$$

The forces are all vertical, so that means we are dealing with a one-dimension problem along the vertical axis; hence, the condition can be written as

$$\text{net}\ F_y = 0$$

Where we again call the vertical axis the y-axis. Choosing upward to be the positive direction, and using plus and minus signs to indicate the directions of the forces, we see that

$$F_p - w_1 - w_2 = 0$$

This equation yields what might have been guessed at the beginning:

$$F_p = w_1 + w_2$$

So, the pivot supplies a supporting force equal to the total weight of the system:

$$F_p = m_1 g + m_2 g$$

Entering known values gives

$$F_p = (26.0\ \text{kg})\left(9.80\ \text{m/s}^2\right) + (32.0\ \text{kg})\left(9.80\ \text{m/s}^2\right)$$

$$= 568N$$

Discussion:

The two results make intuitive sense. The heavier child sits closer to the pivot. The pivot supports the weight of the two children. Part (b) can also be solved using the second condition for equilibrium, since both distances are known, but only if the pivot point is chosen to be somewhere other than the location of the seesaw's actual pivot!

Several aspects of the preceding example have broad implications. First, the choice of the pivot as the point around which torques are calculated simplified the problem. Since F_p is exerted on the pivot point, its lever arm is zero. Hence, the torque exerted by the supporting force F_p is zero relative to that pivot point. The second condition for equilibrium holds for any choice of pivot point, and so we choose the pivot point to simplify the solution of the problem.

Second, the acceleration due to gravity canceled in this problem, and we were left with a ratio of masses. This will not always be the case. Always enter the correct forces – do not jump ahead to enter some ratio of masses.

Third, the weight of each child is distributed over an area of the seesaw, yet we treated the weights as if each force were exerted at a single point. This is not an approximation – the distances r_1 and r_2 are the distances to points directly below the **center of gravity** of each child.

Finally, not that the concept of torque has an importance beyond **static equilibrium**. Torque plays the same role in rotational motion that force plays in linear motion.

Stability

A system is said to be in **stable equilibrium** if, when displaced from equilibrium, it experiences a net force or torque in a direction opposite to the direction of the displacement. For example, a marble at the bottom of a bowl will experience a restoring force when displaced from its equilibrium position. This force moves it back toward the equilibrium position. Most systems are in stable equilibrium, especially for small displacements. For another example of stable equilibrium, see the pencil in Figure 9.

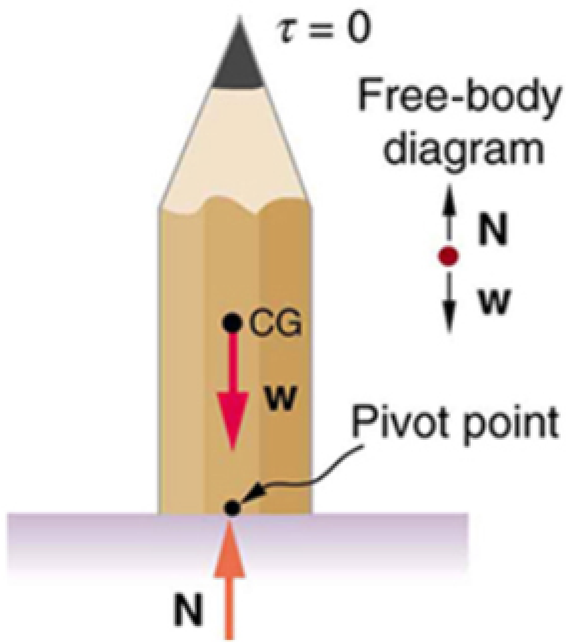

Figure 9: This pencil is in the condition of equilibrium. The net force on the pencil is zero and the total torque about any pivot is zero.

A system is in **unstable equilibrium** if, when displaced, it experiences a net force or torque in the same direction as the displacement from equilibrium. A system in unstable equilibrium accelerates away from its equilibrium position if displaced even slightly. An obvious example is a ball resting on top of a hill. Once displaced, it accelerates away from the crest. See Figures 10-13 for examples of unstable equilibrium.

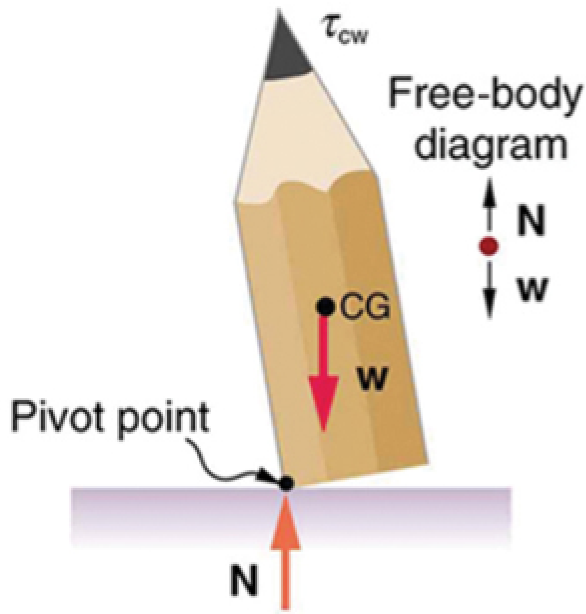

Figure 10: If the pencil is displaced slightly to the side (counterclockwise), it is no longer in equilibrium. Its weight produces a clockwise torque that returns the pencil to its equilibrium position.

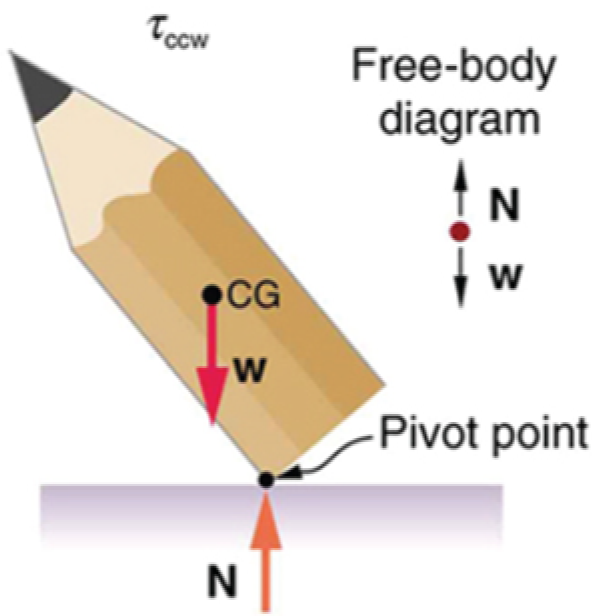

Figure 11: If the pencil is displaced too far, the torque caused by its weight changes to counterclockwise and causes the displacement to increase.

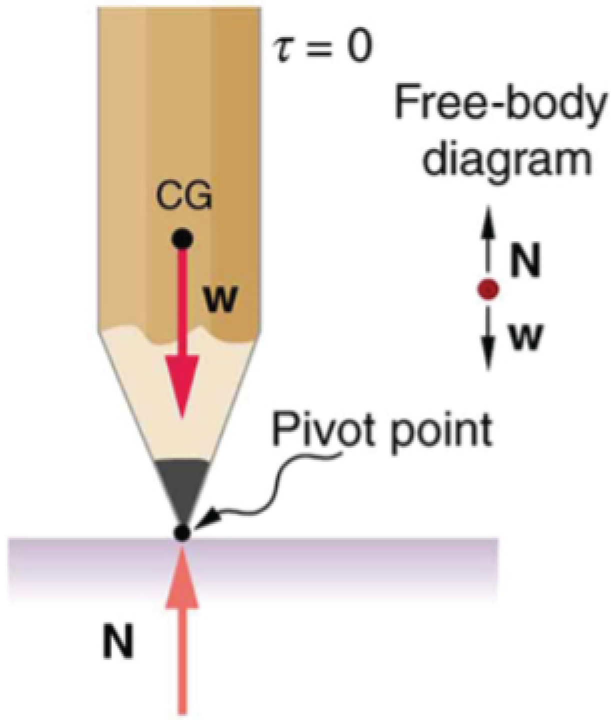

Figure 12: This figure shows unstable equilibrium, although both conditions for equilibrium are satisfied.

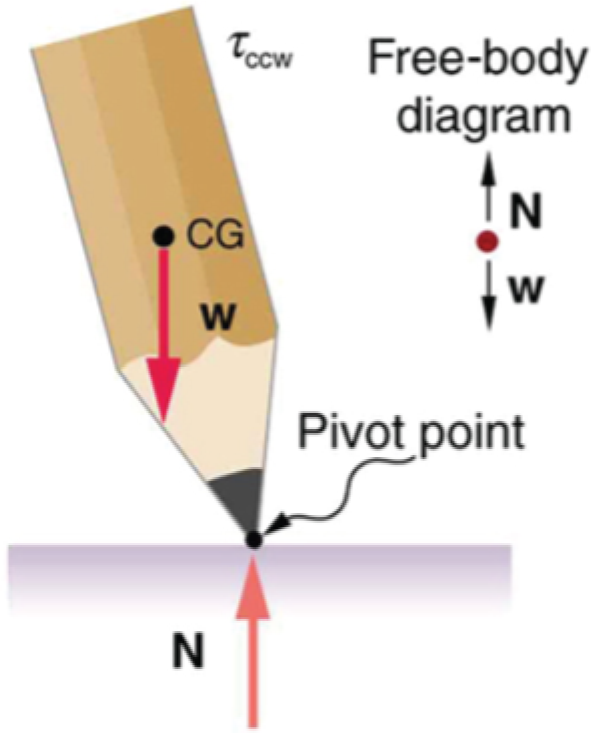

Figure 13: If the pencil is displaced even slightly, a torque is created by its weight that is in the same direction as the displacement, causing the displacement to increase.

A system is in **neutral equilibrium** if its equilibrium is independent of displacements from its original position. A marble on a flat horizontal surface is an example. Combinations of these situations are possible. For example, a marble on a saddle is stable for displacements toward the front or back of the saddle and unstable for displacements to the side. Figures 14-15 show another example of neutral equilibrium.

