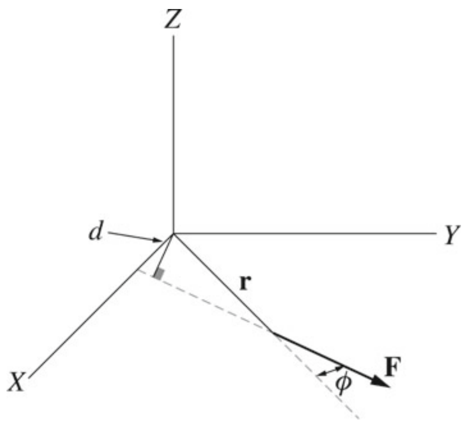


Fig. 5.12 $d = r \sin\phi$ is called the moment arm of F and it represents the perpendicular distance from the axis of rotation to the line of action of F

$$\boldsymbol{\tau} = (2l\mathbf{i} + 25\mathbf{j} - 2\mathbf{k})\ \mathrm{N/m}$$

5.6 Angular Momentum

The angular momentum **L** of a particle of mass m and linear momentum **p** is a vector quantity defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

where r is the position vector of the particle relative to an origin O that is in an inertial frame. Therefore, as $\boldsymbol{\tau}$, **L** also depends on the choice of the origin. Suppose the particle moves in the x–y plane (see Fig. 5.13). The direction of L is then perpendicular to the plane containing r and p and its sense is found by the right-hand rule. The magnitude of **L** is given by

$$L = mvr \sin\phi$$

where ϕ is the smaller angle between **r** and **p**. This quantity is the rotational analog of linear momentum in translational motion. If $\phi = 0$ or 180° the particle will move along a line passing through O and its angular momentum is zero. The SI unit of angular momentum is kg.m^2/s. In terms of rectangular components, we have

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times (p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k})$$

$$= (yp_z - zp_y)\mathbf{i} + (zp_x - xp_z)\mathbf{j} + (xp_y - yp_x)\mathbf{k}$$

5.6.1 Newton's Second Law in Angular Form

From the definition of torque, we have

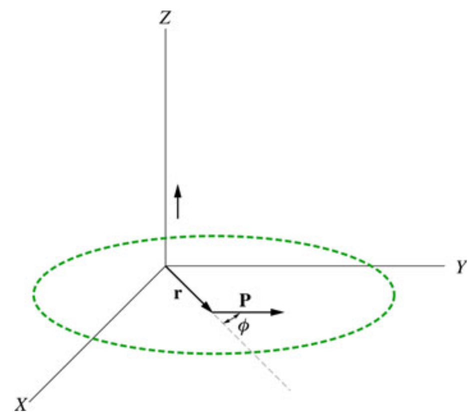


Fig. 5.13 If the particle is moving in the x-y plane, then the direction of L is perpendicular to the plane containing r and p and is found by the right-hand rule

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d(m\mathbf{v})}{dt}$$

$$\frac{d\mathbf{L}}{dt} = \frac{d(\mathbf{r} \times m\mathbf{v})}{dt} = \frac{d\mathbf{r}}{dt} \times (m\mathbf{v}) + \mathbf{r} \times \frac{d(m\mathbf{v})}{dt}$$

$$= \mathbf{v} \times (m\mathbf{v}) + \mathbf{r} \times \frac{d(m\mathbf{v})}{dt} = \mathbf{0} + \mathbf{r} \times \mathbf{F} = \boldsymbol{\tau}$$

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} \tag{5.12}$$

This implies that the torque acting on a particle is equal to the time rate of change of the angular momentum for that particle. This equation is valid only if $\boldsymbol{\tau}$ and **L** are evaluated with respect to the same origin or any other fixed point in an inertial frame. If several forces act on the particle, Eq. 5.12 can be written as

$$\Sigma\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$

where $\Sigma\boldsymbol{\tau}$ is the net torque on the particle. This is the rotational analog of Newton's second law in linear form, which states that the net force acting on a particle is equal to the time rate of change of its linear momentum. In component form, we have $\Sigma\tau_x = dL_x/dt$, $\Sigma\tau_y = dL_y/dt$ and $\Sigma\tau_z = dL_z/dt$.

5.6.2 Conservation of Angular Momentum

The total angular momentum of a particle is constant if the net external torque acting on it is zero:

$$\Sigma\boldsymbol{\tau}_{ext} = \frac{d\mathbf{L}}{dt} = \mathbf{0}$$

$$\mathbf{L} = \text{constant}$$

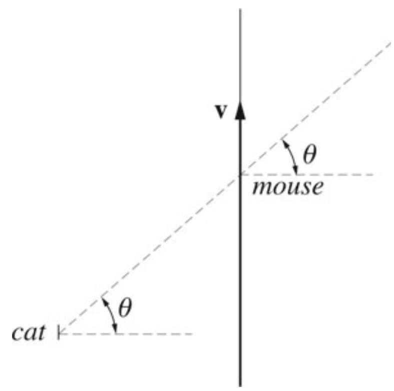


Fig. 5.14 A cat watching a mouse run by

$$m(\mathbf{r} \times \mathbf{v}) = \text{contant}$$

or

$$\mathbf{L}_i = \mathbf{L}_f$$

The law of conservation of angular momentum is a fundamental law of physics and it holds in relativity and quantum mechanics. Thus, for an isolated system, the linear momentum and angular momentum are conserved.

Example 5.15 A cat watches a mouse of mass m run by, as shown in Fig. 5.14. Determine the mouse's angular momentum relative to the cat as a function of time if the mouse has a constant acceleration a and if it starts from rest.

Solution 5.15 Suppose the plane is the x–y plane. Since $v = at$, we have

$$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}) = mrat\cos\theta\mathbf{k}$$

Example 5.16 A 0.2 kg particle is moving in the x–y plane. If at a certain instant $r = 3$ m and $v = 10$ m/s (see Fig. 5.15), find the magnitude and direction of the angular momentum of the particle at that instant relative to the origin.

Solution 5.16

$$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}) = -(mvr\sin\phi)\mathbf{k} = -(0.2\,\text{kg})(10\,\text{m/s})(3\,\text{m})\sin 60°\mathbf{k} = (-5.2\,\mathbf{k})\text{kg.m}^2/\text{s}$$

Example 5.17 A particle is moving under the influence of a force given by $\mathbf{F} = -k\mathbf{r}$. Prove that the angular momentum of the particle is conserved.

Solution 5.17

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = -k(\mathbf{r} \times \mathbf{r}) = \mathbf{0}$$

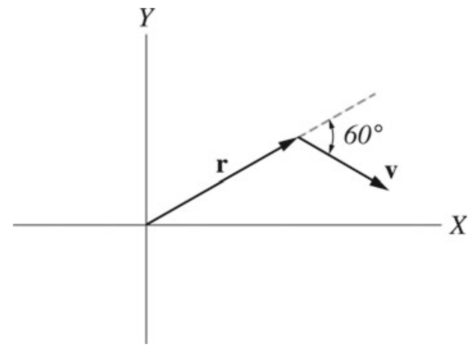


Fig. 5.15 A particle moving in the x-y plane

Since $\boldsymbol{\tau} = d\mathbf{L}/dt$, it follows that the total angular momentum of the particle is conserved. That is,

$$\mathbf{L} = \text{constant}$$

Example 5.18 A particle is moving in a circle where its position as a function of time is given by the expression $\mathbf{r} = a(\cos\omega t\mathbf{i} + \sin\omega t\mathbf{j})$, where ω is a constant. Show that the total angular momentum of the particle is constant.

Solution 5.18

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = a(-\omega\sin\omega t\mathbf{i} + \omega\cos\omega t\mathbf{j})$$

$$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}) = ma^2[(\cos\omega t\mathbf{i} + \sin\omega t\mathbf{j}) \times (-\omega\sin\omega t\mathbf{i} + \omega\cos\omega t\mathbf{j})]$$

$$= ma^2(\omega\cos^2\omega t\mathbf{k} + \omega\sin^2\omega t\mathbf{k})$$

$$= m\omega a^2\mathbf{k} = \text{constant}$$

Problems

1. A tennis ball of mass of 0.06 kg is initially traveling at an angle of 47° to the horizontal at a speed of 45 m/s. It then was shot by the tennis player and return horizontally at a speed of 35 m/s. Find the impulse delivered to the ball.
2. A force on a 0.5 kg particle varies with time according to Fig. 5.16. Find (a) The impulse delivered to the particle, (b) the average force exerted on the particle from $t = 0$ to $t = 6$ s(c). The final velocity of the particle if its initial velocity is 2 m/s.
3. A 1 kg particle moves in a force field given by $\mathbf{F} = (2t^2\mathbf{i} + (5t - 3)\mathbf{j} - 6t\mathbf{k})$ N. Find the impulse delivered to the particle during the time interval from $t = 1$ s to $t = 3$ s.
4. A boy of mass 45 kg runs and jump with a horizontal speed of 4.5 m/s into a 70 kg cart that is initially at rest (see Fig. 5.17). Find the final velocity of the boy and the cart.

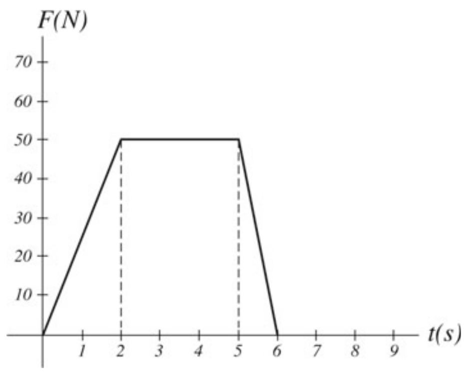


Fig. 5.16 A force acting on a particle varies with time

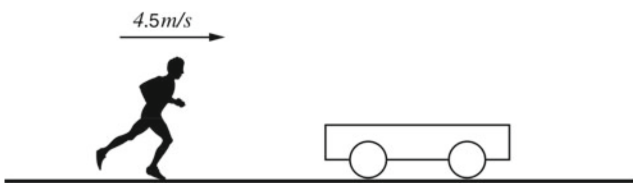


Fig. 5.17 A boy jumps on a cart that is initially at rest

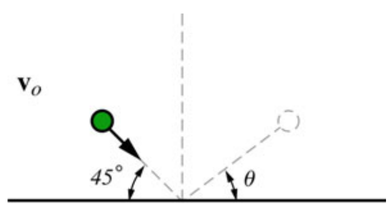


Fig. 5.18 A ball bouncing off a smooth surface

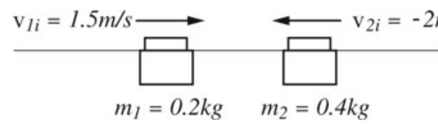


Fig. 5.19 Two gliders moving on a frictionless linear air track experience a perfectly elastic collision

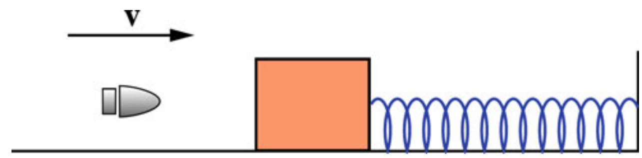


Fig. 5.20 A bullet of mass of m is fired with a horizontal velocity v into a block of mass M

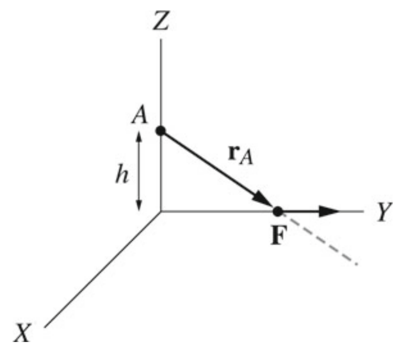
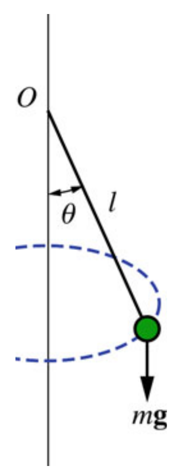


Fig. 5.21 A block moving along the y-axis subject to a force

Fig. 5.22 A conical pendulum of mass m and length L is in uniform circular motion with a velocity v



5. A rubber ball of mass of 0.2 kg is dropped from a height of 2.2 m. It re- bounds to a height of 1.1 m. Find (a) the coefficient of restitution, (b) the energy lost due to impact.
6. A 1200 kg car initially traveling at 12 m/s due east collides with another car of mass of 1600 kg that is initially at rest. If the cars become entangled after the collision, find the common final speed of the cars.
7. Figure 5.18 shows a ball that strikes a smooth surface with a velocity of 20 m/s at an angle of 45° with the horizontal. If the coefficient of restitution for the impact between the ball and the surface is $e = 0.85$, find the magnitude and direction of the velocity in which the ball rebounds from the surface. (Hint: use the velocity components in the direction perpendicular to the surface for the coefficient of restitution).
8. Two gliders moving on a frictionless linear air track experience a perfectly elastic collision (see Fig. 5.19). Find the velocity of each glider after the collision.
9. A bullet of mass of m is fired with a horizontal velocity v into a block of mass M. The block is initially at rest on a frictionless surface and is connected to a spring of force constant of k (see Fig. 5.20). If the bullet embeds itself in the block causing the spring to compress to a maximum distance d, find the initial speed of the bullet.
10. A block moves along the y-axis due to a force given by $\mathbf{F} = a\mathbf{i}$ (see Fig. 5.21). Find the torque on the block about (a) the origin (b) point A.
11. A conical pendulum of mass m and length L is in uniform circular motion with a velocity v (see Fig. 5.22). Find the angular momentum and torque on the mass about O.

6 System of Particles

6.1 System of Particles

In the previous chapters, objects that can be treated as particles were only considered. We have seen that this is possible only if all parts of the object move in exactly the same way. An object that does not meet this condition must be treated as a system of particles. Next, we will see that the complex motion of this object or system of particles can be represented by the motion of a point located at the center of mass of the system. The center of mass moves as if all of the mass of the object is concentrated there and as if the net external force acting on the system is applied there (at the center of mass). As well as representing an object by a particle, the concept of the center of mass is used to analyze the motion of many systems such as a system of two colliding blocks (particle-like objects) and the system of two colliding subatomic particles such as the neutron with the nucleus.

6.2 Discrete and Continuous System of Particles

6.2.1 Discrete System of Particles

A discrete system of particles is a system in which particles are separated from each other.

6.2.2 Continuous System of Particles

A continuous system of particles is a system where the separation of particles is very small such that it approaches zero. An extended object is a continuous system of particles. Now, consider the skateboarder example mentioned in Sect. 4.3. It has been shown that the system (man+skateboard) cannot be treated as a particle since different parts of the system move in different ways. By representing the skateboarder as a system of particles its motion can be represented by the motion of its center of mass, hence, the work–energy theorem can be applied to that point. The work done by the force, exerted on the skateboarder by the bar, is not zero because the point of application of that force (which is at the center of mass) has moved.

6.3 The Center of Mass of a System of Particles

For a system of particles of total mass M the acceleration of its center of mass is given by

$$\mathbf{a} = \frac{\mathbf{F}}{M}$$

6.3.1 Two Particle System

Consider two particles of masses m_1 and m_2 moving in space. Suppose that their position vectors at a particular instant of time are given by $\mathbf{r}_1$ and $\mathbf{r}_2$ as shown in Fig. 6.1. The center of mass of the system lies somewhere along the line joining the two particles and its position vector is given by

$$\mathbf{r}_{cm} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

The x, y and z components of the center of mass is

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}$$

and

$$z_{cm} = \frac{m_1z_1 + m_2z_2}{m_1 + m_2}$$

S. Alrasheed, *Principles of Mechanics*, Advances in Science, Technology & Innovation, https://doi.org/10.1007/978-3-030-15195-9_6

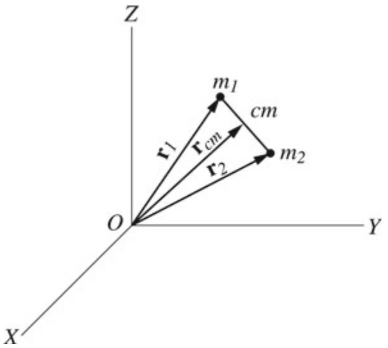


Fig. 6.1 Two particles of masses m_1 and m_2 moving in space. Their position vectors at a particular instant of time are given by $\mathbf{r}_1$ and $\mathbf{r}_2$

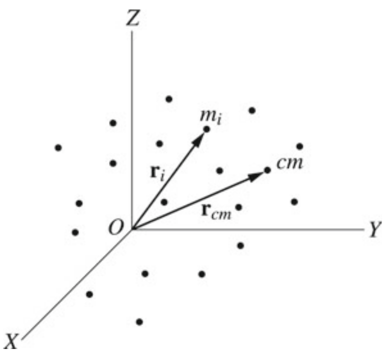


Fig. 6.2 A discrete system of particles consisting of n particles

6.3.2 Discrete System of Particles

Consider a discrete system of particles consisting of n particles (see Fig. 6.2). The position vector of the center of mass at a particular instant is given by

$$\mathbf{r}_{cm} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3. + \cdots\cdots\cdots\cdots \cdot m_n\mathbf{r}_n}{m_1 + m_2 + m_3 + \cdots + m_n} = \frac{\Sigma_{i=1}^{n} m_i\mathbf{r}_i}{M}$$

where $\mathbf{r}_i$ is the position vector of the ith particle and $M = \sum_{i=1}^{n} m_i$ is the total mass of the system. In component form, r_i can be written as

$$\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}$$

The x, y and z components of the center of mass vector are

$$x_{cm} = \frac{\sum_{i=1}^{n} m_i x_i}{M}$$

$$y_{cm} = \frac{\sum_{i=1}^{n} m_i y_i}{M}$$

and

$$z_{cm} = \frac{\sum_{i=1}^{n} m_i z_i}{M}$$

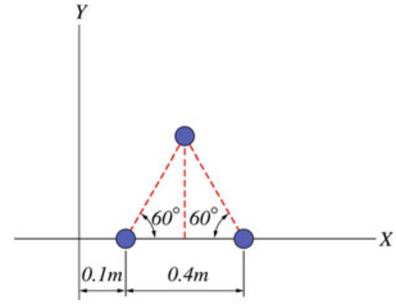


Fig. 6.3 The center of mass of a system in the x-y plane

Example 6.1 Find the center of mass of the system shown in Fig. 6.3 where the three particles have an equal mass of $m = 1\,\mathrm{kg}$.

Solution 6.1

$$x_{cm} = \frac{(1\,\mathrm{kg})((0.1\,\mathrm{m}) + (0.5\,\mathrm{m}) + (0.3\,\mathrm{m}))}{(3\,\mathrm{kg})} = 0.3\,\mathrm{m}$$

$$y_{cm} = \frac{0 + 0 + (1\,\mathrm{kg})(0.2\,\mathrm{m})\tan(60^{o})}{(3\,\mathrm{kg})} = 0.12\,\mathrm{m}$$

$$\mathbf{r}_{cm} = x_{cm}\mathbf{i} + y_{cm}\mathbf{j} = (0.3\,\mathrm{m})\,\mathbf{i} + (0.12\,\mathrm{m})\,\mathbf{j}$$

Example 6.2 A system of particles consists of three masses $m_A = 0.5\,\mathrm{kg}$, $m_B = 2\,\mathrm{kg}$ and $m_C = 5\,\mathrm{kg}$ located at $\mathrm{P_A}(-3, 1, 2)$, $\mathrm{P_B}(0, 1, 2)$ and $\mathrm{P_C}(-1, 3, 0)$, respectively. Find the position vector of the center of mass of the system.

Solution 6.2 The position vector of each particle is

$$\mathbf{r}_A = (-3\mathbf{i} + \mathbf{j} + 2\mathbf{k})\,\mathrm{m}$$

$$\mathbf{r}_B = (\mathbf{j} + 2\mathbf{k})\,\mathrm{m}$$

and

$$\mathbf{r}_C = (-\mathbf{i} + 3\mathbf{j})\,\mathrm{m}$$

The center of mass of the system is

$$\mathbf{r}_{cm} = \frac{\sum_{i=1}^{n} m_i\mathbf{r}_i}{\sum_{i=1}^{n} m_i} = \frac{(0.5\,\mathrm{kg})((-3\mathbf{i} + \mathbf{j} + 2\mathbf{k})\,\mathrm{m}) + (2\,\mathrm{kg})((\mathbf{j} + 2\mathbf{k})\,\mathrm{m}) + (5\,\mathrm{kg})((-\mathbf{i} + 3\mathbf{j})\,\mathrm{m})}{(7.5\,\mathrm{kg})}$$

That gives

$$\mathbf{r}_{cm} = (-0.87\mathbf{i} + 2.3\mathbf{j} + 0.7\mathbf{k})\,\mathrm{m}.$$

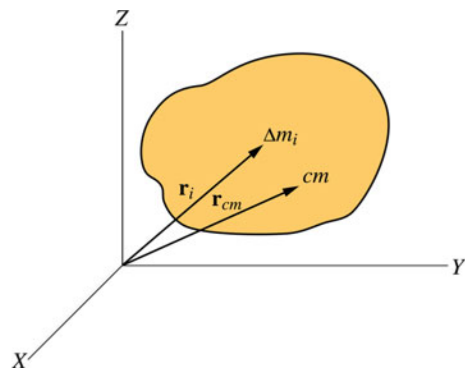


Fig. 6.4 An extended object of mass M divided into small volume elements each of mass $\triangle m_i$ and a vector position r_I

6.3.3 Continuous System of Particles (Extended Object)

A continuous system of particles is a system consisting of a large number of particles separated by very small distances. Consider an extended object of mass M divided into small volume elements each of mass $\triangle m_i$ and a vector position r_i (see Fig. 6.4). The position vector of the center of mass at a particular instant is then approximately given by

$$\mathbf{r}_{cm} \approx \frac{\sum_{i=1}^{n} \mathbf{r}_i \triangle m_i}{M}$$

For a very large number of particles where $n \to \infty$ we have $\triangle m_i \to 0$, that gives

$$\mathbf{r}_{cm} = \lim_{\triangle m_i} \frac{\sum_{i=1}^{n} \mathbf{r}_i \triangle m_i}{M} = \frac{1}{M} \int \mathbf{r} dm$$

Since $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the x, y and z components of the center of mass are given by

$$x_{cm} = \frac{1}{M} \int x dm$$

$$y_{cm} = \frac{1}{M} \int y dm$$

and

$$z_{cm} = \frac{1}{M} \int z dm$$

6.3.4 Elastic and Rigid Bodies

A body is called an elastic (deformable) body if the separation between its particles changes when a force is applied to it. This change or deformation is sometimes so small that it can be neglected. A body that behaves in this way is called a rigid body. A rigid body can be defined as a body in which the separation between its particles remain constant with time despite the applied force, i.e., the body has a constant size and shape. Therefore, the center of mass of a rigid object remains fixed at the same location at all times. In this book, only rigid bodies are discussed. In solving problems, it is common to use the volume density ρ defined as the mass per unit volume given by

$$\rho = \frac{dm}{dV}$$

Therefore, the total mass of a rigid object is

$$M = \int \rho dV$$

The center of mass of a rigid object can thus be written as

$$\mathbf{r}_{cm} = \frac{1}{M} \int \mathbf{r} dm = \frac{\int \rho \mathbf{r} dV}{\int \rho dV}$$

ρ may be a function of position, i.e., it can vary from point to point in the body If the body has a uniform density (homogeneous body), then ρ can be written as

$$\rho = \frac{dm}{dV} = \frac{\text{Tota1Mass}}{\text{Tota1Volume}} = \text{constant}$$

If the continuous distribution of particles occupies a surface, then the surface density σ is used and is given by

$$\sigma = \frac{dm}{dA} \text{ (mass per unit area)}$$

$$\sigma = \frac{\text{Tota1Mass}}{\text{Tota1Area}} = \text{constant (homogeneous body)}$$

If the particles occupy a curve or a line, the linear density λ is used given by

$$\lambda = \frac{dm}{dl} \text{ (mass per unit length)}$$

$$\lambda = \frac{\text{Tota1Mass}}{\text{Tota1Length}} = \text{constant} \quad \text{(homogeneous body)}$$

The center of mass of any homogeneous symmetric object is at its geometrical center and it is not necessarily located within the object.

Example 6.3 A thin rod of length $L = 2$ m has a linear density that increases with x according to the expression

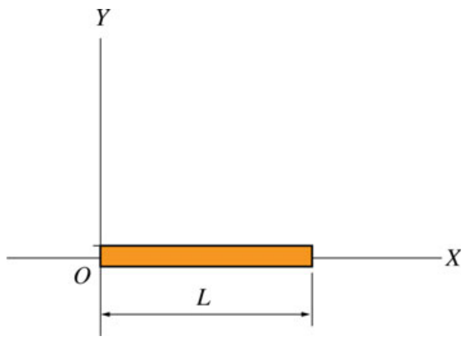


Fig. 6.5 A thin rod of length $L = 2$ m has a linear density that increases with x

$\lambda(x) = (2x - 1)$ kg/m (see Fig. 6.5). Locate the center of mass of the rod relative to O.

Solution 6.3

$$x_{cm} = \frac{1}{M}\int x dm = \frac{\int_0^L x\lambda(x)dx}{\int_0^L \lambda(x)dx} = \frac{\int_0^L (2x^2 - x)dx}{\int_0^L (2x-1)dx}$$

$$= \frac{((2/3)x^3 - x^2/2)|_{x=0}^L}{(x^2 - x)|_{x=0}^L} = \frac{L((2/3)L - 1/2)}{(L-1)}$$

Substituting $L = 2$ m gives $x_{cm} = 1.7$ m.

Example 6.4 A uniform square sheet is suspended by a uniform rod where they both lie in the same plane as shown in Fig. 6.6. Find the center of mass of the system.

Solution 6.4 Because the sheet and the rod are homogeneous, the center of mass of each is at its geometric center. Since the center of the sheet is at the origin we have

$$x_{cm} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{0 + (M_2 L/2)}{M_1 + M_2} = \frac{LM_2}{2(M_1 + M_2)}$$

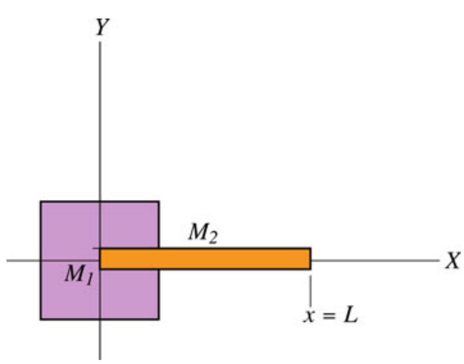


Fig. 6.6 A uniform square sheet suspended by a uniform rod where they both lie in the same plane

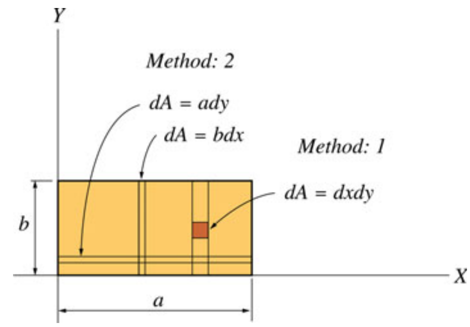


Fig. 6.7 The center of mass of a rectangular plate

Example 6.5 Find the center of mass of the rectangular plate shown in Fig. 6.7. The plate has a uniform surface density σ.

Solution 6.5 • Method 1:

$$x_{cm} = \frac{\int x dm}{M} = \frac{\int x\sigma dA}{\int \sigma dA} = \frac{\int_{y=0}^b \int_{x=0}^a x dx dy}{\int_{y=0}^b \int_{x=0}^a dx dy} = \frac{ba^2}{2ab} = \frac{a}{2}$$

$$y_{cm} = \frac{\int y dm}{M} = \frac{\int x\sigma dA}{\int \sigma dA} = \frac{\int_{x=0}^a \int_{y=0}^b y dx dy}{\int_{x=0}^a \int_{y=0}^b dx dy} = \frac{ab^2}{2ab} = \frac{b}{2}$$

Hence

$$\mathbf{r}_{cm} = \frac{a}{2}\mathbf{i} + \frac{b}{2}\mathbf{j}$$

• Method 2:

Dividing the plate into very thin rods each of mass $\sigma b dx$ gives

$$x_{cm} = \frac{\int x dm}{M} = \frac{1}{M}\int x\sigma dA = \frac{1}{M}\left(\frac{M}{ab}\right)\int_{x=0}^a xbdx = \frac{1}{a}\left[\frac{x^2}{2}\right]_{x=0}^a = \frac{a}{2}$$

Similarly by dividing the plate into thin horizontal rods each of mass $\sigma a dy$ gives

$$y_{cm} = \frac{\int y dm}{M} = \frac{1}{M}\int y\sigma dA = \frac{1}{M}\left(\frac{M}{ab}\right)\int_{y=0}^b aydy = \frac{1}{b}\left[\frac{y^2}{2}\right]_{y=0}^b = \frac{b}{2}$$

and

$$\mathbf{r}_{cm} = \frac{a}{2}\mathbf{i} + \frac{b}{2}\mathbf{j}$$

Example 6.6 An object of uniform surface density σ and mass M has the shape shown in Fig. 6.8 (half of an ellipse). Find the center of mass of the object.

Solution 6.6 The equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

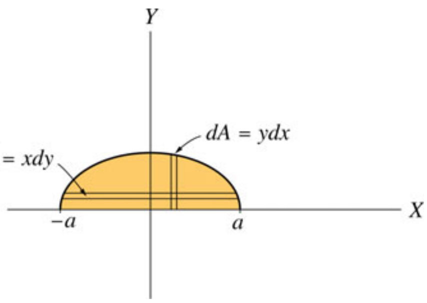


Fig. 6.8 The center of mass of half an ellipse

therefore

$$\frac{2xdx}{a^2} + \frac{2ydy}{b^2} = 0$$

or

$$xdx = \frac{-a^2}{b^2} ydy$$

By dividing the area into very thin rectangles each of mass σydx gives

$$x_{cm} = \frac{\int xdm}{M} = \frac{1}{M}\int x\sigma dA = \frac{1}{M}\int_{x=-a}^{a} x\left(\frac{2M}{\pi ab}\right)ydx$$

$$= \frac{2}{\pi ab}\int_{y=0}^{0}\left(\frac{-a^2}{b^2}\right)y^2dy = \frac{-2a}{\pi b^3}\left[\frac{y^3}{3}\right]_{y=0}^{0} = 0$$

To obtain the y coordinate of the center of mass we divide the area into very thin rectangles each of mass σxdy as in Fig. 6.8. That gives

$$y_{cm} = \frac{1}{M}\int ydm = \frac{1}{M}\int y\sigma dA = \frac{2}{\pi ab}\int_{y=0}^{b} yxdy$$

$$= \frac{2}{\pi ab}\int_{x=a}^{-a}\left(\frac{-b^2}{a^2}\right)x^2dx = \frac{-2b}{\pi a^3}\int_{x=a}^{-a}x^2dx = \frac{-2b}{\pi a^3}\left[\frac{x^3}{3}\right]_{x=a}^{-a}$$

$$\frac{-2b}{\pi a^3}\left[\frac{x^3}{3}\right]_{x=a}^{-a} = \frac{-2b}{\pi a^3}\left(\frac{-a^3}{3} - \frac{a^3}{3}\right) = \frac{4b}{3\pi}$$

Example 6.7 Determine the center of mass of the cylindrical shell shown in Fig. 6.9. The shell has a uniform surface density σ.

Solution 6.7 From symmetry, the center of mass lies on the z-axis. By dividing the shell into very thin rings each of mass $\sigma 2\pi Rdz$ we have

$$z_{cm} = \frac{\int zdm}{M} = \frac{\int z\sigma dA}{M} = \frac{1}{M}\int_{z=0}^{h} z\sigma 2\pi Rdz = \frac{1}{M}\left(\frac{M}{2\pi Rh}\right)\int_{z=0}^{h} 2\pi Rzdz$$

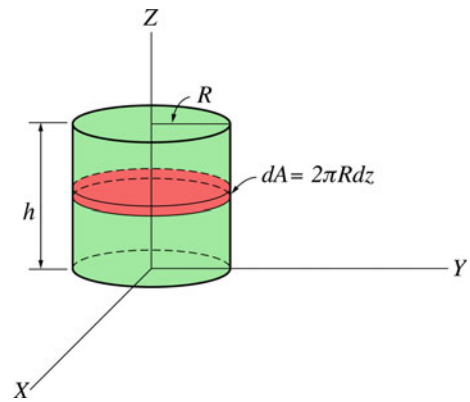


Fig. 6.9 The center of mass of a cylindrical shell

$$= \frac{1}{h}\left[\frac{z^2}{2}\right]_{z=0}^{h} = \frac{h}{2}$$

Example 6.8 A boy standing on a smooth ice surface wants to fetch a container that is at a distance of 10 m away from him. To do that, he throws a rope around the container and start to pull. Because the surface is smooth, both the boy and the container will move until they meet. If the masses of the boy and of the container are 40 kg and 70 kg respectively, how far will the container move when the boy has moved a distance of 2 m?

Solution 6.8 By taking the midpoint between the boy and the container as the origin (see Fig. 6.10) and by neglecting the mass of the rope, the center of mass of the system is

$$x_{cm} = \frac{\Sigma_i m_i x_i}{\Sigma_i m_i} = \frac{(70\,\text{kg})(5\,\text{m}) + (40\,\text{kg})(-5\,\text{m})}{(110\,\text{kg})} = 1.36\,\text{m}$$

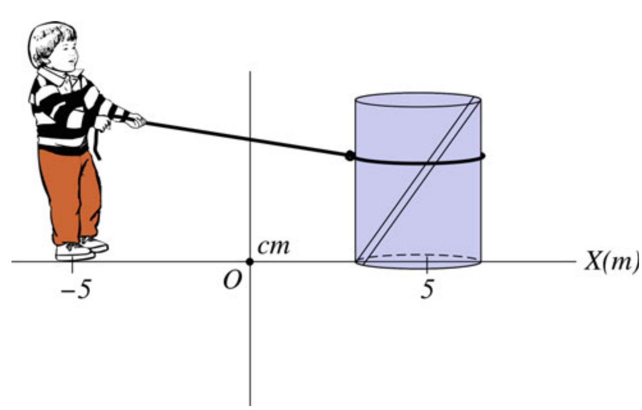


Fig. 6.10 A boy pulling a container on a smooth surface

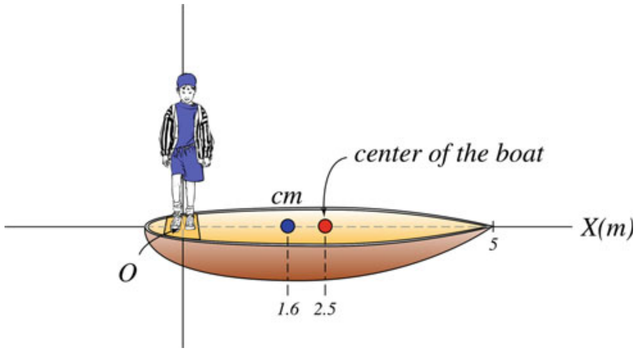


Fig. 6.11 A boy walking on a small boat

Because the surface may be assumed to be frictionless, the resultant external force on the system is zero and therefore the center of mass must remain stationary at all times. Hence, if the boy has moved a distance of 2 m, he will be at a distance of −3 m from the origin. Thus, we have

$$(1.36\,\text{m}) = \frac{(70\,\text{kg})x_c + (40\,\text{kg})(-3\,\text{m})}{(110\,\text{kg})}$$

That gives $x_c = 3.86$ m, therefore the distance moved by the container towards the center of mass is $(5\,\text{m}) - (3.86\,\text{m}) = 1.14$ m.

Example 6.9 A boy is standing at the rear of a boat as shown in Fig. 6.11. The masses of the boy and of the boat are 45 kg and 80 kg respectively Find the distance that the boat would move relative to the origin if the boy moves a distance of lm from the rear of the boat (the length of the boat is 5 m).

Solution 6.9 By neglecting air and water resistance, the net external force on the (boy+ boat) system is zero. Therefore the center of mass of the system must remain at rest. Suppose that the boat is a symmetrical homogeneous object where its center of mass is at its geometrical center. The center of mass of the boat is therefore at a distance of 2.5 m from the origin. Thus, the center of mass of the system is

$$x_{cm} = \frac{\sum_{i=1}^{n} m_i x_i}{M} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$
$$= \frac{(45\,\text{kg})(0) + (80\,\text{kg})(2.5\,\text{m})}{(125\,\text{kg})} = 1.6\,\text{m}$$

If the boy moves a distance of 1 m, the center of mass is still at the same position, and we have

$$(1.6\,\text{m}) = \frac{(45\,\text{kg})(1\,\text{m}) + (80\,\text{kg})x_b}{(125\,\text{kg})}$$

That gives $x_b = 1.94$ m. Thus, the displacement of the center of mass of the boat is $(1.94\,\text{m}) - (2.5\,\text{m}) = -0.56$ m.

6.3.5 Velocity of the Center of Mass

The velocity of the center of mass of a system of particles that has a constant mass M is

$$\mathbf{v}_{cm} = \frac{d\mathbf{r}_{cm}}{dt} = \frac{1}{M}\frac{d}{dt}\left(\sum_{i=1}^{n} m_i \mathbf{r}_i\right) = \frac{1}{M}\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i$$

where $\dot{\mathbf{r}}_i = d\mathbf{r}_i/dt$, or

$$\mathbf{v}_{cm} = \sum_{i=1}^{n} \frac{m_i \mathbf{v} i}{M} \tag{6.1}$$

where $\mathbf{v}_i$ is the ith particle velocity. The acceleration of the center of mass is given by

$$\mathbf{a}_{cm} = \frac{d\mathbf{v}_{cm}}{dt} = \frac{1}{M}\frac{d}{dt}\left(\sum_{i=1}^{n} m_i \mathbf{v}_i\right) = \frac{1}{M}\sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i$$

$$\mathbf{a}_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i \mathbf{a}_i \tag{6.2}$$

where $\mathbf{a}_i$ is the acceleration of the ith particle.

6.3.6 Momentum of a System of Particles

The total linear momentum of a system of particles is the vector sum of the linear momenta of the individual particles:

$$\sum_{i=1}^{n} m_i \mathbf{v}_i = \sum_{i=1}^{n} \mathbf{p}_i = \mathbf{p}_{tot} \tag{6.3}$$

By using Eq. 6.1

$$\mathbf{p}_{tot} = M\mathbf{v}_{cm} \tag{6.4}$$

Example 6.10 Two particles of masses $m_1 = 1$ kg and $m_2 = 2$ kg have position vectors given by $\mathbf{r}_1 = (2t\mathbf{i} - 4\mathbf{j})$ m and $\mathbf{r}_2 = (5t\mathbf{i} - 2t\mathbf{j})$ m respectively where t is time. Determine the velocity and linear momentum of the center of mass of the two- particle system at any time and at $t = 1$ s.

Solution 6.10

$$\mathbf{r}_{cm} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} = \frac{(1\,\text{kg})(2t\mathbf{i} - 4\mathbf{j}) + (2\,\text{kg})(5t\mathbf{i} - 2t\mathbf{j})}{(3\,\text{kg})}$$

That gives

$$\mathbf{r}_{cm} = \left(4t\mathbf{i} - \frac{4}{3}(t+1)\mathbf{j}\right)\,\text{m}$$

$$\mathbf{v}_{cm} = \frac{d\mathbf{r}_{cm}}{dt} = \left(4\mathbf{i} - \frac{4}{3}\mathbf{j}\right)\,\text{m/s}$$

The total linear momentum is

$$\mathbf{p}_{tot} = M\mathbf{v}_{cm} = (3\text{kg})\left(4\mathbf{i} - \frac{4}{3}\mathbf{j}\right) = (12\mathbf{i} - 4\mathbf{j})\ \text{kg.m/s}$$

at $t = 1\text{s}$

$$\mathbf{r}_{cm} = (4\mathbf{i} - \frac{8}{3}\mathbf{j})\ \text{m}$$

$$\mathbf{v}_{cm} = (4\mathbf{i} - \frac{4}{3}\mathbf{j})\ \text{m/s}$$

and

$$\mathbf{p}_{tot} = (12\mathbf{i} - 4\mathbf{j})\ \text{kg.m/s}$$

6.3.7 Motion of a System of Particles

From Newton's second law Eq. 6.2 can be written as

$$\mathbf{a}_{cm} = \frac{1}{M}\sum_{i=1}^{n}\mathbf{F}_i \quad (6.5)$$

where $\mathbf{F}_i$ is the net force acting on the ith particle. If both the external forces on the system and the internal forces between the particles in the system are included, then $\mathbf{F}_i$ may be written as

$$\mathbf{F}_i = \mathbf{F}_{i(ext)} + \sum_j \mathbf{f}_{ij} \quad (6.6)$$

Where $\mathbf{F}_{i(ext)}$ is the resultant external force acting on the ith particle. $\mathbf{f}_{ij}$ is the internal force exerted on the ith particle by the jth particle. Note that it is as- sumed that no force is exerted on the particle by itself, i.e., $\mathbf{f}_{ii} = 0$. Substituting Eq. 6.6 into Eq. 6.5 gives:

$$\mathbf{a}_{cm} = \frac{1}{M}\left(\sum_i \mathbf{F}_{i(ext)} + \sum_i\sum_j \mathbf{f}_{ij}\right) \quad (6.7)$$

Now, from Newton's third law we have

$$\mathbf{f}_{ij} = -\mathbf{f}_{ji}$$

Therefore, the second term in Eq. 6.7 is equal to zero. Hence the net force acting on the system is due only to external forces. That gives

$$\mathbf{F}_{net} = \sum_i \mathbf{F}_{i(ext)} = M\mathbf{a}_{cm}$$

where $\mathbf{F}_{net}$ is the resultant external force on the center of mass, i.e.,

$$\mathbf{F}_{net} = \sum \mathbf{F}_{ext} = M\mathbf{a}_{cm}$$

By differentiating Eq. 6.4 with respect to time we have

$$M\mathbf{a}_{cm} = \frac{d\mathbf{p}_{tot}}{dt}$$

thus

$$\sum \mathbf{F}_{ext} = \frac{d\mathbf{p}_{tot}}{dt}$$

Thus, the net external force acting on a system of particles is equal to the time rate of change of the total linear momentum of the system.

6.3.8 Conservation of Momentum

For an isolated system of particles, we have

$$\sum \mathbf{F}_{ext} = 0$$

Thus

$$\frac{d\mathbf{p}_{tot}}{dt} = 0$$

and

$$\mathbf{p}_{tot} = M\mathbf{v}_{cm} = \text{constant}$$

Which is the law of conservation of linear momentum for a system of particles.

6.3.9 Angular Momentum of a System of Particles

The angular momentum $\mathbf{L}$ of a system of particles about a fixed point is the vector sum of angular momenta of the individual particles:

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3 + \ +\mathbf{L}_n = \sum_{i=1}^{n}\mathbf{L}_i = \sum_{i=1}^{n}(\mathbf{r}_i \times \mathbf{p}_i) = \sum_{i=1}^{n} m_i(\mathbf{r}_i \times \mathbf{v}_i)$$

6.3.10 The Total Torque on a System

The total torque acting on a particle in a system is the sum of torques associated with the internal forces and of torques associated with external forces. Using Eq. 6.6 we have

$$\boldsymbol{\tau_i} = \mathbf{r}_i \times \mathbf{F}_i = \mathbf{r}_i \times \left(\mathbf{F}_{iext} + \sum_j \mathbf{f}_{ij} \right) = \mathbf{r}_i \times \mathbf{F}_{iext} + \sum_j \mathbf{r}_i \times \mathbf{f}_{ij}$$

Summing over i we get

$$\sum_i \boldsymbol{\tau_i} = \sum_i \mathbf{r}_i \times \mathbf{F}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_{iext} + \sum_i \sum_j \mathbf{r}_i \times \mathbf{f}_{ij} \quad (6.8)$$

By using Newton's third law of action and reaction, the double sum in Eq. 6.8 has terms of the form

$$\mathbf{r}_i \times \mathbf{f}_{ij} + \mathbf{r}_j \times \mathbf{f}_{ji} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{f}_{ij}$$

Now, suppose that the internal forces between the two particles lie along the line joining the particles (i.e., the vectors $\mathbf{f}_{ij}$ and $(\mathbf{r}_i - \mathbf{r}_j)$ have the same direction). This condition is known as the strong law of action and reaction. It requires the internal forces to be central. If the internal forces are equal and opposite but not central, then they are said to satisfy the weak law of action and reaction. The force of gravity is an example of a force satisfying the strong law of action and reaction. Some forces such as the forces between two moving charges are not central. From this, it follows that the double summation in Eq. 6.8 is equal to zero.

$$\boldsymbol{\tau_{net}} = \sum_i \boldsymbol{\tau_i} = \sum_i \mathbf{r}_i \times \mathbf{F}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_{iext}$$

Therefore, the total torque on the system about the origin is only the torque associated with external forces

$$\boldsymbol{\tau_{net}} = \sum \boldsymbol{\tau_{ext}} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_{i(ext)} \quad (6.9)$$

6.3.11 The Angular Momentum and the Total External Torque

The angular momentum of the individual particles may change with time. This will change the total angular momentum of the system

$$\frac{d\mathbf{L}}{dt} = \sum_{i=1}^{n} \frac{d\mathbf{L}_i}{dt}$$

Eq. 6.9 may be written as

$$\boldsymbol{\tau_{net}} = \sum \boldsymbol{\tau_{ext}} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_{i(ext)} = \frac{d}{dt}\left\{ \sum_{i=1}^{n} m_i (\mathbf{r}_i \times \mathbf{v}_i) \right\} = \frac{d}{dt}\left\{ \sum_{i=1}^{n} \mathbf{L}_i \right\} = \frac{d\mathbf{L}}{dt}$$

i.e., the net external torque about some origin exerted on a system of particles is equal to the time rate of change of the total angular momentum of the system.

