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Contemporary Mathematics

Contemporary Mathematics

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Contents

Preface

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Contemporary Mathematics is designed to meet the requirements for a liberal arts mathematics course. The textbook covers a range of topics that are typically found in a liberal arts course as well as some topics to connect mathematics to the world around us. The text provides stand-alone sections with a focus on showing relevance in the features as well as the examples, exercises, and exposition.

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Learning Objectives

Every section begins with a set of clear and concise learning objectives, which have been thoroughly revised to be both measurable and more closely aligned with current teaching practice. These objectives are designed to help the instructor decide what content to include or assign and to guide student expectations of learning. After completing the section and end-of-section exercises, students should be able to demonstrate mastery of the learning objectives.

Key Features

Check Your Understanding: Concept checks to confirm students understand content at the end of every section immediately before the exercise sets are provided to help bolster confidence before embarking on homework.

People in Mathematics: A mix of historic and contemporary profiles aimed to incorporate extensive diversity in gender and ethnicity. The profiles incorporate how the person’s contribution has benefitted students or is relevant to their lives in some way.

Who Knew?: A high-interest feature designed to showcase something interesting related to the section contents. These features are crafted to offer something students might be surprised to find is so relevant to them.

Work It Out: Offers some activity ideas in line with the sections to support the learning objectives.

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Section summaries distill the information in each section for both students and instructors down to key, concise points addressed in the section.

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Key terms are bold and are followed by a definition in context.

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About the Authors

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Donna Kirk received her B.S. in Mathematics from the State University of New York at Oneonta and her master's degree from City University – Seattle in Educational Technology and Curriculum Design. After teaching math in higher education for more than twenty years, she joined University of Wisconsin's Education Department in 2021, teaching math education for teacher preparation. She is also the director of a STEM institute focused on connecting underrepresented students with access to engaging and innovative experiences to empower themselves to pursue STEM related careers.

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1 SETS

Figure 1.1 A flatware drawer is like a set in that it contains distinct objects. (credit: modification of work “silverware” by Jo Naylor/Flickr, CC BY 2.0)

Chapter Outline

Introduction

Think of a drawer in your kitchen used to store flatware. This drawer likely holds forks, spoons, and knives, and possibly other items such as a meat thermometer and a can opener. The drawer in this case represents a tool used to group a collection of objects. The members of the group are the individual items in the drawer, such as a fork or a spoon.

The members of a set can be anything, such as people, numbers, or letters of the alphabet. In statistical studies, a **set** is a well-defined collection of objects used to identify an entire population of interest. For example, in a research study examining the effects of a new medication, there can be two sets of people: one set that is given the medication and a different set that is given a placebo (control group). In this chapter, we will discuss sets and Venn diagrams, which are graphical ways to show relationships between different groups.

1.1 Basic Set Concepts

Figure 1.2 A spoon, fork, and knife are elements of the set of flatware. (credit: modification of work "Cupofjoy" Wikimedia CC0 1.0 Public Domain Dedication)

Learning Objectives

After completing this section, you should be able to:

1. Represent sets in a variety of ways.
2. Represent well-defined sets and the empty set with proper set notation.
3. Compute the cardinal value of a set.
4. Differentiate between finite and infinite sets.
5. Differentiate between equal and equivalent sets.

Sets and Ways to Represent Them

Think back to your kitchen organization. If the drawer is the **set**, then the forks and knives are **elements** in the set. Sets can be described in a number of different ways: by roster, by set-builder notation, by interval notation, by graphing on a number line, and by Venn diagrams. Sets are typically designated with capital letters. The simplest way to represent a set with only a few members is the **roster** (or *listing*) **method**, in which the elements in a set are listed, enclosed by curly braces and separated by commas. For example, if F represents our set of flatware, we can represent F by using the following set notation with the roster method:

$F = \{\text{fork, spoon, knife, meat thermometer, can opener}\}$

EXAMPLE 1.1

Writing a Set Using the Roster or Listing Method

Write a set consisting of your three favorite sports and label it with a capital S.

Solution

There are multiple possible answers depending on what your three favorite sports are, but any answer must list three different sports separated by commas, such as the following:

$$S = \{\text{hockey, basketball, soccer}\}$$

YOUR TURN 1.1

1. Write a set consisting of four small hand tools that might be in a toolbox and label it with a capital T.

All the sets we have considered so far have been **well-defined sets**. A well-defined set clearly communicates whether an element is a member of the set or not. The members of a well-defined set are fixed and do not change over time. Consider the following question. What are your top 10 songs of 2021? You could create a list of your top 10 favorite songs from 2021, but the list your friend creates will not necessarily contain the same 10 songs. So, the set of your top 10 songs of 2021 is not a well-defined set. On the other hand, the set of the letters in your name is a well-defined set because it does not vary (unless of course you change your name). The NFL wide receiver, Chad Johnson, famously changed his name to Chad Ochocinco to match his jersey number of 85.

EXAMPLE 1.2

Identifying Well-Defined Sets

For each of the following collections, determine if it represents a well-defined set.

1. The group of all past vice presidents of the United States.
2. A group of old cats.

Solution

1. The group of all past vice presidents of the United States is a well-defined set, because you can clearly identify if any individual was or was not a member of that group. For example, Britney Spears is not a member of this set, but Joe Biden is a member of this set.
2. A group of old cats is not a well-defined set because the word old is ambiguous. Some people might consider a seven-year-old cat to be old, while others might think a cat is not old until it is 13 years old. Because people can disagree on what is and what is not a member of this group, the set is not well-defined.

YOUR TURN 1.2

For each of the following collections, determine if it represents a well-defined set.

1. A collection of medium-sized potatoes.
2. The original members of the Black Eyed Peas musical group.

On January 20, 2021, Kamala Harris was sworn in as the first woman vice president of the United States of America. If we were to consider the set of all women vice presidents of the United States of America prior to January 20, 2021, this set would be known as an **empty set**; the number of people in this set is 0, since there were no women vice presidents before Harris. The empty set, also called the null set, is written symbolically using a pair of braces, $\{\ \}$, or a zero with a slash through it, $\emptyset$.

The set containing the number 0, $\{0\}$, is a set with one element in it. It is not the same as the empty set, $\{\ \}$, which does not have any elements in it. Symbolically: $\{0\} \neq \{\ \}$.

EXAMPLE 1.3

Representing the Empty Set Symbolically

Represent each of the following sets symbolically.

1. The set of prime numbers less than 2.
2. The set of birds that are also mammals.

Solution

1. A prime number is a natural number greater than 1 that is only divisible by one and itself. Since there are no prime numbers less than 2, this set is empty, and we can represent it symbolically as follows: $\{\ \}$ or $\emptyset$. These two different symbols for the empty set can be used interchangeably.
2. The set of birds and the set of mammals do not intersect, so the set of birds that are also mammals is empty, and we can represent it symbolically as $\emptyset$ or $\{\ \}$.

YOUR TURN 1.3

1. Represent the set of all numbers divisible by 0 symbolically.

WHO KNEW?

The Number Zero

We use the number zero to represent the concept of nothing every day. The machine language of computers is binary, consisting only of zeros and ones, and even way before that, the number zero was a powerful invention that allowed our understanding of mathematics and science to develop. The historical record shows the Babylonians first used zeros around 300 B.C., while the Mayans developed and began using zero separately around 350 A.D. What is considered the first formal use of zero in arithmetic operations was developed by the Indian mathematician Brahmagupta around 650 A.D.

Brahmagupta, Mathematician and Astronomer (https://openstax.org/r/brahmagupta.html)

Another interesting feature of the number zero is that although it is an even number, it is the only number that is neither negative nor positive.

For larger sets that have a natural ordering, sometimes an ellipsis is used to indicate that the pattern continues. It is common practice to list the first three elements of a set to establish a pattern, write the ellipsis, and then provide the last element. Consider the set of all lowercase letters of the English alphabet, A. This set can be written symbolically as $A = \{\text{a, b, c}, \ldots, \text{z}\}$.

The sets we have been discussing so far are **finite sets**. They all have a limited or fixed number of elements. We also use an ellipsis for **infinite sets**, which have an unlimited number of elements, to indicate that the pattern continues. For example, in set theory, the set of **natural numbers**, which is the set of all positive counting numbers, is represented as $\mathbb{N} = \left\{1, 2, 3, \ldots\right\}$.

Notice that for this set, there is no element following the ellipsis. This is because there is no largest natural number; you can always add one more to get to the next natural number. Because the set of natural numbers grows without bound, it is an infinite set.

EXAMPLE 1.4

Writing a Finite Set Using the Roster Method and an Ellipsis

Write the set of even natural numbers including and between 2 and 100, and label it with a capital E. Include an ellipsis.

Solution

Write the label, E, followed by an equal sign and then a bracket. Write the first three even numbers separated by commas, beginning with the number two to establish a pattern. Next, write the ellipsis followed by a comma and the last number in the list, 100. Finally, write the closing bracket to complete the set.

Write the label, E, followed by an equal sign and then a bracket.

$$E =\{$$

Write the first three even numbers separated by commas, beginning with the number 2 to establish a pattern.

$$E = \{2, 4, 6,$$

Next, write the ellipsis followed by a comma and the last number in the list, 100.

$$E = \{2, 4, 6, \ldots, 100$$

Finally, write the closing bracket to complete the set.

$$E = \{2, 4, 6, \ldots, 100\}$$

YOUR TURN 1.4

1. Use an ellipsis to write the set of single digit numbers greater than or equal to zero and label it with a capital D.

Our number system is made up of several different infinite sets of numbers. The set of **integers**, $\mathbb{Z}$, is another infinite set of numbers. It includes all the positive and negative counting numbers and the number zero. There is no largest or smallest integer.

EXAMPLE 1.5

Writing an Infinite Set Using the Roster Method and Ellipses

Write the set of integers using the roster method, and label it with a $\mathbb{Z}$.

Solution

Step 1: As always, we write the label and then the opening bracket. Because the negative counting numbers are infinite, to represent that the pattern continues without bound to the left, we must use an ellipsis as the first element in our list.

Step 2: We place a comma and follow it with at least three consecutive integers separated by commas to establish a pattern.

Step 3: Add an ellipsis to the end of the list to show that the set of integers continues without bound to the right.

Complete the list with a closing bracket. The set of integers may be represented as follows:
$\mathbb{Z} = \left\{\ldots, -2, -1, 0, 1, 2, \ldots\right\}$.

YOUR TURN 1.5

1. Write the set of odd numbers greater than 0 and label it with a capital M.

A shorthand way to write sets is with the use of **set builder notation**, which is a verbal description or formula for the set. For example, the set of all lowercase letters of the English alphabet, A, written in set builder notation is:

$A = \{x|x \text{ is a lowercase letter of the English alphabet.}\}$

This is read as, "Set A is the set of all elements x such that x is a lowercase letter of the English alphabet."

EXAMPLE 1.6

Writing a Set Using Set Builder Notation

Using set builder notation, write the set B of all types of balls. Explain what the notation means.

Solution

The verbal description of the set is, "Set B is the set of all elements b such that b is a ball." This set can be written in set builder notation as follows: $B = \{b|b \text{ is a ball.}\}$

YOUR TURN 1.6

1. Using set builder notation, write the set C of all types of cars.

EXAMPLE 1.7

Writing Sets Using Various Methods

Consider the set of letters in the word "happy." Determine the best way to represent this set, and then write the set using either the roster method or set builder notation, whichever is more appropriate.

Solution

Because the letters in the word “happy” consist of a small finite set, the best way to represent this set is with the roster method. Choose a label to represent the set, such as H.

$H = \{\text{h, a, p, y}\}.$

Notice that the letter “p” is only represented one time. This occurs because when representing members of a set, each unique element is only listed once no matter how many times it occurs. Duplicate elements are never repeated when representing members of a set.

YOUR TURN 1.7

1. Use the roster method or set builder notation to represent the collection of all musical instruments.

Computing the Cardinal Value of a Set

Almost all the sets most people work with outside of pure mathematics are finite sets. For these sets, the **cardinal value** or **cardinality** of the set is the number of elements in the set. For finite set A, the cardinality is denoted symbolically as $n(A)$. For example, a set that contains four elements has a cardinality of 4.

How do we measure the cardinality of infinite sets? The ‘smallest’ infinite set is the set of natural numbers, or counting numbers, $\mathbb{N} = \left\{1, 2, 3, \ldots\right\}$. This set has a cardinality of $\aleph_0$ (pronounced "aleph-null"). All sets that have the same cardinality as the set of natural numbers are **countably infinite**. This concept, as well as notation using aleph, was introduced by mathematician Georg Cantor who once said, “A set is a Many that allows itself to be thought of as a One.”

EXAMPLE 1.8

Computing the Cardinal Value of a Set

Write the cardinal value of each of the following sets in symbolic form.

1. $F = \{\text{fork, spoon, knife, meat thermometer, can opener}\}$
2. The empty set.

Solution

1. There are 5 distinct elements in set F: a fork, a spoon, a knife, a meat thermometer, and a can opener. Therefore, the cardinal value of set F is 5 and written symbolically as $n(F) = 5$.
2. Because the empty set does not have any elements in it, the cardinality of the empty set is zero. Symbolically we write this as: $n(\varnothing) = 0$.

YOUR TURN 1.8

Write the cardinal value of each of the following sets in symbolic form.

1. Set P is the set of prime numbers less than 2.
2. Set A is the set of lowercase letters of the English alphabet, $A = \{\text{a,b,c}, \ldots, \text{z}\}$.

Now that we have learned to represent finite and infinite sets using both the roster method and set builder notation, we should also be able to determine if a set is finite or infinite based on its verbal or symbolic description. One way to determine if a set is finite or not is to determine the cardinality of the set. If the cardinality of a set is a natural number, then the set is finite.

EXAMPLE 1.9

Differentiating Between Finite and Infinite Sets
Classify each of the following sets as infinite or finite.

1. $E = \{2, 4, 6, 8, 10\}$
2. A is the set of lowercase letters of the English Alphabet, $A = \{\text{a}, \text{b}, \text{c}, \ldots, \text{z}\}$.
3. $\mathbb{Q} = \left\{ \frac{p}{q} \middle| p \text{ and } q \text{ are integers and } q \neq 0 \right\}$

Solution

1. $n(E) = 5$. Since 5 is a natural number, the set is finite.
2. $n(A) = 26$. Since 26 is a natural number, the set is finite.
3. Set $\mathbb{Q}$ is the set of rational numbers or fractions. Because the set of integers is a subset of the set of rational numbers, and the set of integers is infinite, the set of rational numbers is also infinite. There is no smallest or largest rational number.

YOUR TURN 1.9

Classify each of the following sets as infinite or finite.

1. $B = \{b, a, k, e\}$
2. $\mathbb{R} = \{x | x \text{ is a real number}\}$

Equal versus Equivalent Sets

When speaking or writing we tend to use equal and equivalent interchangeably, but there is an important distinction between their meanings. Consider a new Ford Escape Hybrid and a new Toyota Rav4 Hybrid. Both cars are hybrid electric sport utility vehicles; in that sense, they are **equivalent**. They will both get you from place to place in a relatively fuel-efficient way. In this example we are comparing the single member set {Toyota Rav4 Hybrid} to the single member set {Ford Escape Hybrid}. Since these two sets have the same number of elements, they are also equivalent mathematically, meaning they have the same cardinality. But they are not equal, because the two cars have different looks and features, and probably even handle differently. Each manufacturer will emphasize the features unique to their vehicle to persuade you to buy it; if the SUVs were truly equal, there would be no reason to choose one over the other.

Now consider two Honda CR-Vs that are made with exactly the same parts, on the same assembly line within a few minutes of each other—these SUVs are **equal**. They are identical to each other, containing the same elements without regard to order, and the only differentiator when making a purchasing decision would be varied pricing at different dealerships. The set {Honda CR-V} is equal to the set {Honda CR-V}. Symbolically, we represent equal sets as $A = B$ and equivalent sets as $A \sim B$.

Now, let us consider a Toyota dealership that has 10 RAV4s on the lot, 8 Prii, 7 Highlanders, and 12 Camrys. There is a one-to-one relationship between the set of vehicles on the lot and the set consisting of the number of each type of vehicle on the lot. Therefore, these two sets are equivalent, but not equal. The set {RAV4, Prius, Highlander, Camry} is equivalent to the set {10, 8, 7, 12} because they have the same number of elements.

VIDEO

Equal and Equivalent Sets (https://openstax.org/r/Equal_and_Equivalent_Sets)

If two sets are equal, they are also equivalent, because equal sets also have the same cardinality.

EXAMPLE 1.10

Differentiating Between Equivalent and Equal Sets
Determine if the following pairs of sets are equal, equivalent, or neither.

1. $E = \{2, 4, 6, 8, 10\}$ and $F = \{\text{fork, spoon, knife, meat thermometer, can opener}\}$
2. The empty set and the set of prime numbers less than 2.
3. The set of vowels in the word happiness and the set of consonants in the word happiness.

Solution

1. Sets *E* and *F* both have a cardinal value of 5, but the elements in these sets are different. So, the two sets are equivalent, but they are not equal: $E \sim F$.
2. The set of prime numbers consists of the set of counting numbers greater than one that can only be divided evenly by one and itself. The set of prime numbers less than 2 is an empty set, since there are no prime numbers less than 2. Therefore, these two sets are equal (and equivalent).
3. The set of vowels in the word happiness is $\{\text{a, i, e}\}$ and the set of consonants in the word happiness is $\{\text{h, p, n, s}\}$. The cardinal value of these sets two sets is $n(\{\text{a, i, e}\}) = 3$ and $n(\{\text{h, p, n, s}\}) = 4$, respectively. Because the cardinality of the two sets differs, they are not equivalent. Further, their elements are not identical, so they are also not equal.

YOUR TURN 1.10

Determine if the following pairs of sets are equal, equivalent, or neither.

1. Set $B = \{b, a, k, e\}$ and set $A = \{a, b, e, k\}$
2. Set $B = \{b, a, k, e\}$ and set $F = \{f, l, a, k, e\}$
3. Set $B = \{b, a, k, e\}$ and set $C = \{c, a, k, e\}$

PEOPLE IN MATHEMATICS

Georg Cantor

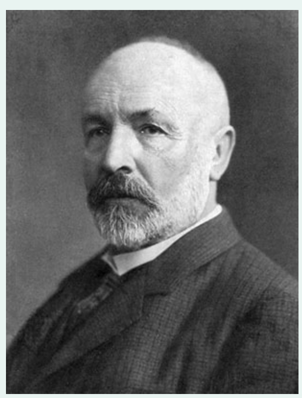

Figure 1.3 Georg Cantor (credit: Wikimedia, public domain)

Georg Cantor, the father of modern set theory, was born during the year 1845 in Saint Petersburg, Russa and later moved to Germany as a youth. Besides being an accomplished mathematician, he also played the violin. Cantor received his doctoral degree in Mathematics at the age of 22.

In 1870, at the age of 25 he established the uniqueness theorem for trigonometric series. His most significant work happened between 1874 and 1884, when he established the existence of transcendental numbers (also called irrational numbers) and proved that the set of real numbers are uncountably infinite—despite the objections of his former professor Leopold Kronecker.

Cantor published his final treatise on set theory in 1897 at the age of 52, and was awarded the Sylvester Medial from the Royal Society of London in 1904 for his contributions to the field. At the heart of Cantor's work was his goal to solve the continuum problem, which later influenced the works of David Hilbert and Ernst Zermelo.

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Check Your Understanding

1. A ____________ is a well-defined collection of objects.
2. The ________________ of a finite set A, denoted $n(A)$, is the number of elements in set A.
3. Determine if the following description describes a well-defined set: “The top 5 pizza restaurants in Chicago.”
4. The United States is the only country to have landed people on the moon as of March 21, 2021. What is the cardinality of the set of all people who have walked on the moon prior to this date?
5. Set A is a set of a dozen distinct donuts, and set B is a set of a dozen different types of apples. Is set A equal to set B, equivalent to set B, or neither?
6. Is the set of all butterflies in the world a finite set or an infinite set?
7. Represent the set of all upper-case letters of the English alphabet using both the roster method and set builder notation.

SECTION 1.1 EXERCISES

For the following exercises, represent each set using the roster method.

1. The set of primary colors: red, yellow, and blue.
2. A set of the following flowers: rose, tulip, marigold, iris, and lily.
3. The set of natural numbers between 50 and 100.
4. The set of natural numbers greater than 17.
5. The set of different pieces in a game of chess.
6. The set of natural numbers less than 21.

For the following exercises, represent each set using set builder notation.

7. The set of all types of lizards.
8. The set of all stars in the universe.
9. The set of all integer multiples of 3 that are greater than zero.
10. The set of all integer multiples of 4 that are greater than zero.
11. The set of all plants that are edible.
12. The set of all even numbers.

For the following exercises, represent each set using the method of your choice.

13. The set of all squares that are also circles.
14. The set of natural numbers divisible by zero.
15. The set of Mike and Carol’s children on the TV show, *The Brady Bunch*.
16. The set of all real numbers.
17. The set of polar bears that live in Antarctica.
18. The set of songs written by Prince.
19. The set of children’s books written and illustrated by Mo Willems.
20. The set of seven colors commonly listed in a rainbow.

For the following exercises, determine if the collection of objects represents a well-defined set or not.

21. The names of all the characters in the book, *The Fault in Our Stars* by John Green.
22. The five greatest soccer players of all time.
23. A group of old dogs that are able to learn new tricks.
24. A list of all the movies directed by Spike Lee as of 2021.
25. The group of all zebras that can fly an airplane.
26. The group of National Baseball League Hall of Fame members who have hit over 700 career home runs.

For the following exercises, compute the cardinal value of each set.

27. $P = \{\text{Snuzzle, Butterscotch, Blue Belle, Minty, Blossom, Cotton Candy}\}$

28. $T = \{\text{pepperoni, sausage, bacon, ham, mushrooms, olives, bell pepper, pineapple}\}$
29. $\varnothing$
30. $B = \{5, 6, 7, \ldots, 20\}$
31. $F = \left\{\frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{6}{9}, \frac{7}{9}, \frac{8}{9}, \frac{9}{9}\right\}$
32. $\{\ \}$
33. $C = \left\{n^3 | n \text{ is a member of } \mathbb{N}\right\}$
34. $S = \{7n | n \text{ is an element of } \mathbb{N}\}$
35. $L = \{l, m, n, \ldots, y\}$
36. The set of numbers on a standard 6-sided die.

For the following exercises, determine whether set A and set B are equal, equivalent or neither.

37. $A = \{\text{right, acute, obtuse}\}$; $B = \{\text{equilateral, scalene, isoceles}\}$.
38. $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}$; $B = \left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$.
39. $A = \{\text{red, orange, yellow}\}$; $B = \{\text{green, blue, indigo, violet}\}$.
40. $A = \{5n | n \in \mathbb{N}\}$; $B = \mathbb{N}$.
41. $A = \{-2, -1, 0, \ldots\}$; $B = \{2, 3, 5, \ldots\}$.
42. $A = \{\text{John, Paul, George, Ringo}\}$; $B = \{\text{Bono, Larry, The Edge, Adam}\}$.
43. $A = \varnothing$; $B = \{\ \}$.
44. $A = \{\text{lemon, lime, orange}\}$; $B = \{\text{orange, lemon, lime, grape}\}$.

For the following exercises, determine if the set described is finite or infinite.

45. The set of natural numbers.
46. The empty set.
47. The set consisting of all jazz venues in New Orleans, Louisiana.
48. The set of all real numbers.
49. The set of all different types of cheeses.
50. The set of all words in *Merriam-Webster's Collegiate Dictionary*, Eleventh Edition, published in 2020.

1.2 Subsets

Figure 1.4 The players on a soccer team who are actively participating in a game are a subset of the greater set of team members. (Credit: “PAFC-Mezokovesd-108” by Puskás Akadémia/Flickr, Public Domain Mark 1.0)

Learning Objectives

After completing this section, you should be able to:

1. Represent subsets and proper subsets symbolically.
2. Compute the number of subsets of a set.
3. Apply concepts of subsets and equivalent sets to finite and infinite sets.

The rules of Major League Soccer (MLS) allow each team to have up to 30 players on their team. However, only 18 of these players can be listed on the game day roster, and of the 18 listed, 11 players must be selected to start the game.

How the coaches and general managers form the team and choose the starters for each game will determine the success of the team in any given year.

The entire group of 30 players is each team's set. The group of game day players is a **subset** of the team set, and the group of 11 starters is a subset of both the team set and the set of players on the game day roster.

Set A is a subset of set B if every member of set A is also a member of set B. Symbolically, this relationship is written as $A \subseteq B$.

Sets can be related to each other in several different ways: they may not share any members in common, they may share some members in common, or they may share all members in common. In this section, we will explore the way we can select a group of members from the whole set.

Every set is also a subset of itself, $B \subseteq B$

Recall the set of flatware in our kitchen drawer from Section 1.1, $F = \{\text{fork, spoon, knife, meat thermometer, can opener}\}$. Suppose you are preparing to eat dinner, so you pull a fork and a knife from the drawer to set the table. The set $D = \{\text{knife, fork}\}$ is a subset of set F, because every member or element of set D is also a member of set F. More specifically, set D is a **proper subset** of set F, because there are other members of set F not in set D. This is written as $D \subset \mathrm{F}$. The only subset of a set that is not a proper subset of the set would be the set itself.

The empty set or null set, $\emptyset$, is a proper subset of every set, except itself.

Graphically, sets are often represented as circles. In the following graphic, set A is represented as a circle completely enclosed inside the circle representing set B, showing that set A is a proper subset of set B. The element x represents an element that is in both set A and set B.

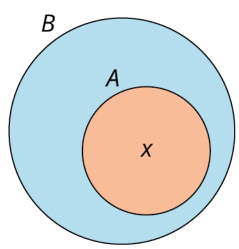


Figure 1.5

While we can list all the subsets of a finite set, it is not possible to list all the possible subsets of an infinite set, as it would take an infinitely long time.

EXAMPLE 1.11

Listing All the Proper Subsets of a Finite Set

Set L is a set of reading materials available in a shop at the airport, $L = \{\text{newspaper, magazine, book}\}$. List all the subsets of set L.

Solution

Step 1: It is best to begin with the set itself, as every set is a subset of itself. In our example, the cardinality of set L is $n(L) = 3$. There is only one subset of set L that has the same number of elements of set L: $\{\text{newspaper, magazine, book}\}$.

Step 2: Next, list all the proper subsets of the set containing $n(L) - 1$ elements. In this case, $3 - 1 = 2$. There are three subsets that each contain two elements: $\{\text{newspaper, magazine}\}$, $\{\text{newspaper, book}\}$, and $\{\text{magazine, book}\}$.

Step 3: Continue this process by listing all the proper subsets of the set containing $n(L) - 2$ elements. In this case, $3 - 2 = 1$. There are three subsets that contain one element: $\{\text{newspaper}\}$, $\{\text{magazine}\}$, and $\{\text{book}\}$.

Step 4: Finally, list the subset containing 0 elements, or the empty set: $\{\ \}$.

YOUR TURN 1.11

1. Consider the set of possible outcomes when you flip a coin, $S = \{\text{heads, tails}\}$. List all the possible subsets of set S.

EXAMPLE 1.12

Determining Whether a Set Is a Proper Subset

Consider the set of common political parties in the United States, $P = \{\text{Democratic, Green, Libertarian, Republican}\}$. Determine if the following sets are proper subsets of P.

1. $M = \{\text{Democratic, Republican}\}$
2. $G = \{\text{Green}\}$
3. $V = \{\text{Republican, Libertarian, Green, Democratic}\}$

Solution

1. M is a proper subset of P, written symbolically as $M \subset P$ because every member of M is a member of set P, but P also contains at least one element that is not in M.
2. G is a single member proper subset of P, written symbolically as $G \subset P$, because Green is a member of set P, but P also contains other members (such as Democratic) that are not in G.
3. V is subset of P because every member of V is also a member of P, but it is not a proper subset of P because there are no members of V that are not also in set P. We can represent the relationship symbolically as $V \subseteq P$, or more precisely, set V is equal to set P, $V = P$.

YOUR TURN 1.12

Consider the set of generation I legendary Pokémon, $L = \{\text{Articuno, Zapdos, Moltres, Mewtwo}\}$. Give an example of a proper subset containing:

1. one member.
2. three members.
3. no members.

EXAMPLE 1.13

Expressing the Relationship between Sets Symbolically

Consider the subsets of a standard deck of cards: $S = \{\text{spades, hearts, diamonds, clubs}\}$; $R = \{\text{hearts, diamonds}\}$; $B = \{\text{spades, clubs}\}$; and $C = \{\text{clubs}\}$.

Express the relationship between the following sets symbolically.

1. Set S and set B.
2. Set C and set B.
3. Set R and R.

Solution

1. $B \subset S$. B is a proper subset of set S.
2. $C \subset B$. C is a proper subset of set B.
3. $R \subseteq R$ or $R = R$. R is subset of itself, but not a proper subset of itself because R is equal to itself.

YOUR TURN 1.13

1. Express the relationship between the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$, and the set of even numbers, $E = \{2, 4, 6, \ldots\}$.

Exponential Notation

So far, we have figured out how many subsets exist in a finite set by listing them. Recall that in Example 1.11, when we listed all the subsets of the three-element set $L = \{\text{newspaper, magazine, book}\}$ we saw that there are eight subsets. In Your Turn 1.11, we discovered that there are four subsets of the two-element subset, $S = \{\text{heads, tails}\}$. A one-element set has two subsets, the empty set and itself. The only subset of the empty set is the empty set itself. But how can we easily figure out the number of subsets in a very large finite set? It turns out that the number of subsets can be found by raising 2 to the number of elements in the set, using **exponential notation** to represent repeated multiplication. For example, the number of subsets of the set $L = \{\text{newspaper, magazine, book}\}$ is equal to $2^3 = 2 \cdot 2 \cdot 2 = 8$. Exponential notation is used to represent repeated multiplication, $b^n = b \cdot b \cdot b \cdot \ldots \cdot b$, where b appears as a factor n times.

FORMULA

The number of subsets of a finite set A is equal to 2 raised to the power of $n(A)$, where $n(A)$ is the number of elements in set A: $\text{Number of Subsets of Set } A = 2^{n(A)}$.

Note that $2^0 = 1$, so this formula works for the empty set, also.

EXAMPLE 1.14

Computing the Number of Subsets of a Set

Find the number of subsets of each of the following sets.

1. The set of top five scorers of all time in the NBA:
   $S = \{\text{LeBron James, Kareem Abdul-Jabbar, Karl Malone, Kobe Bryant, Michael Jordan}\}$.
2. The set of the top four bestselling albums of all time:
   $A = \{\text{Thriller, Hotel California, The Beatles White Album, Led Zepplin IV}\}$.
3. $R = \{\text{Snap, Crackle, Pop}\}$.

Solution

1. $n(S) = 5$. So, the total number of subsets of S is $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$.
2. $n(A) = 4$. Therefore, the total number of subsets of A is $2^4 = 16$.
3. $n(R) = 3$. So, the total number of subsets of R is $2^3 = 8$.

YOUR TURN 1.14

1. Compute the total number of subsets in the set of the top nine tennis grand slam singles winners,
   $T = \{\text{Margaret Court, Serena Williams, Steffi Graff, Rafael Nadal, Novak Djokovic, Roger Federer, Helen Wills, Martina Navratilova, Chris Everett}\}$.

Equivalent Subsets

In the early 17th century, the famous astronomer Galileo Galilei found that the set of natural numbers and the subset of the natural numbers consisting of the set of square numbers, n^2, are equivalent. Upon making this discovery, he conjectured that the concepts of less than, greater than, and equal to did not apply to infinite sets.

Sequences and **series** are defined as infinite subsets of the set of natural numbers by forming a relationship between the sequence or series in terms of a natural number, n. For example, the set of even numbers can be defined using set builder notation as $\{a | a = 2n \text{ where } n \text{ is a natural number}\}$. The formula in this case replaces every natural number with two times the number, resulting in the set of even numbers, $\{2, 4, 6, \ldots\}$. The set of even numbers is also equivalent to the set of natural numbers.

WHO KNEW?

Employment Opportunities

You can make a career out of working with sets. Applications of equivalent sets include relational database design and analysis.

Relational databases that store data are tables of related information. Each row of a table has the same number of columns as every other row in the table; in this way, relational databases are examples of set equivalences for finite sets. In a relational database, a primary key is set up to identify all related information. There is a one-to-one relationship between the primary key and any other information associated with it.

Database design and analysis is a high demand career with a median entry-level salary of about $85,000 per year, according to salary.com.

EXAMPLE 1.15

Writing Equivalent Subsets of an Infinite Set

Using natural numbers, multiples of 3 are given by the sequence $\{3, 6, 9, \ldots\}$. Write this set using set builder notation by expressing each multiple of 3 using a formula in terms of a natural number, n.

Solution

$\{m | m = 3n \text{ where } n \text{ is a natural number}\}$ or $\{m | m = 3n \text{ where } n \in \mathrm{N}\}$. In this example, m is a multiple of 3 and n is a natural number. The symbol $\in$ is read as "is a member or element of." Because there is a one-to-one correspondence between the set of multiples of 3 and the natural numbers, the set of multiples of 3 is an equivalent subset of the natural numbers.

YOUR TURN 1.15

1. Using natural numbers, multiples of 5 are given by the sequence $\{5, 10, 15, \ldots\}$. Write this set using set builder notation by associating each multiple of 5 in terms of a natural number, n.

EXAMPLE 1.16

Creating Equivalent Subsets of a Finite Set That Are Not Equal

A fast-food restaurant offers a deal where you can select two options from the following set of four menu items for $6: a chicken sandwich, a fish sandwich, a cheeseburger, or 10 chicken nuggets. Javier and his friend Michael are each purchasing lunch using this deal. Create two equivalent, but not equal, subsets that Javier and Michael could choose to have for lunch.

Solution

The possible two-element subsets are: {chicken sandwich, fish sandwich}, {chicken sandwich, cheeseburger}, {chicken sandwich, chicken nuggets}, {fish sandwich, cheeseburger}, {fish sandwich, chicken nuggets}, and {cheeseburger, chicken nuggets}. One possible solution is that Javier picked the set {chicken sandwich, chicken nuggets}, while Michael chose the {cheeseburger, chicken nuggets}. Because Javier and Michael both picked two items, but not exactly the same two items, these sets are equivalent, but not equal.

YOUR TURN 1.16

1. Serena and Venus Williams walk into the same restaurant as Javier and Michael, but they order the same pair of items, resulting in equal sets of choices. If Venus ordered a fish sandwich and chicken nuggets, what did Serena order?

EXAMPLE 1.17

Creating Equivalent Subsets of a Finite Set

A high school volleyball team at a small school consists of the following players: {Angie, Brenda, Colleen, Estella, Maya, Maria, Penny, Shantelle}. Create two possible equivalent starting line-ups of six players that the coach could select for the next game.

Solution

There are actually 28 possible ways that the coach could choose his starting line-up. Two such equivalent subsets are {Angie, Brenda, Maya, Maria, Penny, Shantelle} and {Angie, Brenda, Colleen, Estella, Maria, Shantelle}. Each subset has six members, but they are not identical, so the two sets are equivalent but not equal.

YOUR TURN 1.17

1. Consider the same group of volleyball players from above: {Angie, Brenda, Colleen, Estella, Maya, Maria, Penny, Shantelle}. The team needs to select a captain and an assistant captain from their members. List two possible equivalent subsets that they could select.

Check Your Understanding

8. Every member of a ________ of a set is also a member of the set.
9. Explain what distinguishes a proper subset of a set from a subset of a set.
10. The ________ set is a proper subset of every set except itself.
11. Is the following statement true or false? $A \subseteq A$.
12. If the cardinality of set A is $n(A) = 10$, then set A has a total of _________ subsets.
13. Set A is ___________ to set B if $n(A) = n(B)$.
14. If every member of set A is a member of set B and every member of set B is also a member set A, then set A is __________ to set B.

SECTION 1.2 EXERCISES

For the following exercises, list all the proper subsets of each set.

1. $\{\text{chocolate, vanilla, strawberry}\}$
2. $\{\text{true, false}\}$
3. $\{\text{mother, father, daughter, son}\}$
4. $\{7\}$

For the following exercises, determine the relationship between the two sets and write the relationship symbolically.

$D = \{0, 1, 2, \ldots, 9\}$, $A = \{0, 2, 4, 6, 8\}$, $B = \{1, 3, 5, 7, 9\}$, $C = \{8, 6, 4, 2, 0\}$, $Z = \{0\}$, and $\varnothing$

5. D and A
6. B and D
7. C and D
8. Z and C
9. Z and $\varnothing$
10. A and B
11. A and C
12. $\varnothing$ and D
13. B and C
14. A and Z

For the following exercises, calculate the total number of subsets of each set.