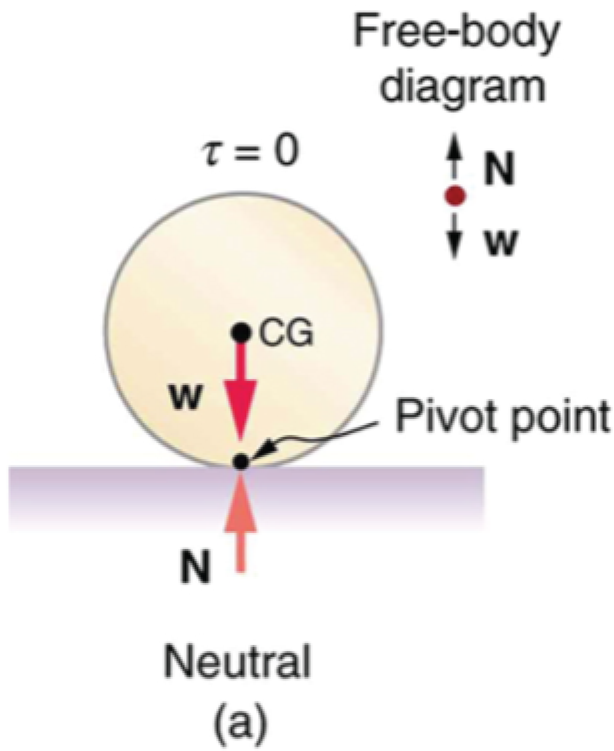

Figure 14: Here we see neutral equilibrium. The cg of a sphere on a flat surface lies directly above the point of support, independent of the position on the surface. The sphere is therefore in equilibrium in any location, and if displaced, it will remain put.

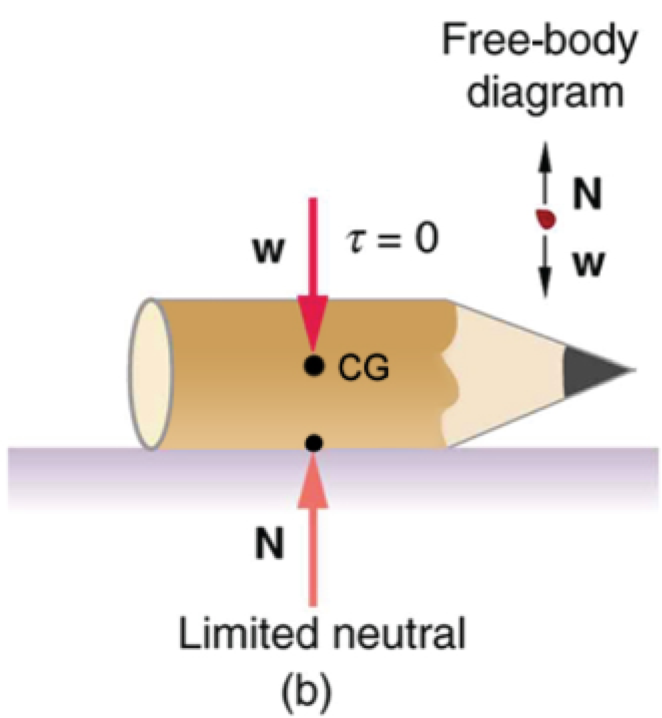

Figure 15: Because it has a circular cross section, the pencil is in neutral equilibrium for displacements perpendicular to its length.

Center of Gravity

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Check Your Understanding: Stability

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RELEVANCE TO TRANSPORTATION ENGINEERING COURSEWORK

This section explains the relevance of traffic safety and superelevation to transportation engineering coursework.

Traffic Safety

Avoiding collisions (or crashes) and in case collisions do occur minimization of harm to human beings involved are of critical importance in transportation engineering practice. Practitioners and researchers recommend use of term "crashes" or "collisions" instead of "accidents" so that they are thought of as preventable events instead of inevitable incidents with no possible avoidance/mitigation measures. Crashes are classified according to their severity from property-damage-only (PDO) to fatal, and the crash severity is highly correlated with the impact speed, and hence the momentum. Mitigation measures, such as guardrails, traffic signs with breakaway poles, crash attenuators, air bags, seat belts, and child seats are all implemented to reduce the severities of crashes according to the momentum principles, described in the section titled "Use of Momentum Preservation to Describe the Post-Collision Movement of Objects."

Superelevation

Superelevation, i.e., raising of outer edge of the pavement relative to the inner edge, is provided to support higher speed travel along a horizontal curve. The banking helps provide the required inward force to keep vehicle on a circular motion. Figure 14 illustrates a superelevated curved segment of racetrack, where the relative elevation of the outer edge is more pronounced (compared to public roadways) due to significantly higher speeds of race cars. The formula for estimating the required superelevation for a given design speed is based on the principles of balancing forces described in the above section titled "Balance Forces on a Stationary Object in Equilibrium."

Figure 16: Superelevated segment of a racetrack with raised outer edge

Key Takeaways

- Transportation engineers and safety professionals recognize that crashes and collisions are not "accidents" in the sense that these events are not inevitable, can be prevented, and their impacts can be mitigated. One key aspect of crash impact mitigation is to reduce the severity of collisions that do occur, especially those involving VRUs. The speed-momentum relationships discussed in this chapter are essential to developing this understanding.
- Superelevation or banking refers to the raised outer edge of a roadway segment on a horizontal

curve. It is provided to counteract the tendency of a vehicle to roll outward on a curve. With a gravitational force component pulling the vehicle inward, the superelevation helps keep the vehicle on a circular path providing stability and control for negotiating circular curve segments at higher speeds.

GLOSSARY: KEY TERMS

Center of Gravity[1] – the point where the total weight of the body is assumed to be concentrated
Change in Momentum[1]– the difference between the final and initial momentum; the mass times the change in velocity
Conservation of Momentum Principle[1] – when the net external force is zero, the total momentum of the system is conserved or constant
Dynamic Equilibrium[1] – a state of equilibrium in which the net external force and torque on a system moving with constant velocity are zero
Elastic Collision[1] – a collision that also conserves internal kinetic energy
Impulse[1] – the average net external force times the time it acts; equal to the change in momentum
Inelastic Collision[1] – a collision in which internal kinetic energy is not conserved
Internal Kinetic Energy[1] – the sum of the kinetic energies of the objects in a system
Isolated System[1] – a system in which the net external force is zero
Linear Momentum[1] – the product of mass and velocity
Neutral Equilibrium[1] – a state of equilibrium that is independent of a system's displacements from its original position
Perfectly Inelastic Collision[1] – a collision in which the colliding objects stick together
Perpendicular Lever Arm[1] – the shortest distance from the pivot point to the line along which � lies
Point Masses[1] – structureless particles with no rotation or spin
Second Law of Motion[1] – physical law that states that the net external force equals the change in momentum of a system divided by the time over which it changes
SI Units of Torque[1]– newton times meters, usually written as �·�
Stable Equilibrium[1] – a system, when displaced, experiences a net force or torque in a direction opposite to the direction of the displacement
Static Equilibrium[1] – a state of equilibrium in which the net external force and torque acting on a system is zero; equilibrium in which the acceleration of the system is zero and accelerated rotation does not occur
Torque[1] – turning or twisting effectiveness of a force
Unstable Equilibrium[1] – a system, when displaced, experiences a net force or torque in the same direction as the displacement from equilibrium

[1] "College Physics for AP® Courses" by Greg Wolfe, Erika Gasper, John Stoke, Julie Kretchman, David Anderson, Nathan Czuba, Sudhi Oberoi, Liza Pujji, Irina Lyublinskaya, Douglas Ingram. Access for free at https://openstax.org/books/college-physics-ap-courses/pages/1-connection-for-ap-r-courses

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- Farid, A. (2022). Highway Design. Personal Collection of Ahmed Farid, California Polytechnic State University, San Luis Obispo, CA.

CHAPTER 7: WAVES AND DOPPLER EFFECT

This chapter discusses the propagation of sound waves because it is critical to understanding and mitigating the noise pollution resulting from transportation projects. A related concept, i.e., the Doppler Effect, is helpful in understanding the use of radar guns for traffic speed enforcement and spot speed studies.

Learning Objectives

At the end of the chapter, the reader should be able to do the following:

- Define a wave and its basic properties (e.g., Period, Wavelength, and Frequency).
- List different wave types.
- Describe the movement of sound waves.
- Describe Doppler Effect and its implications.
- Identify topics in the introductory transportation engineering courses that build on the Physics concept discussed in this course.

WAVE PROPERTIES

What do we mean when we say something is a wave? The most intuitive and easiest wave to imagine is the familiar water wave. More precisely, a **wave** is a disturbance that propagates or moves from the place it was created. For water waves, the disturbance is in the surface of the water, perhaps created by a rock thrown into a pond or by a swimmer splashing the surface repeatedly. For sound waves, the disturbance is a change in air pressure, perhaps created by the oscillating cone inside a speaker. For earthquakes, there are several types of disturbances, including disturbance of Earth's surface and pressure disturbances under the surface. Even radio waves are most easily understood using an analogy with water waves. Visualizing water waves is useful because there is more to it than just a mental image. Water waves exhibit characteristics common to all waves, such as amplitude, period, frequency, and energy. All wave characteristics can be described by a small set of underlying principles.

Wave Movement

A wave is a disturbance that propagates or moves from the place it was created. The simplest waves repeat themselves for several cycles and are associated with simple harmonic motion. Let us start by considering the simplified water wave in Figure 1 below. The wave is an up and down disturbance of the water surface. It causes a sea gull to move up and down in simple harmonic motion as the wave crests and troughs (peaks and valleys) pass under the bird. The time for one complete up and down motion is the wave's **period** ⟐. The wave's frequency $f = \frac{1}{T}$. The wave itself moves to the right in the figure. This movement of the wave is actually the disturbance moving to the right, not the water itself (or the bird would move to the right). We define **wave velocity** $\mathcal{V}_W$ to be the speed at which the disturbance moves. Wave velocity is sometimes also called the propagation velocity or propagation speed because the disturbance propagates from one location to another.