6.3.12 Conservation of Angular Momentum

If

$$\sum \boldsymbol{\tau_{ext}} = \mathbf{0}$$

$$\mathbf{L} = \sum_{i=1}^{n} m_i (\mathbf{r}_i \times \mathbf{v}_i) = \text{constant}$$

or

$$\mathbf{L}_i = \mathbf{L}_f$$

Hence, if the resultant external torque acting on a system is zero, the total angular momentum remains constant.

6.3.13 Kinetic Energy of a System of Particles

The total kinetic energy of a system of particles is the sum of the kinetic energies of the individual particles

$$K = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2$$

6.3.14 Work

Since the total force acting on the ith particle is given by

$$\mathbf{F}_i = \mathbf{F}_{i(ext)} + \sum_j \mathbf{f}_{ij}$$

then the total work done on such particle is given by

$$W_{12} = \sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{s}_i$$

6.3.15 Work–Energy Theorem

The total work done in moving a system from one state to another is

$$W_{12} = \sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{s}_i = \sum_i \int_1^2 \mathbf{F}_i \cdot \frac{d\mathbf{s}_i}{dt} dt = \sum_i \int_1^2 \mathbf{F}_i \cdot \mathbf{v}_i dt$$

$$= \sum_i \int_1^2 \mathbf{v}_i \cdot \mathbf{F}_i dt = \sum_i \int_1^2 \mathbf{v}_i \cdot \frac{d}{dt}(m_i \mathbf{v}_i) dt$$

Since

$$\mathbf{v}_i \frac{d}{dt}(m_i \mathbf{v}_i) = \frac{1}{2}\frac{d}{dt}(m_i(\mathbf{v}_i \cdot \mathbf{v}_i)) = \frac{1}{2}\frac{d}{dt}(m_i v_i^2)$$

it follows that

$$W_{12} = \frac{1}{2}\sum_i \int_1^2 \frac{d}{dt}(m_i v_i^2)dt = \frac{1}{2}\sum_i \left(m_i v_i^2\right)|_1^2 = K_2 - K_1$$

where $\frac{1}{2}\sum_i m_i v_i^2$ is the total kinetic energy of the system.

6.3.16 Potential Energy and Conservation of Energy of a System of Particles

Consider a system of particles in which the external and internal forces acting on the system are conservative. First, let us calculate the work done by the internal conservative forces. Suppose that $\mathbf{f}_{ij}$ is the conservative force acting on the ith particle due to the jth particle and $\mathbf{f}_{ji}$ is the force acting on the jth particle due to the ith particle. Note that $\mathbf{f}_{ij}$ and $\mathbf{f}_{ji}$ form an action and reaction pair, i.e., $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$. Because these forces are conservative there is a potential energy associated with each force. That is,

$$\mathbf{f}_{ij} = -\nabla_i U_{ij}$$

and

$$\mathbf{f}_{ji} = -\nabla_j U_{ij}$$

From the law of action and reaction, U_{ij} is a function only of the distance between the particles. That is

$$U_{ij} = U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) = U_{ji}(|\mathbf{r}_i - \mathbf{r}_j|)$$

or

$$U_{ij}(r_{ij}) = U_{ji}(r_{ji})$$

where $|\mathbf{r}_i - \mathbf{r}_j| = r_{ij} = r_{ji}$ is the distance between the ith and jth particles. The work done by each pair of forces in displacing the ith and jth particles through $d\mathbf{r}_i$ and $d\mathbf{r}_j$, respectively, is

$$\mathbf{f}_{ij}\cdot d\mathbf{r}_i + \mathbf{f}_{ji}\cdot d\mathbf{r}_j = -\nabla_i U_{ij}\cdot d\mathbf{r}_i - \nabla_j U_{ij}\cdot d\mathbf{r}_j$$

$$= -\left[\frac{\partial U_{ij}}{\partial x_i}dx_i + \frac{\partial U_{ij}}{\partial y_i}dy_i + \frac{\partial U_{ij}}{\partial z_i}dz_i + \frac{\partial U_{ij}}{\partial x_j}dx_j + \cdots\cdots\right] = -dU_{ij}$$

Hence, the total work done by the internal conservative forces in moving the system from stage 1 to stage 2 is

$$W_{12(in,c)} = \sum_i\sum_j \int_1^2 \mathbf{f}_{ij}\cdot d\mathbf{r}_i = -\frac{1}{2}\sum_i\sum_j \int_1^2 dU_{ij}$$
$$= -\frac{1}{2}\sum_i\sum_j U_{ij}|_1^2 = U_{1(\text{int})} - U_{2(\text{int})} = -\triangle U_{(\text{int})}$$

The factor 1/2 occurs since each term in the summation appears twice. Now, consider the total work done by the external conservative forces

$$W_{12(ext,c)} = \sum_i \int_1^2 \mathbf{F}_{i(ext)}.d\mathbf{s}_i = -\sum_i \int_1^2 \nabla_i U_i \cdot d\mathbf{s}_i = -\sum_i U_i|_1^2 = U_{1(ext)} - U_{2(ext)}$$

To show that energy is conserved when both the external and internal forces are conservative, we may define a total potential of the system as

$$U = \sum_i U_i + \frac{1}{2}\sum_i\sum_j U_{ij}$$

From the work–energy theorem, the work done by the total force F_i acting on the ith particle is equal to the change in the kinetic energy of that particle

$$W_{12} = \sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i = K_2 - K_1$$

and since

$$W_{12} = W_{12(in,c)} + W_{12(ext,c)}$$

From this, we conclude that for a system of particles in which the internal and external forces are conservative, the total mechanical energy of the system is conserved

$$U_{1(\text{int})} - U_{2(\text{int})} + U_{1(ext)} - U_{2(ext)} = K_2 - K_1$$

or

$$U_1 - U_2 = K_2 - K_1$$

or

$$\triangle K = -\triangle U$$

Thus

$$\triangle K + \triangle U = 0$$

$$\triangle E = 0$$

6.3.17 Impulse

In Sect. 6.3.7, we have seen that the net external force on a system of particles is equal to the rate of change of the total linear momentum of the system

$$\mathbf{F}_{net} = \frac{d\mathbf{p}_{tot}}{dt}$$

The total linear impulse on the system as the system goes from one state to another is defined as

$$\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}_{net} dt = \int_{t_1}^{t_2} \frac{d\mathbf{p}_{tot}}{dt} dt = \mathbf{p}_{tot2} - \mathbf{p}_{tot1}$$

That is, the total linear impulse on the system is equal to the change in the total momentum of the system.

6.4 Motion Relative to the Center of Mass

The motion of a system of particles is sometimes described relative to the center of mass of the system. This method is used in some problems to simplify the analysis and add a particular symmetry to it.

6.4.1 The Total Linear Momentum of a System of Particles Relative to the Center of Mass

The position vector of the center of mass of the system with respect to an origin in an inertial frame of reference (for example, the lab frame) is given by

$$\mathbf{r}_{cm} = \frac{\Sigma_i^n m_i \mathbf{r}_i}{M} \tag{6.10}$$

From Fig. 6.12, the position vector ($\mathbf{r}_i'$) of the ith particle relative to the center of mass is

$$\mathbf{r}_i' = \mathbf{r}_i - \mathbf{r}_{cm}$$

or

$$\mathbf{r}_i = \mathbf{r}_i' + \mathbf{r}_{cm} \tag{6.11}$$

Where $\mathbf{r}_i$ is the position vector of the ith particle relative to the origin O. Substituting Eq. 6.11 into Eq. 6.10 gives

$$\mathbf{r}_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i(\mathbf{r}_i' + \mathbf{r}_{cm}) = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i' + \frac{\sum_{i=1}^{n} m_i}{M}\mathbf{r}_{cm}$$

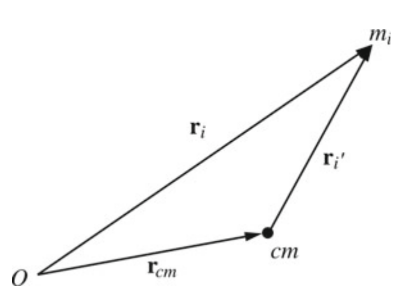


Fig. 6.12 The position vector ($\mathbf{r}_i'$) of the ith particle relative to the center of mass

$$= \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i' + \mathbf{r}_{cm}$$

therefore

$$\frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i' = \mathbf{r}_{cm} - \mathbf{r}_{cm} = 0$$

That gives

$$\sum_{i=1}^{n} m_i\mathbf{r}_i' = \mathbf{0} \tag{6.12}$$

Differentiating Eq. 6.12 with respect to t gives

$$\sum_{i=1}^{n} m\mathbf{v}_i' = \mathbf{0} \tag{6.13}$$

or

$$\sum_{i=1}^{n} \mathbf{p}_i' = \mathbf{0}$$

or

$$\mathbf{p}' = \mathbf{0}$$

That is, the total linear momentum of the system is zero when observed from the center of mass frame.

6.4.2 The Total Angular Momentum About the Center of Mass

By differentiating Eq. 6.11 with respect to time gives

$$\mathbf{v}_i = \mathbf{v}_i' + \mathbf{v}_{cm} \tag{6.14}$$

where $\mathbf{v}_i$ and $\mathbf{v}_i'$ are the velocities of the particle relative to the origin O and the center of mass respectively $\mathbf{v}_{cm}$ is the velocity of the center of mass relative to O. The angular momentum of the system about the origin O is

$$\mathbf{L} = \sum_i m_i(\mathbf{r}_i \times \mathbf{v}_i) = \sum_i m_i\{(\mathbf{r}_i' + \mathbf{r}_{cm}) \times (\mathbf{v}_i' + \mathbf{v}_{cm})\}$$

$$= \sum_i m_i(\mathbf{r}_i' \times \mathbf{v}_i') + \sum_i m_i(\mathbf{r}_i' \times \mathbf{v}_{cm}) + \sum_i m_i(\mathbf{r}_{cm} \times \mathbf{v}_i') + \sum_i m_i(\mathbf{r}_{cm} \times \mathbf{v}_{cm})$$

The second and third terms are zero followed from Eqs. 6.12 and 6.13 where $\left(\sum_i m_i\mathbf{r}_i'\right) \times \mathbf{v}_{cm} = \mathbf{0}$ and $\mathbf{r}_{cm} \times \left(\sum_i m_i\mathbf{v}_i'\right) = \mathbf{0}$, hence

$$\mathrm{L} = \sum_i m_i(\mathbf{r}'_i \times \mathbf{v}'_i) + \sum_i m_i(\mathbf{r}_{cm} \times \mathbf{v}_{cm})$$

Thus, the total angular momentum of the system of particles about an origin O equals the angular momentum of the system about the center of mass plus the angular momentum of the center of mass about O. Therefore, the total angular momentum $\mathbf{L}'$ about the center of mass is

$$\mathbf{L}' = \sum_i m_i(\mathbf{r}'_i \times \mathbf{v}'_i) = \sum_i m_i(\mathbf{r}_i \times \mathbf{v}_i) - M(\mathbf{r}_{cm} \times \mathbf{v}_{cm}) \tag{6.15}$$

6.4.3 The Total Kinetic Energy of a System of Particles About the Center of Mass

The total kinetic energy of a system of particles relative to an origin in an inertial frame of reference is given by

$$K = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i(\mathbf{v}_i \cdot \mathbf{v}_i)$$

From Eq. 6.14 we have

$$K = \frac{1}{2}\sum_i m_i((\mathbf{v}'_i + \mathbf{v}_{cm}) \cdot (\mathbf{v}'_i + \mathbf{v}_{cm}))$$

$$= \frac{1}{2}\sum_i m_i(\mathbf{v}'_i \cdot \mathbf{v}'_i) + \sum_i m_i(\mathbf{v}'_i \cdot \mathbf{v}_{cm}) + \frac{1}{2}\sum_i m_i(\mathbf{v}_{cm} \cdot \mathbf{v}_{cm})$$

$$= \frac{1}{2}\sum_i m_i v_i'^2 + \mathbf{v}_{cm} \cdot \left(\sum_i m_i \mathbf{v}'_i\right) + \frac{1}{2}\left(\sum_i m_i\right) v_{cm}^2$$

From Eq. 6.13, the term in brackets in the second term is equal to zero. Hence

$$K = \frac{1}{2}\sum_i m_i v_i'^2 + \frac{1}{2} M v_{cm}^2$$

That is the total kinetic energy of a system of particles about an origin is equal to the kinetic energy of the system with respect to the center of mass plus the kinetic energy of the center of mass relative to the origin O. Therefore, the total kinetic energy of the system with respect to the center of mass is

$$K' = \frac{1}{2}\sum_i m_i v_i'^2 = \frac{1}{2}\sum_i m_i v_i^2 - \frac{1}{2} M v_{cm}^2$$

6.4.4 Total Torque on a System of Particles About the Center of Mass of the System

The total torque acting on a system of particles about the center of mass is (from theorem (5.6.1)) equal to the time rate of change of the angular momentum of the system about the center of mass. That is,

$$\boldsymbol{\tau}' = \frac{d\mathbf{L}'}{dt}$$

Example 6.11 Two particles of masses $m_1 = 1$ kg and $m_2 = 2$ kg are moving in the x-y plane. Their position vectors relative to the origin are $\mathbf{r}_1 = (t^2\mathbf{i} - 2t\mathbf{j})$ m and $\mathbf{r}_2 = (3t\mathbf{i} + \mathbf{j})$ m where t is time. Find: (a) the total angular momentum of the system; the total external torque acting on the system; and the total kinetic energy of the system all relative to the origin at any time; (b) repeat (a) relative to the center of mass.

Solution 6.11 (a)

$$\mathbf{v}_1 = \frac{d\mathbf{r}_1}{dt} = (2t\mathbf{i} - 2\mathbf{j})\ \mathrm{m/s}$$

$$\mathbf{v}_2 = \frac{d\mathbf{r}_2}{dt} = (3\mathbf{i})\ \mathrm{m/s}$$

The total angular momentum of the system relative to the origin is

$$\mathbf{L} = \sum_i m_i(\mathbf{r}_i \times \mathbf{v}_i) = (1)[(t^2\mathbf{i} - 2t\mathbf{j}) \times (2t\mathbf{i} - 2\mathbf{j})] + (2)[(3t\mathbf{i} + \mathbf{j}) \times (3)\mathbf{i}]$$

that gives

$$\mathbf{L} = ((2t^2 - 6)\mathbf{k})\ \mathrm{kg.m^2/s}$$

The total kinetic energy of the system relative to O is

$$K = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2 = \frac{1}{2}(m_1 v_1^2 + m_2 v_2^2) = \frac{1}{2}[(1)(4t^2 + 4) + (2)(9)] = (2t^2 + 11)\ \mathrm{J}$$

The net external torque about the origin is

$$\sum \boldsymbol{\tau}_{ext} = \frac{d\mathbf{L}}{dt} = ((4t)\mathbf{k})\ \mathrm{N.m}$$

(b) To find the total angular momentum relative to the center of mass let's find first the total angular momentum of the center of mass relative to the origin

$$\mathbf{r}_{cm} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} = \frac{(1)(t^2\mathbf{i} - 2t\mathbf{j}) + (2)(3t\mathbf{i} + \mathbf{j})}{(3)}$$

$$= \left(\left(\frac{t^2}{3} + 2t\right)\mathbf{i} + \left(\frac{2}{3} - \frac{2}{3}t\right)\mathbf{j}\right) \text{m}$$

The velocity of the center of mass is

$$\mathbf{v}_{cm} = \left(\left(\frac{2}{3}t + 2\right)\mathbf{i} - \left(\frac{2}{3}\right)\mathbf{j}\right) \text{m/s}$$

and the total angular momentum of the center of mass relative to O is

$$\mathbf{L}_{cm} = M(\mathbf{r}_{cm} \times \mathbf{v}_{cm}) = (3)\left[\left(\left(\frac{t^2}{3} + 2t\right)\mathbf{i} + \left(\frac{2}{3} - \frac{2}{3}t\right)\mathbf{j}\right) \times \left(\left(\frac{2}{3}t + 2\right)\mathbf{i} - \left(\frac{2}{3}\right)\mathbf{j}\right)\right]$$

$$= \left(-\left(\frac{2}{3}t^2 + \frac{4}{3}t + 4\right)\mathbf{k}\right) \text{kg.m}^2/\text{s}$$

From Eq. 6.15, the total angular momentum relative to the center of mass is

$$\mathbf{L}' = \sum_i m_i(\mathbf{r}'_i \times \mathbf{v}'_i) = \sum_i m_i(\mathbf{r}_i \times \mathbf{v}_i) - M(\mathbf{r}_{cm} \times \mathbf{v}_{cm})$$

$$= (2t^2 - 6)\mathbf{k} + \left(\frac{2t^2}{3} + \frac{4}{3}t + 4\right)\mathbf{k} = \left(\left(\frac{8}{3}t^2 + \frac{4}{3}t - 2\right)\mathbf{k}\right) \text{kg.m}^2/\text{s}$$

The net external torque about the center of mass is

$$\boldsymbol{\tau}' = \frac{d\mathbf{L}'}{dt} = \left(\left(\frac{16}{3}t + \frac{4}{3}\right)\mathbf{k}\right) \text{N.m}$$

The total kinetic energy of the system relative to the center of mass is

$$K' = \frac{1}{2}\sum_i m_i v_i'^2 = \sum_i m_i v_i^2 - \frac{1}{2}Mv_{cm}^2$$

$$= (2t^2 + 11) - \frac{1}{2}(3)\left[\left(\frac{2}{3}t + 2\right)^2 + \frac{4}{9}\right] = \left(\frac{4t^2}{3} - 2t - \frac{13}{3}\right) \text{J}$$

Example 6.12 Two particles of equal mass m are rotating about their center of mass with a constant speed v as in Fig. 6.13. If they are separated by a distance $2d$, find the total angular momentum of the system.

Solution 6.12

$$L = mvd + mvd = 2mvd$$

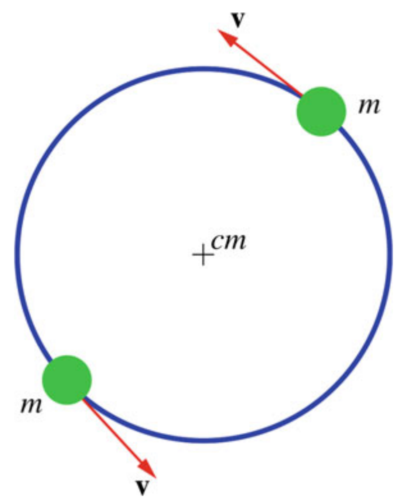


Fig. 6.13 Two particles rotating about their center of mass

6.4.5 Collisions and the Center of Mass Frame of Reference

In problems involving collisions, it is useful to use an inertial frame of reference that is attached to the center of mass to analyze the collision. This method is most commonly used in analyzing collisions between subatomic particles or atoms. In section (6.4.1), we proved that the total linear momentum of a system when observed from the center of mass frame is equal to zero.

$$\mathbf{p}'_i = \mathbf{p}'_f = \mathbf{0} \tag{6.16}$$

Now consider a system consisting of two bodies undergoing a one-dimensional collision (see Fig. 6.14). Then from Eq. 6.16 we have

$$p'_{1i} = -p'_{2i}$$

and

$$p'_{1f} = -p'_{2f}$$

That is, when viewed from the center of mass frame the two objects approach each other with equal and opposite momenta and move away from each other with an equal and opposite momenta. Therefore, the center of mass frame simplifies the analysis since it exhibits a particular symmetry to the problem (see Fig. 6.15).

Example 6.13 A rocket is projected vertically upward and explodes into three fragments of equal mass when it reaches the top of its flight at an altitude of 40 m (see Fig. 6.16). If the two fragments land to the ground after 3 s from the explosion, find the time it takes the third fragment to hit the ground.

Solution 6.13 When the rocket reaches the top its velocity immediately before explosion is zero. Since $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are

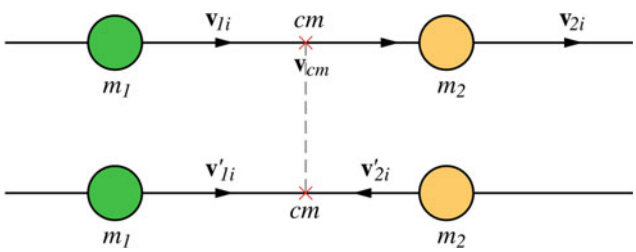


Fig. 6.14 Consider a system consisting of two bodies undergoing a one-dimensional collision

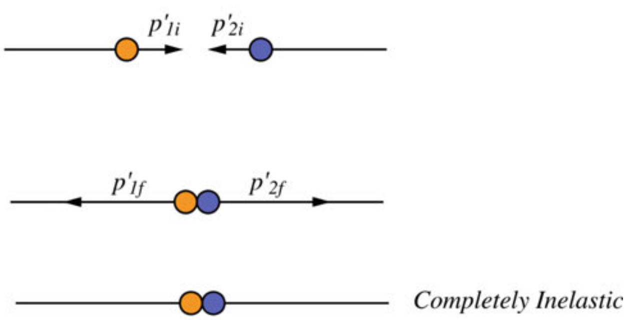
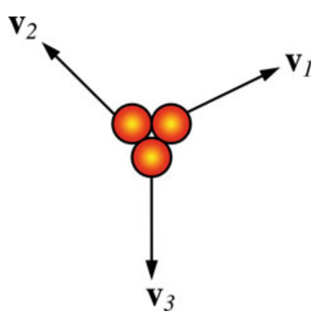


Fig. 6.15 The center of mass frame analysis of a collision

Fig. 6.16 A rocket is projected vertically upward and explodes into three fragments of equal mass when it reaches the top of its flight at an altitude of 40 m



the velocities of the fragments immediately after explosion, we have from the conservation of momentum

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3 = \mathbf{0}$$

Since $m_1 = m_2 = m_3$, then $v_1 + v_2 + v_3 = 0$. The first and second fragments land at the same time t' and hence they have the same vertical velocity initially which is equal to $-v_3/2$. Therefore

$$h = v_3 t + \frac{gt^2}{2}$$

and

$$h = \frac{-v_3 t'}{2} + \frac{gt'2}{2}$$

That gives

$$v_3 = \frac{g(t'^2 - t^2)}{2t + t'}$$

and

$$h = \frac{gtt'(t + 2t')}{2(2t + t')}$$

Substituting the values of h and t' gives

$$29.4t^2 + 160t + 63.6 = 0$$

Thus, $t = 2.3$ s.

Example 6.14 Find the center of mass of the Earth–Moon System and describe its motion around the sun.

Solution 6.14 As we shall see in Chap. 9, the center of mass of two bodies with different masses moving under gravity will trace an ellipse. Since the external forces on the sun can be neglected, we may consider it to be at rest in an inertial frame of reference and at the origin of a coordinate system (see Fig. 6.17). The center of mass of the Earth–Moon system is

$$\mathbf{r}_{cm} = \frac{M_E\mathbf{r}_E + M_M\mathbf{r}_M}{M_E + M_M}$$

where $\hat{\mathbf{r}}_E$ and $\hat{\mathbf{r}}_M$ are unit vectors in the direction of $\mathbf{r}_E$ and $\mathbf{r}_M$ respectively. The equation of motion of the center of mass is

$$\mathbf{F} = (M_E + M_M)\ddot{\mathbf{r}}_{cm}$$

The gravitational force on the Earth–Moon system exerted by the sun is

$$\mathbf{F} = -GM_S\left(\frac{M_E}{r_E^2}\hat{\mathbf{r}}_E + \frac{M_M}{r_M^2}\hat{\mathbf{r}}_M\right)$$

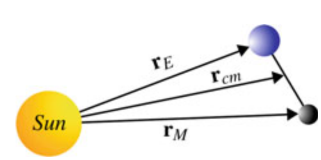
Since the distance between the earth and the moon is so small compared to their distance from the sun we may write $r_E \approx r_M \approx r_{cm}$

$$\mathbf{F} = -\frac{GM_S}{r_{cm}^2}(M_E + M_M)\hat{\mathbf{r}}_{cm} = (M_E + M_M)\ddot{\mathbf{r}}_{cm}$$

Hence, the center of mass of the Earth–Moon system moves as a single planet of mass $(M_E + M_M)$ about the sun as shown in Fig. 6.18.

Example 6.15 Describe the motion of a rocket in space using the law of conservation of momentum.

Fig. 6.17 The center of mass of the Earth-Moon system



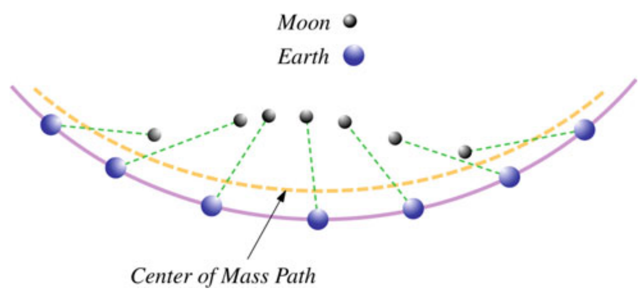


Fig. 6.18 The center of mass of the Earth-Moon system moves as a single planet of mass $(M_E + M_M)$ about the sun

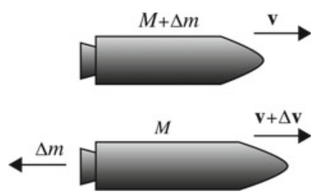


Fig. 6.19 A rocket moving in space is a system with varying mass. Its motion is analyzed using the law of conservation of momentum

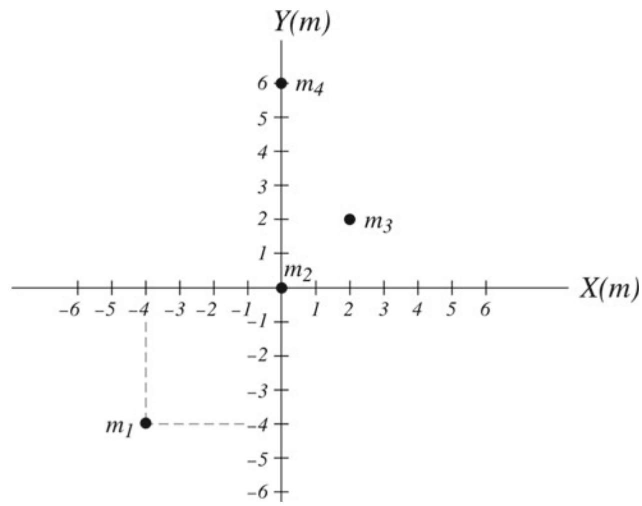


Fig. 6.20 A system of particles in x-y plane

Solution 6.15 A rocket moving in space is a system with varying mass. Its motion is analyzed using the law of conservation of momentum. In order for a rocket to move in space, its fuel is burned and gases are produced and ejected from its rear. This will cause the mass of the rocket to decrease continuously The ejected gases produce momentum in the backward direction and as a result the rocket receives a forward momentum and its velocity increases (see Fig. 6.19). Suppose at an instant t, the rocket has a mass M and velocity v relative to a stationary frame of reference. During a time interval t, a mass $\triangle m$ of the fuel is expelled as gas with a velocity u relative to the rocket. The speed of the rocket increases to $v+\triangle v$ and the speed of the fuel relative to the stationary frame of reference is $v - u$. The initial momentum of the rocket is

$$\mathbf{p}(t) = (M + \triangle m)\mathbf{v}$$

and the final momentum is

$$\mathbf{p}(t + \triangle t) = M(\mathbf{v} + \triangle \mathbf{v}) + \triangle m\ (\mathbf{v} - \mathbf{u})$$

The change in the momentum is

$$\triangle \mathbf{p}(t + \triangle t) = \mathbf{p}(t + \triangle t) - \mathbf{p}(t) = M\triangle \mathbf{v} - (\triangle m)\mathbf{u}$$

Therefore, the force acting on the rocket is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \lim_{\triangle t \to 0} \frac{\triangle \mathbf{p}}{\triangle t} = M\frac{d\mathbf{v}}{dt} - \mathbf{u}\frac{dm}{dt}$$

Since the increase in the exhaust mass produce an equal decrease in the rocket mass, we have

$$dm = -dM$$

Thus

$$\mathbf{F} = M\frac{d\mathbf{v}}{dt} + \mathbf{u}\frac{dM}{dt}$$

If no external forces act on the rocket we have $\mathbf{F} = \mathbf{0}$ and

$$M\frac{d\mathbf{v}}{dt} = -\mathbf{u}\frac{dM}{dt}$$

hence

$$\int_{t_0}^{t} \frac{d\mathbf{v}}{dt}dt = -\mathbf{u}\int_{M_0}^{M} \frac{1}{M}\frac{dM}{dt}dt = -\mathbf{u}\int_{M_0}^{M} \frac{dM}{M}$$

That gives

$$\mathbf{v} - \mathbf{v}_0 = \mathbf{u}\ln\left(\frac{M_0}{M}\right)$$

Therefore, the final speed of the rocket depends on the exhaust speed and on the ratio of the initial and final masses.

Problems

1. Find the coordinate of the center of mass of the system shown in Fig. 6.20.
2. Find the center of mass of a uniform plate bounded by $y = -0.24x^2 + 6$ and the x-axis from $x = -5$ to $x = 5$ m.
3. Find the center of mass of the homogeneous sheet shown in Fig. 6.21.
4. Find the center of mass of the homogeneous sheet shown in Fig. 6.22.
5. Find the center of mass of a uniform solid circular cone of radius a and height h.
6. Find the center of mass of a uniform solid hemisphere of radius R.

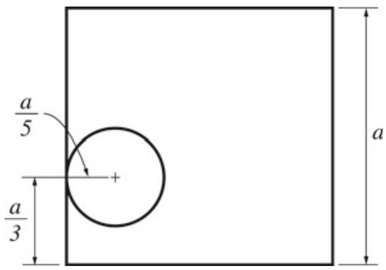


Fig. 6.21 A homogenous sheet with a hole

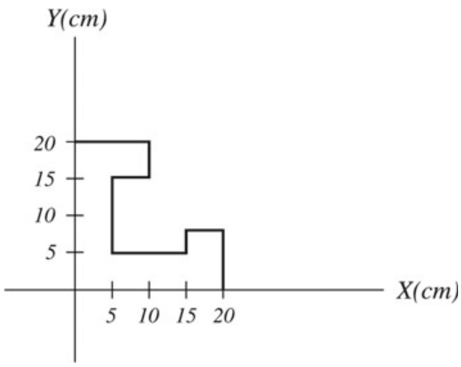


Fig. 6.22 A homogenous sheet in the x-y plane

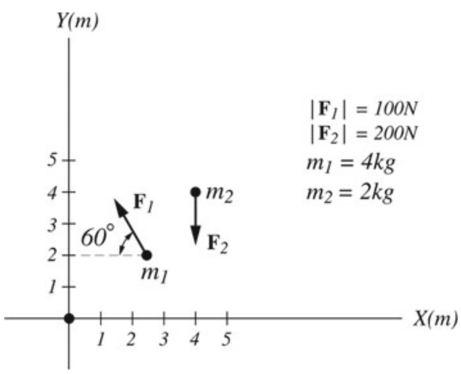


Fig. 6.23 The acceleration of the center of mass of two masses acted upon by different forces

7. Two masses initially at rest are located at the points shown in Fig. 6.23. If external forces act on the particles as in Fig. 6.23, find the acceleration of the center of mass.

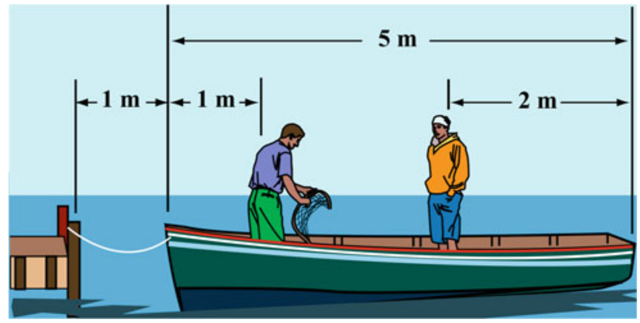


Fig. 6.24 By neglecting friction between the boat and water, the center of mass can be used to find the distance moved by the boat

8. A projectile of mass 15 kg is fired from the ground with an initial velocity of 12 m/s at an angle of 45° to the horizontal. 1 second later, the projectile explodes into two fragments A and B. If immediately after explosion, fragment A has a mass of 5 kg and a speed of 5 m/s at an angle of 30° to the horizontal, find the velocity of fragment B (assuming air resistance is neglected).
9. Two boys of masses 45 and 40 kg are standing on a boat of mass 150 kg and length 5 m as in Fig. 6.24. The boat is initially lm from the pier. Assuming that there is no friction between the boat and the water, find the distance moved by the boat when the two meet at the middle of the boat.
10. Two particles of masses $m_1 = 3$ kg and $m_2 = 5$ kg are moving relative to the lab frame with velocities of 10 m/s along the y-axis and 15 m/s at an angle of 30° to the x-axis. Find (a) the velocity of their center of mass (b) the momentum of each particle in the center of mass frame (c) the total kinetic energy of the particles relative to the lab frame and relative to the center of mass frame.
11. Two particles of masses $m_1 = 1$ kg and $m_2 = 2$ kg are moving relative to the lab frame with velocities of $\mathbf{v}_1 = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v}_2 = 7\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. If at a certain instant they are located at $(-1, 1, 2)$ and $(3, 0, 1)$, find the angular momentum of the system relative to the origin and relative to the center of mass.

7 Rotation of Rigid Bodies

7.1 Rotational Motion

Rotational motion exists everywhere in the universe. The motion of electrons about an atom and the motion of the moon about the earth are examples of rotational motion. Objects cannot be treated as particles when exhibiting rotational motion since different parts of the object move with different velocities and accelerations. Therefore, it is necessary to treat the object as a system of particles.

7.2 The Plane Motion of a Rigid Body

When all parts of a rigid body move parallel to a fixed plane, then the motion of the object is referred to as plane motion. There are two types of plane motion, which are given as follows:

1. The pure rotational motion: The rigid body in such a motion rotates about a fixed axis that is perpendicular to a fixed plane. In other words, the axis is fixed and does not move or change its direction relative to an inertial frame of reference.
2. The general plane motion: The motion here can be considered as a combination of pure translational motion parallel to a fixed plane in addition to a pure rotational motion about an axis that is perpendicular to that plane. This chapter discusses the kinematics and dynamics of pure rotational motion.

7.2.1 The Rotational Variables

Suppose a rigid body of an arbitrary shape is in pure rotational motion about the z-axis (see Fig. 7.1). Let us analyze the motion of a particle that lies in a slice of the body in the x-y plane as in Fig. 7.2. This particle (at point P) will rotate in a circle of fixed radius r which represents the perpendicular distance from P to the axis of rotation. If you look at any other particle in the object you will see that every particle will rotate in its own circle that has the axis of rotation at its center. In other words, different particles move in different circles but the center of all of these circles lies on the rotational axis. Suppose the particle moves through an arc length s starting at the positive x-axis. Its angular position is then given by

$$\theta = \frac{s}{r}$$

r and θ are the polar coordinates of a point in a plane (which was mentioned in Sect. 2.6) where θ is always measured from the positive x-axis. Because θ is the ratio of the arc length to the radius, it is a pure (dimensionless) number. The unit usually used to measure θ is the radians (rad). One radian is defined as the angle subtended by an arc of length that is equal to the radius of the circle. Since one rotation (360°) corresponds to $\theta = 2\pi r/r = 2\pi$ rad, it follows that:

$$1 \text{ rev} = 360^\circ = 2\pi \text{ rad}$$

$$1 \text{ rad} = 57.3^\circ = 0.159 \text{ rev}$$

Note that if the particle completes one revolution, θ will not become zero again, it is then equal to 2π rad. Thus for example for three revolutions the angular position is given by

$$\theta = (2\pi + 2\pi + 2\pi) \text{ rad} = 6\pi \text{ rad}$$

Suppose that the particle in Fig. 7.2 is at point P_1 at t_1 and at point P_2 at t_2 where it changes its angular position from θ_1 to θ_2 (see Fig. 7.3). Its angular displacement is then given by

$$\triangle\theta = \theta_2 - \theta_1$$

$\triangle\theta$ is positive for counterclockwise rotations (increasing θ) and negative for clockwise rotations (decreasing θ). If the particle undergoes this angular displacement during a time interval $\triangle t$, the average angular velocity $\overline{\omega}$ is then defined as

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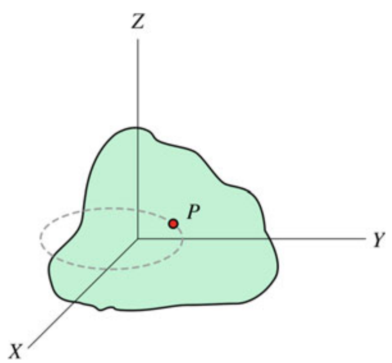


Fig. 7.1 A rigid body of an arbitrary shape is in pure rotational motion about the z-axis

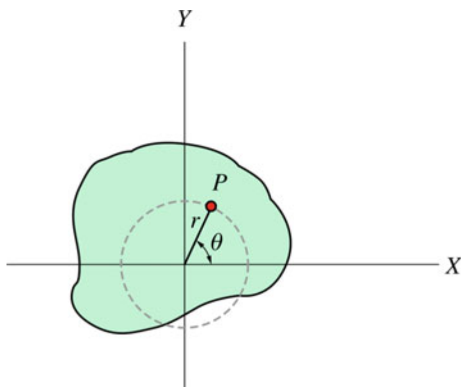


Fig. 7.2 The motion of a particle that lies in a slice of the body in the x-y plane

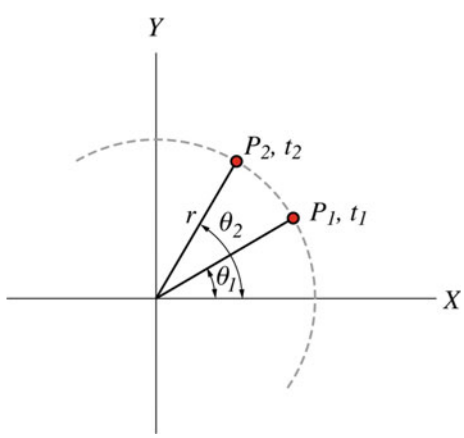


Fig. 7.3 The particle is at point P_1 at t_1 and at P_2 at t_2, where it changes its angular position from θ_1 to θ_2

$$\overline{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

The instantaneous angular velocity is

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

ω has units of rad/s or s^{-1}. The average angular acceleration is defined as

$$\overline{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

The instantaneous angular acceleration is

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

where α is in rad/s^2 or s^{-2}. Note that ω is positive for increasing θ and negative for decreasing θ, while α is positive for increasing ω and negative for decreasing ω. When a rigid body is in pure rotational motion, all particles in the body rotate through the same angle during the same time interval. Thus, all particles have the same angular velocity and the same angular acceleration. Therefore, ω and α describes the motion of the whole body In the case of pure rotational motion, the direction of ω is along the axis of rotation (also see Sect. 7.4), it can be determined by the right-hand rule or of advance of a right-handed screw as in Fig. 7.4. The direction of α is in the same direction of ω if ω is increasing or in the opposite direction if ω is decreasing.

The quantities θ, ω and α in pure rotational motion are the rotational analog of x, v and a in translational one-dimensional motion. The vectors ω and α are not used in the case of pure rotational motion, they are used in the general rotational motion when the axis of rotation changes its direction with time. Note that only the infinitesimal angular displacement $d\theta$ can be represented by a vector but not the finite angular displacement $\Delta\theta$. This is because the finite angular displacement $\Delta\theta$ does not obey the commutative law of vector addition (see Fig. 7.5) and therefore cannot be represented by a vector. Hence, the instantaneous angular velocity and acceleration (ω and α) can be represented by vectors but not their average values ($\overline{\omega}$ and $\overline{\alpha}$).

Example 7.1 Convert each of the following into the other angular units: 15°, 0.25 rev/s^2, 3 rad/s.

Solution 7.1

$$15^{o} = (15\ \text{deg})\left(\frac{1\ \text{rev}}{360\ \text{deg}}\right) = 0.042\ \text{rev}$$

$$15^{o} = (15\ \text{deg})\left(\frac{2\ \pi\,\text{rad}}{360\ \text{deg}}\right) = 0.26\ \text{rad}$$

$$0.25\ \text{rev/s}^2 = \left(0.25\ \frac{\text{rev}}{\text{s}^2}\right)\left(\frac{2\pi\ \text{rad}}{1\text{rev}}\right) = 1.57\ \text{rad/s}^2$$

$$0.25\ \text{rev/s}^2 = \left(0.25\ \frac{\text{rev}}{\text{s}^2}\right)\left(\frac{360\ \text{deg}}{1\ \text{rev}}\right) = 90\ \text{deg/s}^2$$

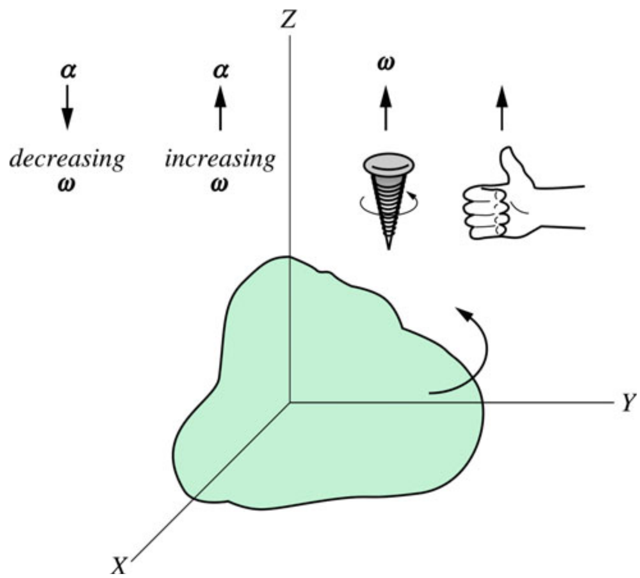


Fig. 7.4 The direction of ω is along the axis of rotation and can be determined by the right-hand rule or of advance of a right-handed screw

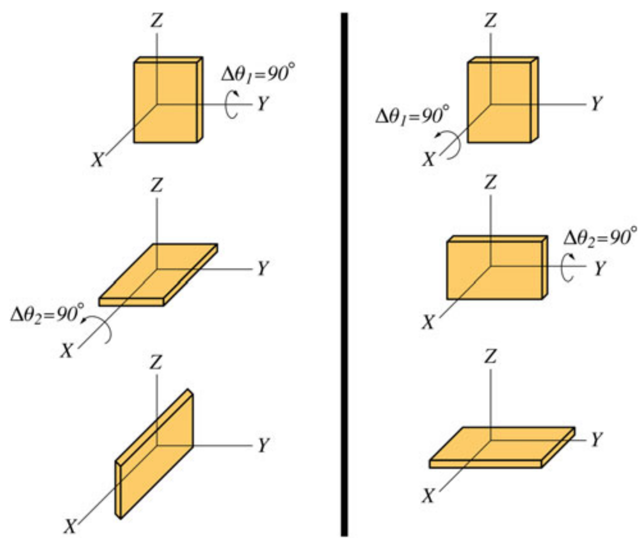


Fig. 7.5 Changing the order of addition will change the final result

$$3\ \text{rad/s} = \left(3\ \frac{\text{rad}}{\text{s}}\right)\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right) = 0.48\ \text{rev/s}$$

$$3\ \text{rad/s} = \left(3\ \frac{\text{rad}}{\text{s}}\right)\left(\frac{360^{\circ}\ \text{deg}}{2\pi\ \text{rad}}\right) = 172\ \text{deg/s}$$

Example 7.2 A rotating rigid object has an angular position given by $\theta(t) = ((0.3)t^2 + (0.4)t^3)$ rad. Determine: (a) the angular displacement of the object and the average angular velocity during the time interval from $t_1 = 1$s to $t_2 = 2$ s. (b) the instantaneous angular velocity and the instantaneous angular acceleration at $t = 5$ s.

