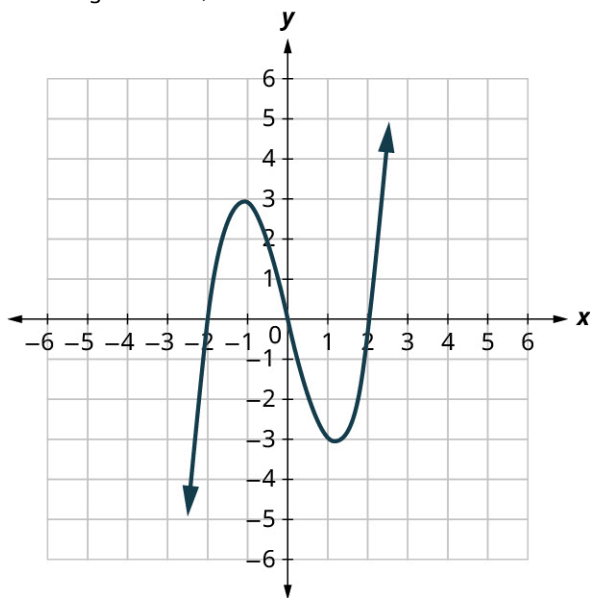
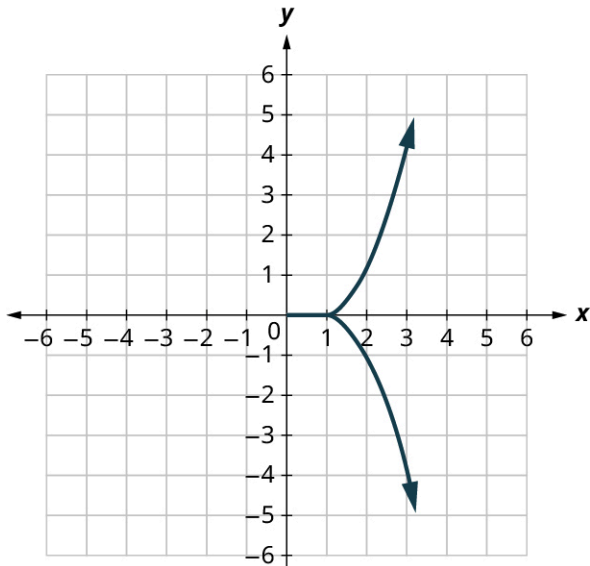


For the following exercises, determine whether the equations represent  $y$  as a function of  $x$ .

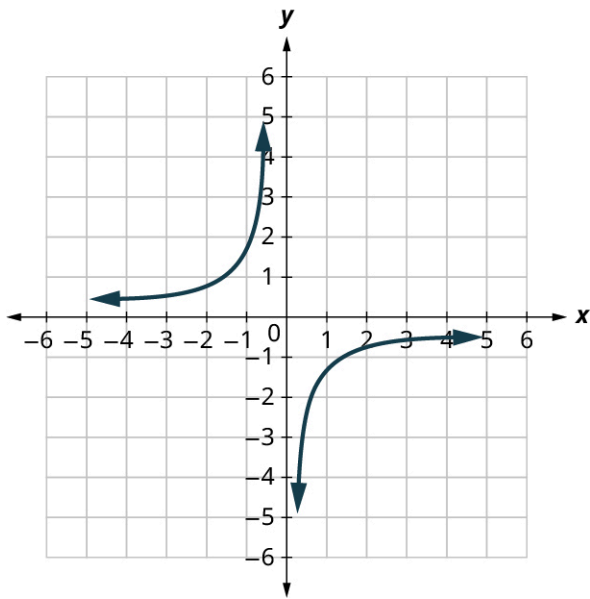
14.  $5x + 2y = 10$
15.  $y = x^2$
16.  $x = y^2$
17.  $3x^2 + y = 14$
18.  $2x + y^2 = 6$
19.  $y = -2x^2 + 40x$

For the following exercises, use the vertical line test to determine which graph represents a function.

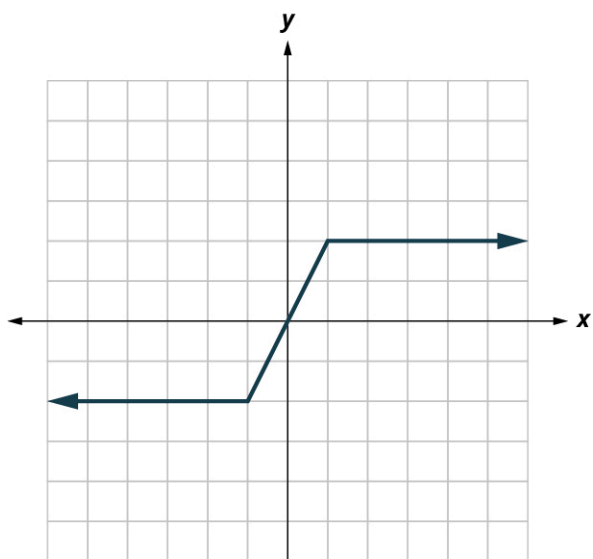




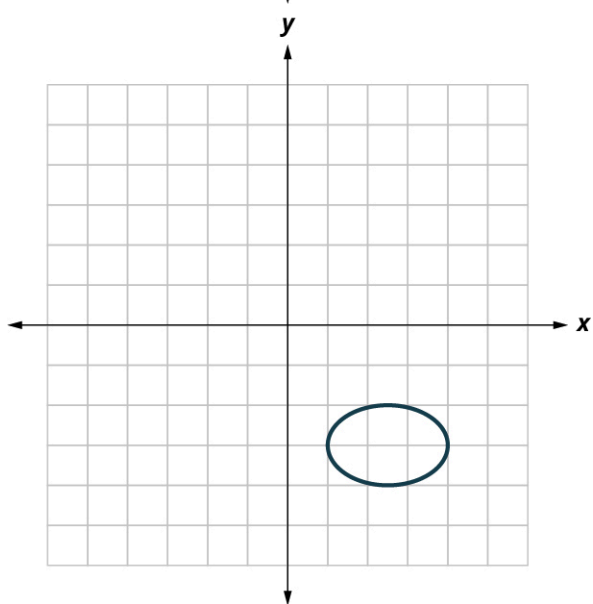
21.



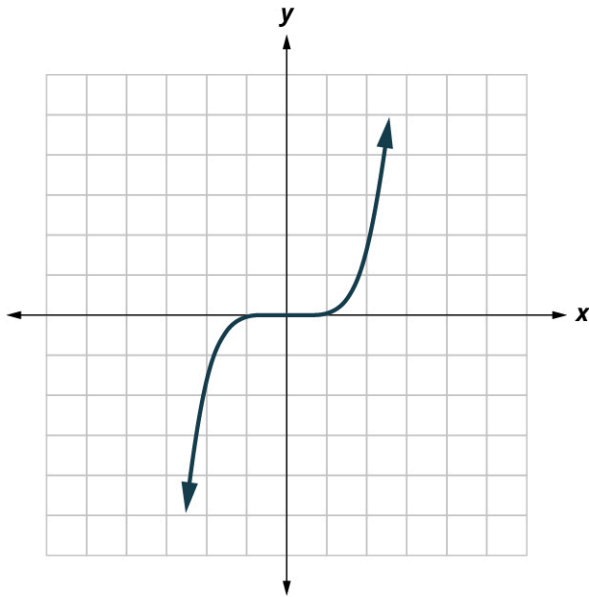
22.



23.



24.

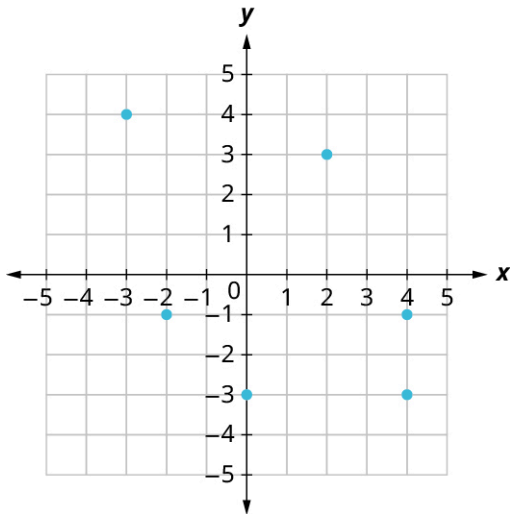


25.

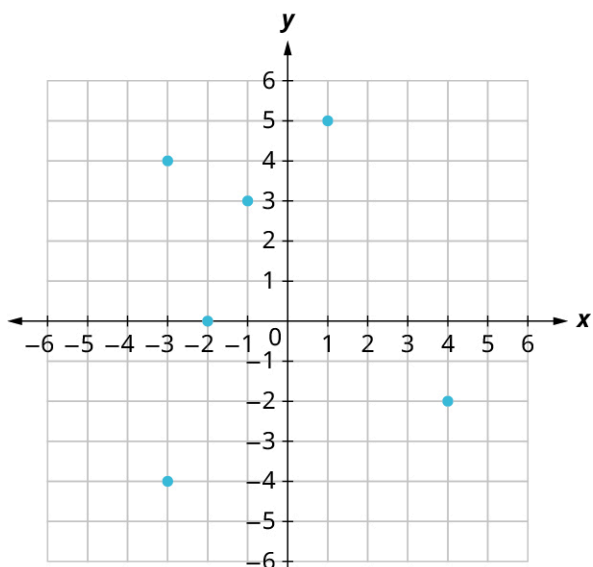
For the following exercises, use the set of ordered pairs to find the domain and the range.

26.  $\{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$   
 27.  $\{(-9, -3), (-4, -2), (-1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$   
 28.  $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$   
 29.  $\{(-3, -27), (-2, -8), (-1, -1), (0, 0), (1, 1), (2, 8), (3, 27)\}$

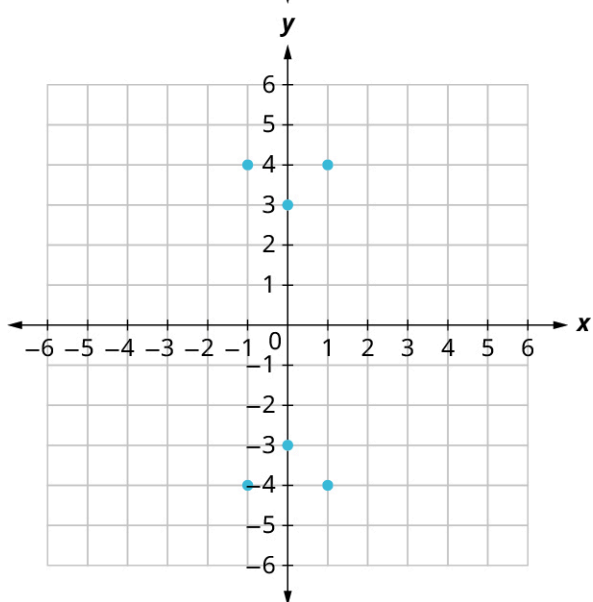
For the following exercises, use the graph to find the domain and the range.



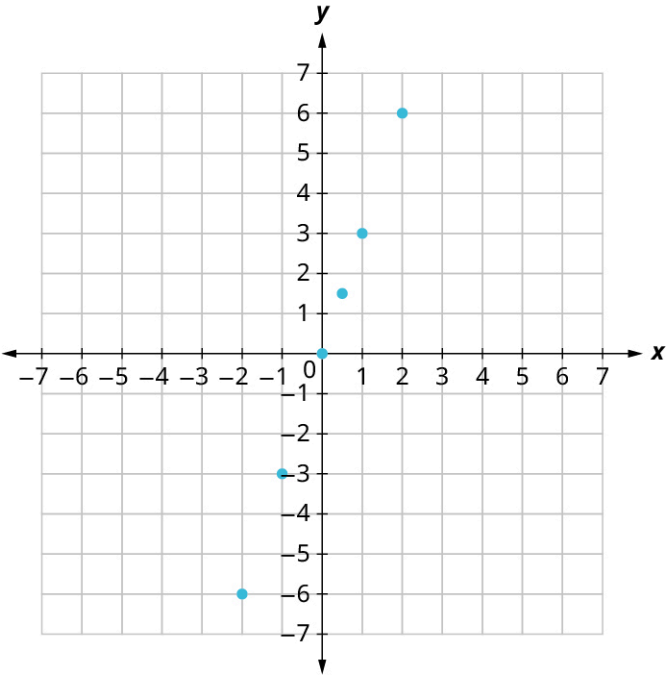
30.



31.



32.



33.

## 5.8 Graphing Functions



**Figure 5.64** The ski lifts and the mountain both have a slope. (credit: modification of work "colorado springs zoo tram" by woodleywonderworks/Flickr, CC BY 2.0)

### Learning Objectives

After completing this module, you should be able to:

1. Graph functions using intercepts.
2. Compute slope.
3. Graph functions using slope and  $y$ -intercept.
4. Graph horizontal and vertical lines.
5. Interpret graphs of functions.
6. Model applications using slope and  $y$ -intercept.

In this section, we will expand our knowledge of graphing by graphing linear functions. There are many real-world scenarios that can be represented by graphs of linear functions. Imagine a chairlift going up at a ski resort. The journey a skier takes travelling up the chairlift could be represented as a linear function with a positive slope. The journey a skier takes down the slopes could be represented by a linear function with a negative slope.

### Graphing Functions Using Intercepts

Every linear equation can be represented by a unique line that shows all the solutions of the equation. We have seen that when graphing a line by plotting points, you can use any three solutions to graph. This means that two people graphing the line might use different sets of three points. At first glance, their two lines might not appear to be the same, since they would have different points labeled. But if all the work was done correctly, the lines should be exactly the same. One way to recognize that they are indeed the same line is to look at where the line crosses the  $x$ -axis and the  $y$ -axis. These points are called the **intercepts** of a line. Let us review the graphs of the lines in [Figure 5.65](#).

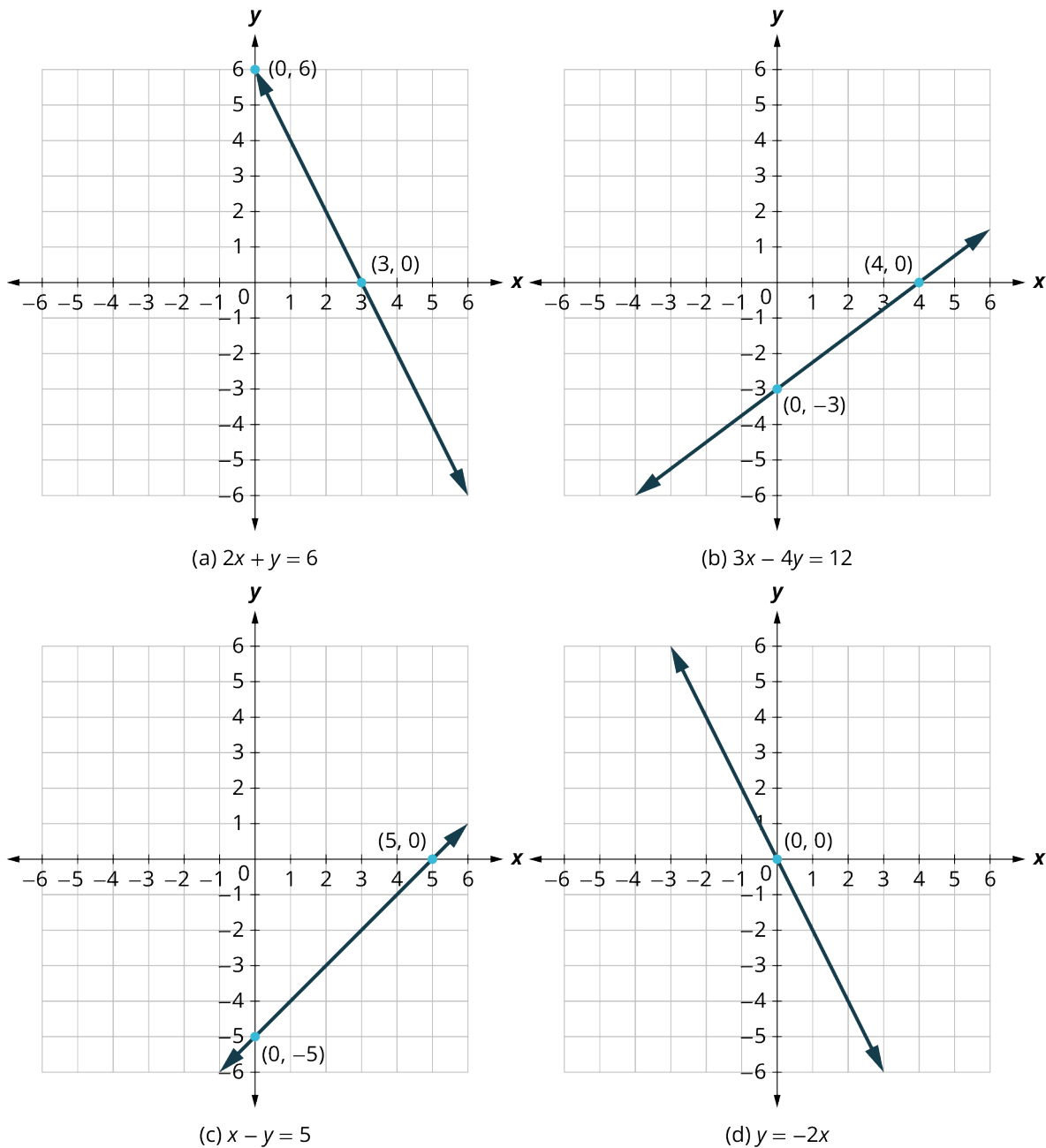


Figure 5.65

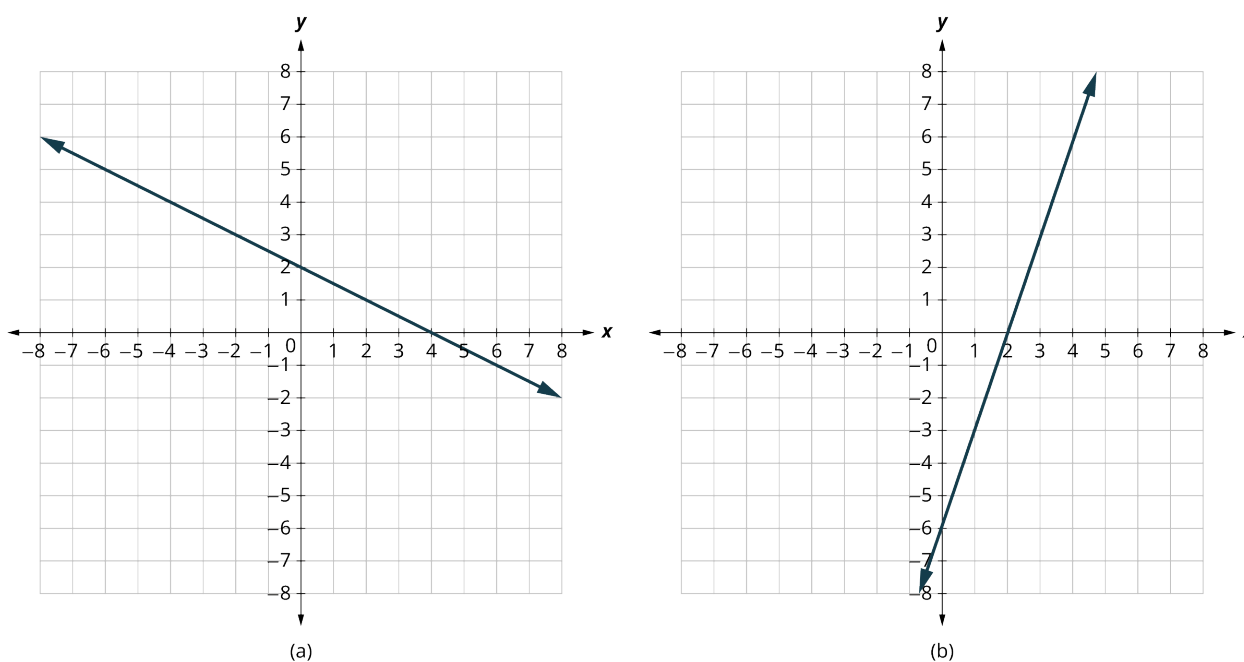
The table below lists where each of these lines crosses the  $x$ - and  $y$ -axis. Do you see a pattern? For each line, the  $y$ -coordinate of the point where the line crosses the  $x$ -axis is zero. The point where the line crosses the  $x$ -axis has the form  $(a, 0)$  and is called the  $x$ -intercept of the line. The  $x$ -intercept occurs when  $y$  is zero. In each line, the  $x$ -coordinate of the point where the line crosses the  $y$ -axis is zero. The point where the line crosses the  $y$ -axis has the form  $(0, b)$  and is called the  $y$ -intercept of the line. The  $y$ -intercept occurs when  $x$  is zero.

Figure	The line crosses the $x$ -axis at:	Ordered Pair for this Point	The line crosses the $y$ -axis at:	Ordered Pair for This Point
Figure (a)	3	$(3, 0)$	6	$(0, 6)$
Figure (b)	4	$(4, 0)$	-3	$(0, -3)$

Figure	The line crosses the $x$ -axis at:	Ordered Pair for this Point	The line crosses the $y$ -axis at:	Ordered Pair for This Point
Figure (c)	5	$(5, 0)$	$-5$	$(0, -5)$
Figure (d)	0	$(0, 0)$	0	$(0, 0)$
General Figure	$a$	$(a, 0)$	$b$	$(0, b)$

**EXAMPLE 5.70****Finding  $x$ - and  $y$ -Intercepts**

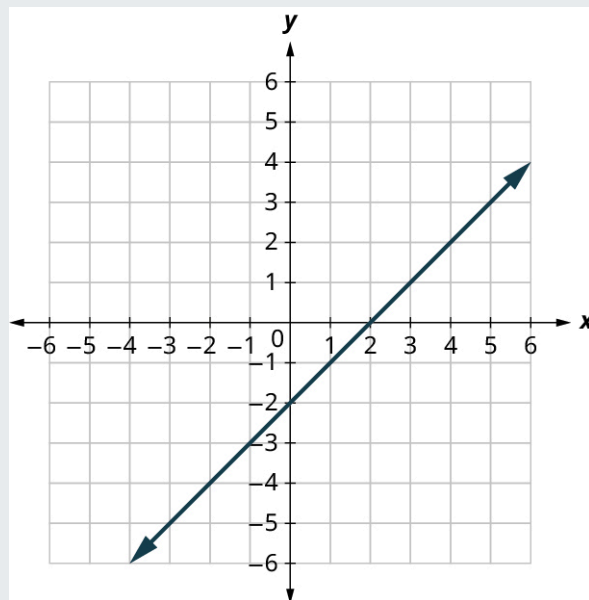
Find the  $x$ -intercept and  $y$ -intercept on the (a) and (b) graphs in [Figure 5.66](#).

**Figure 5.66****✓ Solution**

In [Figure 5.66](#), the graph crosses the  $x$ -axis at the point  $(4, 0)$ . The  $x$ -intercept is  $(4, 0)$ . The graph crosses the  $y$ -axis at the point  $(0, 2)$ . The  $y$ -intercept is  $(0, 2)$ . In [Figure 5.66](#), the graph crosses the  $x$ -axis at the point  $(2, 0)$ . The  $x$ -intercept is  $(2, 0)$ . The graph crosses the  $y$ -axis at the point  $(0, -6)$ . The  $y$ -intercept is  $(0, -6)$ .

**> YOUR TURN 5.70**

1. Find the  $x$ -intercept and  $y$ -intercept on the given graph.

**EXAMPLE 5.71****Graphing a Function Using Intercepts**

Find the intercepts of  $2x + y = 8$ . Then graph the function using the intercepts.

✓ **Solution**

Let  $y = 0$  to find the  $x$ -intercept, and let  $x = 0$  to find the  $y$ -intercept.

	$2x + y = 8$		$2x + y = 8$
To find the $x$ -intercept, let $y = 0$ .	$2x + 0 = 8$	To find the $y$ -intercept, let $x = 0$ .	$2(0) + y = 8$
Simplify.	$2x = 8$ $x = 4$	Simplify.	$y = 8$
The $x$ -intercept is:	$(4, 0)$	The $y$ -intercept is:	$(0, 8)$

Plot the intercepts to get the graph in [Figure 5.67](#).

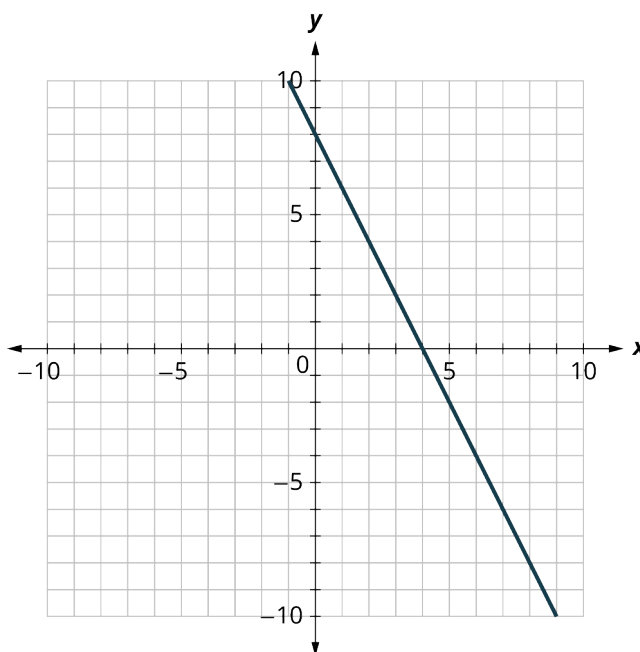


Figure 5.67

### > YOUR TURN 5.71

1. Find the intercepts of  $3x + y = 12$  and use them to graph the equation.

## Computing Slope

When graphing linear equations, you may notice that some lines tilt up as they go from left to right and some lines tilt down. Some lines are very steep and some lines are flatter. In mathematics, the measure of the steepness of a line is called the **slope** of the line. To find the slope of a line, we locate two points on the line whose coordinates are integers. Then we sketch a right triangle where the two points are vertices of the triangle and one side is horizontal and one side is vertical. Next, we measure or calculate the distance along the vertical and horizontal sides of the triangle. The vertical distance is called the **rise** and the horizontal distance is called the **run**.

We can assign a numerical value to the slope of a line by finding the ratio of the rise and run. The rise is the amount the vertical distance changes while the run measures the horizontal change, as shown in this illustration. Slope (Figure 5.68) is a rate of change.

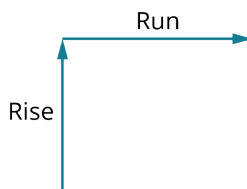


Figure 5.68

### FORMULA

To calculate slope ( $m$ ), use the formula

$$m = \frac{\text{rise}}{\text{run}},$$

where the rise measures the vertical change and the run measures the horizontal change.

The concept of slope has many applications in the real world. In construction, the pitch of a roof, the slant of plumbing pipes, and the steepness of stairs are all applications of slope. As you ski or jog down a hill, you definitely experience slope.

### EXAMPLE 5.72

#### Finding the Slope from a Graph

Find the slope of the line shown in [Figure 5.69](#).

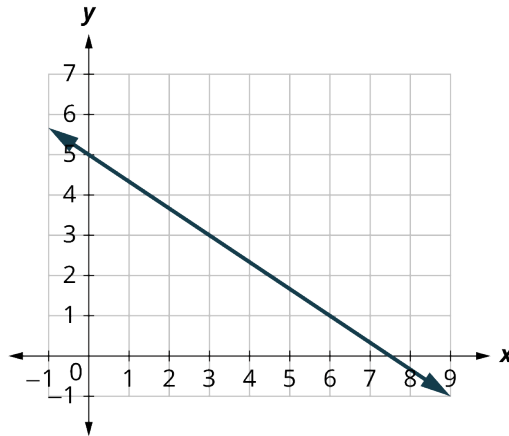


Figure 5.69

#### ✓ Solution

**Step 1:** Locate two points on the graph whose coordinates are integers, such as (0, 5) and (3, 3). Starting at (0, 5), sketch a right triangle to (3, 3) as shown in [Figure 5.70](#).

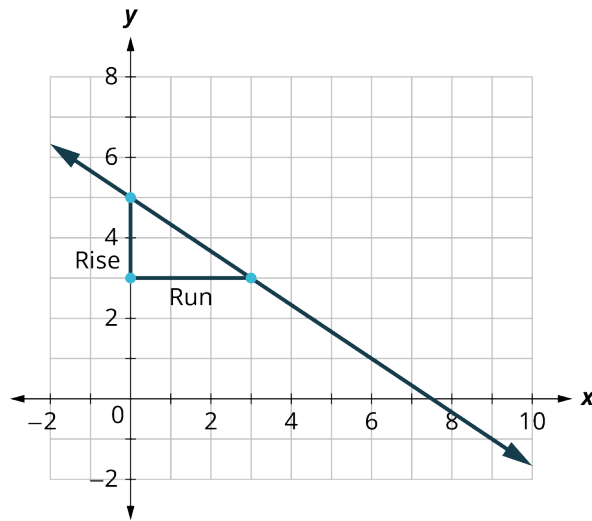


Figure 5.70

**Step 2:** Count the rise; since it goes down, it is negative. The rise is  $-2$ .

**Step 3:** Count the run. The run is 3.

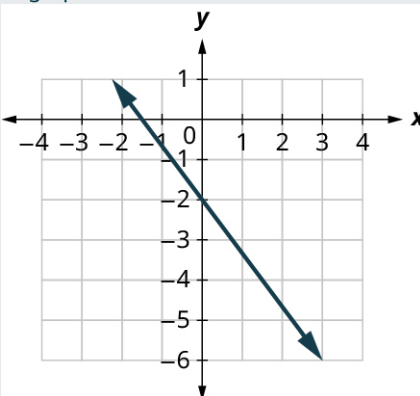
**Step 4:** Use the slope formula  $m = \frac{\text{rise}}{\text{run}}$  substitute the values of the rise and run.  $m = \frac{-2}{3}$

The slope of the line is  $-\frac{2}{3}$ .

The solution is  $y$  decreases by 2 units as  $x$  increases by 3 units.

> YOUR TURN 5.72

1. Find the slope of the line shown in the graph.



Sometimes we will need to find the slope of a line between two points when we don't have a graph to measure the rise and the run. We could plot the points on grid paper, then count out the rise and the run, but there is a way to find the slope without graphing. First, we need to introduce some algebraic notation.

We have seen that an ordered pair  $(x, y)$  gives the coordinates of a point. But when we work with slopes, we use two points. How can the same symbol  $(x, y)$  be used to represent two different points? Mathematicians use subscripts to distinguish such points. For example,  $(x_1, y_1)$  would be said aloud as “x sub 1, y sub 1” and  $(x_2, y_2)$  read “x sub 2, y sub 2.” The “sub” is a short way of saying “subscript.” We will use  $(x_1, y_1)$  to identify the first point and  $(x_2, y_2)$  to identify the second point in our slope equation. If we had more than two points, (if we were finding more than one slope), we could use  $(x_3, y_3)$ ,  $(x_4, y_4)$ , and so on.

Let's review how the rise and run relate to the coordinates of the two points by taking another look at the slope of the line between the points  $(2, 3)$  and  $(7, 6)$ , as shown in [Figure 5.71](#).

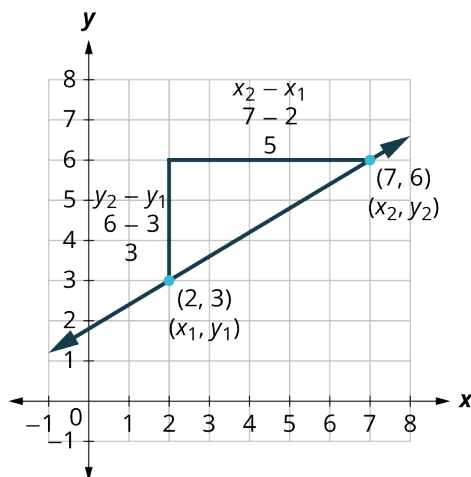


Figure 5.71

On the graph, we count the rise of 3 and the run of 5. Notice on the graph that  $(x_1, y_1)$  is the point  $(2, 3)$  and  $(x_2, y_2)$  is the point  $(7, 6)$ . The rise can be found by subtracting the  $y$ -coordinates, 6 and 3, and the run can be found by subtracting the  $x$ -coordinates 7 and 2.

$$m = \frac{\text{rise}}{\text{run}} = \frac{6 - 3}{7 - 2} = \frac{3}{5}$$

We have shown that  $m = \frac{y_2 - y_1}{x_2 - x_1}$  is really another version of  $m = \frac{\text{rise}}{\text{run}}$ . We can use this formula to find the slope of a line.

**FORMULA**

To find the slope of the line between two points  $(x_1, y_1)$  and  $(x_2, y_2)$ , use the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

**EXAMPLE 5.73****Finding the Slope of the Line Using Points**

Use the slope formula to find the slope of the line through the points  $(-2, -3)$  and  $(-7, 4)$ .

☑ **Solution**

We'll call  $(-2, -3)$  point 1 and  $(-7, 4)$  point 2.

**Step 1:** Use the slope formula:  $m = \frac{y_2 - y_1}{x_2 - x_1}$

**Step 2:** Substitute the values:  $m = \frac{4 - (-3)}{-7 - (-2)}$

**Step 3:** Simplify:  $m = \frac{7}{-5} = -\frac{7}{5}$

**Step 4:** Verify the slope on the graph shown in [Figure 5.72](#).

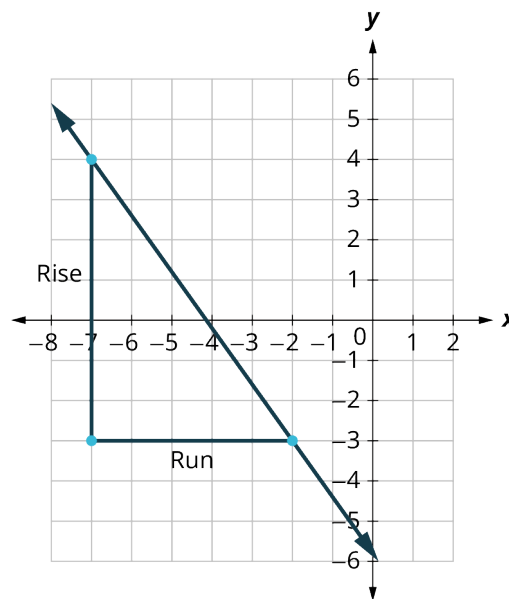


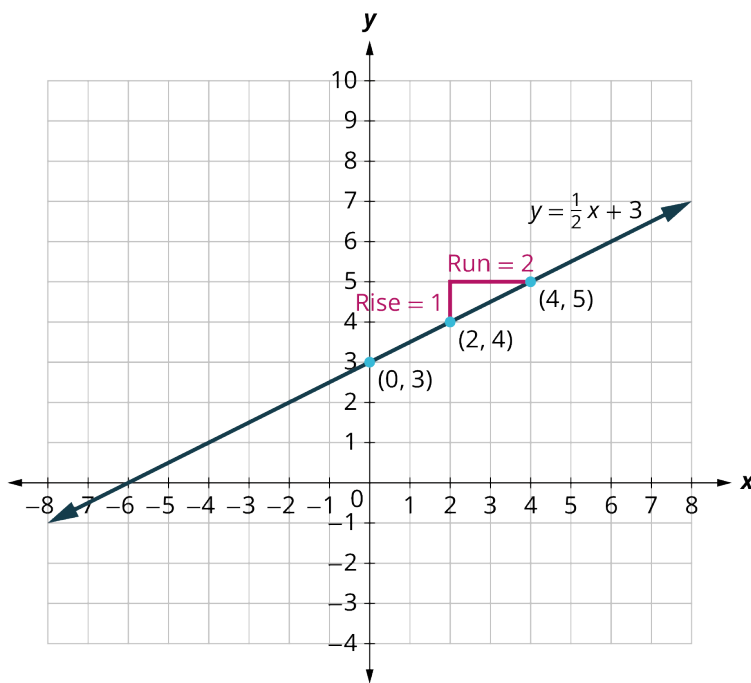
Figure 5.72

> **YOUR TURN 5.73**

1. Use the slope formula to find the slope of the line through the pair of points  $(-3, 4)$  and  $(2, -1)$ .

## Graphing Functions Using Slope and $y$ -Intercept

We have graphed linear equations by plotting points and using intercepts. Once we see how an equation in slope-intercept form and its graph are related, we will have one more method we can use to graph lines. Review the graph of the equation  $y = \frac{1}{2}x + 3$  in [Figure 5.73](#) and find its slope and  $y$ -intercept.



**Figure 5.73** Graph of the equation  $y = \frac{1}{2}x + 3$ .

The vertical and horizontal lines in the graph show us the rise is 1 and the run is 2, respectively.

Substituting into the slope formula:  $m = \frac{\text{rise}}{\text{run}} = \frac{1}{2}$

The  $y$ -intercept is (0, 3). Look at the equation of this line.

$$y = \frac{1}{2}x + 3$$

Look at the slope and  $y$ -intercept.

$$\text{slope } m = \frac{1}{2} \text{ and } y\text{-intercept } (0, 3)$$

When a linear equation is solved for  $y$ , the coefficient of the  $x$  term is the slope and the constant term is the  $y$ -coordinate of the  $y$ -intercept. We say that the equation  $y = \frac{1}{2}x + 3$  is in **slope-intercept form**. Sometimes the slope-intercept form is called the  $y$ -form.

#### EXAMPLE 5.74

##### Finding the Slope and $y$ -Intercept of a Line

Identify the slope and  $y$ -intercept of the line from the equation:

1.  $y = -\frac{4}{7}x - 2$
2.  $x + 3y = 9$

##### ✓ Solution

1. We compare our equation to the slope-intercept form of the equation.

**Step 1:** Write the slope-intercept form of the equation of the line.

$$y = mx + b$$

**Step 2:** Write the equation of the line.

$$y = -\frac{4}{7}x - 2$$

**Step 3:** Identify the slope.

$$m = -\frac{4}{7}$$

**Step 4:** Identify the  $y$ -intercept.

$y$ -intercept is  $(0, -2)$

2. When an equation of a line is not given in slope-intercept form, our first step will be to solve the equation for  $y$ .

**Step 1:** Solve for  $y$ .

$$x + 3y = 9$$

**Step 2:** Subtract  $x$  from each side.

$$3y = -x + 9$$

**Step 3:** Divide both sides by 3.

$$\frac{3y}{3} = \frac{-x + 9}{3}$$

**Step 4:** Simplify.

$$y = -\frac{1}{3}x + 3$$

**Step 5:** Write the slope-intercept form of the equation of the line.

$$y = mx + b$$

**Step 6:** Write the equation of the line.

$$y = -\frac{1}{3}x + 3$$

**Step 7:** Identify the slope.

$$m = -\frac{1}{3}$$

**Step 8:** Identify the  $y$ -intercept.

$y$ -intercept is  $(0, 3)$ .

### YOUR TURN 5.74

Identify the slope and  $y$ -intercept from the equation of the line.

1.  $y = 2x - 1$
2.  $x + 4y = 16$

### EXAMPLE 5.75

#### Graphing the Slope and $y$ -Intercept

Graph the line of the equation  $y = -x + 4$  using its slope and  $y$ -intercept.

#### Solution

The equation is in slope-intercept form  $y = mx + b$ .

$$y = -x + 4.$$

**Step 1:** Identify the slope and  $y$ -intercept.

$m = -1$ ,  $y$ -intercept is  $(0, 4)$ .

**Step 2:** Plot the  $y$ -intercept on the coordinate system (Figure 5.74).

1. Identify the rise over the run.  

$$m = -1 = \frac{-1}{1}$$
2. Count out the rise and run to mark the second point.  
 rise  $-1$ , run  $1$

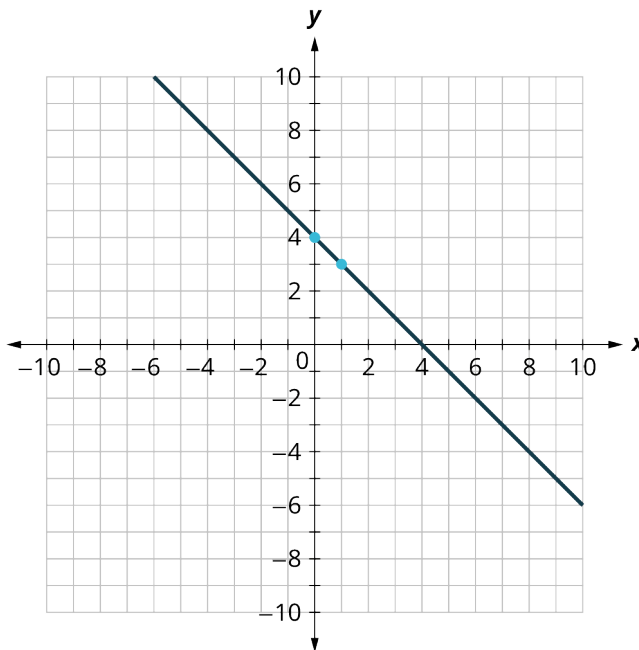


Figure 5.74

### > YOUR TURN 5.75

1. Graph the line of the equation  $y = -x - 3$  using its slope and  $y$ -intercept.

## Graphing Horizontal and Vertical Lines

Some linear equations have only one variable. They may have just  $x$  without the  $y$ , or just  $y$  without an  $x$ . This changes how we make a table of values to get the points to plot. Let us consider the equation  $x = -3$ . This equation has only one variable,  $x$ . The equation says that  $x$  is always equal to  $-3$ , so its value does not depend on  $y$ . No matter what the value of  $y$  is, the value of  $x$  is always  $-3$ . To make a table of values, write  $-3$  in for all the  $x$ -values. Then choose any values for  $y$ . Since  $x$  does not depend on  $y$ , you can choose any numbers you like. But to fit the points on our coordinate graph, we will use  $1, 2,$  and  $3$  for the  $y$ -coordinates in the table below.

$x = -3$		
$x$	$y$	$(x, y)$
$-3$	$1$	$(-3, 1)$
$-3$	$2$	$(-3, 2)$
$-3$	$3$	$(-3, 3)$

Plot the points from the table and connect them with a straight line (Figure 5.75). Notice that we have graphed a vertical line.

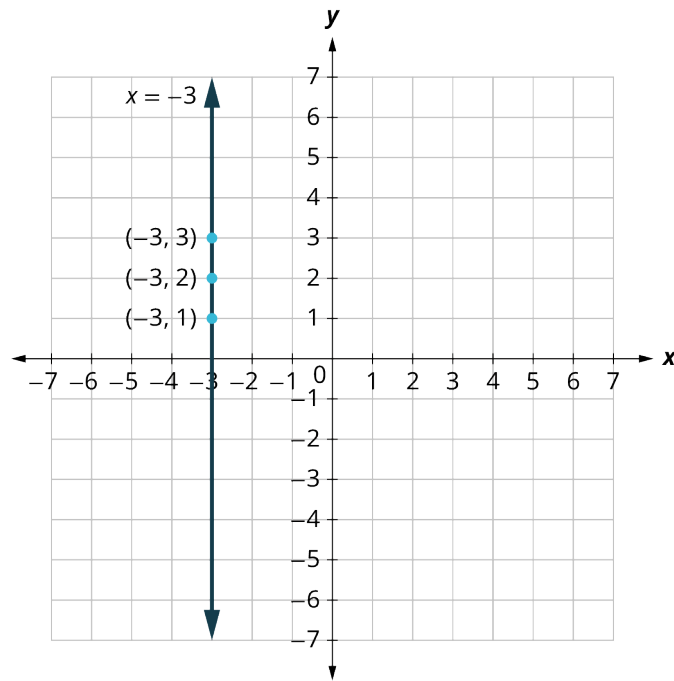


Figure 5.75 Graph of  $x = -3$

What is the slope? If we take the two points  $(-3, 3)$  and  $(-3, 1)$  then the rise is 2 and the run is 0.

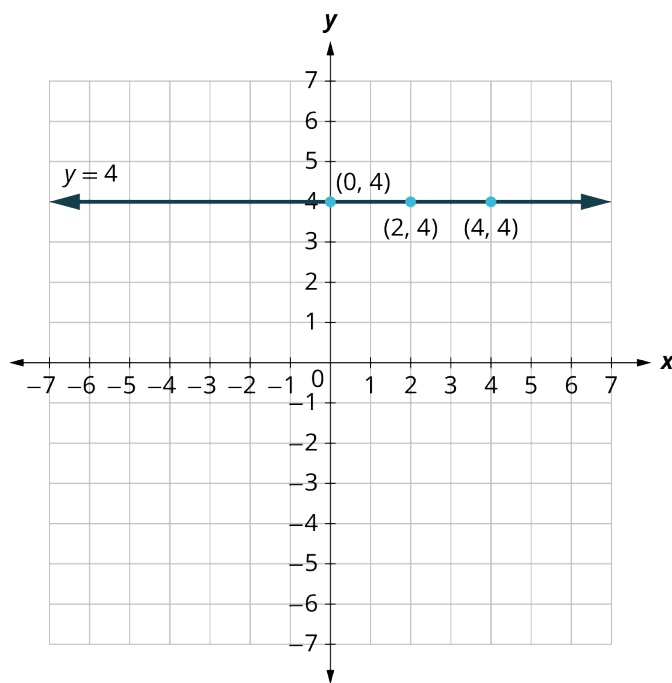
Using the slope formula we get:  $m = \frac{\text{rise}}{\text{run}} = \frac{2}{0}$

The slope is **undefined** since division by zero is undefined. We say that the slope of the vertical line  $x = -3$  is undefined. The slope of any **vertical line**  $x = a$  (where  $a$  is any number) will be undefined.

What if the equation has  $y$  but no  $x$ ? Let's graph the equation  $y = 4$ . This time the  $y$ -value is a constant, so in this equation,  $y$  does not depend on  $x$ . Fill in 4 for all the  $y$  values in the table below and then choose any values for  $x$ . We will use 0, 2, and 4 for the  $x$ -coordinates.

$y = 4$		
$x$	$y$	$(x, y)$
0	4	$(0, 4)$
2	4	$(2, 4)$
4	4	$(4, 4)$

In [Figure 5.76](#), we have graphed a horizontal line passing through the  $y$ -axis at 4.

Figure 5.76 Graph of  $y = 4$ 

What is the slope? If we take the two points  $(2, 4)$  and  $(4, 4)$  then the rise is 0 and the run is 2. Using the slope formula, we get  $m = \frac{\text{rise}}{\text{run}} = \frac{0}{2} = 0$ . The slope of the horizontal line  $y = 4$  is 0. The slope of any **horizontal line**  $y = b$  (where  $b$  is any number) will be 0. When the  $y$ -coordinates are the same, the rise is 0.

**EXAMPLE 5.76****Graphing A Vertical Line**

Graph:  $x = 2$ .

✓ **Solution**

The equation has only one variable,  $x$ , and  $x$  is always equal to 2. We create a table where  $x$  is always 2 and then put in any values for  $y$ . The graph is a vertical line passing through the  $x$ -axis at 2 (Figure 5.77).

$x = 2$		
$x$	$y$	$(x, y)$
2	1	$(2, 1)$
2	2	$(2, 2)$
2	3	$(2, 3)$

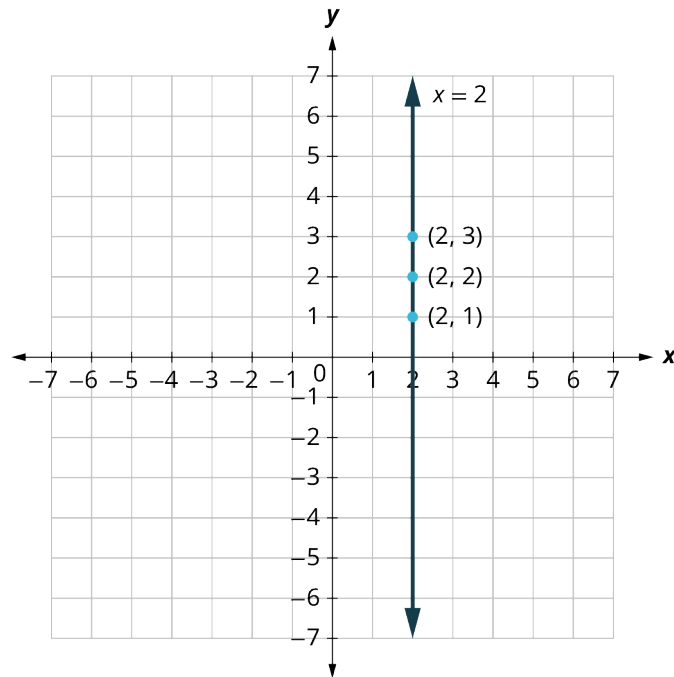


Figure 5.77

> YOUR TURN 5.76

- Graph  $x = 5$ .

EXAMPLE 5.77

**Graphing A Horizontal Line**

Graph:  $y = -1$ .

✓ **Solution**

The equation  $y = -1$  has only one variable,  $y$ . The value of  $y$  is constant. All the ordered pairs in the next table have the same  $y$ -coordinate. The graph is a horizontal line passing through the  $y$ -axis at  $-1$  (Figure 5.78).

$y = -1$		
$x$	$y$	$(x, y)$
0	-1	$(0, -1)$
3	-1	$(3, -1)$
-3	-1	$(-3, -1)$

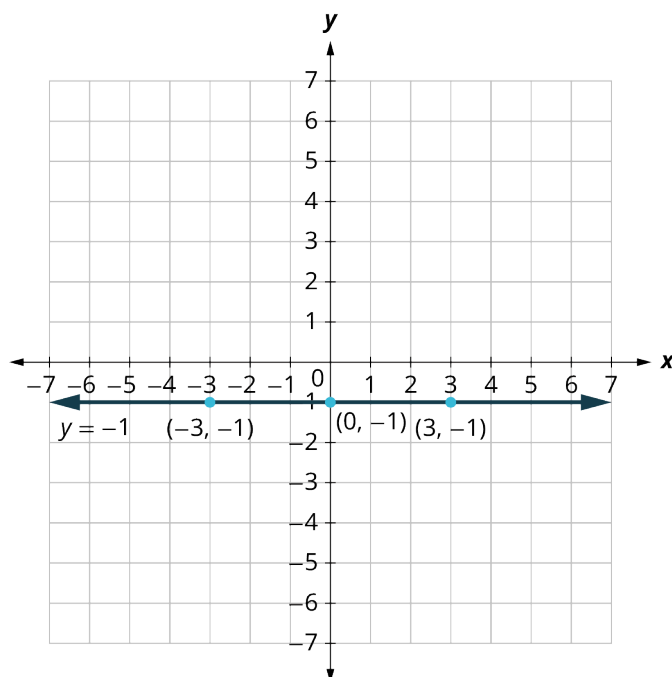


Figure 5.78

### > YOUR TURN 5.77

- Graph the equation  $y = -4$ .