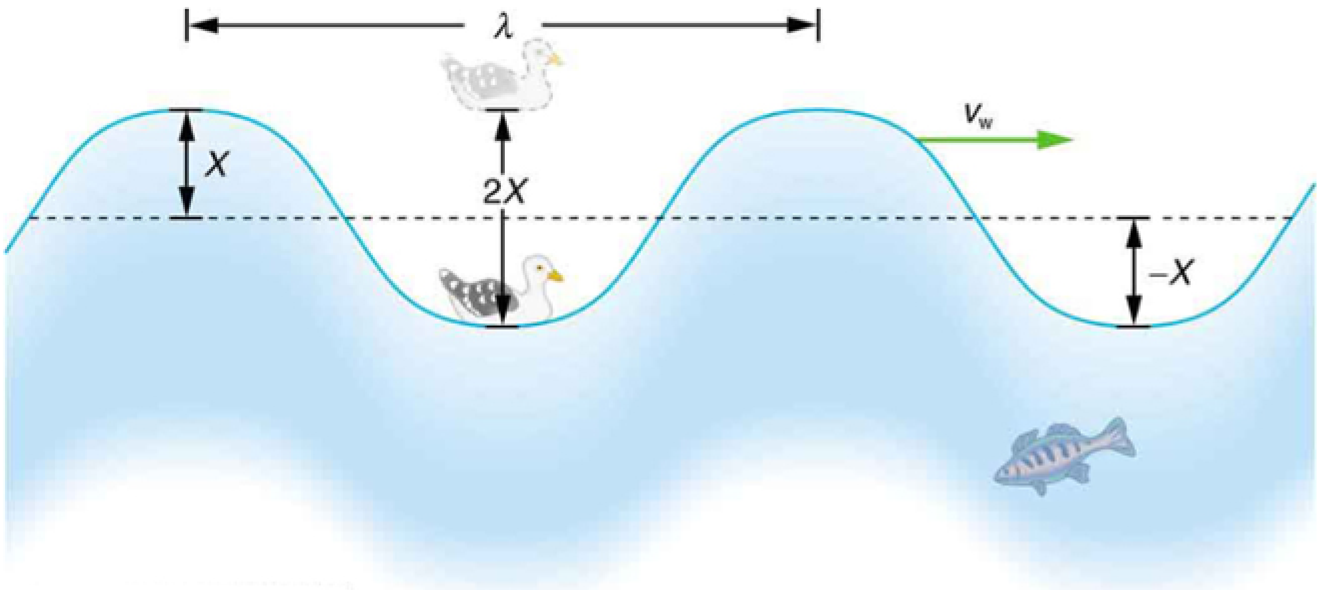

Figure 1: An idealized ocean wave passes under a seagull that bobs up and down in simple harmonic motion. The wave has wavelength λ, which is the distance between adjacent identical parts of the wave. The up and down disturbance of the surface propagates parallel to the surface at a speed $\mathcal{V}_W$.

The water wave in Figure 1 also has a length associated with it, called its **wavelength** ⟐, the distance between adjacent identical parts of a wave. (⟐ is the distance parallel to the direction of propagation.) The speed of propagation $\mathcal{V}_W$ is the distance the wave travels in a given time, which is one wavelength in the time of one period. In equation form, that is

$$v_w = \frac{\lambda}{T}$$

Or

$$v_w = f\lambda$$

This fundamental relationship holds for all types of waves. For water waves, $\mathcal{V}_W$ is the speed of a surface wave; for sound, $\mathcal{V}_W$ is the speed of sound; and for visible light, $\mathcal{V}_W$ is the speed of light, for example.

Applying the Science Practices: Different Types of Waves

Consider a spring fixed to a wall with a mass connected to its end. This fixed point on the wall exerts a force on the complete spring-and-mass system, and this implies that the momentum of the complete system is not conserved. Now, consider energy. Since the system is fixed to a point on the wall, it does not do any work; hence, the total work done is conserved, which means that the energy is conserved. Consequently, we have an oscillator in which energy is conserved but momentum is not. Now, consider a system of two masses connected to each other by a spring. This type of system also forms an oscillator. Since there is no fixed point, momentum is conserved as the forces acting on the two masses are equal and opposite. Energy for such a system will be conserved because there are no external forces acting on the spring-two-masses system. It is clear from above that, for momentum to be conserved, momentum needs to be carried by waves. This is a typical example of a mechanical oscillator producing mechanical waves, which need a medium in which to propagate. Sound waves are also examples of mechanical waves. Some types of waves can travel in the absence of a medium of propagation. Such waves: they are called "electromagnetic waves." Light waves are examples of electromagnetic waves. Electromagnetic waves are created by the vibration of electric charge. which This vibration creates a wave with both electric and magnetic field components.

Traverse and Longitudinal Waves

A simple wave consists of a periodic disturbance that propagates from one place to another. The wave in Figure 2 below propagates in the horizontal direction while the surface is disturbed in the vertical direction. Such a wave is called a transverse wave or shear wave; in such a wave, the disturbance is perpendicular to the direction of propagation. In contrast, in a **longitudinal wave** or compressional wave, the disturbance is parallel to the direction of propagation. Figure 3 below shows an example of a longitudinal wave. The size of the disturbance is its **amplitude** X and is completely independent of the speed of propagation v_W.

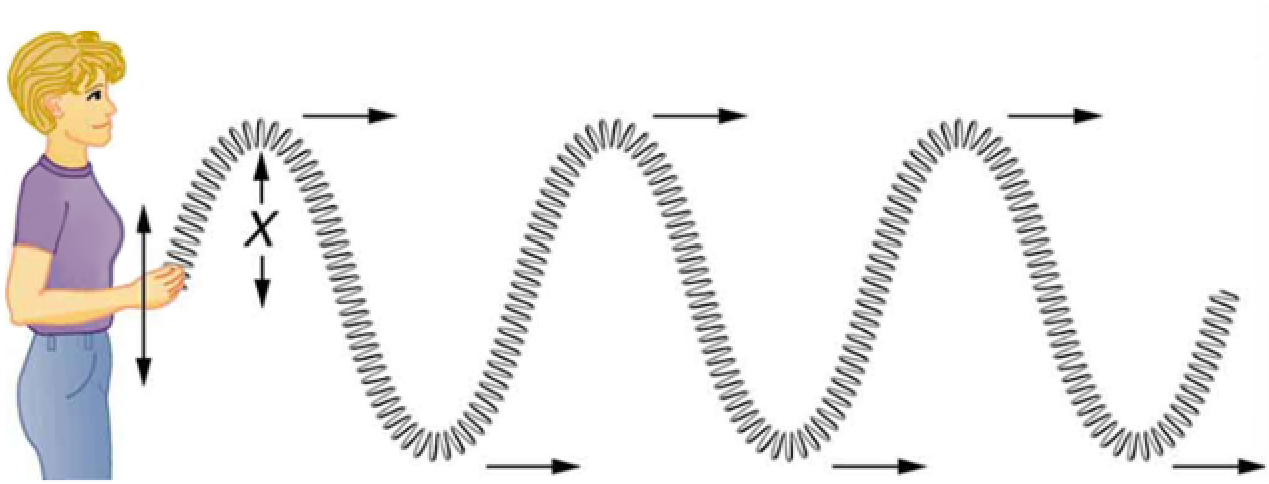

Figure 2: In this example of a transverse wave, the wave propagates horizontally, and the disturbance in the cord is in the vertical direction.

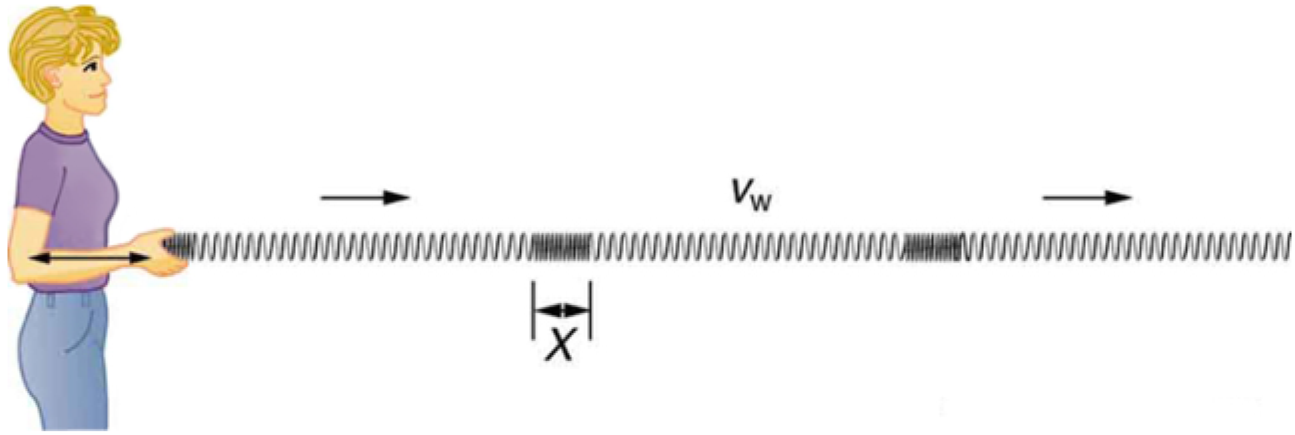

Figure 3: In this example of a longitudinal wave, the wave propagates horizontally, and the disturbance in the cord is also in the horizontal direction.

Waves may be transverse, longitudinal, or a combination of the two. (Water waves are actually a combination of transverse and longitudinal. The simplified water wave illustrated in Figure 1 shows no longitudinal motion of the bird.) The waves on the strings of musical instruments are transverse—so are electromagnetic waves, such as visible light.

Sound waves in air and water are longitudinal. Their disturbances are periodic variations in pressure that are transmitted in fluids. Fluids do not have appreciable shear strength, and thus the sound waves in them must be longitudinal or compressional. Sound in solids can be both longitudinal and transverse.

Earthquake waves under Earth's surface also have both longitudinal and transverse components (called compressional or P-waves and shear or S-waves, respectively). These components have important individual characteristics—they propagate at different speeds, for example. Earthquakes also have surface waves that are similar to surface waves on water.

Basic Properties of Waves

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Check Your Understanding: Basic Properties of Waves

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Wave on a String (Simulation)

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Energy in Waves: Intensity

All waves carry energy. The energy of some waves can be directly observed. Earthquakes can shake whole cities to the ground, performing the work of thousands of wrecking balls.

Loud sounds pulverize nerve cells in the inner ear, causing permanent hearing loss. Ultrasound is used for deep-heat treatment of muscle strains. A laser beam can burn away a malignancy. Water waves chew up beaches.

The amount of energy in a wave is related to its amplitude. Large-amplitude earthquakes produce large ground displacements. Loud sounds have higher pressure amplitudes and come from larger-amplitude source vibrations than soft sounds. Large ocean breakers churn up the shore more than small ones. More quantitatively, a wave is a displacement that is resisted by a restoring force.

The energy effects of a wave depend on time as well as amplitude. For example, the longer deep-heat ultrasound is applied, the more energy it transfers. Waves can also be concentrated or spread out. Sunlight, for example, can be focused to burn wood. Earthquakes spread out, so they do less damage the farther they get from the source. In both cases, changing the area the waves cover has important effects. All these pertinent factors are included in the definition of intensity I as power per unit area:

$$I = \frac{P}{A}$$

Where P is the power carried by the wave through area A. The definition of **intensity** is valis for any energy in transit, including that carried by waves. The SI unit for intensity is watts per square meter $\left(W/m^2\right)$. For example, infrared and visible energy from the Sun impinge on Earth at an intensity of 1300 W/m^2 just above the atmosphere. There are other intensity-related units in use, too. The most common is the decibel. For example, a 90-decibel sound level corresponds to an intensity of $10^{-3} W/m^2$. (This quantity is not much power per unit area considering that 90 decibels is a relatively high sound level. Decibels will be discussed in some detail in a later chapter.)