Solution 7.2 (a)

$$\Delta\theta = \theta_2 - \theta_1$$

$$\theta_1 = ((0.3)(1\ \text{s})^2 + (0.4)(1\ \text{s})^3) = 0.7\ \text{rad}$$

and

$$\theta_2 = ((0.3)(2\ \text{s})^2 + (0.4)(2\ \text{s})^3) = 4.4\ \text{rad}$$

$$\Delta\theta = (4.4\ \text{rad}) - (0.7\ \text{rad}) = 3.7\ \text{rad}$$

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t} = \frac{(3.7\ \text{rad})}{(1\ \text{s})} = 3.7\ \text{rad/s}$$

(b)

$$\omega = \frac{d\theta}{dt} = ((0.6)t + (1.2)t^2)\ \text{rad/s}$$

at $t = 5$ s

$$\omega = (0.6)(5\ \text{s}) + (1.2)(5\ \text{s})^2 = 33\ \text{rad/s}$$

$$\alpha = \frac{d\omega}{dt} = ((0.6) + (2.4)t)\ \text{rad/s}^2$$

at $t = 5$s

$$\alpha = (0.6) + (2.4)(5\ \text{s}) = 12.6\ \text{rad/s}^2$$

Example 7.3 A wheel is rotating with an angular acceleration that is given by $\alpha = (9 - 2t)$ rad/s^2. (a) Find the angular velocity and displacement at any time if at $t = 0$ the wheel has an angular velocity of 2 rad/s and an (initial) angular displacement of 3 rad; (b) at what angular displacement will the wheel reach its maximum angular velocity

Solution 7.3 (a)

$$\omega = \int \alpha dt = \int (9 - 2t)dt = 9t - t^2 + c_1$$

Since at $t = 0$ $\omega = 2$ rad/s, we have $c_1 = 2$ rad/s and hence

$$\omega = (9t - t^2 + 2)\ \text{rad/s}$$

$$\theta = \int \omega dt = \int (9t - t^2 + 2)dt = \frac{9}{2}t^2 - \frac{1}{3}t^3 + 2t + c_2$$

Since at $t = 0$, $\theta = 3$ rad, then $c_2 = 3$ rad and

$$\theta = \left(\frac{9}{2}t^2 - \frac{1}{3}t^3 + 2t + 3\right)\ \text{rad}$$

(b) The maximum velocity is when $\alpha = d\omega/dt = 0$, or $9 - 2t = 0$, i.e. at $t = 4.5$ s The angular displacement at that time is

$$\theta = \frac{9}{2}(4.5\ \text{s})^2 - \frac{1}{3}(4.5\ \text{s})^3 + 2(4.5\ \text{s}) + 3 = 72.8\ \text{rad}$$

7.3 Rotational Motion with Constant Acceleration

A pure rotational motion with constant angular acceleration is the rotational analogue of the pure translational motion with constant acceleration. The corresponding kinematic equations of pure rotational motion can be obtained by using the same method that is used for obtaining the kinematic equations of pure translational motion. To show this, consider a rigid object rotating with a constant angular acceleration during a time interval from t_1 to t_2 through an angle from θ_1 to θ_2. Let $t_1 = 0, t_2 = t, \omega_1 = \omega_o, \omega_2 = \omega, \theta_1 = \theta_o$, and $\theta_2 = \theta$. Because the angular acceleration is constant it follows that the angular velocity changes linearly with time and the average angular velocity is given by

$$\overline{\omega} = \frac{\omega_0 + \omega}{2}$$

Since

$$\alpha = \overline{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\omega - \omega_0}{t}$$

we have

$$\omega = \omega_0 + \alpha t \tag{7.1}$$

Furthermore

$$\overline{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\theta - \theta_0}{t} = \frac{\omega_0 + \omega}{2}$$

Hence

$$\theta = \theta_0 + \frac{1}{2}(\omega_0 + \omega)t \tag{7.2}$$

Substituting Eq. 7.1 into Eq. 7.2 gives

$$\theta = \theta_0 + \frac{1}{2}(\omega_0 + \omega)t = \theta_0 + \frac{1}{2}(\omega_0 + \omega_0 + \alpha t)t$$

or

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \tag{7.3}$$

Finally solving for t from Eq. 7.1 and substituting into Eq. 7.2 gives

$$\theta = \theta_0 + \frac{1}{2}(\omega_0 + \omega)t = \theta_0 + \frac{1}{2}(\omega_0 + \omega)\left(\frac{\omega - \omega_0}{\alpha}\right)$$

or

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \tag{7.4}$$

Note that as mentioned earlier, if a rigid object is in pure rotational motion, all particles in the object have the same angular velocity and angular acceleration. Different particles move in different circles but the center of these circles lies at the axis of rotation. As the rigid body rotates, a particle in the body will move through a distance s along its circular path (see Fig. 7.6). The angular displacement of the particle is related to s by

$$s = r\theta$$

where r is the radius of the circle in which the particle is moving along. Differentiating the above equation with respect to t gives

$$\frac{ds}{dt} = r\frac{d\theta}{dt}$$

Since ds/dt is the magnitude of the linear velocity of the particle and $d\theta/dt$ is the angular velocity of the body we may write

$$v = r\omega \tag{7.5}$$

Therefore, the farther the particle is from the rotational axis the greater its linear speed. The direction of the linear speed of the particles is always tangent to the path (as mentioned in Sect. 2.2.3). In Sect. 2.4.6 we have seen that a particle in nonuniform circular motion has both tangential and radial components of acceleration. The radial component is due to the change in the direction of the velocity and is given by

$$a_r = \frac{v^2}{r} \tag{7.6}$$

Substituting Eq. 7.5 into Eq. 7.6 gives

$$a_r = \frac{v^2}{r} = r\omega^2$$

The tangential component of the acceleration is due to the change in the magnitude of the velocity and it is given by

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt}$$

or

$$a_t = r\alpha$$

The total linear acceleration of the particle (see Fig. 7.7) is given by

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r$$

It's magnitude is given by

$$a = \sqrt{a_t{}^2 + a_r{}^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}$$

Table. 7.1 shows the linear/rotational analogous equations.

Example 7.4 A disc of radius of 10 cm rotates from rest with a constant angular acceleration. If it requires 2 s for it to rotate through an angular displacement of 60°: (a) find the angular

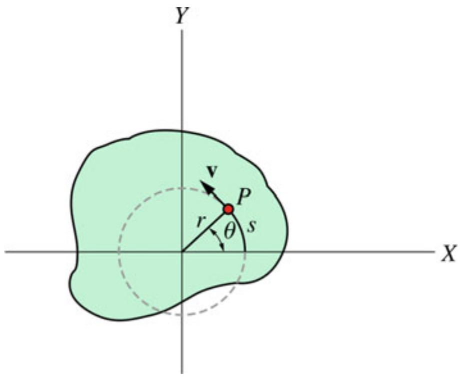


Fig. 7.6 As the rigid body rotates, a particle in the body will move through a distance s along its circular path

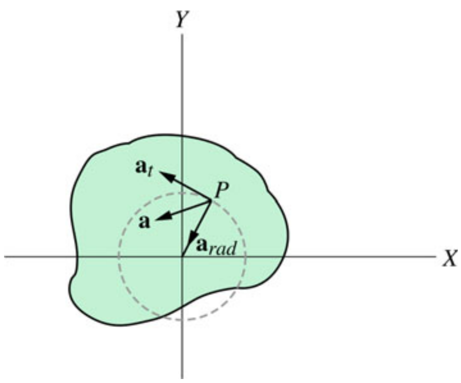


Fig. 7.7 The total acceleration of the particle

Table 7.1 Kinematic equations

Rotational motion about a fixed axis with constant α	Linear motion with constant a
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\theta = \theta_0 + \frac{1}{2}(\omega + \omega_0)t$	$x = x_0 + \frac{1}{2}(v + v)t$
$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	$x = x_0 + v_0 t + \frac{1}{2}at^2$
$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$	$v^2 = v_0^2 + 2a(x - x_0)$

acceleration of the disc; (b) its angular velocity at $t = 2$s and at $t = 6$s, (c) the linear speed at $t = 2$s of a point that is at a distance of 7 cm from the center of the disc; (d) the distance that this point has moved during that time interval.

Solution 7.4 (a) We have $\omega_0 = 0$ and $\theta = (60\ \text{deg})(2\pi \text{rad}/360\ \text{deg}) = 1.05$ rad. By choosing the reference position $\theta_0 = 0$ we have

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\alpha = \frac{2\theta}{t^2} = \frac{2(1.05\ \text{rad})}{(2\ \text{s})^2} = 0.525\ \text{rad/s}^2$$

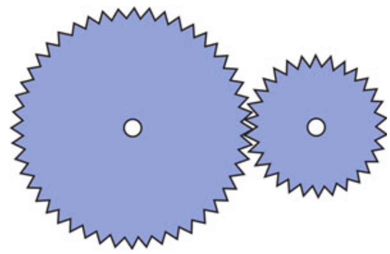


Fig. 7.8 Two sprockets connected at the rim

(b)

$$\omega = \omega_0 + \alpha t = (0.525\ \text{rad/s}^2)(2\ \text{s}) = 1.05\ \text{rad/s}$$

at $t = 6$ s

$$\omega = (0.525\ \text{rad/s}^2)(6\text{s}) = 3.15\ \text{rad/s}$$

(c)

$$v = r\omega = (0.07\ \text{m})(1.05\ \text{rad/s}) = 0.074\ \text{m/s}$$

(d)

$$s = r\theta = (0.07\ \text{m})(1.05\ \text{rad}) = 0.074\ \text{m}$$

Example 7.5 Two sprockets are attached to each other as in Fig. 7.8. There radii are $r_1 = 2$ cm and $r_2 = 5$ cm. If the angular velocity of the smaller sprocket is 2 rad/s, find the angular velocity of the other.

Solution 7.5 A point at the rim of one sprocket has the same linear speed as a point at the rim of the other sprocket since they are attached to each other, i.e.,

$$r_1\omega_1 = r_2\omega_2 = v$$

hence

$$\omega_2 = \frac{r_1}{r_2}\omega_1 = \frac{(2\ \text{cm})}{(5\ \text{cm})}(2\ \text{rad/s}) = 0.8\ \text{rad/s}$$

Example 7.6 Find the angular speed of the moon in its orbit about the earth in rev/day.

Solution 7.6 Assuming that the moon's orbit is circular, the linear speed of the moon is given by $v = 2\pi r/T$, where r is the mean distance from the earth to the moon and T is its period. Thus, the angular velocity of the moon is

$$\omega = rv = \frac{2\pi}{T} = \frac{2(3.14)}{(27.3\ \text{day})} = 0.23\ \text{rad/day}$$

or

$$\omega = \left(0.23\ \frac{\text{rad}}{\text{day}}\right)\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right) = 0.037\ \text{rev/day}$$

7.4 Vector Relationship Between Angular and Linear Variables

Consider a rigid body in pure rotational motion about a fixed axis (for example the z-axis). For any particle in the object, its linear velocity is given by

$$v = r\omega = R\sin\theta\omega$$

where R is the position vector of the particle from the origin (see Fig. 7.9) and θ is the angle between the position vector and the z-axis. As shown in Fig. 7.9, the direction of y is perpendicular to the plane formed by ω and R where it can be verified using the right-hand rule. Therefore, by using the definition of vector product we may write

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{R} \tag{7.7}$$

The total linear acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{R})$$

From Sect. 1.9.1 ($d/dt(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times d\mathbf{B}/dt + d\mathbf{A}/dt \times \mathbf{B}$) we have

$$\mathbf{a} = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{R} + \boldsymbol{\omega} \times \frac{d\mathbf{R}}{dt}$$

$$= \boldsymbol{\alpha} \times \mathbf{R} + \boldsymbol{\omega} \times \mathbf{v}$$

$$|\boldsymbol{\alpha} \times \mathbf{R}| = \alpha R \sin\theta = r\alpha = a_t$$

Furthermore, the direction of $\boldsymbol{\alpha} \times \mathbf{R}$ is tangent to the circular path of the particle at any instant (see Fig. 7.9). Thus the quantity $\boldsymbol{\alpha} \times \mathbf{R}$ is just the tangential component of the total acceleration

$$\mathbf{a_t} = \boldsymbol{\alpha} \times \mathbf{R} \tag{7.8}$$

In addition

$$|\boldsymbol{\omega} \times \mathbf{v}| = \omega v \sin 90^{o} = \omega v = r\omega^2 = a_r$$

The direction of $\boldsymbol{\omega} \times \mathbf{v}$ is along the direction of r (radial direction). Hence, the quantity $\boldsymbol{\omega} \times \mathbf{v}$ is the radial component of the total acceleration

$$\mathbf{a}_r = \boldsymbol{\omega} \times \mathbf{v} \tag{7.9}$$

Equations 7.7–7.9 are the vector relationship between angular and linear quantities.

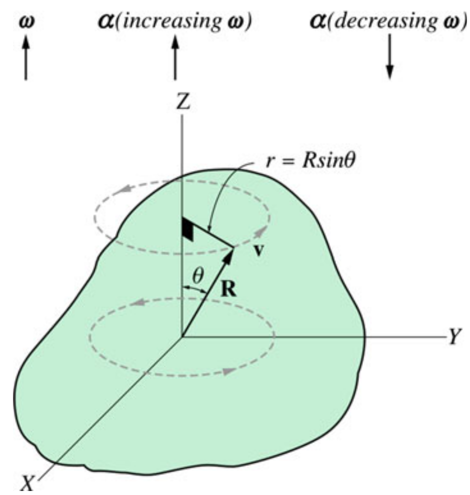


Fig. 7.9 A rigid body in pure rotational motion about a fixed axis (here the z-axis)

7.5 Rotational Energy

In Chap. 6 we have seen that the kinetic energy of a discrete system of particles is $K = \frac{1}{2}\sum_i m_i v_i^2$ where m_i and v_i are the mass and linear velocity of the ith particle respectively (see Fig. 7.10). From Eq. 7.5, we have

$$v_i = r_i\omega$$

where r_i is the perpendicular distance from the particle to the axis of rotation. Therefore the total kinetic energy of the system is

$$K_R = \frac{1}{2}\sum_i (m_i r_i^2)\omega^2$$

The quantity between brackets is known as the moment of inertia of the system

$$I = \sum_i m_i r_i^2$$

This quantity shows how the mass of the system is distributed about the axis of rotation. Thus, to find the rotational inertia, the axis of rotation must be specified. If the rotational axis changes its position or direction, I changes as well. The SI unit of the moment of inertia is kg m^2. The rotational kinetic energy can thus be written as

$$K_R = \frac{1}{2}I\omega^2$$

This quantity is the rotational analogue of the kinetic energy in translational motion. Note that this energy is not a new kind of energy; it is just the sum of the translational kinetic energies

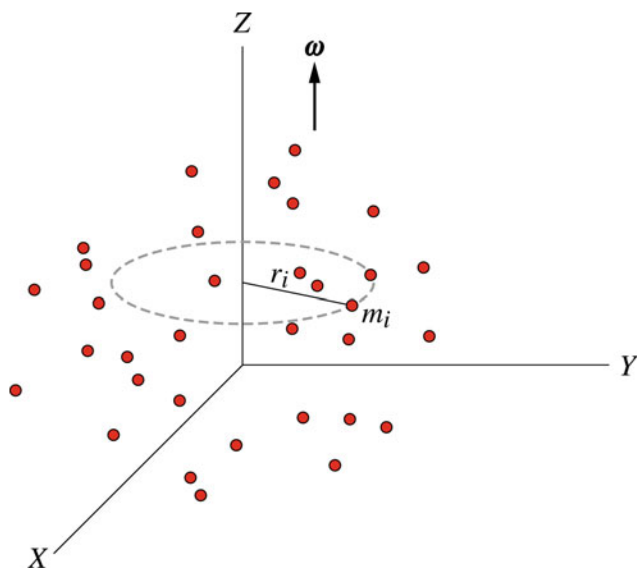


Fig. 7.10 A system of particles rotating about the z-axis

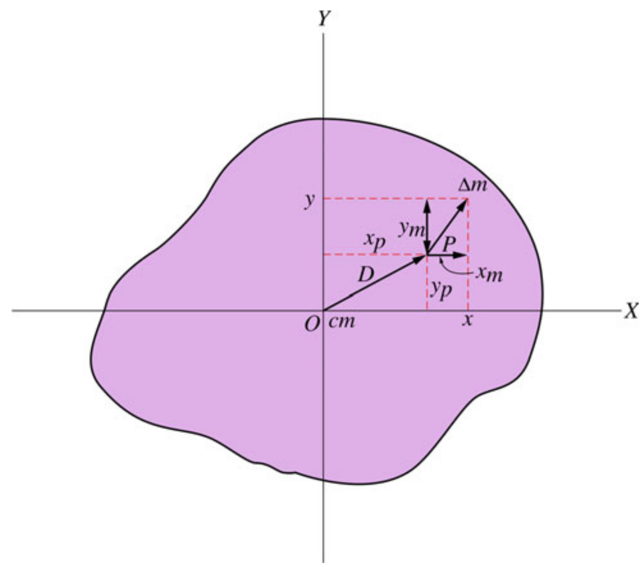


Fig. 7.11 The Parallel-axis Theorem

of the particles. For a rigid body which is a continuous system of particles, the sum is replaced by an integral

$$I = \lim_{\Delta m_i \to 0} \sum_i m_i r_i^2 = \int r^2 dm$$

In solving problems ρ, σ, and λ (see Sect. 6.3.4) are often used to express dm in terms of its position coordinates.

7.6 The Parallel-Axis Theorem

The parallel-axis theorem states that the moment of inertia I of a system about any axis that is parallel to an axis passing through the center of mass is

$$I = I_{cm} + MD^2$$

where I_{cm} is the moment of inertia about an axis passing through the center of mass, M is the total mass of the system, and D is the perpendicular distance between the two parallel axes.

Proof Consider an axis that is perpendicular to the page and passing through the center of mass of the object. Figure 7.11 shows a thin slice of the object that lies in the x-y plane. Because the origin is taken at the center of mass we have

$$z_{cm} = x_{cm} = y_{cm} = 0$$

The moment of inertia of the object about the center of mass axis is

$$I_{cm} = \int r^2 dm = \int (x^2 + y^2) dm$$

where x and y are the coordinates of the mass element dm from the center of mass (the origin). Now consider another axis that is parallel to the first axis and that passes through a point P as shown in Fig. 7.11. Suppose that the x and y coordinates of P from the center of mass are x_p and y_p. The moment of inertia about an axis passing through P is

$$I_P = \int [(x - x_P)^2 + (y - y_P)^2] dm$$

where $(x - x_P)$ and $(y - y_P)$ are coordinates of dm from point P Expanding this equation gives

$$I_P = \int (x^2 + y^2) dm - 2x_P \int xdm - 2y_P \int ydm + \int (x_P^2 + y_P^2) dm$$

Since $x_{cm} = y_{cm} = 0$ and since

$$x_{cm} = \frac{1}{M} \int xdm$$

and

$$y_{cm} = \frac{1}{M} \int ydm$$

it follows that the second and third terms are zero. Thus

$$I_P = I_{cm} + D^2 \int dm$$

where

$$D = \sqrt{(x_P^2 + y_P^2)}$$

is the perpendicular distance between the two parallel axes. Hence

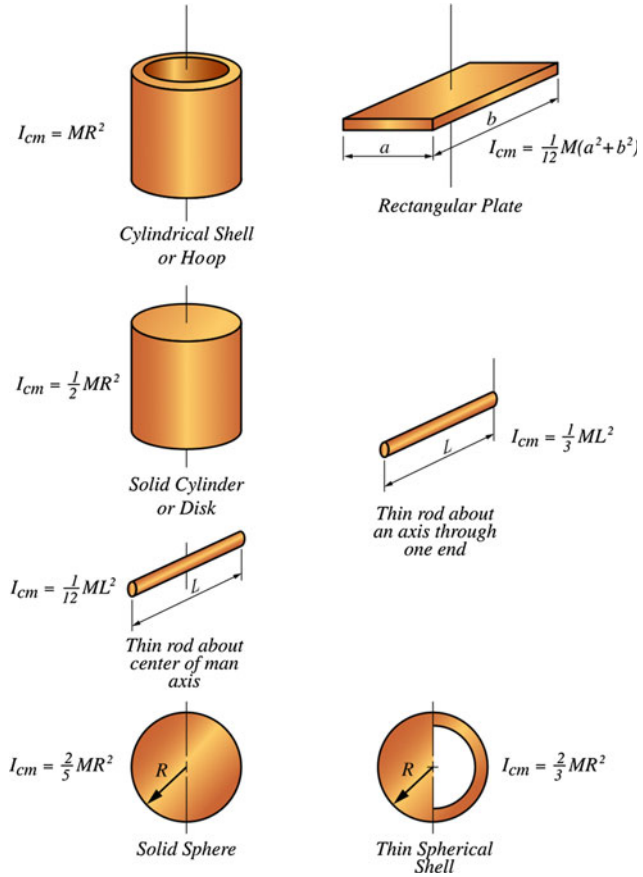


Fig. 7.12 The rotational inertia of various rigid bodies of uniform density

$$I_P = I_{cm} + MD^2 \quad \text{(Parallel–Axis Theorem)}$$

Special Moment of Inertia Fig. 7.12 gives the rotational inertia of various rigid bodies of uniform density.

7.7 Angular Momentum of a Rigid Body Rotating about a Fixed Axis

Consider a rigid body rotating about a fixed axis (the z-axis) with an angular speed ω as shown in Fig. 7.13. The angular momentum of the ith particle with respect to the origin is given by

$$\mathbf{L}_i = \mathbf{R}_i \times \mathbf{p}_i$$

Since the angle between $\mathbf{R}_i$ and $\mathbf{p}_i$ is 90, then $L_i = R_i p_i$. As seen from Fig. 7.13, $\mathbf{L}_i$ is not parallel to $\boldsymbol{\omega}$. $\mathbf{L}_i$ can be analyzed to two components, a component parallel to $\boldsymbol{\omega}$ written ($\mathbf{L}_{iz}$) and a component perpendicular to $\boldsymbol{\omega}$, ($\mathbf{L}_{i\perp}$). The magnitude of $\mathbf{L}_{iz}$ is given by

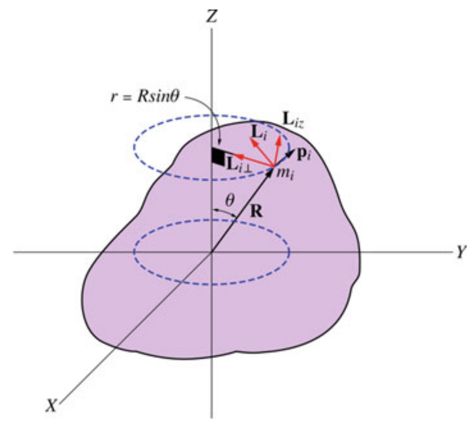


Fig. 7.13 A rigid body rotating about a fixed axis (the z-axis) with an angular speed ω

$$L_{iz} = L_i \sin\theta = R_i p_i \sin\theta = R_i (m_i v_i) \sin\theta$$

$$= R_i m_i (r_i \omega) \sin\theta = m_i r_i^2 \omega$$

where r_i is the radius of the circle in which the particle is moving along and $R_i = r_i \sin\theta$. Therefore, the total angular momentum of the rigid body along the z-direction is

$$L_z = \sum_i m_i r_i^2 \omega = \left(\sum_i m_i r_i^2\right)\omega$$

$$L_z = I\omega$$

where I is the moment of inertia of the rigid body about the rotational axis (z-axis). This equation can also be written in component form since $\mathbf{L}_z$ is parallel to $\boldsymbol{\omega}$, that is,

$$\mathbf{L}_z = I\boldsymbol{\omega} \tag{7.10}$$

Therefore, if a rigid body is rotating about a fixed axis (say the z-axis), the component of the angular momentum along that axis is given by Eq. 7.10. Now suppose that the rigid body is symmetric and homogeneous and that it is rotating about its symmetrical axis (see Fig. 7.14). For any two particles (1 and 2) opposing each other with an equal angular momenta $\mathbf{L}_1$ and $\mathbf{L}_2$, the perpendicular components, $\mathbf{L}_{1\perp}$ and $\mathbf{L}_{2\perp}$, of the angular momenta cancel each other out since they are in opposite directions. That leaves the parallel components $\mathbf{L}_{1z}$ and $\mathbf{L}_{2z}$ which add up since they have the same direction. For all particles in the object the total angular momentum is, therefore, given by

$$\mathbf{L} = \sum_i \mathbf{L}_{iz} = \mathbf{L}_z = I\boldsymbol{\omega}$$

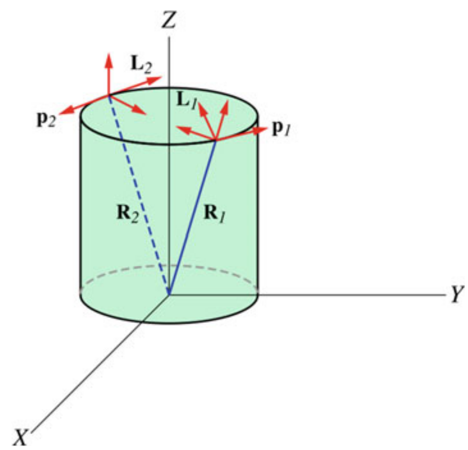


Fig. 7.14 A homogenous symmetrical rigid body rotating about its symmetrical axis

Hence, the total angular momentum of a symmetrical homogeneous body in pure rotation about its symmetrical axis is given by

$$\mathbf{L} = I\boldsymbol{\omega} \tag{7.11}$$

Note that Eq. 7.10 is valid for any rigid object in pure rotation where it only gives the component of the angular momentum that is parallel to the rotational axis. On the other hand, Eq. 7.11 is valid only for a symmetrical homogeneous rigid object rotating about its symmetrical axis, where the angular momentum in the equation is the total angular momentum and it is directed along the axis of rotation. The net external torque acing on the rigid object is equal to the rate of change of the total angular momentum of the object, i.e.,

$$\Sigma \boldsymbol{\tau}_{ext} = \frac{d\mathbf{L}}{dt}$$

In the case of any rigid object symmetrical or not, the net external torque acting on the object about the axis of rotation (say the z-axis) is equal to the rate of change of the component of angular momentum that is along that axis

$$\Sigma \boldsymbol{\tau}_{extz} = \frac{d\mathbf{L}_z}{dt} = \frac{d(I\boldsymbol{\omega})}{dt} = I\boldsymbol{\alpha}$$

However, if the object is symmetric and homogeneous in pure rotation about its symmetrical axis we may write

$$\Sigma \boldsymbol{\tau}_{ext} = \frac{d\mathbf{L}}{dt} = \frac{d(I\boldsymbol{\omega})}{dt} = I\boldsymbol{\alpha}$$

Example 7.7 A 5 kg wheel of radius of 0.1 m decelerates from an angular speed of 5 rad/s to rest after going through an angular displacement of 10 rev If a frictional force causes the wheel to decelerate, find the torque due to this force.

Solution 7.7 The angular displacement is

$$\Delta\theta = (10 \text{ rev})\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right) = 62.8 \text{ rad}$$

The angular acceleration of the wheel is

$$\alpha = \frac{\omega^2 - \omega_0^2}{2\Delta\theta} = \frac{0 - (5 \text{ rad/s})^2}{2(62.8 \text{ rad})} = -0.2 \text{ rad/s}$$

The external torque is

$$\tau = I\alpha = MR^2\alpha = (5 \text{ kg})(0.1 \text{ m})^2(-0.2 \text{ rad/s}^2) = -0.01 \text{ N m}$$

Example 7.8 Three masses are connected by massless rods as in Fig. 7.15. If $m = 0.1$ kg, find the moment of inertia of the system and the corresponding kinetic energy if it rotates with an angular speed of 5 rad/s about: (a) the z-axis; (b) the y-axis and; (c) the x-axis ($a = 0.2$ m).

Solution 7.8 (a)

$$I_z = \sum_i m_i r_i^2 = 2ma^2 + \frac{m}{2}a^2 + ma^2 = \frac{7}{2}ma^2$$
$$= \frac{7}{2}(0.1 \text{ kg})(0.2 \text{ m})^2 = 0.014 \text{ kg m}^2$$

$$K_R = \frac{1}{2}I_z\omega^2 = \frac{1}{2}(0.014 \text{ kg m}^2)(5 \text{ rad/s})^2 = 0.175 \text{ J}$$

(b)

$$I_y = \frac{m}{2}a^2 + 2ma^2 = \frac{5}{2}ma^2 = \frac{5}{2}(0.1 \text{ kg})(0.2 \text{ m})^2 = 0.01 \text{ kg m}^2$$

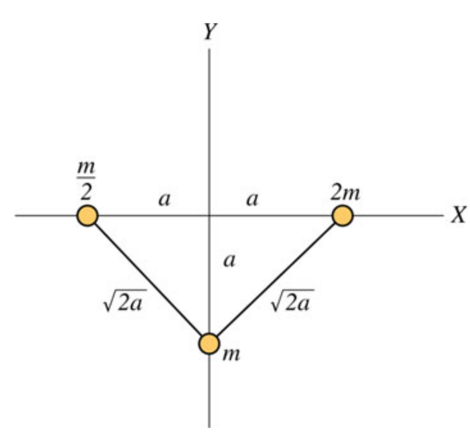


Fig. 7.15 Three masses connected by massless rods

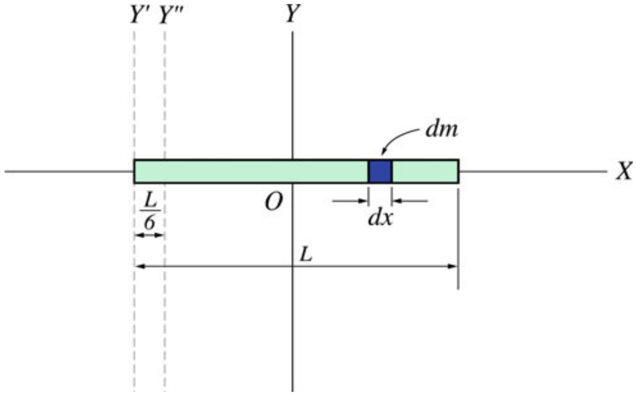


Fig. 7.16 A uniform thin rod of mass M and length L

$$K_R = \frac{1}{2}I_y\omega^2 = \frac{1}{2}(0.01\ \text{kg m}^2)(5\ \text{rad/s})^2 = 0.125\ \text{J}$$

(c)

$$I_x = ma^2 = (0.1\ \text{kg})(0.2\ \text{m})^2 = 4\times10^{-3}\ \text{kg m}^2$$

$$K_R = \frac{1}{2}I_x\omega^2 = \frac{1}{2}(4\times10^{-3}\ \text{kg m}^2)(5\ \text{rad/s})^2 = 0.05\ \text{J}$$

Example 7.9 Fig. 7.16 shows a uniform thin rod of mass M and length L. Find the moment of inertia of the rod about an axis that is perpendicular to it and passing through: (a) the center of mass; (b) at one end; (c) at a distance of $L/6$ from one end.

Solution 7.9 (a) The mass dm of an element in the rod is

$$dm = \lambda dx = \left(\frac{M}{L}\right)dx$$

$$I_{cm} = I_y = \int r^2 dm = \int_{x=-\frac{L}{2}}^{\frac{L}{2}} x^2\left(\frac{M}{L}\right)dx = \frac{M}{L}\left(\frac{x^3}{3}\right)\Big|_{-L/2}^{L/2} = \frac{1}{12}ML^2$$

(b)

$$I_{y'} = I_{cm} + MD^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$

(c)

$$I_{y''} = I_{cm} + MD^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}-\frac{L}{6}\right)^2 = \frac{7}{36}ML^2$$

Example 7.10 Fig. 7.17 shows a uniform thin plate of mass M and surface density σ. Find the moment of inertia of the plate about an axis passing through its center of mass if its length is b and its width is a (the z-axis).

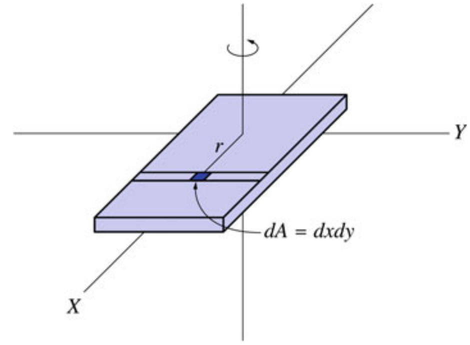


Fig. 7.17 A uniform thin plate of mass M and surface density σ

Solution 7.10 A mass element dm has an area $dxdy$ and is at a distance $r = \sqrt{x^2+y^2}$ from the axis of rotation. Therefore, we have

$$I_{cm} = \int r^2 dm = \int r^2\sigma dA = \int_{y=-a/2}^{a/2}\int_{y=-b/2}^{b/2}(x^2+y^2)\left(\frac{M}{ab}\right)dxdy$$

$$= \frac{M}{ab}\int_{y=-a/2}^{a/2}\left(\frac{x^3}{3}+xy^2\right)\Big|_{x=-b/2}^{b/2}dy = \frac{M}{ab}\int_{y=-a/2}^{a/2}\left(\frac{b^3}{12}+by^2\right)dy$$

$$= \frac{M}{ab}\left(\frac{b^3y}{12}+\frac{y^3b}{3}\right)\Big|_{x=-a/2}^{a/2} = \frac{M}{ab}\left[\frac{ab^3}{12}+\frac{ab^3}{12}\right] = \frac{1}{12}M\left(a^2+b^2\right)$$

Example 7.11 Find the moment of inertia of a uniform solid cylinder of radius R, length L and mass M about its axis of symmetry.

Solution 7.11 Method 1: Using a single integration by dividing the cylinder into thin cylindrical shells each of radius r, length L and thickness dr as in Fig. 7.18, then each volume element is given by

$$dV = 2\pi r dr L$$

and

$$dm = \rho dV = \rho(2\pi r dr L)$$

$$I = \int r^2 dm = \int_0^R r^2(\rho 2\pi r L dr) = 2\pi\rho L\int_0^R r^3 dr = \frac{\pi\rho L}{2}R^4$$

Since

$$\rho = \frac{M}{\pi R^2 L}$$

then

$$I = \frac{1}{2}MR^2$$

Method 2: Using double integration: dividing the cylinder into thin rods each of mass

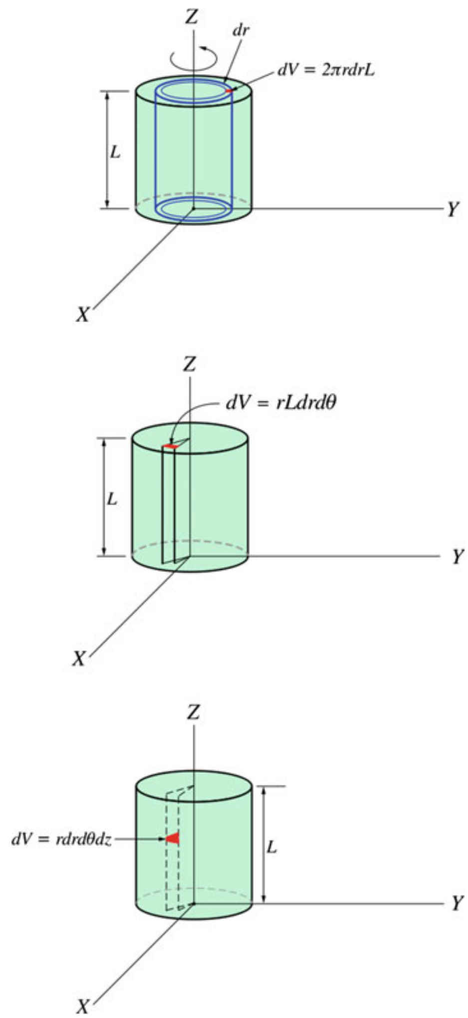


Fig. 7.18 Calculating the moment of inertia of a uniform solid cylinder with the volume element defined in different ways

$$dm = \rho dV = \rho L r dr d\theta$$

$$I = \int r^2 dm = \int_0^{2\pi} \int_{r=0}^{R} r^3 \rho L dr d\theta = \rho \frac{L}{4} R^4 \int_{\theta=0}^{2\pi} d\theta = \frac{\pi \rho L R^4}{2}$$

Since

$$\rho = \frac{M}{\pi R^2 L}$$

We have

$$I = \frac{1}{2} M R^2$$

Method 3: Using triple integration Dividing the cylinder into small cubes each of mass given by

$$dm = \rho r dr d\theta dz$$

$$I = \int r^2 dm = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \int_{z=0}^{L} \rho r^3 dr d\theta dz = \rho L \frac{R^4}{4} \int_{\theta=0}^{2\pi} d\theta = \frac{\pi \rho L R^4}{2}$$

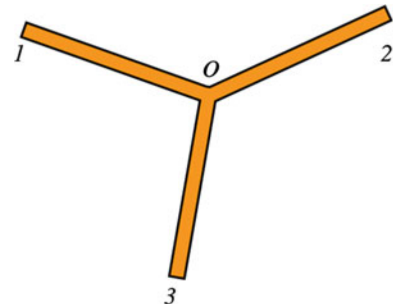


Fig. 7.19 Three rods of length L and mass M are connected together

Since

$$\rho = \frac{M}{\pi R^2 L}$$

Therefore,

$$I = \frac{1}{2} M R^2$$

Example 7.12 Three rods of length L and mass M are connected together as in Fig. 7.19. Determine the moment of inertia of the system about an axis passing through O and perpendicular to the page (the rods lie in the same plane).

Solution 7.12 The moment of inertia of a thin rod about an axis that is perpendicular to it and passing through one end is $1/3ML^2$. The total moment of inertia at O is the sum of the moment of inertias of the rods, i.e.,

$$I = I_1 + I_2 + I_3 = 3\left(\frac{1}{3} M L^2\right) = M L^2$$

Example 7.13 Find the moment of inertia of a spherical shell of radius R and mass M about an axis passing through its center of mass.

Solution 7.13 Let us divide the spherical shell into thin rings each of area (see Fig. 7.20) given by

$$dA = 2\pi R \sin\theta R d\theta = 2\pi R^2 \sin\theta d\theta$$

$$I = \int r^2 dm = \int R^2 \sin^2\theta \sigma 2\pi R^2 \sin\theta d\theta$$

since $\sigma = M/4\pi R^2$, we have

$$I = \frac{M}{2} R^2 \int_{\theta=0}^{\pi} \sin^3\theta d\theta = \frac{M}{2} R^2 \int_{\theta=0}^{\pi} (1 - \cos^2\theta) \sin\theta d\theta$$

$$= \frac{M}{2} R^2 \left[-\cos\theta + \frac{\cos^3\theta}{3} \right]_{\theta=0}^{\pi} = \frac{2}{3} M R^2$$

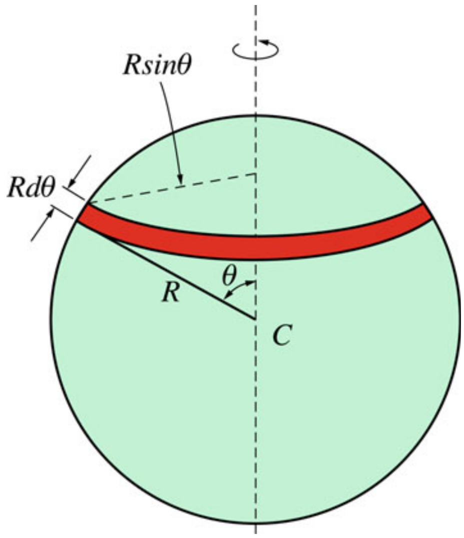


Fig. 7.20 A spherical shell divided into thin rings

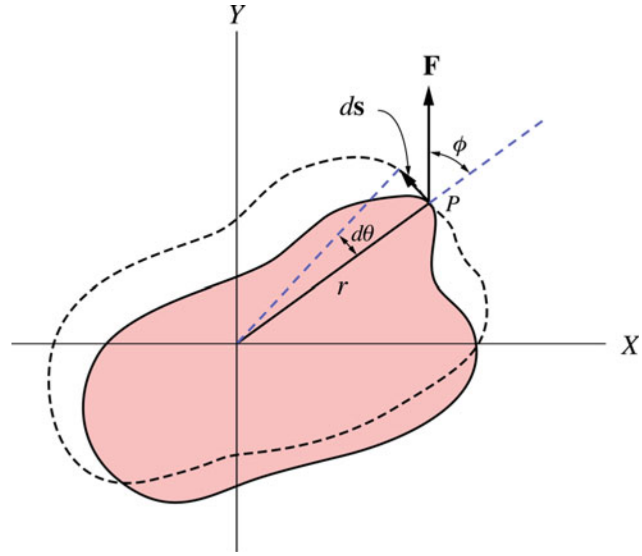


Fig. 7.21 A rigid body rotating about a fixed axis

7.8 Conservation of Angular Momentum of a Rigid Body Rotating About a Fixed Axis

In Chap. 5 we have seen that if the net external torque acting on a system of particles relative to an origin is zero then the total angular momentum of the system about that origin is conserved

$$\mathbf{L}_i = \mathbf{L}_f = \text{constant (isolated system)}$$

In the case of a rigid object in pure rotational motion, if the component of the net external torque about the rotational axis (say the z-axis) is zero then the component of angular momentum along that axis is conserved, i.e., if

$$\tau_z = \frac{dL_z}{dt} = 0$$

then

$$I_i\omega_i = I_f\omega_f$$

That is, the angular momentum is not necessarily conserved in all directions. It is conserved in the direction where the net external torque is equal to zero.

7.9 Work and Rotational Energy

Consider a rigid body rotating about a fixed axis as in Fig. 7.21. If a force that lies in the x-y plane is applied to the body at P, then the work done on the body if it rotates through an angle $d\theta$ is

$$dW = \mathbf{F}\cdot d\mathbf{s} = \mathbf{F}\cdot\frac{d\mathbf{s}}{dt}dt = \mathbf{F}\cdot\mathbf{v}dt = \mathbf{F}\cdot(\boldsymbol{\omega}\times\mathbf{r})dt$$

$$= (\mathbf{r}\times\mathbf{F})\cdot\boldsymbol{\omega}dt = \boldsymbol{\tau}\cdot\boldsymbol{\omega}dt$$

Since $\boldsymbol{\tau}$ and $\boldsymbol{\omega}$ are parallel, (the force lies in the x-y plane therefore the total torque is parallel to the z-axis) we have

$$dW = \tau\omega dt = \tau\frac{d\theta}{dt}dt = \tau d\theta$$

Therefore, the total work done in displacing the body from θ_1 to θ_2 is

$$W = \int_{\theta_1}^{\theta_2}\tau d\theta \qquad (7.12)$$

If this torque is constant we have

$$W = \tau(\theta_2 - \theta_1) = \tau\Delta\theta$$

The Work–Energy Theorem The work–energy theorem states that the work done by an external force while a rigid object rotate from θ_1 to θ_2 is equal to the change in the rotational energy of the object. This follows from Eq. 7.12 and by using the fact that along the axis of rotation the torque is given by $\tau_z = I\alpha$ (see Sect. 7.7), thus

$$W = \int_{\theta_1}^{\theta_2}\tau d\theta = \int_{\theta_1}^{\theta_2}I\alpha d\theta = \int_{\omega_1}^{\omega_2}I\omega\frac{d\omega}{dt}dt = \int_{\omega_1}^{\omega_2}I\omega d\omega = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$$

$$W = \Delta K = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$$

Table 7.2 Analogous Equations in linear Motion and Rotational Motion about a Fixed Axis

Rotational motion	Linear motion
$\tau = I\alpha$	$F = ma$
$W = \int_{\theta_0}^{\theta} \tau d\theta$	$W = \int_{x_0}^{x} F dx$
$K_R = \frac{1}{2} I\omega^2$	$K = \frac{1}{2} mv^2$
$P = \tau\omega$	$P = Fv$

7.10 Power

The instantaneous power delivered to rotate an object about a fixed axis is found from

$$P = \frac{dW}{dt} = \frac{\tau_z d\theta}{dt} = \tau_z \omega_z$$

Table. 7.2 shows analogous equations in linear motion and rotational motion about a fixed axis

Example 7.14 A disc of radius $R = 0.08$ m and mass of 5 kg is rotating about its central axis with an angular speed of 170 rev/min. Find: (a) the rotational kinetic energy of the disc; (b) Suppose that the same disc rotate using a motor that delivers an instantaneous of power 0. 2hp, find in that case the torque applied to the disc.

Solution 7.14 (a) Since the rotational axis is the axis of symmetry of the disc, then the moment of inertia is

$$I = \frac{1}{2} MR^2 = \frac{1}{2}(5 \text{ kg})(0.08 \text{ m})^2 = 0.016 \text{ kg m}^2$$

The angular velocity of the disc is

$$\omega = \left(\frac{170 \text{ rev}}{\text{min}}\right)\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 17.8 \text{ rad/s}$$

$$K = \frac{1}{2} I\omega^2 = \frac{1}{2}(0.016 \text{ kg m}^2)(17.8 \text{ rad/s})^2 = 2.5 \text{ J}$$

(b)

$$P = (0.2 \text{ hp})\left(\frac{746 \text{ W}}{1 \text{hp}}\right) = 149.2 \text{ W}$$

and

$$\tau = \frac{P}{\omega} = \frac{(149.2 \text{ W})}{(17.8 \text{ rad/s})} = 8.4 \text{ N m}$$

Example 7.15 Consider a light rope wrapped around a uniform cylindrical shell of mass 30 kg and radius of 0.2 m as in Fig. 7.22. Suppose that the cylinder is free to rotate about its central axis and that the rope is pulled from rest with a constant force of magnitude of 35 N. Assuming that the rope does not slip, find: (a) the torque applied to the cylinder about its central axis; (b) the angular acceleration of the cylinder; (c) the acceleration of a point in the unwinding rope; (d) the number of revolutions made by the cylinder when it reaches an angular velocity of 12 rad/s, (e) the work done by the applied force when the rope is pulled a distance of 1m, (f) the work done using the work–energy theorem.