The table below summarizes all the methods we have used to graph lines.

Methods to Graph Lines			
<b>Point Plotting</b>  $\begin{array}{c c} x & y \\ \hline & \\ \hline & \\ \hline & \\ \hline & \end{array}$	<b>Slope-Intercept</b>  $y = mx + b$	<b>Intercepts</b>  $\begin{array}{c c} x & y \\ \hline 0 & \\ \hline & 0 \\ \hline & \end{array}$	<b>Recognize Vertical and Horizontal Lines</b>
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and y-intercept. Start at the y-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

## Interpreting Graphs of Functions

An important yet often overlooked area in algebra involves interpreting graphs. Oftentimes in math classes, students are given mathematical functions and can make graphs to represent them. But the interpretation of graphs is a more applicable skill to the real world. Being able to “read” a graph—understanding its domain and range, what the intercepts mean, and what the slope (or curve) means— that’s a real-world skill.

**EXAMPLE 5.78****Interpreting a Graph**

In [Figure 5.79](#) the  $x$ -axis on the graph represents the 120-minute bike ride Juan went on. The  $y$ -axis represents how far away he was from his home.

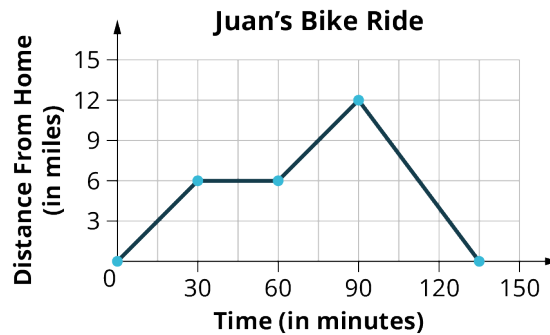


Figure 5.79

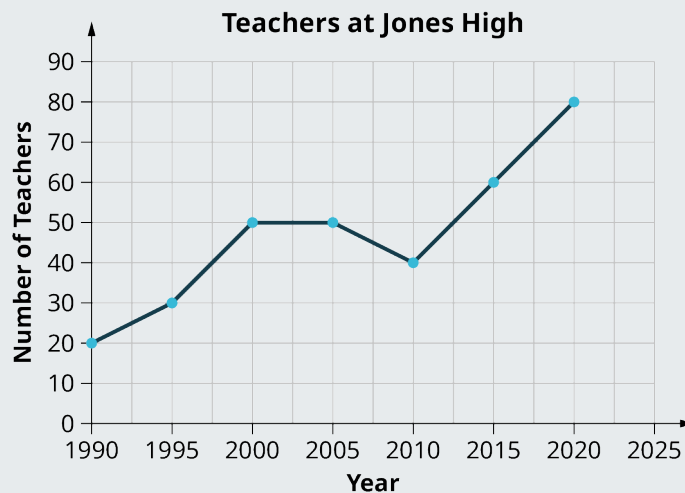
1. Interpret the  $x$ - and  $y$ -intercept.
2. For each segment, find the slope.
3. Create an interpretation of this graph (i.e., make up a story that goes with it).

**✓ Solution**

1.  $(0, 0)$  is the  $x$ - and  $y$ -intercept and represents Juan at home before his bike ride. The distance from home is 0 miles and 0 minutes have passed.
2. In the first 30 minutes, the slope is  $\frac{1}{5}$  and indicates Juan is traveling 1 mile for every 5 minutes. Between 30 and 60 minutes, the slope is 0 and indicates that he's not riding the bike (the distance is not increasing). Then between 60 and 90 minutes, the slope is  $\frac{1}{5}$  again. Finally, after 90 minutes the slope is  $-\frac{4}{15}$ , meaning Juan is getting 4 miles closer to home every 15 minutes.
3. Answers will vary. Juan left his house for a bike ride. After 30 minutes, he was 6 miles from home and he stopped for ice cream at his local ice-cream truck. He enjoyed his ice cream for 30 minutes. He then jumped back on his bike and rode to his friend's house. He arrived there 30 minutes later. His friend's house was 12 miles from his home. His friend was not home so he immediately turned around and quickly rode home in 45 minutes.

**> YOUR TURN 5.78**

In the given figure the  $x$ -axis on the graph represents the years. The  $y$ -axis represents the number of teachers at Jones High School.



Teachers at Jones High

1. Interpret the  $x$ - and  $y$ -intercept.
2. For each segment, find the slope.
3. Create an interpretation of this graph (i.e., make up a story that goes with it).

## Modeling Applications Using Slope and $y$ -Intercept

Many real-world applications are modeled by linear equations. We will review a few applications here so you can understand how equations written in slope-intercept form relate to real-world situations. Usually when a linear equation model uses real-world data, different letters are used for the variables instead of using only  $x$  and  $y$ . The variable names often remind us of what quantities are being measured. Also, we often need to extend the axes in our rectangular coordinate system to bigger positive and negative numbers to accommodate the data in the application.

### EXAMPLE 5.79

#### Converting Temperature

The equation  $F = \frac{9}{5}C + 32$  is used to convert temperatures from degrees Celsius ( $C$ ) to degrees Fahrenheit ( $F$ ).

1. Find the Fahrenheit temperature for a Celsius temperature of  $0^\circ$ .
2. Find the Fahrenheit temperature for a Celsius temperature of  $20^\circ$ .
3. Interpret the slope and  $F$ -intercept of the equation.
4. Graph the equation.

#### ✓ Solution

1. Find the Fahrenheit temperature for a Celsius temperature of  $0^\circ$ .

Find $F$ when $C = 0$ .	$F = \frac{9}{5}(0) + 32$
Simplify.	$F = 32$

2. Find the Fahrenheit temperature for a Celsius temperature of  $20^\circ$ .

Find $F$ when $C = 20$ .	$F = \frac{9}{5}(20) + 32$
Simplify.	$F = 36 + 32$
Simplify.	$F = 68$

3. Interpret the slope and  $F$ -intercept of the equation.  
Even though this equation uses  $F$  and  $C$ , it is still in slope-intercept form.

$$y = mx + b$$

$$F = mC + b$$

$$F = \frac{9}{5}C + 32$$

The slope  $\frac{9}{5}$  means that the temperature Fahrenheit ( $F$ ) increases 9 degrees when the temperature Celsius ( $C$ ) increases 5 degrees.

The  $F$ -intercept means that when the temperature is  $0^\circ$  on the Celsius scale, it is  $32^\circ$  on the Fahrenheit scale.

4. Graph the equation.

We will need to use a larger scale than our usual. Start at the  $F$ -intercept  $(0, 32)$ , and then count out the rise of 9 and the run of 5 to get a second point as shown in [Figure 5.80](#).

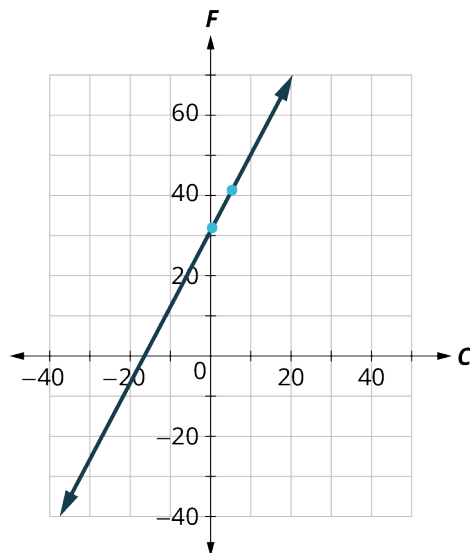


Figure 5.80

### > YOUR TURN 5.79

The equation  $h = 2s + 50$  is used to estimate a person's height in inches,  $h$ , based on women's shoe size,  $s$ .

1. Estimate the height of a child who wears women's shoe size 0.
2. Estimate the height of a woman with shoe size 8.
3. Interpret the slope and  $h$ -intercept of the equation.
4. Graph the equation.

### EXAMPLE 5.80

#### Calculating Driving Costs

Sam drives a delivery van. The equation  $C = 0.5d + 60$  models the relation between his weekly cost,  $C$ , in dollars and the number of miles,  $d$ , that he drives.

1. Find Sam's cost for a week when he drives 0 miles.
2. Find the cost for a week when he drives 250 miles.
3. Interpret the slope and  $C$ -intercept of the equation.
4. Graph the equation.

#### ✓ Solution

1. Find Sam's cost for a week when he drives 0 miles.

Find $C$ when $d = 0$ .	$C = 0.5(0) + 60$
Simplify.	$C = 60$

Sam's costs are \$60 when he drives 0 miles.

2. Find the cost for a week when he drives 250 miles.

Find $C$ when $d = 250$ .	$C = 0.5(250) + 60$
Simplify.	$C = 185$

Sam's costs are \$185 when he drives 250 miles.

3. Interpret the slope and  $C$ -intercept of the equation.

$$y = mx + b$$

$$C = 0.5d + 60$$

The slope, 0.5, means that the weekly cost,  $C$ , increases by \$0.50 when the number of miles driven,  $d$ , increases by 1. The  $C$ -intercept means that when the number of miles driven is 0, the weekly cost is \$60.

4. Graph the equation (Figure 5.81).

We'll need to use a larger scale than usual. Start at the  $C$ -intercept (0, 60). To count out the slope  $m = 0.5$ , we rewrite it as an equivalent fraction that will make our graphing easier.

$$m = 0.5 = \frac{5}{10} = \frac{50}{100}$$

So to graph the next point go up 50 from the intercept of 60 and then to the right 100. The second point will be (100, 110).

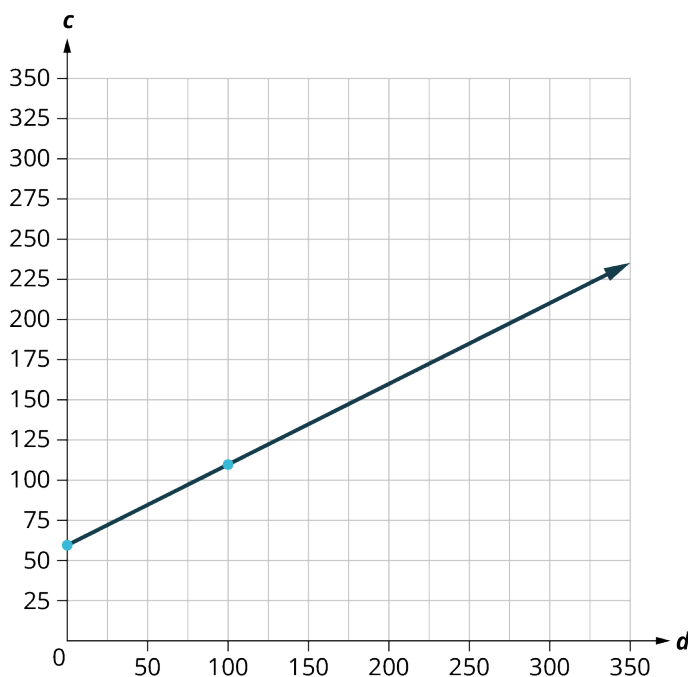


Figure 5.81

> **YOUR TURN 5.80**

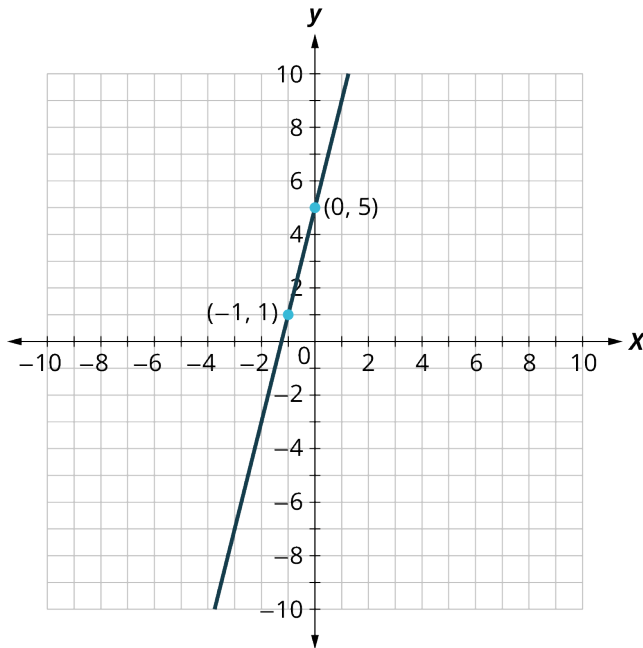
Stella has a home business selling gourmet pizzas. The equation  $C = 4p + 25$  models the relation between her weekly cost,  $C$ , in dollars and the number of pizzas,  $p$ , that she sells.

1. Find Stella's cost for a week when she sells no pizzas.
2. Find the cost for a week when she sells 15 pizzas.
3. Interpret the slope and  $C$ -intercept of the equation.
4. Graph the equation.

## Check Your Understanding

62. True or False. The  $x$ -intercept of  $y = 2x - 8$  is  $(0, -8)$ .
- True
  - False
63. True or False. The slope of the line containing the points  $(1, 2)$  and  $(2, 4)$  is 1.
- True
  - False

For the following exercises, use the graph shown.

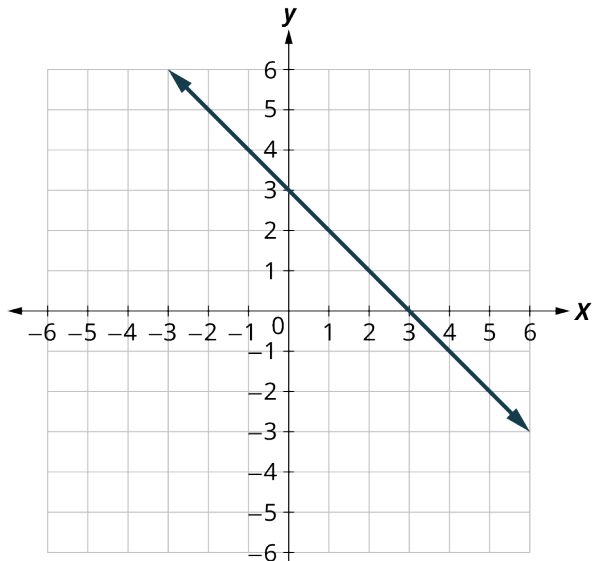


64. True or False. This graph has a slope of 5.
- True
  - False
65. True or False. This is the graph of the equation  $y = 4x + 5$ .
- True
  - False
66. True or False. All vertical lines have a slope of zero.
- True
  - False

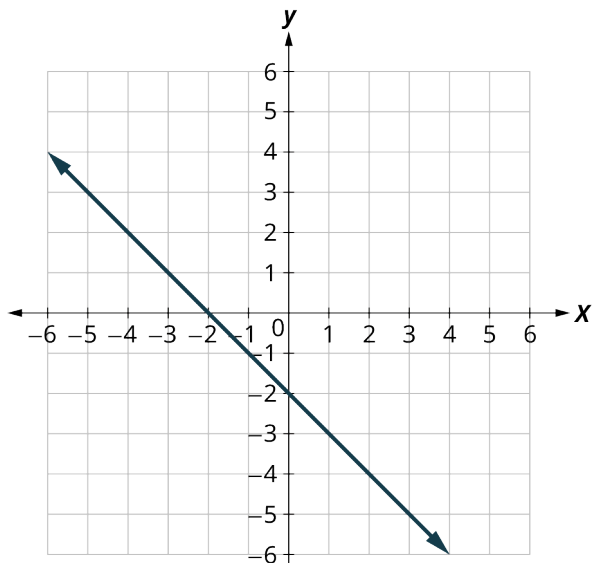


## SECTION 5.8 EXERCISES

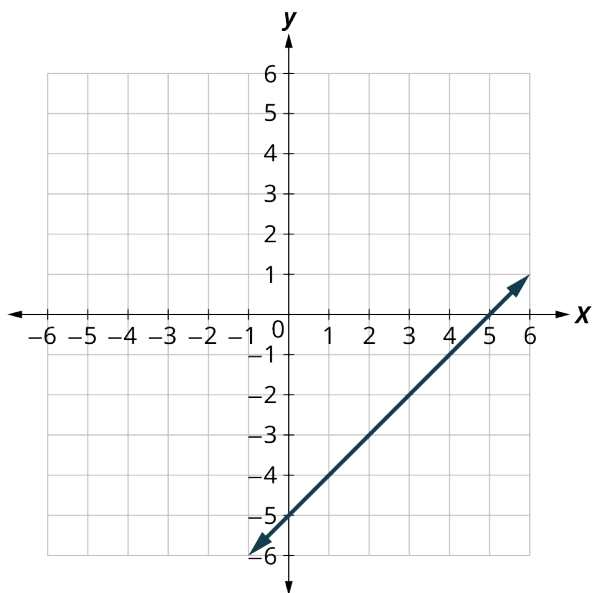
For the following exercises, find the  $x$ - and  $y$ -intercepts on the graph.



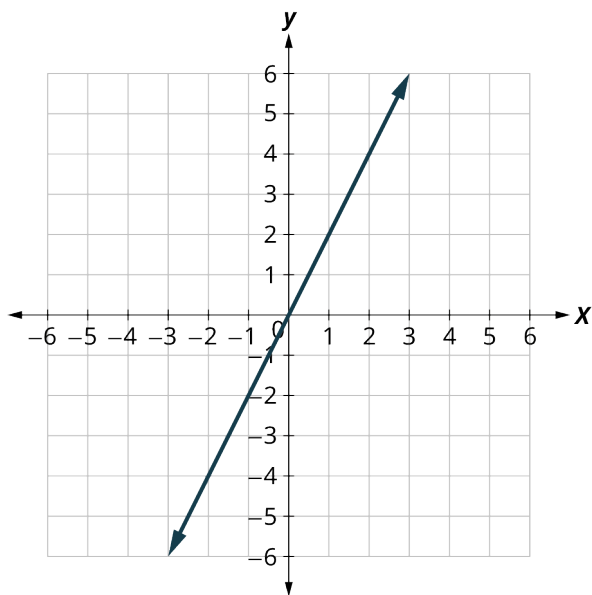
1.



2.



3.

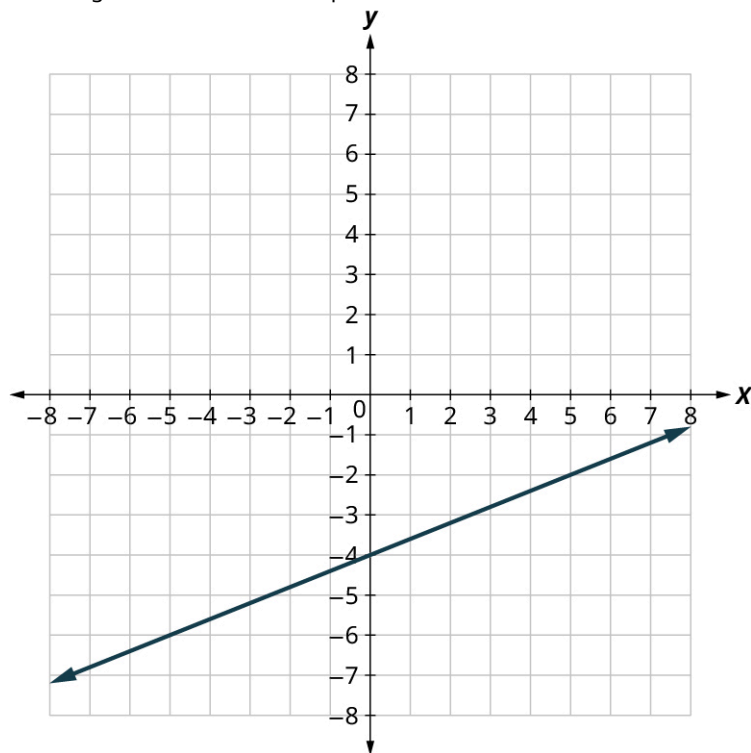


4.

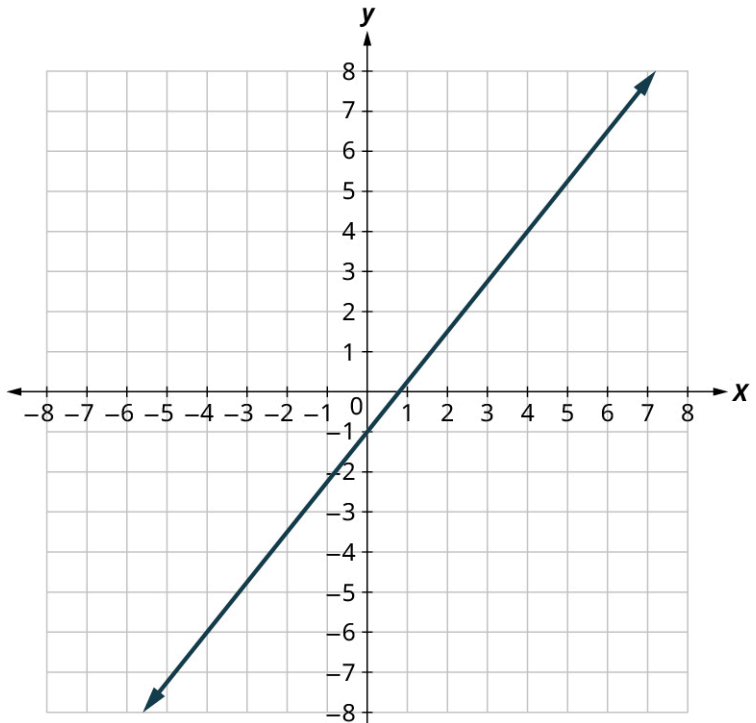
For the following exercises, graph using the intercepts.

5.  $-x + 4y = 8$
6.  $x + y = -3$
7.  $4x + y = 4$
8.  $3x - y = -6$
9.  $2x + 4y = 12$
10.  $2x - 5y = -20$
11.  $y = -2x$
12.  $y = x$

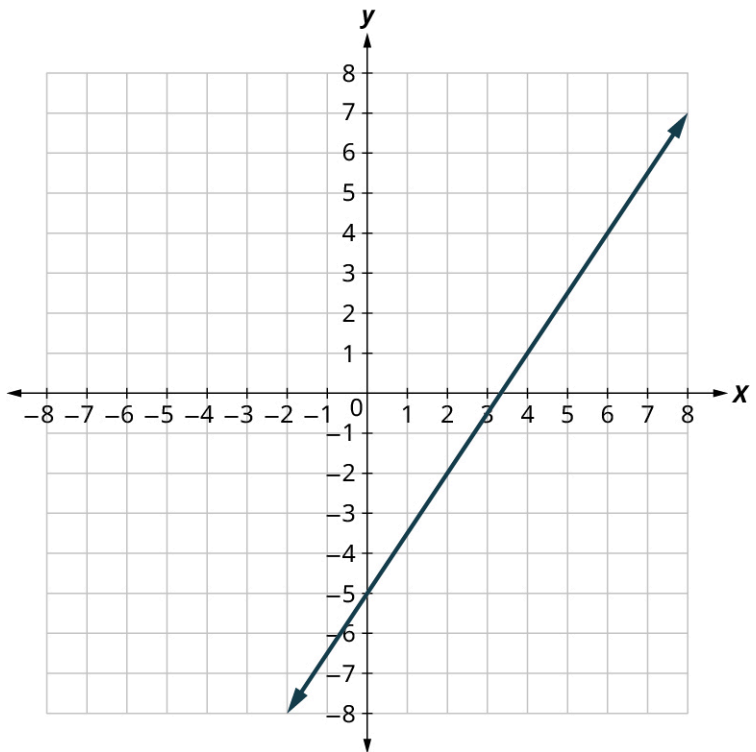
For the following exercises, find the slope of the line.



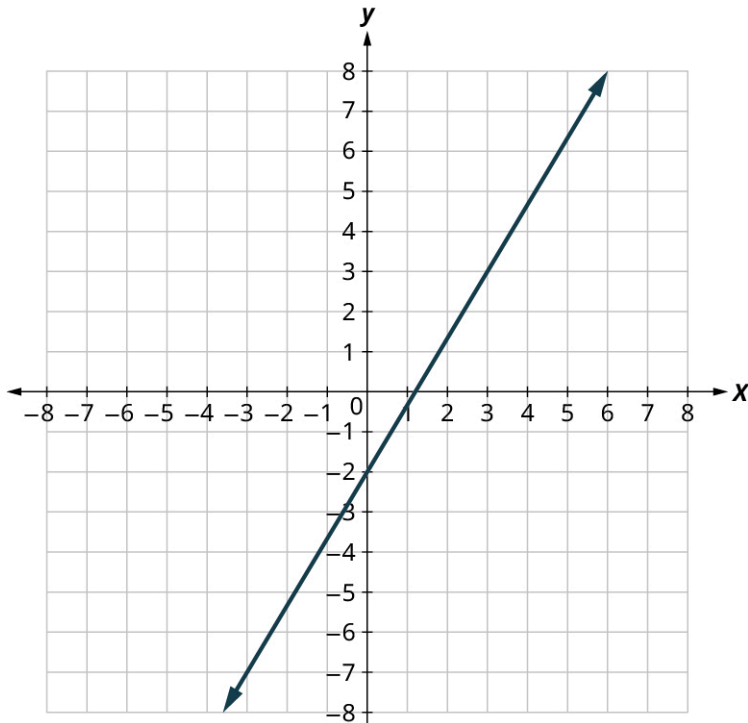
13.



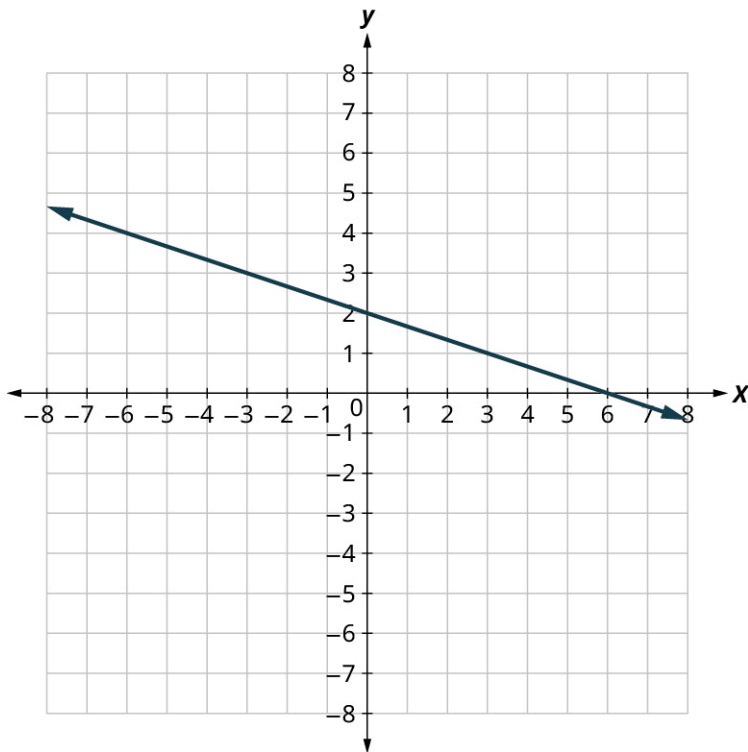
14.



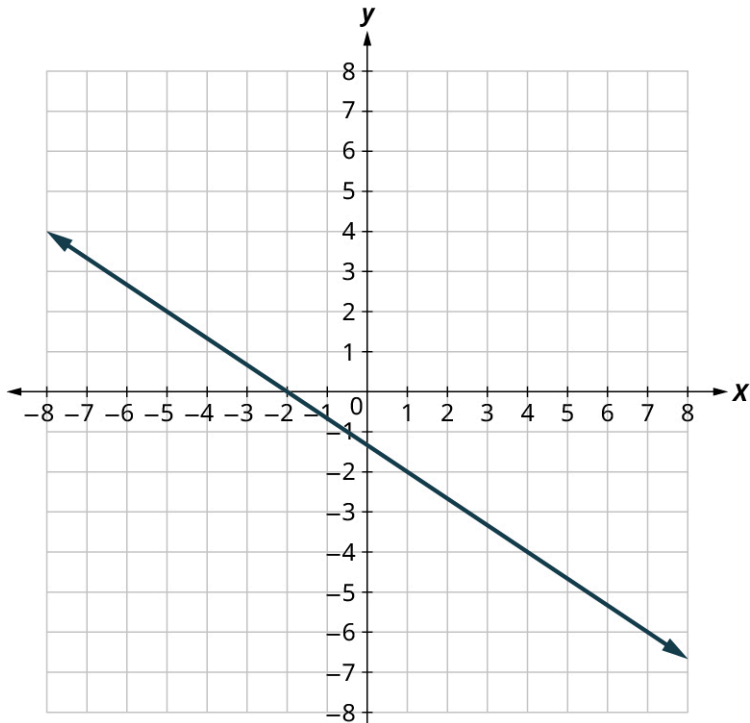
15.



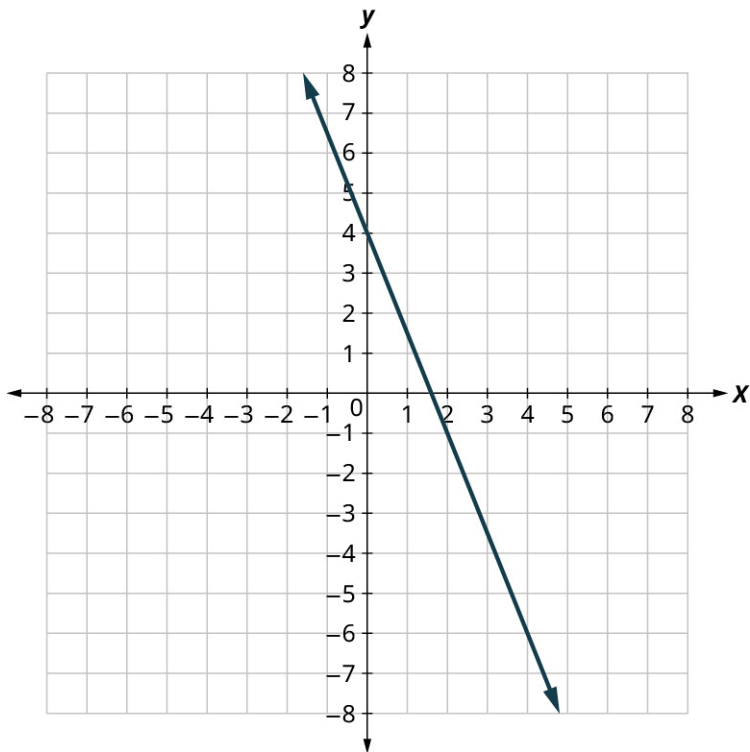
16.



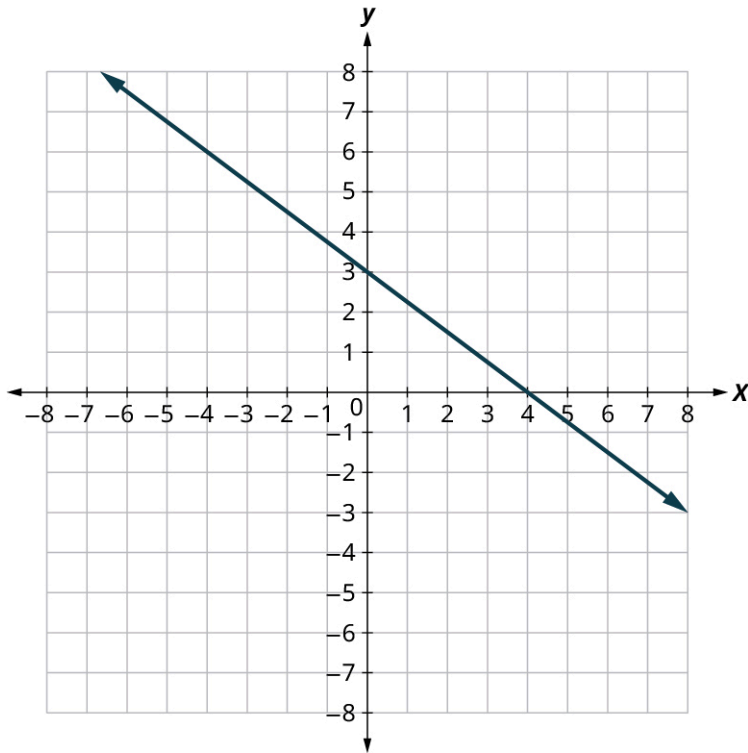
17.



18.



19.



20.

For the following exercises, use the slope formula to find the slope of the line between each pair of points.

21.  $(2, 5), (4, 0)$
22.  $(-3, 3), (4, -5)$
23.  $(-1, -2), (2, 5)$
24.  $(4, -5), (1, -2)$

For the following exercises, identify the slope and y-intercept of each line.

25.  $y = -7x + 3$
26.  $y = 4x - 10$
27.  $3x + y = 5$
28.  $4x + y = 8$
29.  $6x + 4y = 12$
30.  $8x + 3y = 12$
31.  $5x - 2y = 6$
32.  $7x - 3y = 9$

For the following exercises, graph the line of each equation using its slope and y-intercept.

33.  $y = 3x - 1$
34.  $y = 2x - 3$
35.  $y = -x + 3$
36.  $y = -x - 4$
37.  $y = -2x - 3$
38.  $y = -3x + 2$
39.  $3x - 2y = 4$
40.  $3x - 4y = 8$

For the following exercises, find the slope of each line and graph.

41.  $y = 3$
42.  $y = -2$
43.  $x = -5$
44.  $x = 4$

For the following exercises, graph and interpret applications of slope-intercept.

The equation  $P = 31 + 1.75w$  models the relation between the amount of Tuyet's monthly water bill payment,  $P$ , in dollars, and the number of units of water,  $w$ , used.

45. Find Tuyet's payment for a month when 0 units of water are used.
46. Find Tuyet's payment for a month when 12 units of water are used.
47. Interpret the slope and  $P$ -intercept of the equation.
48. Graph the equation.

For the following exercises, graph and interpret applications of slope-intercept.

Bruce drives his car for his job. The equation  $R = 0.575m + 42$  models the relation between the amount in dollars,  $R$ , that he is reimbursed and the number of miles,  $m$ , he drives in one day.

49. Find the amount Bruce is reimbursed on a day when he drives 0 miles.
50. Find the amount Bruce is reimbursed on a day when he drives 220 miles.
51. Interpret the slope and  $R$ -intercept of the equation.
52. Graph the equation.

For the following exercises, graph and interpret applications of slope-intercept.

Cherie works in retail and her weekly salary includes commission for the amount she sells. The equation  $S = 400 + 0.15c$  models the relation between her weekly salary,  $S$ , in dollars and the amount of her sales,  $c$ , in dollars.

53. Find Cherie's salary for a week when her sales were \$0.
54. Find Cherie's salary for a week when her sales were \$3,600.
55. Interpret the slope and  $S$ -intercept of the equation.
56. Graph the equation.

For the following exercises, graph and interpret applications of slope-intercept.

Costa is planning a lunch banquet. The equation  $C = 450 + 28g$  models the relation between the cost in dollars,  $C$ , of the banquet and the number of guests,  $g$ .

57. Find the cost if the number of guests is 40.
58. Find the cost if the number of guests is 80.
59. Interpret the slope and  $C$ -intercept of the equation.
60. Graph the equation.

## 5.9 Systems of Linear Equations in Two Variables



**Figure 5.82** Fruits and vegetables at a farmer's market. (credit: "California Ave. Farmers' Market" by Jun Seita/Flickr, CC BY 2.0)

### Learning Objectives

After completing this section, you should be able to:

1. Determine and show whether an ordered pair is a solution to a system of equations.
2. Solve systems of linear equations using graphical methods.
3. Solve systems of linear equations using substitution.
4. Solve systems of linear equations using elimination.
5. Identify systems with no solution or infinitely many solutions.
6. Solve applications of systems of linear equations.

In this section, we will learn how to solve systems of linear equations in two variables. There are several real-world scenarios that can be represented by systems of linear equalities. Suppose two friends, Andrea and Bart, go shopping at a farmers market to buy some vegetables. Andrea buys 2 tomatoes and 4 cucumbers and spends \$2.00. Bart buys 4 tomatoes and 5 cucumbers and spends \$2.95. What is the price of each vegetable?

### Determining If an Ordered Pair Is a Solution to a System of Equations

When we solved linear equations in [Linear Equations in One Variable with Applications](#) and [Linear Inequalities in One Variable with Applications](#), we learned how to solve linear equations with one variable. Now we will work with two or more linear equations grouped together, which is known as a **system of linear equations**.

In this section, we will focus our work on systems of two linear equations in two unknowns (variables) and applications of systems of linear equations. An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

A linear equation in two variables, such as  $2x + y = 7$ , has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation and every solution to the equation is a point on the line. To solve a system of two linear equations, we want to find the values of the variables that are solutions to both equations. In other words, we are looking for the ordered pairs  $(x, y)$  that make both equations true. These are called the **solutions of a system of equations**.

To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into each equation. If the ordered pair makes both equations true, it is a solution to the system.

**EXAMPLE 5.81****Determining Whether an Ordered Pair Is a Solution to the System**

Determine whether the ordered pair is a solution to the system.

$$\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$$

- $(-2, -1)$
- $(-4, -3)$

✓ **Solution**

$$\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$$

1.

We substitute  $x = -2$  and  $y = -1$  into both equations.

$$\begin{array}{rcl} x - y = -1 & & 2x - y = -5 \\ -2 - (-1) \stackrel{?}{=} -1 & & 2(-2) - (-1) \stackrel{?}{=} -5 \\ -1 = -1 \checkmark & & -3 \neq -5 \end{array}$$

 $(-2, -1)$  does not make both equations true.  $(-2, -1)$  is not a solution.2. We substitute  $x = -4$  and  $y = -3$  into both equations.

$$\begin{array}{rcl} x - y = -1 & & 2x - y = -5 \\ -4 - (-3) \stackrel{?}{=} -1 & & 2 \cdot (-4) - (-3) \stackrel{?}{=} -5 \\ -1 = -1 \checkmark & & -5 = -5 \checkmark \end{array}$$

 $(-4, -3)$  makes both equations true.  $(-4, -3)$  is a solution.**> YOUR TURN 5.81**

Determine whether the ordered pair is a solution to the system.

$$\begin{cases} x - 3y = -8 \\ -3x - y = 4 \end{cases}$$

- $(2, -2)$
- $(-2, 2)$

**EXAMPLE 5.82****Determining Whether an Ordered Pair Is a Solution to the System**

Determine whether the ordered pair is a solution to the system

$$\begin{cases} y = \frac{3}{2}x + 1 \\ 2x - 3y = 7 \end{cases}$$

- $(-4, -5)$
- $(-4, 5)$

✓ **Solution**

$$\begin{cases} y = \frac{3}{2}x + 1 \\ 2x - 3y = 7 \end{cases}$$

1.

Substitute  $-4$  for  $x$  and  $-5$  for  $y$  into both equations.

$$\begin{array}{rcl}
 -5 & \stackrel{?}{=} & \frac{3}{2}(-4) + 1 \quad 2(-4) - 3(-5) \stackrel{?}{=} 7 \\
 -5 & \stackrel{?}{=} & 3(-2) + 1 \quad (-8) - (-15) \stackrel{?}{=} 7 \\
 -5 & \stackrel{?}{=} & -6 + 1 \quad -8 + 15 \stackrel{?}{=} 7 \\
 -5 & = & -5 \checkmark \quad 7 = 7 \checkmark
 \end{array}$$

$(-4, -5)$  is a solution.

2.

$$\begin{cases} y = \frac{3}{2}x + 1 \\ 2x - 3y = 7 \end{cases}$$

Substitute  $-4$  for  $x$  and  $5$  for  $y$  into both equations.

$$\begin{array}{rcl}
 5 & \stackrel{?}{=} & \frac{3}{2}(-4) + 1 \quad 2(-4) - 3(5) \stackrel{?}{=} 7 \\
 5 & \stackrel{?}{=} & 3(-2) + 1 \quad (-8) - (15) \stackrel{?}{=} 7 \\
 5 & \stackrel{?}{=} & -6 + 1 \quad -8 - 15 \stackrel{?}{=} 7 \\
 5 & \neq & -5 \quad -23 \neq 7
 \end{array}$$

$(-4, 5)$  is not a solution.

### > YOUR TURN 5.82

Determine whether the ordered pair is a solution to the system.

$$\begin{cases} y = \frac{2}{3}x + 1 \\ -2x + 3y = 5 \end{cases}$$

1.  $(3, 3)$
2.  $(-3, 3)$

## Solving Systems of Linear Equations Using Graphical Methods

We will use three methods to solve a system of linear equations. The first method we will use is graphing. The graph of a linear equation is a line. Each point on the line is a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what points the lines have in common, we will find the solution to the system.

Most linear equations in one variable have one solution; but for some equations called **contradictions**, there are no solutions, and for other equations called **identities**, all numbers are solutions. Similarly, when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown in [Figure 5.83](#).

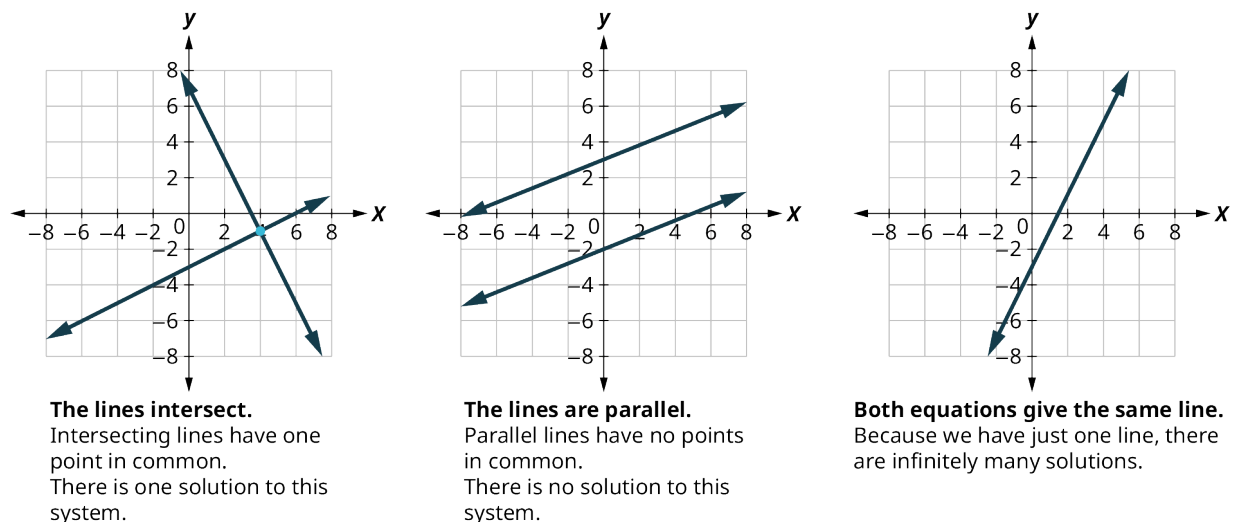


Figure 5.83

Each time we demonstrate a new method, we will use it on the same system of linear equations. At the end you will decide which method was the most convenient way to solve this system.

The steps to use to solve a system of linear equations by graphing are shown here.

**Step 1:** Graph the first equation.

**Step 2:** Graph the second equation on the same rectangular coordinate system.

**Step 3:** Determine whether the lines intersect, are parallel, or are the same line.

**Step 4:** Identify the solution to the system.

If the lines intersect, identify the point of intersection. This is the solution to the system.

If the lines are parallel, the system has no solution.

If the lines are the same, the system has an infinite number of solutions.

**Step 5:** Check the solution in both equations.

### EXAMPLE 5.83

#### Solving a System of Linear Equations by Graphing

Solve this system of linear equations by graphing.

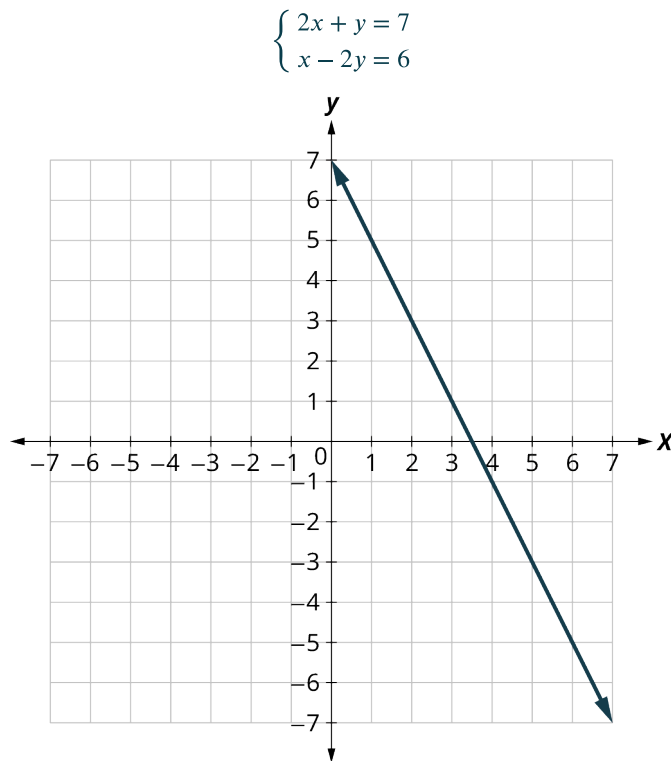
$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

✔ **Solution**

**Step 1:** Graph the first equation.

To graph the first line, write the equation in slope-intercept form.

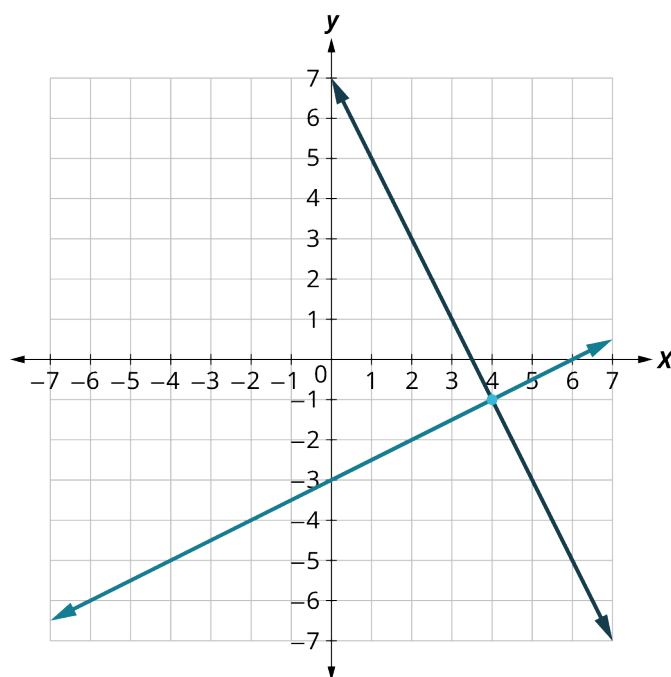
$$\begin{aligned} 2x + y &= 7 \\ y &= -2x + 7 \\ m &= -2 \quad b = 7 \end{aligned}$$



**Step 2:** Graph the second equation on the same rectangular coordinate system.

To graph the second line, use intercepts.

$$\begin{aligned} x - 2y &= 6 \\ (0, -3) \quad (6, 0) \end{aligned}$$



<p><b>Step 3:</b> Determine whether the lines intersect, are parallel, or are the same line.</p>	<p>Look at the graph of the lines.</p>	<p>The lines intersect.</p>
<p><b>Step 4 and Step 5:</b> Identify the solution to the system. If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system. If the lines are parallel, the system has no solution. If the lines are the same, the system has an infinite number of solutions.</p>	<p>Since the lines intersect, find the point of intersection. Check the point in both equations.</p>	<p>The lines intersect at <math>(4, -1)</math>.</p> $2x + y = 7$ $2(4) + (-1) \stackrel{?}{=} 7$ $8 - 1 \stackrel{?}{=} 7$ $7 = 7 \checkmark$ $x - 2y = 6$ $4 - 2(-1) \stackrel{?}{=} 6$ $6 = 6 \checkmark$ <p>The solution is <math>(4, -1)</math>.</p>

### YOUR TURN 5.83

1. Solve this system of linear equations by graphing.

$$\begin{cases} x - 3y = -3 \\ x + y = 5 \end{cases}$$

## Solving Systems of Linear Equations Using Substitution

We will now solve systems of linear equations by the substitution method. We will use the same system we used for graphing.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

We will first solve one of the equations for either  $x$  or  $y$ . We can choose either equation and solve for either variable—but we'll try to make a choice that will keep the work easy. Then, we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it makes both equations true. This process is summarized here:

**Step 1:** Solve one of the equations for either variable.

**Step 2:** Substitute the expression from Step 1 into the other equation.

**Step 3:** Solve the resulting equation.

**Step 4:** Substitute the solution in Step 3 into either of the original equations to find the other variable.

**Step 5:** Write the solution as an ordered pair.

**Step 6:** Check that the ordered pair is a solution to both original equations.

**EXAMPLE 5.84****Solving a System of Linear Equations Using Substitution**

Solve this system of linear equations by substitution:

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

🔍 **Solution**

<b>Step 1:</b> Solve one of the equations for either variable.	We'll solve the first equation for $y$ .	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$ $2x + y = 7$ $y = 7 - 2x$
<b>Step 2:</b> Substitute the expression from Step 1 into the other equation.	We replace $y$ in the second equation with the expression $7 - 2x$ .	$x - 2y = 6$ $x - 2(7 - 2x) = 6$
<b>Step 3:</b> Solve the resulting equation.	Now we have an equation with just 1 variable. We know how to solve this!	$x - 2(7 - 2x) = 6$ $x - 14 + 4x = 6$ $5x = 20$ $x = 4$
<b>Step 4:</b> Substitute the solution from Step 3 into one of the original equations to find the other variable.	We'll use the first equation and replace $x$ with 4.	$2x + y = 7$ $2(4) + y = 7$ $8 + y = 7$ $y = -1$
<b>Step 5:</b> Write the solution as an ordered pair.	The ordered pair is $(x, y)$ .	$(4, -1)$
<b>Step 6:</b> Check that the ordered pair is a solution to <b>both</b> original equations.	Substitute $x = 4, y = -1$ into both equations and make sure they are both true.	$2x + y = 7 \qquad x - 2y = 6$ $2(4) + (-1) \stackrel{?}{=} 7 \qquad 4 - 2(-1) \stackrel{?}{=} 6$ $7 = 7\checkmark \qquad 6 = 6\checkmark$ <p>Both equations are true.</p> <p><math>(4, -1)</math> is the solution to the system.</p>

➤ **YOUR TURN 5.84**

1. Solve this system of linear equations by substitution:

$$\begin{cases} x - 3y = -3 \\ x + y = 5 \end{cases}$$

**Solving Systems of Linear Equations Using Elimination**

We have solved systems of linear equations by graphing and by substitution. Graphing works well when the variable coefficients are small, and the solution has integer values. Substitution works well when we can easily solve one equation for one of the variables and not have too many fractions in the resulting expression.