DESCRIBE THE MOVEMENT OF SOUND WAVES IN THE ENVIRONMENT

In this section, you will learn how to understand the physics, speed, wavelength, intensity, and level of sound by reading each description along with watching the videos included. Also, short problems to check your understanding are included.

Physics of Hearing: Sound

Sound can be used as a familiar illustration of waves. Because hearing is one of our most important senses, it is interesting to see how the physical properties of sound correspond to our perceptions of it. **Hearing** is the perception of sound, just as vision is the perception of visible light. The physical phenomenon of **sound** is defined to be a disturbance of matter that is transmitted from its source outward. Sound is a wave. The amplitude of a sound wave decreases with distance from its source, because the energy of the wave is spread over a larger and larger area. But it is also absorbed by objects, such as the eardrum.

A vibrating string produces a sound wave as illustrated in Figures 4, 5, and 6. As the string oscillates back and forth, it transfers energy to the air, mostly as thermal energy created by turbulence. But a small part of the string's energy goes into compressing and expanding the surrounding air, creating slightly higher and lower local pressures. These compressions (high pressure regions) and rarefactions (low pressure regions) move out as longitudinal pressure waves having the same frequency as the string—they are the disturbance that is a sound wave. (Sound waves in air and most fluids are longitudinal because fluids have almost no shear strength. In solids, sound waves can be both transverse and longitudinal.) Figure 6 shows a graph of gauge pressure versus distance from the vibrating string.

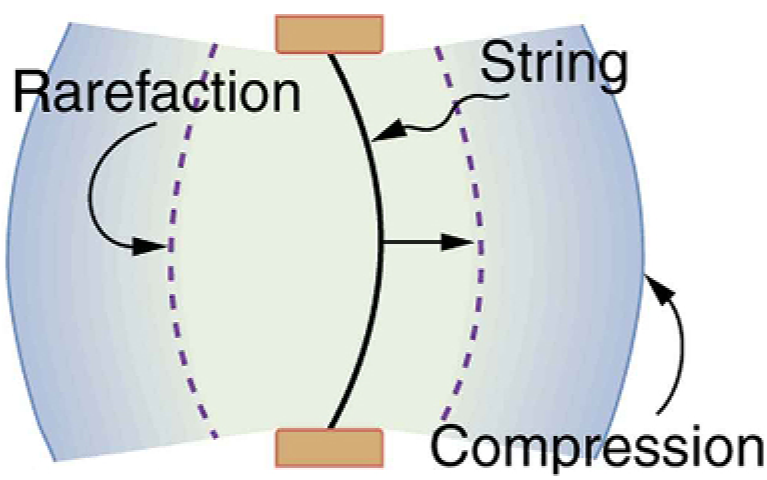

Figure 4: A vibrating string moving to the right compresses the air in front of it and expands the air behind it.

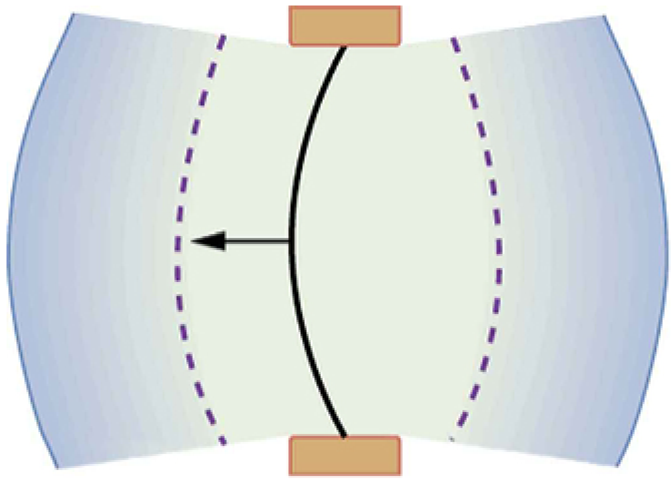

Figure 5: As the string moves to the left, it creates another compression and rarefaction as the ones on the right move away from the string.

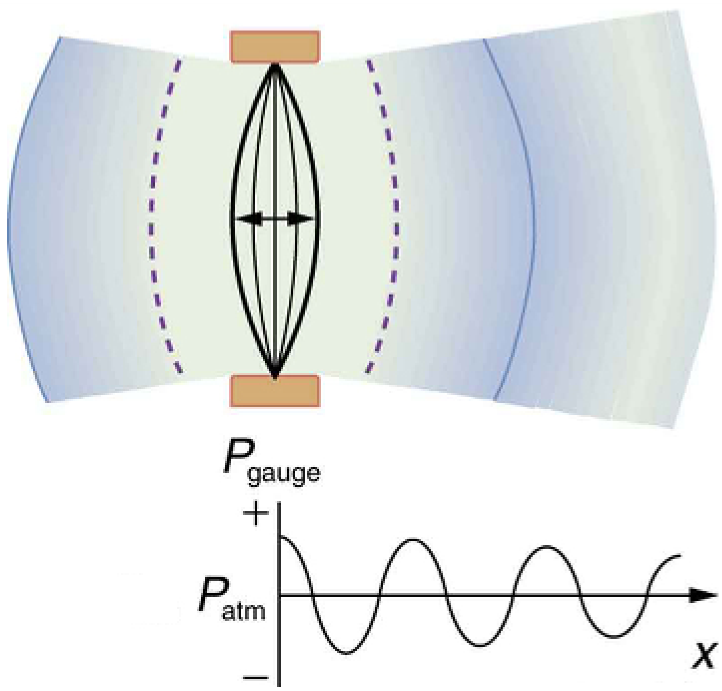

Figure 6: After many vibrations, these are a series of compressions and rarefactions moving out from the string as a sound wave. The graph shows gauge pressure versus distance from the source. Pressures vary only slightly from atmospheric for ordinary sounds.

Speed of Sound, Frequency, and Wavelength

Sound, like all waves, travels at a certain speed and has the properties of **frequency** and wavelength. You can observe direct evidence of the speed of sound while watching a fireworks display. The flash of an explosion is seen well before its sound is heard, implying both that sound travels at a finite speed and that it is much slower than light. You can also directly sense the frequency of a sound. Perception of frequency is called **pitch**. The wavelength of sound is not directly sensed, but indirect evidence is found in the correlation of the size of musical instruments with their pitch. Small instruments, such as a piccolo, typically make high-pitch sounds, while large instruments, such as a tuba, typically make low-pitch sounds. High pitch means small wavelength, and the size of a musical instrument is directly related to the wavelengths of sound it produces. So, a small instrument creates short-wavelength sounds. Similar arguments hold that a large instrument creates long-wavelength sounds.

The relationship of the speed of sound, its frequency, and wavelength is the same as for all waves:

$$v_w = f\lambda$$

Where v_w is the speed of sound, f is its frequency, and �">� is its wavelength. The wavelength of a sound is the distance between adjacent identical parts of a wave – for example, between adjacent compressions as illustrated in Figure 7. The frequency is the same as that of the source and is the number of waves that pass a point per unit of time.

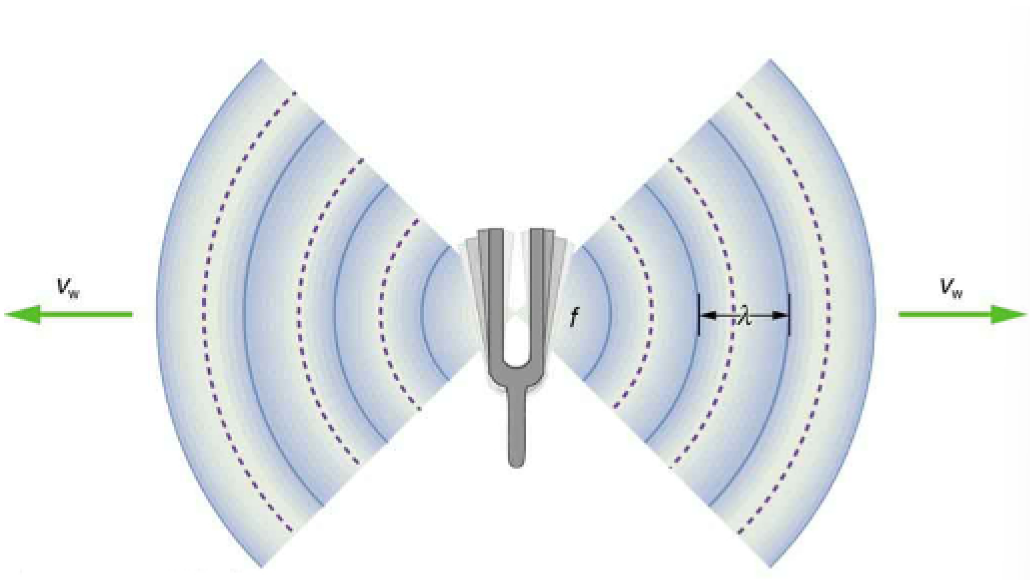

Figure 7: A sound wave emanates from a source vibrating at a frequency f">�*, propagates at* v_w*, and has a wavelength* �

The speed of sound can change when sound travels from one medium to another. However, the frequency usually remains the same because it is like a driven oscillation and has the frequency of the original source. If v_w changes and f remains the same, then the wavelength �">� must change. That is, because $v_w = f\lambda$, the higher the speed of a sound, the greater its wavelength for a given frequency.

Check Your Understanding: Speed of Sound, Frequency, and Wavelength

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Sound Intensity and Sound Level

In a quiet forest, you can sometimes hear a single leaf fall to the ground. After settling into bed, you may hear your blood pulsing through your ears. But when a passing motorist has his stereo turned up, you cannot even hear what the person next to you in your car is saying. We are all very familiar with the **loudness** of sounds and aware that they are related to how energetically the source is vibrating. In cartoons depicting a screaming person (or an animal making a loud noise), the cartoonist often shows a gaping mouth with a vibrating uvula, the hanging tissue at the back of the mouth, to suggest a loud sound coming from the throat (see Figure 8 below). High noise exposure is hazardous to hearing, and it is common for musicians to have hearing losses that are sufficiently severe that they interfere with the musicians' abilities to perform. The relevant physical quantity is sound intensity, a concept that is valid for all sounds whether or not they are in the audible range.