15. $\{\text{Adele, Beyonce, Cher, Madonna, Shakira}\}$
16. $\{\text{Art, Paul}\}$

17. $\{\text{Peter, Paul, Mary}\}$
18. $\varnothing$
19. $\{3\}$
20. $\{l, o, v, e\}$
21. $\{\ \}$
22. $\{\text{football, baseball, basketball, soccer, hockey, tennis, golf}\}$
23. Set A, if $n(A) = 12$.
24. Set B, if $n(B) = 9$.

For the following exercises, use the set of letters in the word largest as the set, $U = \{\text{l, a, r, g, e, s, t}\}$.

25. Find a subset of U that is equivalent, but not equal, to the set: $\{\text{l, a, s, t}\}$.
26. Find a subset of U that is equal to the set: $\{\text{l, a, s, t}\}$.
27. Find a subset of U that is equal to the set: $\{\text{a, r, t}\}$.
28. Find a subset of U that is equivalent, but not equal, to the set $\{\text{a, r, t, s}\}$.
29. Find a subset of U that is equivalent, but not equal, to the set: $\{\text{r, a, t, e, s}\}$.
30. Find a subset of U that is equal to the set: $\{\text{r, a, t, e, s}\}$.
31. Find two three-character subsets of set U that are equivalent, but not equal, to each other.
32. Find two three-character subsets of set U that are equal to each other.
33. Find two five-character subsets of set U that are equal to each other.
34. Find two five-character subsets of set U that are equivalent, but not equal, to each other.

For the following exercises, use the set of integers as the set $U = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

35. Find two equivalent subset of U with a cardinality of 7.
36. Find two equal subsets of U with a cardinality of 4.
37. Find a subset of U that is equivalent, but not equal to, $\{0, 3, 6, 9, \ldots\}$.
38. Find a subset of U that is equivalent, but not equal to, $\{-1, -4, -9, -16, -25, \ldots\}$.
39. True or False. The set of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$, is equivalent to set U.
40. True or False. Set U is an equivalent subset of the set of rational numbers, $\mathbb{Q} = \left\{ \frac{p}{q} \middle| p \text{ and } q \text{ are integers and } q \neq 0. \right\}$.

1.3 Understanding Venn Diagrams

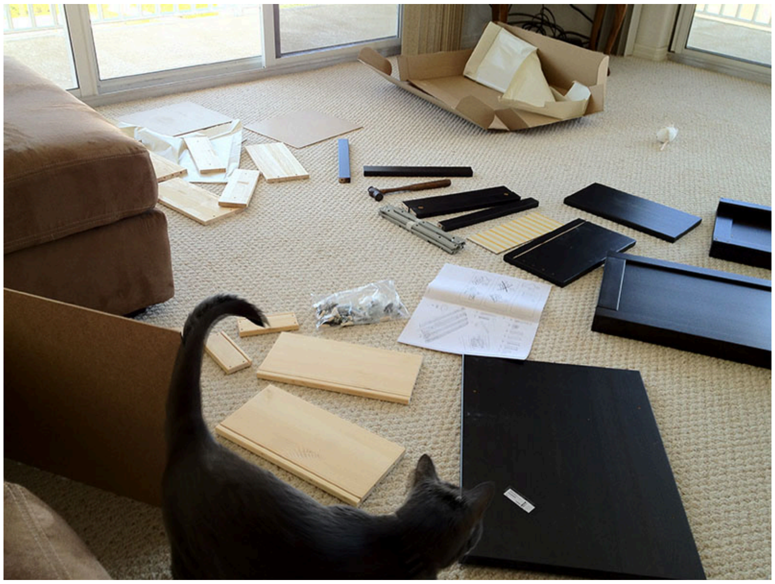


Figure 1.6 When assembling furniture, instructions with images are easier to follow, just like how set relationships are easier to understand when depicted graphically. (credit: "Time to assemble more Ikea furniture!" by Rod Herrea/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Utilize a universal set with two sets to interpret a Venn diagram.
2. Utilize a universal set with two sets to create a Venn diagram.
3. Determine the complement of a set.

Have you ever ordered a new dresser or bookcase that required assembly? When your package arrives you excitedly open it and spread out the pieces. Then you check the assembly guide and verify that you have all the parts required to assemble your new dresser. Now, the work begins. Luckily for you, the assembly guide includes step-by-step instructions with images that show you how to put together your product. If you are really lucky, the manufacturer may even provide a URL or QR code connecting you to an online video that demonstrates the complete assembly process. We can likely all agree that assembly instructions are much easier to follow when they include images or videos, rather than just written directions. The same goes for the relationships between sets.

Interpreting Venn Diagrams

Venn diagrams are the graphical tools or pictures that we use to visualize and understand relationships between sets. Venn diagrams are named after the mathematician John Venn, who first popularized their use in the 1880s. When we use a Venn diagram to visualize the relationships between sets, the entire set of data under consideration is drawn as a rectangle, and subsets of this set are drawn as circles completely contained within the rectangle. The entire set of data under consideration is known as the **universal set**.

Consider the statement: All trees are plants. This statement expresses the relationship between the set of all plants and the set of all trees. Because every tree is a plant, the set of trees is a subset of the set of plants. To represent this relationship using a Venn diagram, the set of plants will be our universal set and the set of trees will be the subset. Recall that this relationship is expressed symbolically as: $\text{Trees} \subset \text{Plants}$. To create a Venn diagram, first we draw a rectangle and label the universal set "$U = \text{Plants}$." Then we draw a circle within the universal set and label it with the word "Trees."

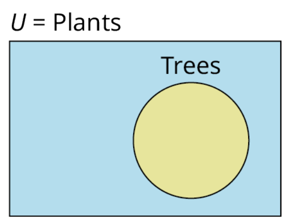


Figure 1.7

This section will introduce how to interpret and construct Venn diagrams. In future sections, as we expand our knowledge of relationships between sets, we will also develop our knowledge and use of Venn diagrams to explore how multiple sets can be combined to form new sets.

EXAMPLE 1.18

Interpreting the Relationship between Sets in a Venn Diagram

Write the relationship between the sets in the following Venn diagram, in words and symbolically.

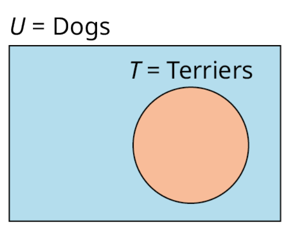


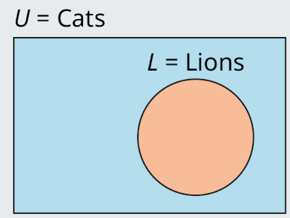
Figure 1.8

Solution

The set of terriers is a subset of the universal set of dogs. In other words, the Venn diagram depicts the relationship that all terriers are dogs. This is expressed symbolically as $T \subset U$.

YOUR TURN 1.18

1. Write the relationship between the sets in the following Venn diagram, in words and symbolically.



So far, the only relationship we have been considering between two sets is the subset relationship, but sets can be related in other ways. Lions and tigers are both different types of cats, but no lions are tigers, and no tigers are lions. Because the set of all lions and the set of all tigers do not have any members in common, we call these two sets **disjoint sets**, or non-overlapping sets.

Two sets A and B are disjoint sets if they do not share any elements in common. That is, if a is a member of set A, then a is not a member of set B. If b is a member of set B, then b is not a member of set A. To represent the relationship between the set of all cats and the sets of lions and tigers using a Venn diagram, we draw the universal set of cats as a rectangle and then draw a circle for the set of lions and a separate circle for the set of tigers within the rectangle, ensuring that the two circles representing the set of lions and the set of tigers do not touch or overlap in any way.

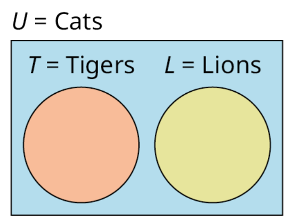


Figure 1.9

EXAMPLE 1.19

Describing the Relationship between Sets

Describe the relationship between the sets in the following Venn diagram.

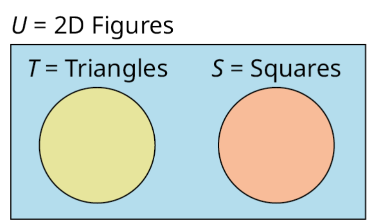


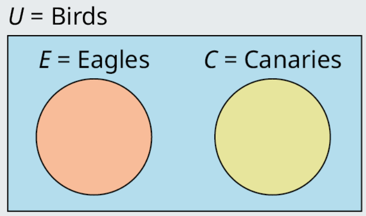
Figure 1.10

Solution

The set of triangles and the set of squares are two disjoint subsets of the universal set of two-dimensional figures. The set of triangles does not share any elements in common with the set of squares. No triangles are squares and no squares are triangles, but both squares and triangles are 2D figures.

YOUR TURN 1.19

1. Describe the relationship between the sets in the following Venn diagram.



Creating Venn Diagrams

The main purpose of a Venn diagram is to help you visualize the relationship between sets. As such, it is necessary to be able to draw Venn diagrams from a written or symbolic description of the relationship between sets.

Procedure

To create a Venn diagram:

1. Draw a rectangle to represent the universal set, and label it $U = \text{set name}$.
2. Draw a circle within the rectangle to represent a subset of the universal set and label it with the set name.

If there are multiple disjoint subsets of the universal set, their separate circles should not touch or overlap.

EXAMPLE 1.20

Drawing a Venn Diagram to Represent the Relationship Between Two Sets

Draw a Venn diagram to represent the relationship between each of the sets.

1. All rectangles are parallelograms.
2. All women are people.

Solution

1. The set of rectangles is a subset of the set of parallelograms.
   First, draw a rectangle to represent the universal set and label it with $U = \text{Parallelograms}$, then draw a circle completely within the rectangle, and label it with the name of the set it represents, $R = \text{Rectangles}$.

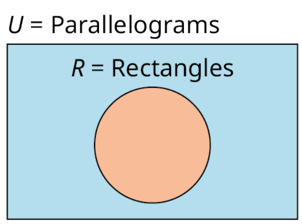


Figure 1.11

In this example, both letters and names are used to represent the sets involved, but this is not necessary. You may use either letters or names alone, as long as the relationship is clearly depicted in the diagram, as shown below.

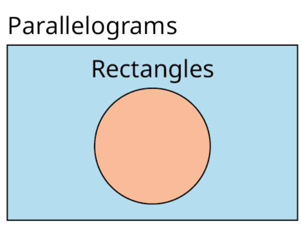


Figure 1.12

or

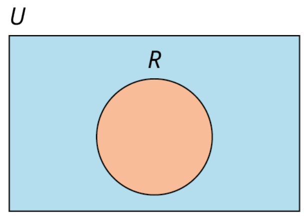


Figure 1.13

2. The universal set is the set of people, and the set of all women is a subset of the set of people.

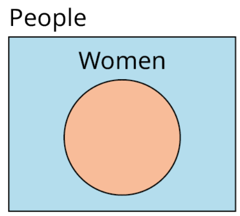


Figure 1.14

YOUR TURN 1.20

1. Draw a Venn diagram to represent the relationship between each of the sets. All natural numbers are integers.
2. $A \subset U$. Draw a Venn diagram to represent this relationship.

EXAMPLE 1.21

Drawing a Venn Diagram to Represent the Relationship Between Three Sets

All bicycles and all cars have wheels, but no bicycle is a car. Draw a Venn diagram to represent this relationship.

Solution

Step 1: The set of bicycles and the set of cars are both subsets of the set of things with wheels. The universal set is the set of things with wheels, so we first draw a rectangle and label it with $U = \text{Things with Wheels}$.

Step 2: Because the set of bicycles and the set of cars do not share any elements in common, these two sets are disjoint and must be drawn as two circles that do not touch or overlap with the universal set.

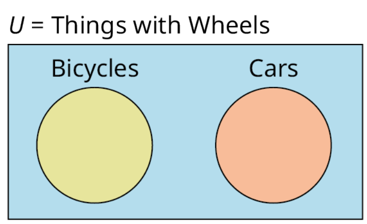


Figure 1.15

YOUR TURN 1.21

1. Airplanes and birds can fly, but no birds are airplanes. Draw a Venn diagram to represent this relationship.

The Complement of a Set

Recall that if set A is a proper subset of set U, the universal set (written symbolically as $A \subset U$), then there is at least one element in set U that is not in set A. The set of all the elements in the universal set U that are not in the subset A is called the **complement** of set A, A'. In set builder notation this is written symbolically as: $A' = \{x \in U | x \notin A\}$. The symbol ∈ is used to represent the phrase, "is a member of," and the symbol ∉ is used to represent the phrase, "is not a member of." In the Venn diagram below, the complement of set A is the region that lies outside the circle and inside the rectangle. The universal set U includes all of the elements in set A and all of the elements in the complement of set A, and nothing else.

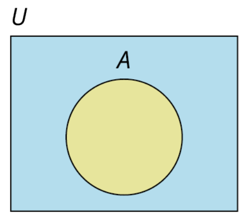


Figure 1.16

Consider the set of digit numbers. Let this be our universal set, $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Now, let set A be the subset of U consisting of all the prime numbers in set U, $A = \{2, 3, 5, 7\}$. The complement of set A is $A' = \{0, 1, 4, 6, 8, 9\}$. The following Venn diagram represents this relationship graphically.

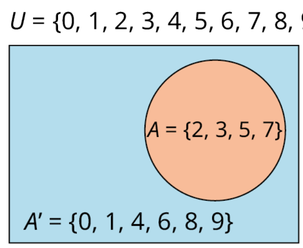


Figure 1.17

EXAMPLE 1.22

Finding the Complement of a Set

For both of the questions below, A is a proper subset of U.

1. Given the universal set $U = \{\text{Billie Eilish, Donald Glover, Bruno Mars, Adele, Ed Sheeran}\}$ and set $A = \{\text{Donald Glover, Bruno Mars, Ed Sheeran}\}$, find A'.
2. Given the universal set $U = \{\text{d}|\text{d is a dog}\}$ and $B = \{\text{b} \in \text{U}|\text{b is a beagle}\}$, find B'.

Solution

1. The complement of set A is the set of all elements in the universal set U that are not in set A. $A' = \{\text{Billie Eilish, Adele}\}$.
2. The complement of set B is the set of all dogs that are not beagles. All members of set B' are in the universal set because they are dogs, but they are not in set B, because they are not beagles. This relationship can be expressed in set build notation as follows: $B' = \{\text{All dogs that are not beagles.}\}$, $B' = \{d \in U|d \text{ is not a beagle.}\}$, or $B' = \{d \in U|d \notin B\}$.

YOUR TURN 1.22

For both of the questions below, A is a proper subset of U.

1. Given the universal set $U = \{\text{red, orange, yellow, green, blue, indigo, violet}\}$ and set $A = \{\text{yellow, red, blue}\}$, find A'.
2. Given the universal set $U = \{c|c \text{ is a cat}\}$ and set $A = \{c \in U|c \text{ is not a lion}\}$, find A'.

Check Your Understanding

15. A Venn diagram is a graphical representation of the ___________ between sets.

16. In a Venn diagram, the set of all data under consideration, the ___________ set, is drawn as a rectangle.

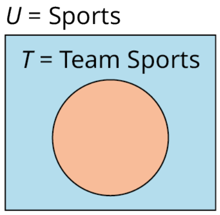
17. Two sets that do not share any elements in common are ___________ sets.

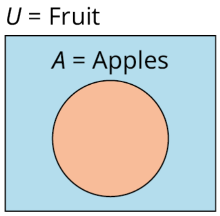
18. The ____________ of a subset A or the universal set, U, is the set of all members of U that are not in A.

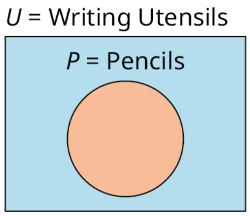
19. The sets A and A' are ____________ subsets of the universal set.

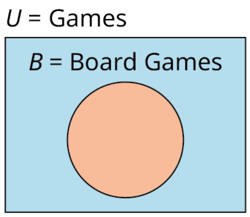
SECTION 1.3 EXERCISES

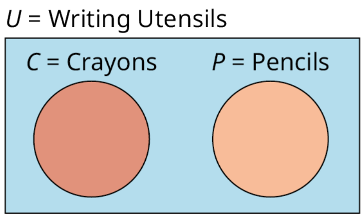
For the following exercises, interpret each Venn diagram and describe the relationship between the sets, symbolically and in words.

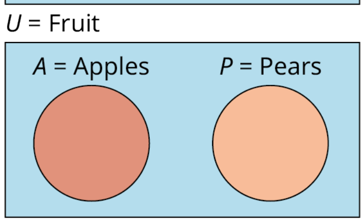
1. 

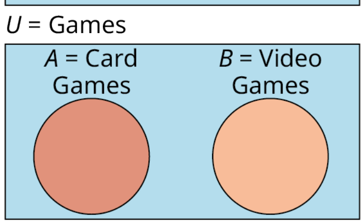
2. 

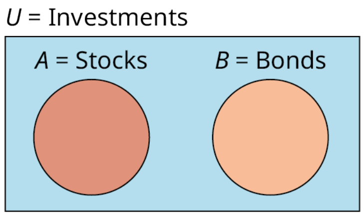
3. 

4. 

5. 

6. 

7. 

8. 

For the following exercises, create a Venn diagram to represent the relationships between the sets.

9. All birds have wings.
10. All cats are animals.
11. All almonds are nuts, and all pecans are nuts, but no almonds are pecans.
12. All rectangles are quadrilaterals, and all trapezoids are quadrilaterals, but no rectangles are trapezoids.
13. Lizards ⊂ Reptiles.
14. Ladybugs ⊂ Insects.
15. Ladybugs ⊂ Insects and Ants ⊂ Insects, but no Ants are Ladybugs.
16. Lizards ⊂ Reptiles and Snakes ⊂ Reptiles, but no Lizards are Snakes.
17. A and B are disjoint subsets of U.
18. C and D are disjoint subsets of U.
19. T is a subset of U.
20. S is a subset of U.
21. $J =$ Jazz, $M =$ Music, and $J \subset M$.
22. $R =$ Reggae, $M =$ Music, and $R \subset M$.
23. $J =$ Jazz, $R =$ Reggae, and $M =$ Music are sets with the following relationships: $J \subset M$, $R \subset M$, and R is disjoint from J.
24. $J =$ Jazz, $B =$ Bebop, and $M =$ Music are sets with the following relationships: $J \subset M$ and $B \subset J$.

For the following exercises, the universal set is the set of single digit numbers, $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Find the complement of each subset of U.

25. $A = \{6, 7, 8\}$
26. $A = \{0, 2, 4, 6, 8\}$
27. $A = \{ \}$
28. $A = \{0, 1, 4, 6, 8, 9\}$
29. $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
30. $A = \{0, 1, 3, 4, 5, 6, 7, 9\}$
31. $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
32. $A = \{0, 3, 6, 9\}$

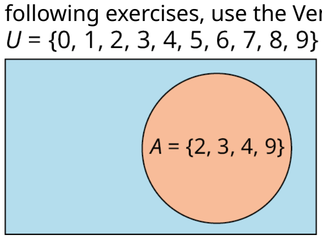
For the following exercises, the universal set is $U =$ {Bashful, Doc, Dopey, Grumpy, Happy, Sleepy, Sneezy}. Find the complement of each subset of U.

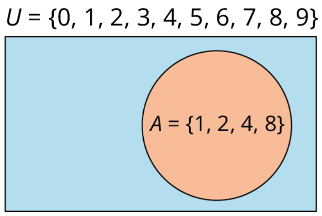
33. $A =$ {Happy, Bashful, Grumpy}
34. $A =$ {Sleepy, Sneezy}
35. $A =$ {Doc}
36. $A =$ {Doc, Dopey}
37. $A = \varnothing$
38. $A =$ {Doc, Grumpy, Happy, Sleepy, Bashful, Sneezy, Dopey}

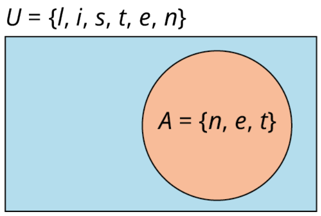
For the following exercises, the universal set is $U = \mathbb{N} = \{1, 2, 3, \dots\}$. Find the complement of each subset of U.

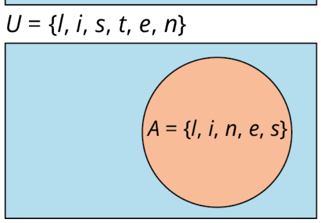
39. $A = \{1, 2, 3, 4, 5\}$
40. $A = \{1, 3, 5, \dots\}$
41. $A = \{1\}$
42. $A = \{4, 5, 6, \dots\}$

For the following exercises, use the Venn diagram to determine the members of the complement of set A, A'.

43. 

44.


45.


46.


1.4 Set Operations with Two Sets

Figure 1.18 A large, multigenerational family contains an intersection and a union of sets. (credit: "Family Photo Shoot Bani Syakur" by Mainur Risyada/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Determine the intersection of two sets.
2. Determine the union of two sets.
3. Determine the cardinality of the union of two sets.
4. Apply the concepts of AND and OR to set operations.
5. Draw conclusions from Venn diagrams with two sets.

The movie *Yours, Mine, and Ours* was originally released in 1968 and starred Lucille Ball and Henry Fonda. This movie, which is loosely based on a true story, is about the marriage of Helen, a widow with eight children, and Frank, a widower with ten children, who then have an additional child together. The movie is a comedy that plays on the interpersonal and organizational struggles of feeding, bathing, and clothing twenty people in one household.

If we consider the set of Helen's children and the set of Frank's children, then the child they had together is the intersection of these two sets, and the collection of all their children combined is the union of these two sets. In this section, we will explore the operations of union and intersection as it relates to two sets.

The Intersection of Two Sets

The members that the two sets share in common are included in the **intersection of two sets**. To be in the intersection of two sets, an element must be in both the first set and the second set. In this way, the intersection of two sets is a logical AND statement. Symbolically, A intersection B is written as: $A \cap B$. A intersection B is written in set builder notation as: $A \cap B = \{x | x \in A \text{ and } x \in B\}$.

Let us look at Helen's and Frank's children from the movie *Yours, Mine, and Ours.* Helen's children consist of the set $H = \{\text{Colleen, Nick, Janette, Tommy, Jean, Phillip, Gerald, Theresa, Joseph}\}$ and Frank's children are included in the set $F = \{\text{Mike, Rusty, Greg, Rosemary, Loise, Susan, Veronica, Mary, Germaine, Joan, Joseph}\}$. H intersection F is the set of children they had together. $H \cap F = \{\text{Joseph}\}$, because Joseph is in both set H and set F.

EXAMPLE 1.23

Finding the Intersection of Set A and Set B

Set $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 5, 7\}$. Find A intersection B.

Solution

The intersection of sets A and B include the elements that set A and B have in common: 3, 5, and 7. $A \cap B = \{3, 5, 7\}$.

YOUR TURN 1.23

1. Set $A = \{h, a, p, y\}$ and $B = \{s, a, d\}$. Find A intersection B.

Notice that if sets A and B are disjoint sets, then they do not share any elements in common, and A intersection B is the empty set, as shown in the Venn diagram below.

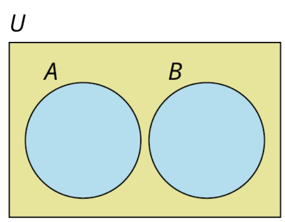


Figure 1.19

EXAMPLE 1.24

Determining the Intersection of Disjoint Sets

Set $A = \{0, 2, 4, 6, 8\}$ and set $B = \{1, 3, 5, 7, 9\}$. Find $A \cap B$.

Solution

Because sets A and B are disjoint, they do not share any elements in common. So, the intersection of set A and set B is the empty set. $A \cap B = \emptyset$.

YOUR TURN 1.24

1. Set $A = \{\text{red, yellow, blue}\}$ and set $B = \{\text{orange, green, purple}\}$. Find $A \cap B$.

Notice that if set A is a subset of set B, then A intersection B is equal to set A, as shown in the Venn diagram below.

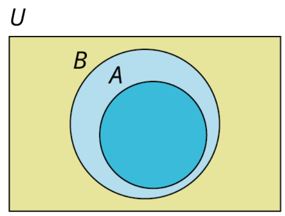


Figure 1.20

EXAMPLE 1.25

Finding the Intersection of a Set and a Subset

Set $A = \{1, 3, 5, \ldots\}$ and set $B = \mathbb{N} = \left\{1, 2, 3, \ldots\right\}$ Find $A \cap B$.

Solution

Because set A is a subset of set B, A intersection B is equal to set A. $A \cap B = A = \{1, 3, 5, \ldots\}$, the set of odd natural numbers.

YOUR TURN 1.25

1. Set $A = \{a, b, c, \ldots, z\}$ and set $B = \{a, e, i, o, u\}$. Find $A \cap B$.

The Union of Two Sets

Like the union of two families in marriage, the **union of two sets** includes all the members of the first set and all the members of the second set. To be in the union of two sets, an element must be in the first set, the second set, or both. In this way, the union of two sets is a logical inclusive OR statement. Symbolically, A union B is written as: $A \cup B$. A union B is written in set builder notation as: $A \cup B = \{x | x \in A \text{ or } x \in B\}$.

Let us consider the sets of Helen's and Frank's children from the movie *Yours, Mine, and Ours* again. Helen's children is set $H = \{\text{Colleen, Nick, Janette, Tommy, Jean, Phillip, Gerald, Theresa, Joseph}\}$ and Frank's children is set $F = \{\text{Mike, Rusty, Greg, Rosemary, Loise, Susan, Veronica, Mary, Germaine, Joan, Joseph}\}$. The union of these two sets is the collection of all nineteen of their children,

$$H \cup F = \{\text{Colleen, Nick, Janette, Tommy, Jean, Phillip, Gerald, Theresa, Joseph,}$$
$$\text{Mike, Rusty, Greg, Rosemary, Loise, Susan, Veronica, Mary, Germaine, Joan}\}.$$

Notice, Joseph is in both set H and set F, but he is only one child, so, he is only listed once in the union.

EXAMPLE 1.26

Finding the Union of Sets A and B When A and B Overlap

Set $A = \{1, 3, 5, 7, 9\}$ and set $B = \{2, 3, 5, 7\}$. Find A union B.

Solution

A union B is the set formed by including all the unique elements in set A, set B, or both sets A and B: $A \cup B = \{1, 3, 5, 7, 9, 2\}$. The first five elements of the union are the five unique elements in set A. Even though 3, 5, and 7 are also members of set B, these elements are only listed one time. Lastly, set B includes the unique element 2, so 2 is also included as part of the union of sets A and B.

YOUR TURN 1.26

1. Set $A = \{h, a, p, y\}$ and set $B = \{s, a, d\}$. Find A union B.

When observing the union of sets A and B, notice that both set A and set B are subsets of A union B. Graphically, A union B can be represented in several different ways depending on the members that they have in common. If A and B are disjoint sets, then A union B would be represented with two disjoint circles within the universal set, as shown in the

Venn diagram below.

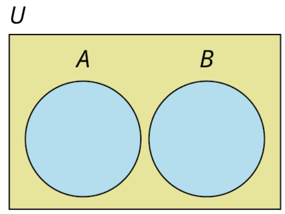


Figure 1.21 $A \cup B$

If sets A and B share some, but not all, members in common, then the Venn diagram is drawn as two separate circles that overlap.

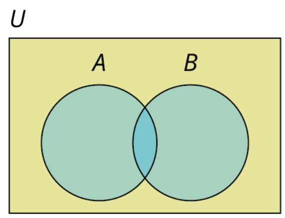


Figure 1.22

If every member of set A is also a member of set B, then A is a subset of set B, and A union B would be equal to set B. To draw the Venn diagram, the circle representing set A should be completely enclosed in the circle containing set B.

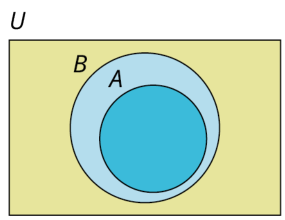


Figure 1.23

EXAMPLE 1.27

Finding the Union of Sets A and B When A and B Are Disjoint

Set $A = \{0, 2, 4, 6, 8\}$ and set $B = \{1, 3, 5, 7, 9\}$. Find $A \cup B$.

Solution

Because sets A and B are disjoint, the union is simply the set containing all the elements in both set A and set B. $A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

YOUR TURN 1.27

1. Set $A = \{\text{red, yellow, blue}\}$ and set $B = \{\text{orange, green, purple}\}$. Find $A \cup B$.

EXAMPLE 1.28

Finding the Union of Sets A and B When One Set is a Subset of the Other

Set $A = \{1, 3, 5, \ldots\}$ and set $B = \mathbb{N} = \left\{1, 2, 3, \ldots\right\}$. Find $A \cup B$.

Solution

Because set A is a subset of set B, A union B is equal to set B. $A \cup B = \mathbb{N} = \left\{1, 2, 3, \ldots\right\} = B$.

YOUR TURN 1.28

1. Set $A = \{a, b, c, \ldots, z\}$ and set $B = \{a, e, i, o, u\}$. Find $A \cup B$.

VIDEO

The Basics of Intersection of Sets, Union of Sets and Venn Diagrams (https://openstax.org/r/operation-on-Sets)

TECH CHECK

Set Operation Practice

Sets Challenge is an application available on both Android and iPhone smartphones that allows you to practice and gain familiarity with the operations of set union, intersection, complement, and difference.

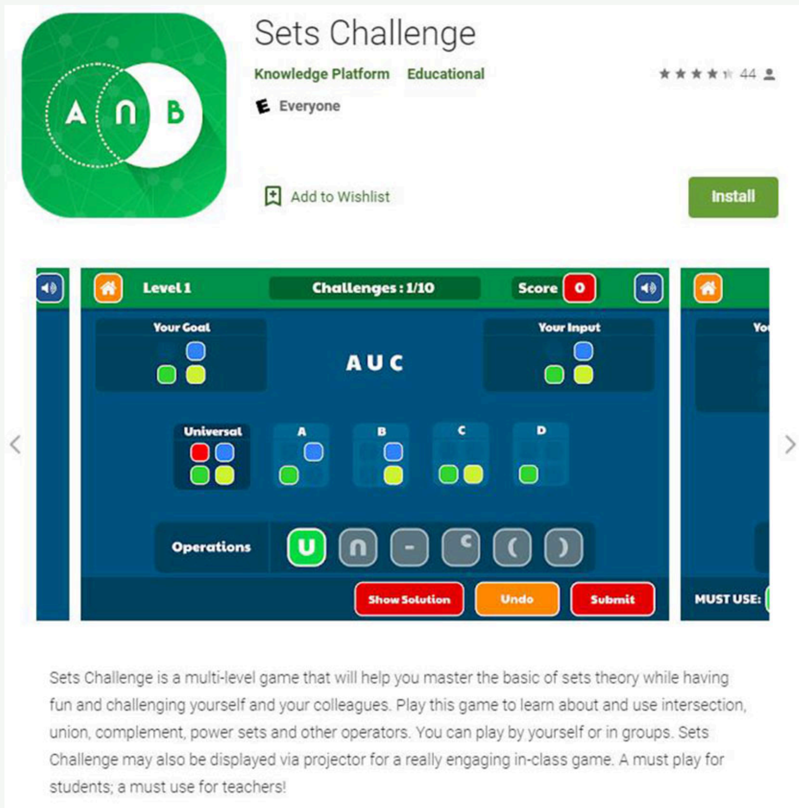


Figure 1.24 Google Play Store image of Sets Challenge game. (credit: screenshot from Google Play)

The Sets Challenge application/game uses some notation that differs from the notation covered in the text.

- The complement of set A in this text is written symbolically as A', but the Sets Challenge game uses A^C to represent the complement operation.
- In the text we do not cover set difference between two sets A and B, represented in the game as $A - B$. In the game this operation removes from set A all the elements in $A \cap B$. For example, if set $A = \{a, b, c, d\}$ and set $B = \{b, d, f, h\}$ are subsets of the universal set $U = \{a, b, c, \ldots, z\}$, then $A - B = \{a, b, c, d\} - \{b, d\} = \{a, c\}$, and $B - A = \{b, d, f, h\} - \{b, d\} = \{f, h\}$. There is a project at the end of the chapter to research the set difference operation.

Determining the Cardinality of Two Sets

The **cardinality of the union of two sets** is the total number of elements in the set. Symbolically the cardinality of A union B is written, $n(A \cup B)$. If two sets A and B are disjoint, the cardinality of A union B is the sum of the cardinality of set A and the cardinality of set B. If the two sets intersect, then A intersection B is a subset of both set A and set B. This means that if we add the cardinality of set A and set B, we will have added the number of elements in A intersection B twice, so we must then subtract it once as shown in the formula that follows.

FORMULA

The **cardinality of A union B** is found by adding the number of elements in set A to the number of elements in set B, then subtracting the number of elements in the intersection of set A and set B.
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$ or $n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$.

If sets A and B are disjoint, then $n(A \cap B) = n(A \text{ and } B) = 0$ and the formula is still valid, but simplifies to $n(A \cup B) = n(A) + n(B)$.

EXAMPLE 1.29

Determining the Cardinality of the Union of Two Sets

The number of elements in set A is 10, the number of elements in set B is 20, and the number of elements in A intersection B is 4. Find the number of elements in A union B.

Solution

Using the formula for determining the cardinality of the union of two sets, we can say $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 10 + 20 - 4 = 26$.

YOUR TURN 1.29

1. If $n(A) = 23$, $n(B) = 17$, and $n(A \cap B) = 7$, then find $n(A \cup B)$.

EXAMPLE 1.30

Determining the Cardinality of the Union of Two Disjoint Sets

If A and B are disjoint sets and the cardinality of set A is 37 and the cardinality of set B is 43, find the cardinality of A union B.

Solution

To find the cardinality of A union B, apply the formula, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. Because sets A and B are disjoint, $A \cap B$ is the empty set, therefore $n(A \cap B) = n(\varnothing) = 0$ and $n(A \cup B) = 37 + 43 - 0 = 80$.

YOUR TURN 1.30

1. If $A \cap B = \varnothing$, $n(A) = 35$, and $n(B) = 78$, then find $n(A \cup B)$.

Applying Concepts of “AND” and “OR” to Set Operations

To become a licensed driver, you must pass some form of written test and a road test, along with several other requirements depending on your age. To keep this example simple, let us focus on the road test and the written test. If you pass the written test but fail the road test, you will not receive your license. If you fail the written test, you will not be allowed to take the road test and you will not receive a license to drive. To receive a driver's license, you must pass the written test AND the road test. For an “AND” statement to be true, both conditions that make up the statement must be true. Similarly, the intersection of two sets A and B is the set of elements that are in both set A and set B. To be a member of A intersection B, an element must be in set A and also must be in set B. The intersection of two sets corresponds to a logical "AND" statement.

The union of two sets is a logical inclusive "OR" statement. Say you are at a birthday party and the host offers Leah, Lenny, Maya, and you some cake or ice cream for dessert. Leah asks for cake, Lenny accepts both cake and ice cream, Maya turns down both, and you choose only ice cream. Leah, Lenny, and you are all having dessert. The “OR” statement is true if at least one of the components is true. Maya is the only one who did not have cake or ice cream; therefore, she did not have dessert and the “OR” statement is false. To be in the union of two sets A and B, an element must be in set A or set B or both set A and set B.

EXAMPLE 1.31

Applying the "AND" or "OR" Operation

$A = \{0, 3, 6, 9, 12\}$, $B = \{0, 4, 8, 12, 16\}$, and $C = \{1, 2, 3, 5, 8, 13\}$.

Find the set consisting of elements in:

1. A and B.
2. A or B.
3. A or C.
4. $(B$ and $C)$ or A.

Solution

1. A and $B = A \cap B = \{0, 12\}$, because only the elements 0 and 12 are members of both set A and set B.
2. A or $B = A \cup B = \{0, 3, 4, 6, 8, 9, 12, 16\}$, because the set A or B is the collection of all elements in set A or set B, or both.
3. A or $C = A \cup C = \{0, 1, 2, 3, 5, 6, 8, 9, 12, 13\}$, because the set A or C is the collection of all elements in set A or set C, or both.
4. $(B$ and $C)$ or $A = (B \cap C) \cup A$. Parentheses are evaluated first: $(B$ and $C) = B \cap C = \{8\}$, because the only member that both set B and set C share in common is 8. So, now we need to find $\{8\}$ or $\{0, 3, 6, 9, 12\}$, Because the word translates to the union operation, the problem becomes $\{8\} \cup \{0, 3, 6, 9, 12\}$, which is equal to $\{0, 3, 6, 8, 9, 12\}$.

YOUR TURN 1.31

$A = \{h, a, p, y\}$, and $B = \{a, w, e, s, o, m\}$, and $C = \{m, a, t, h\}$.
Find the set consisting of elements in:

1. A or B.
2. A and C.
3. B or C.
4. $(A$ and $C)$ and B.

EXAMPLE 1.32

Determine and Apply the Appropriate Set Operations to Solve the Problem

Don Woods is serving cake and ice cream at his Juneteenth celebration. The party has a total of 54 guests in attendance. Suppose 30 guests requested cake, 20 guests asked for ice cream, and 12 guests did not have either cake or ice cream.