The third method of solving systems of linear equations is called the elimination method. When we solved a system by substitution, we started with two equations and two variables and reduced it to one equation with one variable. This is

what we'll do with the elimination method, too, but we'll have a different way to get there.

The elimination method is based on the Addition Property of Equality. The Addition Property of Equality says that when you add the same quantity to both sides of an equation, you still have equality. We will extend the Addition Property of Equality to say that when you add equal quantities to both sides of an equation, the results are equal. For any expressions  $a$ ,  $b$ ,  $c$ , and  $d$ :

$$\text{if } a = b$$

$$\text{and } c = d$$

$$\text{then } a + c = b + d.$$

To solve a system of equations by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites, so that we can add the equations together and eliminate that variable. Notice how that works when we add these two equations together:

$$\begin{cases} 3x + y = 5 \\ 2x - y = 0 \\ \hline 5x = 5 \end{cases}$$

The  $y$ 's add to zero and we have one equation with one variable. Let us try another one:

$$\begin{cases} x + 4y = -2 \\ 2x + 5y = -2 \end{cases}$$

This time we do not see a variable that can be immediately eliminated if we add the equations. But if we multiply the first equation by  $-2$ , we will make the coefficients of  $x$  opposites. We must multiply every term on both sides of the equation by  $-2$ .

$$\begin{cases} -2(x + 4y) = -2(2) \\ 2x + 5y = -2 \end{cases}$$

Then rewrite the system of equations.

$$\begin{cases} -2x - 8y = -4 \\ 2x + 5y = -2 \end{cases}$$

Now we see that the coefficients of the  $x$  terms are opposites, so  $x$  will be eliminated when we add these two equations.

$$\begin{cases} -2x - 8y = -4 \\ 2x + 5y = -2 \\ \hline -3y = -6 \end{cases}$$

Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations. Here's a summary of using the elimination method:

**Step 1:** Write both equations in standard form. If any coefficients are fractions, clear them.

**Step 2:** Make the coefficients of one variable opposites.

Decide which variable you will eliminate.

Multiply one or both equations so that the coefficients of that variable are opposites.

**Step 3:** Add the equations resulting from Step 2 to eliminate one variable.

**Step 4:** Solve for the remaining variable.

**Step 5:** Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.

**Step 6:** Write the solution as an ordered pair.

**Step 7:** Check that the ordered pair is a solution to both original equations.

**EXAMPLE 5.85****Solving a System of Linear Equations Using Elimination**

Solve this system of linear equations by elimination:

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

 **Solution**

<p><b>Step 1:</b> Write both equations in standard form. If any coefficients are fractions, clear them.</p>	Both equations are in standard form, $Ax + By = C$ . There are no fractions.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$
<p><b>Step 2:</b> Make the coefficients of one variable opposites. Decide which variable you will eliminate. Multiply one or both equations so that the coefficients of that variable are opposites.</p>	<p>We can eliminate the <math>y</math>'s by multiplying the first equation by 2. Multiply both sides of <math>2x + y = 7</math> by 2.</p>	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \\ \hline 2(2x + y) = 2(7) \\ x - 2y = 6 \end{cases}$
<p><b>Step 3:</b> Add the equations resulting from Step 2 to eliminate one variable.</p>	We add the $x$ 's, $y$ 's, and constants.	$\begin{array}{r} 4x + 2y = 14 \\ \underline{x - 2y = 6} \\ 5x = 20 \end{array}$
<p><b>Step 4:</b> Solve for the remaining variable.</p>	Solve for $x$ .	$x = 4$
<p><b>Step 5:</b> Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.</p>	Substitute $x = 4$ into the second equation, $x - 2y = 6$ . Then solve for $y$ .	$\begin{array}{r} x - 2y = 6 \\ 4 - 2y = 6 \\ -2y = 2 \\ y = -1 \end{array}$
<p><b>Step 6:</b> Write the solution as an ordered pair.</p>	Write it as $(x, y)$ .	$(4, -1)$
<p><b>Step 7:</b> Check that the ordered pair is a solution to <b>both</b> original equations.</p>	Substitute $x = 4, y = -1$ into $2x + y = 7$ and $x - 2y = 6$ . Do they make both equations true? Yes!	$\begin{array}{ll} 2x + y = 7 & x - 2y = 6 \\ 2(4) + (-1) \stackrel{?}{=} 7 & 4 - 2(-1) \stackrel{?}{=} 6 \\ 7 = 7\checkmark & 6 = 6\checkmark \end{array}$ <p>The solution is <math>(4, -1)</math></p>

 **YOUR TURN 5.85**

1. Solve this system of linear equations by elimination:

$$\begin{cases} x - 3y = -3 \\ x + y = 5 \end{cases}$$

## Identifying Systems with No Solution or Infinitely Many Solutions

In all the systems of linear equations so far, the lines intersected, and the solution was one point. In [Example 5.86](#) and [Example 5.87](#), we will look at a system of equations that has no solution and at a system of equations that has an infinite number of solutions.

### EXAMPLE 5.86

#### Solving a System of Linear Equations with No Solution

Solve the system by a method of your choice:

$$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$$

#### ✓ Solution

Let us solve the system of linear equations by graphing.

$$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$$

To graph the first equation, we will use its slope and  $y$ -intercept.

$$\begin{aligned} y &= \frac{1}{2}x - 3 \\ m &= \frac{1}{2} \\ b &= -3 \end{aligned}$$

To graph the second equation, we will use the intercepts.

$$x - 2y = 4$$

$x$	$y$
0	-2
4	0

Graph the lines ([Figure 5.84](#)).

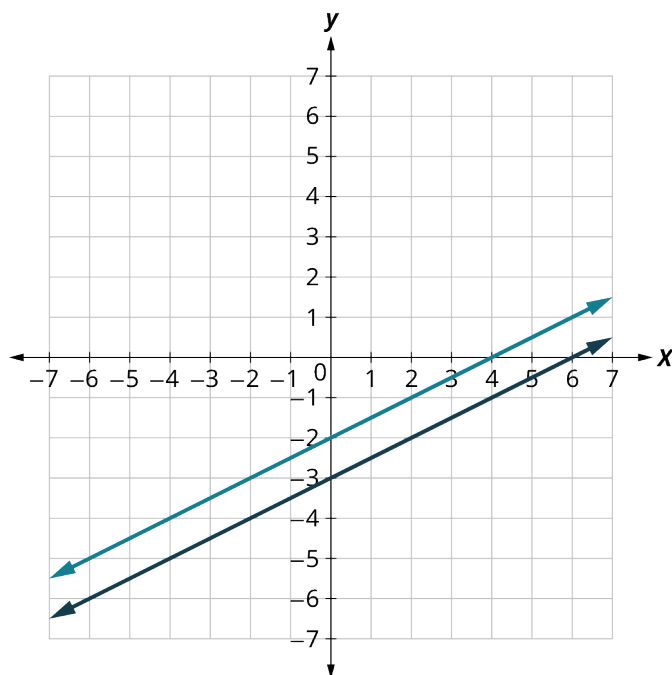


Figure 5.84

Determine the points of intersection. The lines are parallel. Since no point is on both lines, there is no ordered pair that makes both equations true. There is no solution to this system.

#### > YOUR TURN 5.86

1. Solve the system by a method of your choice:

$$\begin{cases} y = \frac{1}{3}x + 2 \\ x - 3y = 7 \end{cases}$$

#### EXAMPLE 5.87

##### Solving a System of Linear Equations with Infinite Solutions

Solve the system by a method of your choice:

$$\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$$

##### ✓ Solution

Let us solve the system of linear equations by graphing.

$$\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$$

Find the slope and  $y$ -intercept of the first equation.

$$y = 2x - 3$$

$$m = 2$$

$$b = -3$$

Find the intercepts of the second equation.

$$-6x + 3y = -9$$

$x$	$y$
0	-3
$\frac{3}{2}$	0

Graph the lines (Figure 5.85).

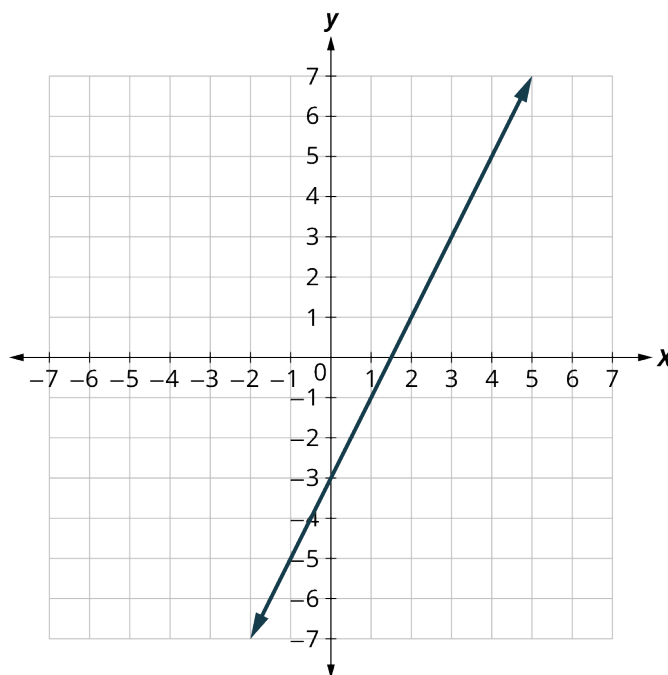


Figure 5.85

The lines are the same! Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true.

There are infinitely many solutions to this system.

**> YOUR TURN 5.87**

1. Solve the system by a method of your choice:

$$\begin{cases} y = 3x - 5 \\ -18x + 6y = -30 \end{cases}$$

In the previous example, if you write the second equation in slope-intercept form, you may recognize that the equations have the same slope and same  $y$ -intercept. Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true. There are infinitely many solutions to the system. We say the two lines are coincident. **Coincident lines** have the same slope and same  $y$ -intercept. A system of equations that has at least one solution is called a **consistent system**. A system with parallel lines has no solution. We call a system of equations like this an **inconsistent system**. It has no solution.

We also categorize the equations in a system of equations by calling the equations independent or dependent. If two equations are independent, they each have their own set of solutions. Intersecting lines and parallel lines are independent. If two equations are dependent, all the solutions of one equation are also solutions of the other equation. When we graph two dependent equations, we get coincident lines. Let us sum this up by looking at the graphs of the three types of systems. See [Figure 5.86](#) and the table that follows

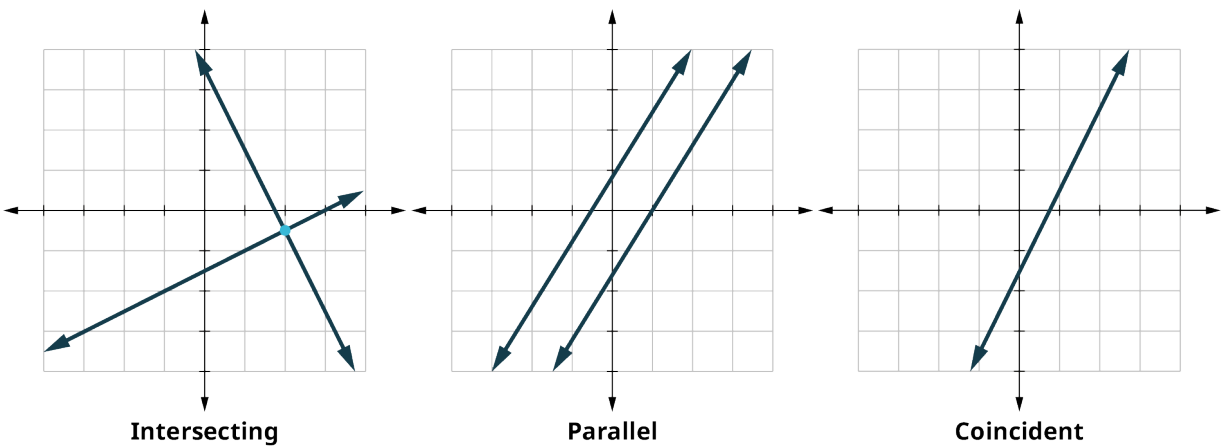


Figure 5.86

Lines	Intersecting	Parallel	Coincident
Number of Solutions	1 point	No solution	Infinitely many
Consistent/Inconsistent	Consistent	Inconsistent	Consistent
Dependent/Independent	Independent	Independent	Dependent

**WORK IT OUT****Using Matrices and Cramer's Rule to Solve Systems of Linear Equations**

An  $m$  by  $n$  matrix is an array with  $m$  rows and  $n$  columns, where each item in the matrix is a number. Matrices are used for many things, but one thing they can be used for is to represent systems of linear equations. For example, the system of linear equations

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

can be represented by the following matrix:

$$\begin{bmatrix} 2 & 1 & 7 \\ 1 & -2 & 6 \end{bmatrix}$$

To use Cramer's Rule, you need to be able to take the determinant of a matrix. The determinant of a 2 by 2 matrix  $A$ , denoted  $|A|$ , is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11} \times a_{22}) - (a_{21} \times a_{12})$$

For example, the determinant of the matrix  $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$  is  $(2 \times -2) - (1 \times 1) = -4 - 1 = -5$ .

Cramer's Rule involves taking three determinants:

1. The determinant of the first two columns, denoted  $|D|$ ;
2. The determinant of the first column and the third column, denoted  $|D_y|$ ;
3. The determinant of the third column and the first column, denoted  $|D_x|$ .

Going back to the original matrix  $\begin{bmatrix} 2 & 1 & 7 \\ 1 & -2 & 6 \end{bmatrix}$

$$|D| = \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -4 - 1 = -5$$

$$|D_y| = \begin{vmatrix} 2 & 7 \\ 1 & 6 \end{vmatrix} = 12 - 7 = 5$$

$$|D_x| = \begin{vmatrix} 7 & 1 \\ 6 & -2 \end{vmatrix} = -14 - 6 = -20$$

Now Cramer's Rule for the solution of the system will be:

$$x = \frac{|D_x|}{|D|}, y = \frac{|D_y|}{|D|}$$

Putting in the values for these determinants, we have  $x = \frac{-20}{-5} = 4$ ;  $y = \frac{5}{-5} = -1$ . The solution to the system is the ordered pair  $(4, -1)$ .

## Solving Applications of Systems of Linear Equations

Systems of linear equations are very useful for solving applications. Some people find setting up word problems with two variables easier than setting them up with just one variable. To solve an application, we will first translate the words into a system of linear equations. Then we will decide the most convenient method to use, and then solve the system.

**Step 1:** Read the problem. Make sure all the words and ideas are understood.

**Step 2:** Identify what we are looking for.

**Step 3:** Name what we are looking for. Choose variables to represent those quantities.

**Step 4:** Translate into a system of equations.

**Step 5:** Solve the system of equations using good algebra techniques.

**Step 6:** Check the answer in the problem and make sure it makes sense.

**Step 7:** Answer the question with a complete sentence.

### EXAMPLE 5.88

#### Applying System to a Real-World Application

Heather has been offered two options for her salary as a trainer at the gym. Option A would pay her \$25,000 a year plus \$15 for each training session. Option B would pay her \$10,000 a year plus \$40 for each training session. How many training sessions would make the salary options equal?

#### Solution

**Step 1:** Read the problem.

**Step 2:** Identify what we are looking for.

We are looking for the number of training sessions that would make the pay equal.

**Step 3:** Name what we are looking for.

Let  $s$  = Heather's salary, and  $n$  = the number of training sessions

**Step 4:** Translate into a system of equations.

Option A would pay her \$25,000 plus \$15 for each training session.

$$s = 25,000 + 15n$$

Option B would pay her \$10,000 + \$40 for each training session.

$$s = 10,000 + 40n$$

The system is shown.

$$\begin{cases} s = 25,000 + 15n \\ s = 10,000 + 40n \end{cases}$$

**Step 5:** Solve the system of equations.

We will use substitution.

Substitute  $25,000 + 15n$  for  $s$  in the second equation

$$\begin{aligned} s &= 25,000 + 15n \\ s &= 10,000 + 40n \end{aligned}$$

Solve for  $n$ .

$$\begin{aligned} 25,000 + 15n &= 10,000 + 40n \\ 25,000 &= 10,000 + 25n \\ 15,000 &= 25n \\ 600 &= n \end{aligned}$$

**Step 6:** Check the answer.

Are 600 training sessions a year reasonable?

Are the two options equal when  $n = 600$ ?

Substitute into each equation.

$$s = 25,000 + 15(600) = 34,000$$

$$s = 10,000 + 40(600) = 34,000$$

**Step 7:** Answer the question.

The salary options would be equal for 600 training sessions.

### YOUR TURN 5.88

1. Translate to a system of equations and then solve.

When Jenna spent 10 minutes on the elliptical trainer and then did circuit training for 20 minutes, her fitness app says she burned 278 calories. When she spent 20 minutes on the elliptical trainer and 30 minutes circuit training, she burned 473 calories. How many calories does she burn for each minute on the elliptical trainer? How many calories for each minute of circuit training?

### VIDEO

[Practice with Solving Applications of Systems of Equations \(https://openstax.org/r/Practice\\_with\\_Solving\)](https://openstax.org/r/Practice_with_Solving)

[Applications of Systems of Linear Equations \(https://openstax.org/r/Applications\\_of\\_Systems\)](https://openstax.org/r/Applications_of_Systems)

## Check Your Understanding

Decide whether it would be more convenient to solve the system of equations by substitution or elimination.

67. 
$$\begin{cases} 8x - 15y = -32 \\ 6x + 3y = -5 \end{cases}$$

68. 
$$\begin{cases} x = 4y - 3 \\ 4x - 2y = -6 \end{cases}$$

69. 
$$\begin{cases} y = 7x - 5 \\ 3x - 2y = 16 \end{cases}$$

70. 
$$\begin{cases} 12x - 5y = -42 \\ 3x + 7y = -15 \end{cases}$$

71. 
$$\begin{cases} y = 4x + 9 \\ 5x - 2y = -21 \end{cases}$$

$$72. \begin{cases} 9x - 4y = 24 \\ 3x + 5y = 14 \end{cases}$$

$$73. \begin{cases} 14x - 15y = -30 \\ 7x + 2y = 10 \end{cases}$$

$$74. \begin{cases} x = 9y - 11 \\ 2x - 7y = -27 \end{cases}$$



## SECTION 5.9 EXERCISES

For the following exercises, determine if the points are solutions to the given system of equations.

$$1. \begin{cases} 2x - 6y = 0 & \text{A: } (3, 1) \\ 3x - 4y = 5 & \text{B: } (-3, 4) \end{cases}$$

$$2. \begin{cases} -3x + y = 8 & \text{A: } (-5, -7) \\ -x + 2y = -9 & \text{B: } (-5, 7) \end{cases}$$

$$3. \begin{cases} x + y = 2 & \text{A: } \left(\frac{8}{7}, \frac{6}{7}\right) \\ y = \frac{3}{4}x & \text{B: } \left(1, \frac{3}{4}\right) \end{cases}$$

$$4. \begin{cases} 2x + 3y = 6 & \text{A: } (-6, 2) \\ y = \frac{2}{3}x + 2 & \text{B: } (-3, 4) \end{cases}$$

For the following exercises, solve the following systems of equations by graphing.

$$5. \begin{cases} -x + y = 2 \\ 2x + y = -4 \end{cases}$$

$$6. \begin{cases} y = x - 2 \\ y = -3x + 2 \end{cases}$$

$$7. \begin{cases} y = \frac{2}{3}x - 2 \\ y = -\frac{1}{3}x - 5 \end{cases}$$

$$8. \begin{cases} -x + 3y = 3 \\ x + 3y = 3 \end{cases}$$

$$9. \begin{cases} 2x - y = 4 \\ 2x + 3y = 12 \end{cases}$$

$$10. \begin{cases} -x + 2y = -6 \\ y = -\frac{1}{2}x - 1 \end{cases}$$

$$11. \begin{cases} 3x + 5y = 10 \\ y = -\frac{3}{5}x + 1 \end{cases}$$

$$12. \begin{cases} 4x = 3y + 7 \\ 8x - 6y = 14 \end{cases}$$

For the following exercises, solve the systems of equations by substitution.

$$13. \begin{cases} 2x + y = -4 \\ 3x - 2y = -6 \end{cases}$$

$$14. \begin{cases} 2x + y = -2 \\ 3x - y = 7 \end{cases}$$

$$15. \begin{cases} x - 2y = -5 \\ 2x - 3y = -4 \end{cases}$$

$$16. \begin{cases} x - 3y = -9 \\ 2x + 5y = 4 \end{cases}$$

$$17. \begin{cases} -2x + 2y = 6 \\ y = -3x + 1 \end{cases}$$

$$18. \begin{cases} 3x + 4y = 1 \\ y = -\frac{2}{5}x + 2 \end{cases}$$

$$19. \begin{cases} x = 2y \\ 4x - 8y = 0 \end{cases}$$

$$20. \begin{cases} y = \frac{7}{8}x + 4 \\ -7x + 8y = 6 \end{cases}$$

For the following exercises, solve the systems of equations by elimination.

$$21. \begin{cases} 5x + 2y = 2 \\ -3x - y = 0 \end{cases}$$

$$22. \begin{cases} 6x - 5y = -1 \\ 2x + y = 13 \end{cases}$$

$$23. \begin{cases} 2x - 5y = 7 \\ 3x - y = 17 \end{cases}$$

$$24. \begin{cases} 5x - 3y = -1 \\ 2x - y = 2 \end{cases}$$

$$25. \begin{cases} 3x - 5y = -9 \\ 5x + 2y = 16 \end{cases}$$

$$26. \begin{cases} \frac{1}{3}x - y = -3 \\ x + \frac{5}{2}y = 2 \end{cases}$$

$$27. \begin{cases} 6x + 3y = 9 \\ 2x + y = 3 \end{cases}$$

$$28. \begin{cases} \frac{2}{3}x + y = 5 \\ 2x + 3y = 11 \end{cases}$$

For the following exercises, solve the system of equations by graphing, substitution, or elimination.

$$29. \begin{cases} 3x + y = -3 \\ 2x + 3y = 5 \end{cases}$$

$$30. \begin{cases} y = x + 2 \\ y = -2x + 2 \end{cases}$$

$$31. \begin{cases} y = \frac{3}{2}x + 1 \\ y = -\frac{1}{2}x + 5 \end{cases}$$

$$32. \begin{cases} x + y = -4 \\ -x + 2y = -2 \end{cases}$$

$$33. \begin{cases} -2x + 3y = 3 \\ x + 3y = 12 \end{cases}$$

$$34. \begin{cases} x + 3y = -6 \\ y = -\frac{4}{3}x + 4 \end{cases}$$

$$35. \begin{cases} 4x - 3y = 8 \\ 8x - 6y = 14 \end{cases}$$

$$36. \begin{cases} x = -3y + 4 \\ 2x + 6y = 8 \end{cases}$$

For the following exercises, translate to a system of equations and solve.

37. Jackie has been offered positions by two cable companies. The first company pays a salary of \$14,000 plus a commission of \$100 for each cable package sold. The second pays a salary of \$20,000 plus a commission of \$25 for each cable package sold. How many cable packages would need to be sold to make the total pay the same?
38. Drew burned 1,800 calories Friday playing 1 hour of basketball and canoeing for 2 hours. Saturday she spent 2 hours playing basketball and 3 hours canoeing and burned 3,200 calories. How many calories did she burn per hour when playing basketball? How many calories did she burn per hour when canoeing?
39. Mitchell currently sells stoves for company A at a salary of \$12,000 plus a \$150 commission for each stove he sells. Company B offers him a position with a salary of \$24,000 plus a \$50 commission for each stove he sells. How many stoves would Mitchell need to sell for the options to equal?
40. The total number of calories in 2 hot dogs and 3 cups of cottage cheese is 960 calories. The total number of calories in 5 hot dogs and 2 cups of cottage cheese is 1,190 calories. How many calories are in a hot dog? How many calories are in a cup of cottage cheese?
41. Andrea and Bart go to the local farmers market to purchase some fruit. Andrea buys 4 apples and 5 oranges, which cost \$3.10; Bart buys 4 apples and 6 oranges, which cost \$3.40. What is the cost of an orange? What is the cost of an apple?

42. Jack and Jill go to a local farmers market to purchase some fruit. Jack buys 3 peaches and 2 limes, which cost \$1.50; Jill buys 6 peaches and 5 limes, which cost \$3.45. What is the cost of a peach? What is the cost of a lime?

## 5.10 Systems of Linear Inequalities in Two Variables



**Figure 5.87** Many college students find part-time jobs at places such as coffee shops to help pay for college. (credit: modification of work “TULLY’S COFFEE” by MIKI Yoshihito/Flickr, CC BY 2.0)

### Learning Objectives

**After completing this section, you should be able to:**

1. Demonstrate whether an ordered pair is a solution to a system of linear inequalities.
2. Solve systems of linear inequalities using graphical methods.
3. Graph systems of linear inequalities.
4. Interpret and solve applications of linear inequalities.

In this section, we will learn how to solve systems of linear inequalities in two variables. In [Systems of Linear Equations in Two Variables](#), we learned how to solve for systems of linear equations in two variables and found a solution that would work in both equations. We can solve systems of inequalities by graphing each inequality (as discussed in [Graphing Linear Equations and Inequalities](#)) and putting these on the same coordinate system. The double-shaded part will be our solution to the system. There are many real-life examples for solving systems of linear inequalities.

Consider Ming who has two jobs to help her pay for college. She works at a local coffee shop for \$7.50 per hour and at a research lab on campus for \$12 per hour. Due to her busy class schedule, she cannot work more than 15 hours per week. If she needs to make at least \$150 per week, can she work seven hours at the coffee shop and eight hours in the lab?

### Determining If an Ordered Pair Is a Solution of a System of Linear Inequalities

The definition of a **system of linear inequalities** is similar to the definition of a system of linear equations. A system of linear inequalities looks like a system of linear equations, but it has inequalities instead of equations. A system of two linear inequalities is shown here.

$$\begin{cases} x + 4y \geq 10 \\ 3x - 2y < 12 \end{cases}$$

To solve a system of linear inequalities, we will find values of the variables that are solutions to both inequalities. We solve the system by using the graphs of each inequality and show the solution as a graph. We will find the region on the plane that contains all ordered pairs  $(x, y)$  that make both inequalities true. The solution of a system of linear inequalities is shown as a shaded region in the  $xy$ -coordinate system that includes all the points whose ordered pairs make the inequalities true.

To determine if an ordered pair is a solution to a system of two inequalities, substitute the values of the variables into each inequality. If the ordered pair makes both inequalities true, it is a solution to the system.

**EXAMPLE 5.89****Determining Whether an Ordered Pair Is a Solution to a System**

Determine whether the ordered pair is a solution to the system:

$$\begin{cases} x + 4y \geq 10 \\ 3x - 2y < 12 \end{cases}$$

1.  $(-2, 4)$
2.  $(3, 1)$

 **Solution**

1. Is the ordered pair  $(-2, 4)$  a solution?

We substitute  $x = -2$  and  $y = 4$  into both inequalities.

$$\begin{array}{rcl} x + 4y \geq 10 & & 3x - 2y < 12 \\ -2 + 4(4) \stackrel{?}{\geq} 10 & & 3(-2) - 2(4) \stackrel{?}{<} 12 \\ 14 \geq 10 \text{ true} & & -14 < 12 \text{ true} \end{array}$$

The ordered pair  $(-2, 4)$  made both inequalities true. Therefore  $(-2, 4)$  is a solution to this system.

2. Is the ordered pair  $(3, 1)$  a solution?

We substitute  $x = 3$  and  $y = 1$  into both inequalities.

$$\begin{array}{rcl} x + 4y \geq 10 & & 3x - 2y < 12 \\ 3 + 4(1) \stackrel{?}{\geq} 10 & & 3(3) - 2(1) \stackrel{?}{<} 12 \\ 7 \geq 10 \text{ false} & & 7 < 12 \text{ true} \end{array}$$

The ordered pair  $(3, 1)$  made one inequality true, but the other one false. Therefore  $(3, 1)$  is not a solution to this system.

 **YOUR TURN 5.89**

Determine whether the ordered pair is a solution to the system:

$$\begin{cases} x - 5y > 10 \\ 2x + 3y > -2 \end{cases}$$

1.  $(3, -1)$
2.  $(6, -3)$

## Solving Systems of Linear Inequalities Using Graphical Methods

The solution to a single linear inequality was the region on one side of the boundary line that contains all the points that make the inequality true. The solution to a system of two linear inequalities is a region that contains the solutions to both inequalities. We will review graphs of linear inequalities and solve the linear inequality from its graph.

**EXAMPLE 5.90****Solving a System of Linear Inequalities by Graphing**

Use [Figure 5.88](#) to solve the system of linear inequalities:

$$\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$$

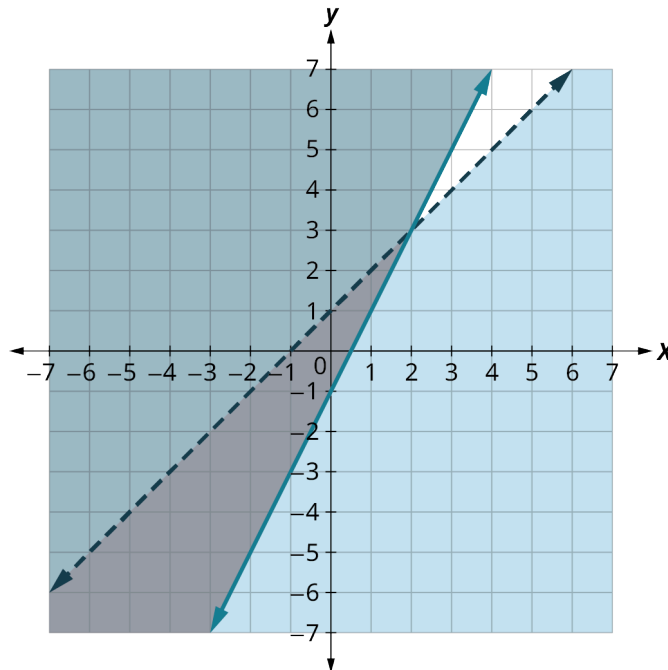


Figure 5.88

✓ **Solution**

To solve the system of linear inequalities we look at the graph and find the region that satisfies BOTH inequalities. To do this we pick a test point and check. Let's us pick  $(-1, -1)$ .

Is  $(-1, -1)$  a solution to  $y \geq 2x - 1$ ?

$$\begin{aligned} -1 &\stackrel{?}{\geq} 2(-1) - 1 \\ -1 &\geq -3 \text{ true} \end{aligned}$$

Is  $(-1, -1)$  a solution to  $y < x + 1$ ?

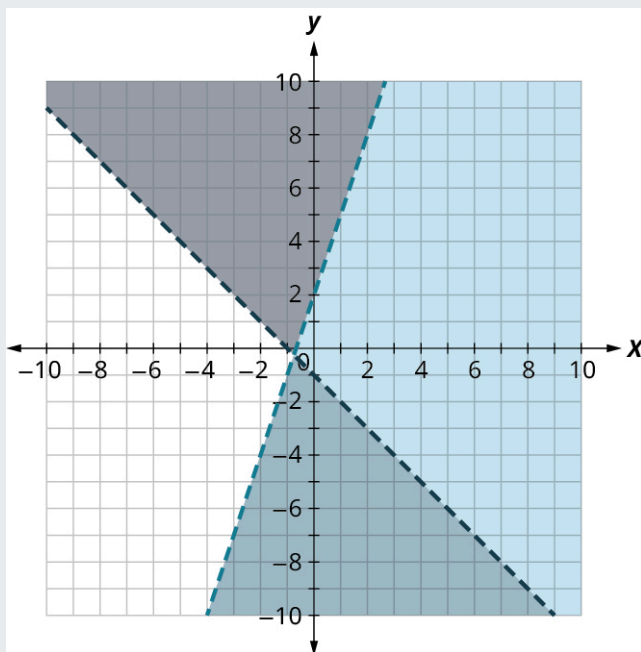
$$\begin{aligned} -1 &\stackrel{?}{<} -1 + 1 \\ -1 &< 0 \text{ true} \end{aligned}$$

The region containing  $(-1, -1)$  is the solution to the system of linear inequalities. Notice that the solution is all the points in the area shaded twice, which appears as the darkest shaded region.

> **YOUR TURN 5.90**

1. Use the graph shown to solve the system of linear inequalities:

$$\begin{cases} y < 3x + 2 \\ y > -x - 1 \end{cases}$$



## Graphing Systems of Linear Inequalities

We learned that the solution to a system of two linear inequalities is a region that contains the solutions to both inequalities. To find this region by graphing, we will graph each inequality separately and then locate the region where they are both true. The solution is always shown as a graph.

**Step 1:** Graph the first inequality.

Graph the boundary line.

Shade in the side of the boundary line where the inequality is true.

**Step 2:** On the same grid, graph the second inequality.

Graph the boundary line.

Shade in the side of that boundary line where the inequality is true.

**Step 3:** The solution is the region where the shading overlaps.

**Step 4:** Check by choosing a test point.

### EXAMPLE 5.91

#### Solving a System of Linear Inequalities by Graphing

Solve the system by graphing:

$$\begin{cases} x - y > 3 \\ y < -\frac{1}{5}x + 4 \end{cases}$$

#### ✓ Solution

Graph  $x - y > 3$  by graphing  $x - y = 3$  and testing a point (Figure 5.89). The intercepts are  $x = 3$  and  $y = -3$  and the boundary line will be dashed. Test  $(0, 0)$  which makes the inequality false so shade the side that does not contain  $(0, 0)$ .

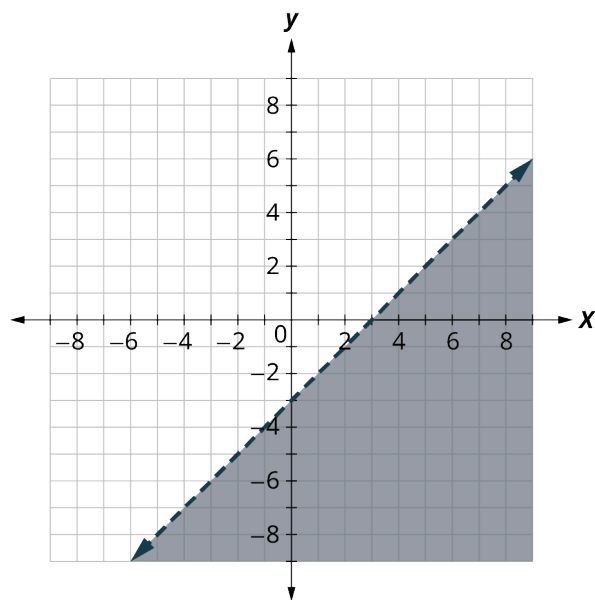


Figure 5.89

Graph  $y < -\frac{1}{5}x + 4$  by graphing  $y = -\frac{1}{5}x + 4$  using the slope  $m = -\frac{1}{5}$  and  $y$ -intercept  $b = 4$  (Figure 5.90). The boundary line will be dashed. Test  $(0, 0)$  which makes the inequality true, so shade the side that contains  $(0, 0)$ .

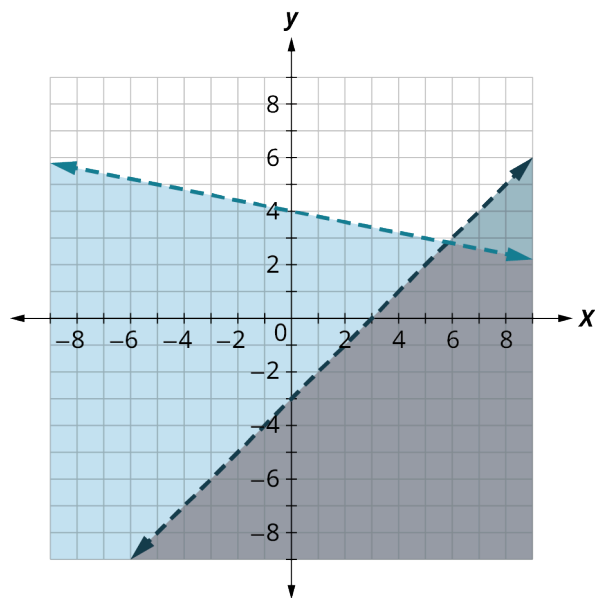


Figure 5.90

Choose a test point in the solution and verify that it is a solution to both inequalities. The point of intersection of the two lines is not included as both boundary lines were dashed. The solution is the area shaded twice—which appears as the darkest shaded region.

### > YOUR TURN 5.91

1. Solve the system by graphing:

$$\begin{cases} x + y \leq 2 \\ y \geq \frac{2}{3}x - 1 \end{cases}$$

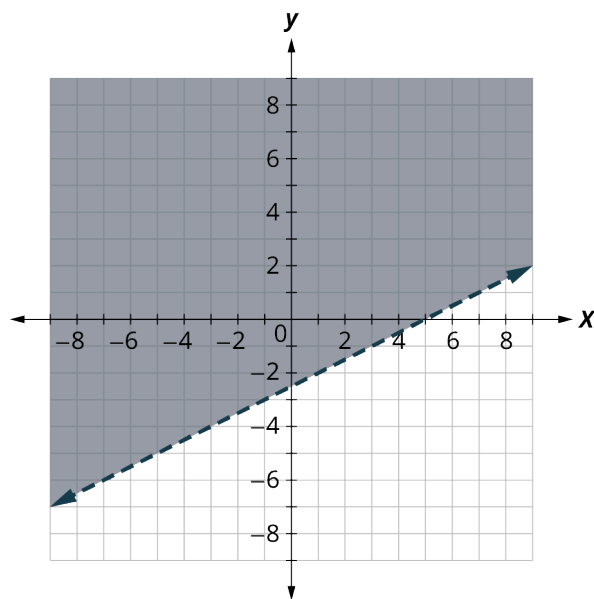
**EXAMPLE 5.92****Graphing a System of Linear Inequalities**

Solve the system by graphing:

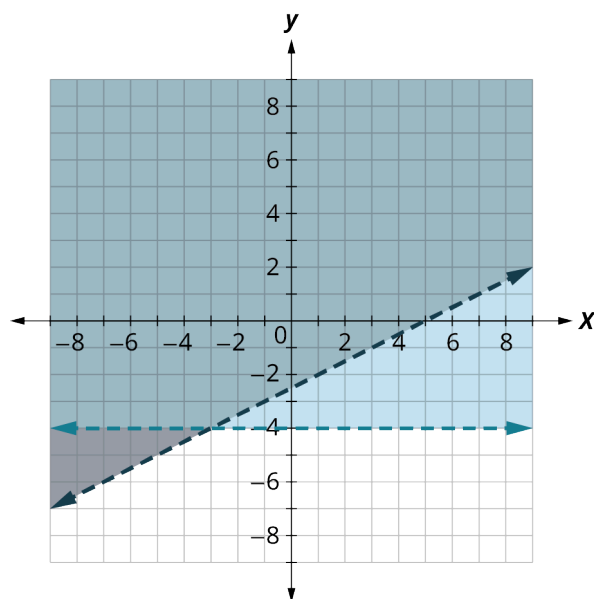
$$\begin{cases} x - 2y < 5 \\ y > -4 \end{cases}$$

**✓ Solution**

Graph  $x - 2y < 5$  by graphing  $x - 2y = 5$  (Figure 5.91) and testing a point. The intercepts are  $x = 5$  and  $y = -2.5$  and the boundary line will be dashed. Test  $(0, 0)$ , which makes the inequality true, so shade the side that contains  $(0, 0)$ .

**Figure 5.91**

Graph  $y > -4$  by graphing  $y = -4$  and recognizing that it is a horizontal line through  $y = -4$  (Figure 5.92). The boundary line will be dashed. Test  $(0, 0)$ , which makes the inequality true so shade the side that contains  $(0, 0)$ .

**Figure 5.92**

The point  $(0, 0)$  is in the solution, and we have already found it to be a solution of each inequality. The point of

intersection of the two lines is not included as both boundary lines were dashed. The solution is the area shaded twice, which appears as the darkest shaded region.

### > YOUR TURN 5.92

1. Solve the system by graphing:

$$\begin{cases} y \geq 3x - 2 \\ y < -1 \end{cases}$$

Systems of linear inequalities where the boundary lines are parallel might have no solution. We will see this in the next example.

### EXAMPLE 5.93

#### Graphing Parallel Boundary Lines with No Solution

Solve the system by graphing:

$$\begin{cases} 4x + 3y \geq 12 \\ y < -\frac{4}{3}x + 1 \end{cases}$$

#### ✓ Solution

Graph  $4x + 3y \geq 12$ , by graphing  $4x + 3y = 12$  (Figure 5.93) and testing a point. The intercepts are  $x = 3$  and  $y = 4$  and the boundary line will be solid. Test  $(0, 0)$ , which makes the inequality false, so shade the side that does not contain  $(0, 0)$ .

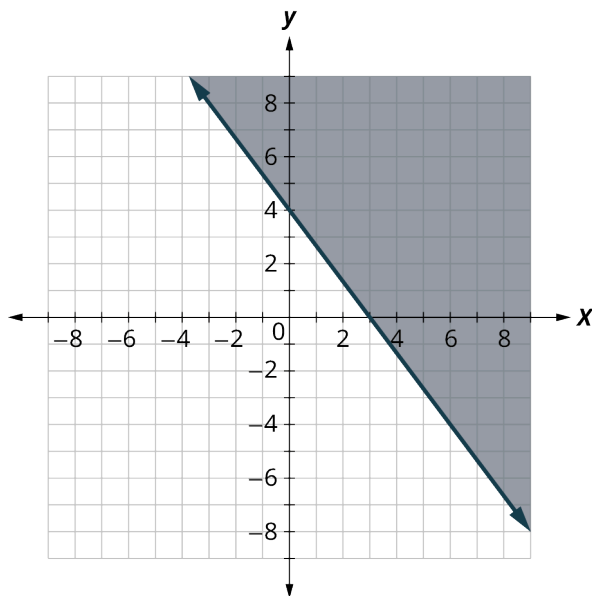


Figure 5.93

Graph  $y < -\frac{4}{3}x + 1$  by graphing  $y = -\frac{4}{3}x + 1$  using the slope  $m = -\frac{4}{3}$  and  $y$ -intercept  $b = 1$  (Figure 5.94). The boundary line will be dashed. Test  $(0, 0)$ , which makes the inequality true, so shade the side that contains  $(0, 0)$ .

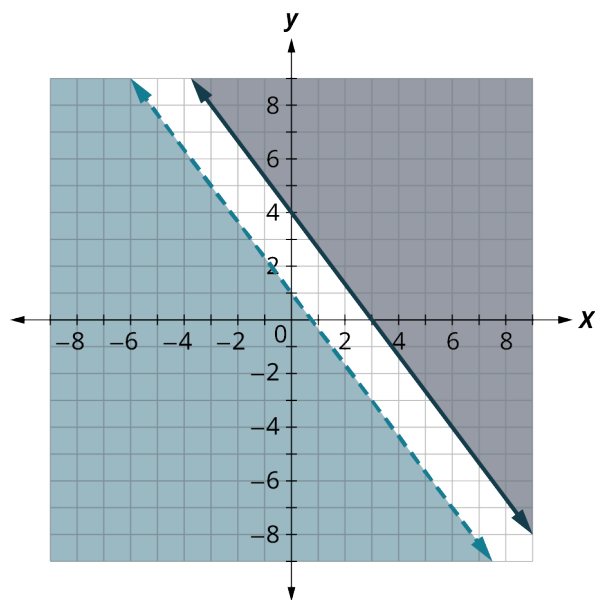


Figure 5.94

No shared point exists in both shaded regions, so the system has no solution.

### > YOUR TURN 5.93

1. Solve the system by graphing:

$$\begin{cases} 3x - 2y \geq 12 \\ y \geq \frac{3}{2}x + 1 \end{cases}$$

Some systems of linear inequalities where the boundary lines are parallel will have a solution. We will see this in the next example.

### EXAMPLE 5.94

#### Graphing Parallel Boundary Lines with a Solution

Solve the system by graphing:

$$\begin{cases} y > \frac{1}{2}x - 4 \\ x - 2y < -4 \end{cases}$$

#### ☑ Solution

Graph  $y > \frac{1}{2}x - 4$  by graphing  $y = \frac{1}{2}x - 4$  using the slope  $m = \frac{1}{2}$  and the  $y$ -intercept  $b = -4$  (Figure 5.95). The boundary line will be dashed. Test  $(0, 0)$ , which makes the inequality true, so shade the side that contains  $(0, 0)$ .

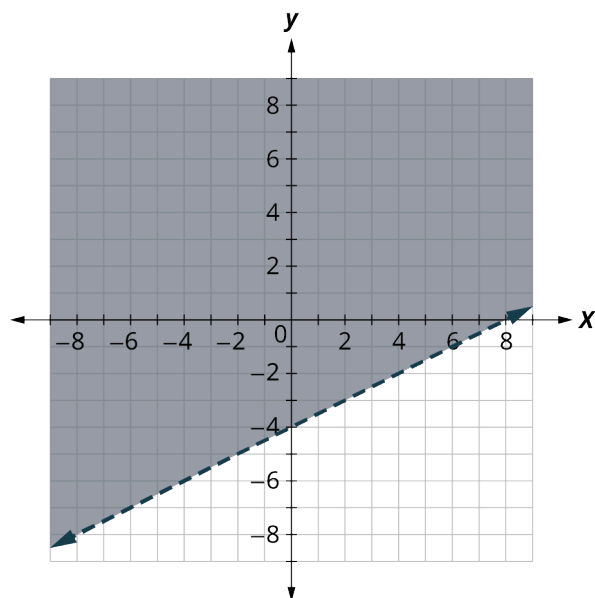


Figure 5.95

Graph  $x - 2y < -4$  by graphing  $x - 2y = -4$  (Figure 5.96) and testing a point. The intercepts are  $x = -4$  and  $y = 2$  and the boundary line will be dashed. Choose a test point in the solution and verify that it is a solution to both inequalities. Test  $(0, 0)$ , which makes the inequality false, so shade the side that does not contain  $(0, 0)$ .

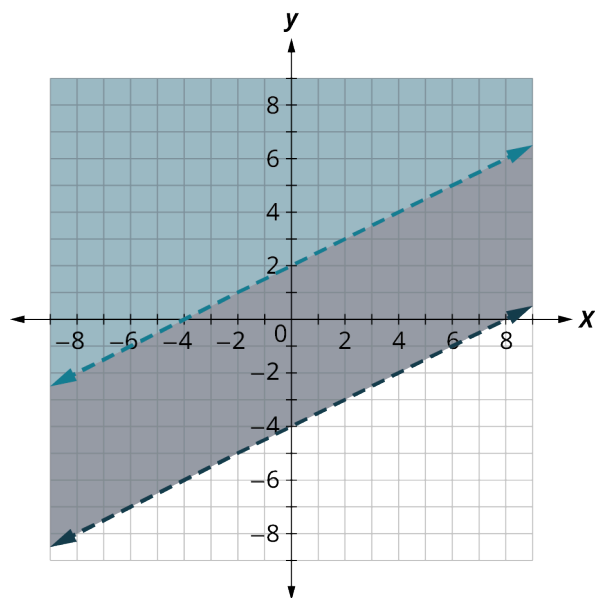


Figure 5.96

No point on the boundary lines is included in the solution as both lines are dashed. The solution is the region that is shaded twice which is also the solution to  $x - 2y < -4$ .

#### YOUR TURN 5.94

1. Solve the system by graphing:

$$\begin{cases} y \geq 3x + 1 \\ -3x + y \geq -4 \end{cases}$$

## Interpreting and Solving Applications of Linear Inequalities

When solving applications of systems of inequalities, first translate each condition into an inequality. Then graph the system, as we did above, to see the region that contains the solutions. Many situations will be realistic only if both variables are positive, so add inequalities to the system as additional requirements.

### EXAMPLE 5.95

#### Applying Linear Inequalities to Calculating Photo Costs

A photographer sells their prints at a booth at a street fair. At the start of the day, they want to have at least 25 photos to display at their booth. Each small photo they display costs \$4 and each large photo costs \$10. They do not want to spend more than \$200 on photos to display.

1. Write a system of inequalities to model this situation.
2. Graph the system.
3. Could they display 10 small and 20 large photos?
4. Could they display 20 large and 10 small photos?

#### ✓ Solution

1. Let  $x$  = the number of small photos and  $y$  = the number of large photos. To find the system of equations translate the information. They want to have at least 25 photos.

The number of small plus the number of large should be at least 25.	$x + y \geq 25$
\$4 for each small and \$10 for each large must be no more than \$200	$4x + 10y \leq 200$
The number of small photos must be greater than or equal to 0.	$x \geq 0$
The number of large photos must be greater than or equal to 0.	$y \geq 0$
We have our system of equations.	$\begin{cases} x + y \geq 25 \\ 4x + 10y \leq 200 \\ x \geq 0 \\ y \geq 0 \end{cases}$

2. Since  $x \geq 0$  and  $y \geq 0$  (both are greater than or equal to) all solutions will be in the first quadrant. As a result, our graph shows only Quadrant I. To graph  $x + y \geq 25$ , graph  $x + y = 25$  as a solid line. Choose  $(0, 0)$  as a test point. Since it does not make the inequality true, shade the side that does not include the point  $(0, 0)$ .