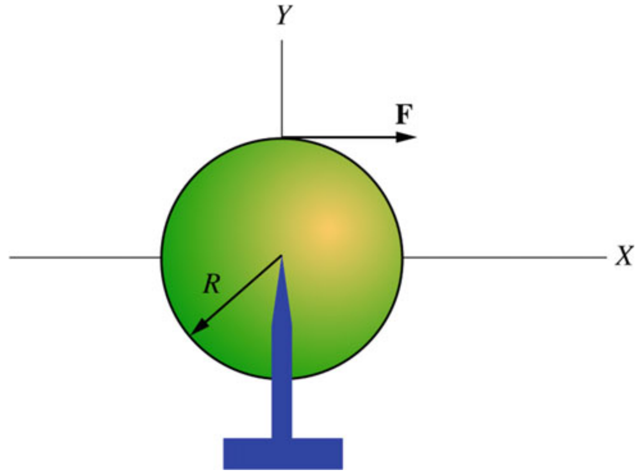


Fig. 7.22 A light rope wrapped around a uniform cylindrical shell

Solution 7.15 (a) Because the line of action of both the weight and the normal forces passes through the central axis of the cylinder, they produce no torque. Hence, the total torque acting on the cylinder is

$$\tau = FR = (35 \text{ N})(0.2 \text{ m}) = 7 \text{ N/m}$$

(b) The moment of inertia of the cylinder is

$$I = MR^2 = (30 \text{ kg})(0.2 \text{ m})^2 = 1.2 \text{ kg m}^2$$

and

$$\alpha = \frac{\tau}{I} = \frac{(7 \text{ N m})}{(1.2 \text{ kg m}^2)} = 5.8 \text{ rad/s}^2$$

(c) The acceleration of a point in the unwinding rope is the same as the acceleration of a point at the rim of the cylinder, i.e.,

$$a = R\alpha = (0.2 \text{ m})(5.8 \text{ rad/s}^2) = 1.2 \text{ m/s}^2$$

(d)

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

Since $\omega_0 = 0$,

$$\theta = \frac{(12 \text{ rad/s})^2}{2(5.8 \text{ rad/s}^2)} = 12.4 \text{ rad}$$

or

$$\theta = (12.4 \text{ rad})\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right) = 2 \text{ rev}$$

(e) If the rope has moved a distance of lm, the angular displacement of the cylinder is

$$\theta = \frac{s}{R} = \frac{(1 \text{ m})}{(0.2 \text{ m})} = 5 \text{ rad}$$

the work done is

$$W = \int_{\theta_0}^{\theta} \tau d\theta = \tau(\theta - \theta_0) = (7 \text{ N m})\ ((5 \text{ rad}) - 0) = 35 \text{ J}$$

(f) The final angular speed when $\theta = 5$ rad is

$$\omega^2 = \omega_0^2 + 2\alpha\theta = 0 + 2(5.8 \text{ rad/s}^2)(5 \text{ rad})$$

That gives $\omega = 7.6$ rad/s. From the work–energy theorem we have

$$W = \triangle K = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2 = \frac{1}{2}(1.2 \text{ kg m}^2)(7.6 \text{ rad/s})^2 - 0 = 35 \text{ J}$$

Example 7.16 A uniform rod of mass $M = 0.75$ kg and length $L = 1$m is hinged at one end and is free to rotate in a vertical plane as in Fig. 7.23. If the rod is released from rest at an angle $\theta = 30^o$ to the horizontal, find; (a) the initial angular acceleration of the rod when it is released; (b) the initial acceleration of a point at the end of the rod; (c) from conservation of energy find the angular speed of the rod at its lowest position (Neglect friction at the pivot).

Solution 7.16 (a) Since the normal force exerted by the pin on the rod passes through O, then the only force that contributes to the torque is the force of gravity This force acts at the center of gravity which is at the center of mass (see Sect. 8.4). Therefore the net external torque is

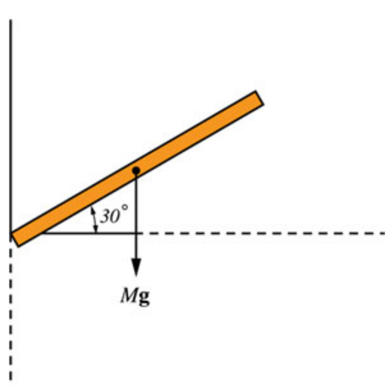


Fig. 7.23 A uniform rod free to rotate at one end

$$\tau = \frac{MgL}{2}\cos\theta = \frac{(0.75 \text{ kg})(9.8 \text{ m/s}^2)(1 \text{ m})}{2}\cos 30^\circ = 3.2 \text{ N m}$$

The moment of inertia about the rotational axis is

$$I = \frac{1}{3}ML^2 = \frac{(0.75 \text{ kg})(1 \text{ m})^2}{3} = 0.25 \text{ kg m}^2$$

and hence

$$\alpha = \frac{\tau}{I} = \frac{(3.2 \text{ N m})}{(0.25 \text{ kg m}^2)} = 12.8 \text{ rad/s}^2$$

(b) The acceleration of a point at the end of the rod is

$$a_t = r\alpha = L\alpha = (1 \text{ m})(12.8 \text{ rad/s}^2) = 12.8 \text{ m/s}^2$$

(c) When the rod reaches its lowest position, the potential energy of its center of mass is transformed into rotational kinetic energy of the rod. From conservation of energy we have $K_i + U_i = K_f + U_f$. Taking the potential energy to be zero at the lowest position, gives

$$0 + Mg\frac{L}{2}(\sin\theta + 1) = \frac{1}{2}I\omega^2 + 0$$

That gives

$$\omega = \sqrt{Mg\frac{L}{I}(\sin\theta + 1)} = \sqrt{\frac{(0.75 \text{ kg})(9.8 \text{ m/s}^2)(1\text{m})}{(0.25 \text{ kg m}^2)}(\sin 30^\circ + 1)} = 6.64 \text{ rad/s}$$

Example 7.17 Find the net torque on the system shown in Fig. 7.24 where $r_1 = 5$ cm, $r_2 = 15$ cm, $F_1 = 10$ N, $F_2 =$

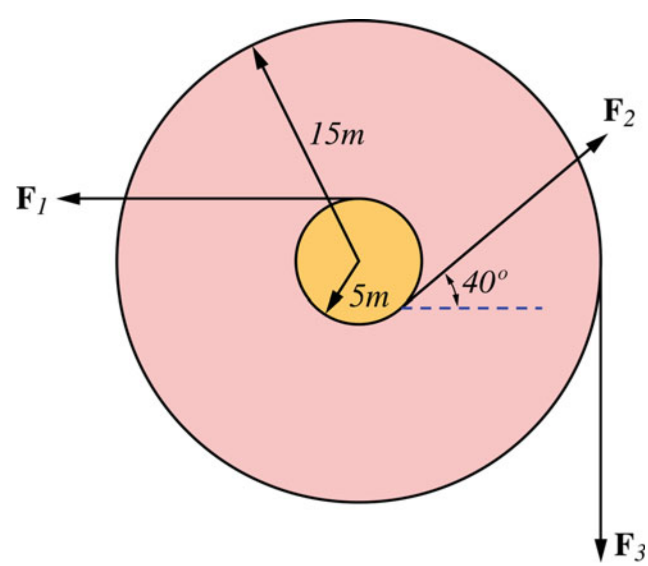
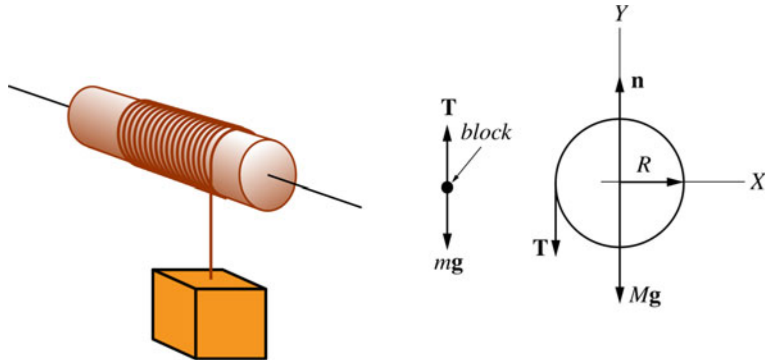


Fig. 7.24 A cylinder with a core section is free to rotate about its center. Ropes wrapped around the inner and outer sections exert different forces

Fig. 7.25 A block of mass m is attached to a light string that is wrapped around the rim of a uniform solid disk of radius R and mass M



20 N and $F_3 = 15$ N. Neglect the mass and friction of the ropes and pulleys.

Solution 7.17 Since all forces lie in the same plane the net torque is

$$\tau_{\text{net}} = \tau_1 + \tau_2 + \tau_3 = (10\text{ N})(0.05\text{ m}) + (20\text{ N})(0.05\text{ m}) - (15\text{ N})(0.15\text{ m}) = -0.75\text{ N m}$$

Example 7.18 A block of mass m is attached to a light string that is wrapped around the rim of a uniform solid disc of radius R and mass M as in Fig. 7.25. Assuming that the string does not slip and that the disc rotates without friction, find: (a) the acceleration of the block; (b) the angular acceleration of the disc, and; (c) the tension in the string when the system is released from rest.

Solution 7.18 The free-body diagrams of the disc and the block are shown in Fig. 7.25. Applying Newton's second law to the block gives

$$T - mg = -ma$$

or

$$a = \frac{mg - T}{m} \tag{7.13}$$

where positive y is chosen to be directed upwards. Applying Newton's second law in angular form to the disc gives

$$\tau = RT = I\alpha$$

or

$$\alpha = \frac{RT}{I}$$

Since the acceleration of the block is equal to the (tangential) acceleration of a point at the rim of the disc we have

$$a = R\alpha = \frac{TR^2}{I} \tag{7.14}$$

Equating Eqs. 7.13 and 7.14 gives

$$\frac{TR^2}{I} = \frac{mg - T}{m}$$

$$T = \frac{g}{1/m + R^2/I} = \frac{g}{1/m + 2R^2/MR^2}$$

that gives

$$T = \frac{mg}{1 + 2m/M}$$

Substituting this into Eq. 7.14

$$a = \frac{TR^2}{I} = \frac{2TR^2}{MR^2}$$

gives

$$a = \frac{g}{1 + M/2m}$$

Finally

$$\alpha = \frac{a}{R} = \frac{g}{R(1 + M/2m)}$$

Example 7.19 A homogeneous solid sphere of mass 4.7 kg and radius of 0.05 m rotate from rest about its central axis with a constant angular acceleration of 3 rad/s^2. Find: (a) the torque that produces this angular acceleration; (b) the work done on the sphere after 7 revolutions; (c) the work done after 7 revolutions using the work–energy theorem.

Solution 7.19 (a)

$$\tau = I\alpha = \frac{2}{5}MR^2\alpha = \frac{2}{5}(4.7\text{ kg})(0.05\text{ m})^2(3\text{ rad/s}^2) = 0.014\text{ N}$$

(b)

$$\theta = (7\text{ rev})\left(\frac{2\pi\,\text{rad}}{1\text{ rev}}\right) = 44\text{ rad}$$

and

$$W = \tau\triangle\theta = (0.014\text{ N/m})(44\text{ rad}) = 0.6\text{ J}$$

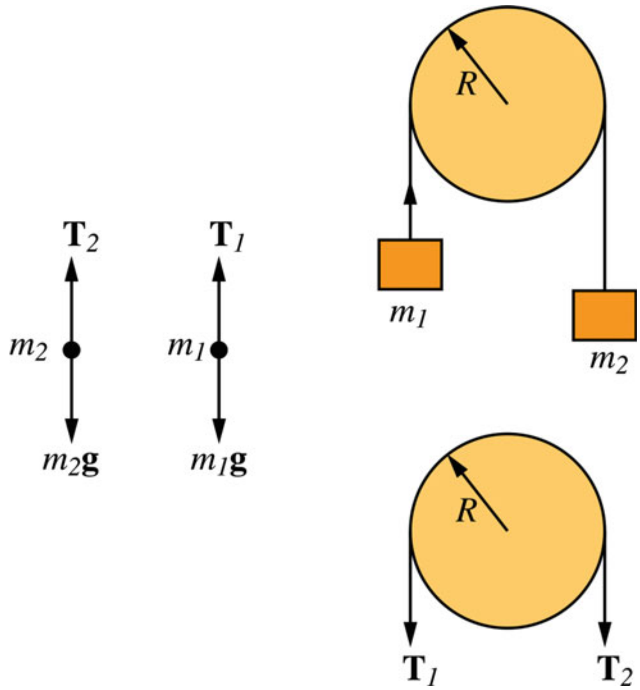


Fig. 7.26 AtwoodÕs machine

assuming $\theta_0 = 0$.

(c) After seven revolutions the angular velocity is

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Since $\omega_0 = 0$, we have

$$\omega^2 = 2\alpha\theta = 2(3 \text{ rad/s}^2)(44 \text{ rad})$$

that gives $\omega = 16.24$ rad/s. Hence

$$W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2 = \frac{1}{2}(4.7 \times 10^{-3} \text{ kg m}^2)(16.24 \text{ rad/s}^2)^2 - 0 = 0.6 \text{ J}$$

Example 7.20 Fig. 7.26 shows Atwood's machine when the mass of the pulley is considered. If the system is released from rest (and assuming that the string does not stretch or slip) and that the friction of the pulley is negligible, find linear acceleration of the blocks and the angular acceleration of the pulley.

Solution 7.20 Fig. 7.26 shows the free-body diagram for each block and for the pulley Applying Newton's second law gives

$$T_1 - m_1 g = m_1 a$$

$$T_2 - m_2 g = -m_2 a$$

$$\tau = (T_1 - T_2)R = -I\alpha$$

and

$$n - T_1 - T_2 - M g = 0$$

The torque is negative because the pulley rotates in the clockwise direction. Therefore we have

$$T_1 - T_2 + g(m_2 - m_1) = a(m_1 + m_2)$$

and

$$T_2 - T_1 = \frac{I\alpha}{R} = \frac{Ia}{R^2}$$

That gives

$$a = \frac{g(m_2 - m_1)}{(m_1 + m_2 + I/R^{2)}}$$

If the pulley is a uniform solid disc then

$$I = \frac{1}{2}MR^2$$

and

$$a = \frac{g(m_2 - m_1)}{(m_1 + m_2 + M/2)}$$

$$\alpha = \frac{g(m_2 - m_1)}{R(m_1 + m_2 + M/2)}$$

Example 7.21 A uniform solid cylinder of radius of 0.2 m and mass of 10 kg is rotating about its central axis. If the angular speed of the cylinder is 5 rad/s:(a) calculate the angular momentum of the cylinder about its central axis; (b) Suppose the cylinder accelerates at a constant rate of 0.5 rad/s^2, find the angular momentum of the cylinder at $t = 3$s(c) find the applied torque; (d) find the work done after 3s.

Solution 7.21 (a) The moment of inertia of the cylinder is

$$I = \frac{1}{2}MR^2 = \frac{1}{2}(10 \text{ kg})(0.2 \text{ m})^2 = 0.2 \text{ kg m}^2$$

for homogeneous symmetrical objects the total angular momentum is

$$L = I\omega = (0.2 \text{ kg m}^2)(5 \text{ rad/s}) = 1 \text{ kg m}^2/\text{s}$$

(b) At $t = 3$ s

$$\omega = \omega_0 + \alpha t = (5 \text{ rad/s}) + (0.5 \text{ rad/s}^2)(3 \text{ s}) = 6.5 \text{ rad/s}$$

at that instant

$$L = I\omega = (0.2 \text{ kg m}^2)(6.5 \text{ rad/s}) = 1.3 \text{ kg m}^2/\text{s}$$

(c)

$$\tau = I\alpha = (0.2 \text{ kg m}^2)(0.5 \text{ rad/s}^2) = 0.1 \text{ N m}$$

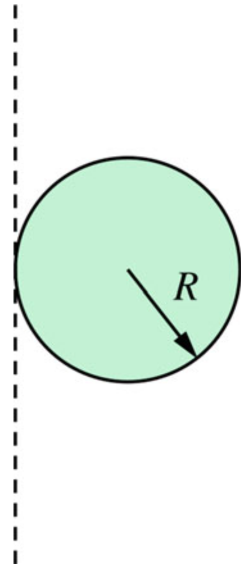


Fig. 7.27 A uniform solid sphere rotating about an axis tangent to the sphere

(d)

$$W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2 = \frac{1}{2}(0.2\,\mathrm{kg\,m^2})((6.5\,\mathrm{rad/s})^2 - (5\,\mathrm{rad/s})^2) = 1.72\,\mathrm{J}$$

Example 7.22 A uniform solid sphere of radius of 5 cm and mass of 4.7 kg is rotating about an axis that is tangent to the sphere (see Fig. 7.27). If its angular acceleration is given by $\alpha = (4t)\,\mathrm{rad/s^2}$ and if at $t = 0, \omega_0 = 0$, find the angular momentum of the sphere and the applied torque as a function of time.

Solution 7.22

$$\omega = \int \alpha dt = \int 4t dt = 2t^2 + c$$

since at $t = 0$, $\omega_0 = 0$ then $c = 0$ and

$$\omega = (2t^2)\,\mathrm{rad/s}$$

The moment of inertia of the sphere is

$$I = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2 = \frac{7}{5}(4.7\,\mathrm{kg})(0.05\,\mathrm{m})^2 = 0.016\,\mathrm{kg\,m^2}$$

and

$$L = I\omega = (0.016\,\mathrm{kg\,m^2})((2t^2)\,\mathrm{rad/s}) = (0.03t^2)\,\mathrm{kg\,m^2/s}$$

$$\tau = \frac{dL}{dt} = (0.06t)\mathrm{N\,m}$$

Example 7.23 In Example 7.8 find the angular momentum in each case.

Solution 7.23 (a)

$$L = I_z\omega = (0.014\,\mathrm{kg\,m^2})(5\,\mathrm{rad/s}) = 0.07\,\mathrm{kg\,m^2/s}$$

(b)

$$L = I_y\omega = (0.01\,\mathrm{kg\,m^2})(5\,\mathrm{rad/s}) = 0.05\,\mathrm{kg\,m^2/s}$$

(c)

$$L = I_x\omega = (4 \times 10^{-3}\,\mathrm{kg\,m^2})(5\,\mathrm{rad/s}) = 0.02\,\mathrm{kg\,m^2/s}$$

Example 7.24 A uniform solid sphere of radius of 0.2 m is rotating about its central axis with an angular speed of 5 rad/s. If an impulsive force that has an average value of 100 N acts at the rim of the sphere at the center level for a short time of 2 ms:(a) find the angular impulse of the force; (b) the final angular speed of the sphere.

Solution 7.24 (a)

$$\triangle L = \int_{t_1}^{t_2} \tau dt = \tau_{ave}\Delta t = \overline{F}Rt = (100\,\mathrm{N})(0.2\,\mathrm{m})(2 \times 10^{-3}\,\mathrm{s}) = 0.04\,\mathrm{kg\,m^2/s}$$

(b)

$$\Delta L = I(\omega_f - \omega_i)$$

$$(0.04\,\mathrm{kg\,m^2/s}) = (0.2\,\mathrm{kg\,m^2})(\omega_f - (5\,\mathrm{rad/s}))$$

That gives $\omega_f = 5.2\,\mathrm{rad/s}$.

Example 7.25 A man stands on a platform that is free to rotate without friction about a vertical axis as in Fig. 7.28. If the system is initially rotating with an angular speed of 0.3 rev/s: (a) find the final angular speed of the system if the man draws the weights in; (b) find the increase in the kinetic energy of the system and its source. ($I_i = 15\,\mathrm{kg\,m^2}$ And $I_f = 3\,\mathrm{kg\,m^2}$).

Solution 7.25 Because the resultant external torque on the system is zero, it follows that the total angular momentum of the system is conserved. That is

$$L_i = L_f$$

$$I_i\omega_i = I_f\omega_f$$

hence

$$\omega_f = \frac{I_i}{I_f}\omega_i = \frac{(15\,\mathrm{kg\,m^2/s})}{(3\,\mathrm{kg\,m^2/s})}(0.3\,\mathrm{rev/s}) = 1.5\,\mathrm{rev/s}$$

(b)

$$\omega_i = \left(0.3\,\frac{\mathrm{rev}}{\mathrm{s}}\right)\left(\frac{2\pi\ \mathrm{rad}}{1\ \mathrm{rev}}\right) = 1.9\,\mathrm{rad/s}$$

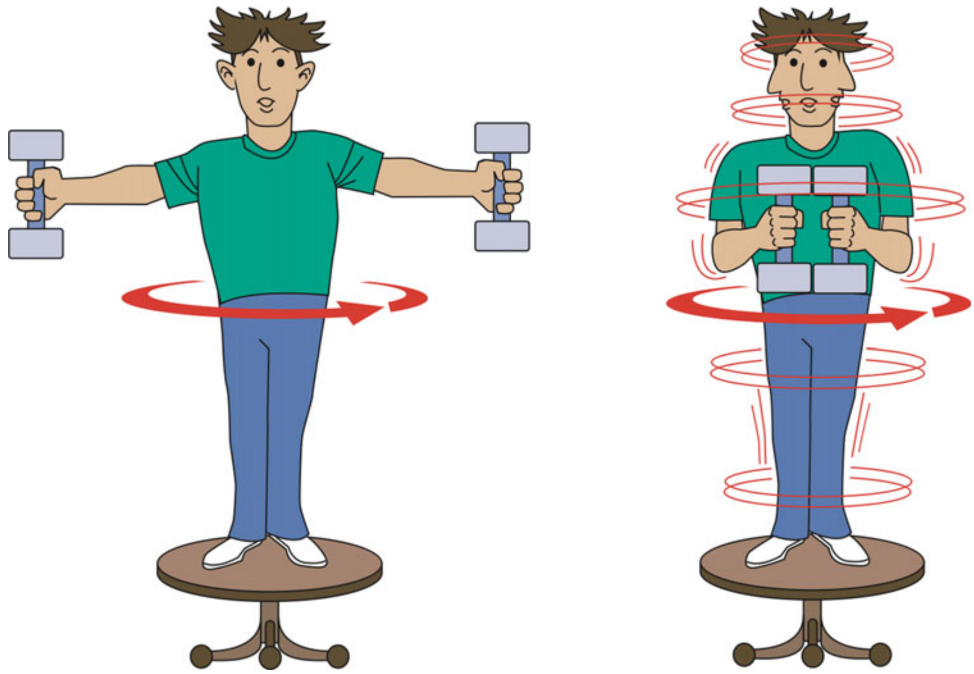


Fig. 7.28 A man stands on a platform that is free to rotate without friction about a vertical axis

$$\omega_f = \left(1.5\ \frac{\text{rev}}{\text{s}}\right)\left(\frac{2\pi\ \text{rad}}{1\ \text{rev}}\right) = 9.4\ \text{rad/s}$$

$$K_i = \frac{1}{2}I_i\omega_i^2 = \frac{1}{2}(15\ \text{kg m}^2)(1.9\ \text{rad/s})^2 = 27\ \text{J}$$

$$K_f = \frac{1}{2}I_f\omega_f^2 = \frac{1}{2}(3\ \text{kg m}^2)(9.4\ \text{rad/s})^2 = 132.5\ \text{J}$$

This increase in the kinetic energy is because the man does work when he moves the dumbbells inwards.

Example 7.26 A uniform disc of moment of inertia of 0.1 kg m^2 is rotating without friction with an angular speed of 3 rad/s about an axle passing through its center of mass as in Fig. 7.29. When another disc of moment of inertia of 0.05 kg m^2 that is initially at rest is dropped on the first, the two will eventually rotate with the same angular speed due to friction between them. Determine (a) the final angular speed; (b) the change in the kinetic energy of the system.

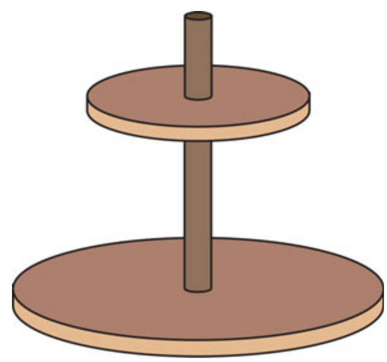


Fig. 7.29 A uniform disc rotating without friction. Another disc that is initially at rest is dropped on the first, the two will eventually rotate with the same angular speed due to friction between them

Solution 7.26 (a) Since the net external torque acting on the system is zero, it follows that the total angular momentum of the system is conserved, i.e.,

$$L_i = L_f$$

or

$$I_1\omega_1 = (I_1 + I_2)\omega$$

hence

$$\omega = \frac{I_1\omega_1}{(I_1 + I_2)} = \frac{(0.1\ \text{kg m}^2)(3\ \text{rad/s})}{(0.15\ \text{kg m}^2)} = 2\ \text{rad/s}$$

(b)

$$K_i = \frac{1}{2}I_1\omega_1^2 = \frac{1}{2}(0.1\ \text{kg m}^2)(3\ \text{rad/s})^2 = 0.45\ \text{J}$$

$$K_f = \frac{1}{2}(I_1 + I_2)\omega^2 = \frac{1}{2}(0.15\ \text{kg m}^2)(2\ \text{rad/s})^2 = 0.3\ \text{J}$$

This decrease in kinetic energy is due to the internal nonconservative (frictional) force that acts within the system.

Problems

1. A wheel is initially rotating at 60 rad/s in the clockwise direction. If a counterclockwise torque acts on the wheel

producing a counterclockwise angular acceleration $\alpha = 2t$ rad/s^2, find the time required for the wheel to reverse its direction of motion.

2. If the angular position of a point on a rotating wheel is given by $\theta = 2t + 5t^2$ rad, find the angular speed and angular acceleration of the point at $t = 2$ s.
3. A wheel of radius of 0.5 m rotates from rest at a constant angular acceleration of 2.5 rad/s^2. At $t = 2$ s Find (a) the angular speed of the wheel (b) the angle in radians through which the wheel rotates (c) the tangential and radial acceleration of a point at the rim of the wheel.
4. Find the angular speed in radians per second of the earth about (a) its axis (b) the sun.
5. An L-shaped bar rotates counterclockwise with an angular acceleration of ω (see Fig. 7.30). Find (in vector form) the linear velocity and acceleration of the point P on the bar.
6. Four masses are connected by light rigid rods as in Fig. 7.31. Calculate the moment of inertia of the system about (a) the x-axis (b) the y-axis (c) the z-axis.
7. Find the moment of inertia of a uniform solid sphere of radius R and mass M about an axis passing through its center of mass.

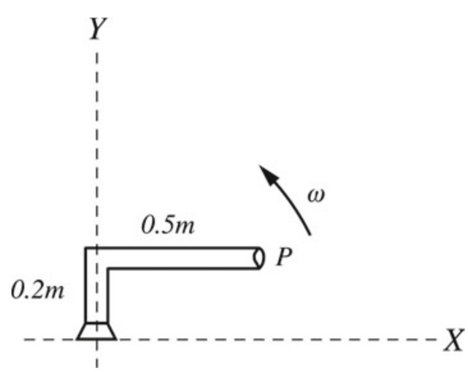


Fig. 7.30 An L-shaped bar rotating counterclockwise

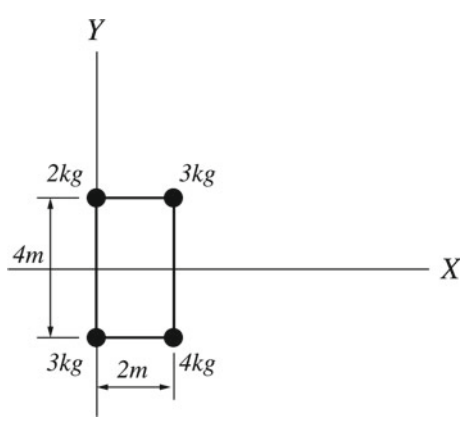


Fig. 7.31 Four masses connected by light rigid rods

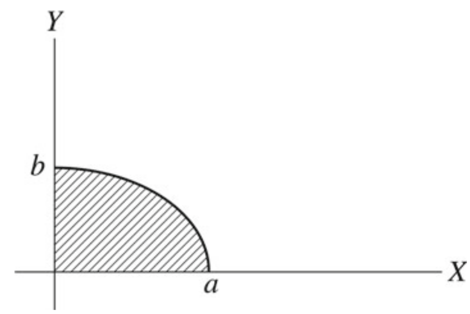
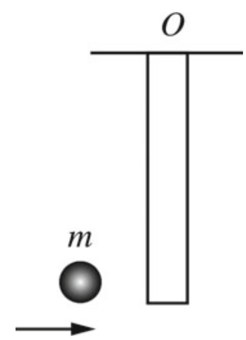


Fig. 7.32 An elliptical quadrant

Fig. 7.33 A uniform rod of length L and mass M is pivoted at O. A projectile of mass m moving at velocity v collides with the rod and sticks to it



8. Find the moment of inertia of an elliptical quadrant about the y-axis (see Fig. 7.32).
9. A 5 kg uniform solid cylinder of radius 0.2 m rotate about its center of mass axis with an angular speed of 10 rev/min. Find (a) its rotational kinetic energy (b) its angular momentum.
10. A wheel of mass of 20 kg and radius of 0.75 m is initially rotating at 120 rev/min. If its angular speed is increased to 300 rev/min in 20 s, find (a) the work done on the wheel (b) the average power delivered to the wheel.
11. A wheel of mass 10 kg and radius 0.4 m accelerates uniformly from rest to an angular speed of 800 rev/min in 20 s. Find (a) the torque applied to the wheel (b) the work done on the wheel (c) the work done using the work–energy theorem.
12. A uniform rod of length L and mass M is pivoted at O (see Fig. 7.33). If a projectile of mass m moving at velocity v collide with the rod and stick to it, find the angular momentum of the system immediately before and immediately after the collision.
13. A disc of radius 2.2 m and mass of 120 kg rotate about a frictionless vertical axle that passes through its center. A man of mass 65 kg walks slowly from the rim of the disc towards the center. Find the angular speed of the disc when the man is at a distance of 0.7 m from the center if its angular speed when the man starts walking is 1.6 rad/s.

8 Rolling and Static Equilibrium

8.1 Rolling Motion

Rolling motion represents the general plane motion of a rigid body It can be considered as a combination of pure translational motion parallel to a fixed plane plus a pure rotational motion about an axis that is perpendicular to that plane. The axis of rotation usually passes through the center of mass. In Sect. 6.4, we've seen that the motion of an object (or a system of particles) can always be considered as a combination of the motion of the object relative to its center of mass plus the motion of its center of mass relative to some origin O. From Sect. 6.4.3, the kinetic energy of an object relative to the origin is

$$K = \frac{1}{2}\sum_i m_i v_i'^2 + \frac{1}{2}Mv_{cm}^2 \quad (8.1)$$

where v_{cm} is the velocity of the center of mass of the object relative to the origin O, m_i is the mass of the ith particle and v_i' is the linear velocity of the ith particle relative to the center of mass. In the case of the general plane motion of a rigid body, the motion can be considered as a combination of pure translational motion of the center of mass plus pure rotational motion about an axis passing through the center of mass and perpendicular to the plane of motion. Therefore, the first term in Eq. 8.1 can be written as

$$v_i' = \omega r_i'$$

where r_i' is the perpendicular distance from the ith particle to the center of mass axis. Hence

$$K = \frac{1}{2}\left(\sum_i m_i r_i'^2\right)\omega^2 + \frac{1}{2}Mv_{cm}^2$$

$$K = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}Mv_{cm}^2$$

Thus, the total kinetic energy of a rolling object is the sum of the translational kinetic energy of its center of mass and the rotational kinetic energy about its center of mass.

8.2 Rolling Without Slipping

An important special case of the general plane motion is rolling without slipping. Such motion occurs if a perfectly rigid body rolls on a perfectly rigid surface. As the object rolls without slipping, the instantaneous s′ point of contact between the object and the surface is at rest relative to the surface since there is no slipping. Now, consider a wheel of radius R rolling without slipping along the straight track shown in Fig. 8.1. The center of mass of the wheel moves along a straight line, while a point on the rim such as P moves in a cycloid path. As the wheel rotates through an angle θ, its center of mass moves through a distance equal to the arc length s (see Fig. 8.2) given by

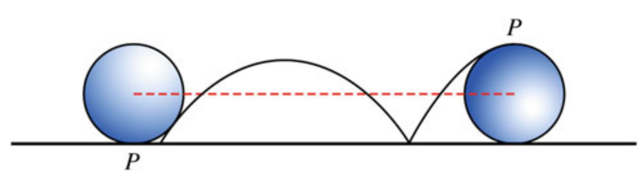


Fig. 8.1 A wheel of radius R rolling without slipping along the straight track

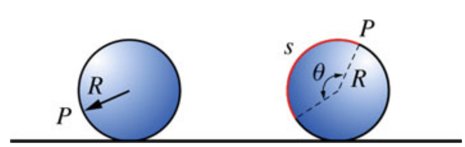


Fig. 8.2 As the wheel rotates through an angle θ, its center of mass moves through a distance equal to the arc length s

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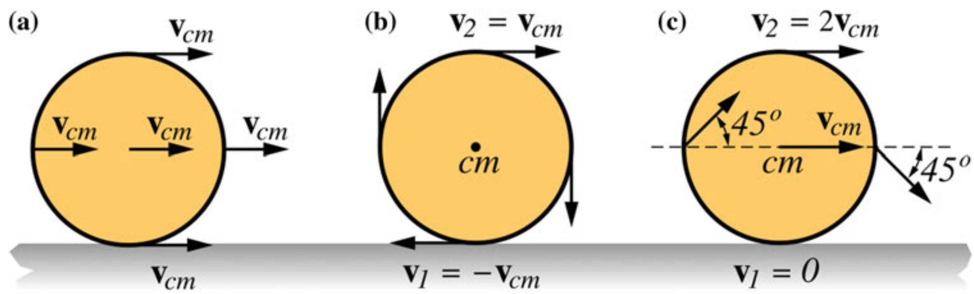


Fig. 8.3 The combination of pure rotational and translational motions

$$s = R\theta$$

Hence, the speed of the center of mass is

$$v_{cm} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

The acceleration of the center of mass is given by

$$a_{cm} = \frac{dv_{cm}}{dt} = R\frac{d\omega}{dt} = R\alpha$$

The combination of pure rotational and translational motions is viewed in Fig. 8.3. In the pure translational motion (see Fig. 8.3 part a) every particle in the wheel moves with the velocity $\mathbf{v}_{cm}$. In pure rotational motion (see Fig. 8.3 part b), each particle moves with an angular speed ω about the center of mass axis and the linear speed of any particle at the rim is

$$v_{cm} = R\omega \tag{8.2}$$

The resulting motion of these two combined motions is shown in Fig. 8.3 part c, where the linear velocity of each particle is the vector sum of its linear velocity in pure translational motion and its linear velocity in pure rotational motion. Therefore, the instantaneous velocity of the point of contact is equal to zero ($\mathbf{v}_1 = 0$) and of a point at the top of the wheel is equal to twice the velocity of the center of mass ($\mathbf{v}_2 = 2\mathbf{v}_{cm}$). Note that Eq. 8.2 is valid only in the special case of rolling without slipping; in the general rolling motion this equation does not hold. The total kinetic energy of a rigid object rolling without slipping is therefore given by

$$K = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}Mv_{cm}^2$$

$$= \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}MR^2\omega^2$$

Another way to view rolling without slipping is to consider the wheel to be in pure rotational motion about an instantaneous axis that passes through the point of contact P (see Fig. 8.4). In that case, the velocity of the point of contact P is zero and

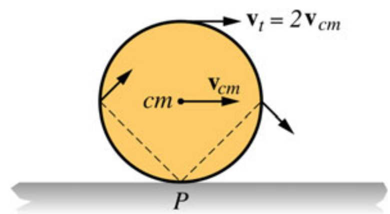


Fig. 8.4 Another way to view rolling without slipping is to consider the wheel to be in pure rotational motion about an instantaneous axis that passes through the point of contact P

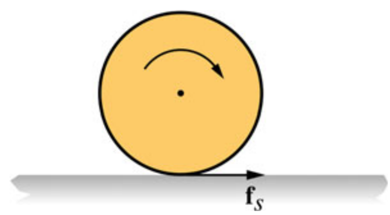


Fig. 8.5 A statistical frictional force acts on it at the instantaneous point of contact producing a torque about the center

the velocity of the center of mass is $v_{cm} = R\omega$ (since it is at a distance R from the axis of rotation) and the velocity of a point at the top is $v_t = 2R\omega = 2v_{cm}$. Note that the angular velocity ω of the wheel is the same as its angular velocity if the axis of rotation is at the center of mass.

For simplicity, only homogeneous symmetrical objects will be considered here such as hoops, cylinders, and spheres. When a rigid body rolls without slipping with a constant speed, there will be no frictional force acting on the body at the instantaneous point of contact. However, if the object is accelerating, then a statistical frictional force acts on it at the instantaneous point of contact producing a torque about the center (see Fig. 8.5). This will cause the object to rotate about its center of mass. The direction of the statistical force opposes the tendency of the object to slide. For example, if a wheel is rolling down an incline, the direction of the frictional force will be opposing the downward motion.

In most situations, the body and the surface are not perfectly rigid. As a result, the normal force would not be a single force; rather it would be a number of forces that are distributed over the area of contact (see Fig. 8.6). Therefore, each normal force will exert an opposing torque since its line of action will

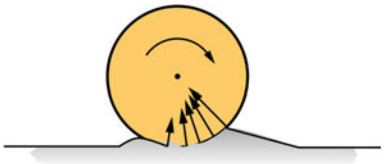


Fig. 8.6 If the body and the surface are not perfectly, the normal force would not be a single force; rather it would be a number of forces that are distributed over the area of contact

not pass through the center of mass. Furthermore, as the object rolls over the surface, both the object and the surface undergo deformation resulting in a loss in the mechanical energy.

Example 8.1 A uniform solid hoop of mass of 32 kg and radius of 1.2 m rolls without slipping on a horizontal track where the center of mass speed is 2 m/s. Find: (a) the total energy of the hoop and compare it with its total energy if it would slide without rolling; (b) the speed of the hoop at its top and bottom.

Solution 8.1 (a) the total energy is given by

$$K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M v_{cm}^2$$

$$= \frac{1}{2}(MR^2)\left(\frac{v_{cm}}{R}\right)^2 + \frac{1}{2} M v_{cm}^2 = M v_{cm}^2 = (32\text{ kg})(2\text{ m/s})^2 = 128\text{ J}$$

If the hoop slides without rolling its total kinetic energy is $\frac{1}{2} M v_{cm}^2$, that is, its value is half of that if the hoop were to roll without slipping.

(b)

$$v_{\text{top}} = 2 v_{cm} = 2(2\text{ m/s}) = 4\text{ m/s}$$

$$v_{\text{bottom}} = 0$$

Example 8.2 A uniform solid cylinder, sphere, and hoop roll without slipping from rest at the top of an incline (see Fig. 8.7). Find out which object would reach the bottom first.

Solution 8.2 For each object, we have

$$K_i + U_i = K_f + U_f$$

$$0 + Mgh = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \left(\frac{v_{cm}}{R}\right)^2$$

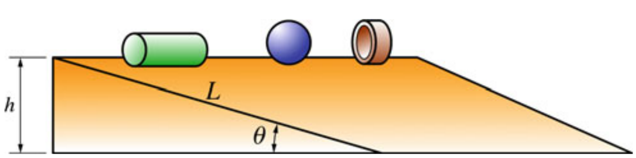


Fig. 8.7 A uniform solid cylinder, sphere and hoop roll without slipping from rest at the top of an incline

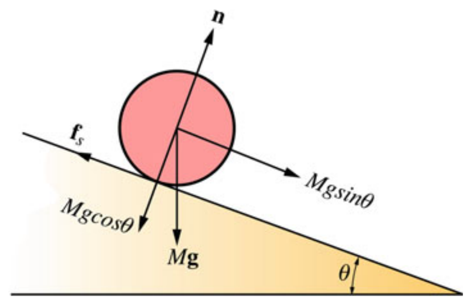


Fig. 8.8 A marble ball of radius R and mass M rolls without slipping down the incline

$$v_{cm} = \sqrt{\frac{2gh}{1 + I_{cm}/MR^2}}$$

Hence, the speed of the center of mass of any object at the bottom of the incline does not depend on its mass or size; it depends only on its shape. Therefore, all objects of the same shape such as spheres (of any mass or size) have the same speed at the bottom. That is, the smaller the ratio I_{cm}/MR^2 the faster the object moves since less of its energy goes to rotational kinetic energy and more goes to translational kinetic energy The ratio I_{cm}/MR^2 is equal to 0.4, 0.5, and 1 for a sphere, cylinder, and hoop, respectively Therefore, these objects will finish in the order of any sphere, any cylinder, and any hoop.

Example 8.3 A marble ball of radius R and mass M rolls without slipping down the incline shown in Fig. 8.8. Find: (a) its acceleration; (b) the minimum coefficient of static friction that is required to prevent slipping.

Solution 8.3 (a) Applying Newton's second law in both linear and angular form (see Fig. 8.7) we have

$$\sum F_x = Mg\ \sin\theta - f_s = M a_{cm} \tag{8.3}$$

$$\sum F_y = n - Mg \cos\theta = 0$$

and

$$\sum \tau = f_s R = I_{cm} \alpha = \left(\frac{2}{5} M R^2\right)\left(\frac{a_{cm}}{R}\right)$$

that gives

$$f_s = \frac{2}{5} M a_{cm} \tag{8.4}$$

Substituting Eq. 8.4 into Eq. 8.3 gives

$$Mg \sin\theta - \frac{2}{5} M a_{cm} = M a_{cm}$$

hence

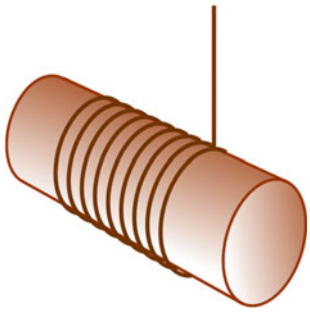


Fig. 8.9 A string wrapped around a uniform solid cylinder of radius of R and mass of M

$$a_{cm} = \frac{5}{7} g \sin\theta$$

and

$$f_s == \frac{2}{7} Mg \sin\theta$$

(b) At the verge of slipping, the statistical frictional force is a maximum given by

$$f_{s\,\max} = \mu_s n = \frac{2}{7} Mg \sin\theta$$

Hence, the coefficient of static friction must be at least as great as $\mu_s = \frac{2}{7}\tan\theta$ in order for the ball not to slip.

Example 8.4 A string is wrapped around a uniform solid cylinder of radius of R and mass of M as in Fig. 8.9. If the cylinder is released from rest while the string is fixed in place and assuming that the string does not slip at the cylinder's surface, find: (a) the acceleration of the center of mass using Newton's laws (b) the acceleration of the center of mass using energy methods if the cylinder descends a distance h(c) the tension in the string.

Solution 8.4 (a) Applying Newton's second law in both the linear and angular form gives

$$\sum F_y = T - Mg = -Ma_{cm} \quad (8.5)$$

$$\sum \tau = TR = I_{cm}\alpha = \frac{1}{2} MR^2 (\frac{a_{cm}}{R})$$

hence

$$T = \frac{1}{2} Ma_{cm} \quad (8.6)$$

Substituting Eq. 8.6 into Eq. 8.5 gives

$$-M_9 + \frac{1}{2} Ma_{cm} = -Ma_{cm}$$

that gives

$$a_{cm} = \frac{2}{3} g$$

(b) Energy Method

$$K_i + U_i = K_f + U_f$$

$$0 + Mgh = \frac{1}{2} Mv_{cm}^2 + \frac{1}{2} I_{cm}\omega^2$$

$$0 + Mgh = \frac{1}{2} Mv_{cm}^2 + \frac{1}{2}\left(\frac{1}{2} MR^2\right)\left(\frac{v_{cm}}{R}\right)^2$$

that gives

$$v_{cm} = \sqrt{\frac{4}{3} gh}$$

From the expression $v^2 = v_0^2 + 2a_{cm}h$, and since $v_0 = 0$ we have

$$a_{cm} = \frac{v_{cm}^2}{2h} = \frac{4gh}{3(2h)} = \frac{2}{3} g$$

(b) From Eq. 8.6,

$$T = \frac{1}{2} Ma_{cm} = \frac{1}{2} M\left(\frac{2}{3} g\right) = \frac{1}{3} Mg$$

Example 8.5 A uniform solid sphere of radius R and mass M is released from rest at the top of an incline at a distance h above the ground. If it rolls without slipping, find the speed of the center of mass at the bottom of the incline.