1. How many guests had cake or ice cream?
2. How many guests had cake and ice cream?

Solution

1. The total number of people at the party is 54, and 12 people did not have cake or ice cream. Recall that the total number of elements in the universal set is always equal to the number of elements in a subset plus the number of elements in the complement of the set, $n(U) = n(A) + n(A')$. That means $54 = n\,(\text{cake or ice cream}) \ + \ n\,(\text{not (cake or ice cream)})$, or equivalently, $n\,(\text{cake} \ \cup \ \text{ice cream}) = \ 54 - n\left((\text{cake} \ \cup \ \text{ice cream})'\right) = 54 - 12 = 42$. A total of 42 people at the party had cake or ice cream.
2. To determine the number of people who had both cake and ice cream, we need to find the intersection of the set of people who had cake and the set of people who had ice cream. From Question 1, the number of people who had

cake or ice cream is 42. This is the union of the two sets. The formula for the union of two sets is $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. Use the information given in the problem and substitute the known values into the formula to solve for the number of people in the intersection: $42 = 30 + 20 - n(A \cap B)$. Adding 30 and 20, the equation simplifies to $42 = 50 - n(\text{cake and ice cream})$. Which means $n(\text{cake and ice cream}) = 50 - 42 = 8$.

YOUR TURN 1.32

Ravi and Priya are serving soup and salad along with the main course at their wedding reception. The reception will have a total of 150 guests in attendance. A total of 92 soups and 85 salads were ordered, while 23 guests did not order any soup or salad.

1. How many guests had soup or salad or both?
2. How many guests had both soup and a salad?

WHO KNEW?

The Real Inventor of the Venn Diagram

John Venn, in his writings, references works by both John Boole and Augustus De Morgan, who referred to the circle diagrams commonly used to present logical relationships as Euler's circles. Leonhard Euler's works were published over 100 years prior to Venn's, and Euler may have been influenced by the works of Gottfried Leibniz.

So, why does John Venn get all the credit for these graphical depictions? Venn was the first to formalize the use of these diagrams in his book *Symbolic Logic,* published in 1881. Further, he made significant improvements in their design, including shading to highlight the region of interest. The mathematician C.L. Dodgson, also known as Lewis Carroll, built upon Venn’s work by adding an enclosing universal set.

Invention is not necessarily coming up with an initial idea. It is about seeing the potential of an idea and applying it to a new situation.

References:

Margaret E. Baron. "A Note on the Historical Development of Logic Diagrams: Leibniz, Euler and Venn." *The Mathematical Gazette*, vol. 53, no. 384, 1969, pp. 113-125. *JSTOR*, www.jstor.org/stable/3614533. Accessed 15 July 2021.

Deborah Bennett. "Drawing Logical Conclusions." *Math Horizons*, vol. 22, no. 3, 2015, pp. 12-15. *JSTOR*, www.jstor.org/stable/10.4169/mathhorizons.22.3.12. Accessed 15 July 2021.

Drawing Conclusions from a Venn Diagram with Two Sets

All Venn diagrams will display the relationships between the sets, such as subset, intersecting, and/or disjoint. In addition to displaying the relationship between the two sets, there are two main additional details that Venn diagrams can include: the individual members of the sets or the cardinality of each disjoint subset of the universal set.

A Venn diagram with two subsets will partition the universal set into 3 or 4 sections depending on whether they are disjoint or intersecting sets. Recall that the complement of set A, written A', is the set of all elements in the universal set that are not in set A.

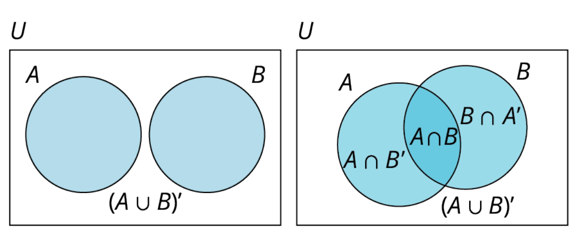


Figure 1.25 Side-by-side Venn diagrams with disjoint and intersecting sets, respectively.

EXAMPLE 1.33

Using a Venn Diagram to Draw Conclusions about Set Membership

$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

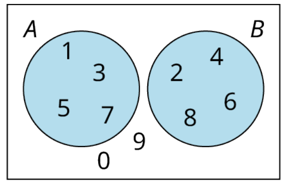


Figure 1.26

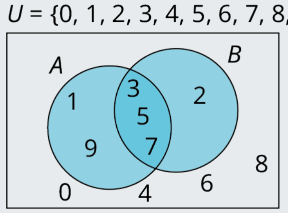
1. Find $A \cup B$.
2. Find $A \cap B$.
3. Find B'.
4. Find $n(B')$.

Solution

1. $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$, because A union B is the collection of all elements in set A or set B or both.
2. Because A and B are disjoint sets, there are no elements that are in both A and B. Therefore, A intersection B is the empty set, $A \cap B = \varnothing$.
3. The complement of set B is the set of all elements in the universal set that are not in set B: $B' = \{0, 1, 3, 5, 7, 9\}$.
4. The cardinality, or number of elements in set B', is $n(B') = 6$.

YOUR TURN 1.33

$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$



Venn diagram with two intersecting sets and members.

1. Find $A \cap B$.
2. Find $A \cup B$.
3. Find $A \cap B'$.
4. Find $n(A \cap B')$.

EXAMPLE 1.34

Using a Venn Diagram to Draw Conclusions about Set Cardinality

U

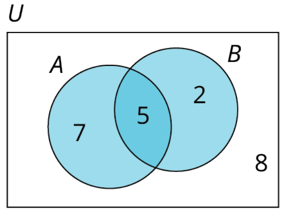


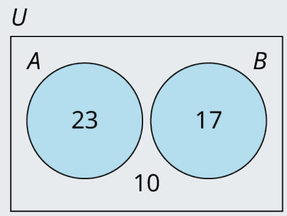
Figure 1.27 Venn diagram with two intersecting sets and number of elements in each section indicated.

1. Find $n(A \text{ or } B)$.
2. Find $n(A \text{ and } B)$.
3. Find $n(A)$.

Solution

1. The number of elements in A or B is the number of elements in A union B: $n(A \cup B) = n(\{2, 5, 7\}) = 14$.
2. The number of elements in A and B is the number of elements in A intersection B: $n(A \cap B) = 5$.
3. The number of elements in set A is the sum of all the numbers enclosed in the circle representing set A: $n(A) = n(\{7, 5\}) = 12$.

YOUR TURN 1.34



Venn diagram with two disjoint sets and number of elements in each section.

1. Find $n(A \text{ or } B)$.
2. Find $n(A \text{ and } B)$.
3. Find $n(A')$.

Check Your Understanding

20. The __________ of two sets A and B is the set of all elements that they share in common.

21. The __________ of two sets A and B is the collection of all elements that are in set A or set B, or both set A and set B.

22. The union of two sets A and B is represented symbolically as __________.

23. The intersection of two sets A and B is represented symbolically as __________.

24. If set A is a subset of set B, then A intersection B is equal to set __________.

25. If set A is a subset of set B, then A union B is equal to set __________.

26. If set A and set B are disjoint sets, then A intersection B is the __________ set.

27. The cardinality of A union B, $n(A \cup B)$, is found using the formula: __________.

SECTION 1.4 EXERCISES

For the following exercises, determine the union or intersection of the sets as indicated.
$A = \{2, 4, 6, 8, 10, 12\}$, $B = \{4, 8, 12, 16, 20\}$, $C = \{8, 16, 24, 32, 40\}$, and $D = \{10, 20, 30, 40, 50\}$.

1. $B \cup C$
2. $A \cap D$
3. $D \cap C$
4. $A \cup D$
5. $A \cap (C \cup D)$
6. $B \cup (A \cap D)$
7. $D \cup (A \cap C)$
8. $C \cap (A \cup D)$
9. $B \cap (A \cap D)$
10. $B \cap (A \cap C)$
11. $B \cup (A \cup D)$
12. $B \cup (A \cup C)$

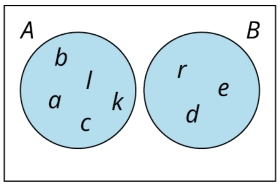
For the following exercises, use the sets provided to apply the “AND” or “OR” operation as indicated to find the resulting set.

$U = \{a, b, c, \ldots, z\}$, $S = \{s, a, m, p, l, e\}$, $M = \{m, a, p\}$, $L = \{l, a, m, p\}$, $D = \{d, o, g\}$, and $P = \{p, l, o, t\}$.

13. Find the set consisting of elements in S and P.
14. Find the set consisting of elements in M or D.
15. Find the set consisting of elements in P or M.
16. Find the set consisting of elements in M and D.
17. Find the set consisting of elements in L and M.
18. Find the set consisting of elements in L or M.
19. Find the set consisting of the elements in D or M or P.
20. Find the set consisting of the elements in S or M or P.
21. Find the set consisting of the elements in (S or D) and P.
22. Find the set consisting of the elements in Sor (D and P).
23. Find the set consisting of elements in U or (P and S).
24. Find the set consisting of elements in (U or P) and S.

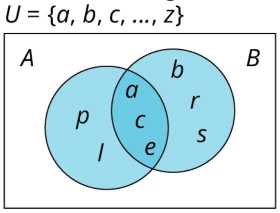
For the following exercises, use the Venn diagram provided to answer the following questions about the sets.

U = {*a*, *b*, *c*, ..., *z*}



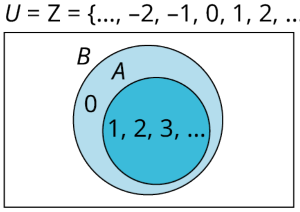
25. Find $A \cup B$.
26. Find $A \cap B$.
27. Find $(A \cap B)'$.
28. Find $(A \cup B)'$.
29. Find $A \cap B'$.
30. Find $B \cap A'$.

For the following exercises, use the Venn diagram provided to answer the following questions about the sets.
U = {*a*, *b*, *c*, ..., *z*}



31. Find $A \cap B$.
32. Find $A \cup B$.
33. Find $(A \cup B)'$.
34. Find $(A \cap B)'$.
35. Find $B \cap A'$.
36. Find $A \cap B'$.

For the following exercises, use the Venn diagram provided to answer the following questions about the sets.
U = Z = {..., –2, –1, 0, 1, 2, ...}



37. Find $A \cup B$.
38. Find $A \cap B$.
39. Find $(A \cup B)'$.
40. Find $(A \cap B)'$.
41. Find $B \cap A'$.
42. Find $A \cap B'$.

For the following exercises, determine the cardinality of the union of set A and set B.

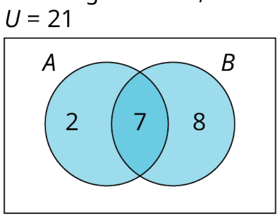
43. If set $A = \{\text{red, white, blue}\}$ and set $B = \{\text{green, white, red}\}$, find $n(A \cup B)$.

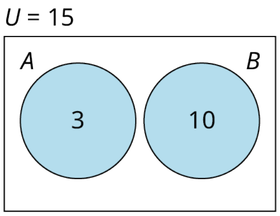
44. If set $A = \{\text{silver, gold, bronze}\}$ and set $B = \{\text{silver, gold}\}$, find the number of elements in A or B.

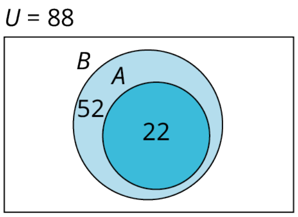
45. If set $A = \{\text{glass, plate, fork, knife}\}$ and set $B = \{\text{bowl, spoon, cup}\}$, find the number of elements in A or B.

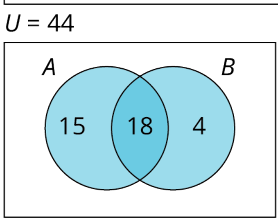
46. If set $A = \{\text{Algebra, Geometry, Biology, Physics, Chemistry, English}\}$ and Set $B = \{\text{Algebra, English, History, Spanish, French, Music}\}$, find $n(A \cup B)$.

For the following exercises, use the Venn diagram to determine the cardinality of A union B.

47.


48.


49.


50.


1.5 Set Operations with Three Sets

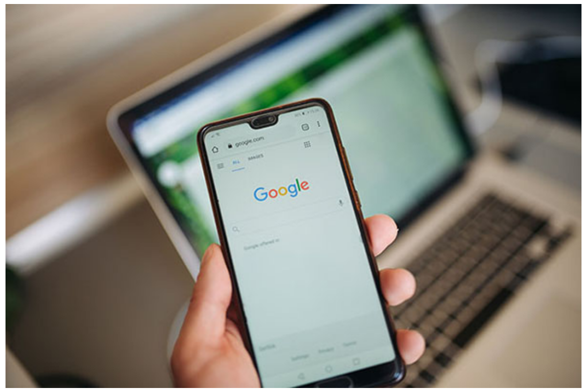


Figure 1.28 Companies like Google collect data on how you use their services, but the data requires analysis to really mean something. (credit: “Man holding smartphone and searches through google” by Nenad Stojkovic/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Interpret Venn diagrams with three sets.
2. Create Venn diagrams with three sets.
3. Apply set operations to three sets.
4. Prove equality of sets using Venn diagrams.

Have you ever searched for something on the Internet and then soon after started seeing multiple advertisements for that item while browsing other web pages? Large corporations have built their business on data collection and analysis. As we start working with larger data sets, the analysis becomes more complex. In this section, we will extend our knowledge of set relationships by including a third set.

A Venn diagram with two intersecting sets breaks up the universal set into four regions; simply adding one additional set will increase the number of regions to eight, doubling the complexity of the problem.

Venn Diagrams with Three Sets

Below is a Venn diagram with two intersecting sets, which breaks the universal set up into four distinct regions.

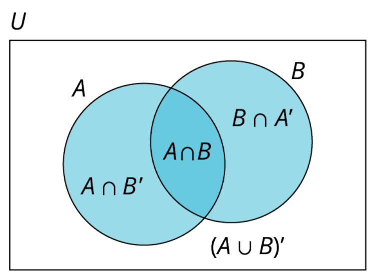


Figure 1.29

Next, we see a **Venn diagram with three intersecting sets**, which breaks up the universal set into eight distinct regions.

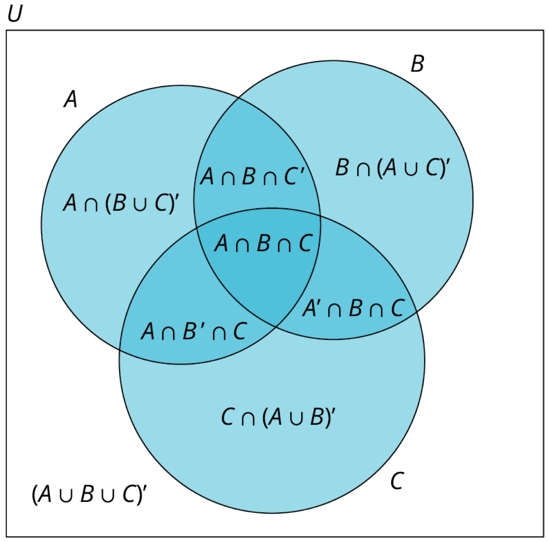


Figure 1.30

TECH CHECK

Shading Venn Diagrams

Venn Diagram is an Android application that allows you to visualize how the sets are related in a Venn diagram by entering expressions and displaying the resulting Venn diagram of the set shaded in gray.

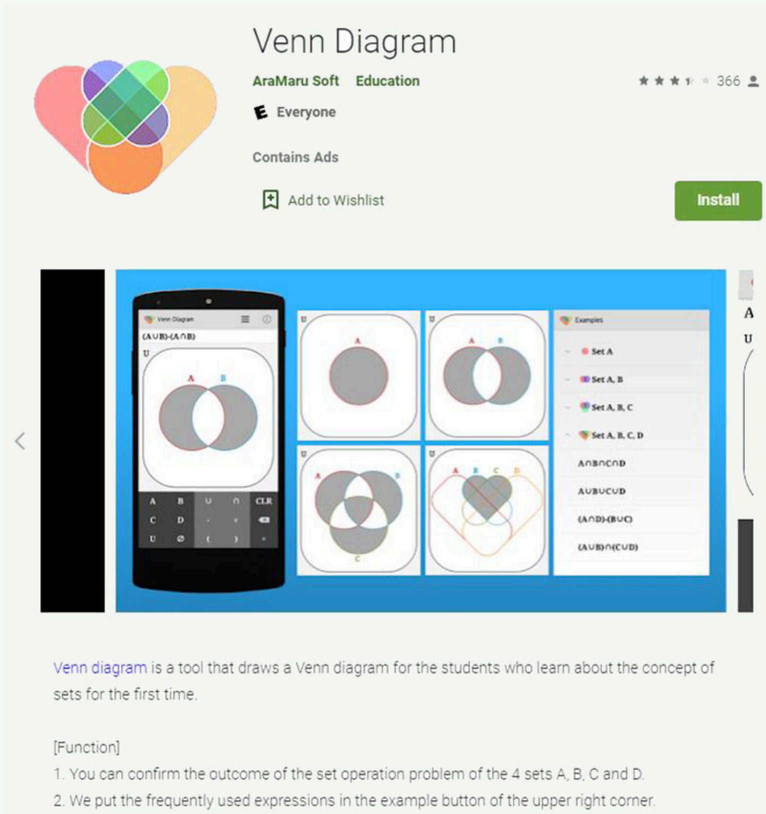


Figure 1.31 Google Play Store image of Venn Diagram app. (credit: screenshot from Google Play)

The Venn Diagram application uses some notation that differs from the notation covered in this text.

a. The complement of set A in this text is written symbolically as A', but the Venn Diagram app uses A^C to represent the complement operation.

b. The set difference operation, −, is available in the Venn Diagram app, although this operation is not covered in the text.

It is recommended that you explore this application to expand your knowledge of Venn diagrams prior to continuing with the next example.

In the next example, we will explore the three main blood factors, A, B and Rh. The following background information about blood types will help explain the relationships between the sets of blood factors. If an individual has blood factor A or B, those will be included in their blood type. The Rh factor is indicated with a + or a −. For example, if a person has all three blood factors, then their blood type would be AB^{+}. In the Venn diagram, they would be in the intersection of all three sets, $A \cap B \cap Rh^{+}$. If a person did not have any of these three blood factors, then their blood type would be O^{-}, and they would be in the set $\left(A \cup B \cup Rh^{+}\right)'$ which is the region outside all three circles.

EXAMPLE 1.35

Interpreting a Venn Diagram with Three Sets

Use the Venn diagram below, which shows the blood types of 100 people who donated blood at a local clinic, to answer the following questions.

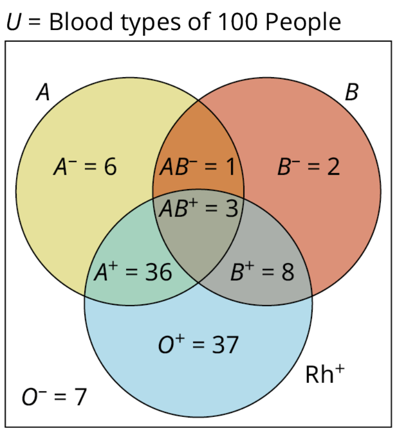


Figure 1.32

1. How many people with a type A blood factor donated blood?
2. Julio has blood type B^{+}. If he needs to have surgery that requires a blood transfusion, he can accept blood from anyone who does not have a type A blood factor. How many people donated blood that Julio can accept?
3. How many people who donated blood do not have the Rh^{+} blood factor?
4. How many people had type A and type B blood?

Solution

1. The number of people who donated blood with a type A blood factor will include the sum of all the values included in the A circle. It will be the union of sets A^{-}, A^{+}, AB^{-} and AB^{+}.
   $n(A) = n(A^{-}) + n(A^{+}) + n(AB^{-}) + n(AB^{+}) = 6 + 36 + 1 + 3 = 46$.
2. In part 1, it was determined that the number of donors with a type A blood factor is 46. To determine the number of people who did not have a type A blood factor, use the following property, A' union is equal to U, which means $n(A) + n(A') = n(U)$, and $n(A') = n(U) - n(A) = 100 - 46 = 54$. Thus, 54 people donated blood that Julio can accept.
3. This would be everyone outside the Rh^{+} circle, or everyone with a negative Rh factor,
   $n(Rh^{-}) = n(O^{-}) + n(A^{-}) + n(AB^{-}) + n(B^{-}) = 7 + 6 + 1 + 2 = 16$.
4. To have both blood type A and blood type B, a person would need to be in the intersection of sets A and B. The two circles overlap in the regions labeled AB^{-} and AB^{+}. Add up the number of people in these two regions to get the total: $1 + 3 = 4$. This can be written symbolically as $n(A \text{ and } B) = n(A \cap B) = n(AB^{-}) + n(AB^{+}) = 1 + 3 = 4$.

YOUR TURN 1.35

Use the same Venn diagram in the example above to answer the following questions.

1. How many people donated blood with a type B blood factor?
2. How many people who donated blood did not have a type B blood factor?
3. How many people who donated blood had a type B blood factor or were Rh^{+}?

WHO KNEW?

Blood Types

Most people know their main blood type of A, B, AB, or O and whether they are Rh^{+} or Rh^{-}, but did you know that the International Society of Blood Transfusion recognizes twenty-eight additional blood types that have important implications for organ transplants and successful pregnancy? For more information, check out this article:

Blood mystery solved: Two new blood types identified (https://openstax.org/r/Two-new-blood-types-identified)

Creating Venn Diagrams with Three Sets

In general, when creating Venn diagrams from data involving three subsets of a universal set, the strategy is to work from the inside out. Start with the intersection of the three sets, then address the regions that involve the intersection of two sets. Next, complete the regions that involve a single set, and finally address the region in the universal set that does not intersect with any of the three sets. This method can be extended to any number of sets. The key is to start with the region involving the most overlap, working your way from the center out.

EXAMPLE 1.36

Creating a Venn Diagram with Three Sets

A teacher surveyed her class of 43 students to find out how they prepared for their last test. She found that 24 students made flash cards, 14 studied their notes, and 27 completed the review assignment. Of the entire class of 43 students, 12 completed the review and made flash cards, nine completed the review and studied their notes, and seven made flash cards and studied their notes, while only five students completed all three of these tasks. The remaining students did not do any of these tasks. Create a Venn diagram with subsets labeled: “Notes,” “Flash Cards,” and “Review” to represent how the students prepared for the test.

Solution

Step 1: First, draw a Venn diagram with three intersecting circles to represent the three intersecting sets: Notes, Flash Cards, and Review. Label the universal set with the cardinality of the class.

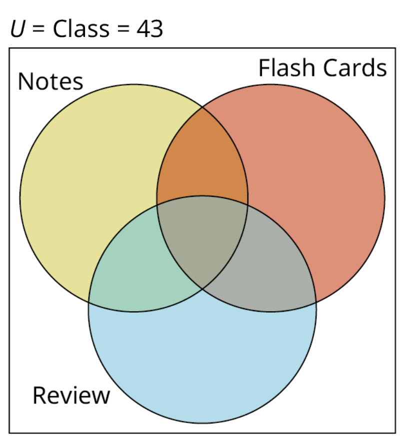


Figure 1.33

Step 2: Next, in the region where all three sets intersect, enter the number of students who completed all three tasks.

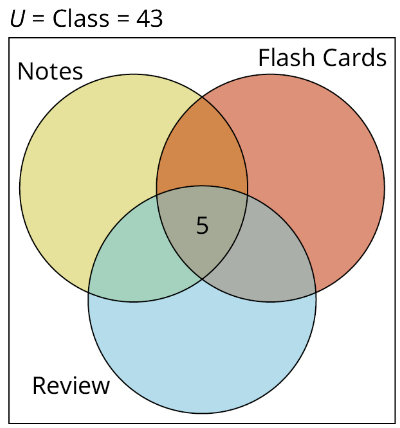


Figure 1.34

Step 3: Next, calculate the value and label the three sections where just two sets overlap.

a. **Review and flash card overlap**. A total of 12 students completed the review and made flash cards, but five of these twelve students did all three tasks, so we need to subtract: $12 - 5 = 7$. This is the value for the region where the flash card set intersects with the review set.
b. **Review and notes overlap**. A total of 9 students completed the review and studied their notes, but again, five of these nine students completed all three tasks. So, we subtract: $9 - 5 = 4$. This is the value for the region where the review set intersects with the notes set.
c. **Flash card and notes overlap**. A total of 7 students made flash cards and studied their notes; subtracting the five students that did all three tasks from this number leaves 2 students who only studied their notes and made flash cards. Add these values to the Venn diagram.

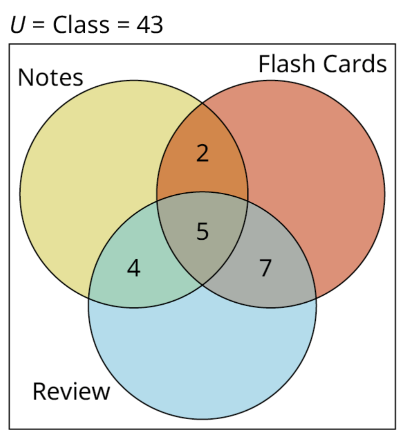


Figure 1.35

Step 4: Now, repeat this process to find the number of students who only completed one of these three tasks.

a. A total of 24 students completed flash cards, but we have already accounted for $2 + 5 + 7 = 14$ of these. Thus, $24 - 14 = 10$ students who just made flash cards.
b. A total of 14 students studied their notes, but we have already accounted for $4 + 5 + 2 = 11$ of these. Thus, $14 - 11 = 3$ students only studied their notes.
c. A total of 27 students completed the review assignment, but we have already accounted for $4 + 5 + 7 = 16$ of these, which means $27 - 16 = 11$ students only completed the review assignment.
d. Add these values to the Venn diagram.

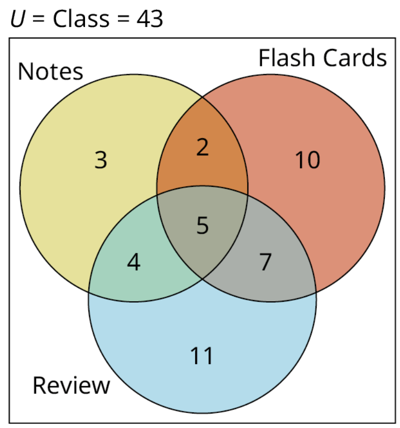


Figure 1.36

Step 5: Finally, compute how many students did not do any of these three tasks. To do this, we add together each value that we have already calculated for the separate and intersecting sections of our three sets: $3+2+4+5+10+7+11=42$. Because there 43 students in the class, and $43-42=1$, this means only one student did not complete any of these tasks to prepare for the test. Record this value somewhere in the rectangle, but outside of all the circles, to complete the Venn diagram.

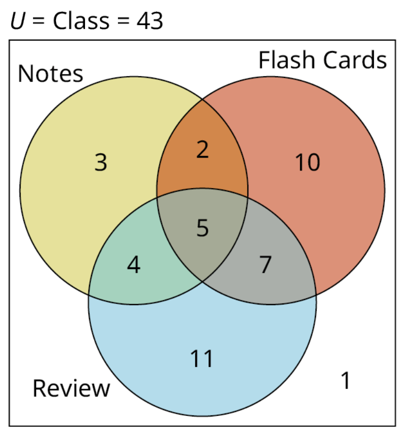


Figure 1.37

YOUR TURN 1.36

1. A group of 50 people attending a conference who preordered their lunch were able to select their choice of soup, salad, or sandwich. A total of 17 people selected soup, 29 people selected salad and 35 people selected a sandwich. Of these orders, 11 attendees selected soup and salad, 10 attendees selected soup and a sandwich, and 18 selected a salad and a sandwich, while eight people selected a soup, a salad, and a sandwich. Create a Venn diagram with subsets labeled “Soup,” “Salad,” and “Sandwich,” and label the cardinality of each section of the Venn diagram as indicated by the data.

Applying Set Operations to Three Sets

Set operations are applied between two sets at a time. Parentheses indicate which operation should be performed first. As with numbers, the inner most parentheses are applied first. Next, find the complement of any sets, then perform any union or intersections that remain.

EXAMPLE 1.37

Applying Set Operations to Three Sets

Perform the set operations as indicated on the following sets: $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $A = \{0, 1, 2, 3, 4, 5, 6\}$, $B = \{0, 2, 4, 6, 8, 10, 12\}$, and $C = \{0, 3, 6, 9, 12\}$.

1. Find $(A \cap B) \cap C$.
2. Find $A \cap (B \cup C)$.
3. Find $(A \cap B) \cup C'$.

Solution

1. Parentheses first, A intersection B equals $A \cap B = \{0, 2, 4, 6\}$, the elements common to both A and B. $(A \cap B) \cap C = \{0, 2, 4, 6\} \cap \{0, 3, 6, 9, 12\} = \{0, 6\}$, because the only elements that are in both sets are 0 and 6.
2. Parentheses first, B union C equals $B \cup C = \{0, 2, 3, 4, 6, 8, 9, 10, 12\}$, the collection of all elements in set B or set C or both. $A \cap (B \cup C) = \{0, 1, 2, 3, 4, 5, 6\} \cap \{0, 2, 3, 4, 6, 8, 9, 10, 12\} = \{0, 2, 3, 4, 6\}$, because the intersection of these two sets is the set of elements that are common to both sets.
3. Parentheses first, A intersection B equals $A \cap B = \{0, 2, 4, 6\}$. Next, find C'. The complement of set C is the set of elements in the universal set U that are not in set C. $C' = \{1, 2, 4, 5, 7, 8, 10, 11\}$. Finally, find $(A \cap B) \cup C' = \{0, 2, 4, 6\} \cup \{1, 2, 4, 5, 7, 8, 10, 11\} = \{0, 1, 2, 4, 5, 6, 7, 8, 10, 11\}$.

YOUR TURN 1.37

Using the same sets from Example 1.37, perform the set operations indicated.

1. Find $A \cap (B \cap C)$.
2. Find $(A \cap B) \cup (A \cap C)$.
3. Find $(A \cup C') \cap (B \cup C')$.

Notice that the answers to the Your Turn are the same as those in the Example. This is not a coincidence. The following equivalences hold true for sets:

- $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$. These are the associative property for set intersection and set union.
- $A \cap B = B \cap A$ and $A \cup B = B \cup A$. These are the commutative property for set intersection and set union.
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. These are the distributive property for sets over union and intersection, respectively.

Proving Equality of Sets Using Venn Diagrams

To prove set equality using Venn diagrams, the strategy is to draw a Venn diagram to represent each side of the equality, then look at the resulting diagrams to see if the regions under consideration are identical.

Augustus De Morgan was an English mathematician known for his contributions to set theory and logic. De Morgan's law for set complement over union states that $(A \cup B)' = A' \cap B'$. In the next example, we will use Venn diagrams to prove De Morgan's law for set complement over union is true. But before we begin, let us confirm De Morgan's law works for a specific example. While showing something is true for one specific example is not a proof, it will provide us with some reason to believe that it may be true for all cases.

Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{2, 3, 4\}$, and $B = \{3, 4, 5, 6\}$. We will use these sets in the equation $(A \cup B)' = A' \cap B'$. To begin, find the value of the set defined by each side of the equation.

Step 1: $A \cup B$ is the collection of all unique elements in set A or set B or both. $A \cup B = \{2, 3, 4, 5, 6\}$. The complement of *A* union *B*, $(A \cup B)'$, is the set of all elements in the universal set that are not in $A \cup B$. So, the left side the equation $(A \cup B)'$ is equal to the set $\{1, 7\}$.

Step 2: The right side of the equation is $A' \cap B'$. A' is the set of all members of the universal set U that are not in set A. $A' = \{1, 5, 6, 7\}$. Similarly, $B' = \{1, 2, 7\}$.

Step 3: Finally, $A' \cap B'$ is the set of all elements that are in both A' and B'. The numbers 1 and 7 are common to both sets, therefore, $A' \cap B' = \{1, 7\}$. Because, $\{1, 7\} = \{1, 7\}$ we have demonstrated that De Morgan's law for set complement over union works for this particular example. The Venn diagram below depicts this relationship.

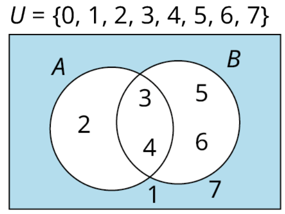


Figure 1.38

EXAMPLE 1.38

Proving De Morgan's Law for Set Complement over Union Using a Venn Diagram

De Morgan's Law for the complement of the union of two sets A and B states that: $(A \cup B)' = A' \cap B'$. Use a Venn diagram to prove that De Morgan's Law is true.

Solution

Step 1: First, draw a Venn diagram representing the left side of the equality. The regions of interest are shaded to highlight the sets of interest. $A \cup B$ is shaded on the left, and $(A \cup B)'$ is shaded on the right.

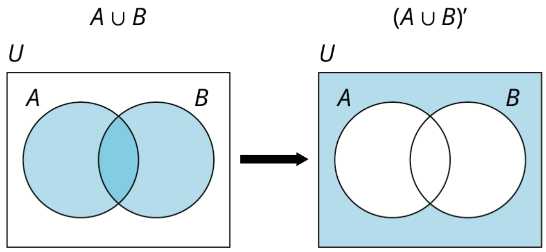


Figure 1.39

Step 2: Next, draw a Venn diagram to represent the right side of the equation. A' is shaded and B' is shaded. Because A' and B' mix to form $A' \cap B'$ is also shaded.

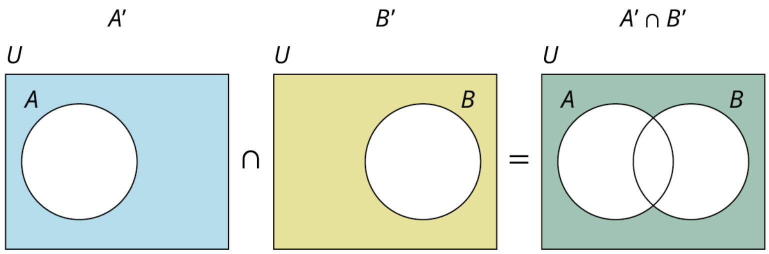


Figure 1.40 Venn diagram of intersection of the complement of two sets.

Step 3: Verify the conclusion. Because the shaded region in the Venn diagram for $(A \cup B)'$ matches the shaded region in the Venn diagram for $A' \cap B$', the two sides of the equation are equal, and the statement is true. This completes the proof that De Morgan's law is valid.

YOUR TURN 1.38

1. De Morgan's Law for the complement of the intersection of two sets A and B states that $(A \cap B)' = A' \cup B'$. Use a Venn diagram to prove that De Morgan's Law is true.

Check Your Understanding

28. When creating a Venn diagram with two or more subsets, you should begin with the region involving the most ____________, then work your way from the center outward.

29. To construct a Venn diagram with three subsets, draw and label three circles that overlap in a common

_____________ region inside the rectangle of the universal set to represent each of the three subsets.

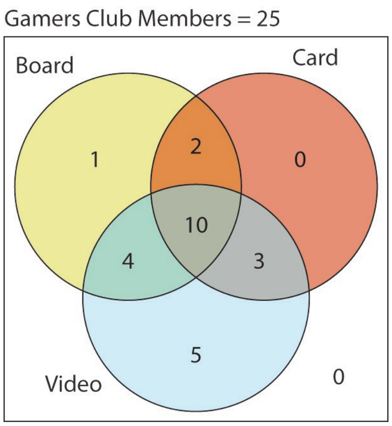
30. In a Venn diagram with three sets, the area where all three sets, A, B, and C overlap is equal to the set _____________.

31. When performing set operations with three or more sets, the order of operations is inner most _____________ first, then find the ___________ of any sets, and finally perform any union or intersection operations that remain.

32. To prove set equality using Venn diagrams, draw a Venn diagram to represent each side of the ______________ and then compare the diagrams to determine if they match or not. If they match, the statement is ____________, otherwise it is not.

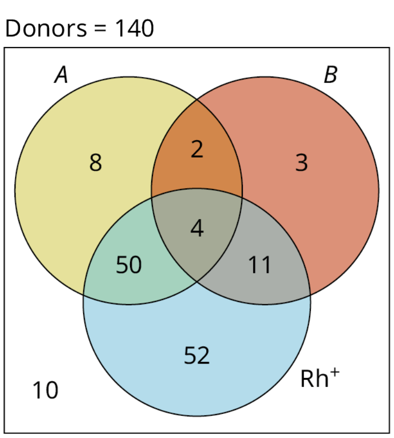
SECTION 1.5 EXERCISES

A gamers club at Baily Middle School consisting of 25 members was surveyed to find out who played board games, card games, or video games. Use the results depicted in the Venn diagram below to answer the following exercises.