To graph  $4x + 10y \leq 200$ , graph  $4x + 10y = 200$  as a solid line. Choose  $(0, 0)$  as a test point. Since it does make the inequality true, shade (bottom left) the side that include the point  $(0, 0)$ .

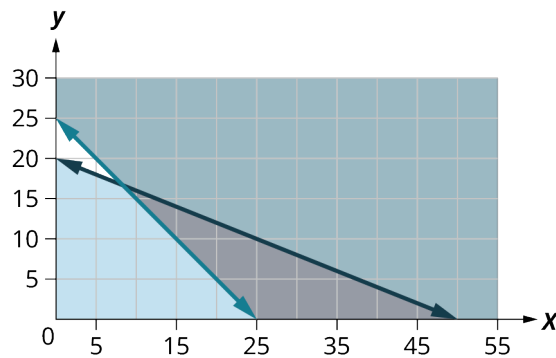


Figure 5.97

The solution of the system is the region of [Figure 5.97](#) that is shaded the darkest. The boundary line sections that border the darkly shaded section are included in the solution as are the points on the  $x$ -axis from  $(25, 0)$  to  $(55, 0)$ .

- To determine if 10 small and 20 large photos would work, we look at the graph to see if the point (10, 20) is in the solution region. We could also test the point to see if it is a solution of both equations. It is not, so the photographer would not display 10 small and 20 large photos.
- To determine if 20 small and 10 large photos would work, we look at the graph to see if the point (20, 10) is in the solution region. We could also test the point to see if it is a solution of both equations. It is, so the photographer could choose to display 20 small and 10 large photos. Notice that we could also test the possible solutions by substituting the values into each inequality.

### > YOUR TURN 5.95

Omar needs to eat at least 800 calories before going to his team practice. All he wants is hamburgers and cookies, and he doesn't want to spend more than \$5. At the hamburger restaurant near his college, each hamburger has 240 calories and costs \$1.40. Each cookie has 160 calories and costs \$0.50.

- Write a system of inequalities to model this situation.
- Graph the system.
- Could he eat 3 hamburgers and 2 cookies?
- Could he eat 2 hamburgers and 4 cookies?

### ▶ VIDEO

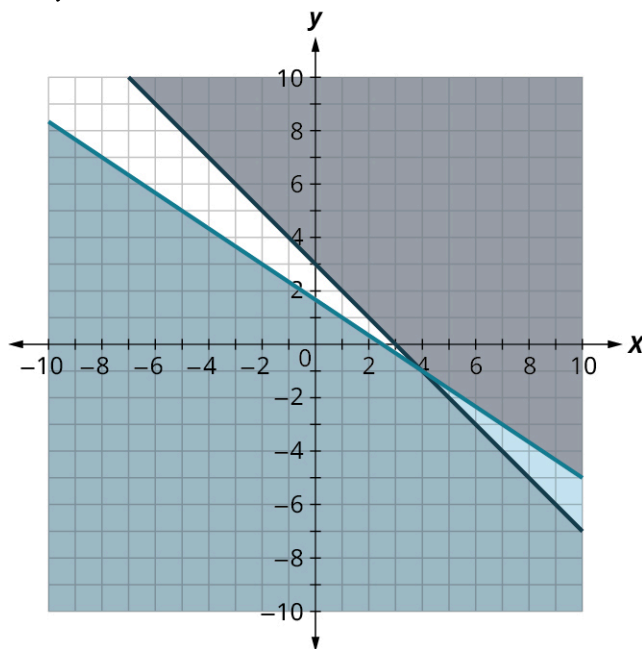
[Solving Systems of Linear Inequalities by Graphing \(https://openstax.org/r/Solving\\_Systems\)](https://openstax.org/r/Solving_Systems)

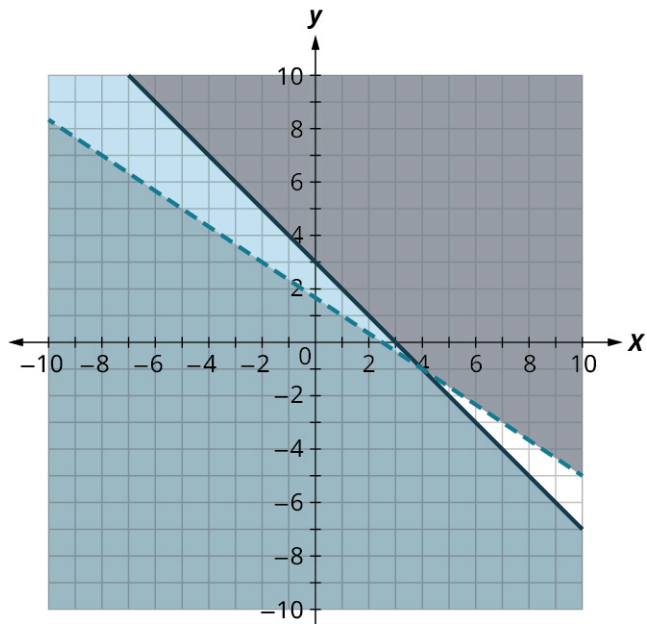
[Systems of Linear Inequalities \(https://openstax.org/r/Systems\\_of\\_Linear\)](https://openstax.org/r/Systems_of_Linear)

## Check Your Understanding

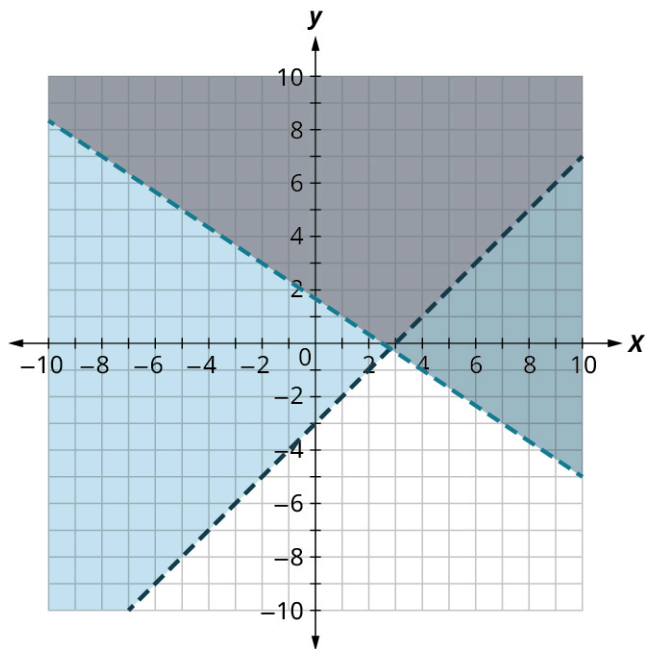
Match the correct graph to its system of inequalities.

75. 
$$\begin{cases} 2x + 3y < 5 \\ x - y > 3 \end{cases}$$

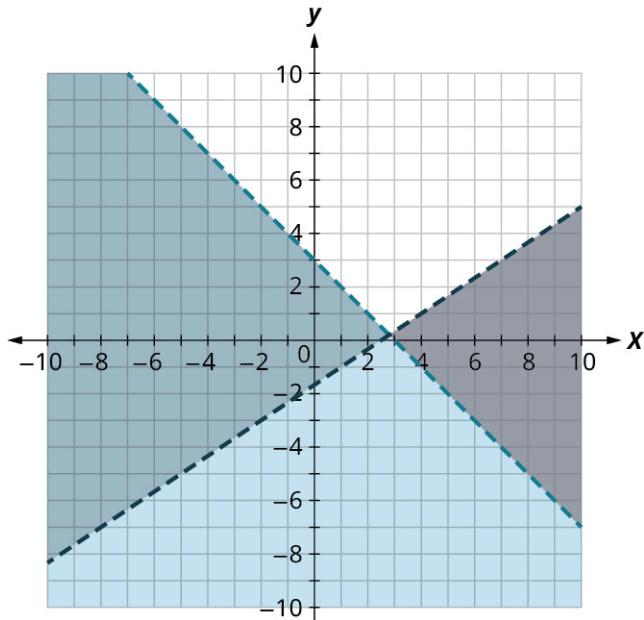




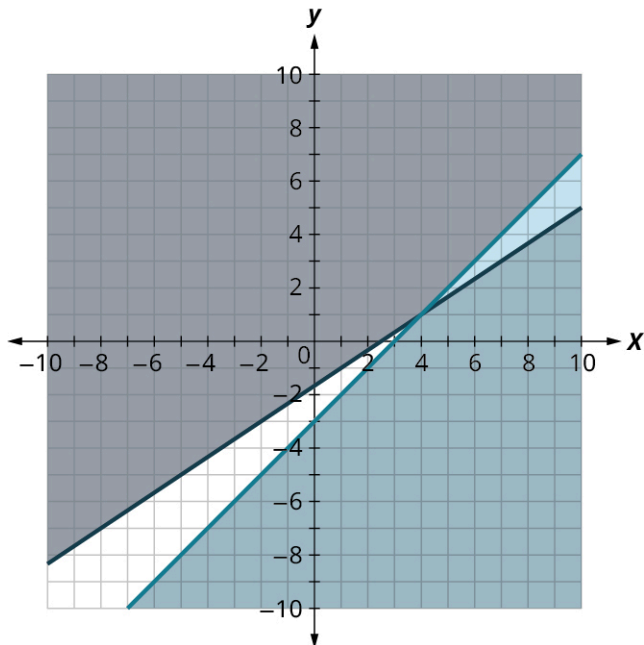
b.



c.

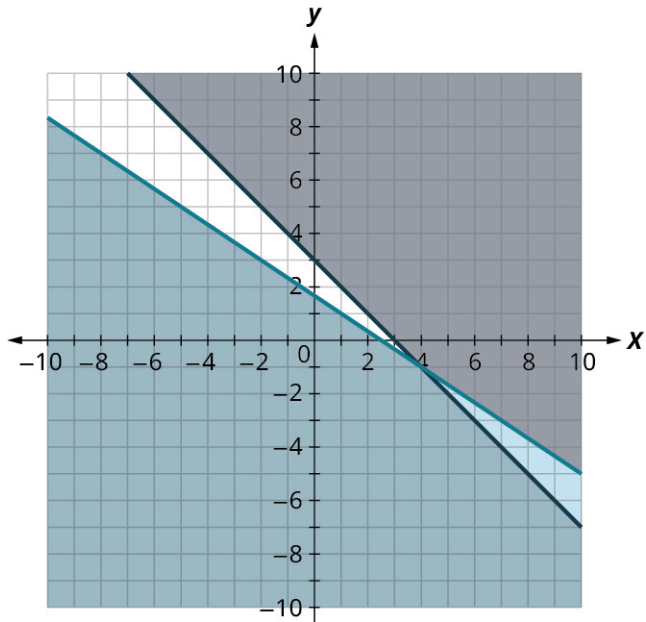


d.

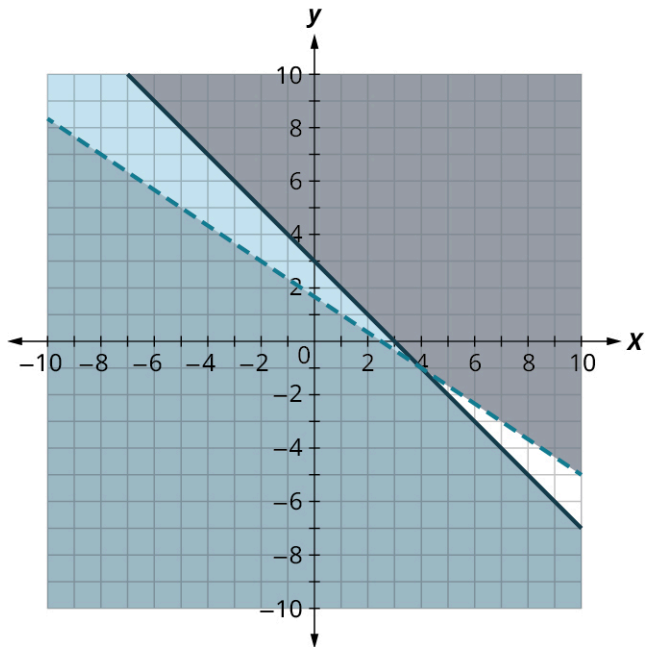


e.

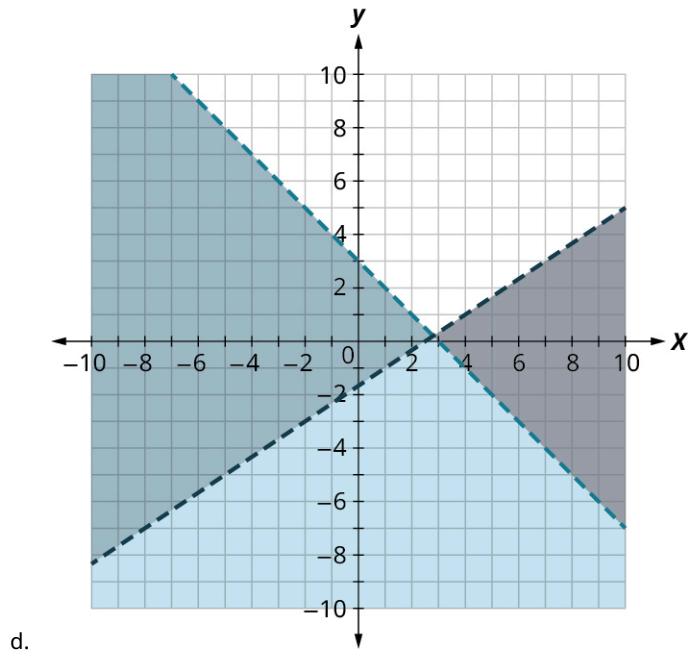
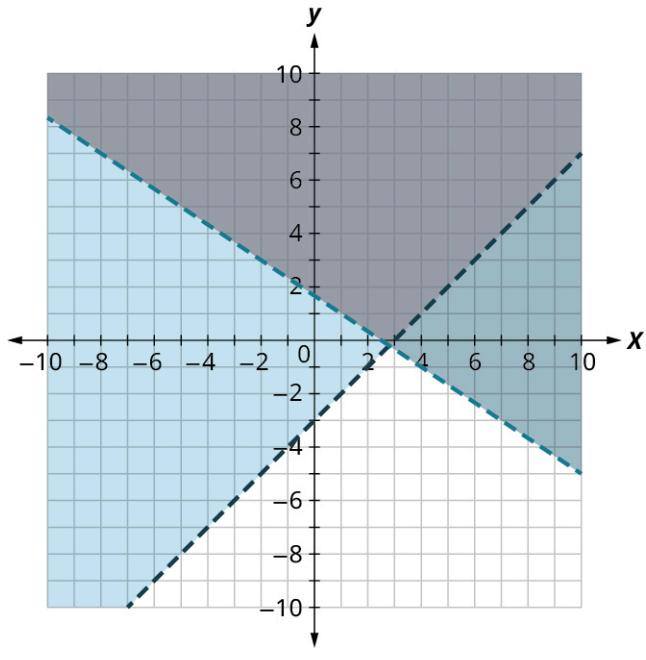
76. 
$$\begin{cases} 2x - 3y \leq 5 \\ x - y \geq 3 \end{cases}$$

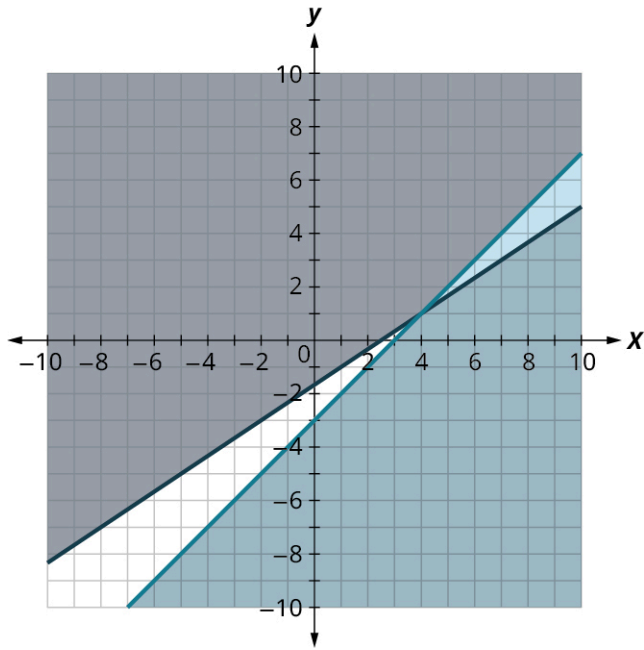


a.



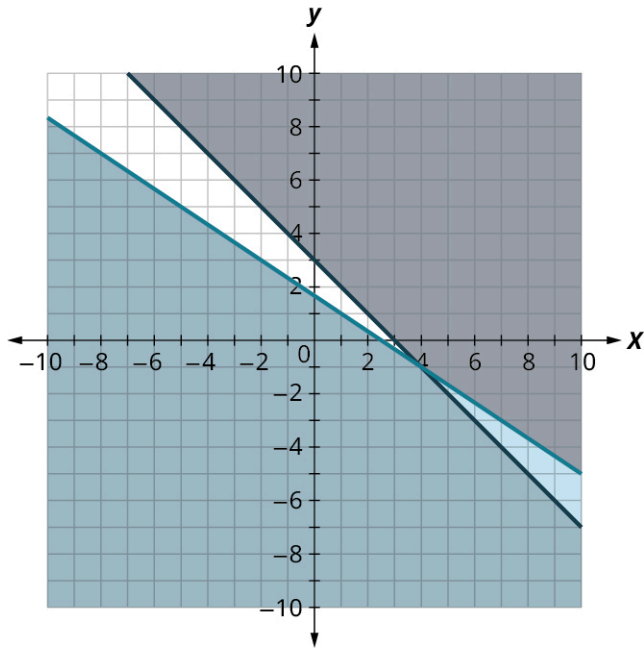
b.



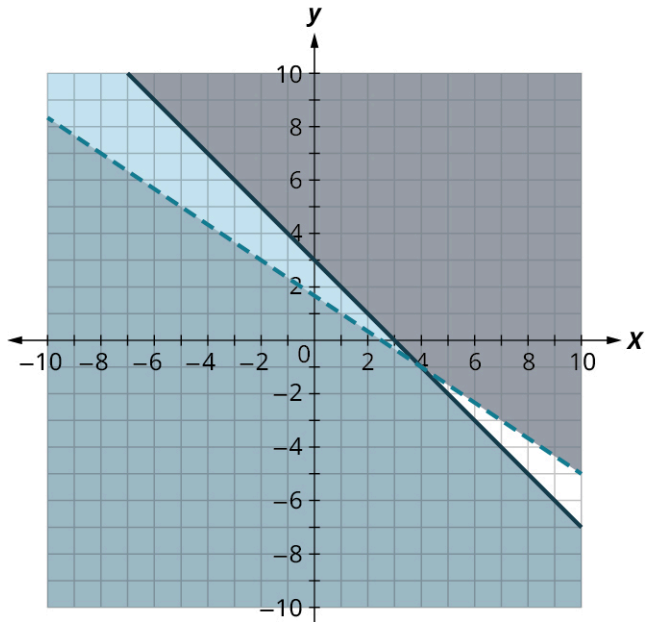


e.

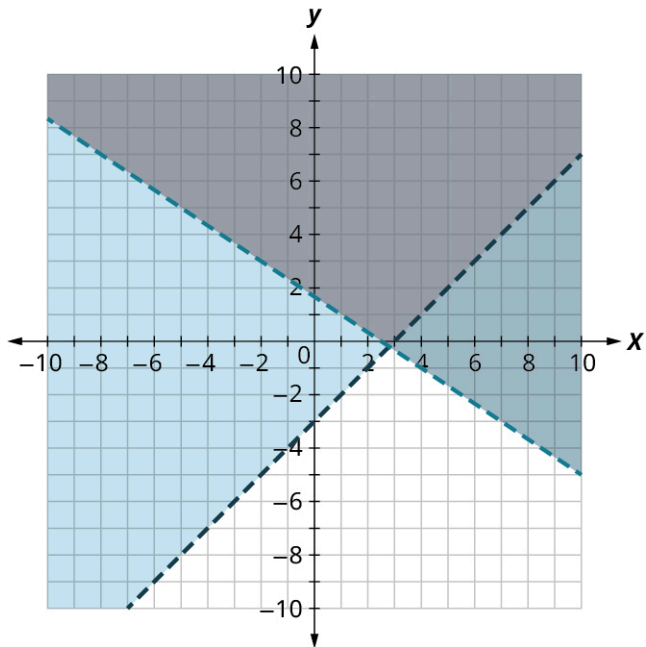
$$77. \begin{cases} 2x - 3y > 5 \\ x + y < 3 \end{cases}$$



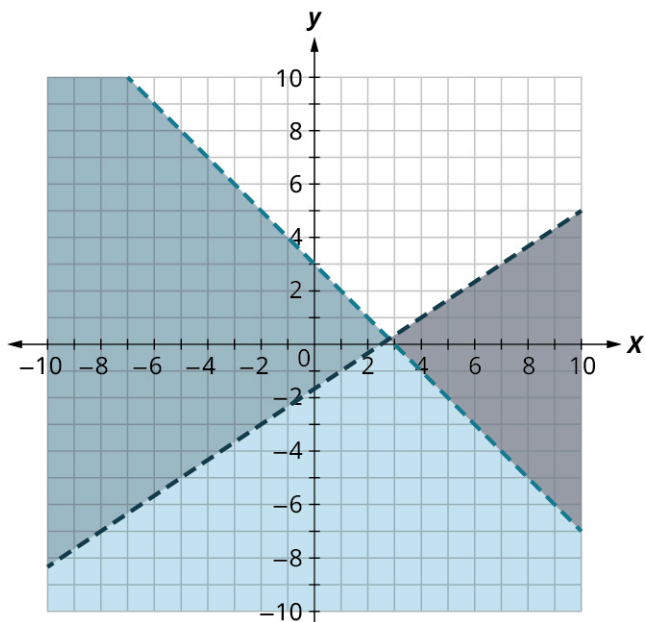
a.



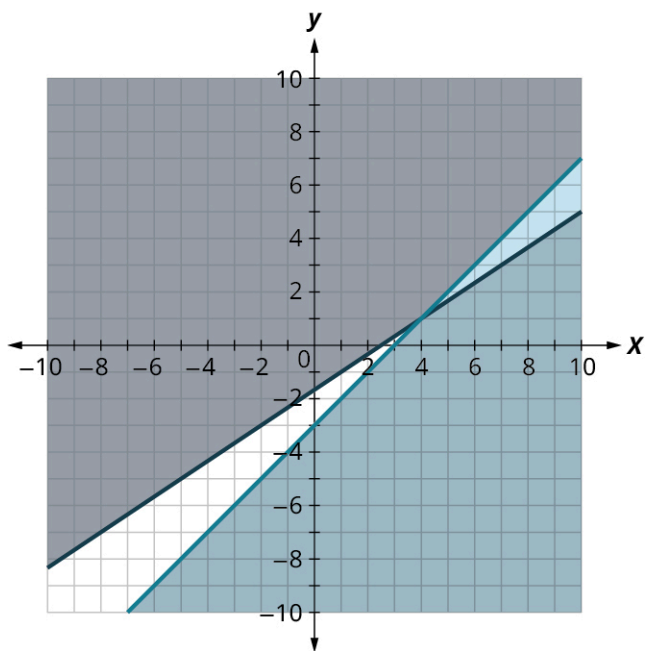
b.



c.

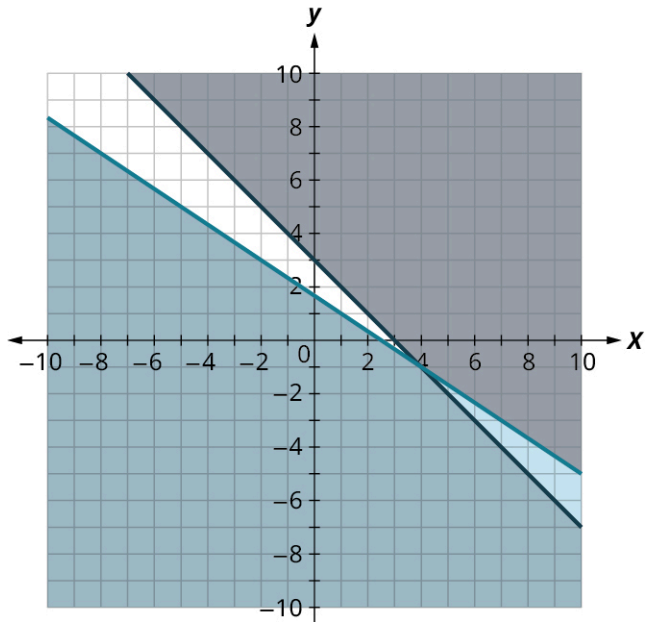


d.

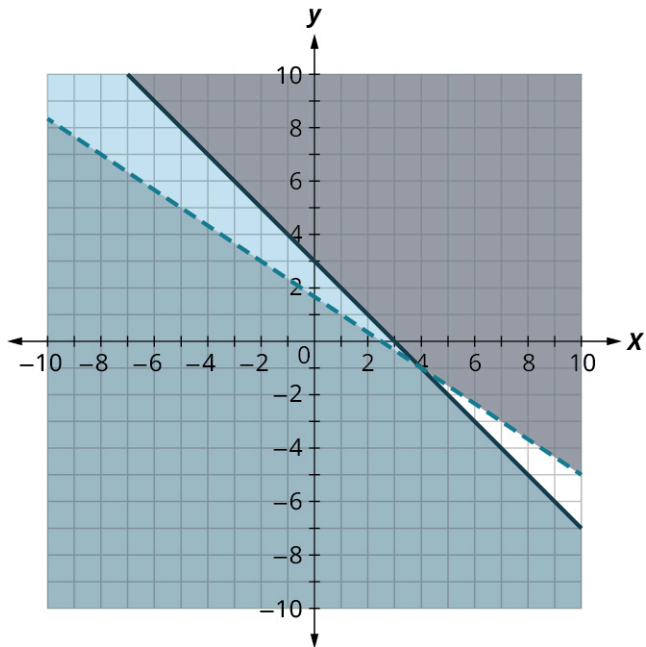


e.

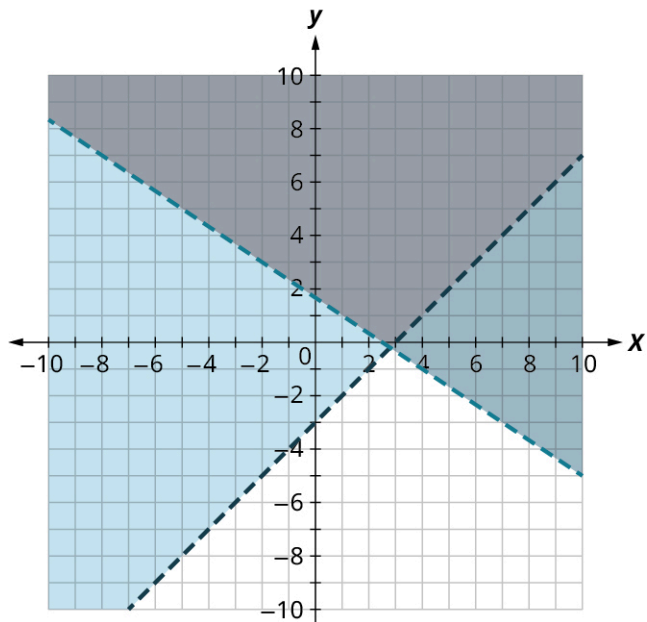
78. 
$$\begin{cases} 2x + 3y > 5 \\ x + y \leq 3 \end{cases}$$



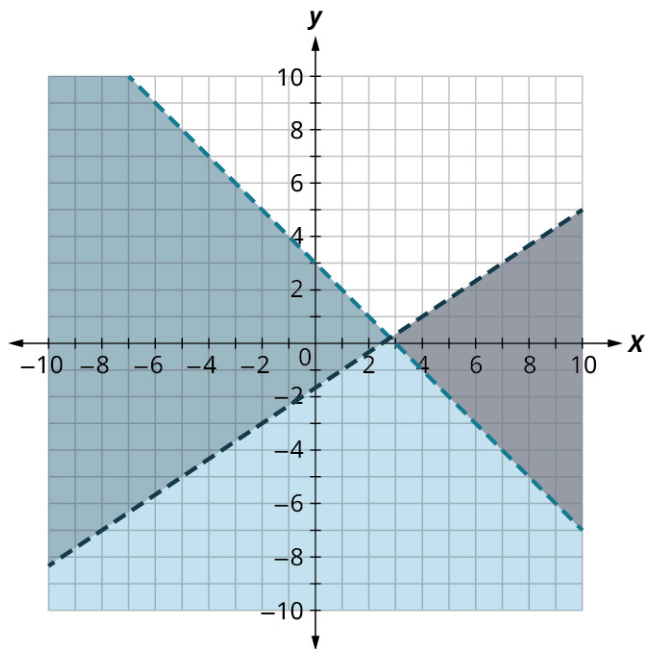
a.



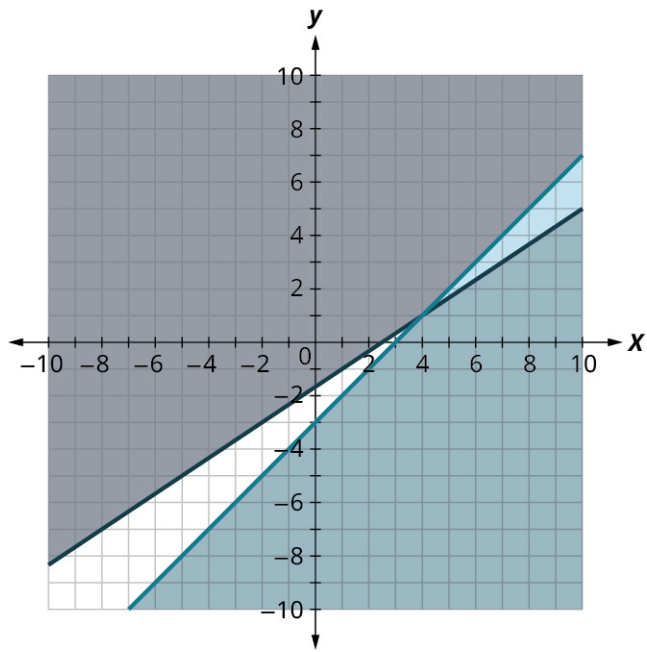
b.



c.

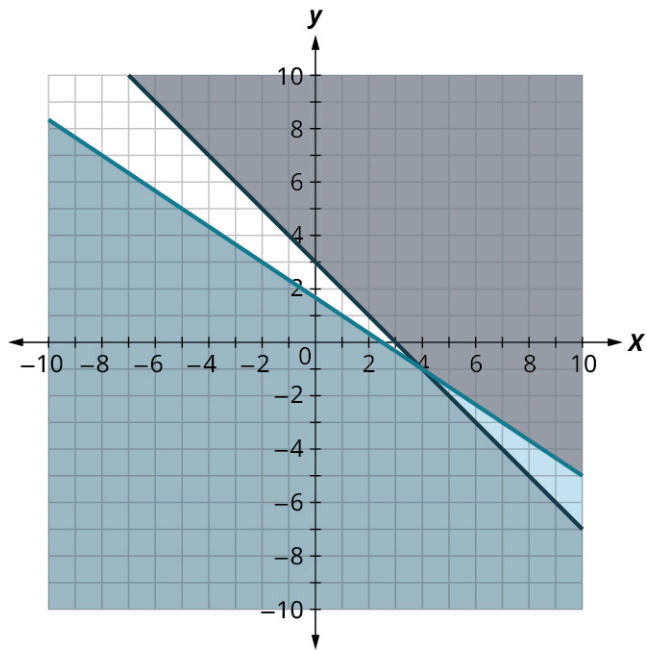


d.

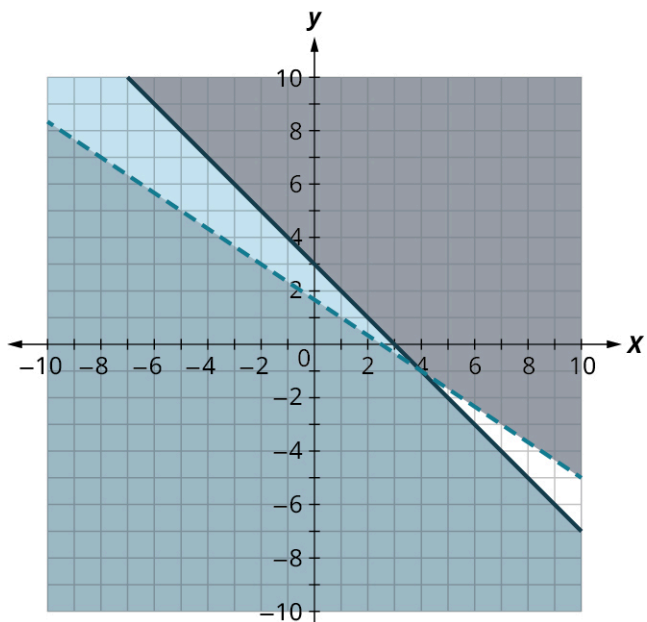


e.

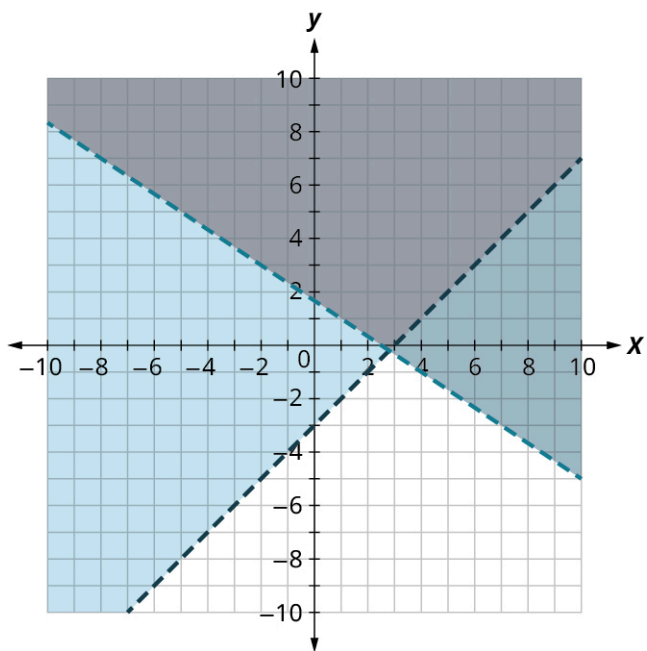
$$79. \begin{cases} 2x + 3y \leq 5 \\ x + y \geq 3 \end{cases}$$



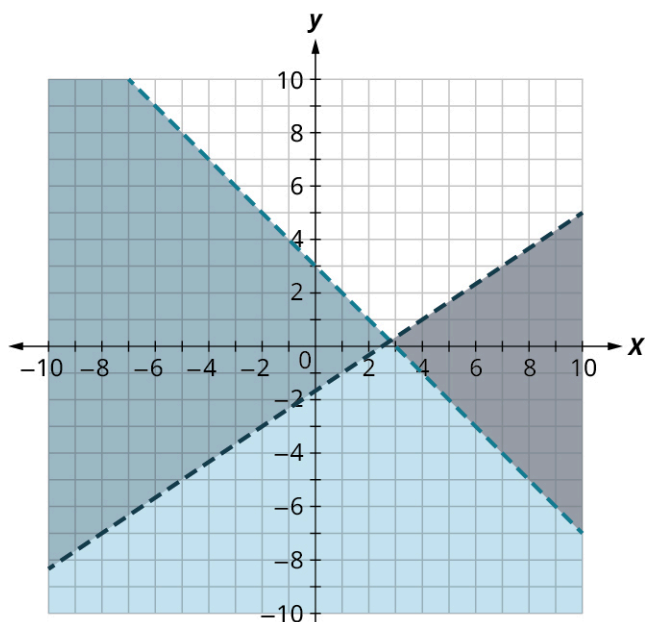
a.



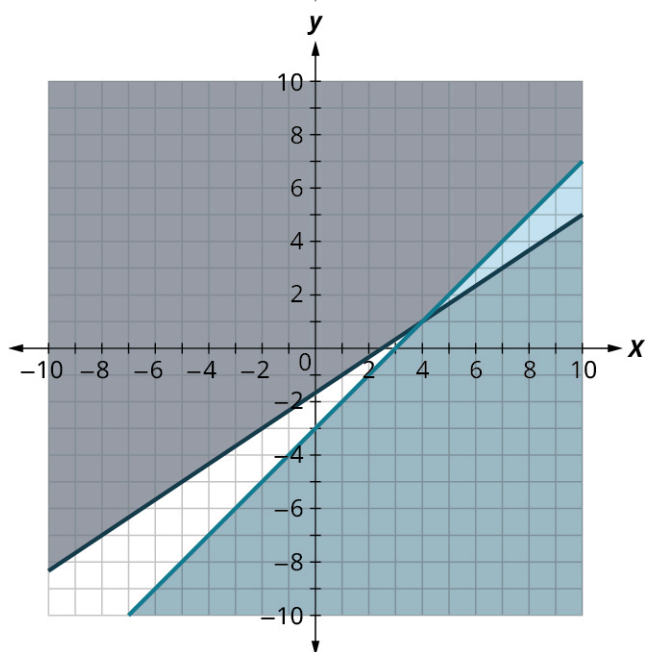
b.



c.



d.



e.



## SECTION 5.10 EXERCISES

For the following exercises, determine whether each ordered pair is a solution to the system.

1. 
$$\begin{cases} 3x + y > 5 \\ 2x - y \leq 10 \end{cases}$$
  
 A: (3, -3)  
 B: (7, 1)

2. 
$$\begin{cases} y < \frac{3}{2}x + 3 \\ \frac{3}{4}x - 2y < 5 \end{cases}$$
  
 A: (-4, -1)  
 B: (8, 3)

$$3. \begin{cases} 4x - y < 10 \\ -2x + 2y > -8 \end{cases}$$

A: (5, -2)  
B: (-1, 3)

$$4. \begin{cases} y > \frac{2}{3}x - 5 \\ x + \frac{1}{2}y \leq 4 \end{cases}$$

A: (6, -4)  
B: (3, 0)

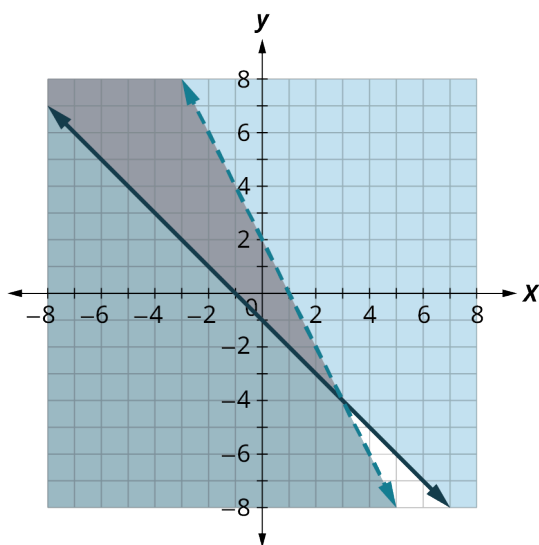
$$5. \begin{cases} 6x - 5y < 20 \\ -2x + 7y > -8 \end{cases}$$

A: (1, -3)  
B: (-4, 4)

$$6. \begin{cases} 7x + 2y > 14 \\ 5x - y \leq 8 \end{cases}$$

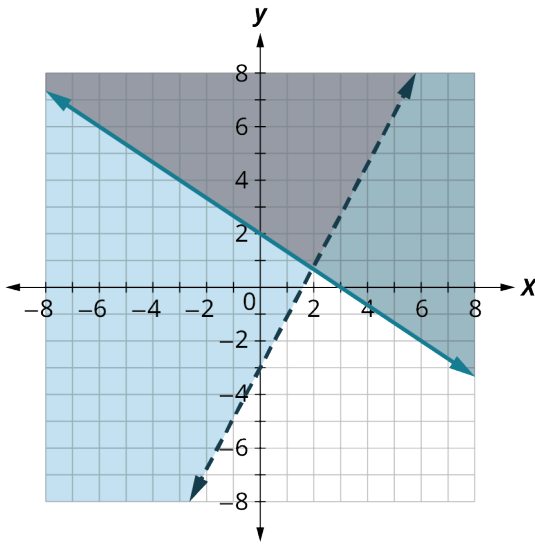
A: (2, 3)  
B: (7, -1)

For the following exercises, determine whether each ordered pair is a solution to the darkest shaded region of the graph.



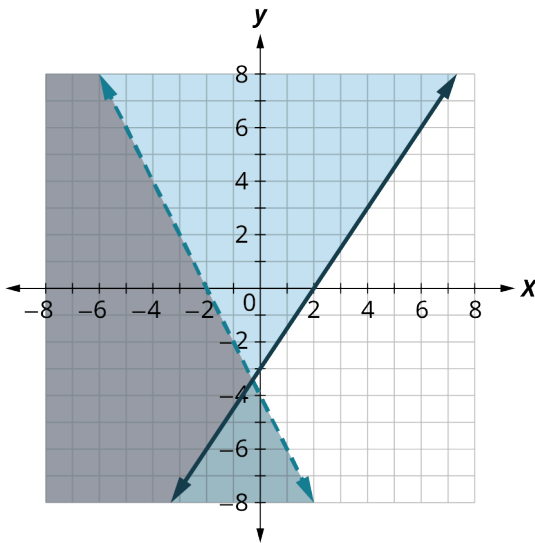
$$7.$$

A: (0, 0)  
B: (6, -8)



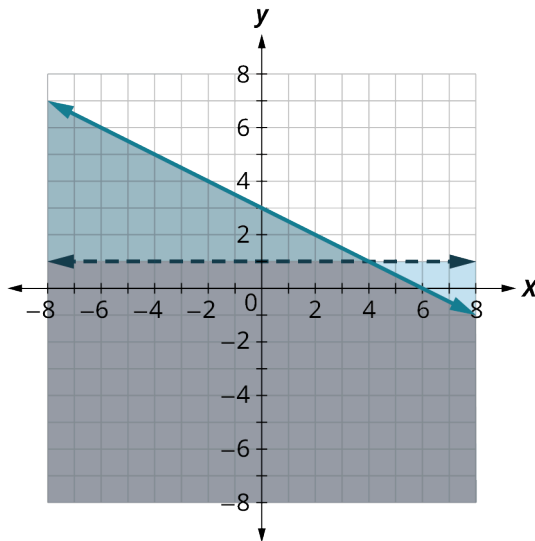
8.

A: (0, 0)  
B: (2, -4)



9.

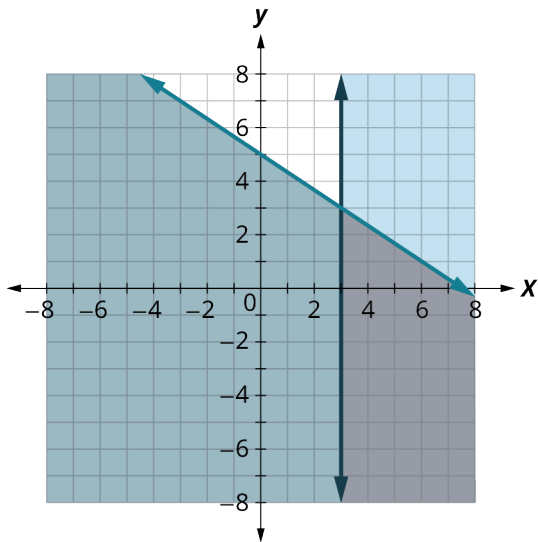
A: (0, 0)  
B: (2, 2)



10.

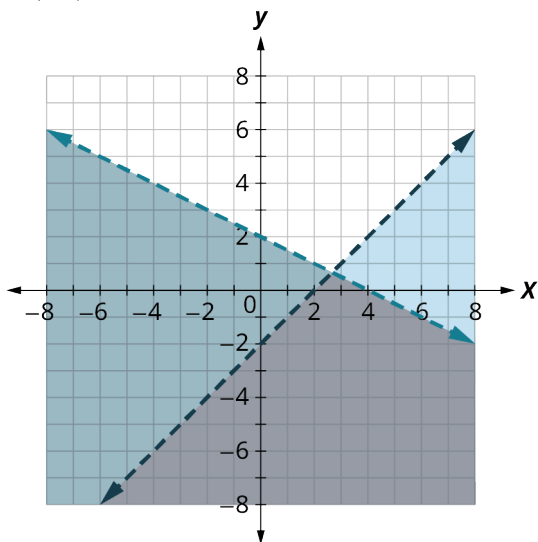
A: (0, 0)

B: (2, 3)



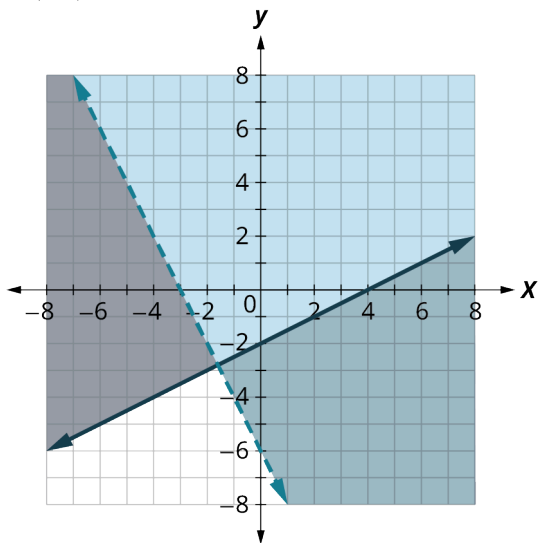
11.

A: (0, 0)  
B: (6, 0)



12.

A: (0, 0)  
B: (6, 2)



13.

A: (0, 0)

B: (2, 2)

For the following exercises, solve the systems of linear equations by graphing.

14. 
$$\begin{cases} y \leq 3x + 2 \\ y > x - 1 \end{cases}$$
15. 
$$\begin{cases} y < 2x - 1 \\ y \leq -\frac{1}{2}x + 4 \end{cases}$$
16. 
$$\begin{cases} x - y > 1 \\ y < -\frac{1}{4}x + 3 \end{cases}$$
17. 
$$\begin{cases} 3x - y \geq 6 \\ y \geq -\frac{1}{2}x \end{cases}$$
18. 
$$\begin{cases} 2x + 4y \geq 8 \\ y \leq \frac{3}{4}x \end{cases}$$
19. 
$$\begin{cases} 2x - 5y < 10 \\ 3x + 4y \geq 12 \end{cases}$$
20. 
$$\begin{cases} 2x + 2y > -4 \\ -x + 3y \geq 9 \end{cases}$$
21. 
$$\begin{cases} x - 2y < 3 \\ y \leq 1 \end{cases}$$
22. 
$$\begin{cases} x - 3y > 4 \\ y \leq -1 \end{cases}$$
23. 
$$\begin{cases} y \geq -\frac{1}{2}x - 3 \\ x \leq 2 \end{cases}$$
24. 
$$\begin{cases} x - 3y \geq 6 \\ y > \frac{1}{3}x + 1 \end{cases}$$
25. 
$$\begin{cases} y \geq \frac{3}{4}x - 2 \\ y < 2 \end{cases}$$
26. 
$$\begin{cases} 3x - 4y < 8 \\ x < 1 \end{cases}$$
27. 
$$\begin{cases} -3x + 5y > 10 \\ x > -1 \end{cases}$$
28. 
$$\begin{cases} x \geq 3 \\ y \leq 2 \end{cases}$$
29. 
$$\begin{cases} x \leq -1 \\ y \geq 3 \end{cases}$$
30. 
$$\begin{cases} 2x + 4y > 4 \\ y \leq -\frac{1}{2}x - 2 \end{cases}$$

For the following exercises, translate to a system of inequalities and solve.

A gardener does not want to spend more than \$50 on bags of fertilizer and peat moss for their garden. Fertilizer costs \$2 a bag and peat moss costs \$5 a bag. The gardener's van can hold at most 20 bags.

31. Write a system of inequalities to model this situation.
32. Graph the system.
33. Can they buy 15 bags of fertilizer and 4 bags of peat moss?
34. Can they buy 10 bags of fertilizer and 10 bags of peat moss?

For the following exercises, translate to a system of inequalities and solve.

A student is studying for their final exams in chemistry and algebra. They only have 24 hours to study, and it will take them at least 3 times as long to study for algebra than chemistry.

35. Write a system of inequalities to model this situation.
36. Graph the system.
37. Can they spend 4 hours on chemistry and 20 hours on algebra?

38. Can they spend 6 hours on chemistry and 18 hours on algebra?

For the following exercises, translate to a system of inequalities and solve.

Mara is attempting to build muscle mass. To do this, she needs to eat an additional 80 grams of protein or more in a day. A bottle of protein water costs \$3.20 and a protein bar costs \$1.75. The protein water supplies 27 grams of protein and the bar supplies 16 grams. Let  $w$  be the number of water bottles Mara can buy, and let  $b$  be the number of protein bars she can buy. If Mara has \$10 dollars to spend:

39. Write a system of inequalities to model this situation.
40. Graph the system.
41. Could she buy 3 bottles of protein water and 1 protein bar?
42. Could she buy no bottles of protein water and 5 protein bars?

For the following exercises, translate to a system of inequalities and solve.

Mark is increasing his exercise routine by running and walking at least 4 miles each day. His goal is to burn a minimum of 1,500 calories from this exercise. Walking burns 270 calories/mile and running burns 650 calories/mile.

43. Write a system of inequalities to model this situation.
44. Graph the system.
45. Could he meet his goal by walking 3 miles and running 1 mile?
46. Could he meet his goal by walking 2 miles and running 2 miles?

For the following exercises, translate to a system of inequalities and solve.

Tension needs to eat at least an extra 1,000 calories a day to prepare for running a marathon. He has only \$25 to spend on the extra food he needs and will spend it on \$0.75 donuts, which have 360 calories each, and \$2 energy drinks, which have 110 calories.