Intensity is defined to be the power per unit area carried by a wave. Power is the rate at which energy is transferred by the wave. In equation form intensity I is $I = \frac{P}{A}$, where P is the power through an area A.

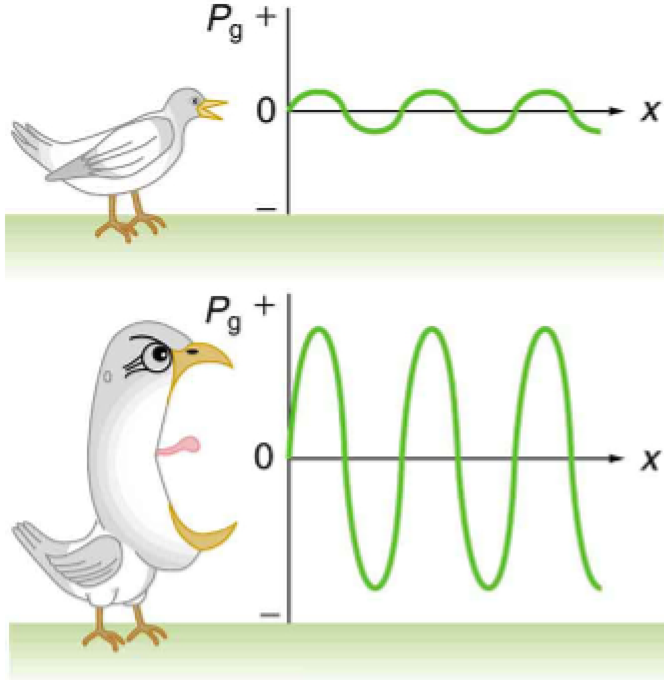

Figure 8: Graphs of the gauge pressures in two sound waves of different intensities. The more intense sound is produced by a source that has larger-amplitude oscillations and has greater pressure maxima and minima. Because pressures are higher in the greater-intensity sound, it can exert larger forces on the objects it encounters.

Sound intensity levels are quoted in decibels (dB) much more often than sound intensities in watts per meter squared. Decibels are the unit of choice in the scientific literature as well as in the popular media. The reasons for this choice of units are related to how we perceive sounds. How our ears perceive sound can be more accurately described by the logarithm of the intensity rather than directly to the intensity. **The sound intensity level $\boldsymbol{\beta}$** in decibels of a sound having an intensity *I* in watts per meter squared is defined to be $\beta(dB) = 10\log_{10}\left(\frac{I}{I_0}\right)$, where $I_0 = 10^{-12}\ \mathrm{W/m^2}$ is a reference intensity. In particular, I_0 is the lowest or threshold intensity of sound a person with normal hearing can perceive at a frequency of 1000 Hz. Sound intensity level is not the same as intensity. Because β is defined in terms of a ratio, it is a unitless quantity telling you the level of the sound relative to a fixed standard $\big(10^{-12}W/m^2$, in this case). The units of decibels (dB) are used to indicate this ratio is multiplied by 10 in its definition. The bel, upon which the decibel is based, is named for Alexander Graham Bell, the inventor of the telephone.

Table 1: Sound Intensity Levels and Intensities [1]

Sound intensity level β (dB)	**Intensity *I*(W/m^2)**	**Example/effect**
00">0	1×10−111×10−11">1×10^{-12}	Threshold of hearing at 1000 Hz
1010">10	1×10−111×10−11">1×10^{-11}	Rustle of leaves
2020">20	1×10−101×10−10">1×10^{-10}	Whisper at 1 m distance
3030">30	1×10−91×10−9">1×10^{-9}	Quiet home
4040">40	1×10−81×10−8">1×10^{-8}	Average home
5050">50	1×10−71×10−7">1×10^{-7}	Average office, soft music
6060">60	1×10−61×10−6">1×10^{-6}	Normal conversation
7070">70	1×10−51×10−5">1×10^{-5}	Noisy office, busy traffic
8080">80	1×10−41×10−4">1×10^{-4}	Loud radio, classroom lecture
9090">90	1×10−31×10−3">1×10^{-3}	Inside a heavy truck; damage from prolonged exposure[1]
100100">100	1×10−21×10−2">1×10^{-2}	Noisy factory, siren at 30 m; damage from 8 h per day exposure
110110">110	1×10−11×10−1">1×10^{-1}	Damage from 30 min per day exposure
120120">120	11">1	Loud rock concert, pneumatic chipper at 2 m; threshold of pain
140140">140	1×1021×102">1×10^{2}	Jet airplane at 30 m; severe pain, damage in seconds
160160">160	1×1041×104">1×10^{4}	Bursting of eardrums

The decibel level of a sound having the threshold intensity of $10^{-12} w/m^2$ is $\beta = 0$dB, because $\log_{10} 1 = 0$. That is, the threshold of hearing is 0 decibels. Table 1 gives levels in decibels and intensities in watts per meter squared for some familiar sounds.

Check Your Understanding: Sound Intensity and Sound Level

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1. Several government agencies and health-related professional associations recommend that 85 dB not be exceeded for 8-hour daily exposures in the absence of hearing protection.

Wave Interference (PhET Simulation)

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Sound Waves and Their Sources (1933 movie clip)

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Acoustics and Your Environment: The Basics of Sounds and Highway Traffic Noise (48 min)

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DESCRIBE THE DOPPLER EFFECT IN THE CONTEXT OF WAVES EMANATING FROM A MOVING SOURCE

The characteristic sound of a motorcycle buzzing by is an example of the **Doppler effect**. The high-pitch scream shifts dramatically to a lower-pitch roar as the motorcycle passes by a stationary observer. The closer the motorcycle brushes by, the more abrupt the shift. The faster the motorcycle moves, the greater the shift. We also hear this characteristic shift in frequency for passing race cars, airplanes, and trains.

The Doppler effect is an alteration in the observed frequency of a sound due to motion of either the source or the observer. Although less familiar, this effect is easily noticed for a stationary source and moving observer. For example, if you ride a train past a stationary warning bell, you will hear the bell's frequency shift from high to low as you pass by. The actual change in frequency due to relative motion of source and observer is called a **Doppler shift**. The Doppler effect and Doppler shift are named for the Austrian physicist and mathematician Christian Johann Doppler (1803–1853), who did experiments with both moving sources and moving observers. Doppler, for example, had musicians play on a moving open train car and also play standing next to the train tracks as a train passed by. Their music was observed both on and off the train, and changes in frequency were measured.

What causes the Doppler shift? Figures 9, 10, and 11 compare sound waves emitted by stationary and moving sources in a stationary air mass. Each disturbance spreads out spherically from the point where the sound was emitted. If the source is stationary, then all of the spheres representing the air compressions in the sound wave centered on the same point, and the stationary observers on either side see the same wavelength and frequency as emitted by the source, as seen in Figure 9. If the source is moving, as in Figure 10, then the situation is difference. Each compression of the air moves out in a sphere from the point where it was emitted, but the point of emission moves. This moving emission point causes the air compressions to be closer together on one side and farther apart on the other. Thus, the wavelength is shorter in the direction the source is moving (on the right in Figure 10), and longer in the opposite direction (on the left in Figure 10). Finally, if the observers move, as in Figure 11, the frequency at which they receive the compressions changes. The observer moving toward the source receives them at a higher frequency, and the person moving away from the source receives them at a lower frequency.

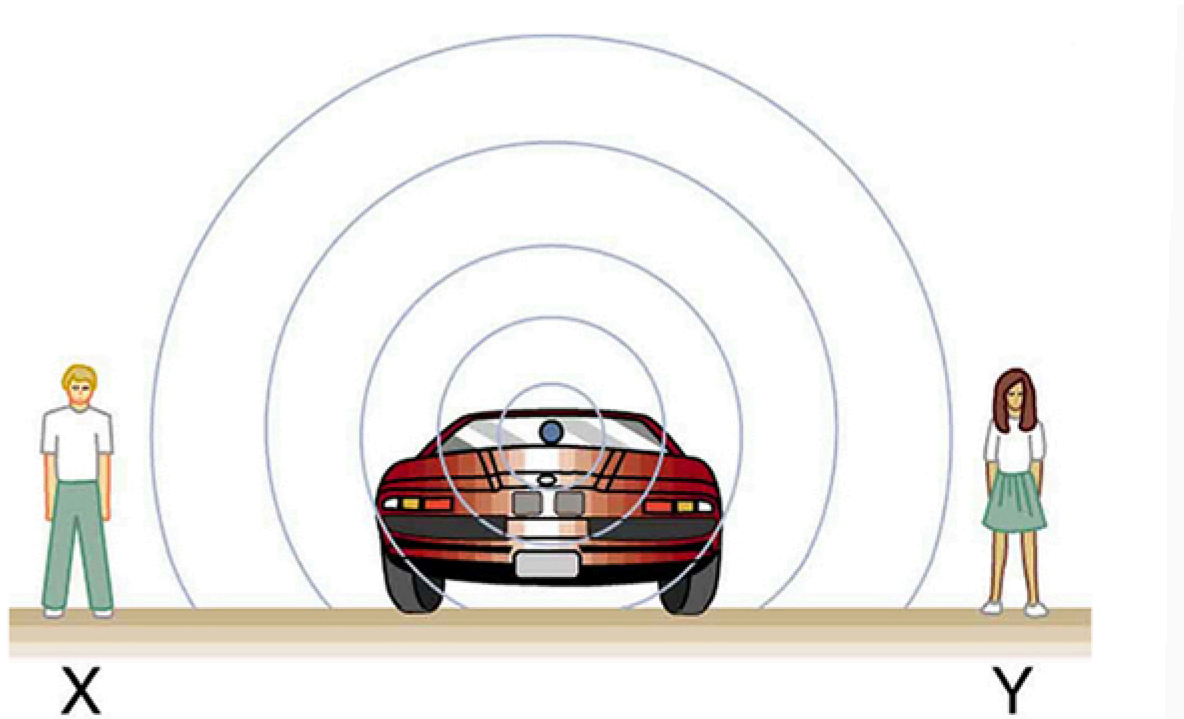

Figure 9: Sounds emitted by a source spread out in spherical waves. Because the source, observers, and air are stationary, the wavelength and frequency are the same in all directions and to all observers.