1. How many gamers club members play all three types of games: board games, card games, and video games?
2. How many gamers are in the set Board ∩ Video?
3. If Javier is in the region with a total of three members, what type of games does he play?
4. How many gamers play video games?
5. How many gamers are in the set Board ∪ Card?
6. How many members of the gamers club do not play video games?
7. How many members of this club only play board games?
8. How many members of this club only play video games?
9. How many members of the gamers club play video and card games?
10. How many members of the gamers club are in the set Card'?

A blood drive at City Honors High School recently collected blood from 140 students, staff, and faculty. Use the results depicted in the Venn diagram below to answer the following exercises.



11. Blood type AB^{+} is the universal acceptor. Of the 140 people who donated at City Honors, how many had blood

type AB^+?

12. Blood type O^- is the universal donor. Anyone needing a blood transfusion can receive this blood type. How many people who donated blood during this drive had O^- blood?
13. How many people donated with a type A blood factor?
14. How many people donated with a type A and type B blood factor (that is, they had type AB blood).
15. How many donors were O^+?
16. How many donors were not Rh^+?
17. Opal has blood type A^+. If she needs to have surgery that requires a blood transfusion, she can accept blood from anyone who does not have a type B blood factor. How many people donated blood during this drive at City Honors that Opal can accept?
18. Find $n(A \cap \mathrm{Rh}^+)$.
19. Find $n(A \cup \mathrm{Rh}^+)$.
20. Find $n(A \cap B \cap \mathrm{Rh}^-)$.

For the following exercises, create a three circle Venn diagram to represent the relationship between the described sets.

21. The number of elements in the universal set, U, is $n(U) = 48$. Sets A, B, and C are subsets of U: $n(A) = 23,\ n(B) = 25$, and $n(C) = 17$. Also, $n(A \cap B) = 15,\ n(B \cap C) = 12,\ n(C \cap A) = 11$, and $n(A \cap B \cap C) = 8$.
22. The number of elements in the universal set, U, is $n(U) = 88$. Sets A, B, and C are subsets of U: $n(A) = 31,\ n(B) = 46$: $n(C) = 33$. Also, $n(A \cap B) = 24,\ n(B \cap C) = 24$, $n(C \cap A) = 26$, and $n(A \cap B \cap C) = 22$.
23. The number of elements in the universal set, U, is $n(U) = 52$. Sets A, B, and C are subsets of U: $n(A) = 23$, $n(B) = 27$, and $n(C) = 29$. Also, $n(A \cap B) = 22,\ n(B \cap C) = 21,\ n(C \cap A) = 19$, and $n(A \cap B \cap C) = 18$.
24. The number of elements in the universal set, U, is $n(U) = 144$. Sets A, B, and C are subsets of U: $n(A) = 36$, $n(B) = 64$, and $n(C) = 81$. Also, $n(A \cap B) = 26,\ n(B \cap C) = 61,\ n(C \cap A) = 29$, and $n(A \cap B \cap C) = 25$.
25. The universal set, U, has a cardinality of 36. $n(A) = 12,\ n(B) = 12,\ n(C) = 15,\ n(A \text{ and } B) = 3,\ n(B \text{ and } C) = 4,\ n(C \text{ and } A) = 5$, $n(C \text{ and } A) = 5$, and $n(A \text{ and } B \text{ and } C) = 1$.
26. The universal set, U, has a cardinality of 63. $n(A) = 29$, $n(B) = 31,\ n(C) = 41,\ n(A \text{ and } B) = 12,\ n(B \text{ and } C) = 16,\ n(C \text{ and } A) = 18$, and $n(A \text{ and } B \text{ and } C) = 5$.
27. The universal set, U, has a cardinality of 72. $n(A) = 32,\ n(B) = 32,\ n(C) = 44,\ n(A \text{ and } B) = 18,\ n(B \text{ and } C) = 22,\ n(C \text{ and } A) = 26$, and $n(A \text{ and } B \text{ and } C) = 14$.
28. The universal set, U, has a cardinality of 81. $n(A) = 54,\ n(B) = 41,\ n(C) = 52,\ n(A \text{ and } B) = 32,\ n(B \text{ and } C) = 28,\ n(C \text{ and } A) = 30$, and $n(A \text{ and } B \text{ and } C) = 21$.
29. The anime drawing club at Pratt Institute conducted a survey of its 42 members and found that 23 of them sketched with pastels, 28 used charcoal, and 17 used colored pencils. Of these, 10 club members used all three mediums, 18 used charcoal and pastels, 11 used colored pencils and charcoal, and 12 used colored pencils and pastels. The remaining club members did not use any of these three mediums.
30. A new SUV is selling with three optional packages: a sport package, a tow package, and an entertainment package. A dealership gathered the following data for all 31 of these vehicles sold during the month of July. A total of 18 SUVs included the entertainment package, 11 included the tow package, and 16 included the sport package. Of these, five SUVs included all three packages, seven were sold with both the tow package and sport package, 11 were sold with the entertainment and sport package, and eight were sold with the tow package and entertainment package. The remaining SUVs sold did not include any of these optional packages.

For the following exercises, perform the set operations as indicated on the following sets: $U = \{\text{red, orange, yellow, green, blue, indigo, violet}\},\ A = \{\text{red, yellow, blue}\},\ B = \{\text{orange, green, violet}\}$, and $C = \{\text{red, green, indigo}\}$.

31. Find $(A \cup B) \cap C$.
32. Find $(A \cap C) \cup B$.
33. Find $U \cap (B \cup C)$.
34. Find $(B \cap A) \cap U$.
35. Find $A \cap (B \cap C)'$.
36. Find $A' \cap (B \cup C)$.

For the following exercises, perform the set operations as indicated on the following sets: $U = \{20, 21, 22, \ldots, 29\},\ A = \{21, 24, 27\},\ B = \{20, 22, 24, 28\}$, and $C = \{21, 23, 25, 27\}$.

37. Find A and B and C'.
38. Find A' or B or C.
39. Find $(A$ or $B)$ and C'.
40. Find $(A$ or $B)$ or C'.
41. Find $(A$ and $C)$ and B'.
42. Find $(A$ or $B)'$ and C.

For the following exercises, use Venn diagrams to prove the following properties of sets:

43. Commutative property for the union of two sets: $A \cup B = B \cup A$.
44. Commutative property for the intersection of two sets: $A \cap B = B \cap A$.
45. Associative property for the intersection of three sets: $(A \cap B) \cap C = A \cap (B \cap C)$.
46. Associative property for the union of three sets: $A \cup (B \cup C) = (A \cup B) \cup C$.
47. Distributive property for set intersection over set union: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
48. Distributive property for set union over set intersection: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Chapter Summary

Key Terms

1.1 Basic Set Concepts

- set
- elements
- well-defined set
- empty set
- roster method
- finite set
- infinite set
- natural numbers
- integer
- set-builder notation
- cardinality of a set
- countably infinite
- equal sets
- equivalent sets

1.2 Subsets

- subset
- proper subset
- equivalent subsets
- exponential notation

1.3 Understanding Venn Diagrams

- Venn diagram
- universal set
- disjoint set
- complement of a set

1.4 Set Operations with Two Sets

- intersection of two sets
- union of two sets

Key Concepts

1.1 Basic Set Concepts

- Identify a set as being a well-defined collection of objects and differentiate between collections that are not well-defined and collections that are sets.
- Represent sets using both the roster or listing method and set builder notation which includes a description of the members of a set.
- In set theory, the following symbols are universally used:
  $\mathbb{N}$ - The set of natural numbers, which is the set of all positive counting numbers.
  $$\mathbb{N} = \left\{1, 2, 3, \ldots\right\}$$
  $\mathbb{Z}$ - The set of integers, which is the set of all the positive and negative counting numbers and the number zero.
  $$\mathbb{Z} = \left\{\ldots, -2, -1, 0, 1, 2, \ldots\right\}$$
  $\mathbb{Q}$ - The set of rational numbers or fractions.
  $$\mathbb{Q} = \left\{\frac{p}{q}\middle| p \text{ and } q \text{ are integers and } q \neq 0\right\}$$
- Distinguish between finite sets, infinite sets, and the empty set to determine the size or cardinality of a set.
- Distinguish between equal sets which have exactly the same members and equivalent sets that may have different members but must have the same cardinality or size.

1.2 Subsets

- Every member of a subset of a set is also a member of the set containing it. $A \subseteq B$
- A proper subset of a set does not contain all the members of the set containing it. There is a least one member of set B that is not a member of set A. $A \subset B$
- The number subsets of a finite set A with $n(A)$ members is equal to 2 raised to the $n(A)$ power.
- The empty set is a subset of every set and must be included when listing all the subsets of a set.
- Understand how to create and distinguish between equivalent subsets of finite and infinite sets that are not equal to the original set.

1.3 Understanding Venn Diagrams

- A Venn diagram is a graphical representation of the relationship between sets.
- In a Venn diagram, the universal set, U is the largest set under consideration and is drawn as a rectangle. All subsets of the universal set are drawn as circles within this rectangle.
- The complement of set A includes all the members of the universal set that are not in set A. A set and its complement are disjoint sets, they do not share any elements in common.
- To find the complement of set A remove all the elements of set A from the universal set U, the set that includes only the remaining elements is the complement of set A, A'.
- Determine the complement of a set using Venn diagrams, the roster method and set builder notation.

1.4 Set Operations with Two Sets

- The intersection of two sets, $A \cap B$ is the set of all elements that they have in common. Any member of A intersection B must be is both set A and set B.
- The union of two sets, $A \cup B$, is the collection of all members that are in either in set A, set B or both sets A and B combined.
- Two sets that share at least one element in common, so that they are not disjoint are represented in a Venn Diagram using two circles that overlap.
  - The region of the overlap is the set A intersection B, $A \cap B$.
  - The regions that include everything in the circle representing set A or the circle representing set B or their overlap is the set A union B, $A \cup B$.
- Apply knowledge of set union and intersection to determine cardinality and membership using Venn Diagrams, the roster method and set builder notation.

1.5 Set Operations with Three Sets

- A Venn diagram with two overlapping sets breaks the universal set up into four distinct regions. When a third overlapping set is added the Venn diagram is broken up into eight distinct regions.
- Analyze, interpret, and create Venn diagrams involving three overlapping sets.
  - Including the blood factors: A, B and Rh
  - To find unions and intersections.
  - To find cardinality of both unions and intersections.
- When performing set operations with three or more sets, the order of operations is inner most parentheses first, then fine the complement of any sets, then perform any union or intersection operations that remain.
- To prove set equality using Venn diagrams the strategy is to draw a Venn diagram to represent each side of the equality or equation, then look at the resulting diagrams to see if the regions under consideration are identical. If they regions are identical the equation represents a true statement, otherwise it is not true.

Videos

1.1 Basic Set Concepts

- Equal and Equivalent Sets (https://openstax.org/r/Equal_and_Equivalent_Sets)

1.4 Set Operations with Two Sets

- The Basics of Intersection of Sets, Union of Sets and Venn Diagrams (https://openstax.org/r/operation-on-Sets)

Formula Review

1.2 Subsets

The number of subsets of a finite set A is equal to 2 raised to the power of $n(A)$, where $n(A)$ is the number of elements in set A: Number of Subsets of Set $A = 2^{n(A)}$.

1.4 Set Operations with Two Sets

The cardinality of A union B : $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Projects

Cardinality of Infinite Sets

In set theory, it has been shown that the set of irrational numbers has a cardinality greater than the set of natural numbers. That is, the set of irrational numbers is so large that it is uncountably infinite.

1. Perform a search with the phrase, "Who first proved that the real numbers are uncountable?"
   a. Who first proved that the real numbers are uncountable?
   b. What was the significance of this proof to the development of set theory and by extension other fields of mathematics?
2. Recent discoveries in the field of set theory include the solution to a 70-year-old problem previously thought to be unprovable. To learn more read this article (https://openstax.org/r/measure-infinities):
   a. What does it mean for two infinite sets to have the same size?
   b. The real numbers are sometimes referred to as what?
   c. Summarize your understanding of the problem known as the "Continuum Hypothesis."
   d. Malliaris and Shelah's proof of this 70-year-old problem is opening up investigation in what two fields of mathematics?
3. Summarize your understanding of infinity.
   a. Define what it means to be infinite.
   b. Explain the difference between countable and uncountable sets.
   c. Research the difference between a discrete set and a continuous set, then summarize your findings.

Set Notation

In arithmetic, the operation of addition is represented by the plus sign, +, but multiplication is represented in multiple ways, including $\cdot, \times, *$, and parentheses, such as 5(3). Several set operations also are written in different forms based on the preferences of the mathematician and often their publisher.

1. Search for "Set Complement" on the internet and list at least three ways to represent the complement of a set.
2. Both the Set Challenge and Venn Diagram smartphone apps highlighted in the Tech Check sections have an operation for set difference. List at least two ways to represent set difference and provide a verbal description of how to calculate the difference between two sets A and B.
3. When researching possible Venn diagram applications, the Greek letter delta, Δ appeared as a symbol for a set operator. List at least one other symbol used for this same operation.
4. Search for "List of possible set operations and their symbols." Find and select two symbols that were not presented in this chapter.

The Real Number System

The set of real numbers and their properties are studied in elementary school today, but how did the number system evolve? The idea of natural numbers or counting numbers surfaced prior to written words, as evidenced by tally marks in cave writing. Create a timeline for significant contributions to the real number system.

1. Use the following phrase to search online for information on the origins of the number zero: "History of the number zero." Then, record significant dates for the invention and common use of the number zero on your timeline.
2. Find out who is credited for discovering that the $\sqrt{2}$ is irrational and add this information to your timeline. Hint: Search for, "Who was the first to discover irrational numbers?"
3. Research Georg Cantor's contribution to the representation of real numbers as a continuum and add this to your timeline.
4. Research Ernst Zermelo's contribution to the real number system and add this to your timeline.

Chapter Review

Basic Set Concepts

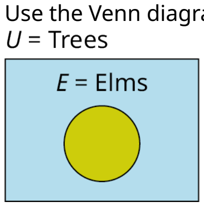
1. A ____________ is a well-defined collection of distinct objects.
2. A collection of well-defined objects without any members in it is called the _______ ______.
3. Write the set consisting of the last five letters of the English alphabet using the roster method.
4. Write the set consisting of the numbers 1 through 20 inclusive using the roster method and an ellipsis.
5. Write the set of all zebras that do not have stripes in symbolic form.
6. Write the set of negative integers using the roster method and an ellipsis.
7. Use set builder notation to write the set of all even integers.
8. Write the set of all letters in the word Mississippi and label it with a capital M.
9. Determine whether the following collection describes a well-defined set: "A group of these five types of apples: Granny Smith, Red Delicious, McIntosh, Fuji, and Jazz."
10. Determine whether the following collection describes a well-defined set: "A group of five large dogs."
11. Determine the cardinality of the set $A = \{\text{Alabama}, \text{ Alaska}, \text{ Arkansas}, \text{ Arizona}\}$.
12. Determine whether the following set is a finite set or an infinite set: $F = \{5, 10, 15, \ldots\}$.
13. Determine whether sets A and B are equal, equivalent, or neither: $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$.
14. Determine if sets A and B are equal, equivalent, or neither: $A = \{a, b, c\}$ and $B = \{c, a, b\}$.
15. Determine if sets A and B are equal, equivalent, or neither: $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$.

Subsets

16. If every member of set A is also a member of set B, then set A is a ________ of set B.
17. Determine whether set A is a subset, proper subset, or neither a subset nor proper subset of set B: $A = \{s, o, n\}$ and $B = \{s, o, n, g\}$.
18. Determine whether set A is a subset, proper subset, or neither a subset nor proper subset of set B: $A = \{s, o, n\}$ and $B = \{s, o, l\}$.
19. Determine whether set A is a subset, proper subset, or neither a subset nor proper subset of set B: $A = \{s, o, n\}$ and $B = \{o, n, s\}$.
20. List all the subsets of the set $\{\text{up}, \text{down}\}$.
21. List all the subsets of the set $\{0\}$.
22. Calculate the total number of subsets of the set {Scooby, Velma, Daphne, Shaggy, Fred}.
23. Calculate the total number of subsets of the set {top hat, thimble, iron, shoe, battleship, cannon}.
24. Find a subset of the set $\{g, r, e, a, t\}$ that is equivalent, but not equal, to $\{t, e, a\}$.
25. Find a subset of the set $\{g, r, e, a, t\}$ that is equal to $\{t, e, a\}$.
26. Find two equivalent finite subsets of the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$, with a cardinality of 4.
27. Find two equal finite subsets of the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$, with a cardinality of 3.

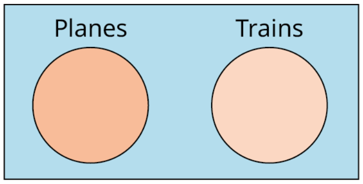
Understanding Venn Diagrams

28. Use the Venn diagram below to describe the relationship between the sets, symbolically and in words:



29. Use the Venn diagram below to describe the relationship between the sets, symbolically and in words:

U = Modes of Transportation



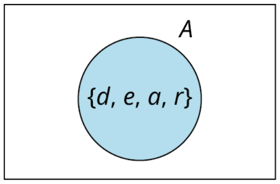
30. Draw a Venn diagram to represent the relationship between the described sets: Falcons ⊂ Raptors.

31. Draw a Venn diagram to represent the relationship between the described sets: Natural numbers ⊂ Integers ⊂ Real numbers.

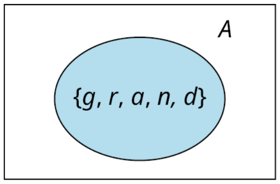
32. The universal set is the set $U = \{s, m, i, l, e\}$. Find the complement of the set $E = \{e, l, m\}$.

33. The universal set is the set $U = \{1, 2, 3, \ldots\}$. Find the complement of the set $V = \{18, 19, 20, \ldots\}$.

34. Use the Venn diagram below to determine the members of the set A'.
U = {*r*, *e*, *a*, *d*, *i*, *n*, *g*}



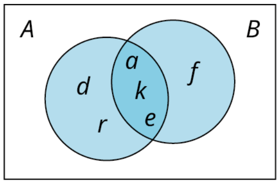
35. Use the Venn diagram below to determine the members of the set A'.
U = {*r*, *e*, *a*, *d*, *i*, *n*, *g*}



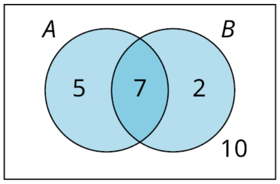
Set Operations with Two Sets

Determine the union and intersection of the sets indicated: $U = \{a, b, c, \ldots, z\}$, $S = \{s, c, r, a, b, l, e\}$, $B = \{b, r, a, c, e\}$, $C = \{c, r, a, b\}$, $R = \{r, i, s, k\}$, and $Q = \{q, u, i, z\}$.

36. What is $S \cap R$?
37. What is $S \cup B$?
38. Write the set containing the elements in sets B or Q.
39. Write the set containing all the elements is both sets B and Q.
40. Find C intersection R.
41. Find C union R.
42. Find the cardinality of $C \cup R$, $n(C \cup R)$.
43. Find $n(S \text{ union } R)$.
44. Use the Venn diagram below to find $A \cap B$.
U = {*a*, *b*, *c*, ..., *z*}

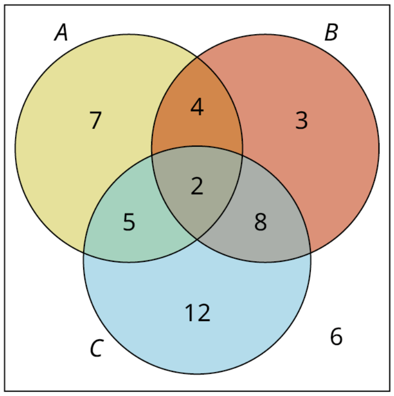


45. Use the Venn diagram below to find $n(A \cup B)$.
U = 24



Set Operations with Three Sets

Use the Venn diagram below to answer the following questions.
U = 47



46. Find $n(A \cup C)$.
47. Find $n(B \cap C)$.
48. A food truck owner surveyed a group of 50 customers about their preferences for hamburger condiments. After tallying the responses, the owner found that 24 customers preferred ketchup, 11 preferred mayonnaise, and 31 preferred mustard. Of these customers, eight preferred ketchup and mayonnaise, one preferred mayonnaise and mustard, and 13 preferred ketchup and mustard. No customer preferred all three. The remaining customers did not select any of these three condiments. Draw a Venn diagram to represent this data.
49. Given $U = \{\text{r, s, t, l, n, e, i, a}\}$, $R = \{\text{r, e, s, t}\}$, $S = \{\text{s, t, a, i, r}\}$, and $L = \{\text{l, i, n, e, s}\}$, find $(S \cup R) \cap L'$.
50. Use Venn diagrams to prove that if $A \subset B$, then $B' \subset A'$.

Chapter Test

1. Determine whether the following collection describes a well-defined set: "A group of small tomatoes."

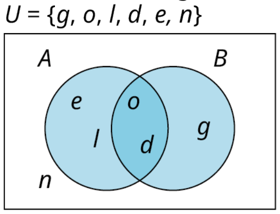
Classify each of the following sets as either finite or infinite.

2. $\{1, 5, 9, \ldots\}$
3. $\{c | c \text{ is a cat}\}$
4. $\{1, 2, 3, \ldots, 1000\}$
5. $\{s, m, i, l, e\}$
6. $\{m \in \mathbb{N} | m = n^2 \text{ where } n \text{ is a natural number}\}$

Use the sets provided to answer the following questions: $U = \{31, 32, 33, \ldots, 50\}$, $A = \{35, 38, 41, 44, 47, 50\}$, $B = \{32, 36, 40, 44, 48\}$, and $C = \{31, 32, 41, 42, 48, 50\}$.

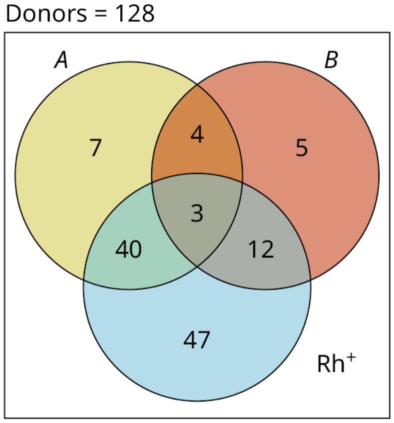
7. Find A or B.
8. Find B and C.
9. Determine if set A is equivalent to, equal to, or neither equal nor equivalent to set C. Justify your answer.
10. Find $n(A \cup C)$.
11. Find $A \cap (B \cap C)$.
12. Find $(A \cup B)' \cap C$.
13. Find $(A \cap B') \cup C$.

Use the Venn diagram below to answer the following questions.
U = {*g*, *o*, *l*, *d*, *e*, *n*}



14. Find B'.
15. Find $A \cup B$.
16. Find $A \cap B'$.
17. Draw a Venn diagram to represent the relationship between the two sets: "All flowers are plants."

For the following questions, use the Venn diagram showing the blood types of all donors at a recent mobile blood drive.



18. Find the number of donors who were O^-; that is, find $n((A \cup B \cup Rh^+)')$.

19. Find the number of donors who were A^+ or B^+ or AB^+.

20. Use Venn diagrams to prove that if $A \subset B$, then $A \cap B = A$.

2 LOGIC

Figure 2.1 Logic is key to a well-reasoned argument, in both math and law. (credit: modification of work "Lady Justicia holding sword and scale bronze figurine with judge hammer on wooden table" by Jernej Furman/Flickr, CC BY 2.0)

Chapter Outline

Introduction

What is logic? **Logic** is the study of reasoning, and it has applications in many fields, including philosophy, law, psychology, digital electronics, and computer science.

In law, constructing a well-reasoned, logical argument is extremely important. The main goal of arguments made by lawyers is to convince a judge and jury that their arguments are valid and well-supported by the facts of the case, so the case should be ruled in their favor. Think about Thurgood Marshall arguing for desegregation in front of the U.S. Supreme Court during the *Brown v. Board of Education of Topeka* lawsuit in 1954, or Ruth Bader Ginsburg arguing for equality in social security benefits for both men and women under the law during the mid-1970s. Both these great minds were known for the preparation and thoroughness of their logical legal arguments, which resulted in victories that advanced the causes they fought for. Thurgood Marshall and Ruth Bader Ginsburg would later become well respected justices on the U.S. Supreme Court themselves.

In this chapter, we will explore how to construct well-reasoned logical arguments using varying structures. Your ability to form and comprehend logical arguments is a valuable tool in many areas of life, whether you're planning a dinner date, negotiating the purchase of a new car, or persuading your boss that you deserve a raise.

2.1 Statements and Quantifiers

Figure 2.2 Construction of a logical argument, like that of a house, requires you to begin with the right parts. (credit: modification of work “Barn Raising” by Robert Stinnett/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Identify logical statements.
2. Represent statements in symbolic form.
3. Negate statements in words.
4. Negate statements symbolically.
5. Translate negations between words and symbols.
6. Express statements with quantifiers of all, some, and none.
7. Negate statements containing quantifiers of all, some, and none.

Have you ever built a club house, tree house, or fort with your friends? If so, you and your friends likely started by gathering some tools and supplies to work with, such as hammers, saws, screwdrivers, wood, nails, and screws. Hopefully, at least one member of your group had some knowledge of how to use the tools correctly and helped to direct the construction project. After all, if your house isn't built on a strong foundation, it will be weak and could possibly fall apart during the next big storm. This same foundation is important in logic.

In this section, we will begin with the parts that make up all logical arguments. The building block of any logical argument is a **logical statement**, or simply a statement. A logical statement has the form of a complete sentence, and it must make a claim that can be identified as being true or false.

When making arguments, sometimes people make false claims. When evaluating the strength or validity of a logical argument, you must also consider the **truth values**, or the identification of true or false, of all the statements used to support the argument. While a false statement is still considered a logical statement, a strong logical argument starts with true statements.

Identifying Logical Statements

Figure 2.3 Not all roses are red. (credit: “assorted pink yellow white red roses macro” by ProFlowers/Flickr, CC BY 2.0)

An example of logical statement with a false truth value is, “All roses are red.” It is a logical statement because it has the form of a complete sentence and makes a claim that can be determined to be either true or false. It is a false statement because not all roses are red: some roses are red, but there are also roses that are pink, yellow, and white. Requests, questions, or directives may be complete sentences, but they are not logical statements because they cannot be determined to be true or false. For example, suppose someone said to you, “Please, sit down over there.” This request does not make a claim and it cannot be identified as true or false; therefore, it is not a logical statement.

EXAMPLE 2.1

Identifying Logical Statements

Determine whether each of the following sentences represents a logical statement. If it is a logical statement, determine whether it is true or false.

1. Tiger Woods won the Master’s golf championship at least five times.
2. Please, sit down over there.
3. All cats dislike dogs.

Solution

1. This is a logical statement because it is a complete sentence that makes a claim that can be identified as being true or false. As of 2021, this statement is true: Tiger Woods won the Master’s in 1997, 2001, 2002, 2005 and 2019.
2. This is not a logical statement because, although it is a complete sentence, this request does not make a claim that can be identified as being either true or false.
3. This is a logical statement because it is a complete sentence that makes a claim that can be identified as being true or false. This statement is false because some cats do like some dogs.

YOUR TURN 2.1

Determine whether each of the following sentences represents a logical statement. If it is a logical statement, determine whether it is true or false.

1. The Buffalo Bills defeated the New York Giants in Super Bowl XXV.
2. Michael Jackson’s album *Thriller* was released in 1982.
3. Would you like some coffee or tea?

Representing Statements in Symbolic Form

When analyzing logical arguments that are made of multiple logical statements, **symbolic form** is used to reduce the amount of writing involved. Symbolic form also helps visualize the relationship between the statements in a more concise way in order to determine the strength or validity of an argument. Each logical statement is represented symbolically as a single lowercase letter, usually starting with the letter p.

To begin, you will practice how to write a single logical statement in symbolic form. This skill will become more useful as you work with compound statements in later sections.

EXAMPLE 2.2

Representing Statements Using Symbolic Form

Write each of the following logical statements in symbolic form.

1. Barry Bonds holds the Major League Baseball record for total career home runs.
2. Some mammals live in the ocean.
3. Ruth Bader Ginsburg served on the U.S. Supreme Court from 1993 to 2020.

Solution

1. p: Barry Bonds holds the Major League Baseball record for total career home runs. The statement is labeled with a p. Once the statement is labeled, use p as a replacement for the full written statement to write and analyze the argument symbolically.
2. q: Some mammals live in the ocean. The letter q is used to distinguish this statement from the statement in question 1, but any lower-case letter may be used.
3. r: Ruth Bader Ginsburg served on the U.S. Supreme Court from 1993 to 2020. When multiple statements are present in later sections, you will want to be sure to use a different letter for each separate logical statement.

YOUR TURN 2.2

Write each of the following logical statements in symbolic form.

1. The movie *Gandhi* won the Academy Award for Best Picture in 1982.
2. Soccer is the most popular sport in the world.
3. All oranges are citrus fruits.

WHO KNEW?

Mathematics is not the only language to use symbols to represent thoughts or ideas. The Chinese and Japanese languages use symbols known as Hanzi and Kanji, respectively, to represent words and phrases. At one point, the American musician Prince famously changed his name to a symbol representing love.

BBC Prince Symbol Article (https://openstax.org/r/magazine-36107590)

Negating Statements

Consider the false statement introduced earlier, “All roses are red.” If someone said to you, “All roses are red,” you might respond with, “Some roses are not red.” You could then strengthen your argument by providing additional statements, such as, “There are also white roses, yellow roses, and pink roses, to name a few.”

The **negation of a logical statement** has the opposite truth value of the original statement. If the original statement is false, its negation is true, and if the original statement is true, its negation is false. Most logical statements can be negated by simply adding or removing the word *not*. For example, consider the statement, “Emma Stone has green eyes.” The negation of this statement would be, “Emma Stone does not have green eyes.” The table below gives some other examples.

Logical Statement	Negation
Gordon Ramsey is a chef.	Gordon Ramsey is not a chef.
Tony the Tiger does not have spots.	Tony the Tiger has spots.

Table 2.1

The way you phrase your argument can impact its success. If someone presents you with a false statement, the ability to rebut that statement with its negation will provide you with the tools necessary to emphasize the correctness of your position.

EXAMPLE 2.3

Negating Logical Statements

Write the negation of each logical statement in words.

1. Michael Phelps was an Olympic swimmer.
2. Tom is a cat.
3. Jerry is not a mouse.

Solution

1. Add the word *not* to negate the affirmative statement: “Michael Phelps was not an Olympic swimmer.”
2. Add the word *not* to negate the affirmative statement: “Tom is not a cat.”
3. Remove the word *not* to negate the negative statement: “Jerry is a mouse.”

YOUR TURN 2.3

Write the negation of each logical statement in words.

1. Ted Cruz was not born in Texas.
2. Adele has a beautiful singing voice.
3. Leaves convert carbon dioxide to oxygen during the process of photosynthesis.

Negating Logical Statements Symbolically

The symbol for negation, or not, in logic is the tilde, ~. So, not p is represented as $\sim p$. To negate a statement symbolically, remove or add a tilde. The negation of not (not p) is p. Symbolically, this equation is $\sim(\sim p) = p$.

EXAMPLE 2.4

Negating Logical Statements Symbolically

Write the negation of each logical statement symbolically.

1. p: Michael Phelps was an Olympic swimmer.
2. r: Tom is not a cat.
3. $\sim q$: Jerry is not a mouse.

Solution

1. To negate an affirmative logical statement symbolically, add a tilde: $\sim p$.
2. Because the symbol for this statement is r, its negation is $\sim r$.
3. The symbol for this statement is $\sim q$, so to negate it we simply remove the ~, because $\sim(\sim q) = q$. The answer is q.

YOUR TURN 2.4

Write the negation of each logical statement symbolically.

1. $\sim p$: Ted Cruz was not born in Texas.
2. q: Adele has a beautiful voice.
3. r: Leaves convert carbon dioxide to oxygen during the process of photosynthesis.

Translating Negations Between Words and Symbols

In order to analyze logical arguments, it is important to be able to translate between the symbolic and written forms of logical statements.

EXAMPLE 2.5

Translating Negations Between Words and Symbols

Given the statements:

r: Elmo is a red Muppet.

p: Ketchup is not a vegetable.

1. Write the symbolic form of the statement, “Elmo is not a red Muppet.”
2. Translate the statement $\sim p$ into words.

Solution

1. “Elmo is not a red muppet” is the negation of “Elmo is a red muppet,” which is represented as r. Thus, we would write “Elmo is not a red muppet” symbolically as $\sim r$.
2. Because p is the symbol representing the statement, “Ketchup is not a vegetable,” $\sim p$ is equivalent to the statement, “Ketchup is a vegetable.”

YOUR TURN 2.5

Given the statements:
r: Woody and Buzz Lightyear are best friends.
$\sim p$: Wonder Woman is not stronger than Captain Marvel.

1. Write the symbolic form of the statement, "Wonder Woman is stronger than Captain Marvel."
2. Translate the statement $\sim r$ into words.

Expressing Statements with Quantifiers of All, Some, or None

A **quantifier** is a term that expresses a numerical relationship between two sets or categories. For example, all squares are also rectangles, but only some rectangles are squares, and no squares are circles. In this example, *all, some,* and *none* are quantifiers. In a logical argument, the logical statements made to support the argument are called **premises**, and the judgment made based on the premises is called the **conclusion**. Logical arguments that begin with specific premises and attempt to draw more general conclusions are called **inductive arguments**.

Consider, for example, a parent walking with their three-year-old child. The child sees a cardinal fly by and points it out. As they continue on their walk, the child notices a robin land on top of a tree and a duck flying across to land on a pond. The child recognizes that cardinals, robins, and ducks are all birds, then excitedly declares, "All birds fly!" The child has just made an inductive argument. They noticed that three different specific types of birds all fly, then synthesized this information to draw the more general conclusion that all birds can fly. In this case, the child's conclusion is false.

The specific premises of the child's argument can be paraphrased by the following statements:

- Premise: Cardinals are birds that fly.
- Premise: Robins are birds that fly.
- Premise: Ducks are birds that fly.

The general conclusion is: “All birds fly!”

All inductive arguments should include at least three specific premises to establish a pattern that supports the general conclusion. To counter the conclusion of an inductive argument, it is necessary to provide a counter example. The parent can tell the child about penguins or emus to explain why that conclusion is false.

On the other hand, it is usually impossible to prove that an inductive argument is true. So, inductive arguments are considered either strong or weak. Deciding whether an inductive argument is strong or weak is highly subjective and often determined by the background knowledge of the person making the judgment. Most hypotheses put forth by scientists using what is called the "scientific method" to conduct experiments are based on inductive reasoning.

In the following example, we will use quantifiers to express the conclusion of a few inductive arguments.

EXAMPLE 2.6

Drawing General Conclusions to Inductive Arguments Using Quantifiers
For each series of premises, draw a logical conclusion to the argument that includes one of the following quantifiers: all, some, or none.

1. Squares and rectangles have four sides. A square is a parallelogram, and a rectangle is a parallelogram. What conclusion can be drawn from these premises?
2. $1 + 2 = 3, 6 + 7 = 13$, and $23 + 24 = 47$. Of these, 1 and 2, 6 and 7, and 23 and 24 are consecutive integers; 3, 13, and 47 are odd numbers. What conclusion can be drawn from these premises?
3. Sea urchins live in the ocean, and they do not breathe air. Sharks live in the ocean, and they do not breathe air. Eels live in the ocean, and they do not breath air. What conclusion can be drawn from these premises?

Solution

1. The conclusion you would likely come to here is "Some four-sided figures are parallelograms." However, it would be incorrect to say that all four-sided figures are parallelograms because there are some four-sided figures, such as trapezoids, that are not parallelograms. This is a false conclusion.
2. From these premises, you may draw the conclusion "All sums of two consecutive counting numbers result in an odd number." Most inductive arguments cannot be proven true, but several mathematical properties can be. If we let n represent our first counting number, then $n + 1$ would be the next counting number and $n + (n + 1) = 2n + 1$. Because $2n$ is divisible by two, it is an even number, and if you add one to any even number the result is always an odd number. Thus, the conclusion is true!
3. Based on the premises provided, with no additional knowledge about whales or dolphins, you might conclude "No creatures that live in the ocean breathe air." Even though this conclusion is false, it still follows from the known premises and thus is a logical conclusion based on the evidence presented. Alternatively, you could conclude "Some creatures that live in the ocean do not breathe air." The quantifier you choose to write your conclusion with may vary from another person's based on how persuasive the argument is. There may be multiple acceptable answers.

YOUR TURN 2.6

For each series of premises, draw a logical conclusion to the argument that includes one of the following quantifiers: all, some, or none.