47. Write a system of inequalities that models this situation.
48. Graph the system.
49. Can he buy 8 donuts and 4 energy drinks and satisfy his caloric needs?
50. Can he buy 1 donut and 3 energy drinks and satisfy his caloric needs?

## 5.11 Linear Programming



**Figure 5.98** The aftermath of an earthquake and tsunami. (credit: modification of work "Earthquake and Tsunami Japan" by Climate and Ecosystems Change Adaptation Research University Network/Flickr, CC BY 2.0)

### Learning Objectives

After completing this section, you should be able to:

1. Compose an objective function to be minimized or maximized.
2. Compose inequalities representing a system application.
3. Apply linear programming to solve application problems.

Imagine you hear about some natural disaster striking a far-away country; it could be an earthquake, a fire, a tsunami, a tornado, a hurricane, or any other type of natural disaster. The survivors of this disaster need help—they especially need food, water, and medical supplies. You work for a company that has these supplies, and your company has decided to help by flying the needed supplies into the disaster area. They want to maximize the number of people they can help. However, there are practical constraints that need to be taken into consideration; the size of the airplanes, how much weight each airplane can carry, and so on. How do you solve this dilemma? This is where linear programming comes into play. **Linear programming** is a mathematical technique to solve problems involving finding maximums or minimums where a linear function is limited by various constraints.

As a field, linear programming began in the late 1930s and early 1940s. It was used by many countries during World War II; countries used linear programming to solve problems such as maximizing troop effectiveness, minimizing their own casualties, and maximizing the damage they could inflict upon the enemy. Later, businesses began to realize they could use the concept of linear programming to maximize output, minimize expenses, and so on. In short, linear programming is a method to solve problems that involve finding a maximum or minimum where a linear function is constrained by various factors.

### WHO KNEW?

#### A Mathematician Invents a “Tsunami Cannon”

On December 26, 2004, a massive earthquake occurred in the Indian Ocean. This earthquake, which scientists estimate had a magnitude of 9.0 or 9.1 on the Richter Scale, set off a wave of tsunamis across the Indian Ocean. The waves of the tsunami averaged over 30 feet (10 meters) high, and caused massive damage and loss of life across the coastal regions bordering the Indian Ocean.

Usama Kadri works as an applied mathematician at Cardiff University in Wales. His areas of research include fluid dynamics and non-linear phenomena. Lately, he has been focusing his research on the early detection and easing of the effects of tsunamis. One of his theories involves deploying a series of devices along coastlines which would fire acoustic-gravity waves (AGWs) into an oncoming tsunami, which in theory would lessen the force of the tsunami. Of course, this is all in theory, but Kadri believes it will work. There are issues with creating such a device: they would take a tremendous amount of electricity to generate an AGW, for instance, but if it would save lives, it may well be worth it.

## Compose an Objective Function to Be Minimized or Maximized

An **objective function** is a linear function in two or more variables that describes the quantity that needs to be maximized or minimized.

### EXAMPLE 5.96

#### Composing an Objective Function for Selling Two Products

Miriam starts her own business, where she knits and sells scarves and sweaters out of high-quality wool. She can make a profit of \$8 per scarf and \$10 per sweater. Write an objective function that describes her profit.

#### Solution

Let  $x$  represent the number of scarves sold, and let  $y$  represent the number of sweaters sold. Let  $P$  represent profit. Since each scarf has a profit of \$8 and each sweater has a profit of \$10, the objective function is  $P = 8x + 10y$ .

### YOUR TURN 5.96

1. For a fundraiser at school, the Robotics Club is selling bags of apples and bunches of bananas during lunch. They will make a profit of \$4 per bag of apples and \$6 per bunch of bananas. Write an objective function that describes the profit the Robotics Club will make.

**EXAMPLE 5.97****Composing an Objective Function for Production**

William's factory produces two products, widgets and wadgets. It takes 24 minutes for his factory to make 1 widget, and 32 minutes for his factory to make 1 wadget. Write an objective function that describes the time it takes to make the products.

**✓ Solution**

Let  $x$  equal the number of widgets made; let  $y$  equal the number of wadgets made; let  $T$  represent total time. The objective function is  $T = 24x + 32y$ .

**> YOUR TURN 5.97**

1. Suppose William has a second factory that can make widgets in 20 minutes and wadgets in 28 minutes. Write an objective function that describes the time it takes to make the products.

**Composing Inequalities Representing a System Application**

For our two examples of profit and production, in an ideal world the profit a person makes and/or the number of products a company produces would have no restrictions. After all, who wouldn't want to have an unrestricted profit? However in reality this is not the case; there are usually several variables that can restrict how much profit a person can make or how many products a company can produce. These restrictions are called **constraints**.

Many different variables can be constraints. When making or selling a product, the time available, the cost of manufacturing and the amount of raw materials are all constraints. In the opening scenario with the tsunami, the maximum weight on an airplane and the volume of cargo it can carry would be constraints. Constraints are expressed as linear inequalities; the list of constraints defined by the problem forms a system of linear inequalities that, along with the objective function, represent a system application.

**EXAMPLE 5.98****Representing the Constraints for Selling Two Products**

Two friends start their own business, where they knit and sell scarves and sweaters out of high-quality wool. They can make a profit of \$8 per scarf and \$10 per sweater. To make a scarf, 3 bags of knitting wool are needed; to make a sweater, 4 bags of knitting wool are needed. The friends can only make 8 items per day, and can use not more than 27 bags of knitting wool per day. Write the inequalities that represent the constraints. Then summarize what has been described thus far by writing the objective function for profit and the two constraints.

**✓ Solution**

Let  $x$  represent the number of scarves sold, and let  $y$  represent the number of sweaters sold. There are two constraints: the number of items the business can make in a day (a maximum of 8) and the number of bags of knitting wool they can use per day (a maximum of 27). The first constraint (total number of items in a day) is written as:

$$x + y \leq 8$$

Since each scarf takes 3 bags of knitting wool and each sweater takes 4 bags of knitting wool, the second constraint, total bags of knitting wool per day, is written as:

$$3x + 4y \leq 27$$

In summary, here are the equations that represent the new business:

$P = 8x + 10y$ ; This is the profit equation: The business makes \$8 per scarf and \$10 per sweater.

$$\begin{aligned} x + y &\leq 8 \\ 3x + 4y &\leq 27 \end{aligned}$$

> **YOUR TURN 5.98**

1. For a fundraiser at school, the Robotics Club is selling bags of apples and bunches of bananas during lunch. They will make a profit of \$4 per bag of apples and \$6 per bunch of bananas. Due to school health regulations, the club is allowed to have only 20 bags and bunches of fruit on school grounds each day to sell. Another regulation: the container where the Robotics Club keeps the fruit has a maximum weight capacity of 70 pounds. Each bag of apples weighs 3 pounds, while each bunch of bananas weighs 5 pounds. Write the inequalities that represent these constraints. Then summarize the equations that represent this system.

**EXAMPLE 5.99**

**Representing Constraints for Production**

A factory produces two products, widgets and wadgets. It takes 24 minutes for the factory to make 1 widget, and 32 minutes for the factory to make 1 wadget. Research indicates that long-term demand for products from the factory will result in average sales of 12 widgets per day and 10 wadgets per day. Because of limitations on storage at the factory, no more than 20 widgets or 17 wadgets can be made each day. Write the inequalities that represent the constraints. Then summarize what has been described thus far by writing the objective function for time and the two constraints.

✓ **Solution**

Let  $x$  equal the number of widgets made; let  $y$  equal the number of wadgets made. Based on the long-term demand, we know the factory must produce a minimum of 12 widgets and 10 wadgets per day. We also know because of storage limitations, the factory cannot produce more than 20 widgets per day or 17 wadgets per day. Writing those as inequalities, we have:

$$x \geq 12$$

$$y \geq 10$$

$$x \leq 20$$

$$y \leq 17$$

The number of widgets made per day must be between 12 and 20, and the number of wadgets made per day must be between 10 and 17. Therefore, we have:

$$12 \leq x \leq 20$$

$$10 \leq y \leq 17$$

The system is:

$$T = 24x + 32y$$

$$12 \leq x \leq 20$$

$$10 \leq y \leq 17$$

$T$  is the variable for time; it takes 24 minutes to make a widget and 32 minutes to make a wadget.

> **YOUR TURN 5.99**

1. Suppose a second factory can make widgets in 20 minutes and wadgets in 28 minutes. Research for this factory indicates that long-term demand for products from this second factory will result in average sales of 15 widgets per day and 13 wadgets per day. Because of limitations on storage at his factory, no more than 22 widgets or 19 wadgets can be made each day. Write the inequalities that represent the constraints. Then summarize what has been described thus far by writing the objective function for time and the two constraints.

## Applying Linear Programming to Solve Application Problems

There are four steps that need to be completed when solving a problem using linear programming. They are as follows:

**Step 1:** Compose an objective function to be minimized or maximized.

**Step 2:** Compose inequalities representing the constraints of the system.

**Step 3:** Graph the system of inequalities representing the constraints.

**Step 4:** Find the value of the objective function at each corner point of the graphed region.

The first two steps you have already learned. Let's continue to use the same examples to illustrate Steps 3 and 4.

### EXAMPLE 5.100

#### Solving a Linear Programming Problem for Two Products

Three friends start their own business, where they knit and sell scarves and sweaters out of high-quality wool. They can make a profit of \$8 per scarf and \$10 per sweater. To make a scarf, 3 bags of knitting wool are needed; to make a sweater, 4 bags of knitting wool are needed. The friends can only make 8 items per day, and can use not more than 27 bags of knitting wool per day. Determine the number of scarves and sweaters they should make each day to maximize their profit.

#### ☑ Solution

**Step 1:** Compose an objective function to be minimized or maximized. From [Example 5.98](#), the objective function is  $P = 8x + 10y$ .

**Step 2:** Compose inequalities representing the constraints of the system. From [Example 5.98](#), the constraints are  $x + y \leq 8$  and  $3x + 4y \leq 27$ .

**Step 3:** Graph the system of inequalities representing the constraints. Using methods discussed in [Graphing Linear Equations and Inequalities](#), the graphs of the constraints are shown below. Because the number of scarves ( $x$ ) and the number of sweaters ( $y$ ) both must be non-negative numbers (i.e.,  $x \geq 0$  and  $y \geq 0$ ), we need to graph the system of inequalities in Quadrant I only. [Figure 5.99](#) shows each constraint graphed on its own axes, while [Figure 5.100](#) shows the graph of the system of inequalities (the two constraints graphed together). In [Figure 5.100](#), the large shaded region represents the area where the two constraints intersect. If you are unsure how to graph these regions, refer back to [Graphing Linear Equations and Inequalities](#).

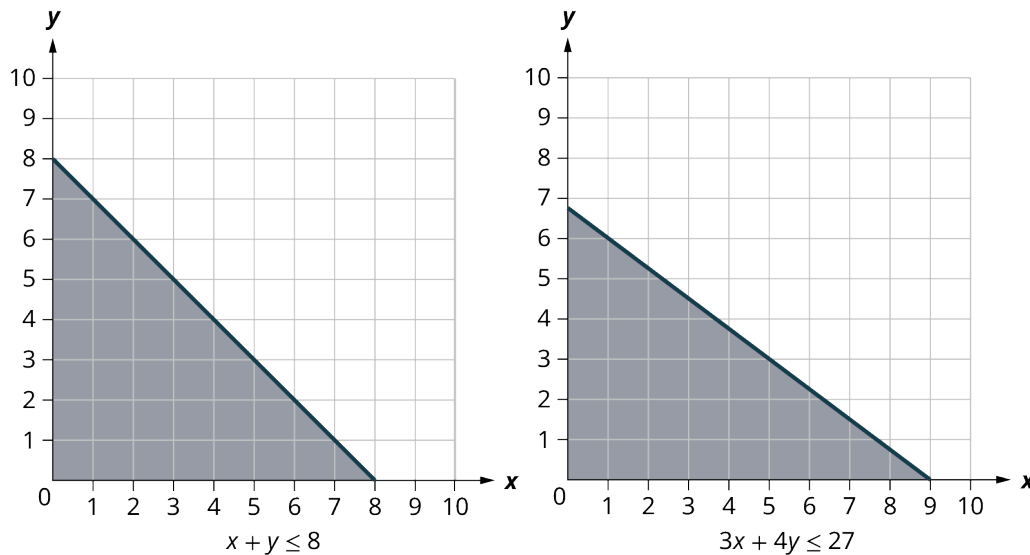
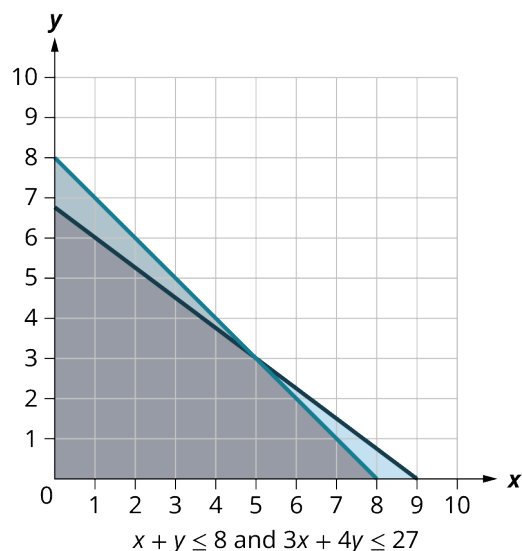
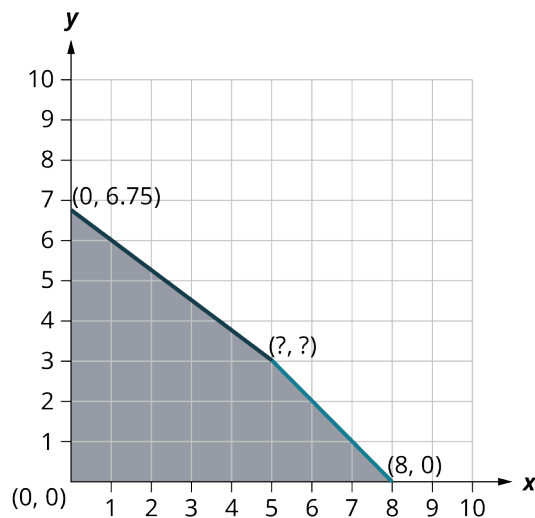


Figure 5.99 Graphs of each constraint



**Figure 5.100** Graph of the System of Inequalities

**Step 4:** Find the value of the objective function at each corner point of the graphed region. The “graphed region” is the area where both of the regions intersect; in [Figure 5.101](#), it is the large shaded area. The “corner points” refer to each vertex of the shaded area. Why the corner points? Because the maximum and minimum of every objective function will occur at one (or more) of the corner points. [Figure 5.101](#) shows the location and coordinates of each corner point.



**Figure 5.101** Graph of Region with Corner Points

Three of the four points are readily found, as we used them to graph the regions; the fourth point, the intersection point of the two constraint lines, will have to be found using methods discussed in [Systems of Linear Equations in Two Variables](#), either using substitution or elimination. As a reminder, set up the two equations of the constraint lines:

$$\begin{aligned} 3x + 4y &= 27 \\ x + y &= 8 \end{aligned}$$

For this example, substitution will be used.

$$\begin{aligned} x + y &= 8 \\ y &= 8 - x. \end{aligned}$$

Substituting  $8 - x$  into the first equation for  $y$ , we have

$$\begin{aligned}
 3x + 4(8 - x) &= 27 \\
 3x + 32 - 4x &= 27 \\
 -x &= -5 \\
 x &= 5
 \end{aligned}$$

Now, substituting the 5 in for  $x$  in either equation to solve for  $y$ . Choosing the second equation, we have:

$$\begin{aligned}
 5 + y &= 8 \\
 y &= 3
 \end{aligned}$$

Therefore,  $x = 5$ , and  $y = 3$ .

To find the value of the objective function,  $P = 8x + 10y$ , put the coordinates for each corner point into the equation and solve. The largest solution found when doing this will be the maximum value, and thus will be the answer to the question originally posed: determining the number of scarves and sweaters the new business should make each day to maximize their profit.

Corner $(x, y)$	Objective Function $P = 8x + 10y$
$(0, 0)$	$P = 8(0) + 10(0) = 0$
$(0, 6.75)$	$P = 8(0) + 10(6.75) = 67.5$
$(5, 3)$	$P = 8(5) + 10(3) = 40 + 30 = 70$
$(8, 0)$	$P = 8(8) + 10(0) = 64$

The maximum value for the profit  $P$  occurs when  $x = 5$  and  $y = 3$ . This means that to maximize their profit, the new business should make 5 scarves and 3 sweaters every day.

### > YOUR TURN 5.100

- For a fundraiser at school, the Robotics Club is selling bags of apples and bunches of bananas during lunch. They will make a profit of \$4 per bag of apples and \$6 per bunch of bananas. Due to school health regulations, the club is allowed to have only 20 bags and bunches of fruit on school grounds each day to sell. Another regulation: the container where the Robotics Club keeps the fruit has a maximum weight capacity of 70 pounds. Each bag of apples weighs 3 pounds, while each bunch of bananas weighs 5 pounds. Determine the number of bags of apples and the number of bags of bananas the Robotics Club should sell each day to maximize their profit.



### PEOPLE IN MATHEMATICS

Leonid Kantorovich

Leonid Vitalyevich Kantorovich was born January 19, 1912, in St. Petersburg, Russia. Two major events affected young Leonid's life: when he was five, the Russian Revolution began, making life in St. Petersburg very difficult; so much so that Leonid's family fled to Belarus for a year. When Leonid was 10, his father died, leaving his mother to raise five children on her own.

Despite the hardships, Leonid showed incredible mathematical ability at a young age. When he was only 14, he enrolled in Leningrad State University to study mathematics. Four years later, at age 18, he graduated with what

would be equivalent to a Ph.D. in mathematics.

Although his primary interests were in pure mathematics, in 1938 he began working on problems in economics. Supposedly, he was approached by a local plywood manufacturer with the following question: how to come up with a work schedule for eight lathes to maximize output, given the five different kinds of plywood they had at the factory. By July 1939, Leonid had come up with a solution, not only to the lathe scheduling problem but to other areas as well, such as an optimal crop rotation schedule for farmers, minimizing waste material in manufacturing, and finding optimal routes for transporting goods. The technique he discovered to solve these problems eventually became known as linear programming. He continued to use this technique for solving many other problems involving optimization, which resulted in the book *The Best Use of Economic Resources*, which was published in 1959. His continued work in linear programming would ultimately result in him winning the Nobel Prize of Economics in 1975.

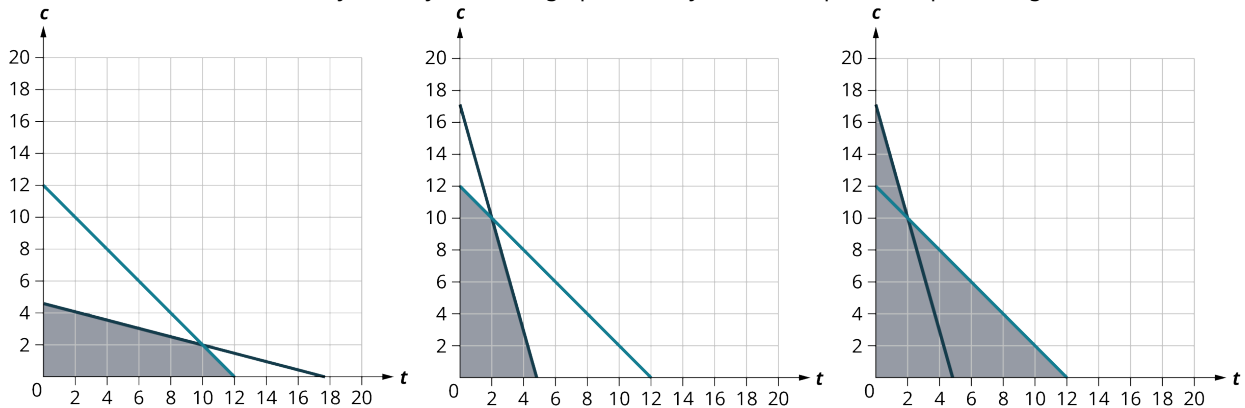
## Check Your Understanding

80. Kellie makes tables and chairs. Kellie profits \$20 from a table ( $t$ ), and \$10 from a chair ( $c$ ). The objective function for profit in this situation is:
- $P = 20t - 10c$
  - $P = 20c + 10t$
  - $P = 20t + 10c$
  - $P = 20c - 10t$
81. Dave grows wheat ( $w$ ) and barley ( $b$ ) on a farm. Dave expects to profit \$150 per acre for wheat and \$180 per acre for barley. The objective function for profit in this situation is:
- $P = 150w + 180b$
  - $P = 150b + 180w$
  - $P = 180w + 150b$
  - $P = 150w - 180b$
82. An antique music store sells two types of vinyl records; 45 rpm records ( $f$ ) and 33 rpm records ( $t$ ). It makes a profit of \$2.50 for each 45 rpm record and \$6.75 for each 33 rpm record. The objective function for profit in this situation is:
- $P = 2.50f + 6.75c$
  - $P = 2.50f + 6.75t$
  - $P = 2.50t + 6.75f$
  - None of these
83. Kellie makes tables and chairs. Kellie profits \$20 from a table ( $t$ ), and \$10 from a chair ( $c$ ). A table requires 15 board feet of wood, while a chair requires 4 board feet of wood. Kellie has 70 board feet available. What is the constraint inequality in this situation?
- $20t + 10c \leq 70$
  - $t + c \leq 70$
  - $15t + 4c \geq 70$
  - $15t + 4c \leq 70$
84. Kellie makes tables and chairs. Kellie profits \$20 from a table ( $t$ ), and \$10 from a chair ( $c$ ). The maximum number of tables and chairs Kellie can make in any one day is 12. What is the constraint inequality in this situation?
- $t + c \leq 12$
  - $20t + 10c \leq 12$
  - $20t + 10c \geq 12$
  - $20t + 10c \leq 70$
85. Dave grows wheat ( $w$ ) and barley ( $b$ ) on a farm. Dave expects to profit \$150 per acre for wheat and \$180 per acre for barley. The cost of seed is \$10 per acre for wheat and \$15 per acre for barley. Dave can only afford to spend \$945 on seed. What is the constraint inequality in this situation?
- $w + b \leq 945$
  - $10w + 15b \geq 945$

- c.  $10w + 15b \leq 945$   
 d.  $150w + 180b \leq 945$

86. Dave grows wheat ( $w$ ) and barley ( $b$ ) on a farm. Dave expects to profit \$150 per acre for wheat and \$180 per acre for barley. The cost of raising each crop is \$30 per acre for wheat and \$25 per acre for barley. Dave budgets \$1,635 for the raising of both crops. What is the constraint inequality in this situation?
- a.  $150w + 180b \leq 1,635$   
 b.  $25w + 30b \leq 1,635$   
 c.  $w + b \leq 1,635$   
 d.  $30w + 25b \leq 1,635$

87. Kellie makes tables and chairs. Kellie profits \$20 from a table ( $t$ ), and \$10 from a chair ( $c$ ). A table requires 15 board feet of wood, while a chair requires 4 board feet of wood. Kellie has 70 board feet available. The maximum number of tables and chairs Kellie can make in any one day is 12. The graph of the system of inequalities representing the constraints is:



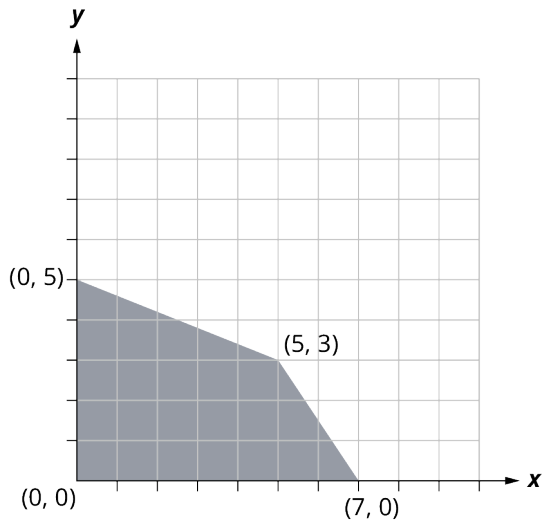
88. Kellie makes tables and chairs. Kellie profits \$20 from a table ( $t$ ), and \$10 from a chair ( $c$ ). A table requires 15 board feet of wood, while a chair requires 4 board feet of wood. Kellie has 70 board feet available. The maximum number of tables and chairs Kellie can make in any one day is 12. The four corner points of the system are:
- a.  $(0, 0), (0, 12), (10, 2), (12, 0)$   
 b.  $(0, 0), (0, 12), (2, 10), (4\frac{2}{3}, 0)$   
 c.  $(0, 0), (17.5, 0), (2, 10), (12, 0)$
89. Kellie makes tables and chairs. Kellie profits \$20 from a table ( $t$ ), and \$10 from a chair ( $c$ ). A table requires 15 board feet of wood, while a chair requires 4 board feet of wood. Kellie has 70 board feet available. The maximum number of tables and chairs Kellie can make in any one day is 12. The maximum profit Kellie can make in one day is:
- \$220    \$120    \$140    \$24



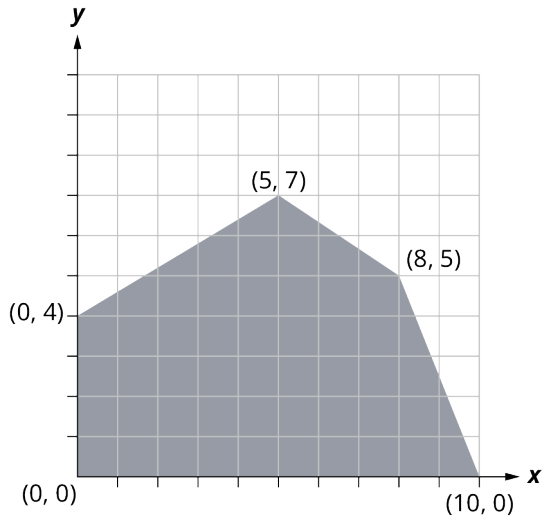
## SECTION 5.11 EXERCISES

For the following exercises, find the value of the objective function at each corner of the graphed region.

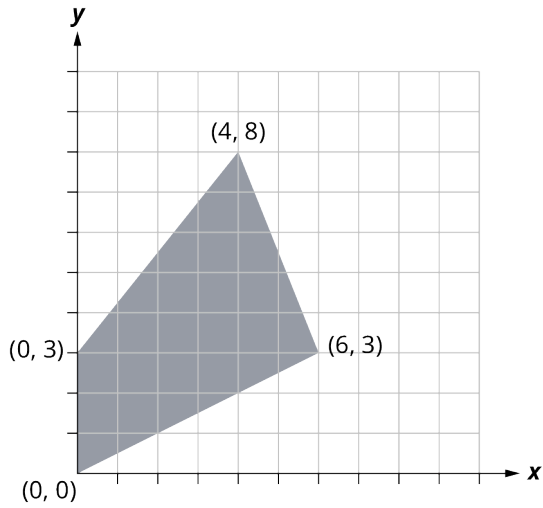
1. Objective Function  $P = 6x + 11y$ .



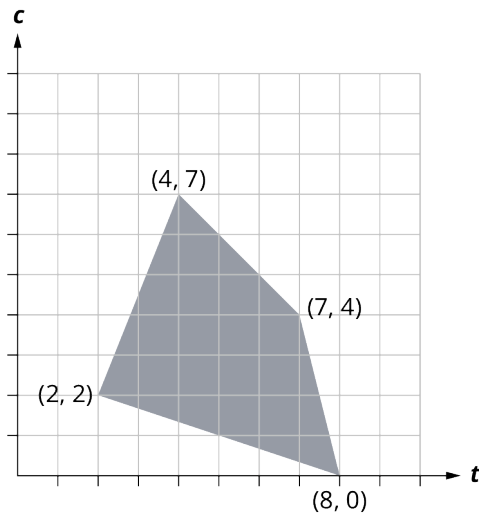
2. Objective Function  $T = 5x + 3y$



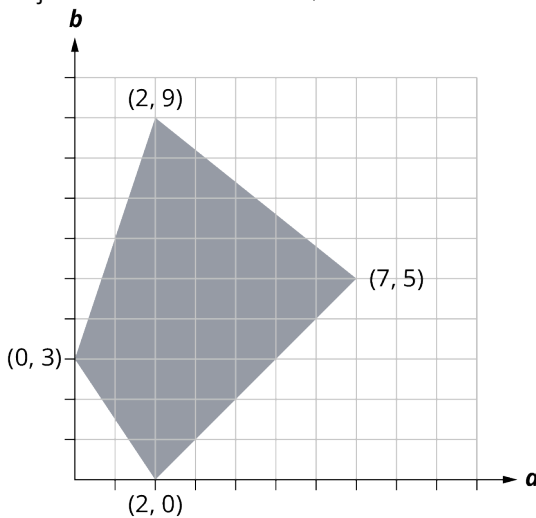
3. Objective Function  $L = 33x + 45y$



4. Objective Function  $P = 2t + 4c$



5. Objective Function  $P = 2.5a + 3.75b$



For the following exercises, write the constraint inequalities. The variables to use are given in parentheses.

6. Fernando builds birdbaths ( $x$ ) and birdhouses ( $y$ ). Fernando can make a total of 7 birdbaths and birdhouses per day. A birdbath costs \$8 to make, while a birdhouse costs \$6 to make. Fernando has \$48 to spend on building materials for the day. When he sells them, Fernando makes \$12 in profit on a birdbath and \$9 in profit on a birdhouse.
7. A fruit pie ( $p$ ) requires 12 ounces of fruit and 15 ounces of dough; a fruit tart ( $t$ ) requires 4 ounces of fruit and 3 ounces of dough. There are 72 ounces of fruit and 60 ounces of dough.
8. One recipe for chocolate cake ( $c$ ) calls for 9 ounces of chocolate chips and 4 eggs; a recipe for dark chocolate cake ( $d$ ) requires 12 ounces of chocolate chips but only 3 eggs. There are 90 ounces of chocolate chips and 36 eggs.
9. To build an outdoor bench ( $b$ ), a carpenter needs 10 pieces of wood and 26 nails; to build an outdoor chair ( $c$ ), the carpenter need 8 pieces of wood and 33 nails. There are 92 pieces of wood and 286 nails.

For the following exercises, graph each of the system of inequalities from Exercises 6–9. Assume all graphs are in the first quadrant.

10. Graph of Exercise 6
11. Graph of Exercise 7
12. Graph of Exercise 8
13. Graph of Exercise 9

For the following exercises, use the four steps for solving linear programming problems to solve.

14. A restaurant sells both regular milk and chocolate milk. To make a glass of regular milk ( $x$ ), it takes 16 ounces of, well, milk. To make a glass of chocolate milk ( $y$ ), it takes 15 ounces of milk and 1 ounce of chocolate flavoring. The restaurant makes a profit of \$1.50 per glass on regular milk and \$1.00 per glass on chocolate

- milk. At the beginning of the day, the restaurant has 600 ounces of milk and 24 ounces of chocolate flavoring. To maximize profits, how much of each should they sell that day?
15. To make a package of all-beef hot dogs ( $x$ ), a factory uses one pound of beef; to make their regular all-meat hot dogs ( $y$ ), they use  $\frac{1}{2}$  pound of beef and  $\frac{1}{2}$  pound of pork. The profit on the package of all-beef hot dogs is \$2.40 per pack; the profit on the all-meat hot dogs is \$3.20 per pack. If there are 400 pounds of beef and 250 pounds of pork available, how many of each product should the factory make to maximize their profit?
  16. A toy maker makes two plastic toys, the Ring ( $x$ ) and the Stick ( $y$ ). The toy maker makes \$5 per Ring and \$4 per Stick. The Ring uses 4 feet of plastic, while the Stick uses 3 feet of plastic. Today the toy maker has 36 feet of plastic available. The toy maker also only makes 10 plastic toys per day. To maximize profit, how many of each toy should the toy maker make?
  17. The toy maker also makes exactly two toys out of wood, the Box ( $x$ ) and the Bat ( $y$ ). The toy maker makes \$6 per Box and \$7 per Bat. Each Box requires 25 ounces of wood, and each Bat requires 40 ounces of wood. Today the toy maker has 260 ounces of wood available. The toy maker also only makes 8 wooden toys per day. To maximize profit, how many of each wooden toy should the toy maker make?
  18. Sara makes two kinds of kites out of fabric and popsicle sticks. Her Famous Flyer ( $x$ ) needs 2 yards of fabric and 9 popsicle sticks; her Gallant Glider ( $y$ ) needs 3 yards of fabric and 18 popsicle sticks. She makes a profit of \$4 on the Famous Flyer and \$6 on the Gallant Glider. Today she has 30 yards of fabric and 153 popsicle sticks. How many of each kite should she make to maximize her profit?
  19. Randy's RV Storage stores two types of Recreational Vehicles (RVs), The Xtra RV ( $x$ ) takes up 400 square feet of space, while the Yosemite RV ( $y$ ) takes up 600 square feet of space. Randy has 55,000 square feet of storage space. By local law, he is only allowed to have a maximum of 100 RVs on his property at any one time. He charges \$60 a month to store an Xtra RV, and \$80 a month to store a Yosemite RV. How many of each should he store in order to maximize his profit?
  20. A Belgian chocolatier wants to introduce two new chocolate bar creations. The first chocolate bar is called Super Dark ( $x$ ), and it consists of 90 grams of chocolate and 10 grams of sugar. The second chocolate bar is called Special Dark ( $y$ ), containing 80 grams of chocolate and 20 grams of sugar. She calculates that her company will make 1 Euro per bar of Super Dark, and 2 Euros per bar on Special Dark. She first will create some samples to sell out of 1,260 grams of chocolate and 240 grams of sugar. How many of each bar should the chocolatier create to maximize profit?
  21. A juice bottler makes two kinds of specialty juices using different mixtures of pineapple ( $x$ ) and orange ( $y$ ) juices. A 16-ounce bottle of Island Delight has 10 ounces of pineapple juice and 6 ounces of orange juice. A 16-ounce bottle of Sun Fun has 4 ounces of pineapple juice and 12 ounces of orange juice. The bottler makes \$1.60 per bottle on Island Delight and \$1.20 per bottle on Sun Fun. The amounts of juice available today are 640 ounces of pineapple juice and 768 ounces of orange juice. To maximize profit, how many of each bottle of juice should the juice bottler make?
  22. Fernando builds birdbaths ( $x$ ) and birdhouses ( $y$ ). Fernando can make a total of 7 birdbaths and birdhouses per day. A birdbath costs \$8 to make, while a birdhouse costs \$6 to make. Fernando has \$48 to spend on building materials for the day. When he sells them, Fernando makes \$12 in profit on a birdbath and \$9 in profit on a birdhouse. Determine how many of each Fernando should make to maximize his profit for the day.
  23. A farmer grows wheat ( $x$ ) and barley ( $y$ ) on his 500 acres of cropland. He expects to profit \$150 per acre for wheat and \$180 per acre for barley. The cost of raising each crop (seed, pesticide, etc.) is \$60 per acre for wheat and \$90 per acre for barley. The farmer can budget \$36,000 for the growing of the crops. To maximize his profit, how many acres of each crop should be grown?
  24. A company is going to ship food ( $x$ ) and water ( $y$ ) to the victims of a tsunami. Each container of food will feed 8 people for a day, and each container of water will give 12 people their daily water. The food containers each weigh 30 pounds and take up 8 cubic feet of space; each container of water weighs 120 pounds, but takes up only 2 cubic feet of space. The airplanes lined up to carry the supplies to the victims cannot have its cargo exceed 24,000 pounds; also, the total cargo area in the airplanes is 4,000 cubic feet. How many containers of food and water can be sent with each plane shipment that maximizes the shipment?
  25. Another company will send clothing ( $x$ ) and medical supplies ( $y$ ) to the victims of the tsunami. Each container of clothing contains enough clothing for 12 people; each container of medical supplies can aid 8 people. The clothing containers each weigh 50 pounds and take up 6 cubic feet of space; each container of medical supplies weighs 20 pounds, and takes up 4 cubic feet of space. The airplanes lined up to carry the supplies to the victims cannot have its cargo exceed 24,000 pounds; also, the total cargo area in the airplanes is 3,000 cubic feet. How many containers of clothing and medical supplies can be sent with each plane shipment that maximizes the shipment?

## Chapter Summary

### Key Terms

#### 5.1 Algebraic Expressions

- variable
- constant
- expression
- equation
- equal sign
- term
- coefficient
- like terms
- Distributive Property

#### 5.2 Linear Equations in One Variable with Applications

- linear equation

#### 5.3 Linear Inequalities in One Variable with Applications

- linear inequality
- Addition and Subtraction Property of Linear Inequalities
- Multiplication and Division Property of Linear Inequalities

#### 5.4 Ratios and Proportions

- ratio
- proportion
- constant of proportionality
- scale
- construct ratios
- solve proportions
- use proportions to solve scaling problems

#### 5.5 Graphing Linear Equations and Inequalities

- ordered pair
- origin
- points on the axes
- linear equation in two variables
- standards form of a linear equation
- solution
- linear inequality in two variables
- solution to a linear inequality
- boundary line

#### 5.6 Quadratic Equations with Two Variables with Applications

- monomial
- polynomial
- binomial
- trinomial
- quadratic equation
- Zero Product Property

#### 5.7 Functions

- relation
- domain
- function
- mapping
- vertical line test

## 5.8 Graphing Functions

- intercepts of a line
- slope
- slope-intercept form

## 5.9 Systems of Linear Equations in Two Variables

- system of linear equations
- solutions of a system of equations
- contradictions
- identities
- coincident lines
- consistent system of linear equations
- inconsistent system of linear equations

## 5.10 Systems of Linear Inequalities in Two Variables

- system of linear inequalities

## 5.11 Linear Programming

- linear programming
- objective function
- constraint

## Key Concepts

### 5.1 Algebraic Expressions

- Algebra is useful because it allows us to understand many situations in real life by modeling them with expressions.
- Algebraic expressions are the building blocks of algebra. From algebraic expressions we can create algebraic equations.
- Algebraic expressions are the building blocks of algebra. From algebraic expressions we can create algebraic equations.
- Algebraic expressions are often simplified and evaluated using the four arithmetic operations.

### 5.2 Linear Equations in One Variable with Applications

- Solving linear equations means discovering what the value of the variable in a linear equation represents in the given conditions.
- When solving a linear equation, most often you will have one solution; however, a linear equation may have no solutions or infinitely many solutions.

### 5.3 Linear Inequalities in One Variable with Applications

- Inequalities can be used when the possible values (answers) in a certain situation are numerous, or when the exact value (answer) is not known, but it is known to be within a range of possible values.
- Linear inequalities can be represented using a number line or using interval notation.

### 5.4 Ratios and Proportions

- A ratio is a comparison of two numbers. The ratio of two numbers  $a$  and  $b$  can be written as:  $a$  to  $b$  OR  $a:b$  OR the fraction  $a/b$ .
- All fractions are ratios, but not all ratios are fractions. Ratios make part to part, part to whole, and whole to part comparisons. Fractions make part to whole comparisons only.
- When two ratios are equal, we say they are in proportion or are proportional.
- Setting up proportions allows us to solve many various situations where three of the four values of the proportion are known.

### 5.5 Graphing Linear Equations and Inequalities

- Linear equations can be represented graphically on a rectangular coordinate system.
- Solving linear equations in two variables means finding the point where two lines intersect. There are three possibilities: The lines intersect at exactly one point; the lines do not intersect (they are parallel); or the lines intersect everywhere (they are the same line).

- Solving linear inequalities in two variables means finding a region of possible answers. Every point in this region will make both inequalities true statements.
- Plotting points is a standard way to help graph linear equations and linear inequalities.

### 5.6 Quadratic Equations with Two Variables with Applications

- A quadratic equation is an algebraic equation where the highest power (degree) of the equation is two.
- To solve a quadratic equation is to find the value(s) that when substituted in for the variables, will make the equation equal to zero.
- There can be two, one, or no solutions to any quadratic equation.
- There are several methods to solve a quadratic equation. These methods include factoring quadratic equations, graphic quadratic equations, using the square root method, and using the quadratic formula.

### 5.7 Functions

- A relation is any set of ordered pairs  $(x, y)$ . All of the  $x$ -values of the set are the domain, and all of the  $y$ -values of the set are the range.
- A relation is a function if each  $x$ -value in the domain is assigned to exactly one element in the range. A  $y$ -value in the range can have more than one  $x$ -value assigned to it; but each  $x$ -value can only be assigned to one  $y$ -value.
- For the function  $y = f(x)$ ,  $f$  is the name of the function,  $x$  is the domain value variable, and  $y = f(x)$  is the range value variable.
- The vertical line test is a test that can be done on the graph of a relation to determine if it is a function.

### 5.8 Graphing Functions

- Every linear function can be graphically represented by a unique line that shows all the solutions of the equation.
- The points where the graph of a line intersects the  $x$ -axis and  $y$ -axis are called the intercepts of the line.
- Most lines will have one  $x$ -intercept and one  $y$ -intercept. Only if the line is straight vertical (no  $y$ -intercept) or straight horizontal (no  $x$ -intercept) will it not have both intercepts. Note that a line that is straight vertical is not a function, but a line that is straight horizontal is a function.
- Since any two points determine a straight line, any linear function can be graphed if both intercepts are known.
- The slope of a linear function is the ratio of the vertical change divided by the horizontal change. It is often referred to as  $\frac{\text{rise}}{\text{run}}$ .
- A formula for finding the slope of linear functions is  $\frac{y_2 - y_1}{x_2 - x_1}$ , for any two points of the linear function  $(x_1, y_1)$  and  $(x_2, y_2)$ .

### 5.9 Systems of Linear Equations in Two Variables

- To solve a system of linear equations means finding the point or points where the two linear equations intersect.
- Two lines can intersect at one point, no points if they are parallel, or every point if they are the same equation.
- Systems of linear equations can be solved by graphing, by using substitution, or by using the elimination method.

### 5.10 Systems of Linear Inequalities in Two Variables

- To solve a system of linear inequalities means to find the area(s) where the points in that area make all the linear inequalities true.
- Systems of linear inequalities can be solved by graphing the linear equations associated with the inequalities, then 'testing' points to see whether the values of the point make the equation true or not.

### 5.11 Linear Programming

- Linear programming is a mathematical technique to solve problems involving finding maximums or minimums where a linear function is limited by various constraints.
- An objective function is a linear function in two or more variables that describes the quantity that needs to be maximized or minimized.
- In linear programming, a constraint is a restriction that affects the maximum or minimum values of an objective function.
- Through the creation of objective functions and restraints, a linear system can be developed and solved through linear programming.

## Videos

### 5.1 Algebraic Expressions

- [Q&A: Why We Teach Algebra \(https://openstax.org/r/Teach\\_Algebra\)](https://openstax.org/r/Teach_Algebra)

### 5.2 Linear Equations in One Variable with Applications

- [Solving for a Variable in an Equation \(https://openstax.org/r/Solving\\_for\\_a\\_variable\)](https://openstax.org/r/Solving_for_a_variable)

### 5.5 Graphing Linear Equations and Inequalities

- [Graphing Linear Inequalities in Two Variables \(https://openstax.org/r/Graphing\\_linear\)](https://openstax.org/r/Graphing_linear)

### 5.6 Quadratic Equations with Two Variables with Applications

- [Factoring with the Box Method \(Area Model\) \(https://openstax.org/r/Factoring\\_with\\_the\\_Box\)](https://openstax.org/r/Factoring_with_the_Box)
- [Solving Quadratics with the Zero Property \(https://openstax.org/r/Zero\\_Property\)](https://openstax.org/r/Zero_Property)
- [Solving Quadratics with the Quadratic Formula \(https://openstax.org/r/Solving\\_Quadratics\)](https://openstax.org/r/Solving_Quadratics)

### 5.7 Functions

- [Relations and Functions \(https://openstax.org/r/Relationsp\\_and\\_Functions\)](https://openstax.org/r/Relationsp_and_Functions)
- [Domain and Range on Graphs \(https://openstax.org/r/Domain\)](https://openstax.org/r/Domain)

### 5.9 Systems of Linear Equations in Two Variables

- [Practice with Solving Applications of Systems of Equations \(https://openstax.org/r/Practice\\_with\\_Solving\)](https://openstax.org/r/Practice_with_Solving)
- [Applications of Systems of Linear Equations \(https://openstax.org/r/Applications\\_of\\_Systems\)](https://openstax.org/r/Applications_of_Systems)

### 5.10 Systems of Linear Inequalities in Two Variables

- [Solving Systems of Linear Inequalities by Graphing \(https://openstax.org/r/Solving\\_Systems\)](https://openstax.org/r/Solving_Systems)
- [Systems of Linear Inequalities \(https://openstax.org/r/Systems\\_of\\_Linear\)](https://openstax.org/r/Systems_of_Linear)

## Formula Review

### 5.1 Algebraic Expressions

- Distributive Property:  $a(b + c) = ab + ac$

### 5.3 Linear Inequalities in One Variable with Applications

- For any numbers  $a$ ,  $b$ , and  $c$ , if  $a < b$ , then  $a + c < b + c$  and  $a - c < b - c$ .
- For any numbers  $a$ ,  $b$ , and  $c$ , if  $a > b$ , then  $a + c > b + c$  and  $a - c > b - c$ .
- For any numbers  $a$ ,  $b$ , and  $c$ , multiply or divide by a positive:
  - if  $a < b$  and  $c > 0$ , then  $ac < bc$  and  $\frac{a}{c} < \frac{b}{c}$
  - if  $a > b$  and  $c > 0$ , then  $ac > bc$  and  $\frac{a}{c} > \frac{b}{c}$
- multiply or divide by a negative:
  - if  $a < b$  and  $c < 0$ , then  $ac > bc$  and  $\frac{a}{c} > \frac{b}{c}$
  - if  $a > b$  and  $c < 0$ , then  $ac < bc$  and  $\frac{a}{c} < \frac{b}{c}$

### 5.8 Graphing Functions

- To calculate slope ( $m$ ), use the formula
 
$$m = \frac{\text{rise}}{\text{run}}$$
 where the rise measures the vertical change and the run measures the horizontal change.
- To find the slope of the line between two points  $(x_1, y_1)$  and  $(x_2, y_2)$ , use the formula
 
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

## Projects

### Ratio and Proportion—Comparing Prices, Part 1

Go to your favorite coffee shops and find out what a same sized drink costs at each. You can do something similar for pizza as well. Find the unit rate (i.e., price per ounce or price per square inch). For example, go to your favorite coffee place and find the price per units on all their large coffee drinks. Or go to your favorite pizza place and compare prices of

all their extra-large pizzas (by price per square inch). Write a report on the best deals.

### **Ratio and Proportion—Comparing Prices, Part 2**

Rather than comparing prices of different, but same sized drinks (or pizzas), compare unit prices of the same drinks but of different sizes. Find out what the best bargain is based on price per ounce, price per square inch, etc. For example, compare the prices of your favorite soft drink sold at a local store, but in various sizes (i.e., 12-ounce can, 16-ounce bottle, 20-ounce bottle, 1-liter bottle, and multipacks). Or go to a pizza place and find out what the best bargain is on their menu, based on price per square inch of pizza. Write a report on the best deals.

### **Systems of Linear Inequalities—Comparing Cell Phone Plans**

Go to the websites of different cell phone companies and compare their plans. Write a report on “the best deals.” “Best Deals” doesn’t necessarily mean “cheapest.” You will need to look at what each company provides concerning restrictions (constraints) on minutes to talk. What are the constraints on the cell phone coverage for each company? Do they cover your area of the country well? Do they cover the entire United States well, or at least areas where you will be travelling? Is this coverage 5G, or is it less? Can you add a phone easily? Can you bring your previous phone number to this plan? The possibilities of constraints affecting each plan are several. So your task is to determine which plan is best, based on not only cost but also all constraints you deem important.

## Chapter Review

### Algebraic Expressions

1. Translate from algebra to words:  $2x + 3$
2. Translate from words to algebra: the quotient of  $5x$  and  $7$ .
3. Translate from an English phrase to an expression: A gym charges \$5.00 per class  $c$  and a \$20 membership fee.
4. Use parentheses to make the following statement true:  $2^2 \div +5 - 4 \times 3 = -5$
5. Evaluate and simplify  $x^2 - 5x$  when  $x = 5$ .
6. Perform the indicated operation for the expression:  $(3x^2 + 2x + 1) + (x^2 - 2x + 2 - (x^2 - 3x + 11))$

### Linear Equations in One Variable with Applications

7. Solve the linear equations using a general strategy:  $3x - 7 = 8$
8. Solve the linear equations using properties of equations:  $(5x - 3) + 7 = 9x - 2(x + 7)$
9. It costs 30 cents for an ear of corn. Construct a linear equation and solve how much it costs to buy 23 ears of corn.
10. State whether the following equation has exactly one solution, no solution, or infinitely many solutions  
 $5x + 4 = 7x - 14$
11. Solve the formula  $A = \frac{(b_1 + b_2)}{2}h$  for  $b_1$

### Linear Inequalities in One Variable with Applications

12. Graph the inequality  $x > 8$  on a number line and write the interval notation.
13. Solve the inequality  $-2x < -6$ , graph the solution on the number line, and write the solution in interval notation.
14. Construct a linear inequality to solve the application: Daniel wants to surprise his girlfriend with a birthday party at her favorite restaurant. It will cost \$42.75 per person for dinner, including tip and tax. His budget for the party is \$500. What is the maximum number of people Daniel can have at the party?

### Ratio and Proportions

Christer opened a bag of marbles and counted the number of each color. They found they had 9 green, 4 yellow, 13 black, 11 orange, 8 blue, and 7 red.