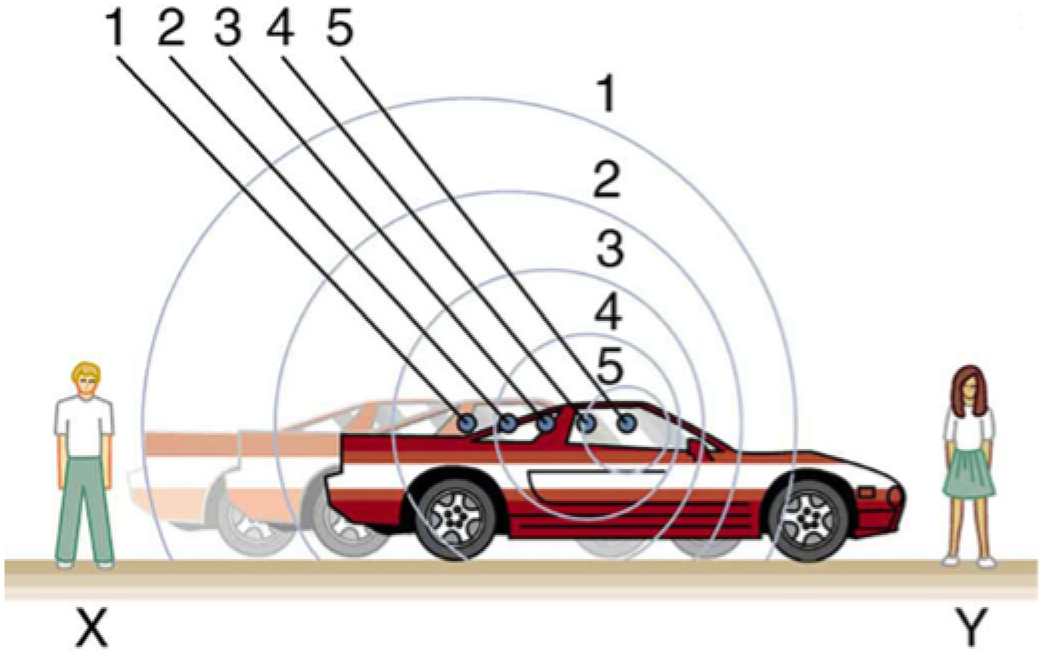

Figure 10: Sounds emitted by a source moving to the right spread out from the points at which they were emitted. The wavelength is reduced and, consequently, the frequency is increased in the direction of motion, so that the observer on the right hears a higher-pitch sound. The opposite is true for the observer on the left, where the wavelength is increased, and the frequency is reduced.

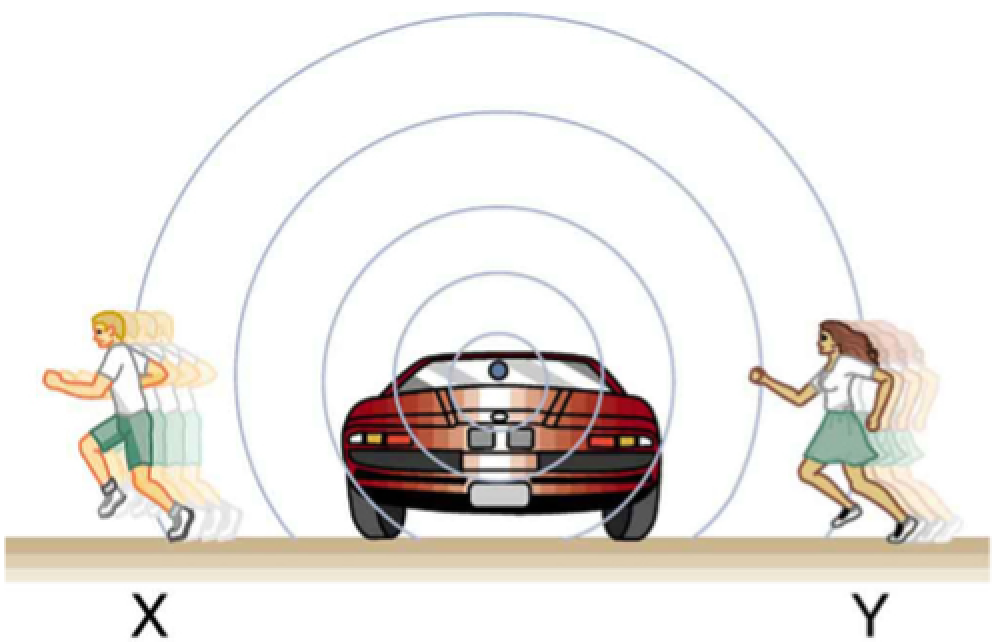

Figure 11: The same effect is produced when the observers move relative to the source. Motion toward the source increases frequency as the observer on the right passes through more wave crests than she would if stationary. Motion away from the source decreases frequency as the observer on the left passes through fewer wave crests, he would if stationary.

We know that wavelength and frequency are related by $v_w = f\lambda$, where v_w is the fixed speed of sound. The sound moves in a medium and has the same speed v_w in that medium whether the source is moving or not. The *f* multiplied by λ is a constant. Because the observer on the fight in the Figure 10 receives a shorter wavelength, the frequency she receives must be higher. Similarly, the observer on the left receives a longer wavelength, and hence he hears a lower frequency. The same thing happens in Figure 11. A higher frequency is received by the observer moving toward the source, and a lower frequency is received by an observer moving away from the source. In general, then, relative motion of source and observer toward one another increases the received frequency. Relative motion apart decreases frequency. The greater the relative speed is, the greater the effect.

What is the Doppler Effect?

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Check Your Understanding: What is the Doppler Effect?

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IDENTIFY THE CAUSES OF NOISE FROM THE TRANSPORTATION SYSTEM

In this section, you will learn about the causes of transportation system noise. To gain a better understanding of the many different noises associated with our environment, please read the article "Traffic Noise and Transportation" by The Center for Environmental Excellence by the American Association of State Highway and Transportation Officials (AASHTO).

Adsorptive Sound Wall Systems

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RELEVANCE TO TRANSPORTATION ENGINEERING COURSEWORK

In this section, you will learn about spot speed study and speed limit enforcement helping ensure that roadway design is consistent with speed on roadways and highways. You will also learn about sustainable transportation planning and development.

Spot speed study and speed limit enforcement

The information from spot speed studies is critical for the enforcement and implementation of speed management strategies if needed. A Doppler radar, either hand-held, vehicle-mounted, or poll-mounted, has been one of the most commonly used tools for speed limit enforcement. These devices are also useful in conducting spot speed studies to ensure that the roadway design is consistent with the desired speed for the street and highways. Newer Lidar-based speed measurement also works based on the Doppler Effect discussed in this chapter.

Transportation Planning and Sustainable Development

As we learn about the ill effects of noise pollution, it becomes a vital part of the environmental impact assessments and mitigation for transportation projects. It remains a critical equity concern since, historically, lower-income and minoritized communities bear the brunt of the impact of traffic-related noise. Traffic noise barriers are one of the key mitigation measures that manipulate sound waves to reduce their intensity and help minimize their impact. Noise barriers reduce the sound entering a community by either absorbing/reflecting the sound of traffic or forcing the waves to take an alternate path over and around the obstacle. The design of noise barriers is based on the wave propagation principles outlined in this chapter.

Key Takeaways

- The technology used for spot speed studies and speed limit enforcement is based on the Doppler effect in wave propagation. Spot speed studies are critical to ensure that the roadway design context aligns with the speed limit posted on the roadway segment.
- The design of noise barriers to protect communities surrounding freeways and railroads from noise pollution is based on the wave propagation principles discussed in this chapter. It has been an equity issue since marginalized lower-income communities have historically borne the most impact of roadway noise pollution.

GLOSSARY: KEY TERMS

Amplitude[1] – the maximum displacement from the equilibrium position of an object oscillating around the equilibrium position
Doppler effect[1] – an alteration in the observed frequency of a sound due to motion of either the source or the observer
Doppler shift[1] – the actual change in frequency due to relative motion of source and observer
Frequency[1] – number of events per unit of time

Hearing[1] – the perception of sound
Intensity[1] – power per unit area; sounds below 20 Hz
Longitudinal Wave[1] – a wave in which the disturbance is parallel to the direction of propagation
Loudness[1] – the perception of sound intensity
Natural Frequency[1] – the frequency at which a system would oscillate if there were no driving and no damping forces
Period[1] – time it takes to complete one oscillation
Pitch[1] – the perception of the frequency of a sound
Sound[1] – a disturbance of matter that is transmitted from its source outward
Sound Intensity Level[1] – a unitless quantity telling you the level of the sound relative to a fixed standard
Sound Pressure Level[1] – the ratio of the pressure amplitude to a reference pressure
Tone[1] – number and relative intensity of multiple sound frequencies
Traverse Wave[1] – a wave in which the disturbance is perpendicular to the direction of propagation
Wave[1] – a disturbance that moves from its source and carries energy
Wave Velocity[1] – the speed at which the disturbance moves. Also called the propagation velocity or propagation speed
Wavelength[1] – the distance between adjacent identical parts of a wave

[1] "College Physics for AP® Courses" by Greg Wolfe, Erika Gasper, John Stoke, Julie Kretchman, David Anderson, Nathan Czuba, Sudhi Oberoi, Liza Pujji, Irina Lyublinskaya, Douglas Ingram. Access for free at https://openstax.org/books/college-physics-ap-courses/pages/1-connection-for-ap-r-courses

MEDIA ATTRIBUTIONS

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Videos

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Figures

- Figure 1: "Chapter 16: Oscillatory Motion and Waves" by Greg Wolfe, Erika Gasper, John Stoke, Julie Kretchman, David Anderson, Nathan Czuba, Sudhi Oberoi, Liza Pujji, Irina Lyublinskaya, and Douglas Ingram is licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)
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- Figure 9: "Chapter 17: Physics of Hearing" by Greg Wolfe, Erika Gasper, John Stoke, Julie Kretchman, David Anderson, Nathan Czuba, Sudhi Oberoi, Liza Pujji, Irina Lyublinskaya, and Douglas Ingram is licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)
- Figure 10: "Chapter 17: Physics of Hearing" by Greg Wolfe, Erika Gasper, John Stoke, Julie

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- Figure 11: "Chapter 17: Physics of Hearing" by Greg Wolfe, Erika Gasper, John Stoke, Julie Kretchman, David Anderson, Nathan Czuba, Sudhi Oberoi, Liza Pujji, Irina Lyublinskaya, and Douglas Ingram is licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)

Tables

- Table 1: Garnett, A. (2022, April 26). Traffic noise overview.Center for Environmental Excellence | AASHTO. Retrieved July 1, 2022, from https://environment.transportation.org/education/environmental-topics/traffic-noise/traffic-noise-overview/

Simulations

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- Simulation 2: "Wave Interference" by PhET Simulations is licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)

REFERENCES

"Traffic Noise and Transportation" by The Center for Environmental Excellence by the American Association of State Highway and Transportation Officials (AASHTO).