1. $1 + 2 = 3, 5 + 6 = 11$, and $14 + 15 = 29$. Of these, 1 and 2 are consecutive integers, 5 and 6 are consecutive integers, and 14 and 15 are consecutive integers. Also, their sums, 3, 11, and 29 are all prime numbers. Prime numbers are positive integers greater than one that are only divisible by one and the number itself. What conclusion can you draw from these premises?
2. A robin is a bird that lays blue eggs. A chicken is a bird that typically lays white and brown eggs. An ostrich is a bird that lays exceptionally large eggs. If a bird lays eggs, then they do not give live birth to their young. What conclusion can you draw from these premises?
3. All parallelograms have four sides. All rectangles are parallelograms. All squares are rectangles. What additional conclusion can you make about squares from these premises?

It is not possible to prove definitively that an inductive argument is true or false in most cases.

Negating Statements Containing Quantifiers

Recall that the negation of a statement will have the opposite truth value of the original statement. There are four basic forms that logical statements with quantifiers take on.

- All A are B.

- Some A are B.
- No A are B.
- Some A are not B.

The negation of logical statements that use the quantifiers *all*, *some*, or *none* is a little more complicated than just adding or removing the word *not*.

For example, consider the logical statement, “All oranges are citrus fruits.” This statement expresses as a subset relationship. The set of oranges is a subset of the set of citrus fruit. This means that there are no oranges that are outside the set of citrus fruit. The negation of this statement would have to break the subset relationship. To do this, you could say, “At least one orange is not a citrus fruit.” Or, more concisely, “Some oranges are not citrus fruit.” It is tempting to say "No oranges are citrus fruit," but that would be incorrect. Such a statement would go beyond breaking the subset relationship, to stating that the two sets have nothing in common. The negation of "A is a subset of B" would be to state that "A is not a subset of B," as depicted by the Venn diagram in Figure 2.4.

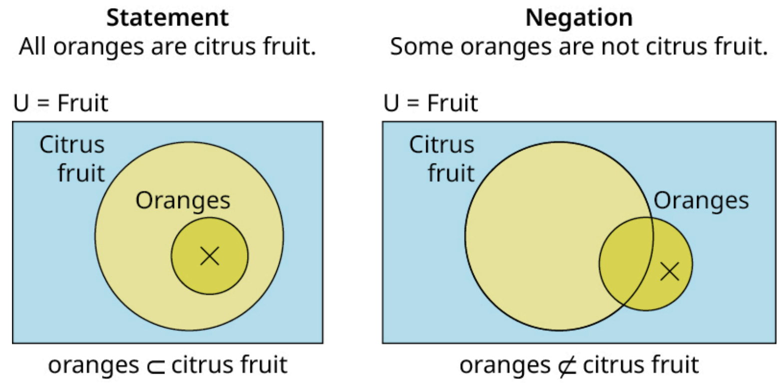


Figure 2.4

The statement, “All oranges are citrus fruit,” is true, so its negation, “Some oranges are not citrus fruit,” is false.

Now, consider the statement, “No apples are oranges.” This statement indicates that the set of apples is disjointed from the set of oranges. The negation must state that the two are not disjoint sets, or that the two sets have a least one member in common. Their intersection is not empty. The negation of the statement, “A intersection B is the empty set,” is the statement that " A intersection B is not empty," as depicted in the Venn diagram in Figure 2.5.

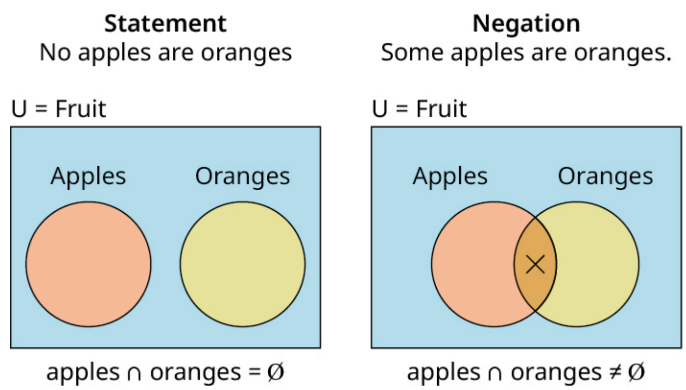


Figure 2.5

The negation of the true statement “No apples are oranges,” is the false statement, “Some apples are oranges.”

Table 2.2 summarizes the four different forms of logical statements involving quantifiers and the forms of their associated negations, as well as the meanings of the relationships between the two categories or sets A and B.

Logical Statements with Quantifiers	Negation of Logical Statements w/Quantifiers
Form: All A are B. Means: A is a subset of B, $A \subset B$. All zebras have stripes. (True)	Form: Some A are not B. Means: A is not a subset of B, $A \not\subset B$. Some zebras do not have stripes. (False)
Form: Some A are B. Means: A intersection B is not empty, $A \cap B \neq \varnothing$. Some fish are sharks. (True)	Form: No A are B. Means: A intersection B is empty, $A \cap B = \varnothing$. No fish are sharks. (False)
Form: No A are B. Means: A intersection B is empty, $A \cap B = \varnothing$. No trees are evergreens. (False)	Form: Some A are B. Means: A intersection B is not empty, $A \cap B \neq \varnothing$. Some trees are evergreens. (True)
Form: Some A are not B. Means: A is not a subset of B, $A \not\subset B$. Some horses are not mustangs. (True)	Form: All A are B. Means: A is a subset of B, $A \subset B$. All horses are mustangs. (False)

Table 2.2

We covered sets in great detail in Chapter 1. To review, " A is a subset of B" means that every member of set A is also a member of set B. The intersection of two sets A and B is the set of all elements that they share in common. If A intersection B is the empty set, then sets A and B do not have any elements in common. The two sets do not overlap. They are disjoint.

VIDEO

Logic Part 1A: Logic Statements and Quantifiers (https://openstax.org/r/Logic_Statements_and_Quantifiers)

EXAMPLE 2.7

Negating Statements Containing Quantifiers *All*, *Some*, or *None*

Given the statements:

p: All leopards have spots.
r: Some apples are red.
s: No lemons are sweet.

Write each of the following symbolic statements in words.

1. $\sim p$
2. $\sim r$
3. $\sim s$

Solution

1. The statement "All leopards have spots" is p and has the form "All A are B." According to Table 2.2, the negation will have the form "Some A are not B." The negation of p is the statement, "Some leopards do not have spots."
2. The statement "Some apples are red" has the form "Some A are B." This indicates that the categories A and B overlap or intersect. According to Table 2.2, the negation will have the form, "No A are B," indicating that A and B do not intersect. This results in the opposite truth value of the original statement, so the negation of "Some apples are red" is the statement: "No apples are red."
3. Because s is the statement: "No lemons are sweet," s is asserting that the set of lemons does not intersect with the set of sweet things. The negation of s, $\sim s$, must make the opposite claim. It must indicate that the set of lemons intersects with the set of sweet things. This means at least one lemon must be sweet. The statement, "Some lemons are sweet" is $\sim s$. The negation of the statement, "No A are B," is the statement, "Some A are B," as indicated in Table 2.2.

> **YOUR TURN 2.7**
>
> Given the statements:
> $\sim p$: Some apples are not sweet.
> r: No triangles are squares.
> s: Some vegetables are green. Write each of the following symbolic statements in words.
> 1. p
> 2. $\sim r$
> 3. $\sim s$

Check Your Understanding

1. A _________ __________ is a complete sentence that makes a claim that may be either true or false.
2. The _________________ of a logical statement has the opposite truth value of the original statement.
3. If p represents the logical statement, “Marigolds are yellow flowers,” then ______ represents the statement, “Marigolds are not yellow flowers.”
4. The statement $\sim(\sim p)$ has the same truth value as the statement _______.
5. The logical statements used to support the conclusion of an argument are called ____________.
6. _______________________ arguments attempt to draw a general conclusion from specific premises.
7. All, some, and none are examples of ______________________, words that assign a numerical relationship between two or more groups.
8. The negation of the statement, “All giraffes are tall,” is ________________________________.

SECTION 2.1 EXERCISES

For the following exercises, determine whether the sentence represents a logical statement. If it is a logical statement, determine whether it is true or false.

1. A loan used to finance a house is called a mortgage.
2. All odd numbers are divisible by 2.
3. Please, bring me that notebook.
4. Robot, what’s your function?
5. In English, a conjunction is a word that connects two phrases or parts of a sentence together.
6. $8 - 3 = 5$.
7. $7 + 3 = 11$.
8. What is 7 plus 3?

For the following exercises, write each statement in symbolic form.

9. Grammy award winning singer, Lady Gaga, was not born in Houston, Texas.
10. Bruno Mars performed during the Super Bowl halftime show twice.
11. Coco Chanel said, “The most courageous act is still to think for yourself. Aloud.”
12. Bruce Wayne is not Superman.

For the following exercises, write the negation of each statement in words.

13. Bozo is not a clown.
14. Ash is Pikachu’s trainer and friend.
15. Vanilla is the most popular flavor of ice cream.
16. Smaug is a fire breathing dragon.
17. Elephant and Piggy are not best friends.
18. Some dogs like cats.
19. Some donuts are not round.
20. All cars have wheels.
21. No circles are squares.
22. Nature’s first green is not gold.
23. The ancient Greek philosopher Plato said, “The greatest wealth is to live content with little.”

24. All trees produce nuts.

For the following exercises, write the negation of each statement symbolically and in words.

25. p: Their hair is red.
26. $\sim q$: My favorite superhero does not wear a cape.
27. s: All wolves howl at the moon.
28. t: Nobody messes with Texas.
29. $\sim u$: I do not love New York.
30. $\sim v$: Some cats are not tigers.
31. $\sim q$: No squares are not parallelograms.
32. $\sim p$: The President does not like broccoli.

For the following exercises, write each of the following symbolic statements in words.

33. Given: p: Kermit is a green frog; translate $\sim p$ into words.
34. Given: $\sim r$: Mick Jagger is not the lead singer for The Rolling Stones; translate r into words.
35. Given: q: All dogs go to heaven; translate $\sim q$ into words.
36. Given: $\sim s$: Some pizza does not come with pepperoni on it; translate s into words.
37. Given: $\sim p$: No pizza comes with pineapple on it; translate $\sim(\sim p)$ into words.
38. Given: r: Not all roses are red; translate $\sim(\sim r)$ into words.
39. Given: $\sim t$: Thelonious Monk is not a famous jazz pianist; translate $\sim(\sim t)$ into words.
40. Given: $\sim v$: Not all violets are blue; translate $\sim(\sim v)$ into words.

For the following exercises, draw a logical conclusion from the premises that includes one of the following quantifiers: all, some, or none.

41. The Ford Motor Company builds cars in Michigan. General Motors builds cars in Michigan. Chrysler builds cars in Michigan. What conclusion can be drawn from these premises?
42. Michelangelo Buonarroti was a great Renaissance artist from Italy. Raphael Sanzio was a great Renaissance artist from Italy. Sandro Botticelli was a great Renaissance artist from Italy. What conclusion can you draw from these premises?
43. Four is an even number and it is divisible by 2. Six is an even number and it is divisible by 2. Eight is an even number and it is divisible by 2. What conclusion can you draw from these premises?
44. Three is an odd number and it is not divisible by 2. Seven is an odd number and it is not divisible by 2. Twenty-one is an odd number and it is not divisible by 2. What conclusion can you draw from these premises?
45. The odd number 5 is not divisible by 3. The odd number 7 is not divisible by 3. The odd number 29 is not divisible by 3. What conclusion can you draw from these premises?
46. Penguins are flightless birds. Emus are flightless birds. Ostriches are flightless birds. What conclusion can you draw from these premises?
47. Plants need water to survive. Animals need water to survive. Bacteria need water to survive. What conclusion can you draw from these premises?
48. A chocolate chip cookie is not sour. An oatmeal cookie is not sour. An Oreo cookie is not sour. What conclusion can you draw from these premises?

2.2 Compound Statements

Figure 2.6 A person seeking their driver's license must pass two tests. A compound statement can be used to explain performance on both tests at once. (credit: modification of work “Drivers License -Teen driver” by State Farm/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Translate compound statements into symbolic form.
2. Translate compound statements in symbolic form with parentheses into words.
3. Apply the dominance of connectives.

Suppose your friend is trying to get a license to drive. In most places, they will need to pass some form of written test proving their knowledge of the laws and rules for driving safely. After passing the written test, your friend must also pass a road test to demonstrate that they can perform the physical task of driving safely within the rules of the law.

Consider the statement: "My friend passed the written test, but they did not pass the road test." This is an example of a **compound statement**, a statement formed by using a **connective** to join two independent clauses or logical statements. The statement, “My friend passed the written test,” is an independent clause because it is a complete thought or idea that can stand on its own. The second independent clause in this compound statement is, “My friend did not pass the road test.” The word "but" is the connective used to join these two statements together, forming a compound statement. So, did your friend acquire their driving license?.

This section introduces common logical connectives and their symbols, and allows you to practice translating compound statements between words and symbols. It also explores the order of operations, or dominance of connectives, when a single compound statement uses multiple connectives.

Common Logical Connectives

Understanding the following logical connectives, along with their properties, symbols, and names, will be key to applying the topics presented in this chapter. The chapter will discuss each connective introduced here in more detail.

The joining of two logical statements with the word "and" or "but" forms a compound statement called a **conjunction**. In logic, for a conjunction to be true, all the independent logical statements that make it up must be true. The symbol for a conjunction is $\wedge$. Consider the compound statement, “Derek Jeter played professional baseball for the New York Yankees, and he was a shortstop.” If p represents the statement, “Derrick Jeter played professional baseball for the New York Yankees,” and if q represents the statement, “Derrick Jeter was a short stop,” then the conjunction will be written symbolically as $p \wedge q$.

The joining of two logical statements with the word “or” forms a compound statement called a **disjunction**. Unless otherwise specified, a disjunction is an inclusive *or* statement, which means the compound statement formed by joining two independent clauses with the word *or* will be true if a least one of the clauses is true. Consider the compound statement, "The office manager ordered cake for for an employee’s birthday or they ordered ice cream.” This is a disjunction because it combines the independent clause, “The office manager ordered cake for an employee’s birthday,” with the independent clause, “The office manager ordered ice cream,” using the connective, *or*. This disjunction is true if the office manager ordered only cake, only ice cream, or they ordered both cake and ice cream. Inclusive *or* means you can have one, or the other, or both!

Joining two logical statements with the word *implies*, or using the phrase "if *first statement*, then *second statement,"* is called a **conditional** or *implication*. The clause associated with the "if" statement is also called the **hypothesis** or *antecedent*, while the clause following the "then" statement or the word *implies* is called the **conclusion** or *consequent*. The conditional statement is like a one-way contract or promise. The only time the conditional statement is false, is if the hypothesis is true and the conclusion is false. Consider the following conditional statement, "If Pedro does his homework, then he can play video games." The hypothesis/antecedent is the statement following the word *if*, which is "Pedro does/did his homework." The conclusion/consequent is the statement following the word *then*, which is "Pedro can play his video games."

Joining two logical statements with the connective phrase "if and only if" is called a **biconditional**. The connective phrase "if and only if" is represented by the symbol, ↔. In the biconditional statement, $p \leftrightarrow q$, p is called the hypothesis or antecedent and q is called the conclusion or consequent. For a biconditional statement to be true, the truth values of p and q must match. They must both be true, or both be false.

The table below lists the most common connectives used in logic, along with their symbolic forms, and the common names used to describe each connective.

Connective	Symbol	Name
and but	∧	conjunction
or	∨	disjunction, inclusive or
not	~	negation
if …, then implies	→	conditional, implication
if and only if	↔	biconditional

Table 2.3

These connectives, along with their names, symbols, and properties, will be used throughout this chapter and should be memorized.

EXAMPLE 2.8

Associate Connectives with Symbols and Names

For each of the following connectives, write its name and associated symbol.

1. or
2. implies
3. but

Solution

1. A compound statement formed with the connective word *or* is called a disjunction, and it is represented by the ∨ symbol.
2. A compound statement formed with the connective word *implies* or phrase "*if …, then"* is called a conditional statement or implication and is represented by the → symbol.
3. A compound statement formed with the connective words *but* or *and* is called a conjunction, and it is represented by the ∧ symbol.

YOUR TURN 2.8

For each connective write its name and associated symbol.

1. not
2. and
3. if and only if

Translating Compound Statements to Symbolic Form

To translate a compound statement into symbolic form, we take the following steps.

1. Identify and label all independent affirmative logical statements with a lower case letter, such as p, q, or r.
2. Identify and label any negative logical statements with a lowercase letter preceded by the negation symbol, such as $\sim p$, $\sim q$, or $\sim r$.
3. Replace the connective words with the symbols that represent them, such as $\wedge, \vee, \rightarrow, \text{or} \leftrightarrow$.

Consider the previous example of your friend trying to get their driver's license. Your friend passed the written test, but they did not pass the road test. Let p represent the statement, "My friend passed the written test." And, let $\sim q$ represent the statement, "My friend did not pass the road test." Because the connective *but* is logically equivalent to the word *and*, the symbol for *but* is the same as the symbol for *and*; replace *but* with the symbol $\wedge$. The compound statement is symbolically written as: $p \wedge \sim q$. My friend passed the written test, but they did not pass the road test.

When translating English statements into symbolic form, do not assume that every connective "and" means you are dealing with a compound statement. The statement, "The stripes on the U.S. flag are red and white," is a simple statement. The word "white" doesn't express a complete thought, so it is not an independent clause and does not get its own symbol. This statement should be represented by a single letter, such as s: The stripes on the U.S. flag are red and white.

EXAMPLE 2.9

Translating Compound Statements into Symbolic Form

Let p represent the statement, "It is a warm sunny day," and let q represent the statement, "the family will go to the beach." Write the symbolic form of each of the following compound statements.

1. If it is a warm sunny day, then the family will go to the beach.
2. The family will go to the beach, and it is a warm sunny day.
3. The family will not go to the beach if and only if it is not a warm sunny day.
4. The family not go to the beach, or it is a warm sunny day.

Solution

1. Replace "it is a warm sunny day" with p. Replace "the family will go to the beach." with q. Next. Next, because the connective is *if ..., then* place the conditional symbol, $\rightarrow$, between p and q. The compound statement is written symbolically as: $p \rightarrow q$.
2. Replace "The family will go to the beach" with q. Replace "it is a warm sunny day." with p. Next, because the connective is *and*, place the $\wedge$ symbol between q and p. The compound statement is written symbolically as: $q \wedge p$.
3. Replace "The family will not go to the beach. with $\sim q$. Replace "it is not a warm sunny day" with $\sim p$. Next, because the connective is *or*, *if and only if*, place the biconditional symbol, $\leftrightarrow$ between $\sim q$ and $\sim p$. The compound statement is written symbolically as: $\sim q \leftrightarrow \sim p$.
4. Replace "The family will not go to the beach" with $\sim q$. Replace "it is a warm sunny day" with p. Next, because the connective is *or*, place the $\vee$ symbol between $\sim q$ and p. The compound statement is written symbolically as: $\sim q \vee p$.

YOUR TURN 2.9

Let p represent the statement, "Last night it snowed," and let q represent the statement, "Today we will go skiing." Write the symbolic form of each of the following compound statements:

1. Today we will go skiing, but last night it did not snow.

2. Today we will go skiing if and only if it snowed last night.
3. Last night is snowed or today we will not go skiing.
4. If it snowed last night, then today we will go skiing.

Translating Compound Statements in Symbolic Form with Parentheses into Words

When parentheses are written in a logical argument, they group a compound statement together just like when calculating numerical or algebraic expressions. Any statement in parentheses should be treated as a single component of the expression. If multiple parentheses are present, work with the inner most parentheses first.

Consider your friend's struggles to get their license to drive. Let p represent the statement, "My friend passed the written test," let q represent the statement, "My friend passed the road test," and let r represent the statement, "My friend received a driver's license." The statement $(p \wedge q) \rightarrow r$ can be translated into words as follows: the statement $p \wedge q$ is grouped together to form the hypothesis of the conditional statement and r is the conclusion. The conditional statement has the form "*if* $p \wedge q$, *then* r." Therefore, the written form of this statement is: "If my friend passed the written test and they passed the road test, then my friend received a driver's license."

Sometimes a compound statement within parentheses may need to be negated as a group. To accomplish this, add the phrase, "it is not the case that" before the translation of the phrase in parentheses. For example, using p, q, and r of your friend obtaining a license, let's translate the statement $\sim(p \wedge q) \rightarrow \sim r$ into words.

In this case, the hypothesis of the conditional statement is $\sim(p \wedge q)$ and the conclusion is $\sim r$. To negate the hypothesis, add the phrase "it is not the case" before translating what is in parentheses. The translation of the hypothesis is the sentence, "It is not the case that my friend passed the written test and they passed the road test," and the translation of the conclusion is, "My friend did not receive a driver's license." So, a translation of the complete conditional statement, $\sim(p \wedge q) \rightarrow \sim r$ is: "If it is not the case that my friend passed the written test and the road test, then my friend did not receive a driver's license."

It is acceptable to interchange proper names with pronouns and remove repeated phrases to make the written statement more readable, as long as the meaning of the logical statement is not changed.

VIDEO

Logic Part 1B: Compound Statements, Connectives and Symbols (https://openstax.org/r/Compound_Statements)

EXAMPLE 2.10

Translating Compound Statements in Symbolic Form with Parentheses into Words

Let p represent the statement, "My child finished their homework," let q represent the statement, "My child cleaned her room," let r represent the statement, "My child played video games," and let s represent the statement, "My child streamed a movie." Translate each of the following symbolic statements into words.

1. $\sim(p \wedge q)$
2. $(p \wedge q) \rightarrow (r \vee s)$
3. $\sim(r \vee s) \leftrightarrow \sim(p \wedge q)$

Solution

1. Replace ~ with "It is not the case," and $\wedge$ with "and." One possible translation is: "It is not the case that my child finished their homework and cleaned their room."
2. The hypothesis of the conditional statement is, "My child finished their homework and cleaned their room." The conclusion of the conditional statement is, "My child played video games or streamed a movie." One possible translation of the entire statement is: "If my child finished their homework and cleaned their room, then they played video games or streamed a movie."
3. The hypothesis of the biconditional statement is $\sim(r \vee s)$ and is written in words as: "It is not the case that my child played video games or streamed a movie." The conclusion of the biconditional statement is $\sim(p \wedge q)$, which translates to: "It is not the case that my child finished their homework and cleaned their room." Because the biconditional, $\leftrightarrow$ translates to *if and only if*, one possible translation of the statement is: "It is not the case that my child played video games or streamed a movie if and only if it is not the case that my child finished their homework and cleaned their room."

> **YOUR TURN 2.10**
>
> Let p represent the statement, "My roommates ordered pizza," let q represent the statement, "I ordered wings," and let r be the statement, "Our friends came over to watch the game." Translate the following statements into words.
>
> 1. $\sim r \rightarrow (p \vee q)$
> 2. $(p \wedge q) \rightarrow r$
> 3. $\sim(p \vee r)$

The Dominance of Connectives

The order of operations for working with algebraic and arithmetic expressions provides a set of rules that allow consistent results. For example, if you were presented with the problem $1 + 3 \times 2$, and you were not familiar with the order of operation, you might assume that you calculate the problem from left to right. If you did so, you would add 1 and 3 to get 4, and then multiply this answer by 2 to get 8, resulting in an incorrect answer. Try inputting this expression into a scientific calculator. If you do, the calculator should return a value of 7, not 8.

Scientific Calculator (https://openstax.org/r/Scientific_Calculator)

The order of operations for algebraic and arithmetic operations states that all multiplication must be applied prior to any addition. Parentheses are used to indicate which operation—addition or multiplication—should be done first. Adding parentheses can change and/or clarify the order. The parentheses in the expression $1 + (3 \times 2)$ indicate that 3 should be multiplied by 2 to get 6, and then 1 should be added to 6 to get 7: $1 + (3 \times 2) = 7$.

As with algebraic expressions, there is a set of rules that must be applied to compound logical statements in order to evaluate them with consistent results. This set of rules is called the **dominance of connectives**. When evaluating compound logical statements, connectives are evaluated from least dominant to most dominant as follows:

1. Parentheses are the least dominant connective. So, any expression inside parentheses must be evaluated first. Add as many parentheses as needed to any statement to specify the order to evaluate each connective.
2. Next, we evaluate negations.
3. Then, we evaluate conjunctions and disjunctions from left to right, because they have equal dominance.
4. After evaluating all conjunctions and disjunctions, we evaluate conditionals.
5. Lastly, we evaluate the most dominant connective, the biconditional. If a statement includes multiple connectives of equal dominance, then we will evaluate them from left to right.

See Figure 2.7 for a visual breakdown of the dominance of connectives.

Dominance	Connective	Symbol	Evaluate
Least Dominant	Parentheses	( )	First
↓	Negation	~	↓
	Disjunction/Conjunction	∨, ∧	Left to right or add parentheses to specify order because or/and have equal dominance.
↓	Conditional	→	↓
Most Dominant	Biconditional	↔	Last

Figure 2.7

Let's revisit your friend's struggles to get their driver's license. Let p represent the statement, "My friend passed the written test," let q represent the statement, "My friend passed the road test," and let r represent the statement, "My friend received a driver's license." Let's use the dominance of connectives to determine how the compound statement $p \wedge \sim q \rightarrow r$ should be evaluated.

Step 1: There are no parentheses, which is least dominant of all connectives, so we can skip over that.

Step 2: Because negation is the next least dominant, we should evaluate $\sim q$ first. We could add parentheses to the statement to make it clearer which connecting needs to be evaluated first: $p \wedge \sim q \rightarrow r$ is equivalent to $p \wedge (\sim q) \rightarrow r$.

Step 3: Next, we evaluate the conjunction, $\wedge$. $p \wedge (\sim q) \rightarrow r$ is equivalent to $(p \wedge (\sim q)) \rightarrow r$.

Step 4: Finally, we evaluate the conditional, $\rightarrow$, as this is the most dominant connective in this compound statement.

WHO KNEW?

When using spreadsheet applications, like Microsoft Excel or Google Sheets, have you noticed that core functions, such as sum, average, or rate, have the same name and syntax for use? This is not a coincidence. Various standards organizations have developed requirements that software developers must implement to meet the conditions set by governments and large customers worldwide. The OpenDocument Format OASIS Standard enables transferring data between different office productivity applications and was approved by the International Standards Organization (ISO) and IEC on May 6, 2006.

Prior to adopting these standards, a government entity, business, or individual could lose access to their own data simply because it was saved in a format no longer supported by a proprietary software product—even data they were required by law to preserve, or data they simply wished to maintain access to.

Just as rules for applying the dominance of connectives help maintain understanding and consistency when writing and interpreting compound logical statements and arguments, the standards adopted for database software maintain global integrity and accessibility across platforms and user bases.

EXAMPLE 2.11

Applying the Dominance of Connectives

For each of the following compound logical statements, add parentheses to indicate the order to evaluate the statement. Recall that parentheses are evaluated innermost first.

1. $p \wedge \sim q \vee r$
2. $q \rightarrow \sim p \wedge r$
3. $\sim(p \vee q) \leftrightarrow \sim p \wedge \sim q$

Solution

1. Because negation is the least dominant connective, we evaluate it first: $p \wedge (\sim q) \vee r$. Because conjunction and disjunction have the same dominance, we evaluate them left to right. So, we evaluate the conjunction next, as indicated by the additional set of parentheses: $(p \wedge (\sim q)) \vee r$. The only remaining connective is the disjunction, so it is evaluated last, as indicated by the third set of parentheses. The complete solution is: $((p \wedge (\sim q)) \vee r)$.
2. Negation has the lowest dominance, so it is evaluated first: $q \rightarrow (\sim p) \wedge r$. The remaining connectives are the conditional and the conjunction. Because conjunction has a lower precedence than the conditional, it is evaluated next, as indicated by the second set of parentheses: $q \rightarrow ((\sim p) \wedge r)$. The last step is to evaluate the conditional, as indicated by the third set of parentheses: $(q \rightarrow ((\sim p) \wedge r))$.
3. This statement is known as De Morgan's Law for the negation of a disjunction. It is always true. Section 2.6 of this chapter will explore De Morgan's Laws in more detail.
   - First, we evaluate the negations on the right side of the biconditional prior to the conjunction.
   - Then, we evaluate the disjunction on the left side of the biconditional, followed by the negation of the disjunction on the left side.
   - Lastly, after completely evaluating each side of the biconditional, we evaluate the biconditional. It does not matter which side you begin with.

   The final solution is: $(\sim (p \vee q)) \leftrightarrow ((\sim p) \wedge (\sim q))$.

YOUR TURN 2.11

For each of the following compound logical statements, add parentheses to indicate the order in which to evaluate the statement. Recall that parentheses are evaluated innermost first.

1. $p \vee q \wedge \sim r$
2. $\sim p \rightarrow q \vee r$

3. $\sim p \vee \sim q \leftrightarrow \sim(p \wedge q)$

WORK IT OUT

Logic Terms and Properties – Matching Game

Materials: For every group of four students, include 30 cards (game, trading, or playing cards), 30 individual clear plastic gaming card sleeves, and 30 card-size pieces of paper. Prepare a list of 60 questions and answers ahead of time related to definitions and problems in Statements and Quantifiers and Compound Statements. Provide each group of four students with 20 questions and their associated answers. Instruct each group to select 15 of the 20 questions, and then, for each problem selected, create one card with the question and one card with the answer. Instruct the groups to then place each problem and answer in a separate card sleeve. Once they create 15 problem cards and 15 answer cards, have students shuffle the set of cards.

To play the game, groups should exchange their card decks to ensure no team begins playing with the deck that they created. Split each four-person group into teams of two students. After shuffling the cards, one team lays cards face down on their desk in a five-by-six grid. The other team will go first.

Each player will have a turn to try matching the question with the correct answer by flipping two cards to the face up position. If a team makes a match, they get to flip another two cards; if they do not get a match, they flip the cards face down and it is the other team's turn to flip over two cards. The game continues in this manner until teams match all question cards with their corresponding answer cards. The team with the most set of matching cards wins.

In the first module of this chapter, we evaluated the truth value of negations. In the following modules, we will discuss how to evaluate conjunctions, disjunctions, conditionals, and biconditionals, and then evaluate compound logical statements using truth tables and our knowledge of the dominance of connectives.

Check Your Understanding

9. A _________ _________ is a logical statement formed by combining two or more statements with connecting words, such as and, or, but, not, and if ..., then.

10. A _____ is a word or symbol used to join two or more logical statements together to form a compound statement.

11. The most dominant connective is the _____.

12. _____ are used to specify which logical connective should be evaluated first when evaluating a compound statement.

13. Both _____ and _____ have equal dominance and are evaluated from left to right when no parentheses are present in a compound logical statement.

SECTION 2.2 EXERCISES

For the following exercises, translate each compound statement into symbolic form.
Given p: "Layla has two weeks for vacation," q: "Marcus is Layla's friend," r: "Layla will travel to Paris, France," and s: "Layla and Marcus will travel together to Niagara Falls, Ontario."

1. If Layla has two weeks for vacation, then she will travel to Paris, France.
2. Layla and Marcus will travel together to Niagara Falls, Ontario or Layla will travel to Paris, France.
3. If Marcus is not Layla's friend, then they will not travel to Niagara Falls, Ontario together."
4. Layla and Marcus will travel to Niagara Falls, Ontario together if and only if Layla and Marcus are friends.
5. If Layla does not have two weeks for vacation and Marcus is Layla's friend, then Marcus and Layla will travel together to Niagara Falls, Ontario.
6. If Layla has two weeks for vacation and Marcus is not her friend, then she will travel to Paris, France.

For the following exercises, translate each compound statement into symbolic form.
Given p: "Tom is a cat," q: "Jerry is a mouse," r: "Spike is a dog," s: "Tom chases Jerry," and t: "Spike catches Tom."

7. Jerry is a mouse and Tom is a cat.
8. If Tom chases Jerry, then Spike will catch Tom.
9. If Spike does not catch Tom, then Tom did not chase Jerry.

10. Tom is a cat and Spike is a dog, or Jerry is not a Mouse.
11. It is not the case that Tom is not a cat and Jerry is not a mouse.
12. Spike is not a dog and Jerry is a mouse if and only if Tom chases Jerry, but Spike does not catch Tom.

For the following exercises, translate the symbolic form of each compound statement into words.
Given p: "Tracy Chapman plays guitar," q: "Joan Jett plays guitar," r: "All rock bands include guitarists," and s: "Elton John plays the piano."

13. $p \vee r$
14. $\sim s \rightarrow \sim q$
15. $(p \wedge q) \leftrightarrow r$
16. $\sim r \wedge (q \vee s)$
17. $\sim(p \wedge \sim q)$
18. $(q \rightarrow \sim r) \leftrightarrow (\sim p \vee \sim r)$

For the following exercises, translate the symbolic form of each compound statement into words.
Given p: "The median is the middle number," q: "The mode is the most frequent number," r: "The mean is the average number," s: "The median, mean, and mode are equal," and t: "The data set is symmetric."

19. $t \rightarrow s$
20. $p \wedge (q \wedge r)$
21. $\sim t \rightarrow \sim s$
22. $(r \wedge p) \leftrightarrow q$
23. $(t \rightarrow \sim q) \vee (r \rightarrow s)$
24. $\sim(q \vee r) \rightarrow t$

For the following exercises, apply the proper dominance of connectives by adding parentheses to indicate the order in which the statement must be evaluated.

25. $p \rightarrow q \vee r$
26. $p \wedge q \leftrightarrow \sim r$
27. $p \vee r \vee \sim q$
28. $p \wedge \sim q \wedge r$
29. $p \wedge r \vee s \wedge t$
30. $q \rightarrow \sim r \leftrightarrow \sim p \vee \sim r$
31. $p \rightarrow r \vee s \leftrightarrow \sim t$
32. $\sim(t \wedge s) \vee (p \rightarrow q) \wedge \sim r$

2.3 Constructing Truth Tables

Figure 2.8 Just like solving a puzzle, a computer programmer must consider all potential solutions in order to account for each possible outcome. (credit: modification of work "Deadline Xmas 2010" by Allan Henderson/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Interpret and apply negations, conjunctions, and disjunctions.
2. Construct a truth table using negations, conjunctions, and disjunctions.
3. Construct a truth table for a compound statement and interpret its validity.

Are you familiar with the Choose Your Own Adventure book series written by Edward Packard? These gamebooks allow the reader to become one of the characters and make decisions that affect what happens next, resulting in different sequences of events in the story and endings based on the choices made. Writing a computer program is a little like what it must be like to write one of these books. The programmer must consider all the possible inputs that a user can put into the program and decide what will happen in each case, then write their program to account for each of these possible outcomes.

A **truth table** is a graphical tool used to analyze all the possible truth values of the component logical statements to determine the validity of a statement or argument along with all its possible outcomes. The rows of the table correspond to each possible outcome for the given logical statement identified at the top of each column. A single logical statement p has two possible truth values, true or false. In truth tables, a capital T will represent true values, and a capital F will represent false values.

In this section, you will use the knowledge built in Statements and Quantifiers and Compound Statements to analyze arguments and determine their truth value and validity. A logical argument is valid if its conclusion follows from its premises, regardless of whether those premises are true or false. You will then explore the truth tables for negation, conjunction, and disjunction, and use these truth tables to analyze compound logical statements containing these connectives.

Interpret and Apply Negations, Conjunctions, and Disjunctions

The negation of a statement will have the opposite truth value of the original statement. When p is true, $\sim p$ is false, and when p is false, $\sim p$ is true.

EXAMPLE 2.12

Finding the Truth Value of a Negation

For each logical statement, determine the truth value of its negation.

1. p: $3 + 5 = 8$.
2. q: All horses are mustangs.
3. $\sim r$: New Delhi is not the capital of India.

Solution

1. p is true because 3 + 5 does equal 8; therefore, the negation of p, $\sim p$: $3 + 5 \neq 8$, is false.
2. q is false because there are other types of horses besides mustangs, such as Clydesdales or Arabians; therefore, the negation of q, $\sim q$, is true.
3. $\sim r$ is false because New Delhi is the capital of India; therefore, the negation of $\sim r$, r, is true.

YOUR TURN 2.12

For each logical statement, determine the truth value of its negation.

1. $\sim p$: $3 \times 5 = 14$.
2. $\sim q$: Some houses are built with bricks.
3. r: Abuja is the capital of Nigeria.

A conjunction is a logical *and* statement. For a conjunction to be true, both statements that make up the conjunction must be true. If at least one of the statements is false, the *and* statement is false.

EXAMPLE 2.13

Finding the Truth Value of a Conjunction

Given $p : 4 + 7 = 11$, $q : 11 - 3 = 7$, and $r : 7 \times 11 = 77$, determine the truth value of each conjunction.

1. $p \wedge q$
2. $\sim q \wedge r$
3. $\sim p \wedge q$

Solution

1. p is true, and q is false. Because one statement is true, and the other statement is false, this makes the complete conjunction false.
2. q is false, so $\sim q$ is true, and r is true. Therefore, both statements are true, making the complete conjunction true.
3. $\sim p$ is false, and q is false. Because both statements are false, the complete conjunction is false.