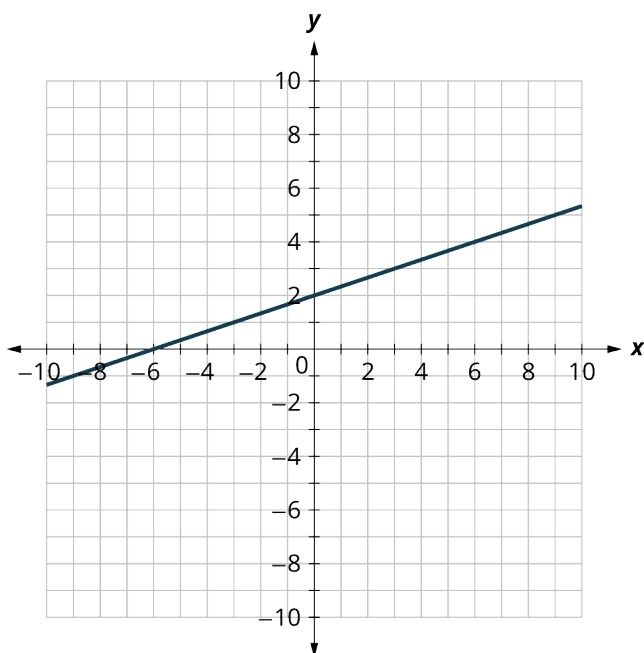
15. What is the ratio of green marbles to orange marbles?
16. What is the ratio of red marbles to blue marbles?
17. What is the ratio of sum of the marbles with an odd number of marbles to the sum of the marbles with an even number of marbles?
18. What is the ratio of black marbles to all marbles?
19. Solve:  $\frac{7}{21} = \frac{21}{x}$
20. Basil the cat is 17 pounds and 24 inches long from head to tail. In his new movie *Claws*, he is supersized to 50 pounds. How many inches long will he be? Round your answer to the nearest tenth.

### Graphing Linear Equations and Inequalities

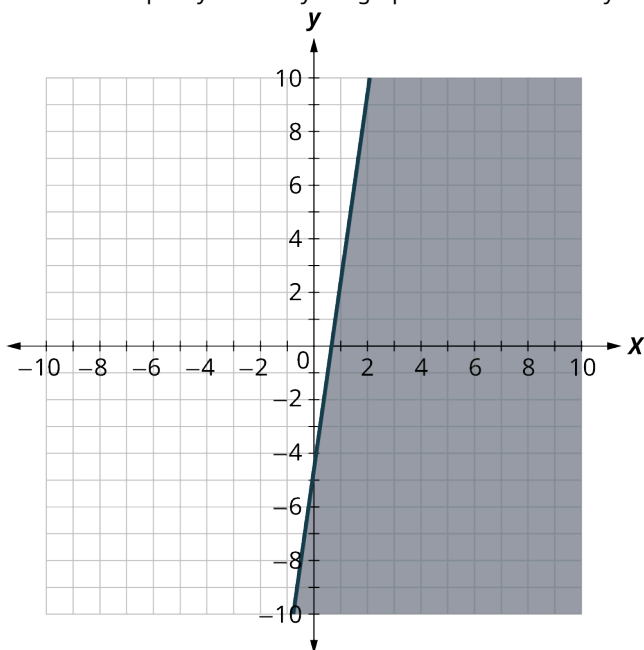
21. For each ordered pair, decide:
  - I. Is the ordered pair a solution to the equation?
  - II. Is the point on the line in the graph shown?

$$y = \frac{1}{3}x + 2$$

A: (0, -3)    B: (-3, 1);    C: (-2, -4);    D: (0, 2)



22. Graph  $y = \frac{1}{2}x + 5$  by plotting points.
23. Determine whether each ordered pair is a solution to the inequality  $y > 2x + 3$ .  
 A: (0, 0)    B: (2, 1)    C: (-1, -5)    D: (-6, -3)    E: (1, 0)
24. Write the inequality shown by the graph with the boundary line  $y = 7x - 5$ .



25. Graph the linear inequality:  $y < \frac{1}{3}x + 7$ .

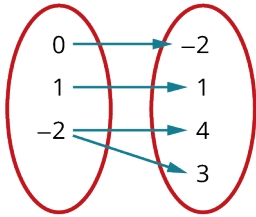
### Quadratic Equations with Two Variables with Applications

26. Multiply  $(3x - 1)(x + 7)$ .
27. Factor  $x^2 + 8x - 9$ .
28. Graph and list the solutions to the quadratic equation  $x^2 + 11x + 18 = 0$ .
29. Solve  $x^2 + 7x + 12 = 15x - 3$  by factoring.

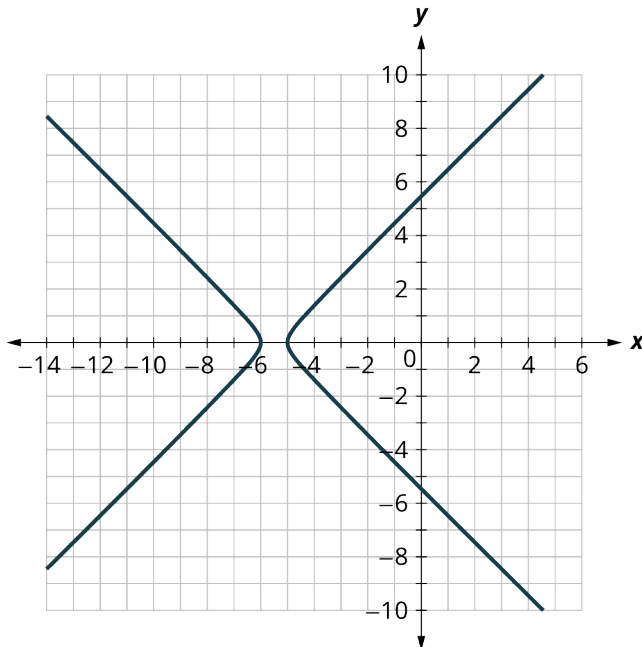
30. Solve  $169x^2 = 25$  using the square root method.
31. Solve  $x^2 + 7x + 3 = 0$  using the quadratic formula.
32. The height in feet,  $h$ , of an object shot upwards into the air with initial velocity,  $v_0$ , after  $t$  seconds is given by the formula  $h = -16t^2 + v_0t$ . A firework is shot upwards with initial velocity 130 feet per second. How many seconds will it take to reach a height of 260 feet? Round to the nearest tenth of a second.

### Functions

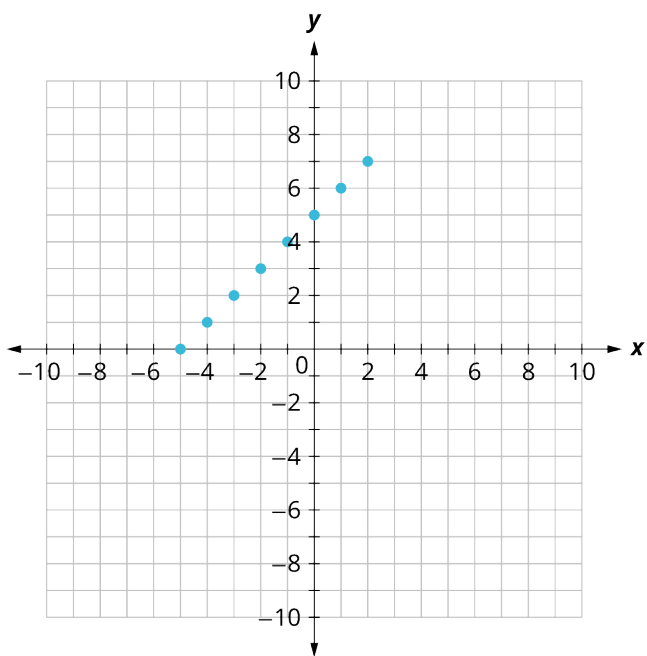
33. Evaluate the function  $f(x) = x^2 - 3x + 21$  at the values  $f(-2)$ ,  $f(-1)$ ,  $f(0)$ ,  $f(1)$ , and  $f(2)$ .
34. Determine whether  $\{(-2, 31), (-1, 25), (1, 19), (2, 19)\}$  represents a function.
35. Determine whether the mappings represent a function:



36. Determine whether  $5x^2 + 2y^2 = 10$  represent  $y$  as a function of  $x$ .
37. Use the vertical line test to determine if the graph represents a function.

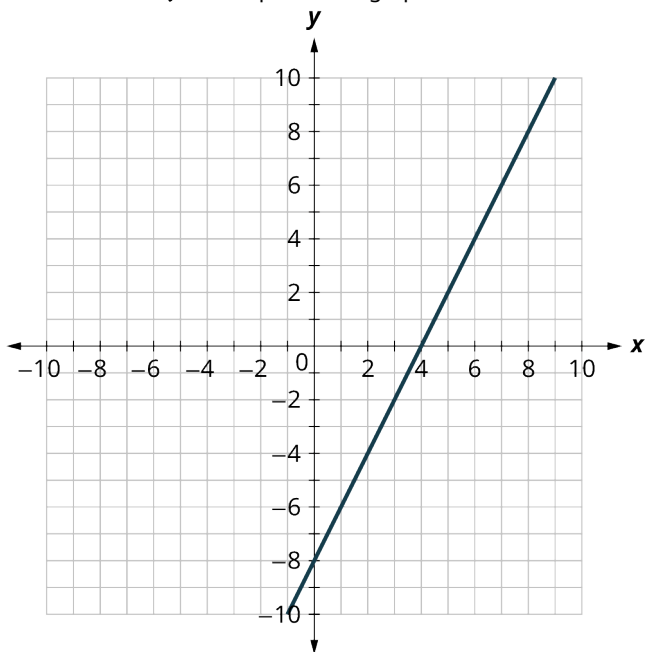


38. Use the set of ordered pairs to find the domain and the range.  
 $\{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$
39. Use the graph shown to find the domain and the range.



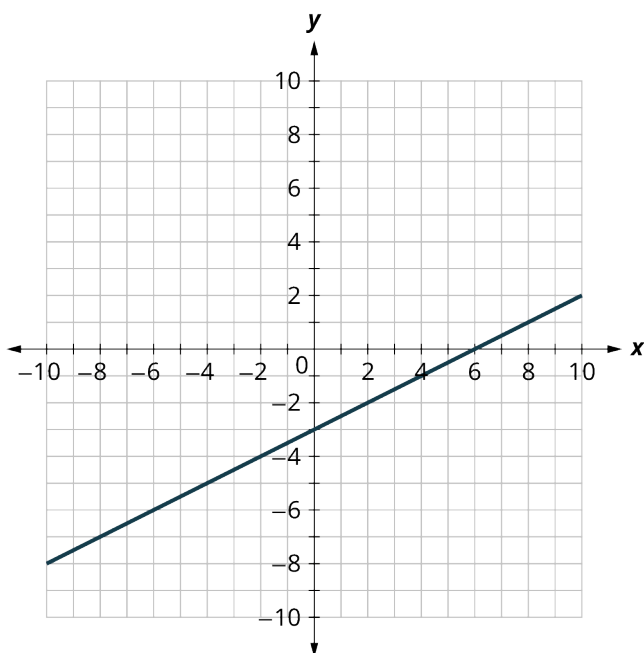
### Graphing Functions

40. Find the x- and y-intercepts on the graph.



41. Graph  $y = 3x - 3$  using the intercepts.

42. Find the slope of the line in the graph shown.



43. Use the slope formula to find the slope of the line between  $(2, 4)$  and  $(5, 7)$ .
44. Identify the slope and  $y$ -intercept of  $2y = -\frac{2}{3}x + 10$ .
45. Graph the line of  $2y = -\frac{2}{3}x - 4$  using its slope and  $y$ -intercept.

The equation  $P = 28 + 2.54w$  models the relation between the monthly water bill payment,  $P$ , in dollars, and the number of units of water,  $w$ , used.

46. Find the payment for a month when 0 units of water were used.
47. Find the payment for a month when 15 units of water were used.
48. Interpret the slope and  $P$ -intercept of the equation.
49. Graph the equation.

#### System of Linear Equations in Two Variables

50. Determine if  $(0, 1)$  and  $(2, 3)$  are solutions to the given system of equations.

$$\begin{cases} x - 2y = 0 \\ 3x - y = 5 \end{cases}$$

51. Solve the system of equations by graphing.

$$\begin{cases} y = 3x + 1 \\ 6x - y = 2 \end{cases}$$

52. Solve the system of equations by substitution.

$$\begin{cases} y = 7x + 2 \\ 3x - y = -6 \end{cases}$$

53. Solve the systems of equations by elimination.

$$\begin{cases} -5x + y = -15 \\ 3x + y = 17 \end{cases}$$

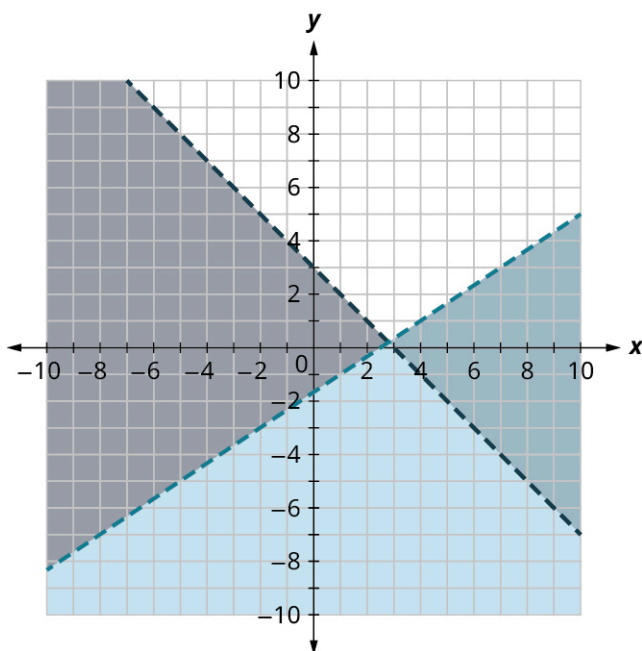
54. Kenneth currently sells suits for company A at a salary of \$22,000 plus a \$10 commission for each suit sold. Company B offers him a position with a salary of \$28,000 plus a \$4 commission for each suit sold. How many suits would Kenneth need to sell for the options to be equal?

#### Systems of Linear Inequalities in Two Variables

55. Determine whether  $(0, 0)$  and  $(2, 3)$  are solutions to the system.

$$\begin{cases} 3x + y > 5 \\ 2x - y \leq 10 \end{cases}$$

56. Determine whether  $(0, 0)$  and  $(6, -8)$  are solutions to the darkest shaded region of the graph.



57. Solve the systems of linear equations by graphing.

$$\begin{cases} y < 3x + 1 \\ 6x - y \geq 2 \end{cases}$$

Jocelyn desires to increase both her protein consumption and caloric intake. She desires to have at least 35 more grams of protein each day and no more than an additional 200 calories daily. An ounce of cheddar cheese has 7 grams of protein and 110 calories. An ounce of parmesan cheese has 11 grams of protein and 22 calories.

58. Write a system of inequalities to model this situation.
59. Graph the system.
60. Could she eat 1 ounce of cheddar cheese and 3 ounces of parmesan cheese?
61. Could she eat 2 ounces of cheddar cheese and 1 ounce of parmesan cheese?

### Linear Programming

A toy maker makes two plastic toys, the Ring ( $x$ ) and the Stick ( $y$ ). The toy maker makes \$4 per Ring and \$6 per Stick. The Ring uses 3 feet of plastic, while the Stick uses 5 feet of plastic. Today the toy maker has 40 feet of plastic available. The toy maker also only makes 10 plastic toys per day. To maximize profit, how many of each toy should the toy maker make?

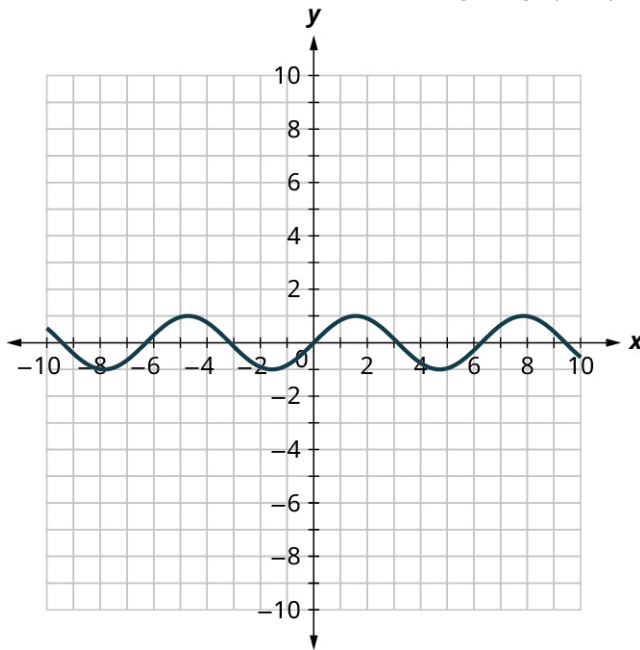
62. Find the objective function.
63. Write the constraints as a system of inequalities.
64. Graph of the system of inequalities.
65. Find the value of the objective function at each corner point of the graphed region.
66. To maximize his profit, how many of each toy should the toymaker make?

### Chapter Test

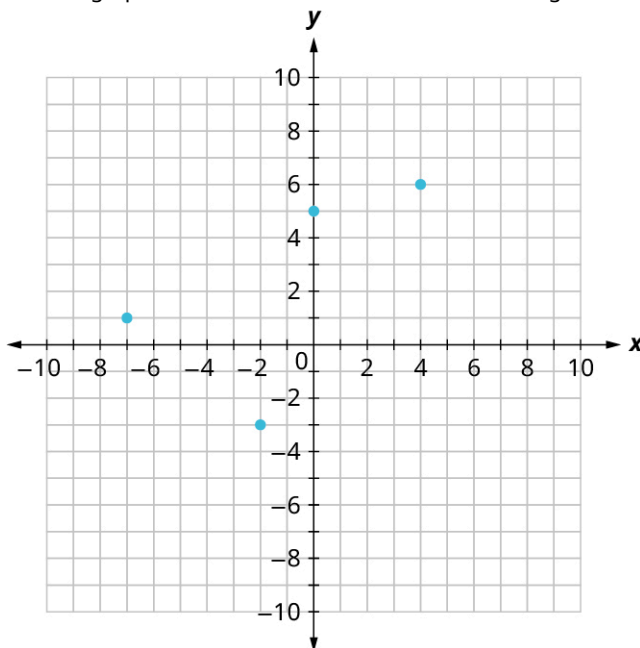
1. Perform the indicated operation for the expression:  $(3x^2 + 2x + 1) - (x^2 - 2 + 2) + x(3x + 11)$
2. Solve the linear equations using properties of equations:  $x(x - 2) + 9x = x(x - 6)$
3. It costs 55 cents for a stamp. Construct a linear equation and solve how much it cost to buy 50 stamps.
4. Solve the formula  $A = \frac{(b_1 + b_2)}{2}h$  for  $h$ .
5. Solve the inequality  $-2x < -6$ , graph the solution on the number line, and write the solution in interval notation.
6. Construct a linear inequality to solve the application: Bella wants to buy a round of shakes for her friends. It will cost \$4.75 per shake, including tip and tax. Her budget is \$50. What is the maximum number of friends Bella can buy shakes for?
7. *Manneken Pis* is a famous statue in Brussels, Belgium. It is 24 inches tall and weighs 37.5 pounds. The average

man is 69 inches tall and weighs 198 pounds. Is *Manneken Pis* proportional to the average male?

8. Graph  $y = \frac{1}{4}x + 1$  by plotting points.
9. Graph the linear inequality:  $y \geq 2x + 7$
10. Graph and list the solutions to the quadratic equation  $x^2 - 36 = 0$ .
11. Solve  $x^2 - 49 = 0$  by factoring.
12. Solve  $x^2 + 5x + 2 = 0$  using the quadratic formula.
13. Evaluate the function  $f(x) = -3x + 21$  at the values  $f(-2)$ ,  $f(-1)$ ,  $f(0)$ ,  $f(1)$ , and  $f(2)$ .
14. Use the vertical line test to determine if the given graph represents a function.



15. Use the graph shown to find the domain and the range.



16. Graph  $2y = x - 4$  using the intercepts.

17. Use the slope formula to find the slope of the line between (1, 4) and (3, 5).
18. Identify the slope and  $y$ -intercept of  $y - \frac{2}{3}x = 1$ .
19. Graph the line of  $y = \frac{2}{3}x + 1$  using its slope and  $y$ -intercept.

The equation  $C = 4.50 + 15p$ , models the cost of visiting the Cat Café in San Diego for one hour.  $C$ , in dollars, is the total cost and the cost per person,  $p$ , is \$15 plus a \$4.50 reservation fee.

20. Find the payment for two people.
21. Find the payment for five people.
22. Interpret the slope and  $C$ -intercept of the equation.
23. Graph the equation.
24. Solve the system of equations by graphing.
- $$\begin{cases} y - 3x = 3 \\ y = -6x + 12 \end{cases}$$
25. Solve the system of equations by substitution.
- $$\begin{cases} y = 5x - 5 \\ -3x + y = -3 \end{cases}$$
26. Solve the systems of equations by elimination.
- $$\begin{cases} y = \frac{1}{3}x - 6 \\ y = x - 3 \end{cases}$$
27. Anna goes to the concession stand at a movie theater. She buys 5 popcorns and 4 large sodas and pays a total of \$60. During intermission, Isabelle goes to the concession stand. She buys 1 popcorn and 2 large sodas and pays a total of \$18. What is the cost of one popcorn, and the cost of one large soda?
28. Solve the systems of linear equations by graphing.
- $$\begin{cases} y > \frac{1}{3}x - 1 \\ y < x - 3 \end{cases}$$

Juliette is selling fresh lemonade and cupcakes. She sells a cup of lemonade for \$2 and a cupcake for \$3. She needs to make at least \$100 to donate to the local cat sanctuary. She needs to sell at least 20 cups of lemonade.

29. Write a system of inequalities to model this situation.
30. Graph the system.
31. Could she sell 30 cups of lemonade and 10 cupcakes and make \$100?
32. Could she sell 20 cups of lemonade and 30 cupcakes and make \$100?

A toy maker makes exactly two toys out of wood; the Box ( $x$ ) and the Bat ( $y$ ). He makes \$5 per Box and \$6 per Bat. Each Box requires 30 ounces of wood, and each Bat requires 45 ounces of wood. Today the toy maker has 270 ounces of wood available. The toy maker also only makes 8 wooden toys per day. To maximize profit, how many of each wooden toy should the toy maker make?

33. Find the objective function.
34. Write the constraints as a system of inequalities.
35. Graph of the system of inequalities.
36. Find the value of the objective function at each corner point of the graphed region.
37. To maximize profit, how many of each toy should the toymaker make?

## 6

## MONEY MANAGEMENT

**Figure 6.1** Financial health helps you realize your goals. (credit: modification of work "Budget and Bills" by Alabama Extension/Flickr, Public Domain)

### Chapter Outline

- 6.1 Understanding Percent
- 6.2 Discounts, Markups, and Sales Tax
- 6.3 Simple Interest
- 6.4 Compound Interest
- 6.5 Making a Personal Budget
- 6.6 Methods of Savings
- 6.7 Investments
- 6.8 The Basics of Loans
- 6.9 Understanding Student Loans
- 6.10 Credit Cards
- 6.11 Buying or Leasing a Car
- 6.12 Renting and Homeownership
- 6.13 Income Tax



## Introduction

The topic of money management is a broad and sometimes complex one. Ultimately, personal money management involves managing both our debt and also our savings and investments.

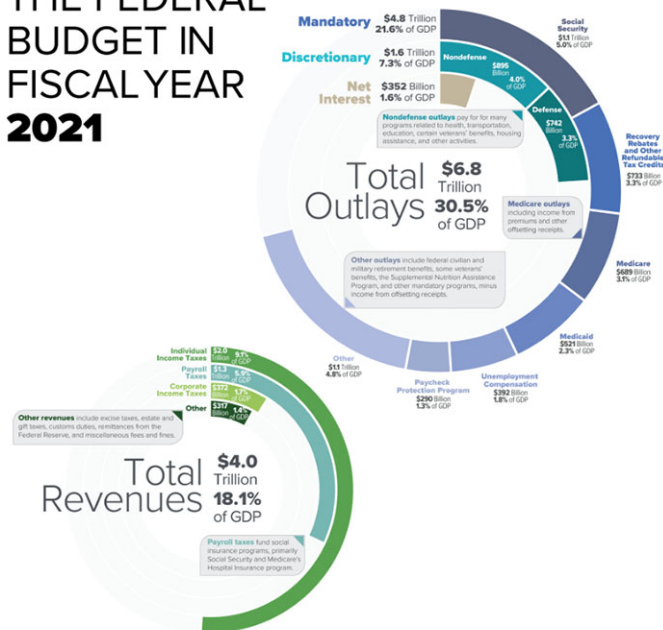
In 2021, the average American had consumer debt balance of \$96,371. Nearly \$100,000 per person. And less than 25% of Americans are debt free. Consumer debt can include mortgages, credit cards, as well as student loans. A key question all consumers should consider is how to manage debt and not become overburdened by it. The first step is to create a budget, which puts earnings into perspective, indicating what we can, and cannot, afford. A budget also entails setting aside certain funds for savings and investment, which help us achieve our short- and long-term goals.

Creating a budget requires an understanding of how money—debt and savings—works. Initially, percentages and interest need to be understood. They drive most of what happens with debt and savings. With that understanding, discussions of buying a house, a car, or incurring credit card debt can be addressed from a financial perspective. All the while, retirement is waiting. Preparing for retirement involves saving and saving earlier rather than later. The power of compound interest is on full display when saving early.

This chapter covers some of the basics of money management: percentages, interest, budgeting, debt (student loans, mortgage, car, credit cards), savings, investments, and taxes.

## 6.1 Understanding Percent

### THE FEDERAL BUDGET IN FISCAL YEAR 2021



**Figure 6.2** The federal budget describes how money is spent and how money is earned. (credit: "Breakdown of revenues and outlays in 2021 US Federal budget" Wikimedia Commons, Public Domain)

### Learning Objectives

After completing this section, you should be able to:


1. Define and calculate percent.
2. Convert between percent, decimal, and fractional values.
3. Calculate the total, percent, or part.
4. Solve application problems involving percents.

In 2020, the U.S. federal government [budgeted \\$3.5 billion for the National Park Service](https://openstax.org/r/National_Park_Service) ([https://openstax.org/r/National\\_Park\\_Service](https://openstax.org/r/National_Park_Service)), which appears to be a very large number (and is!) and a large portion of the total federal budget. However, the [total outlays from the U.S. federal government in 2020 was \\$6.6 trillion](https://openstax.org/r/U.S._federal_government) ([https://openstax.org/r/U.S.\\_federal\\_government](https://openstax.org/r/U.S._federal_government)). So, the amount budgeted for the National Park Service was less than one-tenth of 1 percent, or 1/10%, of the total outlays. This **percent** describes a specific number. Understanding that ratio puts the \$3.5 billion budgeted to the National Park Service in perspective.

This chapter focuses on percent as a primary tool for understanding money management. The interest paid on debt, the interest earned through investments, and even taxes are entirely determined using percent. This section introduces the basics of working with this invaluable tool.

### Define and Calculate Percent

The word **percent** comes from the Latin phrase per centum, which means "by the hundred." So any percent is a number divided by 100. Changing a percent to a fraction is to write the percent in its **fractional form**. To write  $n\%$  in its fractional form is to write the percent as the fraction  $\frac{n}{100}$ .

 A percent need not be an integer and does not have to be less than 100.

#### EXAMPLE 6.1

#### Rewriting a Percent as a Fraction

Rewrite the following as fractions:

1. 18%

2. 84%
3. 38.7%
4. 213%

✓ **Solution**

1. Using the definition and  $n = 18$ , 18% in fractional form is  $\frac{18}{100}$ .
2. Using the definition and  $n = 84$ , 84% in fractional form is  $\frac{84}{100}$ .
3. Using the definition and  $n = 38.7$ , 38.7% in fractional form is  $\frac{38.7}{100}$ .
4. Using the definition and  $n = 213$ , 213% in fractional form is  $\frac{213}{100}$ .

> **YOUR TURN 6.1**

Rewrite the following as fractions:

1. 3%
2. 94%
3. 67.2%
4. 670%

## Convert Between Percent, Decimal, and Fractional Values

When any calculation with a percent is to be performed, the form of the percent must be changed, either to its fractional form or its **decimal form**. We can change a percent into decimal form by dividing the percent by 100 and representing the result as a decimal.

### FORMULA

The decimal form of  $n\%$  is found by calculating the decimal value of  $n \div 100$ .

### EXAMPLE 6.2

#### Converting a Percent to Decimal Form

Convert the following percents to decimal form:

1. 17%
2. 7%
3. 18.45%

✓ **Solution**

1. To convert 17% to its decimal form, divide 17 by 100. This moves the decimal two places to the left, resulting in 0.17. The decimal form of 17% is 0.17.
2. To convert 7% to its decimal form, divide 7 by 100. This moves the decimal two places to the left, resulting in 0.07. The decimal form of 7% is 0.07.
3. To convert 18.45% to its decimal form, divide 18.45 by 100. This moves the decimal two places to the left, resulting in 0.1845. The decimal form of 18.45% is 0.1845.

> **YOUR TURN 6.2**

Convert the following percents to decimal form:

1. 9%
2. 24%
3. 2.18%

You should notice that, to convert from percent to decimal form, you can simply move the decimal two places to the left without performing the division.

#### FORMULA

To convert the number  $x$  from decimal form to percent, multiply  $x$  by 100 and place a percent sign, %, after the number,  $(x \times 100)\%$ .

#### EXAMPLE 6.3

##### Converting the Decimal Form of a Percent to Percent

Convert each of the following to percent:

1. 0.34
2. 4.15
3. 0.0391

#### Solution

1. Using the formula and  $x = 0.34$ , we calculate  $(0.34 \times 100)\%$ , which gives us 34%.
2. Using the formula and  $x = 4.15$ , we calculate  $(4.15 \times 100)\%$ , which gives us 415%.
3. Using the formula and  $x = 0.0391$ , we calculate  $(0.0391 \times 100)\%$ , which gives us 3.91%.

#### YOUR TURN 6.3

Convert the following to percent:

1. 0.41
2. 0.02
3. 9.2481

You should notice that, to convert from decimal form to percent form, you can simply move the decimal two places to the right without performing the multiplication.

### Calculate the Total, Percent, or Part

The word “of” is used to indicate multiplication using fractions, as in “one-fourth of 56.” To find “one-fourth of 56” we would multiply 56 by one-fourth. We can think of percents as fractions with a specific denominator—100. So, to calculate “25% of 52,” we multiply 52 by 25%. But, first we need to convert the percent to either fractional form ( $25/100$ ) or decimal form. Using the decimal form of 25% we have  $0.25 \times 52$ , which equals 13.

In this problem, 52 is the **total** or **base**, 25 is the **percentage**, and 13 is **the percentage of 52**, or the **part** of 52. This is sometimes referred to as the **amount**.

#### FORMULA

The mathematical formula relating the total (base), the percent in decimal form, and the part (amount) is  $part = percent \times total$ , or,  $amount = percent \times base$ .

 In all calculations, the percent is expressed in decimal form.

Knowing any two of the values in our formula allows us to calculate the third value. In the following example, we know the total and the percent, and are asked to find the percentage of the total.

**EXAMPLE 6.4****Finding the Percent of a Total**

1. Determine 70% of 3,500
2. Determine 156% of 720

✓ **Solution**

1. The total is  $x = 3,500$ , and the percent is  $n = 70$ . The decimal form of 70% is 0.70. To find the part, or percent of the total, substitute those values into the formula and calculate.

$$\begin{aligned}\text{part} &= \text{percent} \times \text{total} \\ &= 0.70 \times 3500 \\ &= 2450\end{aligned}$$

From this, we say that 70% of 3,500 is 2,450.

2. The total is  $x = 720$ , and the percent is  $n = 156$ . The decimal form of 156% is 1.56. To find the part, or percent of the total, substitute those values into the formula and calculate.

$$\begin{aligned}\text{part} &= \text{percent} \times \text{total} \\ &= 1.56 \times 720 \\ &= 1,123.2\end{aligned}$$

From this, we say that 156% of 720 is 1,123.2.

▶ **YOUR TURN 6.4**

1. Determine 26% of 1,300.
2. Determine 225% of 915.

▶ **VIDEO**

[Finding Percent of a Total \(https://openstax.org/r/solve\\_percent\\_problem1\)](https://openstax.org/r/solve_percent_problem1)

In the previous example, we knew the total and the percent and found the part using our formula. We may instead know the percent and the part, but not the total. We can use our formula again to solve for the total.

**EXAMPLE 6.5****Finding the Total from the Percent and the Part**

1. What is the total if 35% of the total is 70?
2. What is the total if 10% of the total is 4,000?

✓ **Solution**

1. **Step 1:** The percent is 35, which in decimal form is 0.35. We were given that 35% of the total is 70, so the part is 70. We are to find the total. Substituting into the formula, we have

$$\begin{aligned}\text{part} &= \text{percent} \times \text{total} \\ 70 &= 0.35 \times \text{total}\end{aligned}$$

**Step 2:** To find the total, we solve the equation for the total.

$$\begin{aligned}70 &= 0.35 \times \text{total} \\ \frac{70}{0.35} &= \frac{0.35 \times \text{total}}{0.35} \\ 200 &= \frac{\cancel{0.35} \times \text{total}}{\cancel{0.35}} \\ 200 &= \text{total}\end{aligned}$$

From this we see that 200 is the total, or, that 35% of 200 is 70.

2. **Step 1:** The percent is 10, which in decimal form is 0.1. We were given that 10% of the total is 4,000, so the part is 4,000. Substituting into the formula, we have

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ 4,000 &= 0.1 \times \text{total} \end{aligned}$$

**Step 2:** To find the total, we solve the equation for the total.

$$\begin{aligned} 4,000 &= 0.1 \times \text{total} \\ \frac{4,000}{0.1} &= \frac{0.1 \times \text{total}}{0.1} \\ 40,000 &= \frac{\cancel{0.1} \times \text{total}}{\cancel{0.1}} \\ 40,000 &= \text{total} \end{aligned}$$

From this we see that 40,000 is the total, or that 10% of 40,000 is 4,000.

### > YOUR TURN 6.5

1. What is the total if 18% of the total is 45?
2. What is the total if 15% of the total is 900?

### ▶ VIDEO

[Finding the Total from the Percent and the Part \(https://openstax.org/r/solve\\_percent\\_problem2\)](https://openstax.org/r/solve_percent_problem2)

Similarly, the percent can be found if the total and the percent of the total (the part) are known. This will result in the decimal form of the percent, so it must be converted to percent form.

### EXAMPLE 6.6

#### Finding the Percent from the Total and the Part

1. What percent of 500 is 175?
2. What percent of 228 is 155?

#### ✓ Solution

1. **Step 1:** The total is 500, the percent of the total is 175. Substituting into the formula, we have

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ 175 &= \text{percent} \times 500 \end{aligned}$$

**Step 2:** To find the percent, we solve the equation for the percent.

$$\begin{aligned} 175 &= \text{percent} \times 500 \\ \frac{175}{500} &= \frac{\text{percent} \times 500}{500} \\ 0.35 &= \frac{\text{percent} \times \cancel{500}}{\cancel{500}} \\ 0.35 &= \text{percent} \end{aligned}$$

We see the percent in decimal form is 0.35. Converting from the decimal form yields 35%. We say that 175 is 35% of 500.

2. **Step 1:** The total is 228, the percent of the total is 155. Substituting into the formula, we have

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ 155 &= \text{percent} \times 228 \end{aligned}$$

**Step 2:** To find the percent, we solve the equation for the percent.

$$\begin{aligned} 155 &= \text{percent} \times 228 \\ \frac{155}{228} &= \frac{\text{percent} \times 228}{228} \\ 0.6798 &= \frac{\text{percent} \times \cancel{228}}{\cancel{228}} \\ 0.6798 &= \text{percent} \end{aligned}$$

We see the percent is 0.6798 (rounded to four decimal places). Converting from the decimal form yields 67.98%. We say that 155 is 67.98% of 228.

### > YOUR TURN 6.6

Find the percent in the following:

1. Total is 40, percent of the total is 25
2. Total is 730, percent of the total is 292

### ▶ VIDEO

[Finding the Percent When the Total and the Part Are Known \(https://openstax.org/r/solve\\_percent\\_problem3\)](https://openstax.org/r/solve_percent_problem3)

## Solve Application Problems Involving Percents

Percents are frequently used in finance, research, science experiments, and even casual conversation. Understanding these types of values helps when consuming media or discussing finances, for instance. Effectively working with and interpreting numbers and percents will help you become an informed consumer of this information.

In most cases, working through what is presented requires you to identify that you are indeed working with a question of percents, which two of the three values that are related through percents are known, and which of the three values you need to find.

### EXAMPLE 6.7

#### Retention Rate at College

Justine applies to a medium size university outside her hometown and finds out that the retention rate (percent of students who return for their sophomore year) for the 2021 academic year at the university was 84%. During a visit to the registrar's office, she finds out that 1,350 people had enrolled in academic year 2021. How many students from the academic year 2021 are returning for the 2022 academic year?

#### ✔ Solution

The percent of students who will return for the 2022 academic year (the retention rate) is 84%. The total number of students who enrolled in the 2021 academic year was 1,350. This means the percent is known and the total is known. From this, we can determine the number of students who will return (percent of the total) for the 2022 academic year using the formula  $\text{part} = \text{percent} \times \text{total}$ . Substituting into the formula and calculating, we find that the number of students that are returning is

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ &= 0.84 \times 1,350 \\ &= 1,134 \end{aligned}$$

So 1,134 students will return for the 2022 academic year.

### > YOUR TURN 6.7

1. Harris works the bookstore in their hometown. During one particular day, the store had total sales of \$1,765, of

which Harris sold 30%. What were Harris's total sales that day?

### EXAMPLE 6.8

#### Percent of Chemistry Majors

Cameron enrolls in a calculus class. In this class of 45 students, there are 18 chemistry majors. What percent of the class are chemistry majors?

#### ✓ Solution

In this situation, the percent is to be determined. We know the total number of students, 45, and the part of the students that are chemistry majors, 18. Using that information and the formula  $\text{part} = \text{percent} \times \text{total}$ , the percent can be found. Substituting and solving, we have

$$\begin{aligned} 18 &= \text{percent} \times 45 \\ \frac{18}{45} &= \frac{\text{percent} \times 45}{45} \\ 0.4 &= \frac{\text{percent} \times \cancel{45}}{\cancel{45}} \\ 0.4 &= \text{percent} \end{aligned}$$

Converting the 0.4 from decimal form, we find that 40% of the students in the calculus class are chemistry majors.

### > YOUR TURN 6.8

- At the Fremont County fair, there were 2,532 adult visitors. Of these, 1,679 purchased the Adult Mega Pass. What percent of the adult visitors purchased the Adult Mega Pass?

### EXAMPLE 6.9

#### Total Sales and Commission

Mariel makes a 20% commission on every sale she makes. One week, her commission check is for \$153.00. What were her total sales that week?

#### ✓ Solution

In this problem, Mariel's total sales is to be determined. We know the percent she earns is 20%. We also know that her sales commission was \$153.00, which is the percent of the total. Using this information and the formula  $\text{part} = \text{percent} \times \text{total}$  we can find Mariel's total sales. The decimal form of 20% is 0.2. The part, or percent of the total, is 153. Substituting and solving, we obtain

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ 153 &= 0.2 \times \text{total} \\ \frac{153}{0.2} &= \frac{0.2 \times \text{total}}{0.2} \\ 765 &= \frac{\cancel{0.2} \times \text{total}}{\cancel{0.2}} \\ 765 &= \text{total} \end{aligned}$$

Mariel's total sales were \$765.00.

### > YOUR TURN 6.9

- Mina's family has replaced 65% of their home's older light bulbs with LED bulbs. If they now have 52 LED bulbs, how many total lightbulbs are in Mina's house?

**? WHO KNEW?****LED Lightbulbs**

According to the [energy website from the U.S. government \(https://openstax.org/r/U.S.\\_government\)](https://openstax.org/r/U.S._government), LED lightbulbs use at least 75% less energy than incandescent bulbs. They also last up to 25 times as long as an incandescent bulb. If lighting is a significant percent of your electrical use, replacing all incandescent bulbs with LED bulbs will significantly reduce your electric bill.

**Check Your Understanding**

1. What is the denominator for any percent?
2. Convert 38.7% to decimal form.
3. What is 68% of 280?
4. Find the total if 41% of the total is 342. If necessary, round to two decimal places.
5. TikTok has an estimated 80,000,000 (80 million) registered users in the United States. The population of the United States is 332,403,650. What percent of the U.S. population are registered TikTok users? If necessary, round to two decimal places.
6. An Amazon fulfillment center needs to hire 20% more drivers. If there are currently 80 drivers, how many more drivers will be hired?

**SECTION 6.1 EXERCISES**

For any answer, round to two decimal places, if necessary.

In the following exercises, rewrite the percent as a fraction

1. 45%
2. 9.1%
3. 8%
4. 673%

In the following exercises, rewrite the percent in decimal form.

5. 18%
6. 9%
7. 71.2%
8. 934%
9. Find 35% of 250
10. Calculate 83.1% of 390
11. Calculate 3.1% of 500
12. Calculate 750% of 620
13. If 40% of the total is 32, how much is the total?
14. If 3% of the total is 6.32, how much is the total?
15. If 150% of the total is 61.9, how much is the total?
16. If 18.1% of the total is 18.5, how much is the total?
17. 13 is what percent of 40?
18. 89 is what percent of 500?
19. 31 is what percent of 73?
20. 593.2 is what percent of 184.5?
21. 36 people in a village of 150 want to install a new splashpad at the local playground. What percent of the village

wants to install the new splashpad?

22. Mitena is enrolled in a movie appreciation course. There are 84 students (including Mitena) in the course. After having the students fill out a survey, the professor informs the students that 45.2% chose horror as their favorite movie genre. How many students in Mitena's class chose horror as their favorite movie genre? Round off to the nearest integer.
23. Jadyn's dorm has a "Rick and Morty night" every Wednesday during the semester. One Wednesday, 27 students from the dorm come to watch the TV show Rick and Morty. Jadyn knows this is 30% of the dorm's residents. How many students reside in the dorm?
24. **Percent Error.** When performing a scientific experiment that results in quantities of some sort, such as mass in chemistry or momentum in physics, the percent error is often computed. Percent error, % $E$ , is the percent by which the value obtained in an experiment, the observed value  $O$ , is different than the value that was expected, the expected value  $E$ , in the experiment. The formula is below.
- $$\%E = \frac{|O-E|}{E}$$
- Jim and Kelly are working on a chemistry experiment and expect the result to be 50 grams. However, their result was 48.7 grams. Find Jim and Kelly's percent error.
25. **Percent Error.** See Exercise 24 for the definition of percent error. Hailey and Elsbeth are using an experiment to determine Earth's gravity. The expected value is  $9.807 \text{ m/s}^2$ . Their experiment gives them a value of  $9.457 \text{ m/s}^2$ . Find the percent error for Hailey and Elsbeth's experiment.

## 6.2 Discounts, Markups, and Sales Tax



**Figure 6.3** Sale prices are often described as percent discounts. (credit: "Close-up of a discount sign" by Ivan Radic/Flickr, CC BY 2.0)

### Learning Objectives

After completing this section, you should be able to:

1. Calculate discounts.
2. Solve application problems involving discounts.
3. Calculate markups.
4. Solve application problems involving markups.
5. Compute sales tax.
6. Solve application problems involving sales tax.

Many people first encounter percentages during a retail transaction such as a percent discount (SALE! 25% off!), or through sales tax ("Wait, I thought this was \$1.99?"), a report that something has increased by some percentage of the previous value (NOW! 20% more!!). These are examples of percent decreases and percent increases. In this section, we

discuss decrease, increase, and then the case of sales tax.

## Calculating Discounts

Retailers frequently hold sales to help move merchandise. The sale price is almost always expressed as some amount off the original price. These are **discounts**, a reduction in the price of something. The price after the discount is sometimes referred to as the reduced price or the sale price.

When a reduction is a percent discount, it is an application of percent, which was introduced in [Understanding Percent](#). The formula used was  $\text{part} = \text{percentage} \times \text{total}$ . In a discount application, the discount plays the role of the part, the percent discount is the percentage, and the original price plays the role of the total.

### FORMULA

The formula for a discount based on a percentage is  $\text{discount} = \text{percent discount} \times \text{original price}$ , with the percent discount expressed as a decimal. The price of the item after the discount is  $\text{sale price} = \text{original price} - \text{discount}$ .

These are often combined into the following formula

$$\text{sale price} = \text{original price} - \text{percent discount} \times \text{original price} = \text{original price} \times (1 - \text{percent discount})$$

When the original price and the percent discount are known, the discount and the sale price can be directly computed.

### EXAMPLE 6.10

#### Calculating Discount for a Percent Discount

Calculate the discount for the given price and discount percentage. Then calculate the sale price.

1. Original price = \$75.80; percent discount is 25%
2. Original price = \$168.90; percent discount is 30%

#### ✓ Solution

1. Substituting the values into the formula  $\text{discount} = \text{percent discount} \times \text{original price}$ , we find that the discount is  $\text{discount} = 0.25 \times 75.80 = 18.95$ . The discount is \$18.95.  
The sale price of the item is then  $\text{sale price} = \text{original price} - \text{discount} = 75.80 - 18.95 = 56.85$ , or \$56.85.
2. Substituting the values into the formula  $\text{discount} = \text{percent discount} \times \text{original price}$ , we find that the discount is  $\text{discount} = 0.30 \times 168.90 = 50.67$ . The discount is \$50.67.  
The sale price of the item is then  $\text{sale price} = \text{original price} - \text{discount} = 168.90 - 50.67 = 118.23$ , or \$118.23.

### > YOUR TURN 6.10

Calculate the discount for the given original price and discount percentage. Then calculate the sale price.

1. Original price = \$1,550.00; percent discount is 32%
2. Original price = \$27.50; percent discount is 10%

Sometimes the original price and the sale price of an item is known. From this, the percent discount can be computed using the formula  $\text{discount} = \text{percent discount} \times \text{original price}$ , by solving for the percent discount.

### EXAMPLE 6.11

#### Calculating the Percent Discount from the Original and Sale Prices

Determine the percent discount based on the given original and sale prices.

1. Original price = \$1,200.00; sale price = \$900.00
2. Original price = \$36.70; sale price = \$29.52

✓ **Solution**

1. **Step 1.** Find the discount. Using the original price and the sale price, we can find the discount with the formula  $\text{sale price} = \text{original price} - \text{discount}$ . Substituting and calculating, we find the discount to be  $900.00 = 1,200.00 - \text{discount}$ . Solving for the discount gives \$300.00.

**Step 2.** Find the percent discount. Substituting the discount of \$300.00 and the original price of \$1,200.00, into the formula  $\text{discount} = \text{percent discount} \times \text{original price}$ , we can find the percent discount.

$$\begin{aligned} 300.00 &= \text{percent discount} \times 1,200.00 \\ \frac{300.00}{1,200.00} &= \text{percent discount} \\ 0.25 &= \text{percent discount} \end{aligned}$$

Converting to percent form, the percent discount is 25%.

2. **Step 1.** Find the discount. Using the original price and the sale price, we can find the discount with the formula  $\text{sale price} = \text{original price} - \text{discount}$ . Substituting and calculating, we find the discount to be  $29.52 = 36.70 - \text{discount}$ . Solving for the discount gives \$7.38.

**Step 2.** Find the percent discount. Substituting the discount of \$7.38 and the original price of \$36.70, into the formula  $\text{discount} = \text{percent discount} \times \text{original price}$ , we can find the percent discount.

$$\begin{aligned} 29.52 &= \text{percent discount} \times 36.70 \\ \frac{29.52}{36.70} &= \text{percent discount} \\ 0.2 &= \text{percent discount} \end{aligned}$$

Converting to percent form, the percent discount is 20%.

> **YOUR TURN 6.11**

Determine the percent discount based on the given original and sale prices.

1. Original price = \$250.00; sale price = \$162.50
2. Original price = \$19.50; sale price = \$17.16

Sometimes the sale price and the percent discount of an item are known. From this, the original price can be found. To avoid multiple steps, though, the formula that we will use is  $\text{sale price} = \text{original price} \times (1 - \text{percent discount})$ . The original price can be found by solving this equation for the original price.

**EXAMPLE 6.12**

**Calculating the Original Price from the Percent Discount and Sale Price**

Determine the original price based on the percent discount and sale price.

1. Percent discount 10%; sale price = \$450.00
2. Percent discount 75%, sale price = \$90.00

✓ **Solution**

1. Using the percent discount and the sale price, we can find the original price with the formula  $\text{sale price} = \text{original price} \times (1 - \text{percent discount})$ . Substituting and solving for the original price, we find

$$\begin{aligned} \text{sale price} &= \text{original price} \times (1 - \text{percent discount}) \\ 450.00 &= \text{original price} \times (1 - 0.10) \\ 450.00 &= \text{original price} \times (0.90) \\ 500.00 &= \text{original price} \end{aligned}$$

The original price of the item was \$500.00.

2. Using the percent discount and the sale price, we can find the original price with the formula

$$\begin{aligned} \text{sale price} &= \text{original price} \times (1 - \text{percent discount}). \text{ Substituting and solving for the original price, we find} \\ \text{sale price} &= \text{original price} \times (1 - \text{percent discount}) \\ 90.00 &= \text{original price} \times (1 - 0.75) \\ 90.00 &= \text{original price} \times (0.25) \\ 360.00 &= \text{original price} \end{aligned}$$

The original price of the item was \$360.00.

### YOUR TURN 6.12

Determine the original price based on the percent discount and sale price.

1. Percent discount 15%; sale price = \$11.05
2. Percent discount 9%; sale price = \$200.20

## Solve Application Problems Involving Discounts

In application problems, identify what is given and what is to be found, using the terms that have been learned, such as discount, original price, percent discount, and sale price. Once you have identified those, use the appropriate formula (or formulas) to find the solution(s).

### EXAMPLE 6.13

#### Determine Discount and New Price a Sale Rack Item

The sale rack at a clothing store is marked “All Items 30% off.” Ian finds a shirt that had an original price of \$80.00. What is the discount on the shirt? What is the sale price of the shirt?

#### Solution

We are asked to find the discount, and the sale price. We know the percent discount is 30%, or 0.30 in decimal form. The original price was \$80.

Substituting into the percent discount formula, we find that the discount is  
 $\text{discount} = \text{percent discount} \times \text{original price} = 0.30 \times 80 = 24$ .

The discount is \$24 on that shirt. The sale price is the original price minus the discount, so the sale price is  $\$80 - \$24 = \$56$ .