Solution 8.5

$$K_i + U_i = K_f + U_f$$

$$0 + Mgh = \frac{1}{2} Mv_{cm}^2 + \frac{1}{2} I_{cm}\omega^2$$

$$0 + Mgh = \frac{1}{2} Mv_{cm}^2 + \frac{1}{2}\left(\frac{2}{5} MR^2\right)\left(\frac{v_{cm}}{R}\right)^2$$

That gives

$$v_{cm} = \sqrt{\frac{10}{7} gh}$$

Example 8.6 A block of mass m is attached to a light string that passes over a light pulley and is connected to a uniform solid sphere of radius R and mass M as in Fig. 8.10. Show that the acceleration of the system is $a = \frac{g}{1 + 7/5(M_{/m})}$ when the block is released from rest.

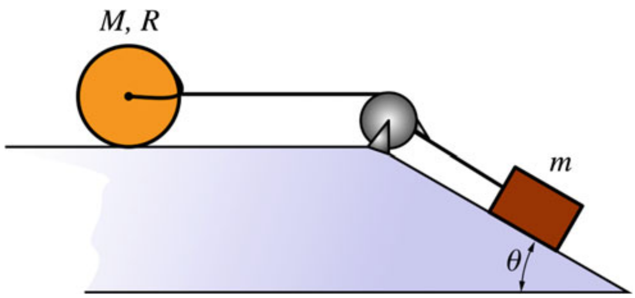


Fig. 8.10 A block of mass m is attached to a light string that passes over a light pulley connected to a uniform solid sphere of radius R and mass M

Solution 8.6 From conservation of energy, we have

$$mgh = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}mv^2$$

Since the block and the sphere are connected, they have the same speed, therefore

$$mgh = \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\left(\frac{v^2}{R}\right)^2 + \frac{1}{2}mv^2$$

Therefore, the speed of the system when the block is at the bottom of the incline is

$$v = \sqrt{\frac{2gh}{1 + 7M/5m}}$$

The acceleration of the system is

$$v^2 - v_0^2 = 2ah$$

or

$$a = \frac{v^2}{2h} = \frac{2gh}{2h(1 + 7/5(M/m))}$$

that gives

$$a = \frac{g}{(1 + 7/5(M/m))}$$

8.3 Static Equilibrium

An extended object is said to be in equilibrium if two conditions are satisfied. First, the net external force acting on the object must be equal to zero. Second, the net external torque on the object about any origin must also be equal to zero. In other words, an object is in equilibrium if its total linear momentum and its total angular momentum (about any origin) are constants. Only the first condition is necessary if the object can be treated as a particle. Thus, the conditions of equilibrium may be written as

$$\sum \mathbf{F} = \mathbf{0} \ (Translational\ Equilibrium) \tag{8.7}$$

$$\sum \boldsymbol{\tau} = \mathbf{0} \ (Rotational\ Equilibrium) \tag{8.8}$$

In terms of components, we may write

$$\sum F_x = 0,\ \sum F_y = 0,\ \sum F_z = 0 \tag{8.9}$$

$$\sum \tau_x = 0,\ \sum \tau_y = 0,\ \sum \tau_z = 0 \tag{8.10}$$

An object is said to be in static equilibrium if it is at rest (there isn't any kind of motion with respect to our inertial frame of reference). Now consider the case in which all external forces acting on the object lie in the same plane (for example the x–y plane). Such forces are called coplanar forces. The net external torque due to these forces is then perpendicular to the x–y plane and parallel to the z-axis. Equations 8.9 and 8.10 are, therefore, reduced to

$$\sum F_x = 0,\ \sum F_y = 0,\ \sum \tau_z = 0$$

Next, we will prove that if the object is in translational equilibrium where ($\Sigma\mathbf{F} = \mathbf{0}$) and the net external torque on the object is equal to zero about some origin, it is also equal to zero about any other origin. Note that the origin may be chosen anywhere inside or outside the object. Suppose that a number of forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots \mathbf{F}_n$ are acting on a rigid object at different points (see Fig. 8.11) and that the object is in translational equilibrium. The point of application of $\mathbf{F}_1$ relative to O is r_1 and of $\mathbf{F}_2$ is r_2 and so on. The net external torque about O is given by

$$\sum \boldsymbol{\tau}_0 = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \cdots + \boldsymbol{\tau}_n = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + + \cdots \mathbf{r}_n \times \mathbf{F}_n$$

The net external torque about O′ (see Fig. 8.12) is

$$\sum \boldsymbol{\tau}_{0'} = \boldsymbol{\tau}_1' + \boldsymbol{\tau}_2' + + \cdots \boldsymbol{\tau}_n' = \mathbf{r}_1' \times \mathbf{F}_1 + \mathbf{r}_2' \times \mathbf{F}_2 + + \cdots \mathbf{r}_n' \times \mathbf{F}_n$$

$$= (\mathbf{r}_1 - \mathbf{r}_{0'}) \times \mathbf{F}_1 + (\mathbf{r}_2 - \mathbf{r}_{0'}) \times \mathbf{F}_2 + \cdot + \cdots (\mathbf{r}_n - \mathbf{r}_{0'}) \times \mathbf{F}_n$$

$$= \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + . + \cdots \mathbf{r}_n \times \mathbf{F}_n - (\mathbf{r}_{0'} \times \mathbf{F}_1 + \mathbf{r}_{0'} \times \mathbf{F}_2 + + \cdots \mathbf{r}_{0'} \times \mathbf{F}_n)$$

$$= \sum \boldsymbol{\tau}_0 - (\mathbf{r}_{0'} \times (\mathbf{F}_1 + \mathbf{F}_2 + + \mathbf{F}_n)) = \sum \tau_0 - (\mathbf{r}_{0'} \times \sum \mathbf{F})$$

Since $\Sigma\mathbf{F} = \mathbf{0}$ we have

$$\sum \boldsymbol{\tau}_{0'} = \sum \boldsymbol{\tau}_0$$

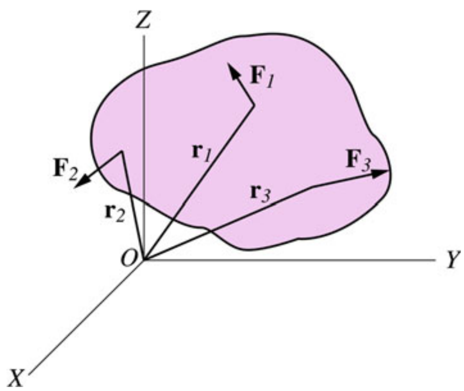


Fig. 8.11 A number of forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, ..\mathbf{F}_n$ act on a rigid object at different points

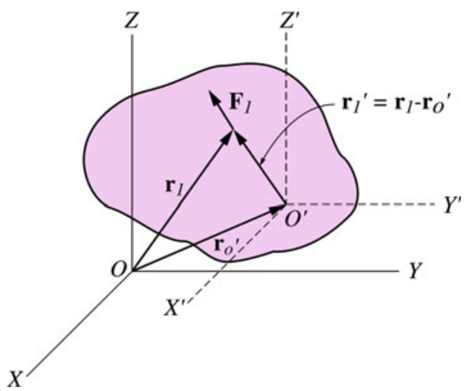


Fig. 8.12 The net external torque on the object about Oõ

8.4 The Center of Gravity

The resultant gravitational force acting on an object is the resultant of the individual gravitational forces acting on different mass elements of the object (see Fig. 8.13), i.e.,

$$\sum \mathbf{F} = \sum m_i \mathbf{g} \tag{8.11}$$

This force can be replaced by a single force that is equal to the weight of the object (Mg) and that acts at a single point called the center of gravity Now consider an object that is near the earth's surface where the force of gravity is assumed to be constant over that range. Equation 8.11 becomes

$$\sum \mathbf{F} = \sum m_i \mathbf{g} = \mathbf{g} \sum m_i = M\mathbf{g} = \mathbf{w}$$

To locate the center of gravity, let us calculate the net torque acting on an object about an origin due to gravity This torque is the vector sum of the individual torques acting on different mass elements. That is,

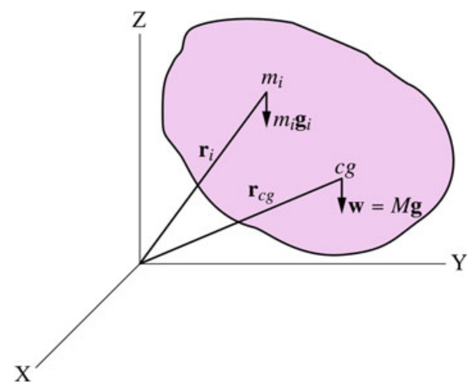


Fig. 8.13 The resultant gravitational force acting on an object is the resultant of the individual gravitational forces acting on different mass elements of the object

$$\boldsymbol{\tau} = \sum_i \boldsymbol{\tau}_i = \sum_i (\mathbf{r}_i \times m_i \mathbf{g}) = \left(\sum_i m_i \mathbf{r}_i\right) \times \mathbf{g}$$

$$\boldsymbol{\tau} = \frac{\left(\sum_i m_i \mathbf{r}_i\right)}{M} \times M\mathbf{g} = \mathbf{r}_{cm} \times \mathbf{w}$$

$$\boldsymbol{\tau} = \mathbf{r}_{cm} \times \mathbf{w}$$

Therefore, we conclude that if the gravitational field (g) is constant over the body, the center of gravity of the object coincides with its center of mass.

Example 8.7 Two blocks of masses $m_2 = 20$ kg and $m_1 = 10$ kg are supported by a uniform horizontal beam of length $L = 1.5$m and mass $M = 6$ kg (see Fig. 8.14). Find: (a) the normal force exerted by the fulcrum (supporting point) on the beam if it is placed under the center of gravity of the beam; (b) the distance x in which m_2 must be placed in order for the system to be balanced.

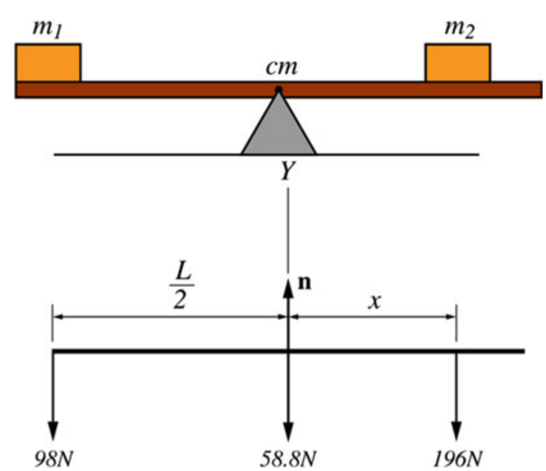


Fig. 8.14 Two blocks supported by a uniform horizontal beam

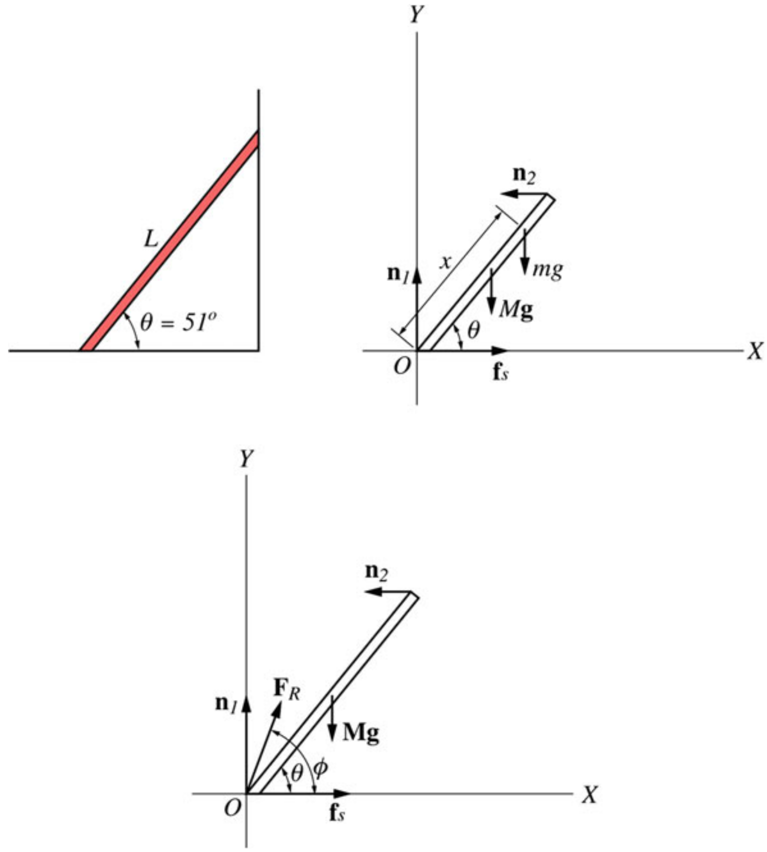


Fig. 8.15 The free-body diagram of a ladder of length L and mass $M = 20$ kg resting against a smooth vertical wall

Solution 8.7 (a) The free-body diagram of the system in shown in Fig. 8.14 where $w_1 = 196$ N, $w_2 = 98$ N, and $w = 58.8$ N. Applying Newton's second law to the beam gives

$$\sum F_y = n - (59 \text{ N}) - (98 \text{ N}) - (196 \text{ N}) = 0$$

and

$$n = 353 \text{ N}$$

(b) The net external torque about an axis passing through the center of the beam and perpendicular to the page is

$$\sum \tau_z = (98 \text{ N})(0.75 \text{ m}) - (196 \text{ N})x = 0$$

$$x = 0.37 \text{ m}$$

Example 8.8 A ladder of length L and mass $M = 20$ kg rests against a smooth vertical wall as shown in Fig. 8.15. If the center of gravity of the ladder is at a distance of $L/3$ from the base, determine: (a) the minimum coefficient of static friction such that the ladder does not slip; (b) the magnitude and direction of the resultant of the contact forces acting on the ladder at the base; (c) if a man of mass of 70 kg climbs up the ladder, what is the maximum distance the man can climb before the ladder slips if $\mu_s = 0.4$.

Solution 8.8 (a) Figure 8.15 shows the free-body diagram of the ladder. Applying Newton's second law to the ladder gives

$$\sum F_x = f_s - n_2 = 0$$

$$f_s = n_2$$

and

$$\sum F_y = n_1 - Mg = 0$$

$$n_1 = Mg$$

Applying Newton's second law in angular form about O (the point must be chosen to give minimum unknowns) we have

$$\sum \tau_z = n_2 L \sin\theta - \frac{1}{3} MgL \cos\theta = 0 \qquad (8.12)$$

If the ladder is at the verge of slipping the statistical frictional force is maximum $f_s = \mu_s n_1$. From Eq. 8.12, we have

$$n_2 = \frac{Mg}{3\tan\theta} = \frac{(196\ \text{N})}{3\tan(51°)} = 53\ \text{N} = f_s$$

hence

$$\mu_s = \frac{f_s}{n_1} = \frac{(53\ \text{N})}{(196\ \text{N})} = 0.27$$

(b) The resultant of the contact forces on the ladder at the base is

$$F_R = \sqrt{f_s^2 + n_1^2} = \sqrt{(53\ \text{N})^2 + (196)^2\ \text{N}} = 203\ \text{N}$$

the direction of F_R is

$$\phi = \tan^{-1}\frac{n_1}{f_s} = \tan^{-1}\frac{(196\ \text{N})}{(52.9\ \text{N})} = 75°$$

(c) The free-body diagram is shown in Fig. 8.15. From the equilibrium condition, we have

$$\sum F_x = f_s - n_2 = 0$$

and

$$\sum F_y = n_1 - mg - Mg = 0$$

or

$$f_s = n_2$$

and

$$n_1 = (m + M)g$$

Furthermore, the resultant external torque about O is

$$\sum \tau_z = n_2 L\sin\theta - \frac{1}{3}MgL\cos\theta - mgx\cos\theta = 0$$

thus

$$n_2 = \frac{g}{\tan\theta}\left(\frac{M}{3} + m\left(\frac{x}{L}\right)\right)$$

at the verge of slipping

$$f_s = \mu_s n_1 = \mu_s g(M + m) = (0.4)(9.8\ \text{m/s}^2)(90\ \text{kg}) = 353\ \text{N} = n_2$$

Hence

$$x = 0.54\ L$$

Example 8.9 A uniform beam of weight w and length L is held by two supports as in Fig. 8.16. A block of weight w_1 is resting on the beam at a distance of $L/6$ from the center of gravity of the beam. Find the magnitude of the forces exerted by the supports on the beam.

Solution 8.9 The free-body diagram of the system is shown in Fig. 8.16. Because the beam has a uniform density its center of mass and gravity are located at its geometrical center. Applying Newton's second law gives

$$\sum F_y = 0$$

$$F_2 + F_1 - w - w_1 = 0 \tag{8.13}$$

Taking the torque about an axis passing through one end (at F_1) gives

$$\sum \tau_z = 0$$

$$F_2 L - \frac{2}{3}Lw_1 - \frac{L}{2}w = 0 \tag{8.14}$$

From Eqs. 8.13 and 8.14 we have

$$F_2 = \frac{2}{3}w_1 + \frac{w}{2}$$

and

$$F_1 = \frac{w_1}{3} + \frac{w}{2}$$

Example 8.10 A man of mass of 80 kg is standing at the end of a uniform beam of mass of 30 kg and length of 12 m as shown in Fig. 8.17. Find the tension in the rope and the reaction force exerted by the hinge on the beam.

Solution 8.10 (a) The free-body diagram is shown in Fig. 8.17. Applying Newton's second law to the beam gives

$$\sum F_y = T\sin 50° + F_R\sin\theta - (294\ \text{N}) - (784\ \text{N}) = 0$$

$$\sum F_x = F_R\cos\theta - T\cos 50° = 0$$

The resultant torque about an axis passing through O is

$$\sum \tau_z = T\sin 50° L - L(784\,\text{N}) - \frac{L}{2}(294\ \text{N}) = 0$$

That gives $T = 1215.3$ N. Hence

$$F_R\cos\theta = T\cos 50° = (1215.3\ \text{N})(0.64) = 781.2\ \text{N} \tag{8.15}$$

and

$$F_R\sin\theta = -T\sin 50° + (294\ \text{N}) + (784\ \text{N}) = -(1215.3\ \text{N})(0.76) + (294\ \text{N}) + (784\ \text{N}) = 147\ \text{N} \tag{8.16}$$

Dividing Eq. 8.16 by Eq. 8.15 gives

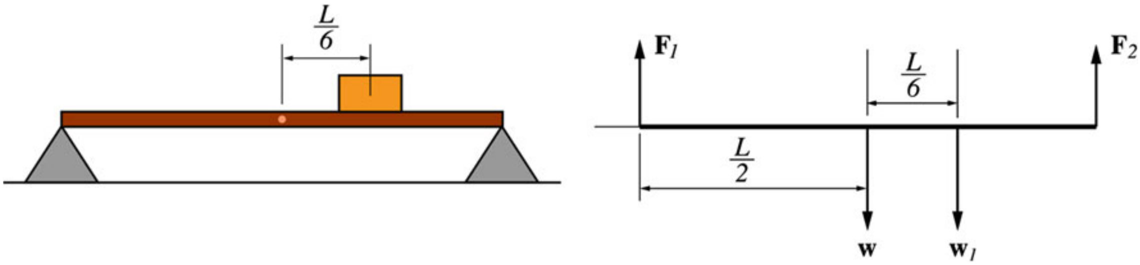
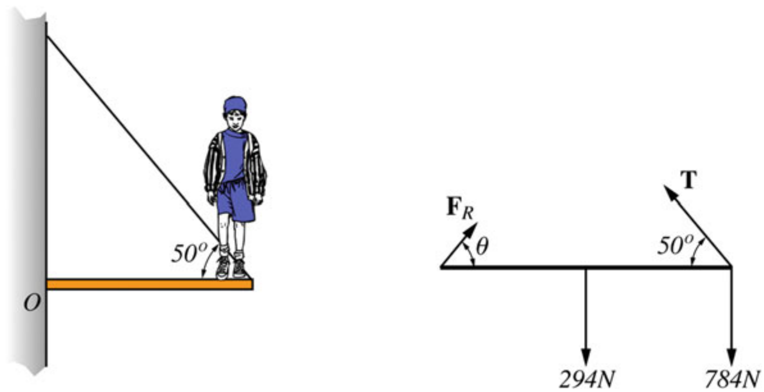


Fig. 8.16 A uniform beam of weight w and length L balanced by two supports

Fig. 8.17 A man standing at the end of a uniform beam



$$\tan\theta = \frac{(147\ \text{N})}{(781.2\ \text{N})} = 0.2$$

$$\theta = 10.6°$$

and

$$F_R = \sqrt{(147)^2 + (7812)^2} = 795\ \text{N}$$

Example 8.11 A uniform beam of weight of 120 N and length of L is in horizontal static equilibrium as in Fig. 8.18. Neglecting the masses of the ropes, find the tension in each string. (The center of mass is at $L/3$ from one end).

Solution 8.11 The free-body diagram is shown in Fig. 8.18. Applying Newton's second law to the beam gives

$$\sum F_y = T_1\cos\theta + T_2\cos 30° - (120\ \text{N}) = 0$$

or

$$T_1\cos\theta + T_2(0.87) = (120\ \text{N}) \tag{8.17}$$

Also

$$\sum F_x = T_1\sin\theta - T_2\sin 30° = 0$$

or

$$T_1\sin\theta = T_2\sin 30° \tag{8.18}$$

Taking the resultant torque on the beam about one end (at T_1) gives

$$\sum \tau = (120\ \text{N})\frac{L}{3} - LT_2\cos 30° = 0$$

or

$$T_2 = 46.2\ \text{N}$$

Substituting T_2 into Eqs. 8.18 and 8.17 gives

$$T_1\sin\theta = (46.2\ \text{N})\sin 30° = 23.1\ \text{N}$$

and

$$T_1\cos\theta + (46.2\ \text{N})(0.87) = (120\ \text{N})$$

$$T_1\cos\theta = 80\ \text{N}$$

Hence

$$\tan\theta = \frac{(23.1\ \text{N})}{(80\ \text{N})} = 0.3$$

That gives $\theta = 16.7°$ and $T_1 = (23.1\ \text{N})/\sin 16.7° = 80.3\ \text{N}$.

Example 8.12 A solid sphere of mass of 12 kg is in static equilibrium inside the wedge shown in Fig. 8.19. If the surface of the wedge is frictionless, find the forces that the wedge exerts on the sphere.

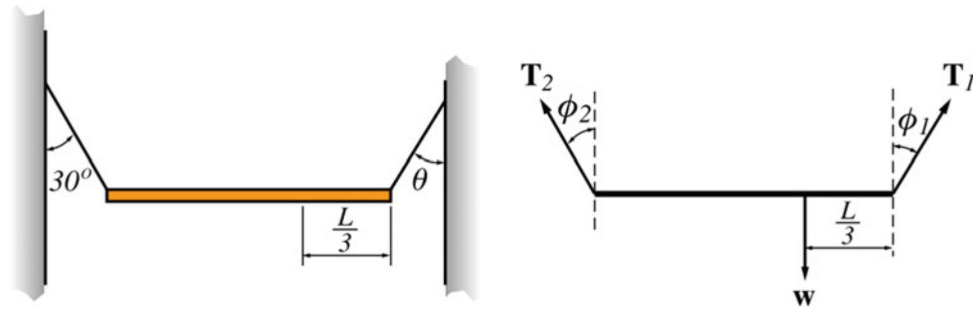


Fig. 8.18 A uniform beam held by ropes in static equilibrium

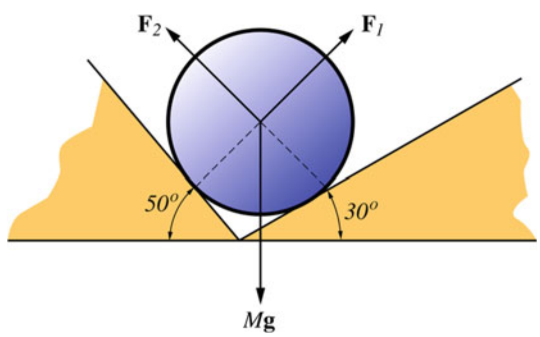


Fig. 8.19 A solid sphere in static equilibrium inside a wedge

Solution 8.12 Applying Newton's second law gives

$$\sum F_x = F_1 \sin 50^\circ - F_2 \sin 30^\circ = 0$$

or

$$F_1 = 0.65 F_2$$

Also we have

$$\sum F_y = F_1 \cos 50^\circ + F_2 \cos 30^\circ - Mg = 0$$

or

$$0.65 F_2 \cos 50^\circ + F_2 \cos 30^\circ - Mg = 0$$

That gives $F_2 = 91.6$ N. Therefore

$$F_1 = 0.65 F_2 = 0.65(91.6 \text{ N}) = 59.5 \text{ N}$$

Problems

1. A uniform cylinder of mass 3 kg and radius of 0.05 m rolls without slipping along a horizontal surface. Find the total energy of the cylinder at the instant its speed is 2 m/s.
2. A uniform solid cylinder of mass 10 kg and radius of 0.2 m rolls up the incline of angle 45° with an initial velocity of 15 m/s. Find the height in which the cylinder will stop.

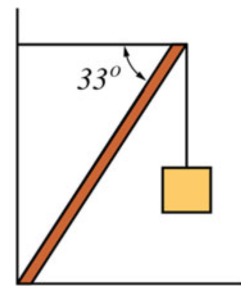


Fig. 8.20 A block suspended by a cable attached to a uniform rod

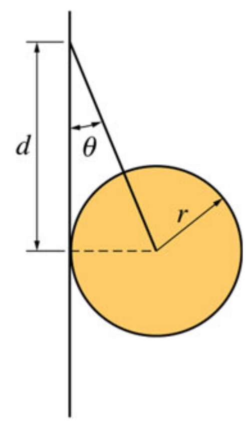


Fig. 8.21 A uniform sphere suspended by a light string and leaning on a frictionless wall

3. A wheel of mass 2 kg and radius of 0.05 m rolls without slipping with an angular speed of 3 rad/s on a horizontal surface. How much work is required to accelerate the wheel to an angular speed of 15 rad/s.
4. A block weighing 1000 N is held by a cable that is attached to a uniform rod of weight 500 N (see Fig. 8.20). Find (a) the tension in the cable, (b) the horizontal and vertical components of the force exerted on the base of the rod.
5. A uniform sphere of radius r and mass m is held by a light string and leans on a frictionless wall as in Fig. 8.21. If the string is attached a distance d above the center of the sphere, find (a) the tension in the string, (b) the reaction force exerted by the wall on the sphere.
6. Find the minimum force applied at the top of a wheel of mass M and radius R to raise it over a step of height h as in Fig. 8.22. Assume that the wheel does not slip on the step.

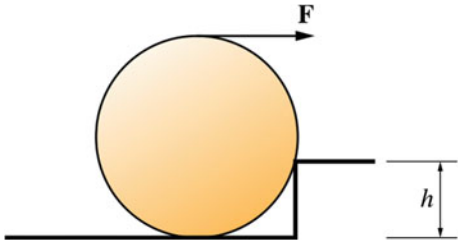


Fig. 8.22 A wheel raised over a step

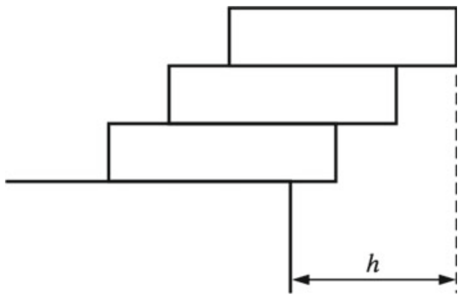


Fig. 8.23 Three identical uniform blocks on top of each other

7. Three identical uniform blocks each of length L are on top of each other as in Fig. 8.23. Find the maximum value of h in order for the stack to be in equilibrium.

9 Central Force Motion

9.1 Motion in a Central Force Field

A force is said to be central under two conditions. First, the direction of the force must always be toward or away from a fixed point (see Fig. 9.1). This point is known as the center of the force. Second, the magnitude of the force should only be proportional to the distance r between the particle and the center of the force. The central force may be written as

$$\mathbf{F} = f(r)\mathbf{r}_1 \tag{9.1}$$

where $\mathbf{r}_1$ is a unit vector in the direction of $\mathbf{r}$. Therefore, if $f(r) < 0$, then the central force is an attractive force since it is directed toward the center of the force O (as shown in Fig. 9.1) and if $f(r) > 0$, the force is repulsively directed away from O.

Example 9.1 Which of the following forces are repulsive and which are attractive? (a)$\mathbf{F} = \frac{-3}{\sqrt{r}}\mathbf{r}_1(b)\mathbf{F} = 4r^2\mathbf{r}_1(c)\mathbf{F} = r(r-2)\mathbf{r}_1$.

Solution 9.1 (a) Attractive, (b) repulsive, and (c) attractive if $0 < r < 2$ and repulsive if $r > 2$.

9.1.1 Properties of a Central Force

1. The resulting motion of the particle takes place in a plane. To show that we have from Eq. 9.1

$$\mathbf{F} = f(r)\mathbf{r}_1 = m\mathbf{a}$$

thus, a is parallel to $\mathbf{r}(\mathbf{r} = \mathrm{r}\mathbf{r}_1)$ and we may write

$$\mathbf{r} \times \mathbf{a} = \mathbf{0}$$

Hence,

$$\mathbf{r} \times \frac{d\mathbf{v}}{dt} = \mathbf{0}$$

or

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{0}$$

Thus,

$$\mathbf{r} \times \mathbf{v} = \mathbf{h} = \text{constant} \tag{9.2}$$

where $\mathbf{h}$ is a constant vector. Therefore, $\mathbf{r}$ and $\mathbf{v}$ always lie in the same plane where $\mathbf{h}$ is perpendicular to that plane for every value of t. As a result, the path of the particle takes place in a plane.

2. The angular momentum of the particle is conserved. From Eq. 9.2, we have

$$m(\mathbf{r} \times \mathbf{v}) = m\mathbf{h}$$

or

$$\mathbf{L} = m\mathbf{h} = \text{constant}$$

Thus, the angular momentum is equal to a constant at all times (conserved).

3. The position vector $\mathbf{r}$ of the particle with respect to the center of force sweeps out equal areas in equal times or in other words, the areal velocity is constant. To show that, consider the plane of motion to be the x–y plane. During an infinitesimally small time interval dt, the radius vector $\mathbf{r}$ sweeps out an area equal to dA. From Fig. 9.2, this area is equal to half of the area of a parallelogram with sides r and dr. That is,

$$d\mathbf{A} = \frac{1}{2}|\mathbf{r} \times d\mathbf{r}|$$

or

$$d\mathbf{A} = \frac{1}{2}|\mathbf{r} \times \mathbf{v}dt|$$

S. Alrasheed, *Principles of Mechanics*, Advances in Science, Technology & Innovation, https://doi.org/10.1007/978-3-030-15195-9_9

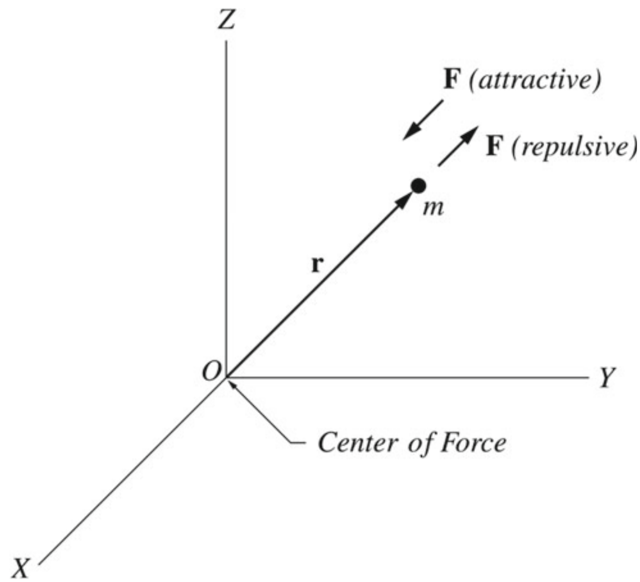


Fig. 9.1 The central force

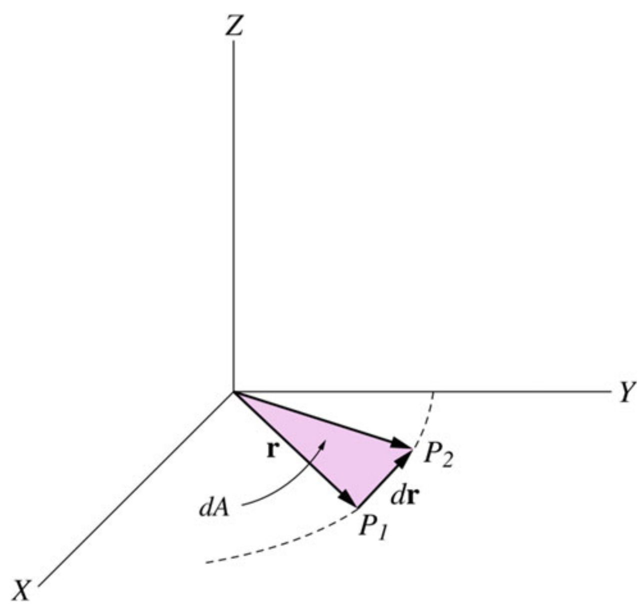


Fig. 9.2 During an infinitesimally small time interval dt, the radius vector $\mathbf{r}$ sweeps out an area equal to dA

or

$$\frac{d\mathbf{A}}{dt} = \frac{1}{2}|\mathbf{r} \times \mathbf{v}|$$

Thus,

$$\frac{dA}{dt} = \frac{h}{2} = \text{constant}$$

9.1.2 Equations of Motion in a Central Force Field

The most convenient coordinate system to describe the motion of a particle, under the influence of a central force, is the polar coordinate system. This convenience lies in the fact that the central force is in the r-direction. In Sect. 2.6, it has been shown that the acceleration of a particle in a plane, in terms of its polar coordinates, is given by

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{r}_1 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{\theta}_1$$

Applying Newton's second law to the particle gives

$$\mathbf{F} = m\mathbf{a}$$

$$f(r)\mathbf{r}_1 = m[(\ddot{r} - r\dot{\theta}^2)\mathbf{r}_1 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{\theta}_1]$$

That gives

$$f(r) = m(\ddot{r} - r\dot{\theta}^2) \tag{9.3}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \tag{9.4}$$

In Sect. 2.6, we've also seen that the velocity of a particle in polar coordinates is given by

$$\mathbf{v} = \dot{r}\mathbf{r}_1 + r\dot{\theta}\boldsymbol{\theta}_1$$

Therefore, we have

$$\mathbf{r} \times \mathbf{v} = r\mathbf{r}_1 \times (\dot{r}\mathbf{r}_1 + r\dot{\theta}\boldsymbol{\theta}_1) = r\dot{r}\,(\mathbf{r}_1 \times \mathbf{r}_1) + r^2\dot{\theta}(\mathbf{r}_1 \times \boldsymbol{\theta}_1)$$

$$= \mathbf{0} + r^2\dot{\theta}(\mathbf{r}_1 \times \boldsymbol{\theta}_1) = \mathbf{h}$$

Taking the plane of motion to be the x–y plane, then $\mathbf{r}_1 \times \boldsymbol{\theta}_1$ is parallel to the z-direction and we have

$$\mathbf{h} = r^2\dot{\theta}\mathbf{k} = h\mathbf{k}$$

Hence,

$$r^2\dot{\theta} = h \tag{9.5}$$

and Eq. 9.2 can be written as

$$\frac{d}{dt}(r^2\dot{\theta}) = 0$$

or

$$r^2\dot{\theta} = \text{constant}$$

Substituting Eq. 9.5 into Eq. 9.3 gives

$$f(r) = m\left(\ddot{r} - \frac{h^2}{r^3}\right) \tag{9.6}$$

Let $u = 1/r$, then $\dot{r} = -\dot{u}(1/u^2)$. Since $r^2\dot{\theta} = h$, we have $u^2 = \dot{\theta}/h$. Thus

$$\dot{r} = -h\left(\frac{\dot{u}}{\dot{\theta}}\right) = -h\left(\frac{du/dt}{d\theta/dt}\right) = -h\left(\frac{du}{d\theta}\right) \tag{9.7}$$

And

$$\ddot{r} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$\ddot{r} = -h\left(\frac{d^2u}{d\theta^2}\right)\dot{\theta} = -h^2u^2\left(\frac{d^2u}{d\theta^2}\right) \tag{9.8}$$

Substituting Eq. 9.8 into Eq. 9.6 gives

$$f(1/u) = m\left(-h^2u^2\left(\frac{d^2u}{d\theta^2}\right) - h^2u^3\right)$$

or

$$\frac{d^2u}{d\theta^2} + u = \frac{-1}{mh^2u^2} f(1/u) \tag{9.9}$$

This is the equation of path in a central force field.

9.1.3 Potential Energy of a Central Force

Consider a particle moving from point P_1 to P_2 (see Fig. 9.3) while a central force that has its center at the origin acts on it. The path of the particle may be considered as a combination of radial and curved segments. The central force is always acting in the direction of the radial segments and is perpendicular to the displacement along any of the curved segments. Thus, the work done by the central force along any curved segment is zero and the total work done in moving the particle along any path is equal to the work done along a radial line from r_i to r_f (see Fig. 9.4). That is, the work done by a central force is independent of path. It depends only on the initial and final positions of the particle.

From this, we conclude that the central force is a conservative force. You may also prove that $\nabla \times \mathbf{F} = \mathbf{0}$. Hence, there exists a potential energy and the work done by the gravitational force may be written as

$$W = -\Delta U$$

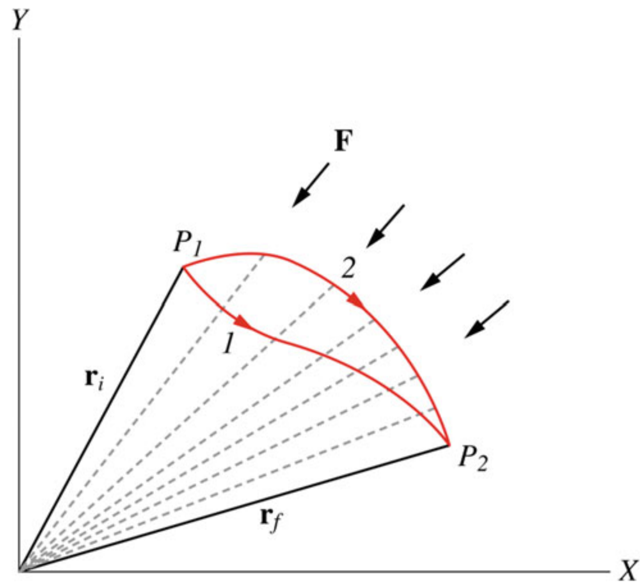


Fig. 9.3 A particle moving from point P_1 to P_2, while a central force that has its center at the origin acts on it

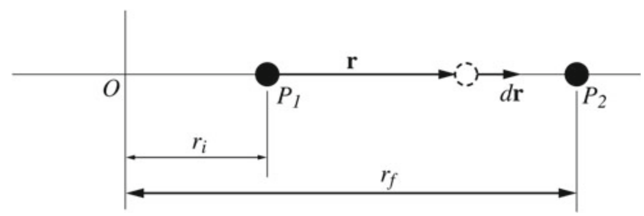


Fig. 9.4 The central force is always acting in the direction of the radial segments and is perpendicular to the displacement along any of the curved segments. Therefore, the total work done in moving the particle along any path is equal to the work done along a radial line from r_i to r_f

The work done in moving the particle from P_1 to P_2 is

$$W = \int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r} = \int_{r_i}^{r_f} f(r)\mathbf{r}_1 \cdot d\mathbf{r} = \int_{r_i}^{r_f} f(r)\frac{\mathbf{r}}{r}\cdot d\mathbf{r}$$

Since $\mathbf{r}\cdot d\mathbf{r} = rdr$, we have

$$W = \int_{r_i}^{r_f} f(r)dr$$

or

$$\Delta U = U_f - U_i = -\int_{r_i}^{r_f} f(r)dr \tag{9.10}$$

9.1.4 The Total Energy

Since F is a conservative force, it follows that the total energy is conserved (constant), that is,

$$E = \frac{1}{2}mv^2 + U(r)$$

Since

$$v^2 = \mathbf{v} \cdot \mathbf{v} = \dot{r}^2 + r^2\dot{\theta}^2$$

we have

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) \qquad (9.11)$$

Substituting Eqs. 9.5 and 9.7 into Eq. 9.11 gives

$$E = \frac{1}{2}m\left(h^2\left(\frac{du}{d\theta}\right)^2 + \left(\frac{1}{u^2}\right)(hu^2)^2\right) + U$$

or

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2(E-U)}{mh^2} \qquad (9.12)$$

9.2 The Law of Gravity

In 1687, Isaac Newton made a remarkable discovery. Newton stated that the force that holds planets in their orbit is the same force that makes an apple fall from a tree. Newton's law of gravity states that *every particle in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them*. The magnitude of this gravitational force is given by

$$F = \frac{Gm_1m_2}{r^2}$$

where m_1 and m_2 are the masses of the particles, r is the distance between them, and G is the universal gravitational constant. G has the same value if the particles (or objects) are located anywhere in the universe and it is given by

$$G = 6.672 \times 10^{-11}\,\mathrm{N.\,m^2/kg^2}$$

The gravitational force is effective when one or both the masses are very large. This is because G is a very small number. Note that, the gravitational force is not a contact force; it is a field force that can act through any medium. The direction of the gravitational force is along the line joining the two particles.

Therefore, the gravitational force is a central force since its magnitude is proportional only to the distance between the two particles (where one of the particles can be considered as the center of force), and its direction is along the line joining them (toward the center of force).

Figure 9.5 shows two particles of masses m_1 and m_2. Each particle exerts a gravitational force on the other. Let the gravitational force exerted on m_2 by m_1 to be $\mathbf{F}_{21}$, and that exerted on m_1 by m_2 to be $\mathbf{F}_{12}$. From Newton's third law of action and reaction, we have

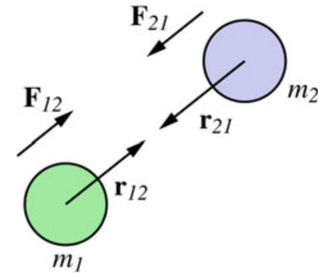


Fig. 9.5 Two particles of masses m_1 and m_2. Each particle exerts a gravitational force on the other

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

That is, the two forces form an action and reaction pair. In terms of unit vectors, we may write

$$\mathbf{F}_{21} = -\frac{Gm_1m_2}{r_{12}^2}\mathbf{r}_{12}$$

and

$$\mathbf{F}_{12} = -\frac{Gm_1m_2}{r_{21}^2}\mathbf{r}_{21}$$

where r_{12} is a unit vector that is directed along the line joining the two particles (directed from m_1 to m_2) and $\mathbf{r}_{21}$ is a unit vector directed from m_2 to m_1. The negative sign indicates that the force is attractive. That is, the force exerted on m_1 by m_2 will move m_1 in the direction opposite of r_{21}, i.e., toward m_2. Where the force exerted on m_2 by m_1 will move m_2 opposite to r_{12} (toward m_1). If particle P of mass of m_P interacts with a system of particles, the resultant gravitational force $\mathbf{F}_P$ exerted on particle P due to all particles in the system is the vector sum of the individual forces that each particle in the system exerts on particle P:

$$\mathbf{F}_P = \sum_{i=1}^{n}\mathbf{F}_{Pi} = \sum_{i=1}^{n}\frac{-Gm_Pm_i}{r_{iP}^2}\mathbf{r}_{iP}$$

where $\mathbf{r}_{iP}$ is a unit vector directed from the ith particle in the system toward the particle P and $\mathbf{F}_{Pi}$ is the force exerted on particle P by the ith particle. If particle P of mass m interacts with an extended body of mass M, the resultant gravitational force $\mathbf{F}_P$ exerted on particle P is the vector sum of the individual forces $d\mathbf{F}$ exerted on particle P due to each mass element dM in the object, but in this case, the sum is replaced by an integral

$$\mathbf{F}_P = \int d\mathbf{F} = -Gm\int\frac{dM}{r^2}\mathbf{r}_1$$

where $\mathbf{r}_1$ is a unit vector directed from the mass element dM to the particle as shown in Fig. 9.6. The force of gravity gives planets and other heavy celestial bodies their spherical shape. That is because as the mass of the body becomes larger the force of gravity becomes stronger and all particles from all

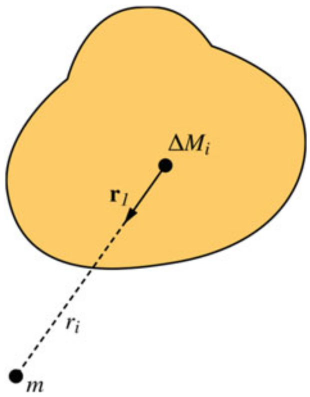


Fig. 9.6 A particle P of mass m interacting with an extended body of mass M

sides are attracted evenly toward the center. As a result, the body tends to have a spherical shape.