YOUR TURN 2.13

Given p: Yellow is a primary color, q: Blue is a primary color, and r: Green is a primary color, determine the truth value of each conjunction.

1. $p \wedge q$
2. $q \wedge r$
3. $\sim r \wedge p$

The only time a conjunction is true is if both statements that make up the conjunction are true.

A disjunction is a logical inclusive *or* statement, which means that a disjunction is true if one or both statements that form it are true. The only way a logical inclusive *or* statement is false is if both statements that form the disjunction are false.

EXAMPLE 2.14

Finding the Truth Value of a Disjunction

Given $p : 4 + 7 = 11$, $q : 11 - 3 = 7$, and $r : 7 \times 11 = 77$, determine the truth value of each disjunction.

1. $p \vee q$
2. $\sim q \vee r$
3. $\sim p \vee q$

Solution

1. p is true, and q is false. One statement is true, and one statement is false, which makes the complete disjunction true.
2. q is false, so $\sim q$ is true, and r is true. Therefore, one statement is true, and the other statement is true, which makes the complete disjunction true.
3. $\sim p$ is false, and q is false. When all of the component statements are false, the disjunction is false.

YOUR TURN 2.14

Given p: Yellow is a primary color, q: Blue is a primary color, and r: Green is a primary color, determine the truth value of each disjunction.

1. $p \vee q$
2. $\sim p \vee r$
3. $q \vee r$

In the next example, you will apply the dominance of connectives to find the truth values of compound statements containing negations, conjunctions, and disjunctions and use a table to record the results. When constructing a truth table to analyze an argument where you can determine the truth value of each component statement, the strategy is to create a table with two rows. The first row contains the symbols representing the components that make up the compound statement. The second row contains the truth values of each component below its symbol. Include as many columns as necessary to represent the value of each compound statement. The last column includes the complete compound statement with its truth value below it.

VIDEO

Logic Part 2: Truth Values of Conjunctions: Is an "AND" statement true or false? (https://openstax.org/r/Truth_Values_of_Conjunctions)

Logic Part 3: Truth Values of Disjunctions: Is an "OR" statement true or false? (https://openstax.org/r/Truth_Values_of_Disjunctions)

EXAMPLE 2.15

Finding the Truth Value of Compound Statements

Given $p : 4 + 7 = 11$, $q : 11 - 3 = 7$, and $r : 7 \times 11 = 77$, construct a truth table to determine the truth value of each compound statement

1. $\sim p \wedge q \vee r$
2. $\sim p \vee q \wedge r$
3. $\sim(p \wedge r) \vee q$

Solution

1. **Step 1:** The statement "$\sim p \wedge q \vee r$" contains three basic logical statements, p, q, and r, and three connectives, $\sim, \wedge, \vee$. When we place parentheses in the statement to indicate the dominance of connectives, the statement becomes $(((\sim p) \wedge q) \vee r)$.
Step 2: After we have applied the dominance of connectives, we create a two row table that includes a column for each basic statement that makes up the compound statement, and an additional column for the contents of each parentheses. Because we have three sets of parentheses, we include a column for $\sim p$, the innermost parentheses, a column for $(\sim p) \wedge q$, the next set of parentheses, and $((\sim p) \wedge q) \vee r$ in the last column for the third parentheses.
Step 3: Once the table is created, we determine the truth value of each statement starting from left to right. The truth values of p, q, and r are true, false, and true, respectively, so we place T, F, and T in the second row of the table. Because p is true, $\sim p$ is false.
Step 4: Next, evaluate $\sim p \wedge q$ from the table: $\sim p$ is false, and q is also false, so $\sim p \wedge q$ is false, because a conjunction is only true if both of the statements that make it are true. Place an F in the table below its heading.
Step 5: Finally, using the table, we understand that $(\sim p \wedge q)$ is false and r is true, so the complete statement $(\sim p \wedge q) \vee r$ is false or true, which is true (because a disjunction is true whenever at least one of the statements that

make it is true). Place a T in the last column of the table. The complete statement $\sim p \wedge q \vee r$ is true.

p	q	r	$\sim p$	$(\sim p) \wedge q$	$((\sim p) \wedge q) \vee r$
T	F	T	F	F	**T**

2. **Step 1:** Applying the dominance of connectives to the original compound statement $\sim p \vee q \wedge r$, we get $(((\sim p) \vee q) \wedge r)$.
Step 2: The table needs columns for $p, q, r, \sim p,\ (\sim p) \vee q$, and $((\sim p) \vee q) \wedge r$.
Step 3: The truth values of p, q, r, and $\sim p$ are the same as in Question 1.
Step 4: Next, $\sim p \vee q$ is false or false, which is false, so we place an F below this statement in the table. This is the only time that a disjunction is false.
Step 5: Finally, $\sim p \vee q$ and r are the conjunction of the statements, $\sim p \vee q$ and r, and so the expressions evaluate to false and true, which is false. Recall that the only time an "and" statement is true is when both statements that form it are also true. The complete statement $\sim p \vee q \wedge r$ is false.

p	q	r	$\sim p$	$(\sim p) \vee q$	$((\sim p) \vee q) \wedge r$
T	F	T	F	F	**F**

3. **Step 1:** Applying the dominance of connectives to the original statement, we have: $((\sim(p \wedge r)) \vee q)$.
Step 2: So, the table needs the following columns: $p, q, r, p \wedge r, \sim(p \wedge r)$, and $\sim(p \wedge r) \vee q$.
Step 3: The truth values of p, q, and r are the same as in Questions 1 and 2.
Step 4: From the table it can be seen that $p \wedge r$ is true and true, which is true. So the negation of p and r is false, because the negation of a statement always has the opposite truth value of the original statement.
Step 5: Finally, $\sim(p \wedge r) \vee q$ is the disjunction of $\sim(p \wedge r)$ with q, and so we have false or false, which makes the complete statement false.

p	q	r	$p \wedge r$	$\sim(p \wedge r)$	$\sim(p \wedge r) \vee q$
T	F	T	T	F	**F**

YOUR TURN 2.15

Given p: Yellow is a primary color, q: Blue is a primary color, and r: Green is a primary color, determine the truth value of each compound statement, by constructing a truth table.

1. $\sim q \wedge p \vee r$
2. $p \vee q \wedge \sim r$
3. $\sim(p \wedge r) \wedge q$

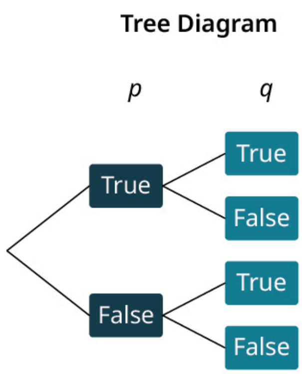
Construct Truth Tables to Analyze All Possible Outcomes

Recall from Statements and Questions that the negation of a statement will always have the opposite truth value of the original statement; if a statement p is false, then its negation $\sim p$ is true, and if a statement p is true, then its negation $\sim p$ is false. To create a truth table for the negation of statement p, add a column with a heading of $\sim p$, and for each row, input the truth value with the opposite value of the value listed in the column for p, as depicted in the table below.

Negation	
p	$\sim p$
T	F
F	T

Conjunctions and disjunctions are compound statements formed when two logical statements combine with the connectives “and” and “or” respectively. How does that change the number of possible outcomes and thus determine the number of rows in our truth table? The **multiplication principle**, also known as the *fundamental counting principle*, states that the number of ways you can select an item from a group of n items and another item from a group with m items is equal to the product of m and n. Because each proposition p and q has two possible outcomes, true or false, the multiplication principle states that there will be $2 \times 2 = 4$ possible outcomes: {TT, TF, FT, FF}.

The tree diagram and table in Figure 2.9 demonstrate the four possible outcomes for two propositions p and q.



Two Statements	
p	q
T	T
T	F
F	T
F	F

Figure 2.9

A conjunction is a logical *and* statement. For a conjunction to be true, both the statements that make up the conjunction must be true. If at least one of the statements is false, the *and* statement is false.

A disjunction is a logical inclusive *or* statement. Which means that a disjunction is true if one or both statements that make it are true. The only way a logical inclusive *or* statement is false is if both statements that make up the disjunction are false.

Conjunction (AND)				Disjunction (OR)		
p	q	$p \wedge q$		p	q	$p \vee q$
T	T	T		T	T	T
T	F	F		T	F	T
F	T	F		F	T	T
F	F	F		F	F	F

VIDEO

Logic Part 4: Truth Values of Compound Statements with "and", "or", and "not" (https://openstax.org/r/opL9I4tZCC0)

Logic Part 5: What are truth tables? How do you set them up? (https://openstax.org/r/-tdSRqLGhaw)

EXAMPLE 2.16

Constructing Truth Tables to Analyze Compound Statements

Construct a truth table to analyze all possible outcomes for each of the following arguments.

1. $p \vee \sim q$
2. $\sim(p \wedge q)$
3. $(p \vee \sim q) \wedge r$

Solution

1. **Step 1:** Because there are two basic statements, p and q, and each of these has two possible outcomes, we will have $2(2) = 4$ rows in our table to represent all possible outcomes: TT, TF, FT, and FF. The columns will include p, q, $\sim q$, and $p \vee \sim q$.
   Step 2: Every value in column $\sim q$ will have the opposite truth value of the corresponding value in column q: F, T, F, and T.
   Step 3: To complete the last column, evaluate each element in column p with the corresponding element in column $\sim q$ using the connective *or*.

p	q	$\sim q$	$p \vee \sim q$
T	T	F	**T**
T	F	T	**T**
F	T	F	**F**
F	F	T	**T**

2. **Step 1:** The columns will include p, q, $p \wedge q$, and $\sim(p \wedge q)$. Because there are two basic statements, p and q, the table will have four rows to account for all possible outcomes.
   Step 2: The $p \wedge q$ column will be completed by evaluating the corresponding elements in columns p and q respectively with the *and* connective.
   Step 3: The final column, $\sim(p \wedge q)$, will be the negation of the $p \wedge q$ column.

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	**F**
T	F	F	**T**
F	T	F	**T**
F	F	F	**T**

3. **Step 1:** This statement has three basic statements, p, q, and r. Because each basic statement has two possible truth values, true or false, the multiplication principal indicates there are $2(2)(2) = 8$ possible outcomes. So eight rows of outcomes are needed in the truth table to account for each possibility. Half of the eight possibilities must be true for the first statement, and half must be false.
   Step 2: So, the first column for statement p, will have four T's followed by four F's. In the second column for statement q, when p is true, half the outcomes for q must be true and the other half must be false, and the same pattern will repeat for when p is false. So, column q will have TT, FF, FF, FF.
   Step 3: The column for the third statement, r, must alternate between T and F. Once, the three basic propositions are listed, you will need a column for$\sim q$, $p \vee \sim q$, and $(p \vee \sim q) \wedge r$.
   Step 4: The column for the negation of q, $\sim q$, will have the opposite truth value of each value in column q.
   Step 5: Next, fill in the truth values for the column containing the statement $p \vee \sim q$. The *or* statement is true if at least one of p or $\sim q$ is true, otherwise it is false.

Step 6: Finally, fill in the column containing the conjunction $(p \vee \sim q) \wedge r$. To evaluate this statement, combine column $p \vee \sim q$ and column r with the *and* connective. Recall, that only time "and" is true is when both values are true, otherwise the statement is false. The complete truth table is:

p	q	r	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \wedge r$
T	T	T	F	T	**T**
T	T	F	F	T	**F**
T	F	T	T	T	**T**
T	F	F	T	T	**F**
F	T	T	F	F	**F**
F	T	F	F	F	**F**
F	F	T	T	T	**T**
F	F	F	T	T	**F**

YOUR TURN 2.16

Construct a truth table to analyze all possible outcomes for each of the following arguments.

1. $p \wedge \sim q$
2. $\sim(p \vee q)$
3. $(p \wedge \sim q) \vee r$

VIDEO

Logic Part 6: More on Truth Tables and Setting Up Rows and Column Headings (https://openstax.org/r/j3kKnUNIt6c)

Determine the Validity of a Truth Table for a Compound Statement

A logical statement is **valid** if it is always true regardless of the truth values of its component parts. To test the validity of a compound statement, construct a truth table to analyze all possible outcomes. If the last column, representing the complete statement, contains only true values, the statement is valid.

EXAMPLE 2.17

Determining the Validity of Compound Statements

Construct a truth table to determine the validity of each of the following statements.

1. $\sim p \wedge q$
2. $\sim(p \wedge \sim p)$

Solution

1. **Step 1:** Because there are two statements, p and q, and each of these has two possible outcomes, there will be $2(2) = 4$ rows in our table to represent all possible outcomes: TT, TF, FT, and FF.
   Step 2: The columns, will include p, q, $\sim p$ and $\sim p \wedge q$. Every value in column $\sim p$ will have the opposite truth value of the corresponding value in column p.
   Step 3: To complete the last column, evaluate each element in column $\sim p$ with the corresponding element in column q using the connective *and* . The last column contains at least one false, therefore the statement $\sim p \wedge q$ is

not valid.

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	**F**
T	F	F	**F**
F	T	T	**T**
F	F	T	**F**

2. **Step 1:** Because the statement $\sim(p \wedge \sim p)$ only contains one basic proposition, the truth table will only contain two rows. Statement p may be either true or false.
Step 2: The columns will include p, $\sim p$, $p \wedge \sim p$, and $\sim(p \wedge \sim p)$. Evaluate column $p \wedge \sim p$ with the *and* connective, because the symbol $\wedge$ represents a conjunction or logical *and* statement. True and false is false, and false and true is also false.
Step 3: The final column is the negation of each entry in the third column, both of which are false, so the negation of false is true. Because all the truth values in the final column are true, the statement $\sim(p \wedge \sim p)$ is valid.

p	$\sim p$	$p \wedge \sim p$	$\sim(p \wedge \sim p)$
T	F	F	**T**
F	T	F	**T**

YOUR TURN 2.17

Construct a truth table to determine the validity of each of the following statements.

1. $p \vee \sim p$
2. $\sim p \vee \sim q$

Check Your Understanding

14. A logical argument is _____ if its conclusion follows from its premises.

15. A logical statement is valid if it is always _____.

16. A _____ _____ is a tool used to analyze all the possible outcomes for a logical statement.

17. The truth table for the conjunction, $p \wedge q$, has _____ rows of truth values.

18. The truth table for the negation of a logical statement, $\sim p$, has _____ rows of truth values.

SECTION 2.3 EXERCISES

For the following exercises, find the truth value of each statement.

1. p: $7 \times 3 = 21$. What is the truth value of $\sim p$?
2. q: The sun revolves around the Earth. What is the truth value of $\sim q$?
3. $\sim r$: The acceleration of gravity is 9.81 m/sec^2. What is the truth value of r?
4. s: Dan Brown is not the author of the book, *The Davinci Code.* What is the truth value of $\sim(\sim s)$?
5. t: Broccoli is a vegetable. What is the truth value of $\sim(\sim t)$?

For the following exercises, given p: $1 + 2 = 3$, q: Five is an even number, and r: Seven is a prime number, find the truth value of each of the following statements.

6. $\sim q$
7. $p \wedge q$
8. $p \vee q$
9. $\sim p \vee \sim q$
10. $p \wedge \sim q$
11. $p \wedge r$
12. $q \wedge r$
13. $q \wedge \sim r$
14. $q \vee \sim r$
15. $\sim(p \wedge r)$
16. $p \vee q \wedge r$
17. $\sim p \vee (q \wedge r)$
18. $\sim(q \wedge r) \vee \sim p$
19. $q \vee r \vee p$
20. $\sim q \wedge r \wedge p$

For the following exercises, complete the truth table to determine the truth value of the proposition in the last column.

21.

p	q	r	$\sim p$	$\sim p \vee q$	$(\sim p \vee q) \wedge r$
T	T	T			

22.

p	q	r	$\sim p$	$\sim p \wedge q$	$(\sim p \wedge q) \wedge r$
F	T	F			

23.

p	q	r	$\sim p$	$\sim r$	$\sim p \wedge q(\sim p \wedge q) \vee \sim r$
F	F	F			

24.

p	q	r	$\sim p$	$\sim r$	$\sim p \vee q$	$(\sim p \vee q) \vee \sim r$
F	F	F				

For the following exercises, given p: All triangles have three sides, q: Some rectangles are not square, and r: A pentagon has eight sides, determine the truth value of each compound statement by constructing a truth table.

25. $\sim r \wedge q \wedge p$
26. $\sim(q \wedge p) \vee r$
27. $\sim p \vee q \wedge r$
28. $\sim p \vee \sim q \vee r$

For the following exercises, construct a truth table to analyze all the possible outcomes for the following arguments.

29. $\sim q \wedge q$
30. $\sim p \vee \sim q$
31. $\sim p \wedge \sim q$
32. $p \wedge q \vee r$

For the following exercises, construct a truth table to determine the validity of each statement.

33. $\sim q \vee q$
34. $p \wedge \sim q$

35. $p \wedge q \vee {\sim}p$

36. $(p \wedge q) \vee ({\sim}p \wedge {\sim}q)$

2.4 Truth Tables for the Conditional and Biconditional

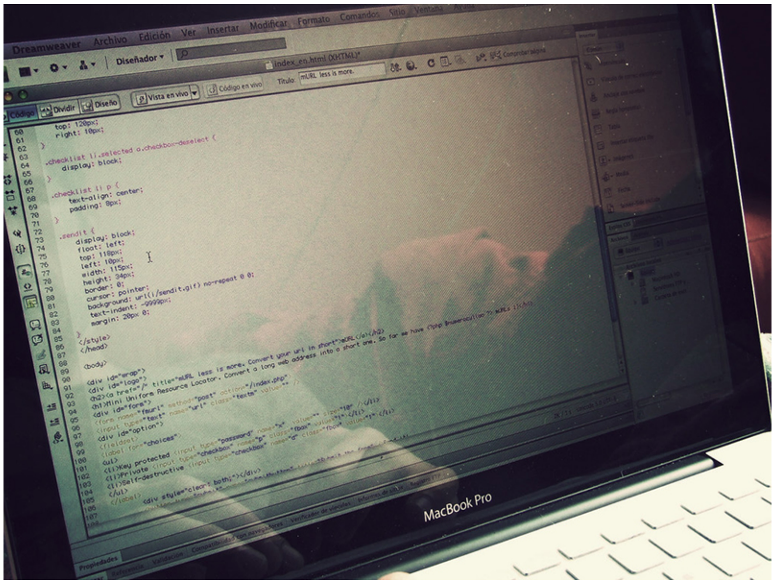


Figure 2.10 If-then statements use logic to execute directions. (credit: "Coding" by Carlos Varela/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Use and apply the conditional to construct a truth table.
2. Use and apply the biconditional to construct a truth table.
3. Use truth tables to determine the validity of conditional and biconditional statements.

Computer languages use if-then or if-then-else statements as decision statements:

- *If* the hypothesis is true, *then* do something.
- Or, *if* the hypothesis is true, *then* do something; else do something else.

For example, the following representation of computer code creates an if-then-else decision statement:

Check value of variable i.

If $i < 1$, then print "Hello, World!" else print "Goodbye".

In this imaginary program, the if-then statement evaluates and acts on the value of the variable i. For instance, if $i = 0$, the program would consider the statement $i < 1$ as true and "Hello, World!" would appear on the computer screen. If instead, $i = 3$, the program would consider the statement $i < 1$ as false (because 3 is greater than 1), and print "Goodbye" on the screen.

In this section, we will apply similar reasoning without the use of computer programs.

PEOPLE IN MATHEMATICS

The Countess of Lovelace, Ada Lovelace, is credited with writing the first computer program. She wrote an algorithm to work with Charles Babbage's Analytical Engine that could compute the Bernoulli numbers in 1843. In doing so, she became the first person to write a program for a machine that would produce more than just a simple calculation. The computer programming language ADA is named after her.

Reference: Posamentier, Alfred and Spreitzer Christian, "Chapter 34 Ada Lovelace: English (1815-1852)" pp. 272-278, *Math Makers: The Lives and Works of 50 Famous Mathematicians*, Prometheus Books, 2019.

Use and Apply the Conditional to Construct a Truth Table

A **conditional** is a logical statement of the form if p, then q. The conditional statement in logic is a promise or contract. The only time the conditional, $p \rightarrow q$, is false is when the contract or promise is broken.

For example, consider the following scenario. A child's parent says, "If you do your homework, then you can play your video games." The child really wants to play their video games, so they get started right away, finish within an hour, and then show their parent the completed homework. The parent thanks the child for doing a great job on their homework and allows them to play video games. Both the parent and child are happy. The contract was satisfied; true implies true is true.

Now, suppose the child does not start their homework right away, and then struggles to complete it. They eventually finish and show it to their parent. The parent again thanks the child for completing their homework, but then informs the child that it is too late in the evening to play video games, and that they must begin to get ready for bed. Now, the child is really upset. They held up their part of the contract, but they did not receive the promised reward. The contract was broken; true implies false is false.

So, what happens if the child does not do their homework? In this case, the hypothesis is false. No contract has been entered, therefore, no contract can be broken. If the conclusion is false, the child does not get to play video games and might not be happy, but this outcome is expected because the child did not complete their end of the bargain. They did not complete their homework. False implies false is true. The last option is not as intuitive. If the parent lets the child play video games, even if they did not do their homework, neither parent nor child are going to be upset. False implies true is true.

The truth table for the conditional statement below summarizes these results.

p	q	$p \rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

Notice that the only time the conditional statement, $p \rightarrow q$, is false is when the hypothesis, p, is true and the conclusion, q, is false.

VIDEO

Logic Part 8: The Conditional and Tautologies (https://openstax.org/r/Conditional_and_Tautologies)

EXAMPLE 2.18

Constructing Truth Tables for Conditional Statements

Assume both of the following statements are true: p: My sibling washed the dishes, and q: My parents paid them $5.00. Create a truth table to determine the truth value of each of the following conditional statements.

1. $p \rightarrow q$
2. $p \rightarrow {\sim}q$
3. ${\sim}p \rightarrow q$

Solution

1. Because p is true and q is true, the statement $p \rightarrow q$ is, "If my sibling washed the dishes, then my parents paid them \$5.00." My sibling did wash the dishes, since p is true, and the parents did pay the sibling \$5.00, so the contract was entered and completed. The conditional statement is true, as indicated by the truth table representing this case: T → T = T.

p	q	$p \rightarrow q$
T	T	**T**

2. $p \rightarrow {\sim}q$ translates to the statement, "If my sibling washed the dishes, then my parents did not pay them \$5.00." p is true, but ${\sim}q$ is false. The sibling completed their end of the contract, but they did not get paid. The contract was broken by the parents. The conditional statement is false, as indicated by the truth table representing this case: T → F = F.

p	q	${\sim}q$	$p \rightarrow {\sim}q$
T	T	F	**F**

3. ${\sim}p \rightarrow q$ translates to the statement, "If my sibling did not wash the dishes, then my parents paid them \$5.00." ${\sim}p$ is false, but q is true. The sibling did not do the dishes. No contract was entered, so it could not be broken. The parents decided to pay them \$5.00 anyway. The conditional statement is true, as indicated by the truth table representing this case: F → T = T.

p	q	${\sim}p$	${\sim}p \rightarrow q$
T	T	F	**T**

YOUR TURN 2.18

Assume p is true and q is false. p: Kevin vacuumed the living room, and q: Kevin's parents did not let him borrow the car. Create a truth table to determine the truth value of each of the following conditional statements.

1. $p \rightarrow q$
2. $p \rightarrow {\sim}q$
3. ${\sim}p \rightarrow q$

EXAMPLE 2.19

Determining Validity of Conditional Statements

Construct a truth table to analyze all possible outcomes for each of the following statements then determine whether they are valid.

1. $p \wedge q \rightarrow {\sim}q$
2. $p \rightarrow {\sim}p \vee q$

Solution

1. Applying the dominance of connectives, the statement $p \wedge q \rightarrow {\sim}q$ is equivalent to $(p \wedge q) \rightarrow ({\sim}q)$. So, the columns of the truth table will include p, q, $p \wedge q$, ${\sim}q$, and $p \wedge q \rightarrow {\sim}q$. Because there are only two basic propositions, p and q, the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$p \wedge q$	$\sim q$	$p \wedge q \rightarrow \sim q$
T	T	T	F	**F**
T	F	F	T	**T**
F	T	F	F	**T**
F	F	F	T	**T**

2. Applying the dominance of connectives, the statement $p \rightarrow \sim p \vee q$ is equivalent to $(p) \rightarrow ((\sim p) \vee q)$. So, the columns of the truth table will include p, q, $\sim p$, $\sim p \vee q$, and $p \rightarrow (\sim p \vee q)$. Because there are only two basic propositions, p and q, the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$\sim p$	$\sim p \vee q$	$p \rightarrow (\sim p \vee q)$
T	T	F	T	**T**
T	F	F	F	**F**
F	T	T	T	**T**
F	F	T	T	**T**

YOUR TURN 2.19

Construct a truth table to analyze all possible outcomes for each of the following statements, then determine whether they are valid.

1. $q \rightarrow \sim p \vee q$
2. $\sim p \rightarrow q \wedge p$

Use and Apply the Biconditional to Construct a Truth Table

The biconditional, $p \leftrightarrow q$, is a two way contract; it is equivalent to the statement $(p \rightarrow q) \wedge (q \rightarrow p)$. A biconditional statement, $p \leftrightarrow q$, is true whenever the truth value of the hypothesis matches the truth value of the conclusion, otherwise it is false.

The truth table for the biconditional is summarized below.

p	q	$p \leftrightarrow q$
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**T**

VIDEO

[Logic Part 11B: Biconditional and Summary of Truth Value Rules in Logic (https://openstax.org/r/omKzui0Fytk)](https://openstax.org/r/omKzui0Fytk)

EXAMPLE 2.20

Constructing Truth Tables for Biconditional Statements

Assume both of the following statements are true: p: The plumber fixed the leak, and q: The homeowner paid the plumber $150.00. Create a truth table to determine the truth value of each of the following biconditional statements.

1. $p \leftrightarrow q$
2. $p \leftrightarrow {\sim}q$
3. ${\sim}p \leftrightarrow {\sim}q$

Solution

1. Because p is true and q is true, the statement $p \leftrightarrow q$ is "The plumber fixed the leak if and only if the homeowner paid them $150.00." Because both p and q are true, the leak was fixed and the plumber was paid, meaning both parties satisfied their end of the bargain. The biconditional statement is true, as indicated by the truth table representing this case: T ↔ T = T.

p	q	$p \leftrightarrow q$
T	T	**T**

2. $p \rightarrow {\sim}q$ translates to the statement, "The plumber fixed the leak if and only if the homeowner did not pay them $150." If the plumber fixed the leak and the homeowner did not pay them, the homeowner will have broken their end of the contract. The biconditional statement is false, as indicated by the truth table representing this case: T ↔ F = F.

p	q	${\sim}q$	$p \leftrightarrow {\sim}q$
T	T	F	**F**

3. ${\sim}p \leftrightarrow {\sim}q$ translates to the statement, "The plumber did not fix the leak if and only if the homeowner did not pay them $150." In this case, neither party—the plumber nor the homeowner—entered into the contract. The leak was not repaired, and the plumber was not paid. No agreement was broken. The biconditional statement is true, as indicated by the truth table representing this case: F ↔ F = T.

p	q	${\sim}p$	${\sim}q$	${\sim}p \leftrightarrow {\sim}q$
T	T	F	F	**T**

YOUR TURN 2.20

Assume p is true and q is false: p: The contractor fixed the broken window, and q: The homeowner paid the contractor $200. Create a truth table to determine the truth value of each of the following biconditional statements.

1. $p \leftrightarrow q$
2. $p \leftrightarrow {\sim}q$
3. ${\sim}p \leftrightarrow q$

The biconditional, $p \leftrightarrow q$, is true whenever the truth values of p and q match, otherwise it is false.

VIDEO

Logic Part 13: Truth Tables to Determine if Argument is Valid or Invalid (https://openstax.org/r/AQB3svnxxiw)

EXAMPLE 2.21

Determining Validity of Biconditional Statements

Construct a truth table to analyze all possible outcomes for each of the following statements, then determine whether they are valid.

1. $p \wedge q \leftrightarrow p \wedge \sim q$
2. $p \vee q \leftrightarrow \sim p \vee q$
3. $p \rightarrow q \leftrightarrow \sim q \rightarrow \sim p$
4. $p \wedge q \rightarrow \sim r \leftrightarrow p \wedge q \wedge r$

Solution

1. Applying the dominance of connectives, the statement $p \wedge q \leftrightarrow p \wedge \sim q$ is equivalent to $(p \wedge q) \leftrightarrow (p \wedge (\sim q))$. So, the columns of the truth table will include p, q, $p \wedge q$, $\sim q$, $p \wedge \sim q$ and $(p \wedge q) \leftrightarrow (p \wedge \sim q)$. Because there are only two basic propositions, p and q, the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$p \wedge q$	$\sim q$	$p \wedge \sim q$	$(p \wedge q) \leftrightarrow (p \wedge \sim q)$
T	T	T	F	F	**F**
T	F	F	T	T	**F**
F	T	F	F	F	**T**
F	F	F	T	F	**T**

2. Applying the dominance of connectives, the statement $p \vee q \leftrightarrow \sim p \vee q$ is equivalent to $(p \vee q) \leftrightarrow ((\sim p) \vee q)$. So, the columns of the truth table will include p, q, $p \vee q$, $\sim p$, $\sim p \vee q$, and $(p \vee q) \leftrightarrow (\sim p \vee q)$. Because there are only two basic propositions, p and q, the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$p \vee q$	$\sim p$	$\sim p \vee q$	$(p \vee q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	**T**
T	F	T	F	F	**F**
F	T	T	T	T	**T**
F	F	F	T	T	**F**

3. Applying the dominance of connectives, the statement $p \rightarrow q \leftrightarrow \sim q \rightarrow \sim p$ is equivalent to $(p \rightarrow q) \leftrightarrow ((\sim q) \rightarrow (\sim p))$. So, the columns of the truth table will include p, q, $p \rightarrow q$, $\sim q$, $\sim p$, $\sim q \rightarrow \sim p$, and $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$. Because there are only two basic propositions, p and q the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is valid because the last column is all true.

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$	$(p \to \mathbf{q}) \leftrightarrow (\sim q \to \sim p)$
T	T	T	F	F	T	**T**
T	F	F	T	F	F	**T**
F	T	T	F	T	T	**T**
F	F	T	T	T	T	**T**

4. Applying the dominance of connectives, the statement $p \wedge q \to \sim r \leftrightarrow p \wedge q \wedge r$ is equivalent to $((p \wedge q) \to (\sim r)) \leftrightarrow ((p \wedge q) \wedge r)$. So, the columns of the truth table will include p, q, r, $\sim r$, $p \wedge q$, $(p \wedge q) \wedge \sim r$, $(p \wedge q) \wedge r$, and $((p \wedge q) \to (\sim r)) \leftrightarrow ((p \wedge q) \wedge r)$. Because there are three basic propositions, p, q, and r, the table will have $2(2)(2) = 8$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	r	$\sim r$	$p \wedge q$	$(p \wedge q) \to \sim r$	$(p \wedge q) \wedge r$	$(p \wedge q \to \sim r) \leftrightarrow (p \wedge q \wedge r)$
T	T	T	F	T	F	T	**T**
T	T	F	T	T	T	F	**T**
T	F	T	F	F	T	F	**F**
T	F	F	T	F	T	F	**F**
F	T	T	F	F	T	F	**F**
F	T	F	T	F	T	F	**F**
F	F	T	F	F	T	F	**F**
F	F	F	T	F	T	F	**F**

YOUR TURN 2.21

Construct a truth table to analyze all possible outcomes for each of the following statements, then determine whether they are valid.

1. $\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
2. $\sim p \leftrightarrow q \wedge p$
3. $p \to q \leftrightarrow \sim p \vee q$
4. $p \wedge q \to r \leftrightarrow \sim p \vee \sim q \vee r$

Check Your Understanding

19. In logic, a conditional statement can be thought of as a ____________.

20. If the hypothesis, p, of a conditional statement is true, the _____, q, must also be true for the conditional statement $p \to q$ to be true.

21. If the ____________ of a conditional statement is false, the conditional statement is true.

22. The symbolic form of the ____________________ statement is $p \leftrightarrow q$.

23. The ____________________ statement is equivalent to the statement $(p \rightarrow q) \wedge (q \rightarrow p)$.

24. p if and only if q is ___________ whenever the truth value of p matches the truth value of q, otherwise it is false.

SECTION 2.4 EXERCISES

For the following exercises, complete the truth table to determine the truth value of the proposition in the last column.

1.

p	q	$\sim p$	$\sim p \to q$
T	T		

2.

p	q	$\sim q$	$p \to \sim q$
T	T		

3.

p	q	$\sim p$	$\sim p \leftrightarrow q$
F	T		

4.

p	q	$\sim q$	$p \leftrightarrow \sim q$
F	T		

5.

p	q	r	$\sim p$	$\sim p \wedge q$	$(\sim p \wedge q) \to r$
F	T	F			

6.

p	q	r	$\sim p$	$\sim r$	$\sim p \wedge q$	$(\sim p \wedge q) \to \sim r$
F	F	F				

7.

p	q	r	$\sim p$	$\sim r$	$\sim p \vee q$	$(\sim p \vee q) \leftrightarrow \sim r$
F	F	F				

8.

p	q	r	$\sim p$	$\sim r$	$\sim p \wedge q$	$(\sim p \wedge q) \leftrightarrow \sim r$
T	F	F				

9.

p	q	r	$\sim p$	$\sim r$	$\sim p \vee r$	$p \to \sim r$	$(\sim p \vee r) \leftrightarrow (p \to \sim r)$
F	F	F					

10.

p	q	r	$\sim p$	$\sim r$	$\sim p \wedge q$	$p \to \sim r$	$(\sim p \wedge q) \leftrightarrow (p \to \sim r)$
T	T	T					

For the following exercises, assume these statements are true: p: Faheem is a software engineer, q: Ann is a project manager, r: Giacomo works with Faheem, and s: The software application was completed on time. Translate each of the following statements to symbols, then construct a truth table to determine its truth value.

11. If Giacomo works with Faheem, then Faheem is not a software engineer.
12. If the software application was not completed on time, then Ann is not a project manager.
13. The software application was completed on time if and only if Giacomo worked with Faheem.
14. Ann is not a project manager if and only if Faheem is a software engineer.
15. If the software application was completed on time, then Ann is a project manager, but Faheem is not a software engineer.
16. If Giacomo works with Faheem and Ann is a project manager, then the software application was completed on time.
17. The software application was not completed on time if and only if Faheem is a software engineer or Giacomo did not work with Faheem.
18. Faheem is a software engineer or Ann is not a project manager if and only if Giacomo did not work with Faheem and the software application was completed on time.
19. Ann is a project manager implies Faheem is a software engineer if and only if the software application was completed on time implies Giacomo worked with Faheem.
20. If Giacomo did not work with Faheem implies that the software application was not completed on time, then Ann was not the project manager.

For the following exercises, construct a truth table to analyze all the possible outcomes and determine the validity of each argument.

21. $p \vee {\sim}q \rightarrow q$
22. ${\sim}q \rightarrow p \wedge {\sim}q$
23. $(p \rightarrow q) \leftrightarrow q$
24. $(p \rightarrow q) \leftrightarrow p$
25. ${\sim}(p \vee q) \leftrightarrow ({\sim}p \wedge {\sim}q)$
26. $(p \rightarrow q) \wedge p \rightarrow q$
27. $p \rightarrow q \rightarrow r$
28. $(p \rightarrow q) \wedge (q \rightarrow r) \leftrightarrow (p \rightarrow r)$
29. $p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$
30. $p \vee (q \vee r) \leftrightarrow (p \vee q) \vee r$

2.5 Equivalent Statements

Figure 2.11 How your logical argument is stated affects the response, just like how you speak when holding a conversation can affect how your words are received. (credit: modification of work by Goelshivi/Flickr, Public Domain Mark 1.0)

Learning Objectives

After completing this section, you should be able to:

1. Determine whether two statements are logically equivalent using a truth table.
2. Compose the converse, inverse, and contrapositive of a conditional statement

Have you ever had a conversation with or sent a note to someone, only to have them misunderstand what you intended to convey? The way you choose to express your ideas can be as, or even more, important than what you are saying. If your goal is to convince someone that what you are saying is correct, you will not want to alienate them by choosing your words poorly.

Logical arguments can be stated in many different ways that still ultimately result in the same valid conclusion. Part of the art of constructing a persuasive argument is knowing how to arrange the facts and conclusion to elicit the desired response from the intended audience.

In this section, you will learn how to determine whether two statements are **logically equivalent** using truth tables, and then you will apply this knowledge to compose logically equivalent forms of the conditional statement. Developing this skill will provide the additional skills and knowledge needed to construct well-reasoned, persuasive arguments that can be customized to address specific audiences.