CHAPTER 8: PROBABILITY: BASIC PRINCIPLES AND DISTRIBUTIONS

This chapter discusses understanding the basic principles of probability because transportation system operations and planning are critically dependent on these basic principles. Several processes are modeled using probability distributions for real-valued random variables. These distributions include normal distribution for the speed of vehicles on the road, Poisson's distribution for gaps in traffic on an uncongested facility, or negative binomial as a distribution for crash frequency on a roadway segment.

Learning Objectives

At the end of the chapter, the reader should be able to do the following:

- Use basic counting techniques (multiplication rule, combinations, permutations) to estimate probability and odds.
- Set up and work with distributions for discrete random variables, including Bernoulli, binomial, geometric, and Poisson distributions.
- Set up and work with distributions for continuous random variables, including uniform, normal and exponential distributions.
- Identify topics in the introductory transportation engineering courses that build on the concepts discussed in this chapter.

USE BASIC COUNTING TECHNIQUES TO ESTIMATE PROBABILITY AND ODDS

This section will explain ways to estimate probability and odds with videos to help your understanding. Also, short problems to check your understanding are included.

Multiplication Rule

Multiplication Rule

The **multiplication rule** says: If there are *n* ways to perform action 1 and then by *m* ways to perform action 2, then there are $n \cdot m$ ways to perform action 1 followed by action 2.

Count Outcomes Using Tree Diagram

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Check Your Understanding: Multiplication Rule

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Permutations

Permutations

A **permutation** of a set is a particular ordering of its elements. For example, the set $\{a, b, c\}$ has six permutations: *abc, acb, bac, bca, cab, cba*. We found the number of permutations by listing them all. We could have also found the number of permutations by using the multiplication rule. That is, there are 3 ways to pick the first element, then 2 ways for the second, and 1 for the third. This gives a total of $3 \cdot 2 \cdot 1 = 6$ permutations.

In general, the multiplication rule tells us that the number of permutations of a set of elements is

$$k! = k \cdot (k-1) \cdots 3 \cdot 2 \cdot 1$$

We also talk about the permutations of *k* things out of a set of *n* things.

Example: List all the permutations of 3 elements out of the set $\{a, b, c, d\}$

Table 1

acd	*adc*	*cad*	*cda*	*dac*	*dca*
abc	***acb***	***bac***	***bca***	***cab***	***cba***
abd	*adb*	*bad*	*bda*	*dab*	*dba*
bcd	*bdc*	*cbd*	*cdb*	*dbc*	*dcb*

Note that *abc* and *acb* count as distinct permutations. That is, for permutations the order matters.

There are 24 permutations. Note that the multiplication rule would have told us there are $4 \cdot 3 \cdot 2 = 24$ permutations without bothering to list them all.

Permutation Formula

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Zero Factorial

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Ways to Pick Officers – Example

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Check Your Understanding: Permutations

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Combinations

Combinations

In contrast to permutations, in combinations order does not matter: permutations are lists and combinations are sets.

Example: List all the combinations of 3 elements out of the set $\{a, b, c, d\}$.

Answer: Such a combination is a collection of 3 elements without regard to order. So, *abc* and *cab* both represent the same combination. We can list all the combinations by listing all the subsets of exactly 3 elements.

$\{a, b, c\} \quad \{a, b, d\} \quad \{a, c, d\} \quad \{b, c, d\}$

There are only 4 combinations. Contrast this with the 24 permutations in the previous example. The factor of 6 comes because every combination of 3 things can be written in 6 different orders.

Introduction to Combinations

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Combination Formula

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Combination Example: 9 Card Hands

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Check Your Understanding: Combinations

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Permutations and Combinations Comparison

We will use the following notations.

${}_nP_k$= number of permutations (list) of *k* distinct elements from a set of size *n*

${}_nC_k = \left(\begin{array}{c} n \\ k \end{array}\right)$ = number of combinations (subsets) of *k* elements from a set of size *n*

We emphasize that by the number of combinations of *k* elements we mean the number of subsets of size *k*.

These have the following notation and formulas:

Permutations: ${}_nP_k = \frac{n!}{(n-k)!} = n(n-1)\ldots(n-k+1)$

Combinations: ${}_nC_k = \frac{n!}{k!(n-k)!} = \frac{{}_nP_k}{k!}$

The notation ${}_nC_k$ is read "*n* choose *k*". The formula for ${}_nP_k$ follows from the multiplication rule. It also implies the formula for ${}_nC_k$ because a subset of size *k* can be ordered in *k!* ways.

We can illustrate the relationship between permutations and combinations by lining up the results of the previous two examples.

Permutations: ${}_4P_3$

Table 2

abc	***acb***	***bac***	***bca***	***cab***	***cba***
abd	*adb*	*bad*	*bda*	*dab*	*dba*
acd	*adc*	*cad*	*cda*	*dac*	*dca*
bcd	*bdc*	*cbd*	*cdb*	*bdc*	*dcb*

Combinations: ${}_4C_3$

$\{a, b, c\}$

$\{a, b, d\}$

$\{a, c, d\}$

$\{b, c, d\}$

Notice that each row in the permutations list consists of all 3! permutations of the corresponding set in the combinations list.

Check Your Understanding: Permutations and Combinations Comparison

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Probability Using the Rules

The General Multiplication Rule

When we calculate probabilities involving one event AND another event occurring, we multiply their probabilities.

In some cases, the first event happening impacts the probability of the second event. We call these **dependent events.**

In other cases, the first event happening does not impact the probability of the seconds. We call these **independent events.**

Independent events: Flipping a coin twice

What is the probability of flipping a fair coin and getting "heads" twice in a row? That is, what is the probability of getting heads on the first flip AND heads on the second flip?

Imagine we had 100 people simulate this and flip a coin twice. On average, 50 people would get heads on the first flip, and then 25 of them would get heads again. So, 25 out of the original 100 people – or 1/4 of them – would get heads twice in a row.

The number of people we start with does not really matter. Theoretically, 1/2 of the original group will get heads, and 1/2 of that group will get heads again. To find a fraction of a fraction, we multiply.

We can represent this concept with a tree diagram like the one shown below in Figure 1.

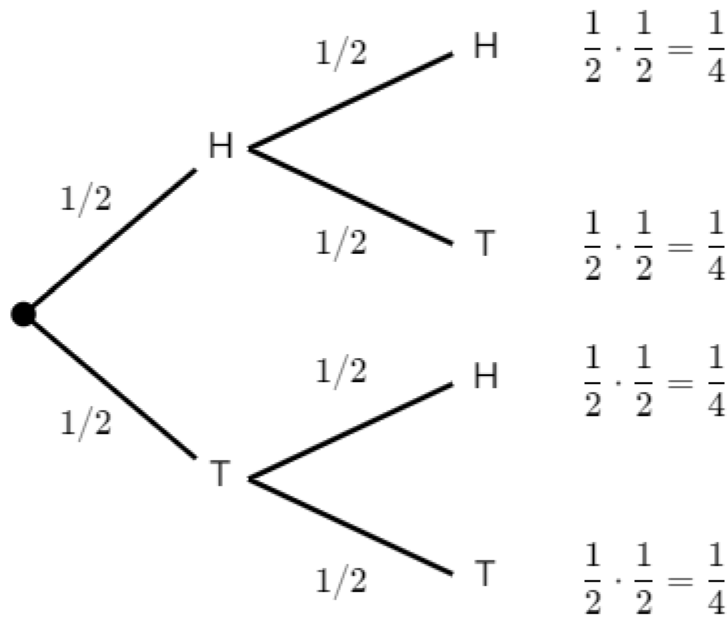

Figure 1

We multiply the probabilities along the branches to find the overall probability of one event AND the next event occurring.

For example, the probability of getting two "tails" in a row would be:

$$P(T \text{ and } T) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

When two events are **independent**, we can say that

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Be careful! This formula only applies to independent events.

Dependent events: Drawing cards

We can use a similar strategy even when we are dealing with dependent events.

Consider drawing two cards, without replacement, from a standard deck of 52 cards. That means we are drawing the first card, leaving it out, and then drawing the second card.

What is the probability that both cards selected are black?

Half of the 52 cards are black, so the probability that the first card is black is 26/52. But the probability of getting a black card changes on the next draw, since the number of black cards and the total number of cards have both been decreased by 1.

Here is what the probabilities would look like in a tree diagram:

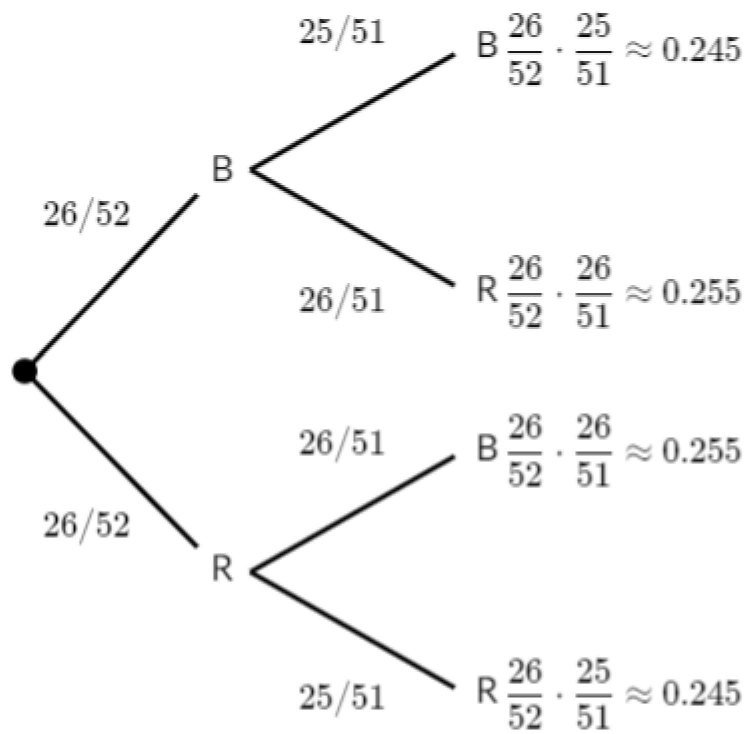

Figure 2

So, the probability that both cards are black is:

$P(\text{ both black }) = \frac{26}{52} \cdot \frac{25}{51} \approx 0.245$

The General Multiplication Rule

For any two events, we can say that

$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$

The vertical bar in $P(B \mid A)$ means "given," so this could also be read as "the probability that B occurs *given* that A has occurred."