Example 9.2 Two particles of masses $m_1 = 0.2$ kg and $m_2 = 0.3$ kg are separated by a distance of 0.05 m. Find (a) the gravitational force that each particle exerts on the other; (b) at what distance a third particle $m_3 = 0.5$ kg must be placed at the other side of m_1 such that the net gravitational force on m_1 is zero. (All particles lie on a straight line).

Solution 9.2 (a)

$$F_{12} = F_{21} = \frac{Gm_1m_2}{r_{12}^2} = \frac{(6.67\times10^{-11}\ \mathrm{N\,m^2/kg^2})(0.2\ \mathrm{kg})(0.3\ \mathrm{kg})}{(0.05\ \mathrm{m})^2} = 1.6\times10^{-9}\mathrm{N}$$

(b)

$$F_{13} = \frac{Gm_1m_3}{r_{31}^2}$$

$$F_{12} = \frac{Gm_1m_2}{r_{21}^2}$$

If the net force on m_1 is zero, we have

$$\sum F_1 = F_{13} - F_{12} = 0$$

or

$$F_{13} = F_{12}$$

$$\frac{Gm_1m_3}{r_{31}^2} = \frac{Gm_1m_2}{r_{21}^2}$$

that gives

$$r_{31}^2 = \frac{m_3}{m_2}r_{21}^2 = \frac{(0.5\ \mathrm{kg})}{(0.3\ \mathrm{kg})}(0.05\ \mathrm{m})^2$$

$$r_{31} = 0.064\ \mathrm{m}$$

9.2.1 The Gravitational Force Between a Particle and a Uniform Spherical Shell

Case I: A Particle outside the Shell Consider a particle of mass m located outside a uniform spherical shell at point P as in Fig. 9.7. Imagine this shell to be made of a large number of thin rings each of outer thickness $Rd\theta$ and inner thickness l. The ring is so thin (since $d\theta$ is used) that every particle in the ring is at a distance s from P Furthermore, each particle in the ring exerts a gravitational force on the particle at P.

From the symmetry of the ring, if a particle (1) on the upper side exerts a gravitational force $\mathbf{F}_1$ on m, there is always another particle (2) at the opposite side of the ring exerting another force ($\mathbf{F}_2$) on the particle. Because $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal in magnitude, then their y components cancel each other out and their x components add up (see Fig. 9.7). Thus, the resultant force exerted on m due to all particles of the sphere is the sum of the x components of their forces. Therefore the resultant force on m is along the x direction (toward the center of the shell). The gravitational force exerted on m by a thin ring of mass dM is

$$dF_g = \frac{GmdM}{s^2}\cos\phi$$

To express dM in terms of the density of the ring, we find the volume of the thin ring

$$dV = (2\pi R\sin\theta)(Rd\theta)l = 2\pi lR^2\sin\theta d\theta$$

Since the shell has a uniform volume density ρ, dM is given by

$$dM = \rho dV = \rho 2\pi lR^2\sin\theta d\theta$$

Thus,

$$dF_g = \frac{2\pi\rho lmGR^2\cos\phi\sin\theta d\theta}{s^2} \tag{9.13}$$

From Fig. 9.7,

$$\cos\phi = \frac{r - R\cos\theta}{s} \tag{9.14}$$

From the cosines law, we have

$$s^2 = R^2 + r^2 - 2Rr\cos\theta \tag{9.15}$$

Substituting Eqs. 9.14 and 9.15 into Eq. 9.13 gives

$$dF_g = \frac{2\pi\rho lmGR^2(r - R\cos\theta)\sin\theta d\theta}{(r^2 + R^2 - 2rR\cos\theta)^{3/2}} \tag{9.16}$$

From Eq. 9.15, we have

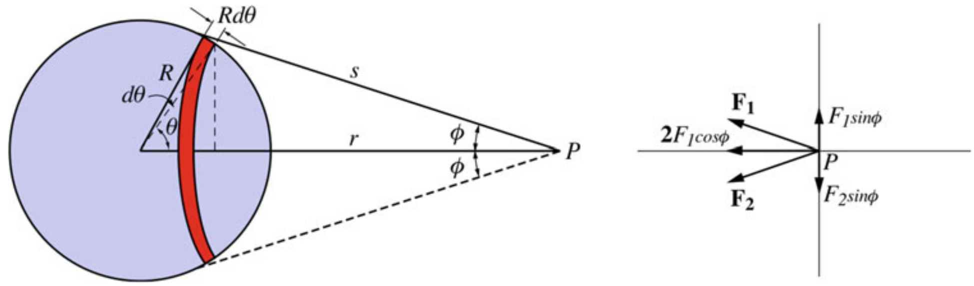


Fig. 9.7 Because $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal in magnitude, then their y components cancel each other out and their x components add up

$$2sds = 2rR\sin\theta d\theta$$

To integrate over all rings, θ will change from $\theta = 0$ to π. From Eq. 9.15, we have at $\theta = 0$, $s = r - R$ since $(r \geq R)$, and at $\theta = \pi$, $s = r + R$. Also, we have from Eq. 9.15

$$\cos\theta = \frac{R^2 + r^2 - s^2}{2rR}$$

Thus

$$r - R\cos\theta = \frac{r^2 + s^2 - R^2}{2r}$$

Substituting this into Eq. 9.16 gives

$$F_g = \frac{\pi G\rho l Rm}{r^2}\int_{r-R}^{r+R}\left(1 + \frac{r^2 - R^2}{s^2}\right)ds = \frac{4\pi G\rho l R^2 m}{r^2} \tag{9.17}$$

Since $4\pi R^2\rho l = M$, it follows that

$$F_g = \frac{GMm}{r^2}$$

That is, the spherical shell behaves as a particle of mass M located at its center.

Case II: A Particle inside the Shell If a particle is inside a uniform spherical shell, the derivation of the gravitational force exerted on the particle by the spherical shell is the same as if the particle were outside the shell, except that the lower integration limit is different. At $\theta = 0$, $s = R - r$ since $r < R$. Thus, we have

$$F_g = \frac{\pi G\rho l Rm}{r^2}\int_{R-r}^{r+R}\left(1 + \frac{r^2 - R^2}{s^2}\right)ds = 0$$

where $r < R$. That is, if the particle is inside the shell, the gravitational force exerted on it by the shell is zero. However, objects outside the shell may still exerts forces on the particle. In summary, we have

$$F_g = \frac{GMm}{r^2}\ (r \geq R)$$

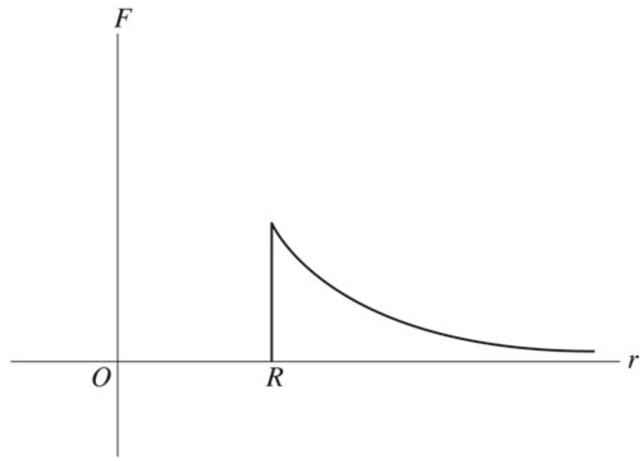


Fig. 9.8 The force exerted on a particle as a function of its r

$$F_g = 0\ (r < R)$$

Figure 9.8 shows the force exerted on a particle as a function of its location.

9.2.2 The Gravitational Force between a Particle and a Uniform Solid Sphere

Case I: A Particle outside the Sphere Consider a particle of mass m located outside a uniform solid sphere. The sphere may be considered to be made of a series of concentric spherical shells. The force exerted on the particle by each shell is given by

$$dF_g = \frac{GdMm}{r^2}$$

The mass of each shell is $dM = \rho dV = \rho 4\pi a^2 da$. Where ρ is the volume density of the sphere and a is the distance from the shell to the center of the sphere and da is the thickness of the shell, Hence,

$$dF_g = \frac{Gm\rho 4\pi a^2 da}{r^2}$$

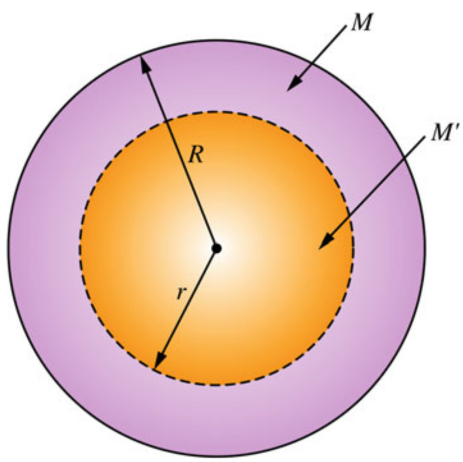


Fig. 9.9 If a particle of mass m is located inside a uniform solid sphere of mass M, then the gravitational force exerted on the particle is due only to the part of the sphere of radius $r < R$ and of mass of M

The total force exerted on m by the sphere is

$$F_g = \frac{Gm\rho 4\pi}{r^2}\int_0^R a^2 da$$

$$F_g = \frac{G(\rho^4/3\pi R^3)m}{r^2}$$

$$F_g = \frac{GMm}{r^2} \tag{9.18}$$

Thus, the solid sphere behaves as a particle of mass M located at the center of the sphere.

Case II: A Particle inside the Sphere If a particle of mass m is located inside a uniform solid sphere of mass M, then the gravitational force exerted on the particle is due only to the part of the sphere of radius $r < R$ and of mass of M (see Fig. 9.9). The remaining part of the sphere is a spherical shell which exerts no force on the particle since the particle is located inside it. From Eq. 9.18, the gravitational force exerted on the particle due to a sphere of radius r and mass M_1 is given by

$$F_g = \frac{GM_1 m}{r^2} \tag{9.19}$$

Since the sphere has a uniform density, we have

$$\rho = \frac{M_1}{V_1} = \frac{M}{V}$$

or

$$\frac{M_1}{M} = \frac{V_1}{V} = \frac{4/3\pi r^3}{4/3\pi R^3} = \frac{r^3}{R^3}$$

or

$$M_1 = M\frac{r^3}{R^3} \tag{9.20}$$

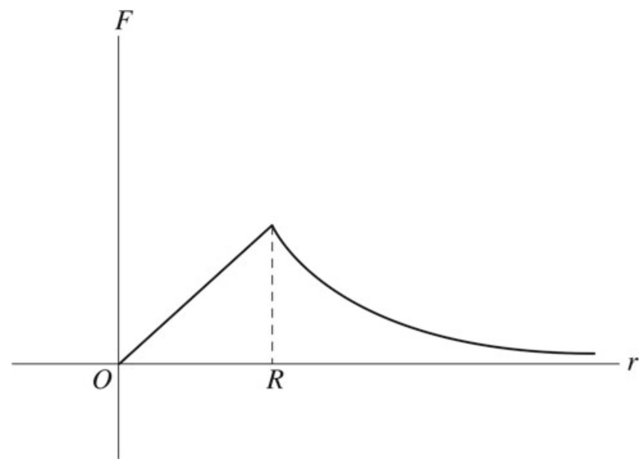


Fig. 9.10 The force exerted on a particle as a function of its r

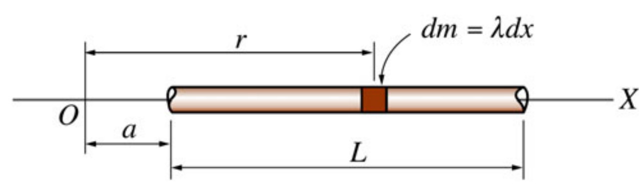


Fig. 9.11 The force exerted on a particle of mass m that is at a distance of a from a thin rod of mass M and length L

Substituting Eq. 9.20 into Eq. 9.19 gives

$$F_g = \frac{GmMr}{R^3}$$

where $r < R$. Therefore at the center of the sphere,

$$F_g = 0$$

Figure 9.10 shows the force exerted on a particle as a function of its location.

Example 9.3 (a) Find the gravitational force exerted on a particle of mass m that is at a distance of a from a thin rod of mass M and length L as in Fig. 9.11; (b) find the force in (a) if $a \gg L$.

Solution 9.3 (a)

$$dF = \frac{GmdM}{x^2}$$

since the rod is uniform we have

$$dM = \lambda dx = \frac{M}{L}dx$$

Thus

$$dF = \frac{GmM}{Lx^2}dx$$

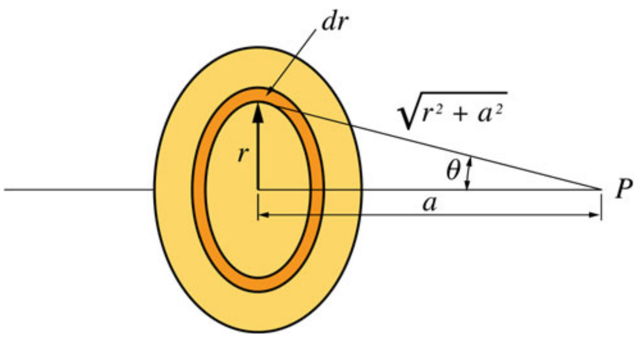


Fig. 9.12 The gravitational force exerted on a particle of mass m that is at a distance a from the center of a uniform solid disk of radius R and mass M

Integrating from a to $a+L$ gives

$$F = \frac{GmM}{L}\int_a^{a+L}\frac{dx}{x^2} = \frac{GmM}{L}\left[\frac{-1}{x}\right]_a^{a+L} = \frac{GmM}{L}\left[\frac{1}{a}-\frac{1}{a+L}\right] = \frac{GmM}{a(a+L)}$$

In vector form,

$$\mathbf{F} = \frac{GmM}{a(a+L)}\mathbf{i}$$

(b) if $a \gg L$, then

$$\mathbf{F} = \frac{GmM}{a^2}\mathbf{i}$$

That is, the rod can be considered as a particle of mass M that is at a distance a from m.

Example 9.4 Find the gravitational force exerted on a particle of mass m that is at a distance a from the center of a uniform solid disk of radius R and mass M as shown in Fig. 9.12.

Solution 9.4 Let us divide the disk into thin concentric rings of radius r and thickness dr. By symmetry, the resultant force on the particle is directed along the axis of the ring, since the y-components of the forces exerted by all particles of the ring will cancel out, where their x-components will add up. That is,

$$dF = \frac{GdMm\cos\theta}{r^2+a^2}$$

Since the mass element dM is given by $dM = \sigma(2\pi r dr)$, we have

$$dF = \frac{G\sigma(2\pi r dr)m\cos\theta}{r^2+a^2}$$

or

$$dF = \frac{G\sigma(2\pi r dr)ma}{(r^2+a^2)^{3/2}}$$

The total force is

$$F = 2\pi G\sigma ma\int_{r=0}^{R}\frac{rdr}{(r^2+a^2)^{3/2}} = \pi G\sigma ma\left[\frac{(r^2+a^2)^{-1/2}}{-1/2}\right]_0^R$$

$$F = 2\pi G\sigma m\left[1-\frac{a}{\sqrt{a^2+R^2}}\right]$$

Example 9.5 A uniform solid sphere has a mass of 4.7 kg and a radius of 0.05 m. Find the magnitude of the gravitational force that the sphere exerts on a 0.02 kg particle located at (a) 0.5 m from the center of the sphere; (b) 0.03 m from the center of the sphere; (c) at the surface of the sphere; (d) at the center of the sphere.

Solution 9.5 (a)

$$F_{1s} = \frac{GmM}{r^2} = \frac{(6.67\times10^{-11}\ \mathrm{Nm^2/kg^2})(0.02\ \mathrm{kg})(4.7\ \mathrm{kg})}{(0.5\ \mathrm{m})^2} = 2.5\times10^{-11}\ \mathrm{N}$$

(b)

$$F_{1s} = \frac{GmMr}{R^3} = \frac{(6.67\times10^{-11}\ \mathrm{Nm^2/kg^2})(0.02\ \mathrm{kg})(4.7\ \mathrm{kg})(0.03\ \mathrm{m})}{(0.05\ \mathrm{m})^3} = 1.5\times10^{-9}\ \mathrm{N}$$

(c)

$$F_{1s} = \frac{GmM}{R^2} = \frac{(6.67\times10^{-11}\ \mathrm{Nm^2/kg^2})(0.02\ \mathrm{kg})(4.7\ \mathrm{kg})}{(0.05\ \mathrm{m^2})} = 2.5\times10^{-9}\ \mathrm{N}$$

(d)

$$F_{1s} = 0$$

Example 9.6 Three concentric spherical shells have masses of M_1, M_2, and M_3 and radius of R_1, R_2, and R_3, respectively, as in Fig. 9.13. Find the gravitational force exerted on a particle of mass m located at (a) $r = a$(b)$r = b$(c)$r = c$(d)$r = d$.

Solution 9.6 (a)

$$F = 0$$

(b)

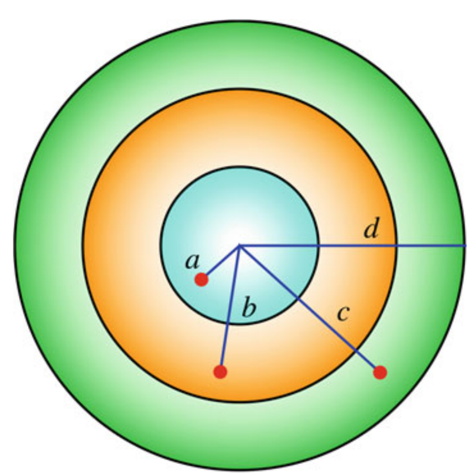


Fig. 9.13 Three concentric spherical shells

$$F = \frac{GM_1m}{b^2}$$

(c)

$$F = \frac{GM_1m}{c^2} + \frac{GM_2m}{c^2} = \frac{Gm}{c^2}(M_1 + M_2)$$

(d)

$$F = \frac{Gm}{d^2}(M_1 + M_2 + M_3)$$

Example 9.7 A spaceship of mass m_1 is moving along a straight line path between the earth and the sun. At what distance from the center of the earth will the gravitational force of the sun balances that of the earth?

Solution 9.7 At that point, we have

$$F_{1E} = F_{1S}$$

$$\frac{Gm_1M_E}{r^2} = \frac{Gm_1M_S}{(d-r)^2}$$

or

$$\frac{(d-r)^2}{r^2} = \frac{M_S}{M_E}$$

$$r = \frac{d[M_E - (M_EM_S)^{1/2}]}{M_E - M_S}$$

Example 9.8 An artificial satellite is moving in a circular orbit about the earth at a distance of 1500 km above the earth's surface. Find its speed and period.

Solution 9.8

$$\frac{Gm_sM_E}{r^2} = \frac{m_sv^2}{r}$$

$$v = \sqrt{\frac{GM_E}{r}}$$

where r is the distance between the center of the earth and the satellite. That is,

$$r = (6.37 \times 10^6\text{m}) + (1500 \times 10^3\text{m}) = 7.9 \times 10^6\text{m}$$

Hence,

$$v = \sqrt{\frac{GM_E}{r}} = \sqrt{\frac{(6.67 \times 10^{-11}\text{Nm}^2/\text{kg}^2)(5.98 \times 10^{24}\text{kg})}{(7.9 \times 10^6\text{m})}} = 7.1 \times 10^3\text{m/s}$$

$$T = \frac{2\pi r}{v} = \frac{2(3.1.4)(7.9 \times 10^6\text{m})}{(71 \times 10^3\text{m/s})} = 6968.8\text{s} = 116.15\,\text{min}$$

9.2.3 Weight and Gravitational Force

In Chap. 4, we've seen that the weight of an object is defined as the gravitational force exerted on the object by the earth (or any other astronomical object) and it is directed toward the center of the earth. The weight of an object is given by $\mathbf{w} = m\mathbf{g}$, where $\mathbf{g}$ is the free-falling acceleration and its value near the earth's surface is 9.8 m/s^2. The exact form of the gravitational force between any two objects was given earlier in this chapter by Newton's law of gravity In the case of an earth–particle system, the gravitational force that each one exerts on the other is

$$F_g = \frac{GM_Em}{r^2}$$

where M_E is the mass of the earth and m is the mass of the particle that is at a distance r from the center of the earth. Note that, it is assumed that the earth is a perfect sphere of uniform mass distribution, and therefore behaves as a particle. In reality, the earth is not a perfect sphere but rather an ellipsoid. Furthermore, the earth's density is not uniform since it varies with the radius of earth.

The earth's density also varies at the earth's surface from one region to another. In addition, if the earth's rotation is included, then the resultant force on an object will be its weight plus the centripetal force exerted on the object due to the rotation. However, these variations are often neglected. From the definition of weight, we have

$$w = mg = F_g = \frac{GM_Em}{r^2}$$

therefore

$$g = \frac{GM_E}{r^2} \tag{9.21}$$

As you can see the free-falling acceleration does not depend on the mass of the object as was predicted before. If the object is falling near the earth's surface, then distance r in Eq. 9.21 can be replaced by R_E which is the radius of the earth and we have

$$g = \frac{GM_E}{R_E^2}$$

If the object is at a distance h from the earth's surface, we may write

$$g = \frac{GM_E}{(R_E + h)^2}$$

Thus, the weight of an object decreases with increasing altitude. Table 9.1 shows the variation of g with altitude.

Table 9.1 Variation of g with altitude

Altitude h (km)	g (m/s^2)
1000	7.34
6000	2.6
10000	1.49
30000	0.3
60000	0.09

Example 9.9 A man can jump vertically upward from the earth's surface and reach an altitude of 0.2 m. Find the altitude the man can reach if he jumps with the same initial velocity on the surface of the moon.

Solution 9.9 Using the formula $y - y_0 = \frac{v^2 - v_0^2}{-2g}$ and by taking $y_0 = 0$ at the earth's surface and $y = h$ at the maximum height and that $v = 0$ there, we have

$$h = \frac{v_0^2}{2g}$$

Since the initial velocity of the man is the same on earth and on moon, we have

$$h_E g_E = h_m g_m$$

At the surface of the moon

$$g_m = \frac{GM_m}{R_m^2} = \frac{(6.67 \times 10^{-11}\ \text{N m}^2/\text{kg}^2)(7.36 \times 10^{22}\ \text{kg})}{(1.74 \times 10^6\ \text{m})^2} = 1.6\ \text{m/s}^2$$

$$h_m = h_E \frac{g_E}{g_m} = (0.2\ \text{m}) \frac{(9.8\ \text{m/s}^2)}{(1.6\ \text{m/s}^2)} = 1.2\ \text{m}$$

That is, the maximum height reached by the man on the moon is six times the height reached on earth.

Example 9.10 A neutron star of radius of 12 km has a gravitational acceleration of $1 \times 10^{12}\ \text{m/s}^2$ at its surface. Calculate its average density.

Solution 9.10 The gravitational acceleration of a particle near the surface of the star is

$$g = \frac{GM_n}{R_n^2}$$

$$M_n = \frac{gR_n^2}{G} = \frac{(1 \times 10^{12}\ \text{m/s}^2)(12 \times 10^3\ \text{m})^2}{(6.67 \times 10^{-11}\ \text{N m}^2/\text{kg}^2)} = 2 \times 10^{30}\ \text{kg}$$

$$\rho = \frac{3M_n}{4\pi R_n^3} = \frac{3(2 \times 10^{30}\ \text{kg})}{4(3.14)(12 \times 10^3\ \text{m})^3} = 2.8 \times 10^{17}\ \text{kg/m}^3$$

Example 9.11 Find the free-fall acceleration of a body that is at a distance of $0.05R_E$ above the surface of the earth.

Solution 9.11

$$g = \frac{GM_E}{(R_E + h)^2} = \frac{GM_E}{(R_E + 0.05R_E)^2} = \frac{GM_E}{(1.05R_E)^2}$$

$$= \frac{(6.67 \times 10^{-11}\ \text{N m}^2/\text{kg}^2)(5.98 \times 10^{24}\ \text{kg})}{(6.7 \times 10^6\ \text{m})^2} = 8.9\ \text{m/s}^2$$

9.2.4 The Gravitational Field

As mentioned previously, the gravitational force is a field force that can act through empty space, i.e., physical contact between objects is not necessary for such a force to act. An alternative way in describing the gravitational attraction is by introducing the concept of the gravitational field. Suppose a test particle of mass m_0 is placed at different points from another mass M (which represents the center of the gravitational force). At each point, the test particle will experience a gravitational force that depends on its distance from M and is given by

$$\mathbf{F}_g = \frac{-GMm_0}{r^2}\mathbf{r}_1$$

where $\mathbf{r}_1$ is a unit vector that points radially outwards. Therefore, M may be considered as producing a gravitational field in the space around it. This field can be sensed by the force that the test particle experience when placed in the vicinity of M. The gravitational field produced by M at any point in space is thus given by

$$\mathbf{g} = \frac{\mathbf{F}_g}{m_0} = \frac{-GM}{r^2}\mathbf{r}_1$$

That is, the gravitational field at a point is defined as the gravitational force per unit mass at that point. A map of the field can be drawn showing the gravitational field at any point in space. Figure 9.14 shows the gravitational field vectors near the earth's surface and at large distances from the earth. Note that, the gravitational field is an example of a static field since the field at any point is constant with time.

Example 9.12 Find the magnitude and direction of the gravitational field at the point P in the arrangement shown in Fig. 9.15, where all particles have equal masses.

Solution 9.12 Since all masses are equal, the net gravitational force at P is due to the sum of the x-components of F_3 and F_2. That is,

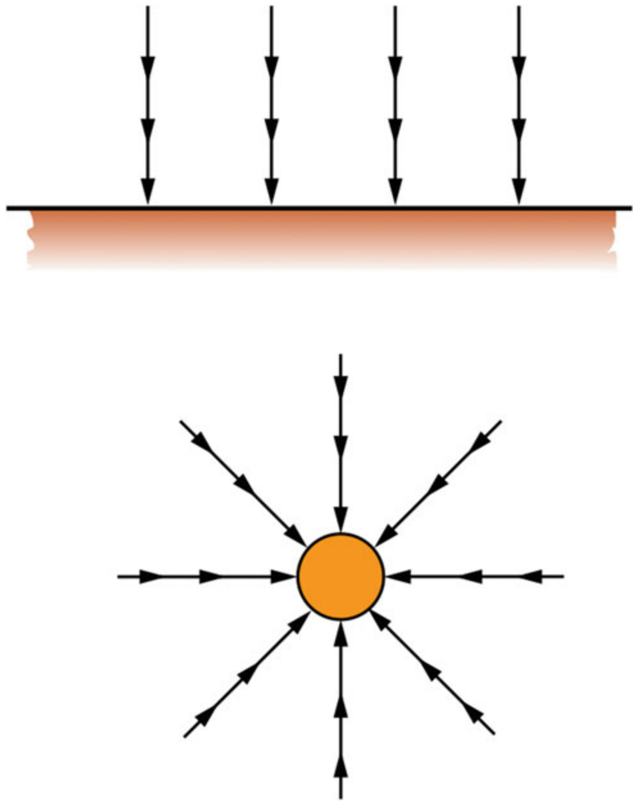


Fig. 9.14 The gravitational field vectors near the earth's surface and at large distances from the earth

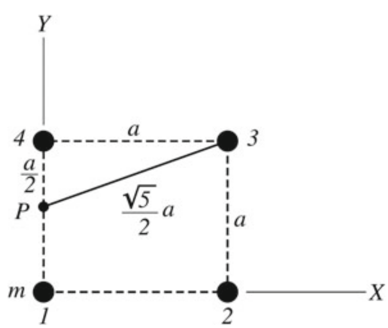


Fig. 9.15 Finding the magnitude and direction of the gravitational field at P

$$\mathbf{F} = 2F_3 \cos\theta \mathbf{i} = \frac{4Gmm_0}{5a^2}\cos\theta \mathbf{i} = \frac{4Gmm_0}{5a^2}\frac{2}{\sqrt{5}}\mathbf{i} = \frac{8Gmm_0}{5\sqrt{5}a^2}\mathbf{i}$$

$$\mathbf{g} = \frac{8Gm}{5\sqrt{5}a^2}\mathbf{i}$$

9.3 Conic Sections

Conic sections are produced if a double right circular cone intersects with a plane. It may be a circle, a parabola, an ellipse, or a hyperbola.

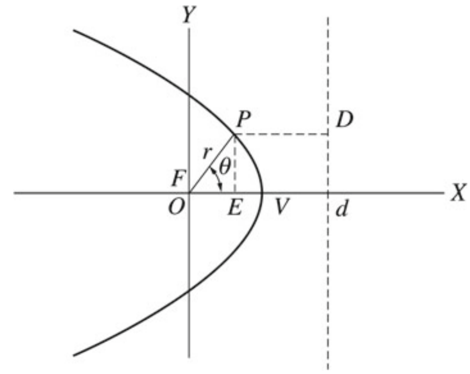


Fig. 9.16 A conic section has the property that the ratio e (called the eccentricity) of the distance between any point on the curve (for example point P) and another point called the focus (F) to the distance between P and a line called the directrix is equal to a constant

9.3.1 The Polar Equation of a Conic Section

A conic section has the property that the ratio e (called the eccentricity) of the distance between any point on the curve (for example point P) and another point called the focus (F) to the distance between P and a line called the directrix is equal to a constant (see Fig. 9.16). This constant differs from one conic section to another. Consider Fig. 9.16 where the focus F is at the origin O of the x and y coordinate system and the directrix is at $x = d$. Since the distance between P and F is

$$PF = r$$

then, the nearest distance between P and the directrix is

$$PD = d - FE = d - r\cos\theta$$

The eccentricity is therefore given by

$$e = \frac{PF}{PD} = \frac{r}{d - r\cos\theta}$$

Hence,

$$r = \frac{ed}{1 + e\cos\theta} \tag{9.22}$$

This equation is the polar equation of a conic section.

1. **Ellipse**: $e < 1$ From Fig. 9.17, you can see that at $\theta = 0$, $r = OV$ and at $\theta = \pi$, $r = OV'$. Substituting this into Eq. 9.22 gives

$$OV = \frac{ed}{1 + e}$$

and

$$OV' = \frac{ed}{1 - e}$$

Since VV' is the length of the major axis which is equal to $2a$, (a is the length of the semimajor axis) we have

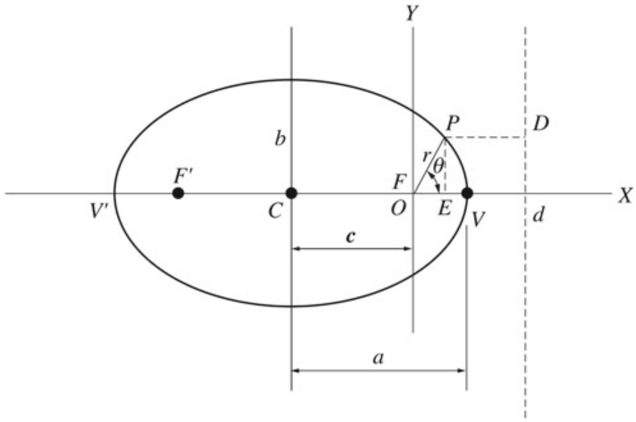


Fig. 9.17 In an ellipse, at $\theta = 0$, $r = OV$ and at $\theta = \pi$, $r = OV'$

$$OV + OV' = 2a \tag{9.23}$$

or

$$\frac{ed}{1+e} + \frac{ed}{1-e} = 2a$$

Hence,

$$a = \frac{ed}{1-e^2}$$

Or

$$ed = a(1-e^2)$$

Substituting into Eq. 9.22, the polar equation of an ellipse is

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

That gives

$$OV = \frac{a(1-e^2)}{1+e} = a(1-e) \tag{9.24}$$

and

$$OV' = \frac{a(1-e^2)}{1-e} = a(1+e) \tag{9.25}$$

The distance C between the center of the ellipse and the focus is

$$C = CV - OV = a - a(1-e) = ae$$

Since from Fig. 9.17, we have $c < a$, i.e., the distance between the foci is less than that between the vertices, then $e < 1$. Furthermore, you can prove that $c = \sqrt{a^2 - b^2}$ or $b = a\sqrt{1-e^2}$ where b is the length of the semiminor axis of the ellipse.

2. **Parabola**: $e = 1$ Since $e = 1$, Eq. 9.22 becomes

$$r = \frac{d}{1+\cos\theta}$$

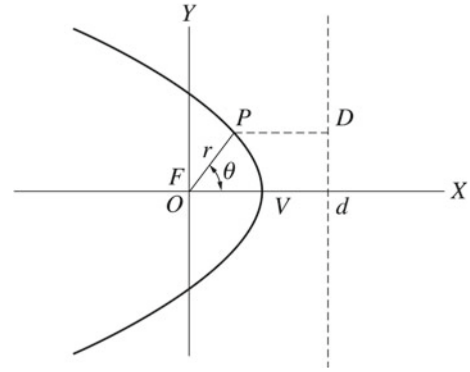


Fig. 9.18 In a parabola, as θ approaches π, r becomes infinite and hence $a \to \infty$

(Polar Equation of a Parabola) As θ approaches π, r becomes infinite and hence $a \to \infty$ (see Fig. 9.18).

3. **Hyperbola**: $e > 1$ The hyperbola has two branches as shown in Fig. 9.19. For the gravitational force, only the first branch (I) represents a possible motion of the particle since GM/h^2 must be positive. The polar equation of a hyperbola is given by

$$r = \frac{a(e^2-1)}{1+e\cos\theta}$$

9.3.2 Motion in a Gravitational Force Field

The path of a particle in any central force field can be found by solving the equation of motion $(d^2u/d\theta^2 + u = -1/(mh^2u^2)f(1/u)$ (Eq. 9.9) if the form of the force is known. In the case of a gravitational force, we have

$$f(r) = \frac{-GMm}{r^2}$$

where M is assumed to be fixed and that it is attracting a particle of mass m and r is the distance between them. In terms of u, we have

$$f(1/u) = -GMmu^2$$

Substituting this into the equation of motion gives

$$\frac{d^2u}{d\theta^2} + u = \frac{-1}{mh^2u^2}(-GMmu^2)$$

or

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2} \tag{9.26}$$

This equation is a nonhomogeneous linear differential equation. Its solution may be given by

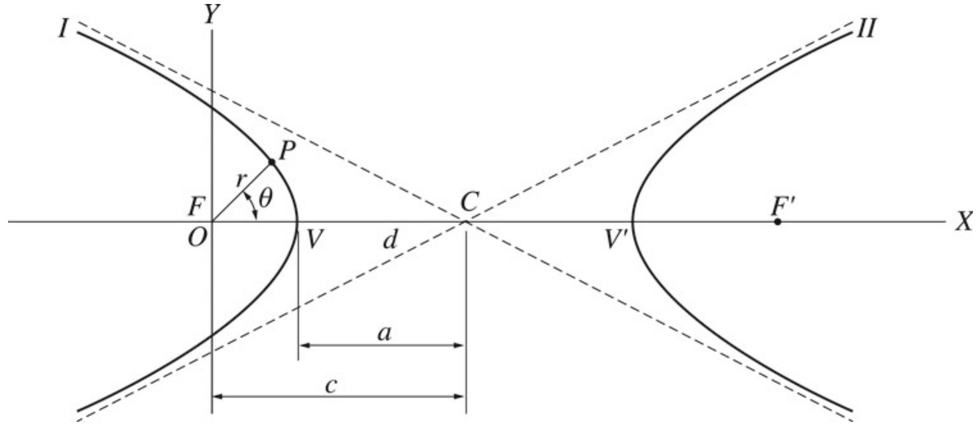


Fig. 9.19 The hyperbola

$$u = \frac{1}{r} = C\cos(\theta - \phi) + \frac{GM}{h^2}$$

where C and ϕ are integration constants. ϕ is known as the phase angle and it can be chosen to be $\phi = 0$ if the x-axis is chosen such that at $\theta = 0$, r is a minimum. That gives

$$u = \frac{1}{r} = C\cos\theta + \frac{GM}{h^2} \qquad (9.27)$$

or

$$r = \frac{h^2/GM}{1 + \frac{Ch^2}{GM}\cos\theta} = \frac{ed}{1 + e\cos\theta}$$

Thus, the path of the particle under the influence of the gravitational force field is a conic with $ed = h^2/GM$ and $d = 1/C$ and $e = h^2C/GM$. If a planet is moving in elliptical orbit about the sun, then the maximum and minimum distances of the planet from the sun (OV and OV') are called the aphelion and perihelion respectively If a satellite is moving about a planet in an elliptical orbit, the maximum and minimum distances of the satellite from the planet are called the apogee and perigee respectively.

9.3.3 The Gravitational Potential Energy

Consider a particle of mass m moving under the influence of a larger particle of mass M ($M \gg m$). By using Eq. 9.10 ($\triangle U = U_f - U_i = -\int_{r_i}^{r_f} f(r)dr$) and noting that $f(r) = -GMm/r^2$, the change in the gravitational potential energy of the system as m moves from r_i to r_f in the field of M is

$$\triangle U_g = U_{gf} - U_{gi} = \int_{r_i}^{r_f} \frac{GMm}{r^2}dr = GMm\int_{r_i}^{r_f}\frac{dr}{r^2}$$

$$= GMm\left[\frac{-1}{r}\right]_{r_i}^{r_f} = GMm\left(\frac{1}{r_i} - \frac{1}{r_f}\right)$$

That is, as the particle of mass m moves toward or away from M, the potential energy of the system decreases and increases respectively Note that, the lighter particle (m) gains most of the kinetic energy as the potential energy changes. By choosing the reference point at infinity ($r_i = \infty$) then $U_i = 0$ and taking $r_f = r$ gives

$$U_g(r) = \frac{-GMm}{r}$$

For more than two-particle systems, there is more than one gravitational force (one for each pair of particles). Hence, there is more than one potential energy The total potential energy is the sum of the potential energies of each pair. For example if there are three particles, the total potential energy is

$$U_{tot} = U_{12} + U_{13} + U_{23} = -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right)$$

Force from Potential Energy The gravitational force may be obtained from its corresponding potential energy. That is,

$$\mathbf{F}_g = -\frac{d}{dr}\left(\frac{-GMm}{r}\right)\mathbf{r}_1 = \frac{-GMm}{r^2}\mathbf{r}_1$$

Example 9.13 Find the potential energy of the system as shown in Fig. 9.20.

Solution 9.13

$$U = U_{12} + U_{13} + U_{23}$$

$$= -G\left(\frac{m_1m_2}{r_{12}} + \frac{m_1m_3}{r_{13}} + \frac{m_2m_3}{r_{23}}\right)$$

$$= -(6.67\times10^{-11}\ \mathrm{N\,m^2/kg^2})\left(\frac{(8\times10^4\ \mathrm{kg})}{(0.3\ \mathrm{m})} + \frac{(12\times10^4\ \mathrm{kg})}{(0.32\ \mathrm{m})} + \frac{(6\times10^4\ \mathrm{kg})}{(0.36\ \mathrm{m})}\right) = -5.4\times10^{-5}\ \mathrm{J}$$

Example 9.14 Two particles of equal masses (3kg) are separated by a distance of 0.3 m : (a) Find the potential energy

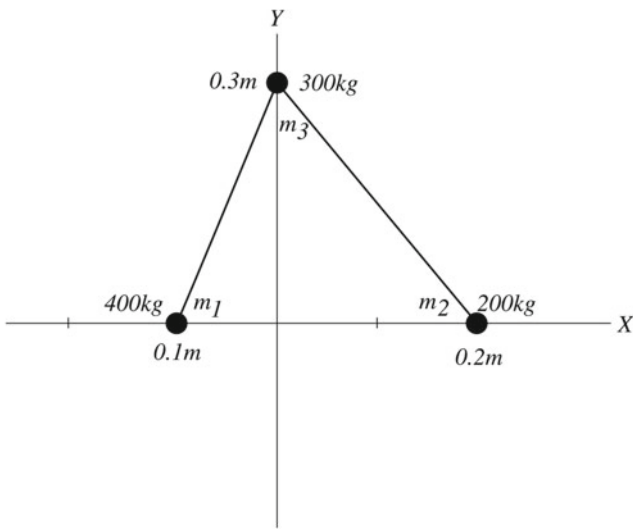


Fig. 9.20 The gravitational potential energy of a system of three particles

of the system; (b) how much work is required to reduce their separation to 0.1 m, (c) to increase their separation to 0.5 m.

Solution 9.14 (a)

$$U = \frac{-Gm^2}{r} = \frac{-(6.67\times 10^{-11}\ \mathrm{N\,m^2/kg^2})(3\ \mathrm{kg})^2}{(0.3\ \mathrm{m})} = -2\times 10^{-9}\ \mathrm{J}$$

(b) The work done by the gravitational force is

$$W = -\triangle U = U_i - U_f = -Gm^2\left(\frac{1}{r_i} - \frac{1}{r_f}\right)$$

$$= -(6.67\times 10^{-11}\ \mathrm{N\,m^2/kg^2})(3\ \mathrm{kg})^2\left(\frac{1}{(0.3\ \mathrm{m})} - \frac{1}{(0.1\ \mathrm{m})}\right)$$

that gives $W = 4\times 10^{-9}$ J. The work done by an external agent is $W = -4\times 10^{-9}$ J.
(c) The work done by the gravitational force is

$$W = -\triangle U = -Gm^2\left(\frac{1}{r_i} - \frac{1}{r_f}\right) = -(6.67\times 10^{-11}\ \mathrm{N\,m^2/kg^2})(3\ \mathrm{kg})^2\left(\frac{1}{(0.3\ \mathrm{m})} - \frac{1}{(0.5\ \mathrm{m})}\right)$$

$$W = -8\times 10^{-10}\ \mathrm{J}$$

The work done by an external agent is $+8\times 10^{-10}$J.

9.3.4 Energy in a Gravitational Force Field

The equation of motion in terms of energy is given by Eq. 9.12:

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2(E-U)}{mh^2}$$

The gravitational potential energy of a two-particle system of masses M and m is given by

$$U_g(r) = \frac{-GMm}{r}$$

In terms of u we may write

$$U_g(1/u) = -GMmu \tag{9.28}$$

Furthermore, the solution of the equation (Eq. 9.26) of motion in the gravitational force field is

$$u = \frac{1}{r} = C\cos\theta + \frac{GM}{h^2} \tag{9.29}$$

Substituting Eqs. 9.28 and 9.29 into Eq. 9.12 gives

$$(C\sin\theta)^2 + \left(C\cos\theta + \frac{GM}{h^2}\right)^2 = \frac{2E}{mh^2} - \frac{2}{mh^2}\left(-GMm\left(C\cos\theta + \frac{GM}{h^2}\right)\right)$$

That gives

$$C^2 = \frac{2E}{mh^2} + \frac{G^2M^2}{h^4}$$

or

$$C = \sqrt{\frac{2E}{mh^2} + \frac{G^2M^2}{h^4}} \quad (\textit{assuming } C > 0)$$

Substituting this value of C into Eq. 9.29 gives

$$u = \frac{GM}{h^2} + \sqrt{\frac{2E}{mh^2} + \frac{G^2M^2}{h^4}}\cos\theta$$

$$= \frac{GM}{h^2} + \frac{GM}{h^2}\sqrt{1 + \frac{2Eh^2}{G^2M^2m}}\cos\theta$$

or

$$u = \frac{GM}{h^2}\left[1 + \sqrt{1 + \frac{2Eh^2}{G^2M^2m}}\cos\theta\right] \tag{9.30}$$

Comparing this equation with the polar equation of a conic section (Eq. 9.22), we have

$$e = \sqrt{1 + \frac{2Eh^2}{G^2M^2m}}$$

Thus the trajectory of the particle is an ellipse if $e < 1$, that is if $E < 0$. Therefore, if the potential energy of the particle is greater than its kinetic energy the particle's path is an ellipse since it does not have enough energy to reach infinity. The trajectory of the particle is a parabola if $e = 1$ and hence

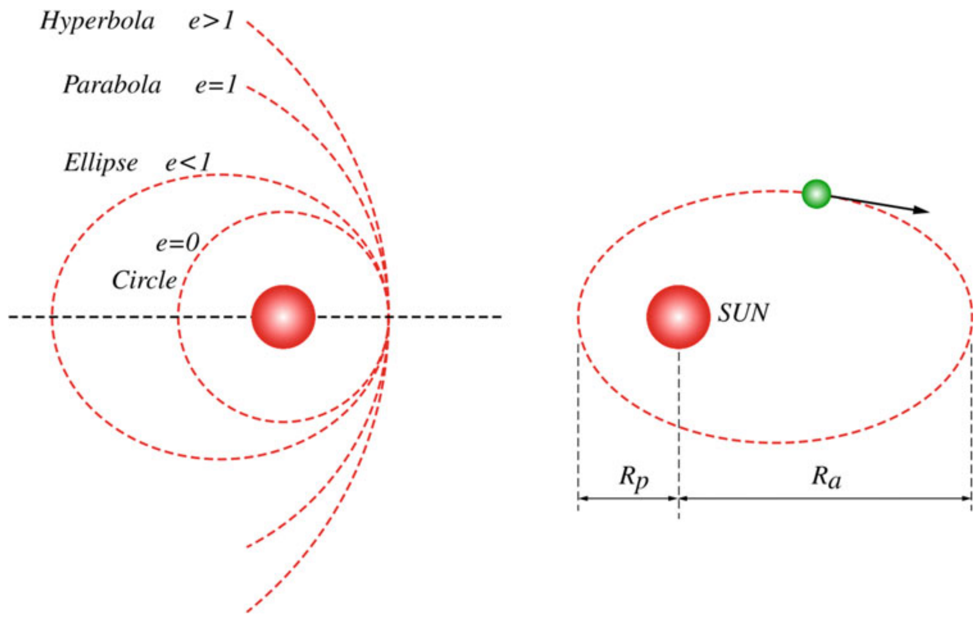


Fig. 9.21 Different paths

if $E = 0$. In that case, the kinetic energy of the particle is equal to its potential energy and thus it can reach infinity with zero kinetic energy. Finally, the trajectory of the particle is a hyperbola if $e > 1$ and therefore if $E > 0$. That is, if the kinetic energy of the particle is greater than its potential energy, then it will reach infinity with positive kinetic energy

- Elliptical Orbit $E < 0$
- Parabolic Orbit $E = 0$
- Hyperbolic Orbit $E > 0$

Different paths are shown in Fig. 9.21.