An alternate way to think about logical equivalence is that the truth values have to match. That is, whenever p is true, q is also true, and whenever p is false, q is also false.

Determine Logical Equivalence

Two statements, p and q, are logically equivalent when $p \leftrightarrow q$ is a valid argument, or when the last column of the truth table consists of only true values. When a logical statement is always true, it is known as a **tautology**. To determine whether two statements p and q are logically equivalent, construct a truth table for $p \leftrightarrow q$ and determine whether it valid. If the last column is all true, the argument is a tautology, it is valid, and p is logically equivalent to q; otherwise, p is not logically equivalent to q.

EXAMPLE 2.22

Determining Logical Equivalence with a Truth Table

Create a truth table to determine whether the following compound statements are logically equivalent.

1. $p \to q$; $\sim p \to \sim q$
2. $p \to q$; $\sim p \vee q$

Solution

1. Construct a truth table for the biconditional formed by using the first statement as the hypothesis and the second statement as the conclusion, $(p \to q) \leftrightarrow (\sim p \to \sim q)$.

p	q	$p \to q$	$\sim p$	$\sim q$	$\sim p \to \sim q$	$(p \to q) \leftrightarrow (\sim p \to \sim q)$
T	T	T	F	F	T	**T**
T	F	F	F	T	T	**F**
F	T	T	T	F	F	**F**
F	F	T	T	T	T	**T**

Because the last column it not all true, the biconditional is not valid and the statement $p \to q$ is not logically equivalent to the statement $\sim p \to \sim q$.

2. Construct a truth table for the biconditional formed by using the first statement as the hypothesis and the second statement as the conclusion, $(p \to q) \leftrightarrow (\sim p \vee q)$.

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	$(p \to q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	**T**
T	F	F	F	F	**T**
F	T	T	T	T	**T**
F	F	T	T	T	**T**

Because the last column is true for every entry, the biconditional is valid and the statement $p \to q$ is logically equivalent to the statement $\sim p \vee q$. Symbolically, $p \to q \equiv \sim p \vee q$.

YOUR TURN 2.22

Create a truth table to determine whether the following compound statements are logically equivalent.

1. $p \to q$; $q \to \sim p$
2. $p \to q$; $p \vee \sim q$

Compose the Converse, Inverse, and Contrapositive of a Conditional Statement

The **converse**, **inverse**, and **contrapositive** are variations of the conditional statement, $p \to q$.

- The converse is if q then p, and it is formed by interchanging the hypothesis and the conclusion. The converse is logically equivalent to the inverse.
- The inverse is if $\sim p$ then $\sim q$, and it is formed by negating both the hypothesis and the conclusion. The inverse is logically equivalent to the converse.
- The contrapositive is if $\sim q$ then $\sim p$, and it is formed by interchanging and negating both the hypothesis and the conclusion. The contrapositive is logically equivalent to the conditional.

The table below shows how these variations are presented symbolically.

				Conditional	Contrapositive	Converse	Inverse
p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$	$q \rightarrow p$	$\sim p \rightarrow \sim q$
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

EXAMPLE 2.23

Writing the Converse, Inverse, and Contrapositive of a Conditional Statement

Use the statements, p: Harry is a wizard and q: Hermione is a witch, to write the following statements:

1. Write the conditional statement, $p \rightarrow q$, in words.
2. Write the converse statement, $q \rightarrow p$, in words.
3. Write the inverse statement, $\sim p \rightarrow \sim q$, in words.
4. Write the contrapositive statement, $\sim q \rightarrow \sim p$, in words.

Solution

1. The conditional statement takes the form, "if p, then q," so the conditional statement is: "If Harry is a wizard, then Hermione is a witch." Remember the *if* ... *then* ... words are the connectives that form the conditional statement.
2. The converse swaps or interchanges the hypothesis, p, with the conclusion, q. It has the form, "if q, then p." So, the converse is: "If Hermione is a witch, then Harry is a wizard."
3. To construct the inverse of a statement, negate both the hypothesis and the conclusion. The inverse has the form, "if $\sim p$, then $\sim q$," so the inverse is: "If Harry is not a wizard, then Hermione is not a witch."
4. The contrapositive is formed by negating and interchanging both the hypothesis and conclusion. It has the form, "if $\sim q$, then $\sim p$," so the contrapositive statement is: "If Hermione is not a witch, then Harry is not a wizard."

YOUR TURN 2.23

Use the statements, p: Elvis Presley wore capes and q: Some superheroes wear capes, to write the following statements:

1. Write the conditional statement, $p \rightarrow q$, in words.
2. Write the converse statement, $q \rightarrow p$, in words.
3. Write the inverse statement, $\sim p \rightarrow \sim q$, in words.
4. Write the contrapositive statement, $\sim q \rightarrow \sim p$, in words.

EXAMPLE 2.24

Identifying the Converse, Inverse, and Contrapositive

Use the conditional statement, "If all dogs bark, then Lassie likes to bark," to identify the following.

1. Write the hypothesis of the conditional statement and label it with a p.
2. Write the conclusion of the conditional statement and label it with a q.
3. Identify the following statement as the converse, inverse, or contrapositive: "If Lassie likes to bark, then all dogs bark."
4. Identify the following statement as the converse, inverse, or contrapositive: "If Lassie does not like to bark, then some dogs do not bark."
5. Which statement is logically equivalent to the conditional statement?

Solution

1. The hypothesis is the phrase following the *if*. The answer is p: All dogs bark. Notice, the word *if* is not included as part of the hypothesis.
2. The conclusion of a conditional statement is the phrase following the *then*. The word *then* is not included when stating the conclusion. The answer is: q: Lassie likes to bark.
3. "Lassie likes to bark" is q and "All dogs bark" is p. So, "If Lassie likes to bark, then all dogs bark," has the form "if q, then p," which is the form of the converse.
4. "Lassie does not like to bark" is $\sim q$ and "Some dogs do not bark" is $\sim p$. The statement, "If Lassie does not like to bark, then some dogs do not bark," has the form "if $\sim q$, then $\sim p$," which is the form of the contrapositive.
5. The contrapositive $\sim q \rightarrow \sim p$ is logically equivalent to the conditional statement $p \rightarrow q$.

YOUR TURN 2.24

Use the conditional statement, "If Dora is an explorer, then Boots is a monkey," to identify the following:

1. Write the hypothesis of the conditional statement and label it with a p.
2. Write the conclusion of the conditional statement and label it with a q.
3. Identify the following statement as the converse, inverse, or contrapositive: "If Dora is not an explorer, then Boots is not a monkey."
4. Identify the following statement as the converse, inverse, or contrapositive: "If Boots is a monkey, then Dora is an explorer."
5. Which statement is logically equivalent to the inverse?

EXAMPLE 2.25

Determining the Truth Value of the Converse, Inverse, and Contrapositive

Assume the conditional statement, $p \rightarrow q$: "If Chadwick Boseman was an actor, then Chadwick Boseman did not star in the movie *Black Panther*" is false, and use it to answer the following questions.

1. Write the converse of the statement in words and determine its truth value.
2. Write the inverse of the statement in words and determine its truth value.
3. Write the contrapositive of the statement in words and determine its truth value.

Solution

1. The only way the conditional statement can be false is if the hypothesis, p: Chadwick Boseman was an actor, is true and the conclusion, q: Chadwick Boseman did not star in the movie *Black Panther*, is false. The converse is $q \rightarrow p$, and it is written in words as: "If Chadwick Boseman did not star in the movie *Black Panther*, then Chadwick Boseman was an actor." This statement is true, because false → true is true.
2. The inverse has the form "$\sim p \rightarrow \sim q$." The written form is: "If Chadwick Boseman was not an actor, then Chadwick Boseman starred in the movie *Black Panther*." Because p is true, and q is false, $\sim p$ is false, and $\sim q$ is true. This means the inverse is false → true, which is true. Alternatively, from Question 1, the converse is true, and because the inverse is logically equivalent to the converse it must also be true.
3. The contrapositive is logically equivalent to the conditional. Because the conditional is false, the contrapositive is also false. This can be confirmed by looking at the truth values of the contrapositive statement. The contrapositive has the form "$\sim q \rightarrow \sim p$." Because q is false and p is true, $\sim q$ is true and $\sim p$ is false. Therefore, $\sim q \rightarrow \sim p$ is true → false, which is false. The written form of the contrapositive is "If Chadwick Boseman starred in the movie *Black Panther*, then Chadwick Boseman was not an actor."

YOUR TURN 2.25

Assume the conditional statement $p \rightarrow q$: "If my friend lives in San Francisco, then my friend does not live in California" is false, and use it to answer the following questions.

1. Write the converse of the statement in words and determine its truth value.
2. Write the inverse of the statement in words and determine its truth value.

3. Write the contrapositive of the statement in words and determine its truth value.

Check Your Understanding

25. Two statements p and q are logically equivalent to each other if the biconditional statement, $p \leftrightarrow q$ is _______________.
26. The _____ statement has the form, "p then q."
27. The contrapositive is ____________ __________ to the conditional statement, and has the form, "if $\sim q$, then $\sim p$."
28. The _______________ of the conditional statement has the form, "if $\sim p$, then $\sim q$."
29. The _______________ of the conditional statement is logically equivalent to the ______________ and has the form, "if q then p."

SECTION 2.5 EXERCISES

For the following exercises, determine whether the pair of compound statements are logically equivalent by constructing a truth table.

1. Converse: $q \rightarrow p$ and inverse: $\sim p \rightarrow \sim q$
2. Conditional: $p \rightarrow q$ and contrapositive: $\sim q \rightarrow \sim p$
3. Inverse: $\sim p \rightarrow \sim q$ and contrapositive: $\sim q \rightarrow \sim p$
4. Conditional: $p \rightarrow q$ and converse: $q \rightarrow p$
5. $\sim p \rightarrow q$ and $p \vee \sim q$
6. $\sim p \rightarrow q$ and $p \vee q$
7. $\sim(p \wedge q)$ and $\sim p \wedge \sim q$
8. $\sim(p \wedge q)$ and $\sim p \vee \sim q$
9. $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$
10. $p \wedge (q \vee r)$ and $(p \wedge q) \vee r$

For the following exercises, answer the following:

a. Write the conditional statement $p \rightarrow q$ in words.
b. Write the converse statement $q \rightarrow p$ in words.
c. Write the inverse statement $\sim p \rightarrow \sim q$ in words.
d. Write the contrapositive statement $\sim q \rightarrow \sim p$ in words.

11. p: Six is afraid of Seven and q: Seven ate Nine.
12. p: Hope is eternal and q: Despair is temporary.
13. p: Tom Brady is a quarterback and q: Tom Brady does not play soccer.
14. p: Shakira does not sing opera and q: Shakira sings popular music.
15. p:The shape does not have three sides and q: The shape is not a triangle.
16. p: All birds can fly and q: Emus can fly.
17. p: Penguins cannot fly and q: Some birds can fly.
18. p: Some superheroes do not wear capes and q: Spiderman is a superhero.
19. p: No Pokémon are little ponies and q: Bulbasaur is a Pokémon.
20. p: Roses are red, and violets are blue and q: Sugar is sweet, and you are sweet too.

For the following exercises,use the conditional statement: "If Clark Kent is Superman, then Lois Lane is not a reporter," to answer the following questions.

21. Write the hypothesis of the conditional statement, label it with a p, and determine its truth value.
22. Write the conclusion of the conditional statement, label it with a q, and determine its truth value.
23. Identify the following statement as the converse, inverse, or contrapositive, and determine its truth value: "If Clark Kent is not Superman, then Lois Lane is a reporter."
24. Identify the following statement as the converse, inverse, or contrapositive, and determine its truth value: "If Lois Lane is a reporter, then Clark Kent is not Superman."
25. Which form of the conditional is logically equivalent to the converse?

For the following exercises, use the conditional statement: "If *The Masked Singer* is not a music competition, then Donnie Wahlberg was a member of New Kids on the Block," to answer the following questions.

26. Write the hypothesis of the conditional statement, label it with a p, and determine its truth value.

27. Write the conclusion of the conditional statement, label it with a q, and determine its truth value.
28. Identify the following statement as the converse, inverse, or contrapositive, and determine its truth value: "If Donnie Wahlberg was a member of New Kids on the Block, then *The Masked Singer* is not a music competition."
29. Identify the following statement as the converse, inverse, or contrapositive, and determine its truth value: "If *The Masked Singer* is a music competition, then Donnie Wahlberg was not a member of New Kids on the Block."
30. Which form of the conditional is logically equivalent to the contrapositive, $\sim q \rightarrow \sim p$?

For the following exercises, use the conditional statement: "If all whales are mammals, then no fish are whales," to answer the following questions.

31. Write the hypothesis of the conditional statement, label it with a p, and determine its truth value.
32. Write the conclusion of the conditional statement, label it with a q, and determine its truth value.
33. Identify the following statement as the converse, inverse, or contrapositive, and determine its truth value: "If some fish are whales, then some whales are not mammals."
34. Write the inverse in words and determine its truth value.
35. Write the converse in words and determine its truth value.

For the following exercises, use the conditional statement: "If some parallelograms are rectangles, then some circles are not symmetrical," to answer the following questions.

36. Write the hypothesis of the conditional statement, label it with a p, and determine its truth value.
37. Write the conclusion of the conditional statement, label it with a q, and determine its truth value.
38. Write the converse in words and determine its truth value.
39. Write the contrapositive in words and determine its truth value.
40. Write the inverse in words and determine its truth value.

2.6 De Morgan's Laws

Figure 2.12 De Morgan's Laws were key to the rise of logical mathematical expression and helped serve as a bridge for the invention of the computer. (credit: modification of work "Golden Gate Bridge (San Francisco Bay, California, USA)" by James St. John/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Use De Morgan's Laws to negate conjunctions and disjunctions.
2. Construct the negation of a conditional statement.
3. Use truth tables to evaluate De Morgan's Laws.

The contributions to logic made by Augustus De Morgan and George Boole during the 19th century acted as a bridge to the development of computers, which may be the greatest invention of the 20th century. **Boolean** logic is the basis for computer science and digital electronics, and without it the technological revolution of the late 20th and early 21st centuries—including the creation of computer chips, microprocessors, and the Internet—would not have been possible. Every modern computer language uses Boolean logic statements, which are translated into commands understood by the underlying electronic circuits enabling computers to operate. But how did this logic get its name?

PEOPLE IN MATHEMATICS

George Boole

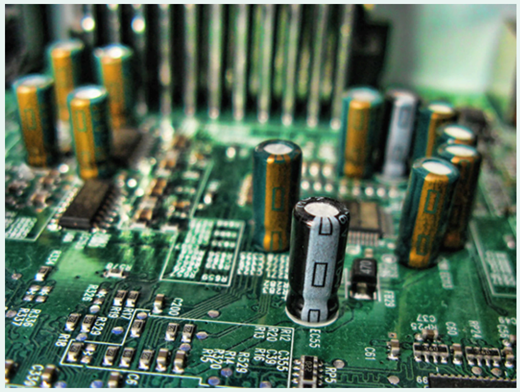


Figure 2.13 Boole's algebra of logic was foundational in the design of digital computer circuits. (credit: "Circuit Board" by Squeezyboy/Flickr, CC BY 2.0)

George Boole was born in Lincolnshire, England in 1815. He was the son of a cobbler who provided him some initial education, but Boole was mostly self-taught. He began teaching at 16 years of age, and opened his own school at the age of 20. In 1849, at the age of 34, he was appointed Professor of Mathematics at Queens College in Cork, Ireland. In 1853, he published the paper, *An Investigation of the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probabilities*, which is the treatise that the field of Boolean algebra and digital circuitry was built on.

Reference: Posamentier, Alfred and Spreitzer Christian, "Chapter 35 George Boole: English (1815-1864)" pp. 279-283, *Math Makers: The Lives and Works of 50 Famous Mathematicians*, Prometheus Books, 2019.

Negation of Conjunctions and Disjunctions

In Chapter 1, Example 1.37 used a Venn diagram to prove **De Morgan's Law** for set complement over union. Because the complement of a set is analogous to negation and union is analogous to an *or* statement, there are equivalent versions of De Morgan's Laws for logic.

FORMULA

De Morgan's Law for negation of a conjunction: $\sim(p \wedge q) \equiv {\sim p} \vee {\sim q}$

De Morgan's Law for the negation of a disjunction: $\sim(p \vee q) \equiv {\sim p} \wedge {\sim q}$

Negation of a conditional: $\sim(p \rightarrow q) \equiv p \wedge {\sim q}$

Writing conditional as a disjunction: $p \rightarrow q \equiv {\sim p} \vee q$

Recall that the symbol for logical equivalence is: $\equiv$.

De Morgan's Laws allow us to write the negation of conjunctions and disjunctions without using the phrase, "It is not the case that ..." to indicate the parentheses. Avoiding this phrase often results in a written or verbal statement that is clearer or easier to understand.

EXAMPLE 2.26

Applying De Morgan's Law for Negation of Conjunctions and Disjunctions

Write the negation of each statement in words without using the phrase, "It is not the case that."

1. Kristin is a biomedical engineer and Thomas is a chemical engineer.
2. A person had cake or they had ice cream.

Solution

1. Kristin is a biomedical engineer and Thomas is a chemical engineer has the form "$p \wedge q$," where p is the statement, "Kristin is a biomedical engineer," and q is the statement, "Thomas is a chemical engineer." By De Morgan's Law, the negation of a conjunction, $\sim(p \wedge q)$, is logically equivalent to $\sim p \vee \sim q$. $\sim p$ is "Kristen is not a biomedical engineer," and $\sim q$ is "Thomas is not a chemical engineer." By De Morgan's Law, the solution has the form "$\sim p \vee \sim q$," so the answer is: "Kristin is not a biomedical engineer or Tom is not a chemical engineer."
2. A person had cake or they had ice cream has the form "$p \vee q$," where p is the statement, "A person had cake," and q is the statement, "A person had ice cream." By De Morgan's Law for the negations of a disjunction, $\sim(p \vee q) \equiv \sim p \wedge \sim q$. The solution is the statement: "A person did not have cake and they did not have ice cream."

YOUR TURN 2.26

Write the negation of each statement in words without using the phrase, *it is not the case that*.

1. Jackie played softball or she ran track.
2. Brandon studied for his certification exam, and he passed his exam.

Negation of a Conditional Statement

The negation of any statement has the opposite truth values of the original statement. The **negation of a conditional**, $\sim(p \to q)$, is the conjunction of p and not q, $p \wedge \sim q$. Consider the truth table below for the negation of the conditional.

p	q	$p \to q$	$\sim(p \to q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

The only time the negation of the conditional statement is true is when p is true, and q is false. This means that $\sim(p \to q)$ is logically equivalent to $p \wedge \sim q$, as the following truth table demonstrates.

p	q	$p \to q$	$\sim(p \to q)$	$\sim q$	$p \wedge \sim q$	$\sim(p \to q) \leftrightarrow (p \wedge \sim q)$
T	T	T	F	F	F	**T**
T	F	F	T	T	T	**T**
F	T	T	F	F	F	**T**
F	F	T	F	T	F	**T**

EXAMPLE 2.27

Constructing the Negation of a Conditional Statement

Write the negation of each conditional statement.

1. If Adele won a Grammy, then she is a singer.
2. If Henrik Lundqvist played professional hockey, then he did not win the Stanley Cup.

Solution

1. The negation of the conditional statement, $p \rightarrow q$, is the statement, $p \wedge {\sim}q$. The hypothesis of the conditional statement is p: "Adele won a Grammy," and conclusion of the conditional statement is q: "Adele is a singer." The negation of the conclusion, ${\sim}q$, is the statement: "She is not a singer." Therefore, the answer is $p \wedge {\sim}q$: "Adele won a Grammy, and she is not a singer."
2. The hypothesis is p: "Henrik Lundqvist played professional hockey," and the conclusion of the conditional statement is q: "He did not win the Stanley Cup." The negation of q is the statement: "He won the Stanley Cup." The negation of the conditional statement is equal to $p \wedge {\sim}q$: "Henrick Lundqvist played professional hockey, and he won the Stanley Cup."

YOUR TURN 2.27

Write the negation of each conditional statement.

1. If Edna Mode makes a new superhero costume, then it will not include a cape.
2. If I had pancakes for breakfast, then I used maple syrup.

EXAMPLE 2.28

Constructing the Negation of a Conditional Statement with Quantifiers

Write the negation of each conditional statement.

1. If all cats purr, then my partner's cat purrs.
2. If a penguin is a bird, then some birds do not fly.

Solution

1. The negation of the conditional statement $p \rightarrow q$ is the statement $p \wedge {\sim}q$. The hypothesis of the conditional statement is p: "All cats purr," and the conclusion of the conditional statement is q: "My partner's cat purrs." The negation of the conclusion, ${\sim}q$, is the statement: "My partner's cat does not purr." Therefore, the answer is $p \wedge {\sim}q$: "All cats purr, but my partner's cat does not purr."
2. The hypothesis is p: "A penguin is a bird," and the conclusion of the conditional statement is q: "Some birds do not fly." The negation of q is the statement: "All birds fly." Therefore, the negation of the conditional statement is equal to $p \wedge {\sim}q$: "A penguin is a bird, and all birds fly."

YOUR TURN 2.28

Write the negation of each conditional statement.

1. If some people like ice cream, then ice cream makers will make a profit.
2. If Raquel cannot play video games, then nobody will play video games.

Many of the properties that hold true for number systems and sets also hold true for logical statements. The following table summarizes some of the most useful properties for analyzing and constructing logical arguments. These properties can be verified using a truth table.

Property	Conjunction (AND)	Disjunction (OR)
Commutative	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$

Property	Conjunction (AND)	Disjunction (OR)
Distributive	$p \wedge (q \wedge r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
De Morgan's	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
Conditional	$\sim(p \rightarrow q) \equiv p \wedge \sim q$	$p \rightarrow q \equiv \sim p \vee q$

EXAMPLE 2.29

Negating a Conditional Statement with a Conjunction or Disjunction

Write the negation of each conditional statement applying De Morgan's Law.

1. If mom needs to buy chips, then Mike had friends over and Bob was hungry.
2. If Juan had pizza or Chris had wings, then dad watched the game.

Solution

1. The conditional has the form "If p then q or r," where p is "Mom needs to buy chips," q is "Mike had friends over," and r is "Bob was hungry." The negation of $p \rightarrow (q \wedge r)$ is $p \wedge \sim(q \wedge r)$. Applying De Morgan's Law to the statement $\sim(q \wedge r)$ the result is $\sim q \vee \sim r$, so our conditional statement becomes $p \wedge (\sim q \vee \sim r)$. By the distributive property for conjunction over disjunction, this statement is equivalent to $(p \wedge \sim q) \vee (p \wedge \sim r)$. Translating the statement $(p \wedge \sim q) \vee (p \wedge \sim r)$ into words, the solution is: "Mom needs to buy chips and Mike did not have friends over, or Mom needs to buy chips and Bob was not hungry."
2. The conditional has the form "If p or q, then r," where p is "Juan had pizza," q is "Chris had wings," and r is "Dad watched the game." The negation of $(p \vee q) \rightarrow r$ is $(p \vee q) \wedge \sim r$. By the distributive property for disjunction over conjuction, the statement is equivalent to $(p \vee \sim r) \wedge (q \vee \sim r)$. Translating the statement $(p \vee \sim r) \wedge (q \vee \sim r)$ into words, the solution is: "Juan had pizza or dad did not watch the game, and Chris had wings or dad did not watch the game."

YOUR TURN 2.29

Write the negation of each conditional statement applying De Morgan's Law.

1. If Eric needs to replace the light bulb, then Marcus left the light on all night or Dan broke the bulb.
2. If Trenton went to school and Regina went to work, then Merika cleaned the house.

Evaluating De Morgan's Laws with Truth Tables

In Chapter 1, you learned that you could prove the validity of De Morgan's Laws using Venn diagrams. Truth tables can also be used to prove that two statements are logically equivalent. If two statements are logically equivalent, you can use the form of the statement that is clearer or more persuasive when constructing a logical argument.

The next example will prove the validity of one of De Morgan's Laws using a truth table. The same procedure can be applied to any two logical statement that you believe are equivalent. If the last column of the truth table is a tautology, then the two statements are logically equivalent.

EXAMPLE 2.30

Verifying De Morgan's Law for Negation of a Conjunction

Construct a truth table to verify De Morgan's Law for the negation of a conjunction, $\sim(p \wedge q) \equiv \sim p \vee \sim q$, is valid.

Solution

Step 1: To verify any logical equivalence, you must first replace the logical equivalence symbol, $\equiv$, with the biconditional symbol, $\leftrightarrow$. The statement $\sim(p \wedge q) \equiv \sim p \vee \sim q$ becomes $\sim(p \wedge q) \leftrightarrow \sim p \vee \sim q$.
Step 2: Next, you create a truth table for the statement. Because we have two basic statements, p, and q, the truth table will have four rows to account for all the possible outcomes. The columns will be p, q, $\sim p$, $\sim q$, $p \wedge q$,$\sim(p \wedge q)$,$\sim p \vee \sim q$, and

the biconditional statement is $\sim(p \wedge q) \leftrightarrow \sim p \vee \sim q$.

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
T	T	T	F	F	F	F	**T**
T	F	F	T	F	T	T	**T**
F	T	F	T	T	F	T	**T**
F	F	F	T	T	T	T	**T**

Step 3: Finally, verify that the statement is valid by confirming it is a tautology. In this instance, the last column is all true. Therefore, the statement is valid and De Morgan's Law for the negation of a conjunction is verified.

YOUR TURN 2.30

1. Construct a truth table to verify De Morgan's Law for the negation of a disjunction, $\sim(p \vee q) \equiv \sim p \wedge \sim q$, is valid.

Check Your Understanding

30. De Morgan's Law for the negation of a conjunction states that $\sim(p \wedge q)$ is logically equivalent to ________________.

31. De Morgan's Law for the negation of a disjunction states that $\sim(p \vee q)$ is logically equivalent to ________________.

32. The negation of the conditional statement, $\sim(p \rightarrow q)$, is logically equivalent to ____________.

33. $\sim(\sim(p \rightarrow q)) \equiv p \rightarrow q$, which means the conditional statement is logically equivalent to $\sim(p \wedge \sim q)$. Apply ______________________ to the statement $\sim(p \wedge \sim q)$ to show that the conditional statement $p \rightarrow q \equiv \sim p \vee q$.

SECTION 2.6 EXERCISES

For the following exercises, use De Morgan's Laws to write each statement without parentheses.

1. $\sim(\sim p \vee q)$
2. $\sim(\sim p \wedge \sim q)$
3. $\sim(p \wedge \sim q)$
4. $\sim(\sim p \vee \sim q)$

For the following exercises, use De Morgan's Laws to write the negation of each statement in words without using the phrase, "It is not the case that, ..."

5. Sergei plays right wing and Patrick plays goalie.
6. Mario is a carpenter, or he is a plumber.
7. Luigi is a plumber, or he is not a video game character.
8. Ralph Macchio was the original Karate Kid, and karate is not for defense only.
9. Some people like broccoli, but my siblings did not like broccoli.
10. Some people do not like chocolate or all people like pizza.

For the following exercises, write each statement as a conjunction or disjunction in symbolic form by applying the property for the negation of a conditional.

11. $\sim(p \rightarrow q)$
12. $\sim(p \rightarrow \sim q)$
13. $\sim(\sim p \rightarrow q)$
14. $\sim(\sim p \rightarrow \sim q)$
15. $\sim(p \wedge q \rightarrow \sim r)$
16. $\sim(p \rightarrow q \vee r)$
17. $\sim(p \vee q \rightarrow \sim r)$

18. $\sim(p \rightarrow q \wedge r)$

For the following exercises, write the negation of each conditional in words by applying the property for the negation of a conditional.

19. If a student scores an 85 on the final exam, then they will receive an A in the class.
20. If a person does not pass their road test, then they will not receive their driver's license.
21. If a student does not do their homework, then they will not play video games.
22. If a commuter misses the bus, then they will not go to work today.
23. If a racecar driver gets pulled over for speeding, then they will not make it to the track on time for the race.
24. If Rene Descartes was a philosopher, then he was not a mathematician.
25. If George Boole invented Boolean algebra and Thomas Edison invented the light bulb, then Pacman is not the best video game ever.
26. If Jonas Salk created the polio vaccine, then his child received the vaccine or his child had polio.
27. If Billie Holiday sang the blues or Cindy Lauper sang about true colors, then John Lennon was not a Beatle.
28. If Percy Jackson is the lightning thief and Artemis Fowl is a detective, then Artemis Fowl will catch Percy Jackson.
29. If all rock stars are men, then Pat Benatar is not a rock star.
30. If Lady Gaga is a rock star, then some rock stars are women.
31. If yellow combined with blue makes green, then all colors are beautiful.
32. If leopards have spots and zebras have stripes, then some animals are not monotone in color.

For the following exercises, construct a truth table to verify that the logical property is valid.

33. $p \rightarrow q \equiv \sim p \vee q$
34. $p \rightarrow \sim q \equiv \sim p \vee \sim q$
35. $\sim p \rightarrow q \equiv p \vee q$
36. $\sim(p \rightarrow \sim q \vee \sim r) \equiv p \wedge q \wedge r$

2.7 Logical Arguments

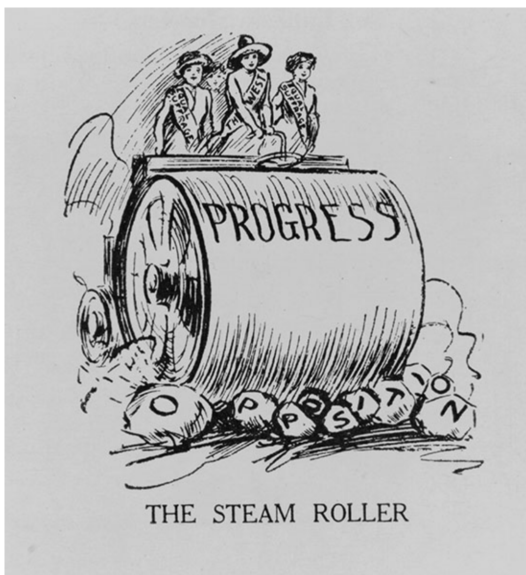


Figure 2.14 Not all logical arguments are valid, and the ongoing fight for equal rights proves that much progress has yet to be made. (credit: "The Steam Roller" by Library of Congress Prints and Photographs Division, public domain)

Learning Objectives

After completing this section, you should be able to:

1. Apply the law of detachment to determine the conclusion of a pair of statements.
2. Apply the law of denying the consequent to determine the conclusion for pairs of statements.
3. Apply the chain rule to determine valid conclusions for pairs of true statements.

The previous sections of this chapter provide the foundational skills for constructing and analyzing logical arguments. All logical arguments include a set of premises that support a claim or conclusion; but not all logical arguments are valid and sound. A logical argument is valid if its conclusion follows from the premises, and it is **sound** if it is valid and all of its premises are true. A false or deceptive argument is called a **fallacy**. Many types of fallacies are so common that they have been named.

WHO KNEW?

In 1936, Dale Carnegie published his first book, titled *How to Win Friends and Influence People*. It was marketed as training materials for the improvement of public speaking and negotiation skills, and the methods it presented are still used today. Carnegie famously said, "When dealing with people, remember you are not dealing with creatures of logic, but creatures of emotion."

People who put forth fallacious logical arguments often take advantage of our susceptibility to emotional appeals, to try to convince us that what they are saying is true. The study of logic helps us combat this weakness through recognition and learning to focus on the facts and structure of the argument.

This section focuses on the two main forms that logical arguments can take. While inductive arguments attempt to draw a more general conclusion from a pattern of specific premises, **deductive arguments** attempt to draw specific conclusions from at least one or more general premises. Deductive arguments can be proven to be valid using Venn diagrams or truth tables.

Inductive arguments generally cannot be proven to be true. They are judged as being strong or weak, but, like any opinion, whether you believe an argument is strong or weak often depends on your knowledge of the topic being discussed along with the evidence being provided in the premises. *Hasty generalization* is the name given to any fallacy that presents a weak inductive argument.

Be careful! Premises may be true or false. If a premise is false, the claims made by the argument should be questioned.

Law of Detachment

The **law of detachment** is a valid form of a conditional argument that asserts that if both the conditional, $p \to q$, and the hypothesis, p, are true, then the conclusion q must also be true. The law of detachment is also called affirming the hypothesis (or antecedent) and modus ponens. Symbolically, it has the form $((p \to q) \wedge p) \to q$.

Law of Detachment	
Premise:	$p \to q$
Premise:	p
Conclusion:	$\therefore q$

The $\therefore$ is read as the word, "therefore."

Looking at the truth table for the conditional statement, the only time the conditional is true is when the hypothesis p is also true. The only place this happens is in the first row, where q is also true, confirming that the law of detachment is a valid argument.

p	q	$p \to q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

Another way to verify that the law of detachment is a valid argument is to construct a truth table for the argument $((p \rightarrow q) \wedge p) \rightarrow q$ and verify that it is a tautology.

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$((p \rightarrow q) \wedge p) \rightarrow q$
T	T	T	T	**T**
T	F	F	F	**T**
F	T	T	F	**T**
F	F	T	F	**T**

Venn diagrams may also be used to verify deductive arguments, which include conditional premises. Consider the statement $p \rightarrow q$: "If you play guitar, then you are a musician." The set of guitarists is a subset of the set of musicians, $p \subset q$. To verify that an argument is valid using a Venn diagram, draw the Venn diagram representing all the premises in the argument only, as shown in Figure 2.15. Then verify if the conclusion is also represented by the Venn diagram of the premises. If it is, the argument is valid. If it is not, the argument is not valid. The set of guitarists is drawn as a subset of the set of musicians to represent the premise, $p \rightarrow q$. The $\times$ represents the premise: p is true. This completes the drawing of the premises.

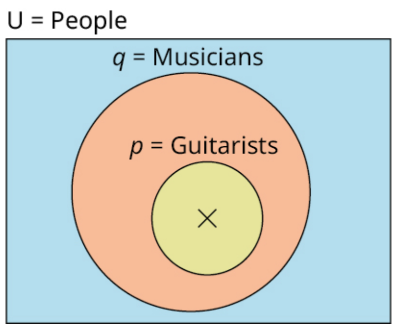


Figure 2.15

Now, examine the Venn diagram to verify if the conclusion is included in the picture. The conclusion is q. Because the $\times$ is in the set p, and p is a subset of q, $\times$ is also in q; therefore, the law of detachment is a valid argument.

Remember that an argument can be valid without being true. For the argument to be proven true, it must be both valid and sound. An argument is sound if all its premises are true.

VIDEO

Logic Part 14: Common Argument Forms like Modus Ponens and Tollens (https://openstax.org/r/Modus_Ponens_and_Tollens)

EXAMPLE 2.31

Applying the Law of Detachment to Determine a Valid Conclusion

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of detachment to determine a valid conclusion.

1. If Leonardo da Vinci was an artist, then he painted the *Mona Lisa*. Leonardo da Vinci was an artist.
2. If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player. Michael Jordan played for the Chicago Bulls.
3. If all fish have gills, then clown fish have gills. All fish have gills.

Solution

1. The premises are $p \rightarrow q$: If Leonardo da Vinci was an artist, then he painted the *Mona Lisa*, and p: Leonardo da

Vinci was an artist. This argument has the form of the law of detachment, so, the conclusion is q: Leonardo da Vinci painted the *Mona Lisa*.

2. The premises follow the form of the law of detachment, so a valid conclusion would be q. The premises are $p \rightarrow q$: If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player, and p: Michael Jordan played for the Chicago Bulls. The conclusion that follows from the premises is q: Michael Jordan was not a soccer player.
3. The premises are $p \rightarrow q$: If all fish have gills, then clown fish have gills, and p: All fish have gills. This argument has the form of the law of detachment, so the conclusion is q: clown fish have gills.

YOUR TURN 2.31

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of detachment to determine a valid conclusion.

1. If my classmate likes history, then some people like history. My classmate likes history.
2. If you do not like to read, then some people do not like reading. You do not like to read.
3. If the polygon has five sides, then it is not an octagon. The polygon has five sides.

Law of Denying the Consequent

Another form of a valid conditional argument is called the **law of denying the consequent**, or modus tollens. Recall, that the conditional statement, $p \rightarrow q$, is logically equivalent to the contrapositive, $\sim q \rightarrow \sim p$. So, if the conditional statement is true, then the contrapositive statement is also true. By the law of detachment, if $\sim q$ is also true, then it follows that $\sim p$ must also be true. Symbolically, it has the form $((p \rightarrow q) \wedge \sim q) \rightarrow \sim p$.

Law of Denying the Consequent	
Premise:	$p \rightarrow q$
Premise:	$\sim q$
Conclusion:	$\therefore \sim p$

The conditional statement can also be described as, "If antecedent, then consequent."This is where the law of denying the consequent gets its name.

To verify if the law of denying the consequent is a valid argument, construct a truth table for the argument, $((p \rightarrow q) \wedge \sim q) \rightarrow \sim p$, and verify that it is a tautology.