This formula says that we can multiply the probabilities of two events, but we need to take the first event into account when considering the probability of the second event.

If the events are independent, one happening does not impact the probability of the other, and in that case, $P(B \mid A) = P(B)$.

Check Your Understanding: Probability Using the Rules

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Probability Using Combinations

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Probability and Combinations

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Conditional Probability and Bayes' Theorem

Conditional Probability

Conditional probability answers the questions 'how does the probability of an event change if we have extra information.'

Example: Toss a fair coin 3 times.

What is the probability of 3 heads?

Answer: Sample space = $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. All outcomes are equally likely, so $P(3 \text{ heads }) = \frac{1}{8}$

Suppose we are told that the first toss was heads. Given this information how should we compute the probability of 3 heads?

Answer: We have a new (reduced) sample space = $\{HHH, HHT, HTH, HTT\}$. All outcomes are equally likely, so $P(3 \text{ heads given that the first toss is heads }) = \frac{1}{4}$

This is called **conditional probability**, since it takes into account additional conditions. To develop the notation, we rephrase (b) in terms of events.

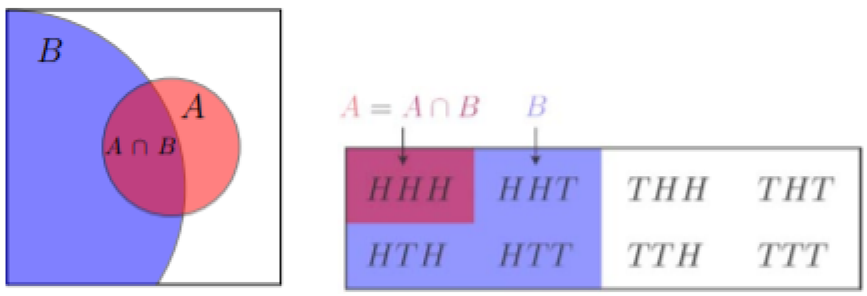
Rephrased (b): Let *A* be the event 'all three tosses are heads' = $\{HHH\}$. Let *B* be the event 'the first toss is heads' = $\{HHH, HHT, HTH, HTT\}$.

The conditional probability of *A* knowing that *B* occurred is written

$$P(A \mid B)$$

This is read as 'the conditional probability of *A* given *B* ' **OR** 'the probability of *A* conditioned on *B* ' **OR simply** 'the probability of *A* given *B*.'

We can visualize conditional probability as follows. Think of $P(A)$ as the proportion of the area of the *whole* sample space taken up by *A*. For $P(A \mid B)$ we restrict our attention to *B*. That is, $P(A \mid B)$ is the proportion of area of *B* taken up by *A*, i.e., $P(A \cap B)/P(B)$.



Conditional probability: Abstract visualization and coin example

Figure 3

The formal definition of conditional probability catches the gist of the above example and visualization.

Formal definition of conditional probability

Let *A* and *B* be events. We define the conditional probability of *A* given *B* as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) \neq 0$$

Let us redo the coin-tossing example using the definition in the equation above. Recall *A* = *3 heads* and *B* = *first toss is heads*. We have $P(A) = 1/8$ and $P(B) = 1/2$. Since $A \cap B = A$, we also have $P(A \cap B) = 1/8$. Now according to the equation, $P(A \mid B) = \frac{1/8}{1/2} = \frac{1}{4}$ which agrees with the answer in Example B.

Bayes' Theorem

Bayes' theorem is a pillar of both probability and statistics. For two events *A* and *B* Bayes' theorem says

$P(B \mid A) = \frac{P(A|B) \cdot P(B)}{P(A)}$

Bayes' rule tells us how to 'invert' conditional probabilities, i.e., to find $P(B \mid A)$ from $P(A \mid B)$.

Proof of Bayes' Theorem

The key point is that $A \cap B$ is symmetric in *A* and *B*. So, the multiplication rule says $P(B \mid A) \cdot P(A) = P(A \cap B) = P(A \mid B) \cdot P(B)$.

Now divide through by *P (A)* to get Bayes' rule.

A common mistake is to confuse $P(A \mid B)$ and $P(B \mid A)$. They can be very different. This is illustrated in the next example.

Example: Toss a coin 5 times. Let $H_1 =$ first toss is heads and let $H_A =$ all 5 tosses are heads. Then $P\left(H_1 \mid H_A\right) = 1$ but $P\left(H_A \mid H_1\right) = 1/16$.

For practice, let us use Bayes' theorem to compute $P\left(H_1 \mid H_A\right)$ using $P\left(H_A \mid H_1\right)$. The terms are $P\left(H_A \mid H_1\right) = 1/16$, $P\left(H_1\right) = 1/2$, $P\left(H_A\right) = 1/32$. So,

$P\left(H_1 \mid H_A\right) = \frac{P(H_A|H_1)P(H_1)}{P(H_A)} = \frac{(1/16) \cdot (1/2)}{1/32} = 1$

Which agrees with our previous calculation.

Conditional Probability and Combinations

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Check Your Understanding: Conditional Probability and Bayes' Theorem

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SET UP AND WORK WITH DISTRIBUTIONS FOR DISCRETE RANDOM VARIABLES

The following sections will help you become familiar with distributions for discrete and continuous random variables. The videos help explain random, discrete, and continuous variables. Probabilities are also explained through watching the videos in this section. Also, short problems to check your understanding are included.

Random Variables

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Discrete and Continuous Random Variables

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Introduction to Discrete Random Variables

A student takes a ten-question, true-false quiz. Because the student had such a busy schedule, he or she could not study and guesses randomly at each answer. What is the probability of the student passing the test with at least a 70?

Small companies might be interested in the number of long-distance phone calls their employees make during the peak time of the day. Suppose the historical average is 20 calls. What is the probability that the employees make more than 20 long-distance phone calls during the peak time?

These two examples illustrate two different types of probability problems involving discrete random variables. Recall that discrete data are data that you can count, that is, the random variable can only take on whole number values. A **random variable** describes the outcomes of a statistical experiment in words. The values of a random variable can vary with each repetition of an experiment, often called a trial.

Random Variable Notation

The upper-case letter X denotes a random variable. Lowercase letters like x or y denote the value of a random variable. If **X is a random variable, then X is written in words, and x is given as a number.**

For example, let X = the number of heads you get when you toss three fair coins. The sample space for the toss of three fair coins is TTT; THH; HTH; HHT; HTT; THT; TTH; HHH. Then, x = 0, 1, 2, 3. X is in words and x is a number. Notice that for this example, the x values are countable outcomes. Because you can count the possible values as whole numbers that X can take on and the outcomes are random (the x values 0, 1, 2, 3), X is a discrete random variable.

Probability Density Functions (PDF) for a Random Variable

A probability density function or **probability distribution function** has two characteristics:

1. Each probability is between zero and one, inclusive.
2. The sum of the probabilities is one.

A probability density function is a mathematical formula that calculates probabilities for specific types of events. There is a sort of magic to a probability density function (Pdf) partially because the same formula often describes very different types of events. For example, the binomial Pdf will calculate probabilities for flipping coins, yes/no questions on an exam, opinions of voters in an up or down opinion poll, and indeed any binary event. Other probability density functions will provide probabilities for the time until a part will fail, when a customer will arrive at the turnpike booth, the number of telephone calls arriving at a central switchboard, the growth rate of a bacterium, and on and on. There are whole families of probability density functions that are used in a wide variety of applications, including medicine, business and finance, physics, and engineering, among others.

Counting Formulas and the Combinational Formula

To repeat, the probability of event A, P(A), is simply the number of ways the experiment will result in A, relative to the total number of possible outcomes of the experiment.

As an equation this is:

$$P(A) = \frac{\text{number of ways to get } A}{\text{Total number of possible outcomes}}$$

When we looked at the sample space for flipping 3 coins, we could easily write the full sample space and thus could easily count the number of events that met our desired result, e.g., x = 1, where X is the random variable defined as the number of heads.

As we have larger numbers of items in the sample space, such as a full deck of 52 cards, the ability to write out the sample space becomes impossible.

We see that probabilities are nothing more than counting the events in each group we are interested in and dividing by the number of elements in the universe, or sample space. This is easy enough if we are counting sophomores in a Stat class, but in more complicated cases listing all the possible outcomes may take a lifetime. There are, for example, 36 possible outcomes from throwing just two six-sided dice where the random variable is the sum of the number of spots on the up-facing sides. If there were four dice, then the total number of possible outcomes would become 1,296. There are more than 2.5 MILLION possible 5-card poker hands in a standard deck of 52 cards. Obviously keeping track of all these possibilities and counting them to get at a single probability would be tedious at best.

An alternative to listing the complete sample space and counting the number of elements we are interested in is to skip the step of listing the sample space, and simply figure out the number of elements in it and do the appropriate division. If we are after a probability, we really do not need to see each and every element in the sample space, we only need to know how many elements are there. Counting formulas were invented to do just this. They tell us the number of unordered subsets of a certain size that can be created from a set of unique elements. By unordered it is meant that, for example, when dealing cards, it does not matter if you got {ace, ace, ace, ace, king} or {king, ace, ace, ace, ace} or {ace, king, ace, ace, ace} and so on. Each of these subsets are the same because they each have 4 aces and one king.

Combinational Formula (Review)

$$\left(\begin{array}{c} n \\ x \end{array}\right) = {}_nC_x = \frac{n!}{x!(n-x)!}$$

It is also sometimes referred to as the Binomial Coefficient.

Let us find the hard way the total number of combinations of the four aces in a deck of cards if we were going to take them two at a time. The sample space would be:

S={(Spade,
Heart),(Spade,Diamond),(Spade,Club),(Diamond,Club),(Heart,Diamond),(Heart,Club)}

There are 6 combinations; formally, six unique unordered subsets of size 2 that can be created from 4 unique elements. To use the combinatorial formula, we would solve the formula as follows:

$$\left(\begin{array}{c} 4 \\ 2 \end{array}\right) = \frac{4!}{2!(4-2)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1} = 6$$

If we wanted to know the number of unique 5-card poker hands that could be created from a 52-card deck, we simply compute:

$$\left(\begin{array}{c} 52 \\ 5 \end{array}\right)$$

where 52 is the total number of unique elements from which we are drawing and 5 is the size group we are putting them into.

With the combinatorial formula we can count the number of elements in a sample space without having to write each one of them down, truly a lifetime's work for just the number of 5 card hands from a deck of 52 cards.

Remember, a probability density function computes probability for us. We simply put the