### YOUR TURN 6.13

1. A bed originally priced at \$550, but is on sale, with a 60% discount. What is the discount on the bed? What is the sale price of the bed?

### EXAMPLE 6.14

#### Determine the Percent Discount of a Bus Pass

An annual pass on the city bus is priced at \$240. The student price, though, is \$168. What is the percent discount for students for the bus pass?

#### Solution

We know the original price of the item, \$240. We also know the sale price of the item, \$168. From this we know the discount is  $\$240 - \$168 = \$72$ . Substituting these values into the formula  $\text{discount} = \text{percent discount} \times \text{original price}$ , we can find the percent discount.

$$\begin{aligned} \text{discount} &= \text{percent discount} \times \text{original price} \\ 72 &= \text{percent discount} \times 240 \\ \frac{72}{240} &= \frac{\text{percent discount} \times 240}{240} \\ 0.3 &= \text{percent discount} \end{aligned}$$

The student percent discount on the bus pass is 30%.

#### YOUR TURN 6.14

1. A pharmacy offers students at a nearby college a discount. Jerry purchases ibuprofen, which had an original price of \$15.80. The cost to Jerry after the student discount was \$13.43. What is the percent discount for students at the pharmacy?

#### EXAMPLE 6.15

##### Finding the Original Price of a New Pair of Tires

Kendra's car developed a flat, and the tire store told her that two tires had to be replaced. She got a 10% discount on the pair of tires, and the sale price came to \$189.00. What was the original price of the tires?

##### Solution

Using the percent discount and the sale price, we can find the original price with the formula  $\text{sales price} = \text{original price} \times (1 - \text{percent discount})$ . Substituting and solving for the original price, we find

$$\begin{aligned} \text{sales price} &= \text{original price} \times (1 - \text{percent discount}) \\ 189.00 &= \text{original price} \times (1 - 0.10) \\ 189.00 &= \text{original price} \times (0.90) \\ 210.00 &= \text{original price} \end{aligned}$$

The original price of the two tires Kendra bought was \$210.00.

#### YOUR TURN 6.15

1. Marisol needed to buy a new microwave. She got a 26% discount. The sale price Marisol paid was \$43.66. What was the original price of the microwave?

#### VIDEO

[Computing Price Based on a Percent Off Coupon \(https://openstax.org/r/Computing\\_Price\\_Based\)](https://openstax.org/r/Computing_Price_Based)

#### WORK IT OUT

There are cases where retailers allow multiple discounts to be applied. However, it is rare that the discount percentages are added together. For example, if you have a 15% coupon and qualify for a 20% price reduction, the retailer typically does not add those two percentages together to determine the new price. The retailer instead applies one discount, then applies the second discount to the price obtained after the first discount was deducted.

Research the original prices of two different laptops offered by one retail outlet. Assume you will receive a student discount of 12% and your outlet of choice is having a 15% off sale on all laptops.

For each laptop:

1. List the original price and calculate the price after applying the student discount (12%) only.

2. Then find the price after applying the sale discount (15% off) to the price found in Step 1.
3. Determine the total saved on the laptop and what percent discount the total savings represents.
4. Now, apply the discounts in reverse order (first the sale discount, then the student discount).
5. Note anything interesting about your findings.

## Calculate Markups

When retailers purchase goods to sell, they pay a certain price, called the **cost**. The retailer then charges more than that amount for the goods. This increase is called the **markup**. This selling price, or **retail price**, is what the retailer charges the consumer in order to pay their own costs and make a profit. Markup, then, is very similar to discount, except we add the markup, while we subtract the discount.

### FORMULA

The formula for a markup based on a percentage is  $\text{markup} = \text{percent markup} \times \text{cost}$ , with the percent markup expressed as a decimal. The price of the item after the markup is  $\text{retail price} = \text{cost} + \text{markup}$ .

These are often combined into the following formula

$$\text{retail price} = \text{cost} + \text{percent markup} \times \text{cost} = \text{cost} \times (1 + \text{percent markup})$$

It should be noted that the formulas used for a markup are very similar to those for a discount, with addition replacing the subtraction.

### EXAMPLE 6.16

#### Determining the Retail Price Based on the Cost and the Percent Markup

Calculate the markup for the given cost and markup percentage. Then calculate the retail price.

1. Cost = \$62.00; percent markup is 15%
2. Cost = \$750.00; percent markup is 45%

#### Solution

1. Substituting the values into the formula  $\text{markup} = \text{percent markup} \times \text{cost}$ , we find that the markup is  $\text{markup} = 0.15 \times 62.00 = 9.30$ . The markup is \$9.30.  
The retail price of the item is then  $\text{retail price} = \text{cost} + \text{markup}$ , or  $\$62.00 + \$9.30 = \$71.30$ .
2. Substituting the values into the formula  $\text{markup} = \text{percent markup} \times \text{cost}$ , we find that the markup is  $\text{markup} = 0.45 \times 750.00 = 337.50$ . The markup is \$337.50.  
The retail price of the item is then  $\text{retail price} = \text{cost} + \text{markup}$ , or  $\$750.00 + \$337.50 = \$1,087.50$ .

### YOUR TURN 6.16

Calculate the markup for the given cost and markup percentage. Then calculate the retail price.

1. Cost = \$1,800.00; percent markup is 22%
2. Cost = \$10.50; percent markup is 10%

Sometimes the cost and the retail price of an item are known. From this, the percent markup can be computed using the formula  $\text{markup} = \text{percent markup} \times \text{cost}$ , by solving for the percent markup.

### EXAMPLE 6.17

#### Calculating the Percent Markup from the Cost and Retail Price

Determine the percent markup based on the given cost and retail price. Round percentages to two decimal places.

1. Cost = \$90.00; retail price = \$103.50

2. Cost = \$5.20; retail price = \$9.90

✓ **Solution**

1. **Step 1:** Using the cost and the retail price, we can find the markup with the formula retail price = cost + markup. Substituting and calculating, we find the markup to be  $103.50 = 90.00 + \text{markup}$ . Solving for the markup gives \$13.50.

**Step 2:** After substituting the markup, \$13.50, and the original price, \$90.00, into the formula markup = percent markup  $\times$  cost, we can find the percent markup.

$$\begin{aligned} 13.50 &= \text{percent markup} \times 90.00 \\ \frac{13.50}{90.00} &= \text{percent markup} \\ 0.15 &= \text{percent markup} \end{aligned}$$

Converting to percent form, the percent markup is 15%.

2. **Step 1:** Using the cost and the retail price, we can find the markup with the formula retail price = cost + markup. Substituting and calculating, we find the markup to be  $9.90 = 5.20 + \text{markup}$ . Solving for the markup gives \$4.70.

**Step 2:** After substituting the markup, \$4.70, and the original price, \$5.20, into the formula markup = percent markup  $\times$  cost, we can find the percent markup.

$$\begin{aligned} 4.70 &= \text{percent markup} \times 5.20 \\ \frac{4.70}{5.20} &= \text{percent markup} \\ 0.9038 &= \text{percent markup} \end{aligned}$$

Converting to percent form, the percent markup is 90.38%.

> **YOUR TURN 6.17**

Determine the percent markup based on the given cost and retail price. Round percentages to two decimal places.

1. Cost = \$120.00; retail price = \$190.00
2. Cost = \$0.38; retail price = \$1.14

Sometimes the retail price and the percent markup of an item are known. From this, the cost can be found. To avoid multiple steps, though, the formula that we will use is retail price = cost  $\times$  (1 + percent markup). The cost can be found by solving this equation for the cost.

**EXAMPLE 6.18**

**Calculating the Cost from the Percent Markup and Retail Price**

Determine the cost based on the percent markup and retail price.

1. Percent markup 20%; retail price = \$10.62
2. Percent markup 125%; retail price = \$26.55

✓ **Solution**

1. Using the percent markup and the retail price, we can find the cost with the formula retail price = cost  $\times$  (1 + percent markup). Substituting and solving for the cost, we find

$$\begin{aligned} \text{retail price} &= \text{cost} \times (1 + \text{percent markup}) \\ 10.62 &= \text{cost} \times (1 + 0.2) \\ 10.62 &= \text{cost} \times (1.2) \\ 8.85 &= \text{cost} \end{aligned}$$

The cost of the item was \$8.85.

2. Using the percent markup and the retail price, we can find the cost with the formula  $\text{retail price} = \text{cost} \times (1 + \text{percent markup})$ . Substituting and solving for the original price, we find
- $$\text{retail price} = \text{cost} \times (1 + \text{percent markup})$$
- $$26.55 = \text{cost} \times (1 + 2.25)$$
- $$26.55 = \text{cost} \times (3.25)$$
- $$11.80 = \text{cost}$$

The cost of the item was \$11.80.

### YOUR TURN 6.18

Determine the cost based on the percent markup and retail price.

1. Percent markup 15%; retail price = \$40.25
2. Percent markup 300%; retail price = \$35.96

## Solve Application Problems Involving Markups

As before when working with application problems, be sure to look for what is given and identify what you are to find. Once you have evaluated the problem, use the appropriate formula to find the solution(s). These application problems address markups.

### EXAMPLE 6.19

#### Determine Retail Price of a Power Bar

Janice works at a convenience store near campus. It sells protein bars at a 60% markup. If a bar costs the store \$1.30, how much is the retail price at the convenience store?

#### Solution

We are asked to find the retail price. We know the percent markup is 60%. The cost of the bar was \$1.30. Substituting into the percent markup formula, we find that the markup is  $\text{markup} = \text{percent markup} \times \text{cost} = 0.60 \times 1.30 = 0.78$ . The markup is \$0.78 on that protein bar. The retail price is the cost plus the markup, so the retail price is  $\text{retail price} = \text{cost} + \text{markup} = 1.30 + 0.78 = 2.08$ . The retail price is \$2.08.

### YOUR TURN 6.19

1. A furniture outlet spends \$360.00 to buy a bed. The store marks up the bed by 250%. What is the retail price of the bed?

### EXAMPLE 6.20

#### Determine the Percent Markup of a Phone

Javi began working at a phone outlet. In a recent shipment, he noticed that the cost of the phone to the store was \$480.00. The phone sells for \$840.00 in the store. What is the percent markup on the phone?

#### Solution

We know the cost of the phone, \$480. We also know the retail price of the phone, \$840.00. From this we know the markup is  $\$840.00 - \$480.00 = \$360.00$ . Substituting these values into the formula  $\text{markup} = \text{percent markup} \times \text{cost}$ , we can find the percent markup.

$$\begin{aligned} \text{markup} &= \text{percent markup} \times \text{cost} \\ 360 &= \text{percent markup} \times 480 \\ \frac{360}{480} &= \frac{\text{percent markup} \times 480}{480} \\ 0.75 &= \text{percent markup} \end{aligned}$$

The markup on the phone is 75%.

### > YOUR TURN 6.20

1. Maggie does some research into textbook costs. The *Sociology of the Family* text she finds sells for \$234.36 but costs the store only \$189.00. What is the percent markup on the sociology book?

### EXAMPLE 6.21

#### Finding the Cost of a T-Shirt

Bob decided to order a t-shirt for his gaming friend online for \$29.50. He knows the markup on such t-shirts is 18%. What was the t-shirt's cost before the markup?

#### ✓ Solution

Using the percent markup and the retail price, \$29.50, we can find the cost with the formula  $\text{retail price} = \text{original price} \times (1 + \text{percent markup})$ . Substituting and solving for cost we find

$$\begin{aligned} \text{retail price} &= \text{cost} \times (1 + \text{percent markup}) \\ 29.50 &= \text{cost} \times (1 + 0.18) \\ 29.50 &= \text{cost} \times (1.18) \\ 25.00 &= \text{cost} \end{aligned}$$

The cost of the t-shirt was \$25.00.

### > YOUR TURN 6.21

1. Tina has opened a retail shop and purchased a unique hat for resale. Tina uses a 50% markup and sells the hat for \$57.00. How much did the hat cost Tina?

## Compute Sales Tax

Sales tax is applied to the sale or lease of some goods and services in the United States but is not determined by the federal government. It is most often set, collected, and spent by individual states, counties, parishes, and municipalities. None of these sales tax revenues go to the federal government.

For example, North Carolina has a state sales tax of 4.75% while New Mexico has a state sales tax of 5%. Additionally, many counties in North Carolina charge an additional 2% sales tax, bringing the total sales tax for most (72 of the 100) counties in North Carolinians to 6.75%. However, in Durham, the county sales tax is 2.25% plus an additional 0.5% tax used to fund public transportation, bringing Durham County's sales tax to 7%. To find the sales tax in a particular place, then, add other locality sales taxes to the base state sales tax rate.


How much we pay in sales tax depends on where we are, and what we are buying.

To determine the amount of sales tax on taxable purchase, we need to find the product of the purchase price, or marked price, and the sales tax rate for that locality.

**FORMULA**

To calculate the amount of sales tax paid on the purchase price in a locality with sales tax given in decimal form, calculate  $\text{sales tax} = \text{purchase price} \times \text{tax rate}$ . The total price is then

$\text{Total price} = \text{purchase price} + \text{purchase price} \times \text{tax rate} = \text{purchase price} \times (1 + \text{tax rate})$

 When the sales tax calculation results in a fraction of a penny, then normal rounding rules apply, round up for half a penny or more, but round down for less than half a penny.

You should notice that this is the same as markup, except using a different term. Sales tax plays the role of markup, the purchase price plays the role of cost, and the tax rate plays the role of percent markup. This means all the strategies developed for markups apply to this situation, with the changes indicated.


**EXAMPLE 6.22****Sales Tax in Kankakee Illinois**

The sales tax in Kankakee, Illinois, is 8.25%. Find the sales tax and total price of items based on the purchase price listed.

1. Purchase price = \$428.99
2. Purchase price = \$34.88

 **Solution**

1. The sales tax is found using  $\text{sales tax} = \text{purchase price} \times \text{tax rate}$ . The purchase price is \$428.99 and the tax rate is 8.25%. Substituting and calculating, the sales tax is  $\text{sales tax} = \$428.99 \times 0.0825 = \$35.391675$ . The sales tax needs to be rounded off. Since the third decimal place (fraction of a penny) is 1, we round down and the sales tax is \$35.39. The total price is the sales tax plus the purchase price, so is  $\$428.99 + \$35.39 = \$464.38$ .
2. The sales tax on the item is found using  $\text{sales tax} = \text{purchase price} \times \text{tax rate}$ . The purchase price is \$34.88 and the tax rate is 8.25%. Substituting and calculating, the sales tax is  $\text{sales tax} = \$34.88 \times 0.0825 = \$2.8776$ . The sales tax needs to be rounded off. Since the third decimal place (fraction of a penny) is 7, we round up and the sales tax is \$2.88. The total price of the item is the sales tax plus the purchase price, so is  $\$34.88 + \$2.88 = \$37.76$ .

 **YOUR TURN 6.22**

The sales tax in Union County, Oregon, is 7%. Find the sales tax and total price of items based on the purchase price listed.

1. Purchase price = \$1,499.00
2. Purchase price = \$26.89

As before, the information available might be different than only the purchase price and the sales tax rate. In these cases, use either  $\text{sales tax} = \text{purchase price} \times \text{tax rate}$  or  $\text{Total price} = \text{purchase price} \times (1 + \text{tax rate})$  and solve for the indicated tax, price, or rate. These problems mirror those for percent markup.

Be aware, almost all sales tax rates are structured as full percentages, or half percent, or one-quarter percent, or three-quarter percent. This means the decimal value of the sales tax rate, written as a percent, will be either 0, as in 5.0%, 5 as in 7.5%, 25 as in 3.25%, or 75 as in 4.75%. When rounding for the sales tax percentage, be sure to use this guideline.

**EXAMPLE 6.23****Calculating the Sales Tax from the Purchase Price and the Total Price**

Find the sales tax rate for the indicated purchase price and total price. Round using the guideline for sales tax percentages.

1. Purchase price = \$329.50; total price = \$354.21
2. Purchase Price = \$13.77; total price = \$14.39

✓ **Solution**

1. **Step 1.** Find the sales tax paid. First, the amount of sales tax must be found. Subtracting the purchase price from the total price, the amount of sales tax is \$24.71.

**Step 2.** Find the sales tax rate. Using the purchase price, the sales tax, and the formula  $\text{sales tax} = \text{purchase price} \times \text{tax rate}$ , the sales tax rate can be found. Substituting and solving yields

$$\text{Sales Tax} = \text{purchase price} \times \text{tax rate}$$

$$\$24.71 = \$329.50 \times \text{tax rate}$$

$$\frac{\$24.71}{\$329.50} = \text{tax rate}$$

$$0.07499 = \text{tax rate}$$

Keeping in mind the guideline for rounding sales tax rate, the sales tax rate is 7.5%.

2. **Step 1.** Find the sales tax paid. First, the amount of sales tax must be found. Subtracting the purchase price from the total price, the amount of sales tax is \$0.62.

**Step 2.** Find the sales tax rate. Using the purchase price, the sales tax, and the formula  $\text{sales tax} = \text{purchase price} \times \text{tax rate}$ , the sales tax rate can be found. Substituting and solving yields

$$\text{sales tax} = \text{purchase price} \times \text{tax rate}$$

$$\$0.62 = \$13.77 \times \text{tax rate}$$

$$\frac{\$0.62}{\$13.77} = \text{tax rate}$$

$$0.04503 = \text{tax rate}$$

Keeping in mind the guideline for rounding sales tax rate, the sales tax rate is 4.5%.

> **YOUR TURN 6.23**

Find the sales tax rate for the indicated purchase price and total price. Round using the guideline for sales tax percentages.

1. Purchase price = \$83.90; total price = \$88.30
2. Purchase price = \$477.00; total price = \$509.20

**EXAMPLE 6.24**

**Calculating the Purchase Price from the Sales Tax and Total Price**

Find the purchase price for the indicated sales tax rate and total price.

1. Sales tax rate = 5.75%; total price = \$36.56
2. Sales tax rate = 4.25%; total price = \$97.17

✓ **Solution**

1. When the sales tax rate and the total price are known, the formula  $\text{total price} = \text{purchase price} \times (1 + \text{tax rate})$  can be used to find the purchase price. Substituting the tax rate and total price into the formula and solving, we find

$$\text{Total price} = \text{purchase price} \times (1 + \text{tax rate})$$

$$\$36.56 = \text{purchase price} \times (1 + 0.0575)$$

$$\frac{\$36.56}{1.0575} = \text{purchase price}$$

$$\$34.57 = \text{purchase price}$$

The purchase price, the price before tax, was \$34.57.

2. When the sales tax rate and the total price are known, the formula  $\text{total price} = \text{purchase price} \times (1 + \text{tax rate})$  can be used to find the purchase price. Substituting the tax rate and total price into the formula and solving, we find

$$\begin{aligned} \text{Total price} &= \text{purchase price} \times (1 + \text{tax rate}) \\ \$97.17 &= \text{purchase price} \times (1 + 0.0425) \\ \frac{\$97.17}{1.0425} &= \text{purchase price} \\ \$93.21 &= \text{purchase price} \end{aligned}$$

The purchase price, the price before tax, was \$93.21.

### > YOUR TURN 6.24

Find the purchase price for the indicated sales tax rate and total price.

1. Sales tax rate = 8.25%; total price = \$157.81
2. Sales tax rate = 6.75%; total price = \$522.01

## Solve Application Problems Involving Sales Tax

Solving problems involving sales tax follows the same ideas and steps as solving problems for markups. But here we will use the following formula:

$$\text{total price} = \text{purchase price} + \text{sales tax}$$

We can also use the formula:

$$\text{total price} = \text{purchase price} \times (1 + \text{sales tax rate}).$$

This can be seen in the following examples.

### EXAMPLE 6.25

#### Compute Sales Tax for Denver, Colorado

The sales tax rate in Denver Colorado is 8.81%. Keven buys a TV in Denver, and the purchase price (before taxes) is \$499.00. How much will Keven pay in sales tax and what will be the total amount he spends when he buys the TV?

#### ✓ Solution

The sales tax rate in Denver is 8.81%. To find the sales tax Keven will pay, find 8.81% of the purchase price. In decimal form, that sales tax rate is 0.0881. Using the formula and substituting 499.00 for purchase price, we find that Keven will pay purchase price  $\times$  tax rate =  $\$499 \times 0.0881 = \$43.96$  in sales tax for the TV.

The total price that Keven will pay is the purchase price plus the sales tax, or  $\$499.00 + \$43.96 = \$542.96$ .

### > YOUR TURN 6.25

1. Daryl decides to buy a new scooter in St. Louis, Missouri, where the sales tax is 9.68%. The scooter he chooses has a purchase price of \$1,149. How much will Daryl pay in sales tax and what is the total price he spends on the scooter?

### EXAMPLE 6.26

#### Compute Sales Tax for Austin, Texas

Jillian visits Austin, Texas, and purchases a new set of weights for her home. She spends, including sales tax, \$467.64. The sales tax rate in Austin Texas is 8.25%. How much of the total price is sales tax?

#### ✓ Solution

The sales tax paid for this purchase is the difference in the total price and the purchase price. We know the total price is \$467.64. We also know the sales tax rate, which is 8.25%. In decimal form, this is 0.0825. Using these values and the

formula total price = purchase price  $\times$  (1 + tax rate) to find the purchase price.

$$\text{total price} = \text{purchase price} \times (1 + \text{tax rate})$$

$$\$467.64 = \text{purchase price} \times (1 + 0.0825)$$

$$\$467.64 = \text{purchase price} \times (1.0825)$$

$$\$432 = \text{purchase price}$$

Knowing both the total price and the now the purchase price, we can find the difference, which is the sales tax.

The total price was \$467.64. The purchase price was \$432. The difference of the total price and the purchase price, or the sales tax, is then  $\$467.64 - \$432.00$ , which is \$35.64. Jillian pays \$35.64 in sales tax.

### YOUR TURN 6.26

1. Elizabeth decides to buy new running shoes in her hometown of Springfield, Illinois, where the sales tax rate is 6.25%. If her total bill comes to \$153, how much of the total price is sales tax?

### VIDEO

[Finding Sales Tax Percentage \(https://openstax.org/r/Finding\\_Sales\\_Tax\)](https://openstax.org/r/Finding_Sales_Tax)

### WHO KNEW?

West Virginia was the first state to impose a sales tax. This happened on May 3, 1921.

Look up your locality on [this website that lists standard state-level sales tax rates \(https://openstax.org/r/resources\\_rates\)](https://openstax.org/r/resources_rates) and compare the sales tax structure in your state to two nearby states (for the lower 48) and for any two states (Alaska and Hawaii).

## Check Your Understanding

7. What is a discount?
8. What is a markup?
9. An item has a retail price of \$45.00. What is the sale price after a 32% discount?
10. A retailer buys an item for \$311.00. What is the retail price if their markup is 60%?
11. Does sales tax have the same formula as markup?
12. If the sales tax is 6.8%, what is the total price for an item that has a purchase price of \$39.95?



## SECTION 6.2 EXERCISES

For the following exercises, use the given values to find the indicated value. Round percent results to 2 decimal places. Round money results to the penny (2 decimal places).

1. Retail price = \$399.00, percent discount = 30%, find the sale price.
2. Retail Price = \$75.00, percent discount = 65%, find the sale price.
3. Retail price = \$125.00, sale price = \$90.00, find the percent discount.
4. Retail price = \$47.00, sale price = \$41.50, find the percent discount.
5. Sale price = \$145.70, percent discount = 20%, find the retail price.
6. Sale price = \$1,208.43, percent discount = 13%, find the retail price.
7. Retail price = \$26,790.00, percent discount = 8%, find the sale price.
8. Sale price = \$314.06, percent discount = 33%, find the retail price.
9. Retail price = \$145.50, sale price = \$117.90, find the percent discount.
10. Retail price = \$28.90, percent discount = 18%, find the sale price.
11. Sale price = \$17.59, percent discount = 12%, find the retail price.

12. Retail price = \$57.50, sale price = \$46.00, find the percent discount.
13. Cost = \$130.00, percent markup = 34%, find the retail price.
14. Cost = \$2.27, percent markup = 42%, find the retail price.
15. Cost = \$68.45, retail price = \$109.90, find the percent markup.
16. Cost = \$466.16, retail price = \$699.00, find the percent markup.
17. Retail price = \$98.99, percent markup = 25%, find the cost.
18. Retail price = \$799.00, percent markup = 55%, find the cost.

In the following exercises, find the sales tax and total paid.

19. Retail price = \$17.99; sales tax = 7.5%
20. Retail price = \$799.00; sales tax = 8.5%
21. Retail price = \$176.83; sales tax = 6.25%
22. Retail price = \$223.93; sales tax = 4.5%

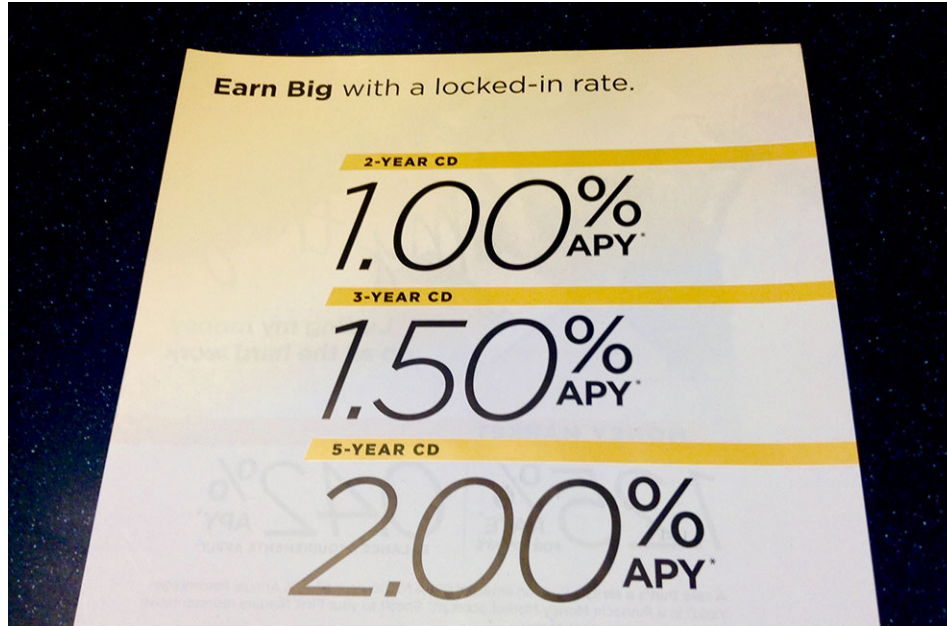
In the following exercises, find the sales tax rate.

23. Purchase price = \$257.45; total price = \$273.54
24. Purchase price = \$14.99; total price = \$15.74
25. Purchase price = \$26.83; total price = \$28.84
26. Purchase price = \$2,399.90; total price = \$2,609.89

In the following exercises, find the purchase price.

27. Sales tax rate = 4.75%; total price = \$50.15
  28. Sales tax rate = 8%; total price = \$1,069.20
  29. Sales tax rate = 9.5%; total price = \$51.45
  30. Sales tax rate = 5.75%; total price = \$3,065.69
31. Harris has a coupon for 20% off for any purchase. She finds a new tennis racket for \$278.00. How much is the price after the coupon is applied?
  32. After the employee discount, Mariam will pay \$46.55. What is her employee discount rate if the retail price was \$53.50? Round to nearest full percent.
  33. Resa purchased a new game for her cousin. After sales tax, she paid \$41.13. Find the sales tax rate she paid if the purchase price of the game was \$38.99.
  34. Larissa opens a new secondhand bookstore. She buys a book for \$2.75. What is her percent markup if she sells the book for \$8.50. Round to the nearest percent.
  35. Doug opens a used auto parts store. He pays \$30 for a car door. How much will he charge if his percent markup is 60%?
  36. Gaia and Seth live in Osceola County in Florida, where the sales tax rate is 7.5%. They purchased some new camping gear. The price before taxes came to \$784.62. How much do they pay after the sales tax is applied?
  37. Theresa decides to purchase a new phone, which has a retail price of \$799.00. Her discount is 20% through a friends and family plan. The sales tax in her county is 6.75%. How much will she pay after the discount? How much will she pay after the tax is applied?
  38. Sakari manages a retail outlet. They receive a shipment of shirts. She sees on the shipping list that each shirt cost 24.50 to the store. The store marks up the shirts by 45%. The county in which she lives charges sales tax of 6.5%. What is the retail price of one of the shirts? After sales tax, how much will a customer pay for the shirt?

## 6.3 Simple Interest



**Figure 6.4** Interest is how savings earns money. (credit: "Interest Rates" by Mike Mozart/Flickr, CC BY 2.0)

### Learning Objectives

After completing this section, you should be able to:

1. Compute simple interest.
2. Understand and compute future value.
3. Compute simple interest loans with partial payments.
4. Understand and compute present value.

There is truth in the phrase "You need to have money to make money." In essence, if you have money to lend, you can lend it at a cost to a borrower and make money on that transaction.

When money is borrowed, the person borrowing the money (borrower) typically has to pay the person or entity that lent the money (the lender) more than the amount of money that was borrowed. This extra money is the **interest** that is to be paid. Interest is sometimes referred to as the cost to borrow, the cost of the loan, or the finance cost.

This idea also applies when someone deposits money in a bank account or some other form of investment. That person is essentially lending the money to the bank or company. The money earned by the depositor is also called interest. The interest is typically based on the amount borrowed, or the **principal**.

The pairing of borrower and lender can take various forms. The borrower may be a consumer using a credit card or taking out a loan from a bank, the lender. Companies also borrow from lending banks. Someone who invests in a company's stock is the lender in this case; the company is essentially the borrower.

In this section, we examine the basic building block of interest paid on loans and borrowed credit and also the returns on investments like bank accounts, simple interest.

### Compute Simple Interest

Let's get some terminology understood. Interest to be paid by a borrower is often expressed as an **annual percentage rate**, which is the percent of the principal that is paid as interest for each year the money is borrowed. This means that the more that is borrowed, the more that must be paid back. Sometimes, the interest to be paid back is **simple interest**, which means that the interest is calculated on the amount borrowed only.

The length of time until the loan must be paid off is the **term** of the loan. The date when the loan must be paid off is when the loan is **due**. The day that the loan is issued is the **origination date**. We'll put this terminology to use in the following examples. Note that in this section we will use letters, called variables, to represent the different parts of the formulas we'll be using. This will help keep our formulas and calculations manageable.

### Simple Interest Loans with Integer Year Terms

Calculating simple interest is similar to the percent calculations we made in [Understanding Percent](#) and [Discounts, Markups, and Sales Tax](#), but must be multiplied by the term of the loan (in years, if dealing with an annual percentage rate).

#### FORMULA

The simple interest,  $I$ , to be paid on a loan with annual interest rate  $r$  for a number of years (term of the loan)  $t$ , with principal  $P$ , is found using  $I = P \times r \times t$ , where the decimal form of the interest rate,  $r$ , is used. The total repaid, then is  $T = P + I$  or, more directly,  $T = P + P \times r \times t$ . This total is often referred to as the loan **payoff amount**, or more simply just the payoff.

When the annual interest rate, the principal, and the number of years that the money is borrowed is known, the interest to be paid can be found and from there the total to be repaid can be calculated.

 Be aware, interest paid to a lender is almost uniformly rounded up to the next cent.

#### EXAMPLE 6.27

##### Simple Interest on Loans with Integer Year Terms

Calculate the simple interest to be paid on a loan with the given principal, annual percentage rate, and number of years. Then, calculate the loan payoff amount.

1. Principal  $P = \$4,000$ , annual interest rate  $r = 5.5\%$ , and number of years  $t = 4$
2. Principal  $P = \$14,800$ , annual interest rate  $r = 7.9\%$ , and number of years  $t = 7$

#### Solution

1. Substitute the principal  $P = \$4,000$ , the decimal form of the annual interest rate  $r = 0.055$ , and number of years  $t = 4$  into the formula for simple interest, and calculate.  

$$I = P \times r \times t = 4,000 \times 0.055 \times 4 = 880.$$
 The simple interest, or cost of the loan, to be paid on the loan is \$880.  
 The loan payoff amount, or the total to be repaid, is  $T = P + I = 4,000 + 880 = 4,880$ , or \$4,880.00.
2. Substitute the principal  $P = \$14,800$ , the decimal form of the annual interest rate  $r = 0.079$ , and number of years  $t = 7$  into the formula for simple interest, and calculate.  

$$I = P \times r \times t = 14,800 \times 0.079 \times 7 = 8,184.4.$$
 The simple interest, or cost to borrow, to be paid on the loan is \$8,184.40.  
 The loan payoff amount, or the total to be repaid, is  $T = P + I = 14,800 + 8,184.4 = 22,984.4$ , or \$22,984.40.

#### YOUR TURN 6.27

Calculate the simple interest to be paid on a loan with the given principal, annual percentage rate, and number of years. Then calculate the loan payoff, or total to be repaid.

1. Principal  $P = \$6,700$ , annual interest rate  $r = 11.99\%$ , and number of years  $t = 3$
2. Principal  $P = \$25,800$ , annual interest rate  $r = 6.9\%$ , and number of years  $t = 5$

#### EXAMPLE 6.28

##### Simple Interest Equipment Loan

Riley runs an auto repair shop, and needs to purchase a new brake lathe, which costs \$11,995. She takes out a two-year, simple interest loan at an annual interest rate of 14.9%. How much interest will she pay and how much total will she repay on the loan?

#### Solution

**Step 1.** Determine the variables, or parts of the formula. The principal  $P$  is the cost of the brake lathe, so  $P = \$11,995$ .

The interest rate Riley pays is 14.9%, or  $r = 0.149$  in decimal form. The length of the loan is two years, so  $t = 2$ . We are first asked to find  $I$ , the interest Riley will pay.

**Step 2.** Substitute the known variables into the formula for simple interest  $I = P \times r \times t$  and solve for  $I$ .

From Step 1 we have  $I = P \times r \times t = 11995 \times 0.149 \times 2 = 3,574.51$ .

This tells us that the simple interest, or cost to borrow, to be paid on the loan is \$3,574.51.

**Step 3.** Use the formula  $T = P + I$  to determine the total amount Riley will repay,  $T$ .

The total to be repaid is  $T = P + I = 11,995 + 3,574.51 = 15,569.51$ , or \$15,569.51.

### > YOUR TURN 6.28

1. Beth is the owner of a small retail store in downtown St. Louis. The windows in the storefront need replacing, so she needs to take out a \$9,500 loan to get the repairs done. The rate she secures is 9.25% and the term of the loan is one year. How much interest will she pay and how much total will she repay on the loan?

### Simple Interest Loans with Other Lengths of Terms

In the previous example and Your Turn exercise, the loans were paid back in one payment after an integer number of years. However, there are also loans lasting a length of time not equal to an integer number of years (like 1, 2, or 3 years or more), but in a number of months (like 4 months, 18 months, and so on). What model would apply to these situations?

When the loan is paid back after a term that is not an integer number of years but is instead a number of months, the term of the loan, or time,  $t$ , is expressed as a fraction of the year. So for a 2-month loan, the time, in years, is  $2/12 = 1/6$ . For a 5-month loan, the time in years is  $5/12$ . For an 18-month term, the term in years is  $18/12 = 1.5$ .

### EXAMPLE 6.29

#### Loan to Purchase Equipment

Abeje needs a loan to purchase equipment for the gym she is going to open. She visits the bank and secures a 4-month loan of \$20,000. Her annual percentage rate is 6.75%. How much interest will Abeje pay and what is her loan payoff amount?

#### ✓ Solution

Abeje's loan is for \$20,000, so her principal is  $P = 20,000$ . The interest rate Abeje will pay is 6.75%, or  $r = 0.0675$  in decimal form. The length of the loan is 4 months, so  $t = \frac{4}{12}$ . Substituting these in the formula for simple interest, we find her interest to be  $I = P \times r \times t = 20,000 \times 0.0675 \times 4/12 = 450$

The simple interest, or cost to borrow, to be paid on the loan is \$450.00.

The payoff is  $T = P + I = 20,000 + 450 = 20,450$ , or \$20,450.00.

### > YOUR TURN 6.29

1. Samuel needs to borrow \$8,400 to pay for repairs to his small manufacturing facility. He manages to get a simple interest loan at 17.33%, to be paid after 6 months. How much interest will Samuel pay and what is Samuel's loan payoff amount?

Those examples dealt in months. However, some loans are for days only (45 days, 60 days, 120 days). In such cases, we find the daily interest rate. The fraction we will use for the daily interest rate is the interest rate (as a decimal) divided by 365. This may be referred to as Actual/365. In order to find the term of the loan, divide the number of days in the term of the loan by 365.

**FORMULA**

To determine the interest,  $I$ , on a loan with term  $t$  expressed in days, with principal of  $P$ , and interest rate in decimal form of  $r$ , calculate  $I = P \times \frac{r}{365} \times t$ . Here,  $\frac{r}{365}$  represents the daily interest rate.

Alternately, the above formula is equivalent to  $I = P \times r \times \frac{t}{365}$ , where the interest rate remains an annual rate, but the time is expressed as a fraction of the year.

**? WHO KNEW?**

It seems reasonable to use 365 as the number of days in the year, since there are 365 days in most years. However, sometimes, banks have used (and continue to use) 360 as the number of days in a year. They may also treat all months as if they have 30 days. These differences lead to (sometimes small) differences in how much interest is paid. Since the number of days is in the denominator, a smaller denominator (360) will result in larger numbers (interest) that is 365 is used for the denominator. See [this page from ACRE \(https://openstax.org/r/ACRE\)](https://openstax.org/r/ACRE) for a comparison.

**EXAMPLE 6.30****Loan for Moving Costs**

David plans to move his family from Raleigh, North Carolina to Tempe, Arizona. His company will reimburse (pay after the move) David for the move. David does research and determines that movers will cost \$5,600 to move his family's belongings to Tempe. He takes out a simple interest, 45-day loan at 11.75% interest to pay this cost. How much interest will be paid on this 45-day loan, and what is David's loan payoff amount?

**✓ Solution**

This loan is in terms of days, so we will use the formula  $I = P \times \frac{r}{365} \times t$ , where  $t$  is the number of days and  $r$  is the annual interest rate.

The principal for the loan is the moving cost, or  $P = 5,600$ . The annual interest rate that David will pay is 11.75%, which in decimal is 0.1175. The length of time for the loan is 45 days, so  $t = 45$ .

Substituting these values into the formula and calculating, we find that the interest to be paid is

$$I = 5,600 \times \frac{0.1175}{365} \times 45 = 81.13, \text{ or } \$81.13 \text{ (remember, interest is almost always rounded up to the next cent).}$$

The payoff for the loan is \$5,681.13.

**> YOUR TURN 6.30**

1. Heather runs a silk screen t-shirt shop, and seeks a short-term loan to pay for new inventory (paints, blades, shirts). They secure a \$3,700 simple interest loan for 60 days, at an annual rate of 18.99%. How much will borrowing the money cost Heather, and what is her loan payoff amount?

**Understand and Compute Future Value**

Money can be invested for a specific amount of time and earn simple interest while invested. The terminology and calculations are the same as we've already seen. However, instead of the total to be paid back, the investor is interested in the total value of the investment after the interest is added. This is called the **future value** of the investment.

**FORMULA**

The future value,  $FV$ , of an investment that yields simple interest is  $FV = P + I = P + P \times r \times t$ , where  $P$  is the principal (amount invested at the start),  $r$  is the annual interest rate in decimal form, and  $t$  is the length of time the

money is invested. The time  $t$  will be an integer if the term of the deposit is an integer number of years, will be number of months/12 if the term is in months, will be actual/365 if the deposit is for a number of days.

### EXAMPLE 6.31

#### Simple Interest on a Deposit

In the following, determine how much interest was earned on the investment and the future value of the investment, if the investment yields simple interest.

1. Principal is \$1,000, annual interest rate is 2.01%, and time is 5 years
2. Principal is \$5,000, annual interest rate is 1.85%, and time is 30 years
3. Principal is \$10,000, annual interest rate is 1.25%, and time is 18 months
4. Principal is \$7,000, annual interest rate is 3.26%, and time is 100 days

#### Solution

1. The principal is  $P = \$1,000$ , the annual interest rate, in decimal form, is 0.0201, and the term is 5 years, or  $t = 5$ . Since the term is an integer number of years, the interest earned on the investment is  $I = P \times r \times t = 1,000 \times 0.0201 \times 5 = 100.5$ , or the interest earned was \$100.50. To find the future value, we use the formula  $FV = P + I$ . Substituting the values and calculating, we find the future value of the investment to be  $FV = P + I = 1,000 + 100.5 = 1100.5$ . The future value of the investment at the end of 5 years is \$1,100.50. Notice that the future value could have been calculated directly with  $FV = P + P \times r \times t$ .
2. The principal is  $P = \$5,000$ , the annual interest rate, in decimal form, is 0.0185, and the term is 30 years, or  $t = 30$ . Since the term is an integer number of years, the interest earned on the investment is  $I = P \times r \times t = 5,000 \times 0.0185 \times 30 = 2,775$ , or the interest earned was \$2,775.00. To find the future value, we use the formula  $FV = P + I$ . Substituting the values and calculating, we find the future value of the investment to be  $FV = P + I = 5,000 + 2,775 = 7,775$ . The future value of the investment at the end of 30 years is \$7,775.00.
3. The principal is  $P = \$10,000$ , the annual interest rate, in decimal form, is 0.0125, and the term is 18 months. Since the term is in months, we have to write the months in terms of years. For 18 months, we use  $18/12$  as  $t$ . The interest earned on the investment is  $I = P \times r \times t = 10,000 \times 0.0125 \times \frac{18}{12} = 187.5$ , or the interest earned was \$187.50. To find the future value, we use the formula  $FV = P + I$ . Substituting the values and calculating, we find the future value of the investment to be  $FV = P + I = 10,000 + 187.5 = 10,187.5$ . The future value of the investment at the end of 18 months is \$10,187.50.
4. Principal is \$7,000, annual interest rate is 3.26%, and time is 100 days. The principal is  $P = \$7,000$ , the annual interest rate, in decimal form, is 0.0326, and the term is 100 days. Since the term is in days, we have to write the time using actual/365, or  $t = 100/365$ . The interest earned on the investment is  $I = P \times r \times t = 7,000 \times 0.0326 \times \frac{100}{365} = 62.52$ , or the interest earned was \$62.52. To find the future value, we use the formula  $FV = P + I$ . Substituting the values and calculating, we find the future value of the investment to be  $FV = P + I = 7,000 + 62.52 = 7,062.52$ . The future value of the investment at the end of 100 days is \$7,062.52.

#### YOUR TURN 6.31

In the following, determine how much interest was earned on the investment and the future value of the investment if the investment yields simple interest.

1. Principal is \$4,500, annual interest rate is 1.88%, and time is 3 years
2. Principal is \$2,000, annual interest rate is 2.03%, and time is 10 years
3. Principal is \$120,000, annual interest rate is 3.1%, and time is 100 days
4. Principal is \$4,680, annual interest rate is 1.55%, and time is 42 months

You may have noticed that for these problems, the future value was rounded down. When the future value is paid, the amount is typically rounded down.

A certificate of deposit (CD) is a savings account that holds a single deposit (the principal) for a fixed term at a fixed interest rate. Once the term of the CD is over, the CD may be redeemed (cashed in or withdrawn) and the owner of the

CD receives the original principal plus the interest earned. The deposit often cannot be withdrawn until the term is up; if it can be withdrawn early, there is often a penalty imposed to do so.

### EXAMPLE 6.32

#### Certificate of Deposit

Jonas deposits \$2,500 in a CD bearing 3.25% simple interest for a term of 3 years. When he redeems his CD at the end of the 3 years, how much will he receive?

#### ✓ Solution

This is a future value example. We know that  $P = \$2,500$  is the amount deposited. The annual simple interest rate in decimal form is  $r = 0.0325$ . The term of the investment is  $t = 3$  years.

Substituting those values into the future value formula, we have  
 $FV = P + P \times r \times t = 2,500 + 2,500 \times 0.0325 \times 3 = 2,500 + 243.75 = 2,743.75$ .

When the CD is redeemed, Jonas will receive \$2,743.75.

### > YOUR TURN 6.32

1. Mia deposits \$4,900 in a CD bearing 3.95%. The CD term is 7 years. When she redeems the CD, how much will Mia receive?

### WORK IT OUT

The reason CD (certificate of deposit) rates look so small is because they are extremely safe investments. Though overall interest rates for CDs change over time and individual returns vary with the terms of the CD, investors are offered predictable interest income for their investments.

To investigate this yourself, search online to determine the strengths and weaknesses of CDs ([investopedia.com](http://investopedia.com) offers good, basic information on investing). Then, online, identify five national banks and two local banks who offer CDs.

- Track the interest rates for the CDs at various terms (1 year, 3 years, 5 years) for each of the banks you found that offer CDs.
- Calculate the amount of interest earned for a \$10,000 deposit for each CD at each of the terms.
- Compare the results from the various banks, CDs, and terms and decide which is the best investment. You may want to consider both the length of time that the money is locked up, and the return.

## Paying Simple Interest Loans with Partial Payments

In every example above, there was one payment for the loan, or one withdrawal for the investment. However, for many loans (house, car, in-ground swimming pool), the loan will be paid back in two or more payments. Such a payment is called a **partial payment**, because they only pay off part of the loan.

When a partial payment is made, some of the payment pays for the principal, but the rest of the payment pays for interest on the principal. When making the first partial payment, the interest is calculated on the principal for the time between the origination date of the loan and the date of the payment. If another partial payment is made, the interest is calculated based on the remaining principal and the time between the previous partial payment and the current partial payment date.

### EXAMPLE 6.33

#### Interest Paid in a Partial Payment on a Loan

1. A simple interest loan for \$6,500 is taken out at 12.6% annual percentage rate. A partial payment is made 45 days into the loan period. How much of the partial payment will be for interest?

2. A simple interest loan for \$13,700 is taken out at 6.55% annual interest rate. A partial payment is to be made after 60 days. How much of the partial payment will be for interest?

✓ **Solution**

1. To find the interest paid in this partial payment, we calculate the interest on the principal for the time between the origination of the loan and the payment day, or 45 days.

The principal is \$6,500. The annual interest rate, in decimal form, is 0.126.

The interest paid for 45 days is found by substituting the values for principal  $P$ , rate  $r$ , and time  $t$  into the formula

$$I = P \times \frac{r}{365} \times t.$$

Calculating, we have  $I = P \times \frac{r}{365} \times t = 6,500 \times \frac{0.126}{365} \times 45 = 100.9726$ . Rounding up, the portion of the partial payment that will be paid for interest is \$100.98.

2. To find the interest paid in this partial payment, we calculate the interest on the principal for the time between the origination of the loan and the payment day, or 60 days.

The principal is \$13,700. The annual interest rate, in decimal form, is 0.0655.

The interest paid for those 60 days is found by substituting those values into the formula  $I = P \times \frac{r}{365} \times t$ .

Calculating, we have  $I = P \times \frac{r}{365} \times t = 13,700 \times \frac{0.0655}{365} \times 60 = 147.5096$ . Rounding up, the portion of the partial payment that will be paid for interest is \$147.51.

> **YOUR TURN 6.33**

1. A simple interest loan for \$50,000 is taken out at 5.15% annual percentage rate. A partial payment is made 120 days into the loan period. How much of the partial payment will be for interest?
2. A simple interest loan for \$8,500 is taken out at 9.9% annual interest rate. A partial payment is to be made after 75 days. How much of the partial payment will be for interest?

### Remaining Balance

The previous examples demonstrated how to determine the interest paid in a partial payment. Using this, we can determine the remaining balance after a partial payment.

**Step 1:** determine the amount of the payment,  $P$ , that is applied to interest,  $I$ .

**Step 2:** subtract the amount paid in interest from the payment,  $(P - I)$ . This is the amount applied to the balance.

**Step 3:** subtract the amount applied to the balance (the value obtained in Step 2) from the balance of the loan,  $B - (P - I)$ . This is the remaining balance after the partial payment.

### EXAMPLE 6.34

#### Determining the Remaining Balance on a Loan After a Partial Payment

1. A simple interest loan for \$45,500 is taken out at 11.8% annual percentage rate. A partial payment of \$20,000 is made 50 days into the loan period. After this payment, what will the remaining balance of the loan be?
2. A simple interest loan for \$150,000 is taken out at 5.85% annual percentage rate. A partial payment of \$50,000 is made 70 days into the loan period. After this payment, what will the remaining balance of the loan be?

✓ **Solution**

1. The principal is \$45,500, which will be treated as the balance,  $B$ , of the loan. The annual simple interest rate, in decimal form, is 0.118. The time is  $t = 50$  days.

**Step 1:** Determine the amount of the partial payment that is applied to interest. To find this, substitute the values above into the formula  $I = P \times \frac{r}{365} \times t$  and calculate. Calculating, the amount of the payment that is applied to interest is  $I = 45,500 \times \frac{0.118}{365} \times 50 = 735.4795$ . Rounding up, we have  $I = \$735.48$ .

**Step 2:** The amount of the payment that is to be applied to the balance of the loan is partial payment minus the amount of the partial payment that is applied to the interest. The payment is \$2,000. The amount that is applied to the balance is  $P - I = \$20,000 - \$735.48 = \$19,264.52$ .

**Step 3:** The remaining balance is found by subtracting the amount applied to the balance from the previous balance, or  $B - (P - I) = \$45,500 - \$19,264.52 = \$26,235.48$ .

The remaining balance after the partial payment is \$26,235.48.

- The principal is \$150,000, which will be treated as the balance,  $B$ , of the loan. The annual simple interest rate, in decimal form, is 0.0585. The time is  $t = 70$  days.

**Step 1:** Determine the amount of the partial payment that is applied to interest. To find this, substitute the values above into the formula  $I = P \times \frac{r}{365} \times t$  and calculate. Calculating, the amount of the payment that is applied to interest is  $I = 150,000 \times \frac{0.0585}{365} \times 70 = 1,682.8767$ . Rounding up, we have  $I = \$1,682.88$ .

**Step 2:** The amount of the payment that is to be applied to the balance of the loan is partial payment minus the amount of the partial payment that is applied to the interest. The payment is \$50,000. The amount that is applied to the balance is  $P - I = \$50,000 - \$1,682.88 = \$48,317.12$ .