9.4 Kepler's Laws

After analyzing the astronomical data of the Danish astronomer Tycho Brahe, the German astronomer Johannes Kepler formulated his three laws of planetary motion.

9.4.1 Kepler's First Law

Every planet moves in an elliptical orbit with the sun at one focus as shown in Fig. 9.21.

Proof The gravitational force between the sun and a planet is

$$\mathbf{F} = \frac{-GM_SM_P}{r^2}\mathbf{r}_1$$

where M_S and M_P are the masses of the sun and the planet, respectively The acceleration of the planet is

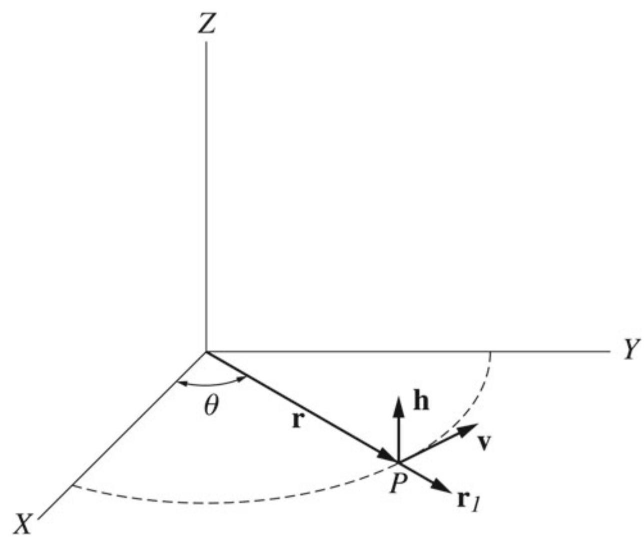


Fig. 9.22 From the first property of a central force we have $\mathbf{r} \times \mathbf{v} = \mathbf{h}$ =constant, where **h** is a constant vector perpendicular to the x-y plane

$$\mathbf{a} = \frac{-GM_S}{r^2}\mathbf{r}_1$$

From the first property of a central force, we have $\mathbf{r} \times \mathbf{v} = \mathbf{h}$ =constant, where **h** is a constant vector perpendicular to the x–y plane (see Fig. 9.22). Since $\mathbf{r} = r\mathbf{r}_1$ and $\mathbf{v} = d\mathbf{r}/dt = dr\mathbf{r}_1/dt = rd\mathbf{r}_1/dt + (dr/dt)\mathbf{r}_1$ we have

$$\mathbf{h} = r\mathbf{r}_1 \times \left(r\frac{d\mathbf{r}_1}{dt} + \frac{dr}{dt}\mathbf{r}_1\right) = r^2\left(\mathbf{r}_1 \times \frac{d\mathbf{r}_1}{dt}\right) + r\frac{dr}{dt}\left(\mathbf{r}_1 \times \mathbf{r}_1\right)$$

$$= r^2\left(\mathbf{r}_1 \times \frac{d\mathbf{r}_1}{dt}\right)$$

$$\mathbf{a}\times\mathbf{h} = \left(\frac{-GM_S}{r^2}\mathbf{r}_1\right)\times\left(r^2\left(\mathbf{r}_1\times\frac{d\mathbf{r}_1}{dt}\right)\right) = -GM_S\left[\left(\mathbf{r}_1\frac{d\mathbf{r}_1}{dt}\right)\mathbf{r}_1 - (\mathbf{r}_1\cdot\mathbf{r}_1)\frac{d\mathbf{r}_1}{dt}\right]$$

Using

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

Since $\mathbf{r}_1 \cdot d\mathbf{r}_1/dt = 0$ and $\mathbf{r}_1 \cdot \mathbf{r}_1 = r_1^2 = 1$, we have

$$\mathbf{a}\times\mathbf{h} = GM_S\frac{d\mathbf{r}_1}{dt} = \frac{d}{dt}(GM_S\mathbf{r}_1)$$

Also we have

$$\mathbf{a}\times\mathbf{h} = \frac{d\mathbf{v}}{dt}\times\mathbf{h} = \frac{d}{dt}(\mathbf{v}\times\mathbf{h})$$

since $\mathbf{h}$ is a constant vector. That gives

$$\frac{d}{dt}(\mathbf{v}\times\mathbf{h}) = \frac{d}{dt}(GM_S\mathbf{r}_1)$$

or

$$\mathbf{v}\times\mathbf{h} = GM_S\mathbf{r}_1 + \mathbf{C}$$

where $\mathbf{C}$ is a constant vector. Since

$$h^2 = \mathbf{h}\cdot\mathbf{h} = (\mathbf{r}\times\mathbf{v})\cdot\mathbf{h} = \mathbf{r}\cdot(\mathbf{v}\times\mathbf{h})$$

$$= (r\mathbf{r}_1)\cdot(GM_S\mathbf{r}_1 + \mathbf{C}) = rGM_S(\mathbf{r}_1\cdot\mathbf{r}_1) + r(\mathbf{r}_1\cdot\mathbf{C})$$

and since

$$\mathbf{r}_1\cdot\mathbf{C} = C\cos\theta$$

we have

$$h^2 = rGM_S + rC\cos\theta$$

or

$$r = \frac{h^2}{GM_S + C\cos\theta} = \frac{h^2/GM_S}{1 + C/GM_S\cos\theta}$$

This equation is of a conic section and since the only closed conic section is an ellipse the law is proved.

9.4.2 Kepler's Second Law

The radius vector drawn from the sun to the planet sweeps out equal areas in equal periods of time.

Proof This was proved in Sect. 9.1 as a property of a central force, where we've seen that for any central force, the position vector r of the particle from the center of force O sweeps out equal areas in equal times. That is,

$$\frac{dA}{dt} = \frac{h}{2} = \text{constant}$$

or

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant}$$

Here, the center of force is the sun and the particle is the planet, hence we have

$$\frac{dA}{dt} = \frac{L}{2M_P}$$

9.4.3 Kepler's Third Law

The square of the period of revolution of any planet about the sun is proportional to the cube of the semimajor axis of its orbit.

Proof The area of an ellipse is given by $A = \pi ab$, where a and b are the semimajor and semiminor axis of the ellipse, respectively. From Kepler's second law, the areal velocity is a constant given by

$$\frac{dA}{dt} = \frac{h}{2} = \text{constant}$$

Therefore, the period of revolution may be considered as the time it takes the radius vector to sweep an area of πab

$$T = \frac{\pi ab}{h/2}$$

From Sect. 9.3, we have $b = a\sqrt{1-e^2}$. That gives

$$T = \frac{\pi a^2\sqrt{1-e^2}}{h/2}$$

Also, we've seen that the eccentricity for the gravitational force is given by $e = h^2C/GM$ or $e = h^2C/GM_S$ in the case of the planet–sun system. Since $ed = a(1-e^2)$, we have

$$\frac{h^2}{GM_S} = a(1-e^2)$$

or

$$\sqrt{1-e^2} = \frac{h}{\sqrt{GM_S a}}$$

Thus,

$$T = \frac{2\pi a^2 h}{h\sqrt{GM_S a}} = \frac{2\pi}{\sqrt{GM_S}}a^{3/2}$$

or

$$T^2 = \left(\frac{4\pi^2}{GM_S}\right)a^3 = K_S a^3$$

where K_S is a constant that has a value given by

$$K_S = \frac{4\pi^2}{GM_S} = 2.97 \times 10^{-19}\ \mathrm{s^2/m^3}$$

This proves Kepler's third law. Note that, Kepler's laws apply also for satellites. In such cases, the mass of the sun in the previous equations is replaced by the earth or any other planet about which the satellite revolves.

9.5 Circular Orbits

The orbits of most planets in our solar system are almost circular. Next, we will find the total energy of a body of mass m moving in a circular orbit about a massive body of mass M that is assumed to be fixed (at rest) in an inertial frame of reference. From that energy, we will find the eccentricity and prove that the orbit is circular. The potential energy of such system is

$$U = \frac{-GMm}{r}$$

where r is the radius of the circular orbit. Applying Newton's second law to m gives

$$\frac{GMm}{r^2} = m\frac{v^2}{r} \tag{9.31}$$

Therefore, the kinetic energy of the particle is

$$K = \frac{1}{2}mv^2 = \frac{GMm}{2r}$$

The total energy of m is therefore given by

$$E = K + U = \frac{GMm}{2r} - \frac{GMm}{r}$$

or

$$E = -\frac{GMm}{2r} \tag{9.32}$$

In Sect. 9.4, the eccentricity of orbit in terms of energy was given by

$$e = \sqrt{1 + \frac{2Eh^2}{G^2M^2m}} \tag{9.33}$$

Substituting Eq. 9.32 into Eq. 9.33 gives

$$e = \sqrt{1 + \left(\frac{-GMm}{2r}\frac{2h^2}{G^2M^2m}\right)}$$

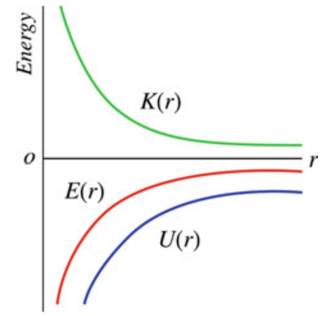


Fig. 9.23 The potential, kinetic and total energy as functions of r of an object in a circular orbit

Since $h = rv$ for a circular orbit and since $GMm/r^2 = mv^2/r$ and thus $v = \sqrt{GM/r}$, we have

$$h = \sqrt{rGM}$$

and

$$e = \sqrt{1 + \left(\frac{-GMm}{2r}\frac{2rGM}{G^2M^2m}\right)} = 0$$

Hence the orbit is circular. The potential, kinetic, and total energy as functions of r of an object in circular orbit are shown in Fig. 9.23.

Example 9.15 A satellite of mass of 1000 kg is in circular orbit about the earth at an altitude of $R_E/2$. What is the amount of work required to move the satellite to an altitude of $2R_E$.

Solution 9.15

$$W = \Delta E = E_f - E_i = GM_E m_s\left(\frac{-1}{2r_f} - \left(\frac{-1}{2r_i}\right)\right) = GM_E m_s\left(\frac{-1}{4R_E} + \frac{1}{R_E}\right)$$

$$= \frac{3GM_E m_s}{4R_E} = \frac{3(6.67\times10^{-11}\ \mathrm{Nm^2/kg^2})(5.98\times10^{24}\ \mathrm{kg})(1000\ \mathrm{kg})}{4(6.37\times10^{6}\ \mathrm{m})} = 4.7\times10^{10}\ \mathrm{J}$$

9.6 Elliptical Orbits

For an elliptical orbit, we have

$$ed = a(1-e^2) = \frac{h^2}{GM} \tag{9.34}$$

Substituting Eq. 9.33 into Eq. 9.34 gives

$$a\left(1 - \left(1 + \frac{2Eh^2}{G^2M^2m}\right)\right) = \frac{h^2}{GM}$$

That gives

$$E = \frac{-GMm}{2a}$$

The speed of an object in an elliptical orbit can be found from

$$K = E - U$$

$$\frac{1}{2}mv^2 = \frac{-GmM}{2a} + \frac{GmM}{r}$$

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

9.7 The Escape Speed

The escape speed v_{esc} is the speed required for an object to escape from the influence of the gravitational field of an astronomical object or system. Suppose an object of mass m is projected from the surface of a planet of mass M. The minimum speed for the object to escape the gravitational field of the planet is that in which the object has zero total mechanical energy at infinity. From conservation of energy, we have

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}mv_{esc}^2 + \left(\frac{-GMm}{R}\right) = 0$$

Hence

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

where R is the radius of the planet. If the object's initial speed is greater than the escape speed from that planet, then the object will still have some kinetic energy at infinity. Table.9.2 shows planetary data escape speeds

Example 9.16 What is the escape speed from the surface of: (a) Earth; (b) Mars; (c) Pluto.

Solution 9.16 (a)

$$v_{esc} = \sqrt{\frac{2GM_E}{R_E}} = \sqrt{\frac{2(6.67\times10^{-11}\ \text{Nm}^2/\text{kg}^2)(5.98\times10^{24}\ \text{kg})}{(6.37\times10^6\ \text{m})}} = 1.12\times10^4\ \text{m/s}$$

(b)

$$v_{esc} = \sqrt{\frac{2GM_M}{R_M}} = \sqrt{\frac{2(6.67\times10^{-11}\ \text{Nm}^2/\text{kg}^2)(6.42\times10^{23}\ \text{kg})}{(3.37\times10^6\ \text{m})}} = 5\times10^3\ \text{m/s}$$

(c)

$$v_{esc} = \sqrt{\frac{2GM_P}{R_P}} = \sqrt{\frac{2(6.67\times10^{-11}\ \text{Nm}^2/\text{kg}^2)(1.4\times10^{22}\ \text{kg})}{(1.5\times10^6\ \text{m})}} = 1.1\times10^3\ \text{m/s}$$

Example 9.17 What must be the minimum speed of a spacecraft that is at a distance of $3R_E$ from the center of the earth in order for it to escape the gravitational field of the earth?

Solution 9.17 The minimum speed is that in which the spacecraft has zero total mechanical energy at infinity,

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}mv_{esc}^2 + \left(\frac{-GM_E m}{3R_E}\right) = 0$$

$$v_{esc} = \sqrt{\frac{2GM_E}{3R_E}} = \sqrt{\frac{2(6.67\times10^{-11}\ \text{Nm}^2/\text{kg}^2)(5.98\times10^{24}\ \text{kg})}{3(6.37\times10^6\ \text{m})}} = 6.46\times10^3\ \text{m/s}$$

Example 9.18 Given that the period of Mars in its orbit about the sun is 1.88 years and its semimajor axis of the orbit is 22.8×10^{10} m, find the mass of the sun.

Solution 9.18 The period in seconds is

$$T = 5.94\times10^7\text{s}$$

From Kepler's second law, we have

$$M_S = \frac{4\pi^2a^3}{GT^2} = \frac{4(3.14)^2(22.8\times10^{10}\ \text{m})^3}{(6.67\times10^{-11}\ \text{N m}^2/\text{kg}^2)(5.94\times10^7\ \text{s})^2} = 1.99\times10^{30}\ \text{kg}$$

Example 9.19 Halley's Comet moves in an elliptical orbit about the sun. Its semimajor axis of orbit is 2.7×10^{12} m and its farthest distance ($OV' = R_a$) from the sun (the aphelion) is 5.3×10^{12} m. Find its period and its closest approach to the sun (the perihelion $OV = R_p$).

Solution 9.19 From Kepler's third law,

$$T^2 = K_S a^3 = (2.97\times10^{-19}\ \text{s}^2/\text{m}^3)(2.7\times10^{12}\ \text{m})^3$$

$$T = 2.4\times10^9\ \text{s} = 76\ \text{years}$$

From Eq. 9.23, we have

$$OV + OV' = 2a$$

or

$$R_p + R_a = 2a$$

$$R_p = 2a - R_a = 2(2.7\times10^{12}\ \text{m}) - (5.3\times10^{12}\ \text{m}) = 1\times10^{11}\ \text{m}$$

Example 9.20 If Pluto's distance from the sun at perihelion is 4.43×10^{12}m, find (a) the ratio of its speed at perihelion to its speed at aphelion; (b) the eccentricity of orbit; (c) the total energy.

Table 9.2 Planetary data escape speeds

Body	Mass (kg)	Radius (m)	Semimajor axis a (m)	Escape speed (km/s)
Mercury	3.18×10^{23}	2.43×10^{6}	5.79×10^{10}	4.3
Venus	4.88×10^{24}	6.06×10^{6}	1.08×10^{11}	10.3
Earth	5.98×10^{24}	6.37×10^{6}	1.496×10^{11}	11.2
Mars	6.42×10^{23}	3.37×10^{6}	2.28×10^{11}	5
Jupiter	1.90×10^{27}	6.99×10^{7}	7.78×10^{11}	60
Saturn	5.68×10^{26}	5.85×10^{7}	1.43×10^{12}	36
Uranus	8.68×10^{25}	2.33×10^{7}	2.87×10^{12}	22
Neptune	1.03×10^{26}	2.21×10^{7}	4.5×10^{12}	24
Pluto	1.4×10^{22}	1.5×10^{6}	5.91×10^{12}	1.1
Moon	7.36×10^{22}	1.74×10^{6}		2.3
Sun	1.99×10^{30}	6.96×10^{8}		618

Solution 9.20 From Table. 9.2, we have $a = 5.9 \times 10^{12}$ m, therefore

$$R_a = 2a - R_p = 2(5.9 \times 10^{12}\text{ m}) - (4.43 \times 10^{12}\text{ m}) = 7.37 \times 10^{12}\text{ m}$$

From the conservation of angular momentum,

$$M_P v_a R_a = M_P v_p R_p$$

hence,

$$\frac{v_p}{v_a} = \frac{R_a}{R_p} = \frac{(7.37 \times 10^{12}\text{m})}{(4.43 \times 10^{12}\text{m})} = 1.7$$

(b) From Eq. 9.24 ($OV = R_p = a(1-e)$), we have

$$e = 1 - \frac{R_p}{a} = 1 - \frac{(4.43 \times 10^{12}\text{ m})}{(5.9 \times 10^{12}\text{ m})} = 0.25$$

(c)

$$E = \frac{-GMm}{2a} = \frac{-(6.67 \times 10^{-11}\text{ Nm}^2/\text{kg}^2)(1.99 \times 10^{30}\text{ kg})(1.4 \times 10^{22}\text{ kg})}{2(5.9 \times 10^{12}\text{ m})} = -1.6 \times 10^{29}\text{ J}$$

Example 9.21 Two stars of equal mass M revolve about their center of mass with a speed v as shown in Fig. 9.24. Find the period of motion of each star.

Solution 9.21 The gravitational force that one star exerts on the other is

$$F = \frac{GM^2}{4r^2} = \frac{Mv^2}{r}$$

where r is the radius of orbit. Therefore,

$$v = \sqrt{\frac{GM}{4r}}$$

and

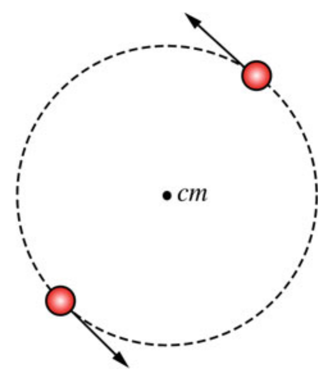


Fig. 9.24 Two stars of equal mass M revolve about their center of mass with a speed v

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{4r}{GM}} = 4\pi\sqrt{\frac{r^3}{GM}}$$

Example 9.22 A spaceship is fired from the surface of Mars with a speed of 12×10^3 m/s, find its speed at a very far distance from Mars.

Solution 9.22

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}mv_i^2 - \left(\frac{GmM_M}{R_M}\right) = \frac{1}{2}mv_f^2 + 0$$

$$v_f^2 = v_i^2 - \frac{2GM_M}{R_M}$$

$$= (12 \times 10^3\text{ m/s})^2 - \frac{2(6.67 \times 10^{-11}\text{ N m}^2/\text{kg}^2)(6.42 \times 10^{23}\text{ kg})}{(3.37 \times 10^6\text{ m})}$$

That gives $v_f = 1.1 \times 10^4$ m/s.

Problems

1. Calculate the gravitational force between the earth and (a) the sun, (b) the moon.
2. Calculate the gravitational acceleration at the surface of Mars.
3. Three particles of masses $m_1 = 2$ kg, $m_2 = 6$ kg, and $m_3 = 3$ kg are located at the points $(0, 0)$, $(0, 5)$, and $(5, 0)$, respectively. Find magnitude and direction of the resultant gravitational force exerted on m_3.
4. The Geosynchronous satellites move in a circular orbit in the equatorial plane of the earth. They move in such a way that they always remain over the same point on the earth. Find the height and velocity of this satellite.
5. If the eccentricity of the orbit of Mercury about the sun is $e = 0.206$ and its semimajor axis is $a = 0.387$ AU, find (a) the distance of its farthest and closest approach to the sun (the aphelion and perihelion), (b) its period, (c) its total energy, (d) its angular momentum. (1 AU $= 1.495 \times 10^{11}$m).
6. A body is released at a distance r from the center of the earth. Find its velocity just as it hits the surface of the earth.
7. Show that the speed of a satellite in an elliptical orbit about the earth at apogee and perigee are given by

$$v_p = \sqrt{\frac{GM}{a}}\sqrt{\frac{1+e}{1-e}} = \sqrt{\frac{GM}{a}}\sqrt{\frac{R_a}{R_p}}$$

and

$$v_a = \sqrt{\frac{GM}{a}}\sqrt{\frac{1-e}{1+e}} = \sqrt{\frac{GM}{a}}\sqrt{\frac{R_p}{R_a}}$$

8. An artificial satellite moves in an elliptical orbit about the earth. Its perigee and apogee altitudes are 1100 km and 4100 km respectively Find (a) the velocity of the satellite at perigee and apogee, (b) its semimajor axis, (c) its eccentricity, (d) the equation of its orbit, (e) its period, (f) its speed when it is at a distance of 3000 km above the earth's surface.
9. A satellite is at a distance of $1.2R_E$ from the center of the earth. Find the speed required for the satellite at this altitude (where it represents the orbit perigee) to be in (a) circular orbit, (b) parabolic orbit, (c) elliptical orbit of eccentricity of $e = 0.7$.
10. Suppose the earth suddenly stops moving about the sun, find the time it would take the earth to fall to the sun.

10 Oscillatory Motion

10.1 Oscillatory Motion

A motion repeating itself is referred to as periodic or oscillatory motion. An object in such motion oscillates about an equilibrium position due to a restoring force or torque. Such force or torque tends to restore (return) the system toward its equilibrium position no matter in which direction the system is displaced. This motion is important to study many phenomena including electromagnetic waves, alternating current circuits, and molecules. For a vibration to occur, two quantities are necessary to be present—stiffness and inertia.

10.2 Free Vibrations

When a system vibrates, a restoring force must be present. In addition to that force, there is always a retarding or damping force such as friction. If the effect of the damping force is small and can be neglected, then the motion is classified as free and undamped motion. Otherwise, the motion is classified as free damped motion. In both cases, the motion is known as free vibration since no forces other than the restoring and damping forces exist during vibration. If a driving force that does positive work on the system exists, the motion is classified as forced vibration.

This force may be applied externally to the system or sometimes is produced within the system. In this chapter, the case in which a restoring force is directly proportional to the displacement is considered. The resulting motion is then known as a harmonic vibration and the system is said to be linear. If the restoring force depends on the displacement in some other way, the resulting motion is known as anharmonic vibration and the system is said to be nonlinear.

10.3 Free Undamped Vibrations

This kind of motion is known as the simple harmonic motion. Next, we will examine examples of such motion in physics.

10.3.1 Mass Attached to a Spring

Consider a block of mass m attached to a light spring of spring constant k that is fixed at the other end (see Fig. 10.1). Suppose that the system lies on a frictionless horizontal surface. For small displacements, the restoring force acting on the block by the spring is given by Hook's law

$$F_s = -kx$$

As we've mentioned in Sect. 4.1, if the block is displaced slightly to the right (for example to $x = A$), the restoring spring force will accelerate the block to the left transferring its potential energy into kinetic energy As the block reaches its equilibrium position $x = 0$, all of its potential energy will be transformed into kinetic energy and it will overshoot to the other side. Again, as it moves left, the spring force decelerates the block to the right, transferring its kinetic energy into potential energy until all of its energy is potential at $x = -A$ where it comes to rest. At that point, it accelerates back to $x = 0$ and regains all of its kinetic energy where it overshoots again to $x = A$. Therefore, stiffness restores the mass where inertia is responsible for the mass to overshoot. From Newton's second law we, have

$$ma = -kx$$

or

$$m\frac{d^2x}{dt^2} + kx = 0$$

or

$$\frac{d^2x}{dt^2} + \omega_n^2 x = 0 \qquad (10.1)$$

where $\omega_n = \sqrt{k/m}$ is called the natural angular frequency of the system. The general solution of this equation is of the form

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t \qquad (10.2)$$

S. Alrasheed, *Principles of Mechanics*, Advances in Science, Technology & Innovation, https://doi.org/10.1007/978-3-030-15195-9_10

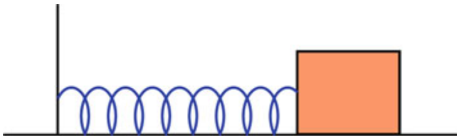


Fig. 10.1 A block of mass m attached to a light spring of spring constant k that is fixed at the other end

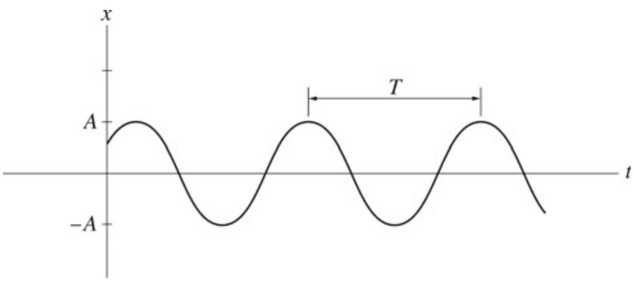


Fig. 10.2 Plot of x versus t for a simple harmonic oscillator

where A_1 and A_2 are arbitrary constants that can be found from the initial conditions. Therefore, there are many possible motions with the same angular frequency ω_n. By multiplying and dividing Eq. 10.2 by $\sqrt{A_1^2 + A_2^2}$, you can show that the solution may be written as

$$x(t) = A\cos(\omega_n t - \phi) \tag{10.3}$$

where $A = \sqrt{A_1^2 + A_2^2}$ is called the amplitude of motion and $\phi = \tan^{-1} A_2/A_1$ is called the phase constant. In general, ϕ is chosen such that $0 \le \phi \le \pi$.A and ϕ can be determined from the initial conditions, i.e., from the values of the displacement and velocity when the motion starts. The mass therefore oscillates between A and $-A$. The quantity $(\omega_n t - \phi)$ is called the phase angle. If this angle is increased by 2π, all physical quantities such as the displacement, velocity, and acceleration repeat themselves. The plot of x versus t is shown in Fig. 10.2. If A is fixed and ϕ is changed the motion will be the same except that the same physical quantities will appear either earlier or later than the preceding motion.

10.3.1.1 The Period and Frequency of Motion

The period of motion is the time required for one complete cycle or oscillation. Since the phase angle is changed by 2π after one complete cycle, we have for the mass–spring system,

$$\omega_n t + 2\pi = \omega_n(t + T)$$

or

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}}$$

The frequency is defined as the number of complete cycles per unit time

$$f_n = \frac{1}{T} = \frac{\omega_n}{2\pi}$$

This frequency is called the natural frequency of the motion. The unit of the frequency is cycles/s or hertz (Hz).

10.3.1.2 The Phase Difference

The phase constant ϕ is important when comparing two or more oscillations of the same frequency Suppose a certain vibration has $\phi = 0$, this means that at $t = 0$ the displacement is maximum $x = A$. If a second vibration has also $\phi = 0$, then the two vibrations are said to be in phase (see Fig. 10.3 part a). Otherwise, the two vibrations are out of phase. If the phase constant of the second vibration is $\phi > 0$, then the second vibration is leading the first vibration in phase by ϕ. If $\phi < 0$, then the second vibration is lagging the first by ϕ. If $\phi = \pm\pi$, the two vibrations are said to be in antiphase with each other (see Fig. 10.3 part b).

10.3.1.3 The Velocity and Acceleration

The velocity of the mass is

$$v(t) = \frac{dx}{dt} = -\omega_n A\sin(\omega_n t - \phi) \tag{10.4}$$

This can also be written as

$$v(t) = \omega_n A\cos\left(\omega_n t - \phi + \frac{\pi}{2}\right) \tag{10.5}$$

The acceleration of the mass is

$$a(t) = \frac{dv}{dt} = -\omega_n^2 A\cos(\omega_n t - \phi) \tag{10.6}$$

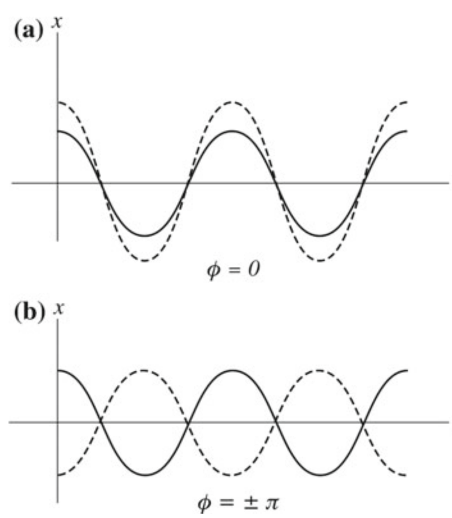


Fig. 10.3 **a** Two simple harmonic motions of the same frequency and same phase constant $\pi = 0$ but differing in amplitude. **b** Two simple harmonic motions of the same frequency and amplitude but differing in phase by $\phi = \pm\pi$

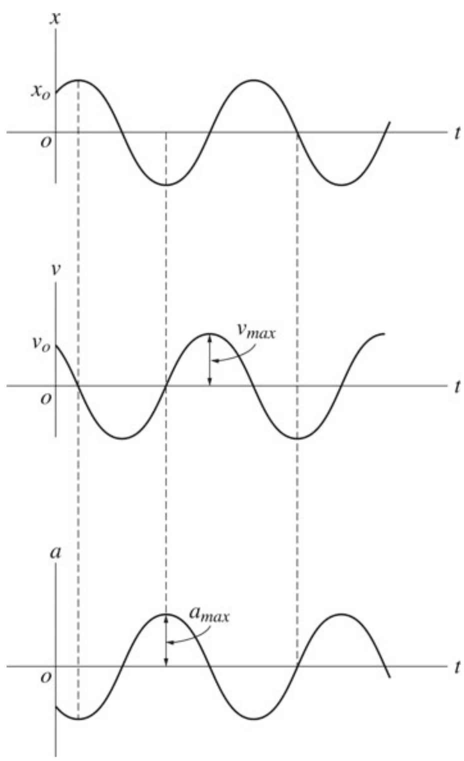


Fig. 10.4 The displacement, velocity and acceleration versus time

or

$$a(t) = \frac{dv}{dt} = \omega_n^2 A \cos(\omega_n t - \phi + \pi) \qquad (10.7)$$

Hence, the velocity and acceleration also vary harmonically with time with amplitudes $\omega_n A$ and $\omega_n^2 A$, respectively, but they all have the same angular frequency From Eqs. 10.5 and 10.7 you can see that the velocity leads the displacement by $\pi/2$ or 90. The acceleration on the other hand leads the velocity by $\pi/2$ and the displacement by π or 180. Figure 10.4 shows the displacement, velocity, and acceleration versus time.

10.3.1.4 Boundary Conditions

Boundary conditions are used to find A and ϕ for a specific vibration. Suppose that the vibration is measured when the stopwatch is set to zero, i.e., at $t = 0$ and that at that instant the mass is released from rest at a distance of $x = A_1$ from its equilibrium position. Substituting these conditions into Eqs. 10.3 and 10.4, we have

$$x = A\cos\phi = A_1 \qquad (10.8)$$

$$v = v_0 = -\omega_n A \sin\phi \qquad (10.9)$$

Dividing Eq. 10.9 by Eq. 10.8 gives

$$\tan\phi = \frac{-v_0}{\omega_n A_1}$$

Squaring and adding Eqs. 10.9 and 10.8 gives

$$A_1^2 + \left(\frac{v_0}{\omega_n}\right)^2 = A^2\cos^2\phi + A^2\sin^2\phi$$

or

$$A = \sqrt{A_1^2 + \left(\frac{v_0}{\omega_n}\right)^2}$$

Example 10.1 An object oscillates in simple harmonic motion according to the expression $x = (3\text{m})\cos(\pi t + \pi/3)$. Find (a) the amplitude, phase constant, period, and frequency of motion; (b) the displacement, velocity, and acceleration of the object at $t = 0.5$s(c) the time when the object first reach $x = -1.5$ m.

Solution 10.1 (a)

$$A = 3\,\text{m}$$

$$\phi = \frac{\pi}{3}$$

$$T = \frac{2\pi}{\omega_n} = \frac{(2\pi)}{\pi} = 2\text{ s}$$

and

$$f_n = \frac{1}{T} = \frac{1}{(2\text{s})} = 0.5\text{ Hz}$$

(b) At $t = 0.5$ s

$$x = (3\text{ m})\cos\left(\pi(0.5\text{ s}) + \frac{\pi}{3}\right) = -2.6\text{ m}$$

$$v = -(3\pi\text{ m/s})\sin\left(\pi t + \frac{\pi}{3}\right)$$

At $t = 0.5$ s

$$v = (-3\pi\text{ m/s})\sin\left(\pi(0.5\text{ s}) + \frac{\pi}{3}\right) = -4.7\text{ m/s}$$

$$a = (-3\pi^2\text{ m/s}^2)\cos\left(\pi t + \frac{\pi}{3}\right)$$

at $t = 0.5$ s

$$a = (-3\pi^2\text{ m/s}^2)\cos\left(\pi(0.5\text{ s}) + \frac{\pi}{3}\right) = 25.6\text{ m/s}^2$$

(c) at $x = -1.5$ m

$$(-1.5\text{ m}) = (3\text{ m})\cos\left(\pi t + \frac{\pi}{3}\right)$$

or

$$\frac{2\pi}{3} = \pi t + \frac{\pi}{3}$$

that gives $t = 0.3$ s.

Example 10.2 A 9 kg object is moving along the x-axis under the influence of a force given by $F = (-3x)$ N. Find (a) the equation of motion; (b) the displacement of the mass at any time if at $t = 0$, $x = 5$ m and $v = 0$.

Solution 10.2 (a)

$$F = -3x = ma = m\frac{d^2x}{dt^2}$$

hence,

$$\frac{d^2x}{dt^2} + 3x = 0$$

(b) The general solution of this equation is

$$x = A\cos\sqrt{3}t + B\sin\sqrt{3}t$$

Since at $t = 0$, $x = 5$ m, then $A = 5$ m and

$$x = (5\text{m})\cos\sqrt{3}t + B\sin\sqrt{3}t$$

also we have at $t = 0$, $dx/dt = 0$, or

$$-5\sqrt{3}\sin\sqrt{3}t + \sqrt{3}B\cos\sqrt{3}t = 0$$

and therefore $B = 0$. Thus,

$$x = (5\text{m})\cos\sqrt{3}t$$

Example 10.3 A 0.3 kg block is attached to a spring of force constant 20 N/m on a frictionless horizontal surface. If the initial displacement and velocity of the system is 0.02 m and 0.2 m/s, respectively, find the period, amplitude, and phase constant of motion.

Solution 10.3

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{(20\text{ N/m})}{(0.3\text{ kg})}} = 8.2\text{ rad/s}$$

$$A = \sqrt{A_1^2 + \left(\frac{v_0}{\omega_n}\right)^2} = \sqrt{(0.02\text{ m})^2 + \left(\frac{(0.2\text{ m/s})}{(82\text{ rad/s})}\right)^2} = 0.03\text{ m}$$

$$\tan\phi = \frac{-v_0}{\omega_n A_1} = \frac{-(0.2\text{ m/.s})}{(8.2\text{ rad/s})(0.03\text{ m})} = -0.8$$

$$\phi = -38.7^\circ$$

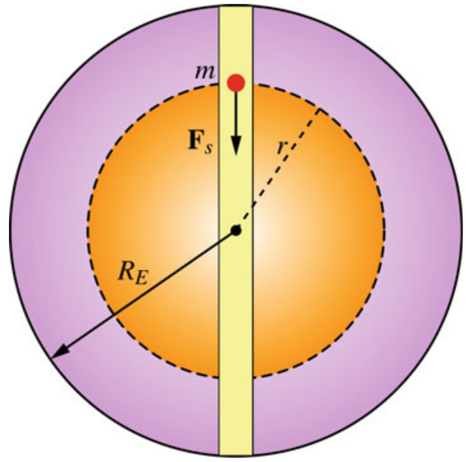


Fig. 10.5 A particle of mass m is dropped in a straight tunnel that is drilled through the earth and which passes through the center of earth

Example 10.4 A particle of mass m is dropped in a straight tunnel that is drilled through the earth and which passes through the center of earth as shown in Fig. 10.5. Show that the motion of the particle is simple harmonic motion and find its period.

Solution 10.4 Assuming that the earth is a perfect sphere of uniform density and since the particle is inside the earth, then from Sect. 9.2, the gravitational force exerted on the particle by the earth is

$$F = -\left(\frac{GmM_E}{R_E^3}\right)r = -kr$$

Because this force is directly proportional to the displacement and is opposite to it, then the particle will move in simple harmonic motion about the center of the earth. The equation of motion is

$$\frac{dr^2}{dt^2} + \left(\frac{GM_E}{R_E^3}\right)r = 0$$

hence,

$$\omega_n = \sqrt{\frac{GM_E}{R_E^3}} = \sqrt{\frac{(6.67\times10^{-11}\text{ Nm}^2/\text{kg}^2)(5.98\times10^{24}\text{ kg})}{(6.37\times10^6\text{ m})^3}} = 1.24\times10^{-3}\text{ rad/s}$$

$$T = \frac{2\pi}{\omega_n} = \frac{2(3.14)}{(1.24\times10^{-3}\text{ rad/s})} = 5055.4\text{ s} = 84.25\text{ min}$$

Example 10.5 A 0.4 kg block is connected to two springs of force constants $k_1 = 20$ N/m and $k_2 = 50$ N/m as in

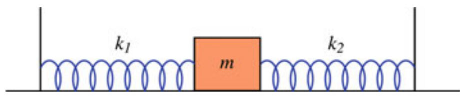


Fig. 10.6 A block connected to two springs

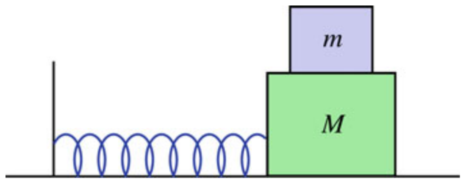


Fig. 10.7 A second block on top of a block connected to a spring

Fig. 10.6. Find (a) the total force acting on the block; (b) the period of motion.

Solution 10.5 The force that each spring exerts on the block acts in the opposite direction of the displacement, therefore we have

$$\sum F = -k_1 x - k_2 x = -(k_1 + k_2)x = -(70 \text{ N/m})x$$

Thus the two springs can be considered as one spring of a force constant of $(k_1 + k_2)$. The period of motion is therefore

$$T = 2\pi\sqrt{\frac{m}{k_1 + k_2}} = 2(3.14)\sqrt{\frac{(0.4 \text{ kg})}{(70 \text{ N/m})}} = 0.5 \text{ s}$$

Example 10.6 A 6 kg block is connected to a light spring of force constant of 300 N/m on a frictionless horizontal surface. On top of it a second block of mass of 2 kg is placed. If the coefficient of static friction between the two blocks is 0.4 (see Fig. 10.7), find the maximum amplitude the system can have when it is in simple harmonic motion such that there is no slipping between the blocks.

Solution 10.6 The maximum acceleration of the lower block is $a_{\max} = \omega_n^2 A$. In order for the upper block not to slip, the force of static friction between the two blocks must produce the same acceleration as the lower block. The maximum statistical frictional force that can be exerted on the upper block is $\mu_s mg$ and hence, the maximum acceleration that the force of static friction can produce is $\mu_s g$. Therefore, $\mu_s g = a_{\max} = \omega_n^2 A$. Since

$$\omega_n = \sqrt{\frac{k}{(m+M)}}$$

we have

$$A = \frac{\mu_s g}{\omega_n^2} = \frac{\mu_s g(m+M)}{k} = \frac{(0.4)(9.8 \text{ m/s}^2)(8 \text{ kg})}{(300 \text{ N/m})} = 0.1 \text{ m}$$

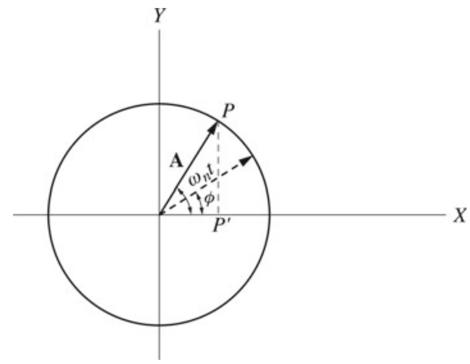


Fig. 10.8 A particle in uniform circular motion

10.3.2 Simple Harmonic Motion and Uniform Circular Motion

Consider a circle of radius A centered at the x and y axes as shown in Fig. 10.8. Let A be the position vector of a particle P rotating with a constant angular speed ω_n in the anticlockwise direction. The particle is thus in uniform circular motion. Suppose P starts the rotation at $t = 0$ at an angle of ϕ measured from the positive x-axis. At any time, the angular position of the particle is given by $(\omega_n t + \phi)$, therefore the vector position of the particle at any time is

$$\mathbf{A} = x\mathbf{i} + y\mathbf{j} = A\cos(\omega_n t + \phi)\mathbf{i} + A\sin(\omega_n t + \phi)\mathbf{j}$$

Hence,

$$x = A\cos(\omega_n t + \phi)$$

and

$$y = A\sin(\omega_n t + \phi)$$

That is, as P moves in uniform circular motion, its projection P′ on the x-axis moves in simple harmonic motion where the radius of the circle is equal to the amplitude of motion. The projection of P along the y-axis also undergoes simple harmonic motion. Thus, uniform circular motion may be considered as a combination of the simple harmonic motions of the projections of P on each axis. These two simple harmonic motions have equal amplitudes and angular frequencies but are in quadrature with each other (they differ in phase by $\pi/2$). The linear tangential velocity of the particle in this uniform circular motion is given by

$$v = A\omega_n$$

The x component of the velocity is from Fig. 10.9 given by

$$v_x = -\omega_n A\sin(\omega_n t + \phi)$$

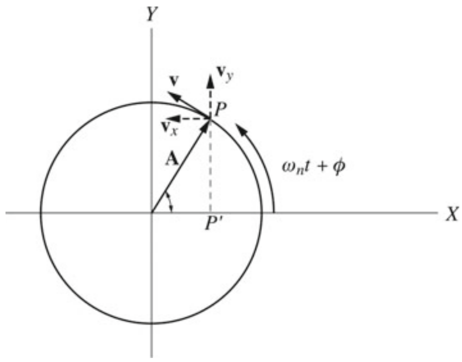


Fig. 10.9 The velocity components of the particle

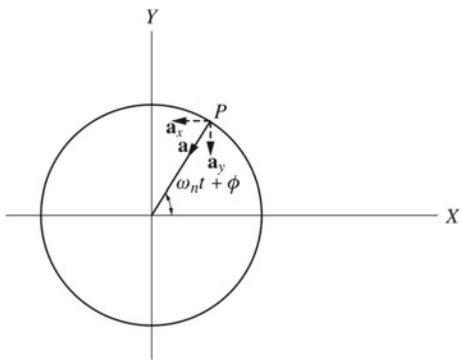


Fig. 10.10 The acceleration components of the particle

The acceleration of the particle in uniform circular motion is just the radial (centripetal) acceleration that is given by

$$a = \frac{v^2}{A} = A\omega_n^2$$

The x components of the acceleration (see Fig. 10.10) is

$$a_x = -\omega_n^2 A\cos(\omega_n t + \phi)$$

Hence as you can see, the displacement, velocity, and acceleration of the projection of P onto the x (or y axis) are the same as that of a simple harmonic motion. From this, we conclude that the simple harmonic motion can be represented as the projection of uniform circular motion along a diameter of the circle.