**Step 3:** The remaining balance is found by subtracting the amount applied to the balance from the previous balance, or  $B - (P - I) = \$150,000 - \$48,317.12 = \$101,682.88$ .  
The remaining balance after the partial payment is \$101,682.88.

### > YOUR TURN 6.34

- A simple interest loan for \$1,400 is taken out at 12.5% annual percentage rate. A partial payment of \$700 is made 20 days into the loan period. After this payment, what will the remaining balance of the loan be?
- A simple interest loan for \$23,000 is taken out at 7.25% annual percentage rate. A partial payment of \$10,000 is made 40 days into the loan period. After this payment, what will the remaining balance of the loan be?

### Loan Payoff

Finally, we will determine the amount to be paid at the end of the loan. To do so, we apply the formula for the loan payoff to the remaining balance. However, the length of time for that remaining balance is the time between the partial payment and the day the loan is paid off.

**Step 1:** Determine the remaining balance after the partial payment.

**Step 2:** Calculate the number of days between the partial payment and the date the loan is paid off. This will be the time  $t$  in the payment formula.

**Step 3:** Calculate the amount to be paid at the end of the loan, or the payoff amount, using  $\text{Payoff} = P + P \times \frac{r}{365} \times t$ , where  $P$  is the remaining balance and  $t$  is the time found in Step 2.

### EXAMPLE 6.35

#### Finding Loan Pay Off After a Partial Payment

Laura takes out an \$18,400 loan for 120 days at 17.9% simple interest. She makes a partial payment of \$7,500 after 45 days. What is her payoff amount at the end of the loan?

#### ✓ Solution

The initial balance, or principal, of her loan is \$18,400. The interest rate in decimal form is 0.179. Her partial payment of \$7,500 is made after 45 days. Using these values, we can determine how much of the partial payment is applied to the balance. From there, we can determine her final loan payoff after 120 days.

**Step 1:** Determine the remaining balance after the partial payment. Using the partial payment process outlined in the previous example, we first find that the amount of the partial payment that is applied to the balance. Their interest paid in the partial payment is  $I = P \times \frac{r}{365} \times t = 18,400 \times \frac{0.179}{365} \times 45 = 406.0603$ , or \$406.07 (remember to round up!). Using this and that the loan amount was for \$18,400, the remaining balance on the loan after the partial payment is  $B - (P - I) = \$18,400 - (\$7,500 - \$406.07) = \$11,306.07$ .

**Step 2:** The number of days between the partial payment and the date that the loan is to be paid off is  $120 - 45 = 75$ . This means that the time between the partial payment and the final payment is 75 days.

**Step 3:** To calculate the payoff amount, use  $\text{payoff} = P + P \times \frac{r}{365} \times t$ , with  $P = \$11,306.07$  (the remaining balance),  $t = 75$  (from Step 2) and  $r = 0.179$ . The payoff amount, then, is  $\text{payoff} = 11,306.07 + 11,306.07 \times \frac{0.179}{365} \times 75 = 11,721.9165$ . Rounding up, the payoff amount is \$11,721.92.

> YOUR TURN 6.35

1. Paola takes out a 75-day loan for \$3,500.00. Her interest rate is 11.2%. If she makes a partial payment of \$1,250.00 after 30 days, what will her payoff be at the end of the loan?

### Repeated Partial Payments

Car loans and mortgages (loans for homes) are paid off through repeated partial payments, most often monthly payments. Since car loans are often 3 to 6 years, and mortgages 15 to 30 years, calculating each individual monthly payment one at a time is time consuming and tedious. Even a 3-year loan would involve applying the above steps 36 times! Fortunately, there is a formula for determining the amount of each partial payment for monthly payments on a simple interest loan.

#### FORMULA

The amount of monthly payments,  $A$ , for a loan with principal  $P$ , monthly simple interest rate  $r$  (in decimal form), for  $t$  number of months is found using the formula  $A = P \times \frac{r \times (1 + r)^t}{(1 + r)^t - 1}$ . The monthly interest rate is the annual rate divided by 12. The number of months is the number of years times 12.

#### EXAMPLE 6.36

#### Calculating Car Payments

Desiree buys a new car, by taking a loan out from her credit union. The balance of her loan is \$27,845.00. The annual interest rate that Desiree will pay is 7.3%. She plans to pay this off over 4 years. How much will Desiree's monthly payment be?

✓ **Solution**

To use the formula for monthly payments, we need the principal, the interest rate, and the number of years. The principal is \$27,845. The annual rate, in decimal form, is 0.073. Dividing 0.073 by 12 gives the monthly interest rate  $0.073/12 = 0.00608\bar{3}$ . She takes the loan out for 4 years, which is  $t = 12 \times 4 = 48$  months. Substituting these values into the formula,  $A = P \times \frac{r \times (1 + r)^t}{(1 + r)^t - 1}$ , we calculate:

$$\begin{aligned} A &= 27,845 \times \frac{0.00608\bar{3} \times (1 + 0.00608\bar{3})^{48}}{(1 + 0.00608\bar{3})^{48} - 1} \\ &= 27,845 \times \frac{0.00608\bar{3} \times (1.00608\bar{3})^{48}}{(1.00608\bar{3})^{48} - 1} \\ &= 27,845 \times \frac{0.00608\bar{3} \times 1.337918996}{1.337918996 - 1} \\ &= 27,845 \times \frac{0.008139007}{0.337918996} \\ &= 27,845 \times 0.024085675 \\ &= 670.6656 \end{aligned}$$

Using the formula and rounding up to the next cent, we see that Desiree's monthly payment will be \$670.67.

> YOUR TURN 6.36

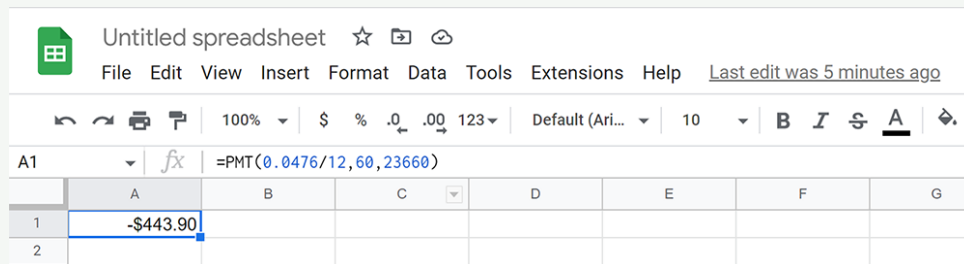
1. Russell buys a new car, by taking a loan out from the dealership. After all of the discussions are over, he finances (gets a loan for) \$23,660. The annual interest rate that Russell will pay is 4.76%. He plans to pay this off over 5 years. How much will Russell's monthly payment be?

 **TECH CHECK**

The calculation of payments is long, and involves many steps. However, most spreadsheet programs, including Google Sheets, have a payment function. In Google Sheets, that function is PMT. To find the payment for an installment loan (like for a car), you need to enter the interest rate per period, the number of payments, and the loan amount. From [Your Turn 6.36](#), the rate was  $0.0476/12$ , the number of payments was 60, and the loan amount was \$23,660. In Google Sheets, select any cell and enter the following:

```
=PMT(0.0476/12,60,23660)
```

And click the enter key. Immediately, in the cell you selected, the payment of \$443.90 appears, though with a negative sign. The negative sign indicates it is a payment out of an account. Since we want to know the payment amount, we ignore the negative sign. The result, with the formula in the formula bar, is shown in [Figure 6.5](#).



**Figure 6.5** Google Sheets payment function

In general, to use the PMT function in Google Sheets, enter

```
=PMT( $r/12, t*12, P$ )
```

where  $r$  is the annual interest rate,  $t$  is the number of years, and  $P$  is the principal of the loan.

## Understand and Compute Present Value for Simple Interest Investments

When finding the future value of an investment, we know how much is deposited, but we have no idea how much that money will be worth in the future. If we set a goal for the future, it would be useful to know how much to deposit now so an account reaches the goal. The amount that needs to be deposited now to hit a goal in the future is called the **present value**.

### FORMULA

The present value,  $PV$ , of money deposited at an annual, simple interest rate of  $r$  (in decimal form) for time  $t$  (in years) with a specified future value of  $FV$ , is calculated with the formula  $PV = \frac{FV}{(1 + rt)}$ .

Note: Present value, in this calculation, is always rounded up. Otherwise, future value may fall short of the target future value.

Understanding what this tells you is important. When you find the present value, that is how much you need to invest now to reach the goal  $FV$ , under the conditions (time and rate) at which the money will be invested.

### EXAMPLE 6.37

Compute the present value of the investment described. Interpret the result.

- $FV = \$10,000$ ,  $t = 15$  years, annual simple interest rate of 5.5%
- $FV = \$150,000$ ,  $t = 20$  years, annual simple interest rate of 6.25%
- $FV = \$250,000$ ,  $t = 486$  months, annual simple interest rate of 4.75%

✔ **Solution**

1. The future value is  $FV = \$10,000$ . The time of the investment is in years, so  $t = 15$ . The annual, simple interest rate is 5.5%, which in decimal form is 0.055. We substitute those values into the formula and calculate.

$$PV = \frac{FV}{(1 + rt)} = \frac{10,000}{(1 + (0.055) \times (15))} = \frac{10,000}{(1 + 0.825)} = \frac{10,000}{(1.825)} = 5,479.452.$$
 Rounding up, we see that the present value of \$10,000 invested at a simple annual interest rate of 5.5% for 15 years is \$5,479.46. This means that \$5,479.46 needs to be invested so that, after 15 years at 5.5% interest, the investment will be worth \$10,000.

2. The future value is  $FV = \$150,000$ . The time of the investment is in years, so  $t = 20$ . The annual, simple interest rate is 6.25%, which in decimal form is 0.0625. We substitute those values into the formula and calculate.

$$PV = \frac{FV}{(1 + rt)} = \frac{150,000}{(1 + (0.0625) \times (20))} = \frac{150,000}{(1 + 1.25)} = \frac{150,000}{(2.25)} = 66,666.\bar{6}.$$
 Rounding up, we see that the present value of \$150,000 invested at a simple annual interest rate of 6.25% for 20 years is \$66,666.67. This means that \$66,666.67 needs to be invested so that, after 20 years at 6.25% interest, the investment will be worth \$150,000.

3. The future value is  $FV = \$250,000$ . The time of the investment is 486 months. This needs to be converted to years. To do so, divide the number of months by 12, giving  $years = \frac{486}{12} = 40.5$ , so  $t = 40.5$  years. The annual, simple interest rate is 4.75%, which in decimal form is 0.0475. We substitute those values into the formula and calculate.

$$PV = \frac{FV}{(1 + rt)} = \frac{250,000}{(1 + (0.0475) \times (40.5))} = \frac{250,000}{(1 + 1.92375)} = \frac{250,000}{(2.92375)} = 129,954.5159.$$
 Rounding up, we see that the present value of \$250,000 invested at a simple annual interest rate of 4.75% for 486 months is \$129,954.52. This means that \$129,954.52 needs to be invested so that, after 486 months at 4.75% interest, the investment will be worth \$250,000.

> **YOUR TURN 6.37**

Compute the present value of the investment described. Interpret the result.

- $FV = \$25,000$ ,  $t = 10$  years, annual simple interest rate of 7.5%
- $FV = \$320,000$ ,  $t = 35$  years, annual simple interest rate of 6.5%
- $FV = \$90,000$ ,  $t = 270$  months, annual simple interest rate of 3.75%

**EXAMPLE 6.38**

**Present Value of a CD**

Beatriz will invest some money in a CD that yields 3.99% simple interest when invested for 30 years. How much must Beatriz invest so that after those 30 years, her CD is worth \$300,000?

✔ **Solution**

Beatriz needs to know how much to deposit now so that her CD is worth \$300,000 after 30 years. This means she needs to know the present value of that \$300,000. The time is 30 years and the annual simple interest rate, in decimal form, is 0.0399. Using that information and the formula for present value, we calculate the present value of that \$300,000.

$$PV = \frac{FV}{(1 + rt)} = \frac{300,000}{(1 + (0.0399) \times (30))} = \frac{300,000}{(1 + 1.197)} = \frac{300,000}{(2.197)} = 136,549.8407.$$
 Rounding up, Beatriz needs to invest \$136,549.85 so that she has \$300,000 in 30 years.

> **YOUR TURN 6.38**

- Kentaro will invest some money in a CD that yields 2.75% simple interest when invested for 5 years. How much must Kentaro invest so that after those 5 years his CD is worth \$12,000?

**Check Your Understanding**

- What is interest?
- What is the principal in a loan?
- Calculate the simple interest to be paid for a 6-year loan with principal \$1,500.00 and annual interest rate of 12.99%

16. A simple interest loan for \$24,200 is taken out at 10.55% annual percentage rate. A partial payment of \$13,000 is made 25 days into the loan period. After this payment, what will the remaining balance of the loan be?
17. Find the monthly payment for a \$9,800.00 loan at a 13.8% interest for 4 years.
18. Find the present value of an investment with future value \$30,000 with a simple interest rate of 3.75% invested for 10 years.



### SECTION 6.3 EXERCISES

1. If \$1,500.00 is invested in an account bearing 3.5% interest, what is the principal?
2. If \$1,500 is invested in an account bearing 3.5% interest, what is the interest rate?
3. What is simple interest?
4. What is the present value of an investment?
5. What is the future value of an investment?
6. What is a partial payment on a loan?

In the following exercises, calculate the simple interest and payoff for the loan with the given principal, simple interest rate, and time.

7. Principal  $P = \$5,000$ , annual interest rate  $r = 6.5\%$ , and number of years  $t = 6$
8. Principal  $P = \$3,500$ , annual interest rate  $r = 12\%$ , and number of years  $t = 7$
9. Principal  $P = \$7,800$ , annual interest rate  $r = 11.5\%$ , and number of years  $t = 10$
10. Principal  $P = \$62,500$ , annual interest rate  $r = 4.88\%$ , and number of years  $t = 4$
11. Principal  $P = \$4,600$ , annual interest rate  $r = 9.9\%$ , for 18 months
12. Principal  $P = \$19,000$ , annual interest rate  $r = 16.9\%$ , for 14 months
13. Principal  $P = \$8,500$ , annual interest rate  $r = 10.66\%$ , for 6 months
14. Principal  $P = \$17,600$ , annual interest rate  $r = 17.9\%$ , for 20 months
15. Principal  $P = \$4,000$ , annual interest rate  $r = 8.5\%$ , for 130 days
16. Principal  $P = \$9,900$ , annual interest rate  $r = 15.9\%$ , for 90 days
17. Principal  $P = \$600$ , annual interest rate  $r = 16.8\%$ , for 25 days
18. Principal  $P = \$890$ , annual interest rate  $r = 9.75\%$ , for 200 days

In the following exercises, find the future value of the investment with the given principal, simple interest rate, and time.

19. Principal is \$5,300, annual interest rate is 2.07%, and time is 18 years.
20. Principal is \$14,700, annual interest rate is 3.11%, and time is 10 years.
21. Principal is \$5,600, annual interest rate is 2.55%, for 30 months.
22. Principal is \$10,000, annual interest rate is 1.99%, for 15 months.
23. Principal is \$2,000, annual interest rate is 3.22%, for 100 days.
24. Principal is \$900, annual interest rate is 3.75%, for 175 days.

In the following exercises, determine the amount applied to principal for the indicated partial payment on the loan with the given principal, interest rate, and time when the partial payment was made.

25. A simple interest loan for \$2,700 is taken out at 11.6% annual percentage rate. A partial payment of \$1,500 is made 28 days into the loan period.
26. A simple interest loan for \$900 is taken out at 18.9% annual percentage rate. A partial payment of \$400 is made 30 days into the loan period.
27. A simple interest loan for \$13,500 is taken out at 14.8% annual percentage rate. A partial payment of \$8,000 is made 75 days into the loan period.
28. A simple interest loan for \$9,900 is taken out at 9.875% annual percentage rate. A partial payment of \$4,000 is made 65 days into the loan period.

In the following exercises, determine the remaining principal for the indicated partial payment on the loan with the given principal, interest rate, and time when the partial payment was made.

29. A simple interest loan for \$2,700 is taken out at 11.6% annual percentage rate. A partial payment of \$1,500 is made 28 days into the loan period.
30. A simple interest loan for \$900 is taken out at 18.9% annual percentage rate. A partial payment of \$400 is made 30 days into the loan period.

31. A simple interest loan for \$13,500 is taken out at 14.8% annual percentage rate. A partial payment of \$8,000 is made 75 days into the loan period.
32. A simple interest loan for \$9,900 is taken out at 9.875% annual percentage rate. A partial payment of \$4,000 is made 65 days into the loan period.

In the following exercises, find the payoff value of the loan with the given principal, annual simple interest rate, term, partial payment, and time at which the partial payment was made.

33. Principal = \$1,500, rate = 6.99%, term is 5 years, partial payment of \$900 made 2 years into the loan.
34. Principal = \$21,500, rate = 7.44%, term is 10 years, partial payment of 15,000 made after 6 years.
35. Principal = \$6,800, rate = 11.9%, term is 200 days, partial payment of \$4,000 made after 100 days.
36. Principal = \$800, rate = 13.99%, term is 150 days, partial payment of \$525 made after 50 days.

In the following exercises, find the monthly payment for a loan with the given principal, annual simple interest rate and number of years.

37. Principal = \$4,500, rate = 8.75%, years = 3
38. Principal = \$2,700, rate = 15.9%, years = 5
39. Principal = \$13,980, rate = 10.5%, years = 4
40. Principal = \$8,750, rate = 9.9%, years = 10

In the following exercises, find the present value for the given future value,  $FV$ , annual simple interest rate  $r$ , and number of years  $t$ .

41.  $FV = \$25,000$ ,  $t = 15$  years, annual simple interest rate of 6.5%
42.  $FV = \$12,000$ ,  $t = 10$  years, annual simple interest rate of 4.5%
43.  $FV = \$15,000$ ,  $t = 16$  years, annual simple interest rate of 3.5%
44.  $FV = \$100,000$ ,  $t = 30$  years, annual simple interest rate of 5.5%
45. Rita takes out a simple interest loan for \$4,000 for 5 years. Her interest rate is 7.88%. How much will Rita pay when the loan is due?
46. Humberto runs a private computer networking company, and needs a loan of \$31,500 for new equipment. He shops around for the lowest interest rate he can find. He finds a rate of 8.9% interest for a 10-year term. How much will Humberto's payoff be at the end of the 10 years?
47. Jaye needs a short-term loan of \$3,500. They find a 75-day loan that charges 14.9% interest. What is Jaye's payoff?
48. Theethat's car needs new struts, which cost \$1,189.50 installed, but he doesn't have the money to do so. He asks the repair shop if they offer any sort of financing. It offers him a short-term loan at 18.9% interest for 60 days. What is Theethat's payoff for the struts?
49. Michelle opens a gaming shop in her small town. She takes out an \$8,500 loan to get started. The loan is at 9.5% interest and has a term of 5 years. Michelle decides to make a partial payment of \$4,700 after 3 years. What will Michelle pay when the loan is due?
50. A small retailer borrows \$3,750 for a repair. The loan has a term of 100 days at 13.55% interest. If the retailer pays a partial payment of \$2,000 after 30 days, what will the loan payoff be when the loan is due?
51. Sharon invests \$2,500 in a CD for her granddaughter. The CD has a term of 5 years and has a simple interest rate of 3.11%. After that 5-year period, how much will the CD be worth?
52. Jen and Fred have a baby, and deposit \$1,500 in a savings account bearing 1.76% simple interest. How much will the account be worth in 18 years?
53. Yasmin decides to buy a used car. Her credit union offers 7.9% interest for 5-year loans on used cars. The cost of the car, including taxes and fees, is \$11,209.50. How much will Yasmin's monthly payment be?
54. Cleo runs her own silk-screening company. She needs new silk-screening printing machines, and finds two that will cost her, in total, \$5,489.00. She takes out a 3-year loan at 8.9% interest. What will her monthly payments be for the loan?
55. Kylie wants to invest some money in an account that yields 4.66% simple interest. Her goal is to have \$20,000 in 15 years. How much should Kylie invest to reach that goal?
56. Ishraq wants to deposit money in an account that yields 3.5% simple interest for 10 years, to help with a down payment for a home. Her goal is to have \$25,000 for the down payment. How much does Ishraq need to deposit to reach that goal?

In the following exercises, use the **Cost of Financing**. The difference between the total paid for a loan, along with all other charges paid to obtain the loan, and the original principal of the loan is the cost of financing. It measures how much more you paid for an item than the original price. In order to find the cost of financing, find the total paid over the life of the loan. Add to that any fees paid for the loan. Then subtract the principal.

Cost of Financing = Total of payments + fees – principal.

57. Yasmin decides to buy a used car. Her credit union offers 7.9% interest for 5-year loans on used cars. The cost of the car, including taxes and fees, is \$11,209.50. How much did she pay the credit union over the 5 years? What was the cost of financing for Yasmin?
58. Cleo runs her own silk-screening company. She needs new silk-screening printing machines, and finds two that will cost her, in total, \$5,489.00. She takes out a 3-year loan at 8.9% interest. What was the cost of financing for Cleo?

## 6.4 Compound Interest



**Figure 6.6** The impact of compound interest (credit: "English Money" by Images Money/Flickr, CC BY 2.0)

### Learning Objectives

After completing this section, you should be able to:

1. Compute compound interest.
2. Determine the difference in interest between simple and compound calculations.
3. Understand and compute future value.
4. Compute present value.
5. Compute and interpret effective annual yield.

For a very long time in certain parts of the world, interest was not charged due to religious dictates. Once this restriction was relaxed, loans that earned interest became possible. Initially, such loans had short terms, so only simple interest was applied to the loan. However, when loans began to stretch out for years, it was natural to add the interest at the end of each year, and add the interest to the principal of the loan. After another year, the interest was calculated on the initial principal plus the interest from year 1, or, the interest earned interest. Each year, more interest was added to the money owed, and that interest continued to earn interest.

Since the amount in the account grows each year, more money earns interest, increasing the account faster. This growth follows a geometric series ([Geometric Sequences](#)). It is this feature that gives compound interest its power. This module covers the mathematics of compound interest.

### Understand and Compute Compound Interest

As we saw in [Simple Interest](#), an account that pays simple interest only pays based on the original principal and the term of the loan. Accounts offering **compound interest** pay interest at regular intervals. After each interval, the interest is added to the original principal. Later, interest is calculated on the original principal plus the interest that has been added previously.

After each period, the interest on the account is computed, then added to the account. Then, after the next period, when interest is computed, it is computed based on the original principal AND the interest that was added in the previous periods.

The following example illustrates how compounded interest works.

### EXAMPLE 6.39

#### Interest Compounded Annually

Abena invests \$1,000 in a CD (certificate of deposit) earning 4% compounded annually. How much will Abena's CD be worth after 3 years?

#### ✓ Solution

Since the interest is compounded annually, the interest will be computed at the end of each year and added to the CD's value. The interest at the end of the following year will be based on the value found from the previous year.

**Step 1:** After the first year, the interest in Abena's CD is computed using the interest formula  $I = P \times r \times t$ . The principal is  $P = 1,000$ , the rate, as a decimal, is 0.04, and the time is one year, so  $t = 1$ . Using that, the interest earned in the first year is  $I = P \times r \times t = 1,000 \times 0.04 \times 1 = 40$ , so the interest earned in the first year was \$40.00. This is added to the value of the CD, making the CD worth  $\$1,000 + \$40 = \$1,040$ .

**Step 2:** At the end of the second year, interest is again computed, but is computed based on the CD's new value, \$1,040. Using this new value and the interest formula ( $r$  and  $t$  are still 0.04 and 1, respectively), we see that the CD earned  $I = P \times r \times t = 1,040 \times 0.04 \times 1 = 41.6$ , or \$41.60. This is added to the value of the CD, making the CD now worth  $\$1,040.00 + \$41.60 = \$1,081.60$ .

**Step 3:** At the end of the third year, interest is again computed, but is computed based on Abena's CD's new value, \$1,081.60. Using this value and the interest formula ( $r$  and  $t$  are still 0.04 and 1, respectively), we see that the CD earned  $I = P \times r \times t = 1,081.60 \times 0.04 \times 1 = 43.264$ , or \$43.26 (remember to round down). This is added to the value of the CD, making the CD now worth  $\$1,081.60 + \$43.26 = \$1,124.86$ .

After 3 years, Abena's CD is worth \$1,124.86.

#### > YOUR TURN 6.39

1. Oksana deposits \$5,000 in a CD that earns 3% compounded annually. How much is the CD worth after 4 years?

## Determine the Difference in Interest Between Simple and Compound Calculations

It is natural to ask, does compound interest make much of a difference? To find out, we revisit Abena's CD.

### EXAMPLE 6.40

#### Comparing Simple to Compound Interest on a 3-Year CD

Abena invested \$1,000 in a CD that earned 4% compounded annually, and the CD was worth \$1,124.86 after 3 years. Had Abena invested in a CD with simple interest, how much would the CD have been worth after 3 years? How much more did Abena earn using compound interest?

#### ✓ Solution

Had Abena invested \$1,000 in a 4% simple interest CD for 3 years, her CD would have been worth  $P + P \times r \times t = 1,000 + 1,000 \times 0.04 \times 3 = 1,120$ , or \$1,120.00. With interest compounded annually, Abena's CD was worth \$1,124.86. The difference between compound and simple interest is  $\$1,124.86 - \$1,120.00 = \$4.86$ . So compound interest earned Abena \$4.86 more than the simple interest did.

#### > YOUR TURN 6.40

1. Oksana deposits \$5,000 in a CD that earned 3% compounded annually and was worth \$5,627.54 after 4 years. Had Oksana invested in a CD with simple interest, how much would the CD have been worth after 4 years? How much more did Oksana earn using compound interest?

 VIDEO


Compound Interest ([https://openstax.org/r/compound\\_interest\\_beginners](https://openstax.org/r/compound_interest_beginners))

## Understand and Compute Future Value

Imagine investing for 30 years and compounding the interest every month. Using the method above, there would be 360 periods to calculate interest for. This is not a reasonable approach. Fortunately, there is a formula for finding the future value of an investment that earns compound interest.

### FORMULA

The future value of an investment,  $A$ , when the principal  $P$  is invested at an annual interest rate of  $r$  (in decimal form), compounded  $n$  times per year, for  $t$  years, is found using the formula  $A = P\left(1 + \frac{r}{n}\right)^{nt}$ . This is also referred to as the future value of the investment.

 Note, sometimes the formula is presented with the total number of periods,  $n$ , and the interest rate per period,  $r$ . In that case the formula becomes  $A = P(1 + r)^n$ .

### EXAMPLE 6.41

#### Computing Future Value for Compound Interest

In the following, compute the future value of the investment with the given conditions.

- Principal is \$5,000, annual interest rate is 3.8%, compounded monthly, for 5 years.
- Principal is \$18,500, annual interest rate is 6.25%, compounded quarterly, for 17 years.

#### Solution

- The principal is  $P = \$5,000$ , interest rate, in decimal form,  $r = 0.038$ , compounded monthly so  $n = 12$ , and for  $t = 5$  years. Substituting these values into the formula, we find

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} = 5,000\left(1 + \frac{0.038}{12}\right)^{12 \times 5} \\ &= 5,000(1 + 0.0031\bar{6})^{60} \\ &= 5,000(1.0031\bar{6})^{40} \\ &= 5,000 \times 1.20888663572 \\ &= 6,044.4332 \end{aligned}$$

The future value of the investment is \$6,044.43.

- The principal is  $P = \$18,500$ , interest rate, in decimal form,  $r = 0.0625$ , compounded quarterly so  $n = 4$ , and for  $t = 17$  years. Substituting these values into the formula, we

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} = 18,500\left(1 + \frac{0.0625}{4}\right)^{4 \times 17} \\ &= 18,500(1 + 0.015625)^{68} \\ &= 18,500(1.015625)^{68} \\ &= 18,500 \times 2.86992151999 \\ &= 53,093.5481 \end{aligned}$$

The future value of the investment is \$53,093.54.

### YOUR TURN 6.41

In the following, compute the future value of the investment with the given conditions.

- Principal is \$7,600, annual interest rate is 4.1%, compounded monthly, for 10 years.

2. Principal is \$13,250, annual interest rate is 2.79%, compounded quarterly, for 25 years.

**EXAMPLE 6.42****Interest Compounded Quarterly**

Cody invests \$7,500 in an account that earns 4.5% interest compounded quarterly (4 times per year). Determine the value of Cody's investment after 10 years.

**✓ Solution**

Cody's initial investment is \$7,500, so  $P = \$7,500$ . The annual interest rate is 4.5%, which is 0.045 in decimal form. Compounding quarterly means there are four periods in a year, so  $n = 4$ . He invests the money for 10 years. Substituting those values into the formula, we calculate

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} = 7,500\left(1 + \frac{0.045}{4}\right)^{4 \times 10} \\ &= 7,500(1 + 0.01125)^{40} \\ &= 7,500(1.01125)^{40} \\ &= 7,500 \times 1.564376865391 \\ &= 11,732.8265 \end{aligned}$$

After 10 years, Cody's initial investment of \$7,500 is worth \$11,732.82.

**> YOUR TURN 6.42**

1. Maggie invests \$3,000 in an account that earns 5.1% interest compounded monthly. How much is the account worth after 13 years?

**EXAMPLE 6.43****Interest Compounded Daily**

Kathy invests \$10,000 in an account that yields 5.6% compounded daily. How much money will be in her account after 20 years?

**✓ Solution**

Kathy's initial investment is \$10,000, so  $P = \$10,000$ . The annual interest rate is 5.6%, which is 0.056 in decimal form. Compounding daily means there are 365 periods in a year, so  $n = 365$ . She invests the money for 20 years, so  $t = 20$ . Substituting those values into the formula, we calculate

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} = 10,000\left(1 + \frac{0.056}{365}\right)^{365 \times 20} \\ &= 10,000(1 + 0.000153424657534)^{7300} \\ &= 10,000(1.000153424657534)^{7300} \\ &= 10,000 \times 3.06459091598 \\ &= 30,645.909 \end{aligned}$$

After 20 years, Kathy's initial investment of \$10,000 is worth \$30,645.90.

**> YOUR TURN 6.43**

1. Jacob invests \$3,000 in a CD that yields 3.4% compounded daily for 5 years. How much is his CD worth after 5 years?

 VIDEO

[Compare Simple Interest to Interest Compounded Annually \(https://openstax.org/r/compare\\_simple\\_compound\\_interest1\)](https://openstax.org/r/compare_simple_compound_interest1)

[Compare Simple Interest and Compound Interest for Different Number of Periods Per Year \(https://openstax.org/r/compare\\_simple\\_compound\\_interest2\)](https://openstax.org/r/compare_simple_compound_interest2)

### WORK IT OUT

To truly grasp how compound interest works over a long period of time, create a table comparing simple interest to compound interest, with different numbers of periods per year, for many years would be useful. In this situation, the principal is \$10,000, and the annual interest rate is 6%.

1. Create a table with five columns. Label the first column YEARS, the second column SIMPLE INTEREST, the third column COMPOUND ANNUALLY, the fourth column COMPOUND MONTHLY and the last column COMPOUND DAILY, as shown below.

YEARS	SIMPLE INTEREST	COMPOUND ANNUALLY	COMPOUND MONTHLY	COMPOUND DAILY

2. In the years column, enter 1, 2, 3, 5, 10, 20, and 30 for the rows.
3. Calculate the account value for each column and each year.
4. Compare the results from each of the values you find. How do the number of periods per year (compounding per year) impact the account value? How does the number of years impact the account value?
5. Redo the chart, with an interest rate you choose and a principal you choose. Are the patterns identified earlier still present?

## Understand and Compute Present Value

When investing, there is often a goal to reach, such as “after 20 years, I’d like the account to be worth \$100,000.” The question to be answered in this case is “How much money must be invested now to reach the goal?” As with simple interest, this is referred to as the present value.

### FORMULA

The money invested in an account bearing an annual interest rate of  $r$  (in decimal form), compounded  $n$  times per year for  $t$  years, is called the present value,  $PV$ , of the account (or of the money) and found using the formula  $PV = \frac{A}{(1 + \frac{r}{n})^{n \times t}}$ , where  $A$  is the value of the account at the investment’s end. Always round this value up to the nearest penny.

### EXAMPLE 6.44

#### Computing Present Value

Find the present value of the accounts under the following conditions.

1.  $A = \$250,000$ , invested at 6.75 interest, compounded monthly, for 30 years.
2.  $A = \$500,000$ , invested at 7.1% interest, compounded quarterly, for 40 years.

#### Solution

1. To reach a final account value of  $A = \$250,000$ , invested at 6.75% interest, in decimal form  $r = 0.0675$  (decimal form!), compounded monthly, so  $n = 12$ , for 30 years, substitute those values into the formula for present value.

Calculating, we find the present value of the \$250,000.

$$\begin{aligned}
 PV &= \frac{A}{\left(1 + \frac{r}{n}\right)^{n \times t}} = \frac{250,000}{\left(1 + \frac{0.0675}{12}\right)^{12 \times 30}} \\
 &= \frac{250,000}{(1 + 0.005625)^{360}} \\
 &= \frac{250,000}{(1.005625)^{360}} \\
 &= \frac{250,000}{7.5332454772} \\
 &= 33,186.2277
 \end{aligned}$$

In order for this account to reach \$250,000 after 30 years, \$33,186.23 needs to be invested.

2. To reach a final account value of  $A = \$500,000$ , invested at 7.1% interest, in decimal form  $r = 0.071$ , compounded quarterly, so  $n = 4$ , for 40 years, substitute those values into the formula for present value. Calculating, we find the present value of the \$500,000.

$$\begin{aligned}
 PV &= \frac{A}{\left(1 + \frac{r}{n}\right)^{n \times t}} = \frac{500,000}{\left(1 + \frac{0.071}{4}\right)^{4 \times 40}} \\
 &= \frac{500,000}{(1 + 0.01775)^{160}} \\
 &= \frac{500,000}{(1.01775)^{160}} \\
 &= \frac{500,000}{16.6946672846} \\
 &= 29,949.6834
 \end{aligned}$$

In order for this account to reach \$500,000 after 40 years, \$29,949.69 needs to be invested.

#### YOUR TURN 6.44

Find the present value of the accounts under the following conditions.

1.  $A = \$1,000,000$ , invested at 5.75% interest, compounded monthly, for 40 years.
2.  $A = \$175,000$ , invested at 3.8% interest, compounded quarterly, for 20 years.

#### EXAMPLE 6.45

##### Investment Goal with Compound Interest

Pilar plans early for retirement, believing she will need \$1,500,000 to live comfortably after the age of 67. How much will she need to deposit at age 23 in an account bearing 6.35% annual interest compounded monthly?

##### Solution

Knowing how much to deposit at age 23 to reach a certain value later is a present value question. The target value for Pilar is \$1,500,000. The interest rate is 6.35%, which in decimal form is 0.0635. Compounded monthly means  $n = 12$ . She's 23 and will leave the money in the account until the age of 67, which is 44 years, making  $t = 44$ . Using this information and substituting in the formula for present value, we calculate

$$\begin{aligned}
 PV &= \frac{A}{\left(1 + \frac{r}{n}\right)^{n \times t}} = \frac{1,500,000}{\left(1 + \frac{0.0635}{12}\right)^{12 \times 44}} \\
 &= \frac{1,500,000}{\left(1 + 0.005291\bar{6}\right)^{528}} \\
 &= \frac{1,500,000}{\left(1.005291\bar{6}\right)^{528}} \\
 &= \frac{1,500,000}{16.226302189} \\
 &= 92,442.5037
 \end{aligned}$$

Pilar will need to invest \$92,442.51 in this account to have \$1,500,000 at age 67.

### > YOUR TURN 6.45

- Hajun turns 30 this year and begins to think about retirement. He calculates that he will need \$1,200,000 to retire comfortably. He finds a fund to invest in that yields 7.23% and is compounded monthly. How much will Hajun need to invest in the fund when he turns 30 so that he can reach his goal when he retires at age 65?

## Compute and Interpret Effective Annual Yield

As we've seen, quarterly compounding pays interest 4 times a year or every 3 months; monthly compounding pays 12 times a year; daily compounding pays interest every day, and so on. **Effective annual yield** allows direct comparisons between simple interest and compound interest by converting compound interest to its equivalent simple interest rate. We can even directly compare different compound interest situations. This gives information that can be used to identify the best investment from a yield perspective.

Using a formula, we can interpret compound interest as simple interest. The effective annual yield formula stems from the compound interest formula and is based on an investment of \$1 for 1 year.

Effective annual yield is  $Y = \left(1 + \frac{r}{n}\right)^n - 1$  where  $Y$  = effective annual yield,  $r$  = interest rate in decimal form, and  $n$  = number of times the interest is compounded in a year.  $Y$  is interpreted as the equivalent annual simple interest rate.

### EXAMPLE 6.46

#### Determine and Interpret Effective Annual Yield for 6% Compounded Quarterly

Suppose you have an investment paying a rate of 6% compounded quarterly. Determine and interpret that effective annual yield of the investment.

#### ✓ Solution

Here,  $n = 4$  (quarterly) and  $r = 0.06$  (decimal form). Substituting into the formula we find that the effective annual yield is

$$\begin{aligned}
 Y &= \left(1 + \frac{0.06}{4}\right)^4 - 1 \\
 &= (1.015)^4 - 1 \\
 &= 1.06136 - 1 \\
 &= 0.0614 \\
 &= 6.14\%
 \end{aligned}$$

Therefore, a rate of 6% compounded quarterly is equivalent to a simple interest rate of 6.14%.

### > YOUR TURN 6.46

- Calculate and interpret the effective annual yield for an investment that pays at a 7% interest compounded quarterly.

**EXAMPLE 6.47****Determine and Interpret Effective Annual Yield for 5% Compounded Daily**

Calculate and interpret the effective annual yield on a deposit earning interest at a rate of 5% compounded daily.

**✓ Solution**

In this case, the rate is  $r = 0.05$  and  $n = 365$  (daily). Using the formula  $Y = \left(1 + \frac{r}{n}\right)^n - 1$ , we have

$$\begin{aligned} Y &= \left(1 + \frac{0.05}{365}\right)^{365} - 1 \\ &= (1.0001369863)^{365} - 1 \\ &= 1.051267 - 1 \\ &= 0.0513 \end{aligned}$$

This tells us that an account earning 5% compounded daily is equivalent to earning 5.13% as simple interest.

**> YOUR TURN 6.47**

1. Calculate and interpret the effective annual yield on a deposit earning 2.5% compounded daily.

**EXAMPLE 6.48****Choosing a Bank**

Minh has a choice of banks in which he will open a savings account. He will deposit \$3,200 and he wants to get the best interest he can. The banks advertise as follows:

Bank	Interest Rate
ABC Bank	2.08% compounded monthly
123 Bank	2.09% compounded annually
XYZ Bank	2.05% compounded daily

Which bank offers the best interest?

**✓ Solution**

To compare these directly, Minh could change each interest rate to its effective annual yield, which would allow direct comparison between the rates. Computing the effective annual yield for all three choices gives:

$$\text{ABC Bank: } Y = \left(1 + \frac{0.0208}{12}\right)^{12} - 1 = 0.0210 = 2.10\%$$

$$\text{123 Bank: } Y = \left(1 + \frac{0.0209}{1}\right)^1 - 1 = 0.0209 = 2.09\%$$

$$\text{XYZ Bank: } Y = \left(1 + \frac{0.0205}{365}\right)^{365} - 1 = 0.0207 = 2.07\%$$

ABC Bank has the highest effective annual yield, so Minh should choose ABC bank.

**> YOUR TURN 6.48**

1. Isabella decides to deposit \$5,500 in a CD but needs to choose between banks that offer CDs. She identifies four banks and finds out the terms of their CDs. Her findings are in the table below.

Bank	Interest Rate
Smith Bank	3.08% compounded quarterly
Park Bank	3.11% compounded annually
Town Bank	3.09% compounded daily
Community Bank	3.10% compounded monthly

Which bank has the best yield?

## Check Your Understanding

19. What is compound interest?
20. Which yields more money, simple interest or compound interest?
21. Find the future value after 15 years of \$8,560.00 deposited in an account bearing 4.05% interest compounded monthly.
22. \$10,000 is deposited in an account bearing 5.6% interest for 5 years. Find the difference between the future value when the interest is simple interest and when the interest is compounded quarterly.
23. Find the present value of \$75,000 after 28 years if money is invested in an account bearing 3.25% interest compounded monthly.
24. What can be done to compare accounts if the rates and number of compound periods per year are different?
25. Find the effective annual yield of an account with 4.89% interest compounded quarterly.



## SECTION 6.4 EXERCISES

1. What is the difference between simple interest and compound interest?
2. What is a direct way to compare accounts with different interest rates and number of compounding periods?
3. Which type of account grows in value faster, one with simple interest or one with compound interest?

How many periods are there if interest is compounded?

4. Daily
5. Weekly
6. Monthly
7. Quarterly
8. Semi-annually

In the following exercises, compute the future value of the investment with the given conditions.

9. Principal = \$15,000, annual interest rate = 4.25%, compounded annually, for 5 years
10. Principal = \$27,500, annual interest rate = 3.75%, compounded annually, for 10 years
11. Principal = \$13,800, annual interest rate = 2.55%, compounded quarterly, for 18 years
12. Principal = \$150,000, annual interest rate = 2.95%, compounded quarterly, for 30 years
13. Principal = \$3,500, annual interest rate = 2.9%, compounded monthly, for 7 years
14. Principal = \$1,500, annual interest rate = 3.23%, compounded monthly, for 30 years
15. Principal = \$16,000, annual interest rate = 3.64%, compounded daily, for 13 years
16. Principal = \$9,450, annual interest rate = 3.99%, compounded daily, for 25 years

In the following exercises, compute the present value of the accounts with the given conditions.

17. Future value = \$250,000, annual interest rate = 3.45%, compounded annually, for 25 years
18. Future value = \$300,000, annual interest rate = 3.99%, compounded annually, for 15 years
19. Future value = \$1,500,000, annual interest rate = 4.81%, compounded quarterly, for 35 years

- 20. Future value = \$750,000, annual interest rate = 3.95%, compounded quarterly, for 10 years
- 21. Future value = \$600,000, annual interest rate = 3.79%, compounded monthly, for 17 years
- 22. Future value = \$800,000, annual interest rate = 4.23%, compounded monthly, for 35 years
- 23. Future value = \$890,000, annual interest rate = 2.77%, compounded daily, for 25 years
- 24. Future value = \$345,000, annual interest rate = 2.99%, compounded daily, for 19 years

In the following exercises, compute the effective annual yield for accounts with the given interest rate and number of compounding periods. Round to three decimal places.

- 25. Annual interest rate = 2.75%, compounded monthly
- 26. Annual interest rate = 3.44%, compounded monthly
- 27. Annual interest rate = 5.18%, compounded quarterly
- 28. Annual interest rate = 2.56%, compounded quarterly
- 29. Annual interest rate = 4.11%, compounded daily
- 30. Annual interest rate = 6.5%, compounded daily

The following exercises explore what happens when a person deposits money in an account earning compound interest.

- 31. Find the present value of \$500,000 in an account that earns 3.85% compounded quarterly for the indicated number of years.
  - a. 40 years
  - b. 35 years
  - c. 30 years
  - d. 25 years
  - e. 20 years
  - f. 15 years
- 32. Find the present value of \$1,000,000 in an account that earns 6.15% compounded monthly for the indicated number of years.
  - a. 40 years
  - b. 35 years
  - c. 30 years
  - d. 25 years
  - e. 20 years
  - f. 15 years
- 33. In the following exercises, the number of years can reflect delaying depositing money. 40 years would be depositing money at the start of a 40-year career. 35 years would be waiting 5 years before depositing the money. Thirty years would be waiting 10 years before depositing the money, and so on. What do you notice happens if you delay depositing money?
- 34. For each 5-year gap for exercise 32, compute the difference between the present values. Do these differences remain the same for each of the 5-year gaps, or do they differ? How do they differ? What conclusion can you draw?
- 35. Daria invests \$2,500 in a CD that yields 3.5% compounded quarterly for 5 years. How much is the CD worth after those 5 years?
- 36. Maurice deposits \$4,200 in a CD that yields 3.8% compounded annually for 3 years. How much is the CD worth after those 3 years?
- 37. Georgita is shopping for an account to invest her money in. She wants the account to grow to \$400,000 in 30 years. She finds an account that earns 4.75% compounded monthly. How much does she need to deposit to reach her goal?
- 38. Zak wants to create a nest egg for himself. He wants the account to be valued at \$600,000 in 25 years. He finds an account that earns 4.05% interest compounded quarterly. How much does Zak need to deposit in the account to reach his goal of \$600,000?
- 39. Eli wants to compare two accounts for their money. They find one account that earns 4.26% interest compounded monthly. They find another account that earns 4.31% interest compounded quarterly. Which account will grow to Eli's goal the fastest?
- 40. Heath is planning to retire in 40 years. He'd like his account to be worth \$250,000 when he does retire. He wants to deposit money now. How much does he need to deposit in an account yielding 5.71% interest compounded semi-

annually to reach his goal?

41. Jo and Kim want to set aside some money for a down payment on a new car. They have 6 years to let the money grow. If they want to make a \$15,000 down payment on the car, how much should they deposit now in an account that earns 4.36% interest compounded monthly?
42. A newspaper's business section runs an article about savings at various banks in the city. They find six that offer accounts that offer compound interest.  
Bank A offers 3.76% compounded daily.  
Bank B offers 3.85% compounded annually.  
Bank C offers 3.77% compounded weekly.  
Bank D offers 3.74% compounded daily.  
Bank E offers 3.81% compounded semi-annually.  
To earn the most interest on a deposit, which bank should a person choose?
43. Paola reads the newspaper article from exercise 32. She really wants to know how different they are in terms of dollars, not effective annual yield. She decides to compute the future value for accounts at each bank based on a principal of \$100,000 that are allowed to grow for 20 years. What is the difference in the future values of the account with the highest effective annual yield, and the account with the second highest effective annual yield?
44. Paola reads the newspaper article from exercise 32. She really wants to know how different they are in terms of dollars, not effective annual yield. She decides to compute the future value for accounts at each bank based on a principal of \$100,000 that are allowed to grow for 20 years. What is the difference in the future values of the account with the highest effective annual yield, and the account with the lowest effective annual yield?
45. Jesse and Lila need to decide if they want to deposit money this year. If they do, they can deposit \$17,400 and allow the money to grow for 35 years. However, they could wait 12 years before making the deposit. At that time, they'd be able to collect \$31,700 but the money would only grow for 23 years. Their account earns 4.63% interest compounded monthly. Which plan will result in the most money, depositing \$17,400 now or depositing \$31,700 in 12 years?
46. Veronica and Jose are debating if they should deposit \$15,000 now in an account or if they should wait 10 years and deposit \$25,000. If they deposit money now, the money will grow for 35 years. If they wait 10 years, it will grow for 25 years. Their account earns 5.25% interest compounded weekly. Which plan will result in the most money, depositing \$15,000 now or depositing \$25,000 in 10 years?

## 6.5 Making a Personal Budget



**Figure 6.7** Calculating a budget is important to your financial health. (credit: "Budget planning concept on white desk" by Marco Verch Professional Photographer/Flickr, CC BY 2.0)

## Learning Objectives

After completing this section, you should be able to:

1. Create a personal budget with the categories of expenses and income.
2. Apply general guidelines for a budget.

“That doesn’t fit in the budget.”

“We didn’t budget for that.”

“We need to figure out our budget and stick to it.”

A **budget** is an outline of how money and resources should be spent. Companies have them, individuals have them, your college has one. But do you have one?

Creating a realistic budget is an important step in careful stewardship of your financial health. Designing your budget will help understand the financial priorities you have, and the constraints on your life choices. You want to have enough income to pay not only for the necessities, but also for things that represent your wants, like trips or dinner out. You also may want to save money for large purchases or retirement. You do not want to just get by, and you do not want the problems associated with overdue balances, rising debt, and possibly losing something you have worked hard to obtain.


While creating a budget may seem intimidating at first, coming up with your basic budget outline is the hardest part. Over time, you will adjust not only the numbers, but the categories.

## Creating a Budget


You should view creating a **budget** as a financial tool that will help you achieve your long-term goals. A budget is an estimation of income and expenses over some period of time. You will be able to track your progress, which will help you to prepare for the future by making smart investment decisions.

There are several budget-creating tools available, such as the apps Good budget and Mint, and Google Sheets. Getting started, though, begins well before you find an app. The following are steps that can be used to create your monthly budget.

1. **Track your income and expenses** Review your income and expenses for the past 6 months to a year. This will give you an idea of your current habits.
2. **Set your income baseline** Determine all the sources of income you will have. This income may come from paychecks, investments, or freelance work. It even includes child support and gifts. Be sure to use income after taxes. This allows you to determine your maximum expenditures per month.

 *For income that is not steady, such as gig work or freelance work, use the previous 6 to 12 months of income to find an average income from that gig or freelance work. Use this average in the budgeting process.*

3. **Determine your expenses** Review your bills from the past 6 months. You should include mortgage payments or rent, insurance, car payments, utilities, groceries, transportation expenses, personal care, entertainment, and savings. Using your credit card statements and bank statements will help you determine these amounts. Be aware that some of the expenses will not change over time. These are referred to as **fixed expenses**, like rent, car payments, insurance, internet service, and the like. Other expenses may vary widely from month to month and are appropriately called **variable expenses**, and include such expenses as gasoline, groceries.

 *Some expenses are yearly, such as insurance or property taxes. Other expenses may be quarterly (four times per year) or semiannual (twice per year). To budget for such bills by month, divide the bill total by the number of months the bill covers.*

4. **Categorize your expenses** These categories may be housing, transportation, or food, for broad categories, or may get more specific, where you categorize car payments, car insurance, and gasoline separately. The categories are your choices. Be sure to account for the cost of maintaining a vehicle or home. The more specific you are, the better you’ll understand your spending needs and habits.
5. **Total your monthly income and monthly expenses and compare** These values should be compared. If your expenses are higher than your income, then adjustments have to be made. Decisions of what to do with any extra income is part of the planning process also.
6. **Make plans for unplanned expenses** Ask anyone, an unexpected car repair can ruin a carefully crafted budget. Have a plan for how you can be ready for these random expenses. This often means creating a cushion in your