10.3.3 Energy of a Simple Harmonic Oscillator

Since in a simple harmonic oscillator, there aren't any dissipative forces, the total mechanical energy of the system is conserved and is equal to the sum of its kinetic and potential energies, that is

$$E = K + U$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega_n^2 A^2\sin^2(\omega_n t + \phi)$$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_n t + \phi)$$

Thus,

$$E = \frac{1}{2}kA^2[\sin^2(\omega_n t + \phi) + \cos^2(\omega_n t + \phi)]$$

or

$$E = \frac{1}{2}kA^2 = \text{constant}$$

The equation of motion of a simple harmonic oscillator can be obtained from the total mechanical energy of the system as follows:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \qquad (10.10)$$

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = 0$$

or

$$m\ddot{x} + kx = 0$$

Hence

$$\ddot{x} + \omega_n^2 x = 0$$

where $\omega_n = \sqrt{k/m}$. As the mass moves, its kinetic energy is transformed into potential energy and vice versa. Figure 10.11 shows the kinetic energy and potential energy of the system as a function of time and as a function of the displacement respectively Note that the variation of U and K with time is at twice the angular frequency of the variation of x, v, and a with time. This is because the potential energy is converted to kinetic energy twice in each cycle. The velocity of the simple harmonic oscillator can be obtained from the total energy of the system. From Eq. 10.10, we have

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)}$$

Hence, the maximum speed is at $x = 0$ and is zero at $x = \pm A$ which are called the turning points as discussed in Chap. chap444.

Example 10.7 A 0.3 kg mass is attached to a light spring. If the total energy of the system is 0.025 J and the amplitude of motion is 5 cm, find the period and frequency of motion.

Solution 10.7

$$E = (0.025\,\text{J}) = \frac{1}{2}kA^2 = \frac{1}{2}k(0.05\,\text{m})^2$$

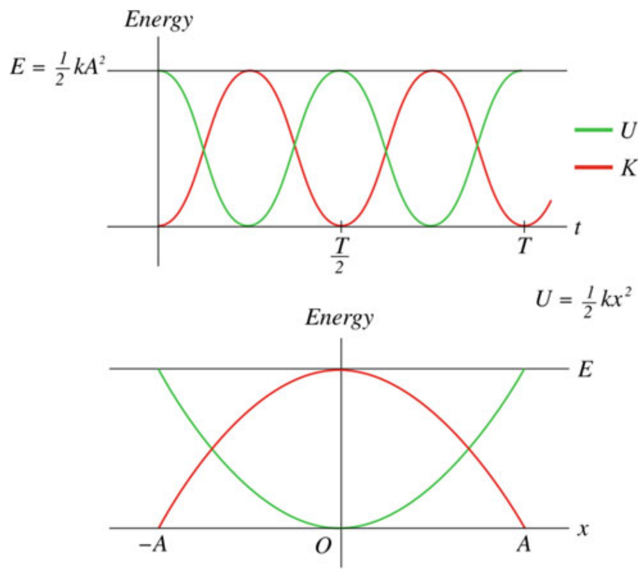


Fig. 10.11 As the mass moves, its kinetic energy is transformed into potential energy and vice versa

hence

$$k = 20 \text{ N/m}$$

The period of motion is therefore

$$T = 2\pi\sqrt{\frac{m}{k}} = 2(3.14)\sqrt{\frac{(0.3 \text{ kg})}{(20 \text{ N/m})}} = 0.8 \text{ s}$$

and the frequency is

$$f_n = \frac{1}{T} = \frac{1}{(0.8 \text{ s})} = 1.25 \text{ Hz}$$

Example 10.8 A 0.2 kg block is attached to a light spring of force constant of 11 N/m on a horizontal frictionless surface. If the block is displaced a distance of 8 cm from its equilibrium position, find (a) the amplitude, the angular frequency, the period and the frequency of motion when the block is released; (b) the maximum force exerted on the block; (c) the total mechanical energy of the system; (d) the maximum speed and maximum acceleration of the block; (e) the velocity of the block when its displacement is 2 cm; (f) the acceleration of the block when its displacement is 3 cm.

Solution 10.8 (a)

$$A = 8 \text{ cm}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{(11 \text{ N/m})}{(0.2 \text{ kg})}} = 7.4 \text{ rad/s}$$

$$T = \frac{2\pi}{\omega_n} = \frac{2(3.14)}{(7.4 \text{ rad/s})} = 0.85 \text{ s}$$

$$f_n = \frac{1}{T} = \frac{1}{(0.85 \text{ s})} = 1.2 \text{ Hz}$$

(b)

$$|F| = kA = (11 \text{ N/m})(0.08 \text{ m}) = 0.9 \text{ N}$$

(c)

$$E = \frac{1}{2}kA^2 = \frac{1}{2}(11 \text{ N/m})(0.08 \text{ m})^2 = 0.035 \text{ J}$$

(d)

$$v_{\max} = \omega_n A = (7.4 \text{ rad/s})(0.08 \text{ m}) = 0.6 \text{ m/s}$$

$$a_{\max} = \omega_n^2 A = (7.4 \text{ rad/s})^2(0.08 \text{ m}) = 4.4 \text{ m/s}^2$$

(e)

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} = \sqrt{\frac{(11 \text{ N/m})}{(0.2 \text{ kg})}((0.08 \text{ m})^2 - (0.02 \text{ m})^2)} = 1.8 \text{ m/s}$$

(f)

$$a = -\omega_n^2 x = -(7.4 \text{ rad/s})^2(0.03 \text{ m}) = -1.6 \text{ m/s}^2$$

Example 10.9 An object connected to a spring is in simple harmonic motion on a frictionless surface. If the object's displacement when $(2v_{\max}/3)$ is ± 0.015 m, find the amplitude of motion.

Solution 10.9

$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\frac{4\omega_n^2 A^2}{9} + \frac{1}{2}kx^2$$

therefore

$$A^2 = \frac{9}{5}x^2 = \frac{9}{5}(0.015 \text{ m})^2$$

$$A = 0.02 \text{ m}$$

Example 10.10 A solid cylinder is connected to a light spring as in Fig. 10.12. If the cylinder rolls without slipping along the surface, show that the motion of the cylinder is simple harmonic motion and find its frequency.

Solution 10.10 At any instant the total mechanical energy is

$$E = \frac{1}{2}kx^2 + \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}Mv_{cm}^2 = \frac{1}{2}kx^2 + \frac{1}{2}I_{cm}\frac{v_{cm}^2}{R^2} + \frac{1}{2}Mv_{cm}^2$$

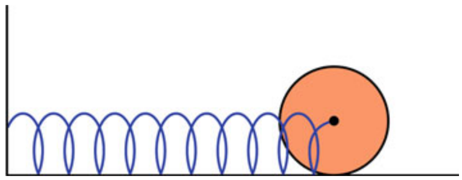


Fig. 10.12 A solid cylinder connected to a light spring

$$= \frac{1}{2}kx^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\frac{v_{cm}^2}{R^2} + \frac{1}{2}Mv_{cm}^2$$

Since the total mechanical energy is conserved

$$\frac{dE}{dt} = kv_{cm}x + \frac{1}{2}Mv_{cm}a_{cm} + Mv_{cm}a_{cm} = 0$$

$$kv_{cm}x = \frac{-3}{2}Mv_{cm}a_{cm}$$

or

$$a_{cm} = \frac{-2}{3}\frac{k}{M}x$$

$$\frac{d^2x}{dt^2} + \frac{2}{3}\frac{k}{M}x = 0$$

this equation is of a simple harmonic motion with

$$\omega_n = \sqrt{\frac{2}{3}\frac{k}{M}}$$

10.3.4 The Simple Pendulum

The simple pendulum is an example of an angular vibration in which the restoring effect is due to a restoring torque. A simple pendulum consists of a mass (called the bob) suspended by a light string of length L that is fixed at the other end (see Fig. 10.13). If the mass is pulled to the right or left from its equilibrium position and released, then the pendulum will swing in a vertical plane about an axis passing through O. The resulting motion is then a periodic or oscillatory motion. The restoring torque is due to gravity and is given by

$$\tau = -(mg\sin\theta)L$$

The minus sign indicates that the torque is a restoring torque, since it always tends to decrease θ. The moment of inertia of the bob about an axis passing through O is

$$I = mL^2$$

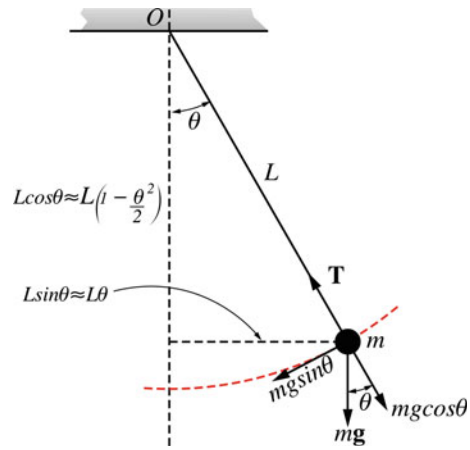


Fig. 10.13 The simple pendulum

From Newton's second law in angular form, we have

$$\tau = I\alpha = I\ddot{\theta}$$

Hence,

$$-mg\sin\theta L = mL^2\ddot{\theta}$$

or

$$\ddot{\theta} + \left(\frac{g}{L}\right)\sin\theta = 0 \quad (10.11)$$

This equation does not represent a harmonic motion. That is because the torque is not directly proportional to the angular displacement. Thus, the system is nonlinear. However for small angular displacements, we have $\sin\theta \approx \theta$(since $\sin\theta = \theta - \theta^3/3! + \theta^5/5!\ldots$) and Eq. 10.11 becomes

$$\ddot{\theta} + \left(\frac{g}{L}\right)\theta = 0$$

or

$$\ddot{\theta} + \omega_n^2\theta = 0 \quad (10.12)$$

where $\omega_n = \sqrt{g/L}$. Hence for small angular displacements, the motion is a simple harmonic motion. The solution of Eq. 10.12 is of the form

$$\theta = \theta_m\cos(\omega_n t - \phi)$$

where θ_m is the maximum angular displacement and ϕ is the phase constant. The plot of this equation is shown in Fig. 10.14. The period of the simple pendulum is therefore given by

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{L}{g}}$$

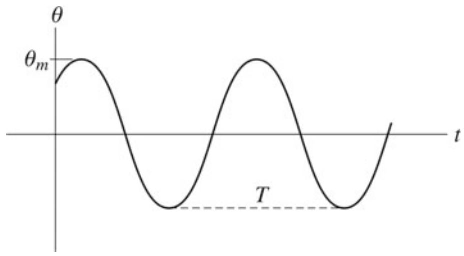


Fig. 10.14 The displacement versus time of a simple pendulum

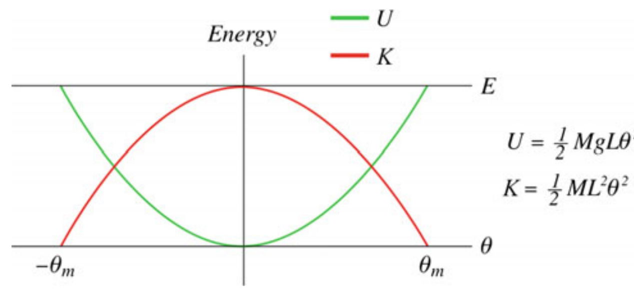


Fig. 10.15 The total energy of a simple pendulum

10.3.4.1 Energy

The kinetic energy of the simple pendulum is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\omega_n^2 = \frac{1}{2}mL\dot{\theta}^2$$

Taking the reference point of potential energy of the system to be zero when the bob is at the bottom, we have

$$U = MgL(1 - \cos\theta)$$

The total energy is therefore given by

$$E = K + U = \frac{1}{2}ML^2\dot{\theta}^2 + MgL(1 - \cos\theta)$$

For small θ, we have $\cos\theta \approx 1 - \frac{\theta^2}{2}$ since $\cos\theta = 1 - \theta^2/2! + \theta^4/4! \ldots$) thus

$$E = \frac{1}{2}ML^2\dot{\theta}^2 + \frac{1}{2}MgL\theta^2$$

Since

$$\dot{\theta} = -\theta_m\omega_n \sin(\omega_n t - \phi)$$

we have

$$E = \frac{1}{2}ML^2\theta_m^2\omega_n^2 \sin^2(\omega_n t - \phi) + \frac{1}{2}MgL\theta_m^2 \cos^2(\omega_n t - \phi)$$

or

$$E = \frac{1}{2}MgL\theta_m^2$$

Therefore, the total energy of the system is constant. Figure 10.15 shows the variation of the kinetic and potential energies with the displacement.

The equation of motion may also be obtained from energy as follows:

$$\frac{dE}{dt} = ML^2\dot{\theta}\ddot{\theta} + MgL\theta\dot{\theta} = 0$$

or

$$\ddot{\theta} + \left(\frac{g}{L}\right)\theta = 0$$

Example 10.11 A simple pendulum is 0.5 m long. Find its period at the surface of Mars and compare it to its period at the earth's surface.

Solution 10.11 At Mars's surface, the gravitational acceleration is

$$g_M = \frac{GM_M}{R_M^2} = \frac{(6.67 \times 10^{-11}\ \text{N}\,mathrmm^2/\text{kg}^2)(6.42 \times 10^{23}\ \text{kg})}{(3.37 \times 10^6\ \text{m})^2} = 3.8\ \text{m/s}^2$$

The period at Mars is therefore

$$T_M = 2\pi\sqrt{\frac{L}{g_M}} = 2(3.14)\sqrt{\frac{(0.5\ \text{m})}{(3.8\ \text{m/s}^2)}} = 2.3\ \text{s}$$

At the earth's surface,

$$T_E = 2\pi\sqrt{\frac{L}{g_E}} = 2(3.14)\sqrt{\frac{(0.5\ \text{m})}{(9.8\ \text{m/s}^2)}} = 1.4\ \text{s}$$

Thus, $T_M = 1.6T_E$.

Example 10.12 A simple pendulum of length of 2 m is displaced through an angle of 12° and released. Find (a) the angular frequency of motion; (b) the maximum angular speed and maximum angular acceleration.

Solution 10.12 (a) The amplitude of motion is

$$\theta_{\max} = (12°)\left(\frac{2\pi\ \text{rad}}{360°\ \text{deg}}\right) = 0.21\ \text{rad}$$

The angular frequency is

$$\omega_n = \sqrt{\frac{g}{L}} = \sqrt{\frac{(9.8\ \text{m/s}^2)}{(2\ \text{m})}} = 2.2\ \text{rad/s}$$

(b) The maximum angular speed is

$$\dot{\theta}_{max} = \omega_n A = (2.2 \text{ rad/s})(0.21 \text{ rad}) = 0.5 \text{ rad/s}$$

The maximum angular acceleration is

$$\ddot{\theta}_{max} = \omega_n^2 A = (2.2 \text{ rad/s})^2(0.21 \text{ rad}) = 1 \text{ rad/s}^2$$

Example 10.13 A simple pendulum 1.4 m in length is displaced through an angle of 10° and released. Find the velocity of the bob when it reaches the bottom.

Solution 10.13

$$\theta = (10°)\left(\frac{2\pi \text{ rad}}{360° \text{ deg}}\right) = 0.17 \text{ rad}$$

Taking the potential energy to be zero at the bottom, we have

$$mgL(1 - \cos\theta) = \frac{1}{2}mv^2$$

Since θ is small, $\cos\theta \approx 1 - \theta^2/2$ and therefore

$$mgL\frac{\theta^2}{2} = \frac{1}{2}mv^2$$

and

$$v = \sqrt{gL}\theta = \sqrt{(9.8 \text{ m/s}^2)(14 \text{ m})}(0.17 \text{ rad}) = 0.63\text{m/s}$$

10.3.5 The Physical Pendulum

The physical pendulum is a rigid body that oscillates about an axis passing through a point in the body other than its center of mass (the center of mass is assumed to be located at the center of gravity). Figure 10.16 shows a rigid body pivoted at point O that is at a distance d from the center of mass. The equilibrium position of the body is when its center of mass is directly below the pivot O. If the body is displaced either to the right or left from the equilibrium position, a restoring torque

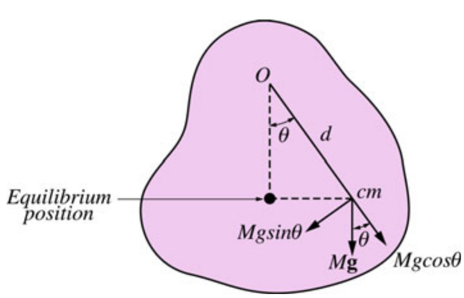


Fig. 10.16 The physical pendulum

due to gravity will act on it. As a result, the body will oscillate in a vertical plane where the axis of rotation is perpendicular to the page. The restoring torque is given by

$$\tau = -Mgd \sin\theta$$

where M is the mass of the body and d is the moment arm of the tangential component of the weight ($Mg \sin\theta$). From Newton's second law, we have

$$\tau = I\alpha$$

$$-Mgd \sin\theta = I\ddot{\theta}$$

For small angular displacements $\sin\theta \approx \theta$ and hence

$$\ddot{\theta} + \left(\frac{Mgd}{I}\right)\theta = 0$$

or

$$\ddot{\theta} + \omega_n^2\theta = 0$$

This equation is of a simple harmonic motion with an angular frequency of

$$\omega_n = \sqrt{\frac{Mgd}{I}}$$

and a period of motion of

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{I}{Mgd}}$$

Thus,

$$I = \frac{T^2 Mgd}{4\pi^2}$$

Therefore, the moment of inertia of a body can be found by measuring its period when it is in simple harmonic motion as a physical pendulum. Note that, the simple pendulum is a special case of the physical pendulum since for a simple pendulum of mass m, the moment of inertia is

$$I = md^2$$

and thus, the angular frequency is

$$\omega_n = \sqrt{\frac{mgd}{md^2}} = \sqrt{\frac{g}{d}}$$

This angular frequency is of a simple pendulum where d represents the length of the string.

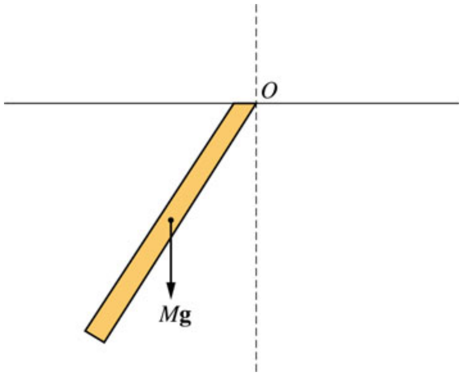


Fig. 10.17 A uniform rod suspended at one end oscillated with a small amplitude

Example 10.14 A uniform rod of length of 0.6 m that is suspended at one end oscillates with a small amplitude as in Fig. 10.17. Find the frequency of motion.

Solution 10.14

$$f_n = \frac{1}{2\pi}\sqrt{\frac{Mgd}{I}} = \frac{1}{2\pi}\sqrt{\frac{Mg(L/2)}{(1/3)ML^2}} = \frac{1}{2\pi}\sqrt{\frac{3g}{2L}} = \frac{1}{2(3.14)}\sqrt{\frac{3(9.8\text{m/s}^2)}{2(0.6\text{m})}} = 0.8\,\text{Hz}$$

Example 10.15 A uniform square plate of length a is pivoted at one of its corners and oscillates in a vertical plane as in Fig. 10.18. Find the period of motion if the amplitude is small.

Solution 10.15 The moment of inertia of a uniform rectangular plate about its center of mass is

$$I_{cm} = \frac{1}{12}M(a^2 + b^2)$$

Thus for a uniform square plate, we have

$$I_{cm} = \frac{1}{6}Ma^2$$

From the parallel axis theorem, the moment of inertia of the plate about an axis that is parallel to the center of mass axis

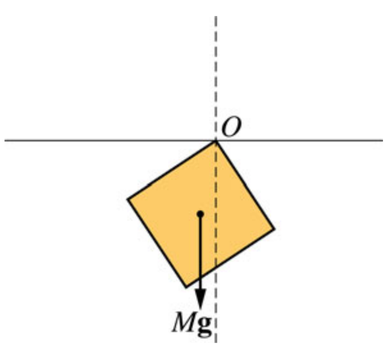


Fig. 10.18 A uniform square plate pivoted at one of its corners and oscillates in a vertical plane

and passing through one corner ($D = \sqrt{2}a$) is

$$I = I_{cm} + MD^2 = \frac{1}{6}Ma^2 + 2Ma^2 = \frac{13}{6}Ma^2$$

and hence

$$T = 2\pi\sqrt{\frac{I}{Mgd}} = 2\pi\sqrt{\frac{(13/6)Ma^2}{Mg\sqrt{2}a}} = 2\pi\sqrt{1.5\frac{a}{g}}$$

10.3.6 The Torsional Pendulum

The torsional pendulum consists of a rigid body suspended by a wire from its center of mass where the other end of the wire is fixed as shown in Fig. 10.19. The body is in equilibrium if the wire is untwisted. If the body is rotated through an angle θ it will oscillate about its equilibrium position (the line OP) due to a restoring torque exerted by the twisted wire on the body. This torque is found to be directly proportional to the angular displacement of the body. That is

$$\tau = -k\theta$$

where k is called the torsional constant. Its value depends on the property of the wire. Note that this equation is the rotational analogue of Hook's law in linear form ($F = -kx$). From Newton's second law, we have

$$\tau = I\alpha$$

or

$$-k\theta = I\ddot{\theta}$$

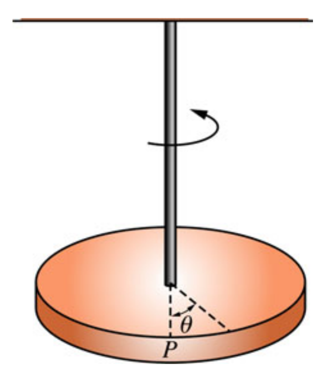
That gives

$$\ddot{\theta} + \left(\frac{k}{I}\right)\theta = 0$$

or

$$\ddot{\theta} + \omega_n^2\theta = 0$$

Fig. 10.19 The torsional pendulum



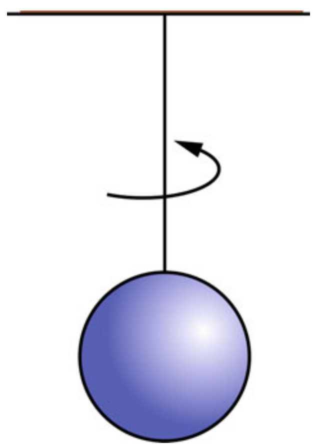


Fig. 10.20 A uniform solid sphere suspended at its midpoint by a light string

where $\omega_n = \sqrt{k/I}$ and the period is $T = 2\pi\sqrt{I/k}$.

Example 10.16 A uniform solid sphere of mass of 4.7 kg and radius of 5 cm is suspended at its midpoint by a light string (see Fig. 10.20) where it oscillates as a torsional pendulum. If the period of motion is 3.5 s, find the torsion constant.

Solution 10.16

$$T = 2\pi\sqrt{\frac{I}{k}}$$

for a uniform solid sphere

$$I_{cm} = \frac{2}{5}MR^2 = \frac{2}{5}(4.7\ \text{kg})(0.05\ \text{m})^2 = 4.7 \times 10^{-3}\ \text{kg m}^2$$

hence,

$$k = \frac{4\pi^2 I_{cm}}{T} = \frac{4(3.14)^2(4.7 \times 10^{-3}\ \text{kg m}^2)}{(3.5\ \text{s})} = 0.05\ \text{kg m}^2/\text{s}^2$$

10.4 Damped Free Vibrations

In this section, we will discuss the case in which the effect of damping that is due to a nonconservative force cannot be neglected. An example of such a force in mechanical systems is the force of friction. In this case, the mechanical energy of the system will be lost, the amplitude of motion will decrease to zero, and the oscillation dies out eventually. Here, we will discuss damping due to friction in the simplest case, where the frictional force is proportional to the first power of the velocity of the oscillating body. An example of such a frictional force is the force that an object experience when moving in a fluid with a low speed and is given by

$$F_D = -bv$$

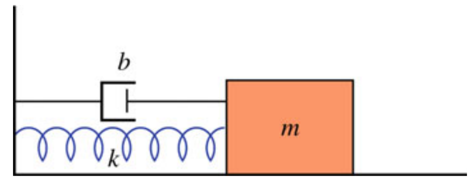


Fig. 10.21 A mass-spring system with damping

where b is a positive constant called the damping coefficient. Its SI units is $\text{N}(\text{m s}^{-1}) = \text{kg s}^{-1}$. The negative sign shows that the direction of the force is always opposite to the velocity. Now consider the spring–mass system as shown in Fig. 10.21, the cylinder shown in the figure contains a viscous fluid and a piston moving in it. Such device is known as the viscous damper. The net force on the oscillating body is

$$\sum F = F_s + F_D = -kx - bv$$

hence

$$m\ddot{x} + b\dot{x} + kx = 0$$

or

$$\ddot{x} + \gamma\dot{x} + \omega_n^2 x = 0 \qquad (10.13)$$

where $\gamma = b/m$ and $\omega_n = \sqrt{k/m}$. The units of γ is s^{-1}. This equation is a second order linear differential equation of constant coefficients. We may assume a solution of the form

$$x = Ce^{\lambda t}$$

Substituting this solution into Eq. 10.13 gives the characteristic (auxiliary) equation given by

$$\lambda^2 + \gamma\lambda + \omega_n^2 = 0$$

The roots of this equation are given by

$$\lambda_1 = -\frac{\gamma}{2} + \sqrt{\left(\frac{\gamma^2}{4} - \omega_n^2\right)}$$

and

$$\lambda_2 = -\frac{\gamma}{2} - \sqrt{\left(\frac{\gamma^2}{4} - \omega_n^2\right)}$$

From superposition, the general solution is given by

$$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \qquad (10.14)$$

Three possible solutions arise depending on whether the sign of the bracket $(\gamma^2/4 - \omega_n^2)$ is positive, negative or zero, i.e., depending on the size of the damping force. The roots λ_1 and λ_2 are either distinct real roots, equal real roots or a conjugate complex roots. Therefore, there are three possible motions of the system.

10.4.1 Light Damping (Under-Damped) ($\gamma < 2\omega_n$)

If $\gamma < 2\omega_n$ the resulting roots are complex roots given by

$$\lambda_1 = -\frac{\gamma}{2} + i\omega_D$$

and

$$\lambda_2 = -\frac{\gamma}{2} - i\omega_D$$

where

$$\omega_D = \left(\omega_n^2 - \frac{\gamma^2}{4}\right)^{1/2}$$

Hence, Eq. 10.14 may be written as

$$x = \left[C_1 e^{i\omega_D t} + C_2 e^{-i\omega_D t}\right] e^{\frac{-\gamma}{2}t}$$

Since $e^{\pm ix} = \cos x \pm i \sin x$ we have

$$x = [C_1(\cos\omega_D t + i\sin\omega_D t) + C_2(\cos\omega_D t - i\sin\omega_D t)]e^{\frac{-\gamma}{2}t}$$

$$= [(C_1 + C_2)\cos\omega_D t + i(C_1 - C_2)\sin\omega_D t]e^{\frac{-\gamma}{2}t}$$

$$= [A_1\cos\omega_D t + A_2\sin\omega_D t]e^{\frac{-\gamma}{2}t} \quad (10.15)$$

where $A_1 = C_1 + C_2$ and $A_2 = i(C_1 - C_2)$. As mentioned earlier Eq. 10.15 can be written as

$$x = A\cos(\omega_D t - \phi)e^{\frac{-\gamma}{2}t} \quad (10.16)$$

where A is the initial amplitude of motion. $Ae^{\frac{-\gamma}{2}t}$ is called the amplitude of motion and ϕ is the phase constant and ω_D is the angular frequency of the damped motion. This equation shows that the system oscillates in a decreasing harmonic motion where the amplitude of motion decreases exponentially with time until eventually the oscillation dies out (see Fig. 10.22). The dashed lines in Fig. 10.22 are called the envelope of the oscillation curve. The period of motion in light damping is therefore given by

$$\tau_D = \frac{2\pi}{\omega_D} = \frac{2\pi}{\sqrt{\omega_n^2 - \frac{\gamma^2}{4}}}$$

If $b = 0$ and thus $\gamma = 0$ the period of motion is reduced to that of a simple harmonic oscillator. If $\gamma \ll \omega_D$, the situation is referred to as very light damping and $\omega_D \approx \omega_n$. Furthermore

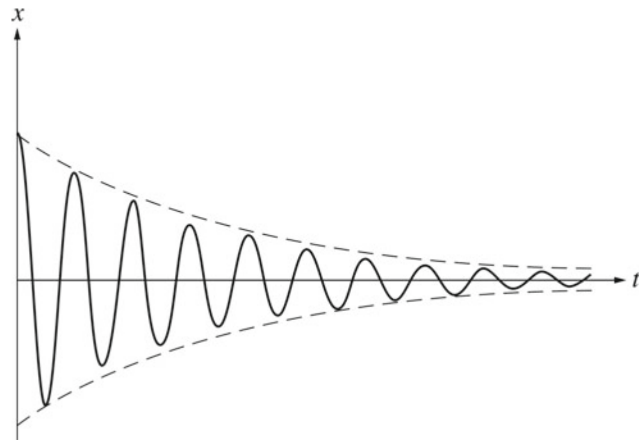


Fig. 10.22 In A lightly damped oscillator, the system oscillates in a decreasing harmonic motion where the amplitude of motion decreases exponentially with time until eventually the oscillation dies out

if there are two amplitudes A_a and A_b separated by the period of motion, then their ratio is given by

$$\frac{A_a}{A_b} = \frac{Ae^{-\frac{\gamma}{2}t_1}}{Ae^{-\frac{\gamma}{2}(t_1+\tau_D)}} = e^{\frac{\gamma}{2}\tau_D}$$

A quantity known as the logarithmic decrement is defined as

$$\delta = \ln\left(\frac{A_a}{A_b}\right) = \frac{\gamma}{2}\tau_D$$

Example 10.17 An 8 kg block is attached to a light spring and a light viscous damper. If at $t = 0$, $x = 0.12$ m and $v = 0$, find (a) the displacement at any time; (b) the logarithmic decrement. ($k = 30$ N/m, $b = 20$ N s/m).

Solution 10.17 (a)

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{(30 \text{ N/m})}{(8 \text{ kg})}} = 1.9 \text{ rad/s}$$

$$\gamma = \frac{b}{m} = \frac{(20 \text{ N s/m})}{(8 \text{ kg})} = 2.5 \text{ s}^{-1}$$

and

$$\omega_D = \left(\omega_n^2 - \frac{\gamma^2}{4}\right)^{1/2} = ((1.9 \text{ rad/s})^2 - (2.5 \text{ Ns/m kg})^2 4)^{1/2} = 1.43 \text{ rad/s}$$

since $\gamma < 2\omega_n$, the damping is light. The displacement as a function of time is given by

$$x = A\cos(\omega_D t - \phi)e^{\frac{-\gamma}{2}t}$$

or

$$x = A\cos(1.43t - \phi)e^{-1.25t}$$

since at $t = 0$, $x = 0.12$ m, then

$$(0.12 \text{ m}) = A\cos\phi \tag{10.17}$$

the velocity of the block at any time is

$$\dot{x} = -1.43A\sin(1.43t - \phi)e^{-1.25t} - 1.25A\cos(1.43t - \phi)e^{-1.25t}$$

at $t = 0$, $v = 0$ and thus

$$0 = -1.43A\sin\phi - 1.25A\cos\phi \tag{10.18}$$

Solving Eqs. 10.17 and 10.18 for A and ϕ gives $\phi = -0.7$ rad and $A = 0.17$ m. Therefore,

$$x = 0.17\cos(1.43t - 0.7)e^{-1.25t}$$

(b)

$$\tau_D = \frac{2\pi}{\omega_D} = \frac{2\pi}{(1.43 \text{ rad/s})} = 4.4 \text{ s}$$

$$\delta = \frac{\gamma}{2}\tau_D = (1.25 \text{ s}^{-1})(4.4 \text{ s}) = 5.5$$

10.4.2 Critically Damped Motion ($\gamma = 2\omega_n$)

If $\gamma = 2\omega_n$, then the roots are equal real roots

$$\lambda_1 = \lambda_2 = -\frac{\gamma}{2} = -\omega_n$$

In that case, the motion decays without oscillation (see Fig. 10.23) and the general solution of Eq. 10.13 is

$$x = (C_1 + C_2\omega_n t)e^{-\omega_n t}$$

C_1 and C_2 are found from boundary conditions. If at $t = 0$, $x = A$, and $v = 0$, then

$$x(0) = C_1 = A$$

and

$$v(0) = \omega_n C_2 - \omega_n C_1 = 0$$

or

$$C_1 = C_2 = A$$

That gives

$$x = A(1 + \omega_n t)e^{-\omega_n t}$$

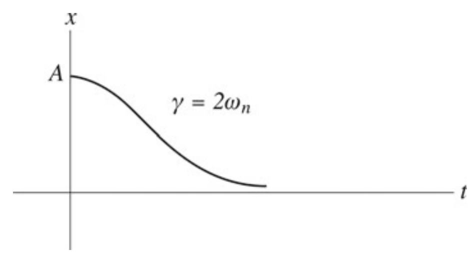


Fig. 10.23 In a critically damped motion, the motion decays without oscillation

10.4.3 Over Damped Motion (Heavy Damping) ($\gamma > 2\omega_n$)

If $\gamma > 2\omega_n$, the roots are distinct real roots given by

$$\lambda_1 = -\frac{\gamma}{2} + \sqrt{\left(\frac{\gamma^2}{4} - \omega_n^2\right)}$$

and

$$\lambda_2 = -\frac{\gamma}{2} - \sqrt{\left(\frac{\gamma^2}{4} - \omega_n^2\right)}$$

The general solution is given by

$$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

or

$$x = (C_1 e^{\alpha t} + C_2 e^{-\alpha t})e^{-\frac{\gamma}{2}t}$$

where

$$\alpha = \sqrt{\left(\frac{\gamma^2}{4} - \omega_n^2\right)}$$

C_1 and C_2 are found from boundary conditions. As critical damping, the resulting motion here is nonperiodic but the system returns to its equilibrium position at large values of t unlike critical damping (see Fig. 10.24).

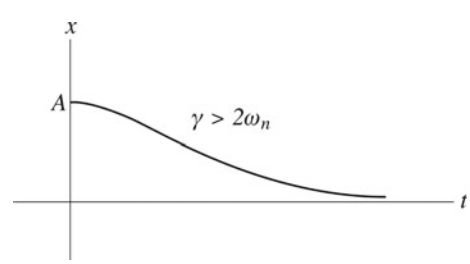


Fig. 10.24 As critical damping, the resulting motion here is non-periodic but the system returns to its equilibrium position at large values of t unlike critical damping

Example 10.18 In Example 10.17, find the range of values of the damping coefficient for the system to be: (a) over damped; (b) critically damped.

Solution 10.18 (a) over damped if $\gamma > 2\omega_n$, i.e., if $\gamma > 3.8\text{s}^{-1}$(b) critically damped if $\gamma = 3.8\text{s}^{-1}$.

10.4.4 Energy Decay

In damped free vibrations, the total mechanical energy is not constant since the damping force opposes the motion and dissipates the energy of the system. Now, consider the mass–spring system, the total mechanical energy of the system is

$$E = K + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

The rate of change of energy is

$$\frac{dE}{dt} = (m\ddot{x} + kx)\dot{x}$$

For damped vibrations in which the damping force is directly proportional to the velocity, we have

$$m\ddot{x} + kx = -b\dot{x}$$

Hence,

$$\frac{dE}{dt} = -b\dot{x}^2 \leq 0$$

Thus, the energy decreases with time in any damped motion and the rate in which it decreases is not uniform.

10.5 Forced Vibrations

In the previous sections, only free vibrations have been considered (i.e., vibrations in which only a restoring and damping force act within the system during motion). This section considers the case in which an external driving force is applied to the vibrator. This force is given as a function of time and we have

$$m\ddot{x} + b\dot{x} + kx = F(t) \tag{10.19}$$

Here, we will consider the case in which the force is a simple periodic force given by

$$F(t) = F_0 \cos\omega t \tag{10.20}$$

where F_0 is the amplitude and ω is the driving frequency. This force does positive work on the system to balance the energy loss due to damping. Substituting Eq. 10.20 into Eq. 10.19 gives

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos\omega t \tag{10.21}$$

or

$$\ddot{x} + \gamma\dot{x} + \omega_n^2 x = \frac{F_0 \cos\omega t}{m}$$

Let us assume that the solution of Eq. 10.19 is given by

$$x = C_1 \cos\omega t + C_2 \sin\omega t$$

then, we have

$$\dot{x} = -\omega C_1 \sin\omega t + \omega C_2 \cos\omega t$$

and

$$\ddot{x} = -\omega^2 C_1 \cos\omega t - \omega^2 C_2 \sin\omega t$$

Substituting into Eq. 10.19 gives

$$(-\omega^2 C_1 \cos\omega t - \omega^2 C_2 \sin\omega t) + \gamma(-\omega C_1 \sin\omega t + \omega C_2 \cos\omega t)$$
$$+ \omega_n^2 (C_1 \cos\omega t + C_2 \sin\omega t) = \frac{F_0 \cos\omega t}{m}$$

That gives

$$-\omega^2 C_1 + \gamma\omega C_2 + \omega_n^2 C_1 = \frac{F_0}{m}$$

and

$$-\omega^2 C_2 - \gamma\omega C_1 + \omega_n^2 C_2 = 0$$

Solving for C_1 and C_2 gives

$$C_1 = \frac{(F_0/m)(\omega_n^2 - \omega^2)}{(\omega^2 - \omega_n^2)^2 + \gamma^2\omega^2}$$

and

$$C_2 = \frac{(F_0/m)\gamma\omega}{(\omega^2 - \omega_n^2)^2 + \gamma^2\omega^2}$$

Hence,

$$x = \frac{(F_0/m)[(\omega_n^2 - \omega^2)\cos\omega t + \gamma\omega \sin\omega t]}{(\omega^2 - \omega_n^2)^2 + \gamma^2\omega^2}$$

The term in brackets is of the form $A_1 \cos\omega t + A_2 \sin\omega t$ and thus it can be written as $A' \cos(\omega t - \phi)$ where

$$A' = \sqrt{A_1^2 + A_2^2}$$

i.e.,

$$A' = ((\omega_n^2 - \omega^2)^2 + \gamma^2\omega^2)^{\frac{1}{2}}$$

and

$$\phi = \tan^{-1}\frac{A_2}{A_1} = \tan^{-1}\frac{\gamma\omega}{(\omega^2 - \omega_n^2)}$$

where $0 \le \phi \le \pi$. Hence,

$$x = \frac{(F_0/m)}{\sqrt{(\omega^2 - \omega_n^2)^2 + \gamma^2\omega^2}} \cos(\omega t - \phi) \qquad (10.22)$$

If the driving force is applied for a long time compared with the time that the damped vibration dies out, then the system will eventually vibrate at the same frequency of the deriving force. Therefore, the general solution of Eq. 10.13 is called the transient solution since it approaches zero in a relativity short time whereas Eq. 10.21 is called the steady-state solution where the system oscillates with the same frequency as the deriving force. Therefore, the amplitude of a steady-state vibration is

$$A = \frac{(F_0/m)}{\sqrt{(\omega^2 - \omega_n^2)^2 + \gamma^2\omega^2}}$$

When the deriving frequency ω approaches the natural frequency of the system ω_D, the amplitude of the resulting forced oscillation will increase. This is known as resonance. If the damping is very light, the amplitude reaches its peak when the deriving frequency is nearly equal to the natural frequency ω_n. As the damping becomes heavier, the maximum amplitude shifts to lower frequencies (see Fig. 10.25). In the case where there is no damping at all ($b = 0$), the amplitude of resonance is infinite at $\omega = \omega_n$.

Example 10.19 In Example 10.17, if a driving force of the form $F(t) = 5\cos 4t$ is applied to the system, find the steady-state displacement as a function of time.

Solution 10.19

$$A = \frac{(F_0/m)}{\sqrt{(\omega^2 - \omega_n^2)^2 + \gamma^2\omega^2}} = \frac{(5/8)}{\sqrt{((4)^2 - (1.9)^2)^2 + (2.5)^2(4)^2}} = 0.04 \text{ m}$$

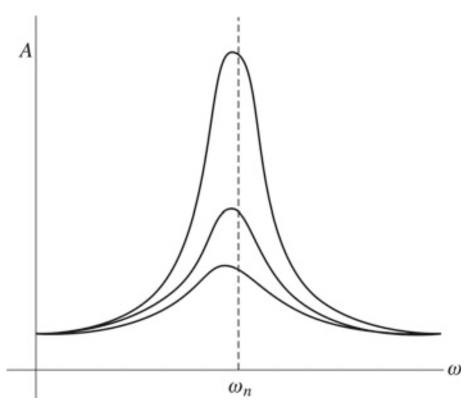


Fig. 10.25 When the deriving frequency ω approaches the natural frequency of the system ω_D, the amplitude of the resulting forced oscillation will increase. This is known as resonance. If the damping is very light the amplitude reaches its peak when the deriving frequency is nearly equal to the natural frequency ω_n. As the damping becomes heavier, the maximum amplitude shifts to lower frequencies

$$\phi = \tan^{-1}\frac{\gamma\omega}{(\omega^2 - \omega_n^2)} = \tan^{-1}\frac{(2.5)(4)}{((4)^2 - (1.9)^2)} = 0.8^\circ$$

Hence,

$$x = 0.04\cos(4t - 0.8)$$

Therefore, the forced vibration has the same frequency as the deriving force but lag in phase by 0.8°

Example 10.20 In Example (10.17), find the steady-state displacement as a function of time if there is no damping.

Solution 10.20 The amplitude of the forced oscillation when the angular frequency ω of the deriving force is varied.

$$A = \frac{(F_0/m)}{\sqrt{(\omega^2 - \omega_n^2)^2 + \gamma^2\omega^2}} = \frac{(5/8)}{\sqrt{((4)^2 - (1.9)^2)^2}} = 0.05 \text{ m}$$

$x = 0.05\cos 4t, \ \phi = 0.$

Problems

1. A 2 kg block is fastened to a spring of force constant 98 N/m on a horizontal frictionless surface. If the block is released a distance of 6 cm from its equilibrium position, find (a) the angular frequency, the frequency and the period of the resulting motion, (b) the time it takes the block to first reach $x = -5$ cm and its velocity at that time, (c) the maximum speed and maximum acceleration of the oscillating block, (d) the total mechanical energy of the oscillator.

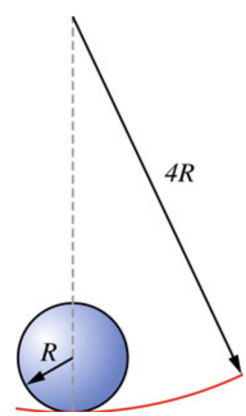


Fig. 10.26 A uniform solid cylinder of radius R and mass M rolls without slipping on a track of radius $4R$

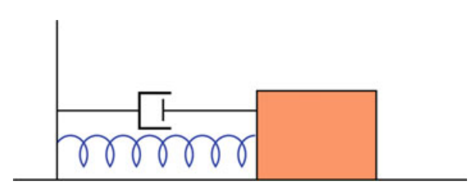


Fig. 10.27 A damped oscillator

2. A 10 kg block is attached to a light spring of force constant 200 N/m on a smooth horizontal surface. Find the amplitude of motion if at $x = 0.06$ m the velocity of the block is $v = 0.5$ m/s.
3. A particle rotate counterclockwise in a circle of radius 0.2 m with a constant angular speed of 2 rad/s. If at $t = 0$ the x-coordinate of the particle is 0.14 m, find the displacement, velocity and acceleration of the particle at any time.
4. If a simple pendulum has a period of 2 s, find its period when its length is increased by 20%.
5. A simple pendulum of length lm and mass of 0.4 kg oscillates in a region where $g = 9.8$ m/s^2. If the amplitude of oscillation is 10°, find (a) the angular displacement, angular velocity and angular acceleration of the pendulum as a function of time.
6. A uniform solid cylinder of radius R and mass M rolls without slipping on a track of radius $4R$ as shown in Fig. 10.26. Find the period of oscillation when the cylinder is displaced slightly from its equilibrium position.
7. A planer body of mass 3 kg oscillates as a physical pendulum. If the period of oscillation is 3 s and if the pivot point is at 0.2 m from the center of mass, find the moment of inertia of the body.
8. A uniform hollow cylinder of radius R and mass M is suspended at its midpoint from a wire and form a torsional pendulum. If the period of motion is T, find the torsion constant.
9. For the system shown in Fig. 10.27, determine the displacement of the block at any time if at $t = 0$, $x = 0$ and $v = 0$. ($k = 200$ N/m, $b = 200$ N s/m).
10. For the system shown in Fig. 10.28, find the steady-state displacement as a function of time.

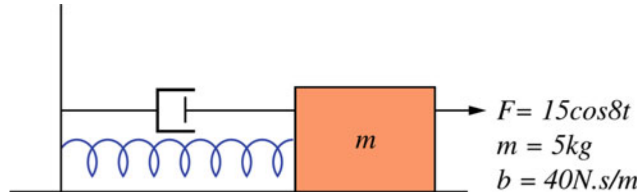


Fig. 10.28 A forced oscillator

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