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$(p \rightarrow q) \wedge \sim q$	$((p \rightarrow q) \wedge \sim q) \rightarrow \sim p$
T	T	F	F	T	F	**T**
T	F	F	T	F	F	**T**
F	T	T	F	T	F	**T**
F	F	T	T	T	T	**T**

To verify an argument of this form using a Venn diagram, again consider the premise: $p \rightarrow q$: "If you play guitar, then you are a musician." We will change the second premise to $\sim q$. In this case, the × represents the premise, $\sim q$. So, it will be placed inside the universal set of all people, but outside the set of musicians, as depicted in the Venn diagram in Figure

2.16.

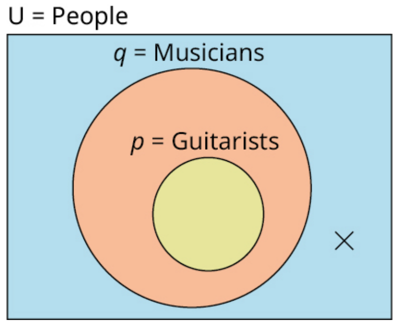


Figure 2.16

Because the × is also outside the set of guitarists, the statement $\sim p$ follows from the premises and the argument is valid.

EXAMPLE 2.32

Applying the Law of Denying the Consequent to Determine a Valid Conclusion

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of denying the consequent to determine a valid conclusion.

1. If Leonardo da Vinci was an artist, then he painted the *Mona Lisa*. Leonardo da Vinci did not paint the *Mona Lisa*.
2. If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player. Michael Jordan was a soccer player.
3. If all fish have gills, then clown fish have gills. Clown fish do not have gills.

Solution

1. The premises are $p \rightarrow q$: If Leonardo da Vinci was an artist, then he painted the *Mona Lisa*, and $\sim q$: Leonardo da Vinci did not paint the *Mona Lisa*. This argument has the form of the law of denying the consequent, so the conclusion is $\sim p$: Leonardo da Vinci was not an artist.
2. The premises follow the form of the law of denying the consequent, so a valid conclusion would be $\sim p$. The premises are: $p \rightarrow q$: If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player, and $\sim q$: Michael Jordan was a soccer player. The conclusion that follows from the premises is $\sim p$: Michael Jordan did not play for the Chicago Bulls.
3. The premises are $p \rightarrow q$: If all fish have gills, then clown fish have gills, and $\sim q$: Clown fish do not have gills. This argument has the form of the law denying the consequent, so the conclusion is $\sim p$: Some fish do not have gills.

YOUR TURN 2.32

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of denying the consequent to determine a valid conclusion.

1. If my classmate likes history, then some people like history. Nobody likes history.
2. If Homer does not like to read, then some people do not like reading. All people like reading.
3. If the polygon has five sides, then it is not an octagon. The polygon is an octagon.

Chain Rule for Conditional Arguments

The **chain rule for conditional arguments** is another form of a valid conditional argument. It is also called hypothetical syllogism or the transitivity of implication. Recall that the conditional statement $p \rightarrow q$ can also be read as p implies q. This is where the name *transitivity of implication* comes from. The transitive property for numbers states that, if $3 < 4$ and $4 < 5$, then it follows that $3 < 5$. The chain rule extends this property to conditional statements. If the premises of the argument consist of two conditional statements, with the form "$p \rightarrow q$" and "$q \rightarrow r$," then it follows that $p \rightarrow r$. Symbolically, it has the form $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$.

Chain Rule for Conditional Arguments	
Premise:	$p \rightarrow q$
Premise:	$q \rightarrow r$
Conclusion:	$\therefore p \rightarrow r$

To verify the chain rule for conditional arguments, construct a truth table for the argument, $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$, and verify that it is a tautology.

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$(p \rightarrow q) \wedge (p \rightarrow r)$	$p \rightarrow r$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	**T**
T	T	F	T	F	F	F	**T**
T	F	T	F	T	F	T	**T**
T	F	F	F	T	F	F	**T**
F	T	T	T	T	T	T	**T**
F	T	F	T	F	F	T	**T**
F	F	T	T	T	T	T	**T**
F	F	F	T	T	T	T	**T**

To verify an argument of this form using a Venn diagram, again consider the premise $p \rightarrow q$: "If you play guitar, then you are a musician," but change the second premise to $q \rightarrow r$: "If you are a musician, then you are an artist." In this case, the set p of guitarists is a subset of the set r of artists, and it follows that if you are a guitarist, then you are an artist. Therefore, the conclusion $p \rightarrow r$ follows from the premises and the chain rule for logical arguments is valid. See Figure 2.17.

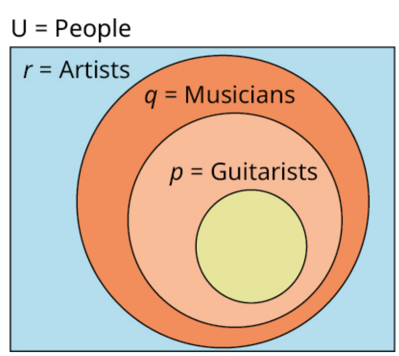


Figure 2.17

EXAMPLE 2.33

Applying the Chain Rule for Conditional Arguments to Determine a Valid and Sound Conclusion

Each pair of statements represents true premises in a logical argument. Based on these premises, apply the chain rule for conditional arguments to determine a valid and sound conclusion.

1. If my roommate goes to work, then my roommate will get paid. If my roommate gets paid, then my roommate will pay their bills.
2. If robins can fly, then some birds can fly. If some birds can fly, then we will watch birds fly.
3. If Irma is a teacher, then Irma has a college degree. If Irma has a college degree, then Irma graduated from college.

Solution

1. The premises are $p \rightarrow q$: "If my roommate goes to work, then they will get paid," and $q \rightarrow r$: "If my roommate gets paid, then my roommate will pay their bills." This argument has the form of the chain rule for conditional arguments, so the valid conclusion will have the form "$p \rightarrow r$." Because all the premises are true, the valid and sound conclusion of this argument is: "If my roommate goes to work, then my roommate will pay their bills."
2. The premises are $p \rightarrow q$: "If robins can fly, then some birds can fly," and $q \rightarrow r$: "If some birds can fly, then we will watch them fly." This argument has the form of the chain rule for conditional arguments, so, the valid conclusion will have the form "$p \rightarrow r$." Because all the premises are true, the valid and sound conclusion of this argument is: "If robins can fly, then we will watch birds fly."
3. The premises are $p \rightarrow q$: (see line 1 of solution 1 and 2 above) "If Irma is a teacher, then Irma has a college degree," and $q \rightarrow r$: "If Irma has a college degree, then Irma graduated from college." This argument has the form of the chain rule for conditional arguments, so the valid conclusion will have the form "$p \rightarrow r$." Because all the premises are true, the valid and sound conclusion of this argument is: "If Irma is a teacher, then Irma graduated from college."

YOUR TURN 2.33

Each pair of statements represent true premises in a logical argument. Based on these premises, apply the chain rule for conditional arguments to determine a valid and sound conclusion.

1. If my roommate does not go to work, then my roommate will not get paid. If my roommate does not get paid, then they will not be able to pay their bills.
2. If penguins cannot fly, then some birds cannot fly. If some birds cannot fly, then we will watch the news.
3. If Marcy goes to the movies, then Marcy will buy popcorn. If Marcy buys popcorn, then she will buy water.

Check Your Understanding

34. A ________________ is a logical statement used as a fact to support the conclusion of an argument.

35. A logical argument is _____________ if its conclusion follows from the premises.

36. A logical argument that attempts to draw a more general conclusion from a pattern of specific premises is called an _____________________ argument.

37. A _____________________ argument draws specific conclusions from more general premises.

38. Not all arguments are true. A false or deceptive argument is called a _________________.

39. If an argument is valid and all of its premises are true, then it is considered ______________.

SECTION 2.7 EXERCISES

For the following exercises, analyze the argument and identify the form of the argument as the law of detachment, the law of denying the consequent, the chain rule for conditional arguments, or none of these.

1. If Apple Inc. releases a new iPhone, then customers will buy it. Customers did not buy a new iPhone. Therefore, Apple Inc. did not release a new iPhone.
2. In the animated movie *Toy Story*, if Paul Newman turned down the role of voicing Woody, then Tom Hanks was chosen for the role. Tom Hanks was chosen as the voice for Woody, therefore, Paul Newman turned down the role of voicing Woody in *Toy Story*.
3. $p \rightarrow q$ and $q \rightarrow r$. $\therefore$ $p \rightarrow r$.
4. $p \rightarrow q$ and p. $\therefore$ q.
5. $p \rightarrow q$ and $\sim q$. $\therefore$ $\sim p$.
6. If all people are created equal, then all people are the same with respect to the law. If all people are the same with respect to the law, then justice is blind. Therefore, if all people are created equal, then justice is blind.

7. If I mow the lawn, then my caregiver will pay me twenty dollars. I mowed the lawn. Therefore, my caregiver paid me twenty dollars.
8. If Robin Williams was a comedian, then some comedians are funny. No comedians are funny. Therefore, Robin Williams was not a comedian.

For the following exercises, each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of detachment to determine and write a valid conclusion.

9. $p \rightarrow \sim q$ and p.
10. $\sim p \rightarrow q$ and $\sim p$.
11. If Richard Harris played Dumbledore, then Daniel Radcliffe played Harry Potter. Richard Harris played Dumbledore.
12. If Emma Watson is an actor, then Emma Watson starred as Belle in the movie *Beauty and the Beast*. Emma Watson is an actor.
13. If some Granny Smiths are available, then we will make an apple pie. Some Granny Smiths are available.
14. If Peter Rabbit lost his coat, then all rabbits must avoid Mr. McGregor's garden. Peter Rabbit lost his coat.

For the following exercises, each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of denying the consequent to determine and write a valid conclusion.

15. If Greg and Ralph are friends, then Greg will not play a prank on Ralph. Greg played a prank on Ralph.
16. If Drogon is not a dragon, then Daenerys ruled Westeros. Daenerys did not rule Westeros.
17. $p \rightarrow \sim q$ and q.
18. $\sim p \rightarrow q$ and $\sim q$.
19. If all dragons breathe fire, then rainwings are not dragons. Rainwings are dragons.
20. If some pirates have parrots as pets, then some parrots do not like crackers. All parrots like crackers.

For the following exercises, each pair of statements represent true premises in a logical argument. Based on these premises, apply the chain rule for conditional arguments to determine a valid and sound conclusion.

21. $\sim p \rightarrow q$ and $q \rightarrow \sim r$.
22. $r \rightarrow \sim q$ and $\sim q \rightarrow \sim p$.
23. $q \rightarrow r$ and $p \rightarrow q$.
24. $\sim r \rightarrow p$ and $q \rightarrow \sim r$.
25. If Mr. Spock is a science officer, then Montgomery Scott is an engineer. If Montgomery Scott is an engineer, then James T. Kirk is the captain.
26. If Prince Charles is a character from *Star Wars*, then Luke Skywalker is not a Jedi. If Luke Skywalker is not a Jedi, then Darth Vader is not his father.

For the following exercises, each pair of statements represent true premises in a logical argument. Based on these premises, state a valid conclusion based on the form of the argument.

27. If my siblings drink milk out of the carton, then they will leave the carton on the counter. My siblings did not leave the carton on the counter.
28. If my friend likes to bowl, then my partner does not like to play softball. My friend likes to bowl.
29. If mathematics is fun, then students will study algebra. If students study algebra, then they will score a 100 on their final exam.
30. If all fleas bite and our dog has fleas, then our dog will scratch a lot. Our dog will not scratch a scratch a lot.
31. If the toddler is not tall, then they will use a stepladder to reach the cookie jar. If the toddler will use a stepladder to reach the cookie jar, then they will drop the jar. If they drop the cookie jar, then they will not eat any cookies.
32. If you do not like to dance, then you will not go to the club. You went to the club.

For the following exercises, use a truth table or construct a Venn diagram to prove whether the following arguments are valid.

33. Denying the hypothesis: $p \rightarrow q$ and $\sim p$. Therefore, $\sim q$.
34. Affirming the consequent: $p \rightarrow q$ and q. Therefore, p.
35. $\sim p \vee q$ and p. Therefore, q.
36. $p \wedge q \rightarrow r$ and $\sim r$. Therefore, $\sim p \vee \sim q$.

Chapter Summary

Key Terms

2.1 Statements and Quantifiers

- logic
- logical statement
- truth values
- symbolic form
- negation of a logical statement
- quantifier
- premises
- conclusion
- inductive logical arguments

2.2 Compound Statements

- compound statement
- connective
- conjunction
- disjunction
- conditional
- hypothesis
- conclusion
- biconditional
- dominance of connectives

2.3 Constructing Truth Tables

- Truth table
- Multiplication principle
- Valid

2.5 Equivalent Statements

- logically equivalent
- tautology
- inverse
- converse
- contrapositive

2.6 De Morgan's Laws

- Boolean logic
- negation of a conditional

2.7 Logical Arguments

- sound
- fallacy
- deductive arguments
- law of detachment
- law of denying the consequent
- chain rule for conditional arguments

Key Concepts

2.1 Statements and Quantifiers

- Logical statements have the form of a complete sentence and make claims that can be identified as true or false.
- Logical statements are represented symbolically using a lowercase letter.
- The negation of a logical statement has the opposite truth value of the original statement.
- Be able to
  - Determine whether a sentence represents a logical statement.

- Write and translate logical statements between words and symbols.
- Negate logical statements, including logical statements containing quantifiers of *all, some, and none*.

2.2 Compound Statements

- Logical connectives are used to form compound logical statements by using words such as *and, or, and if ..., then*.
- A conjunction is a compound logical statement formed by combining two statements with the words “and” or “but.” If the two independent clauses are represented by p and q, respectively, then the conjunction is written symbolically as $p \wedge q$. For the conjunction to be true, both p and q must be true.
- A disjunction joins two logical statements with the *or* connective. In, logic *or* is inclusive. For an *or* statement to be true at least one statement must be true, but both may also be true.
- A conditional statement has the form if p, then q, where p and q are logical statements. The only time the conditional statement is false is when p is true, and q is false.
- The biconditional statement is formed using the connective if and only if for the biconditional statement to be true, the true values of p and q, must match. If p is true then q must be true, if p is false, then q must be false.
- Translate compound statements between words and symbolic form.

Connective	Symbol	Name
and but	∧	conjunction
or	∨	disjunction, inclusive or
not	~	negation
if ..., then implies	→	conditional, implication
if and only if	↔	biconditional

- The dominance of connectives explains the order in which compound logical statements containing multiple connectives should be interpreted.
- The dominance of connectives should be applied in the following order
  - Parentheses
  - Negations
  - Disjunctions/Conjunctions, left to right
  - Conditionals
  - Biconditionals

Dominance	Connective	Symbol	Evaluate
Least Dominant	Parentheses	( )	First
↓	Negation	~	↓
↓	Disjunction/Conjunction	∨, ∧	Left to right or add parentheses to specify order because or/and have equal dominance.
↓	Conditional	→	↓
Most Dominant	Biconditional	↔	Last

Figure 2.18

2.3 Constructing Truth Tables

- Determine the true values of logical statements involving negations, conjunctions, and disjunctions.
  - The negation of a logical statement has the opposite true value of the original statement.
  - A conjunction is true when both p and q are true, otherwise it is false.

- A disjunction is false when both p and q are false, otherwise it is true.

- Know how to construct a truth table involving negations, conjunctions, and disjunctions and apply the dominance of connectives to determine the truth value of a compound logical statement containing, negations, conjunctions, and disjunctions.

Negation			Conjunction (AND)				Disjunction (OR)		
p	$\sim p$		p	q	$p \wedge q$		p	q	$p \vee q$
T	F		T	T	T		T	T	T
F	T		T	F	F		T	F	T
			F	T	F		F	T	T
			F	F	F		F	F	F

- A logical statement is valid if it is always true. Know how to construct a truth table for a compound statement and use it to determine the validity of compound statements involving negations, conjunctions, and disjunctions.

2.4 Truth Tables for the Conditional and Biconditional

- The conditional statement, if p then q, is like a contract. The only time it is false is when the contract has been broken. That is, when p is true, and q is false.

Conditional		
p	q	$p \rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

- The biconditional statement, p if and only if q, it true whenever p and q have matching true values, otherwise it is false.

Biconditional		
p	q	$p \leftrightarrow q$
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**T**

- Know how to construct truth tables involving conditional and biconditional statements.
- Use truth tables to analyze conditional and biconditional statements and determine their validity.

2.5 Equivalent Statements

- Two statements p and q are logically equivalent if the biconditional statement, $p \leftrightarrow q$ is a valid argument. That is, the last column of the truth table consists of only true values. In other words, $p \leftrightarrow q$ is a tautology. Symbolically, p is logically equivalent to q is written as: $p \equiv q$.
- A logical statement is a tautology if it is always true.
- To be valid a local argument must be a tautology. It must always be true.
- Know the variations of the conditional statement, be able to determine their truth values and compose statements with them.

- The converse of a conditional statement, if p then q, is the statement formed by interchanging the hypothesis and conclusion. It is the statement if q then p.
- The inverse of a conditional statement if formed by negating the hypothesis and the conclusion of the conditional statement.
- The contrapositive negates and interchanges the hypothesis and the conclusion.

				Conditional	Contrapositive	Converse	Inverse
p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$	$q \rightarrow p$	$\sim p \rightarrow \sim q$
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

- The conditional statement is logically equivalent to the contrapositive.
- The converse is logically equivalent to the inverse.
- Know how to construct and use truth tables to determine whether statements are logically equivalent.

2.6 De Morgan's Laws

- De Morgan's Law for the negation of a disjunction states that, $\sim(p \vee q)$ is logically equivalent to $\sim p \wedge \sim q$.
- De Morgan's Law The negation of a conjunction states that, $\sim(p \wedge q) \equiv \sim p \vee \sim q$.
- Use De Morgan's Laws to negate conjunctions and disjunctions.
- The negation of a conditional statement, if p then q is logically equivalent to the statement p and not q. Use this property to write the negation of conditional statements.
- Use truth tables to evaluate De Morgan's Laws.

2.7 Logical Arguments

- A logical argument uses a series of facts or premises to justify a conclusion or claim. It is valid if its conclusion follows from the premises, and it is sound if it is valid, and all of its premises are true.
- The law of detachment is a valid form of a conditional argument that asserts that if both the conditional, $p \rightarrow q$ is true and the hypothesis, p is true, then the conclusion q must also be true.

Law of Detachment	
Premise:	$p \rightarrow q$
Premise:	p
Conclusion:	$\therefore q$

- Know how to apply the law of detachment to determine the conclusion of a pair of statements.
- The law of denying the consequent is a valid form of a conditional argument that asserts that if both the conditional, $p \rightarrow q$ is true and the negation of the conclusion, $\sim q$ is true, then the negation of the hypothesis $\sim p$

must also be true.

Law of Denying the Consequent	
Premise:	$p \to q$
Premise:	$\sim q$
Conclusion:	$\therefore \sim p$

- Know how to apply the law of denying the consequent to determine the conclusion for pairs of statements.
- The chain rule for conditional arguments is a valid form of a conditional argument that asserts that if the premises of the argument have the form, $p \to q$ and $q \to r$, then it follows that $p \to r$.

Chain Rule for Conditional Arguments	
Premise:	$p \to q$
Premise:	$q \to r$
Conclusion:	$\therefore p \to r$

- Know how to apply the chain rule to determine valid conclusions for pairs of true statements.

Videos

2.1 Statements and Quantifiers

- Logic Part 1A: Logic Statements and Quantifiers (https://openstax.org/r/Logic_Statements_and_Quantifiers)

2.2 Compound Statements

- Logic Part 1B: Compound Statements, Connectives and Symbols (https://openstax.org/r/Compound_Statements)

2.3 Constructing Truth Tables

- Logic Part 2: Truth Values of Conjunctions: Is an "AND" statement true or false? (https://openstax.org/r/Truth_Values_of_Conjunctions)
- Logic Part 3: Truth Values of Disjunctions: Is an "OR" statement true or false? (https://openstax.org/r/Truth_Values_of_Disjunctions)
- Logic Part 4: Truth Values of Compound Statements with "and", "or", and "not" (https://openstax.org/r/opL9I4tZCC0)
- Logic Part 5: What are truth tables? How do you set them up? (https://openstax.org/r/-tdSRqLGhaw)
- Logic Part 6: More on Truth Tables and Setting Up Rows and Column Headings (https://openstax.org/r/j3kKnUNIt6c)

2.4 Truth Tables for the Conditional and Biconditional

- Logic Part 8: The Conditional and Tautologies (https://openstax.org/r/Conditional_and_Tautologies)
- Logic Part 11B Biconditional and Summary of Truth Value Rules in Logic (https://openstax.org/r/omKzui0Fytk)
- Logic Part 13: Truth Tables to Determine if Argument is Valid or Invalid (https://openstax.org/r/AQB3svnxxiw)

2.7 Logical Arguments

- Logic Part 14: Common Argument Forms like Modus Ponens and Tollens (https://openstax.org/r/Modus_Ponens_and_Tollens)

Projects

Logic Gates

Logic gates are the basis for all digital circuits.

1. Research and document the following terms: logic gate, OR gate, AND gate, and NOT gate.
2. Construct a diagram of a NAND gate, NOR gate, and a XOR gate by using at least two of the following gates: AND, OR, and NOT.

3. Digital electronics use a 1 for true or on, and a 0 for false or off. Create a truth table documenting all possible cases using 0s and 1s for the NAND gate, NOR gate and XOR gate.
4. Use a truth table to explain how XOR is related to the biconditional statement.

Logical Fallacies

Fallacies are false or deceptive logical arguments.

1. Research and document the structure of five of the following named fallacies: hasty generalization, limited choice, false cause, appeal to popularity, appeal to emotion, appeal to authority, personal attack, gamblers' ruin, slippery slope, and circular reasoning.
2. Create a presentation highlighting one of the five fallacies researched in the previous question. The presentation must include an introductory slide with the title of the fallacy and the form or structure of the argument. The second slide must include an example of this fallacy as used in a commercial, a political cartoon or a current event or new article. The third slide must include an explanation of why the example on slide to is a representative example of the fallacy. The last slide must include citations for any materials used. No textbooks should be used as reference.

Careers in Logic

Lawyers, mathematicians, and computer programmers are a few of the careers that require knowledge of logic.

1. What career are you interested in? Research how knowledge of logic applies to your chosen field of study. Then, write a cover letter for a position in your field you'd like to apply to. In the cover letter, include how your knowledge of logic qualifies you for the position you are applying for. If you do not think logic is important for your given career choice, find a position where logic is an essential element of the position and complete the project by pretending you are writing a cover letter for that job.

Chapter Review

Statements and Quantifiers

Fill in the blanks to complete the following sentences.

1. The ____________ of a logical statement has the opposite truth value of the original statement.
2. ______________ are logical statements presented as the facts used to support the conclusion of a logical argument.

Determine whether each of the following sentences represents a logical statement, also called a proposition. If it is a logical statement, determine whether it is true or false.

3. Where is the restroom?
4. No even numbers are odd numbers.
5. $4 + 3 = 8$.

Write the negation of each following statement symbolically and in words.

6. $\sim p$: Pink Floyd's album *The Wall* is not a rock opera.
7. q: Some dogs are Labrador retrievers.
8. $\sim r$: Some universities are not expensive.

Draw a logical conclusion to the following arguments, and include in both one of the following quantifiers: all, some, or none.

9. Spaghetti noodles are made with wheat, ramen noodles are made with wheat, and lo mein noodles are made with wheat.
10. A Porsche Boxster does not have four doors, a Volkswagen Beetle does not have four doors, and a Mazda Miata does not have four doors.

Compound Statements

Fill in the blanks to complete the following sentences.

11. _________________ are words or symbols used to join two or more logical statement together to from a compound statement.
12. ________________ and ________________ have equal dominance and are evaluated from left to right when no parentheses are present in a compound logical statement.

Translate each compound statement below into symbolic form.
Given: p: "Tweety Bird is a bird," q: "Bugs is a bunny," r: "Bugs says, 'What's up, Doc?'," s: "Sylvester is a cat," and t: "Sylvester chases Tweety Bird."

13. If Tweety Bird is a bird, then Sylvester will not chase him.
14. Tweety Bird is a bird and Sylvester chases him if and only if Bugs says, "What's up Doc?"

Translate the symbolic form of each compound logical statement below into words.
Given: p: "Tweety Bird is a bird," q: "Bugs is a bunny," r: "Bugs says, 'What's up, Doc?'," s: "Sylvester is a cat," and t: "Sylvester chases Tweety Bird."

15. $\sim q \vee p \rightarrow \sim s$
16. $\sim(p \wedge \sim s) \leftrightarrow q \rightarrow r$

For each of the following compound logical statements, apply the proper dominance of connectives by adding parentheses to indicate the order to evaluate the statement.

17. $p \vee q \wedge r \rightarrow \ \sim s \wedge t$
18. $\sim p \rightarrow q \vee r \leftrightarrow p \wedge s \rightarrow \sim t$

Constructing Truth Tables

Fill in the blanks to complete the sentences.

19. A ____________ is true if at least one of its component statements is true.
20. For a _________________ to be true, all of its component statements must be true.

Given the statements, p: "No fish are mammals," q: "All lions are cats," and $\sim r$: "Some birds do not lay eggs," construct a truth table to determine the truth value of each compound statement below.

21. $p \wedge \sim r$
22. $\sim(p \vee \sim r)$
23. $\sim p \vee q \wedge \sim(\sim r)$

Construct a truth table to analyze all the possible outcomes of the following statements, and determine whether the statements are valid.

24. $\sim p \vee q \wedge p$

25. $\sim p \vee q \vee \sim q$

Truth Tables for the Conditional and Biconditional

Fill in the blanks to complete the following sentences.

26. If the ________, p, of a conditional statement is true, then the conclusion, q, must also be true for the conditional statement $p \rightarrow q$ to be true.

27. The biconditional statement $p \leftrightarrow q$ is _________ whenever the truth value of p matches the truth value of q, otherwise it is _________.

Complete the truth tables below to determine the truth value of the proposition in the last column.

28.

p	q	r	$p \vee q$	$\sim(p \vee q)$	$\sim r$	$\sim(p \vee q) \rightarrow \sim r$
F	F	T				

29.

p	q	$\sim q$	$p \rightarrow q$	$\sim(p \rightarrow q)$	$p \wedge \sim q$	$\sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q)$
T	F					

Assume the following statements are true. p: “Poof is a baby fairy,” q: “Timmy Turner has fairly odd parents,” r: “Cosmo and Wanda will grant Timmy’s wishes,” and t: “Timmy Turner is 10 years old.” Translate each of the following statements into symbolic form, then determine its truth value.

30. If Timmy Turner is 10 years old and Poof is not a baby fairy, then Timmy Turner has fairly odd parents.

31. Cosmos and Wanda will not grant Timmy’s wishes if and only if Timmy Turner is 10 years old or he does not have fairly odd parents.

32. Construct a truth table to analyze all the possible outcomes and determine the validity of the following argument.
$\sim p \vee q \leftrightarrow \sim q \rightarrow \sim p$

Equivalent Statements

Fill in the blanks to complete the sentences below.

33. The _______________ is logically equivalent to the inverse $\sim p \rightarrow \sim q$.

34. The _______________ is logically equivalent to the conditional $p \rightarrow q$.

Use the conditional statement, $p \rightarrow q$: “If Novak makes the basket, then Novak’s team will win the game," to answer the following questions.

35. Write the conclusion of the conditional statement in words and label it appropriately.

36. Write the hypothesis of the conditional statement in words and label it appropriately.

37. Identify the following statement as the converse, inverse, or contrapositive: “If Novak does not make the basket, then his team will not win the game.”

38. Identify the following statement as the converse, inverse, or contrapositive: “If Novak’s team wins the game, then he made the basket.”

De Morgan’s Laws

Fill in the blanks to complete the sentences.

39. De Morgan’s Law for the negation of a disjunction states that $\sim(p \vee q) \equiv$___________.

40. De Morgan’s Law for the negation of a conjunctions states that ____________ $\equiv \sim p \vee \sim q$.

41. Apply De Morgan’s Law to write the statement without parentheses: $\sim(\sim p \wedge q)$.

42. Apply the property for the negation of a conditional to write the statement as a conjunction or disjunction: $\sim(\sim p \wedge q \rightarrow \sim r)$.

43. Write the negation of the conditional statement in words: If Thomas Edison invented the phonograph, then albums are made of vinyl, or the transistor radio was the first portable music device.

44. Construct a truth table to verify that the logical property is valid: $\sim(\sim p \rightarrow \sim q) \equiv \sim p \wedge q$.

Logical Arguments

Fill in the blanks to complete the sentences below.

45. The _____ ___ __________ is a valid logical argument with premises, $p \rightarrow q$ and p, used to support the conclusion, q.
46. The chain rule for conditional arguments states that the ___________________ property applies to conditional arguments, so that: $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$.

Assume each pair of statements represents true premises in a logical argument. Based on these premises, state a valid conclusion that is consistent with the form of the argument.

47. If the Tampa Bay Buccaneers did not win Super Bowl LV, then Tom Brady was not their quarterback. Tom Brady was the Tampa Bay Buccaneers quarterback.
48. If $\sim q$, then p and if r, then $\sim q$.
49. If Kamala Harris is the vice president of the United States, then Kamala Harris is the president of the U.S. Senate. Kamala Harris is the vice president of the United States.

50. Construct a truth table or Venn diagram to prove whether the following argument is valid. If the argument is valid, determine whether it is sound.
    If all frogs are brown, then Kermit is not a frog. Kermit is a frog. Therefore, some frogs are not brown.

Chapter Test

Determine whether each of the following sentences represent a logical statement. If it is a logical statement, determine whether it is true or false.

1. $1 + 2 - 3 = 0$
2. Please, sit down over there.
3. All mammals lay eggs.

Write the negation of each statement below.

4. Some monkeys do not have tails.
5. $p \wedge \sim q$
6. If the plumber does not remove the clog, then the homeowner will not pay the plumber.

Given: p: Frodo is a hobbit, q: Gandalf is a wizard, r: Frodo and Samwise will take the ring to Mordor, and s: Gollum will help Frodo get into Mordor.
Translate the symbolic form of each compound logical statement into words.

7. $p \vee q \leftrightarrow r$
8. $\sim(\sim r \vee \sim s)$

Translate the written form of each compound logical statement into symbolic form.

9. Frodo and Samwise will take the ring to Mordor or Gandalf is not a wizard and Frodo is a hobbit.
10. If Gollum will not help Frodo get into Mordor, then Gandalf is not a wizard and Frodo is not a hobbit.

For each of the following compound logical statements, apply the proper dominance of connectives by adding parentheses to indicate the order in which the statement must be evaluated.

11. $\sim p \rightarrow q \leftrightarrow r$
12. $\sim(p \wedge q) \leftrightarrow \ \sim p \vee \sim q$

13. Complete the truth table to determine the truth value of the proposition in the last column.

p	q	$p \wedge q$	$\sim q$	$p \vee \sim q$	$(p \wedge q) \leftrightarrow (p \vee \sim q)$
F	T				

Given the true statements p: “A right triangle has one 90-degree angle," q: "The triangle is a right triangle," r: " $a^2 + b^2 = c^2$," and s: "The longest side of a triangle is c implies $a + b$ must be $> c$." Write each of the following compound statements in symbolic form, then construct a truth table to determine the truth value of the compound statement.

14. If a triangle is a right triangle, then it does not have one 90-degree angle or $a^2 + b^2 = c^2$.
15. The triangle is a right triangle, or a right triangle does not have a 90-degree angle, if and only if it is not the case that the longest side of a triangle is c implies $a + b$ must be $> c$.

Use the conditional statement, $p \rightarrow q$: "If Phil Mickelson is 50 years old, then Phil Mickelson won the Player's Championship," to answer the following questions.

16. Write the converse statement in words.

17. If the conditional statement is true, and the hypothesis is true, what is a valid conclusion to the argument?

18. If the conditional statement is true, and the conclusion is false, what is a valid conclusion to the argument?

19. Construct a truth table to analyze all the possible outcomes and determine the validity of the following argument.
$\sim p \vee q \leftrightarrow q \rightarrow p$

20. Construct a truth table or Venn diagram to prove whether the following argument is valid. If the argument is valid, determine whether it is sound.
If John Mayer played *MTV unplugged*, then some guitars are acoustic. John Mayer played *MTV unplugged*.
Therefore, some guitars are acoustic.

3 REAL NUMBER SYSTEMS AND NUMBER THEORY

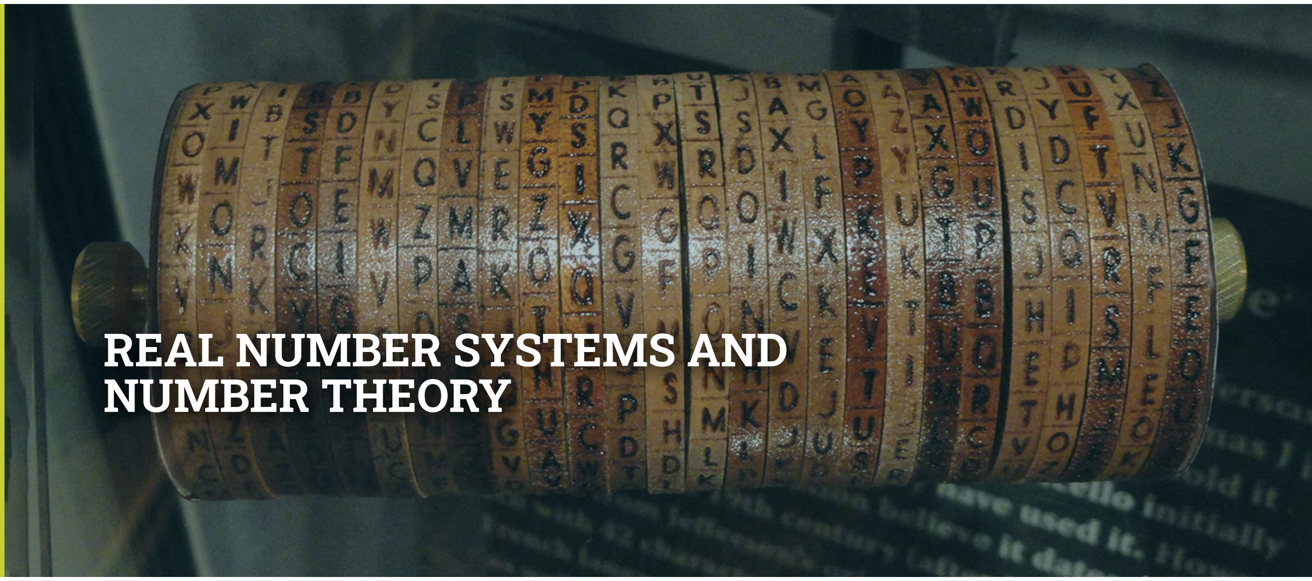


Figure 3.1 Encryption of computers and messages use very large prime numbers. (credit: modification of work "Jefferson cylinder cipher (replica)" by Daderot/Wikimedia Commons, Public Domain)

Chapter Outline

Introduction

Encryption is used to secure online banking, for secure online shopping, and for browsing privately using VPNs (Virtual Private Networks). We need encryption (using prime numbers) for a secure exchange of information. For a prime number to be useful for encryption, though, it has to be large. Encryption uses a composite number that is the product of two very large primes. In order to break the encryption, one must determine the two primes that were used to form the composite number. If the two prime numbers used are sufficiently large, even the fastest computer cannot determine those two prime numbers in a reasonable amount of time. It would take a computer 300 trillion years to crack the current encryption standard.

3.1 Prime and Composite Numbers

Figure 3.2 Computers are protected using encryption based on prime numbers. (credit: "Data Security" by Blogtrepreneur/Flickr, CC BY 2.0)

After completing this section, you should be able to:

1. Apply divisibility rules.
2. Define and identify numbers that are prime or composite.
3. Find the prime factorization of composite numbers.
4. Find the greatest common divisor.
5. Use the greatest common divisor to solve application problems.
6. Find the least common multiple.
7. Use the least common multiple to solve application problems.

Encryption, which is needed for the secure exchange of information (i.e., online banking or shopping) is based on prime numbers. Encryption uses a composite number that is the product of two very large prime numbers. To break the encryption, the two primes that were used to form the composite number need to be determined. If the two prime numbers used are sufficiently large, even the fastest computer cannot determine those two prime numbers in a reasonable amount of time. It would take a computer 300 trillion years to crack the current encryption standard.

Applying Divisibility Rules

Before we begin our investigation of divisibility, we need to know some facts about important sets of numbers:

- The counting numbers are referred to as the **natural numbers**. This set of numbers, $\{1, 2, 3, 4, \ldots\}$, is denoted with the symbol $\mathbb{N}$.
- Another important set of numbers is the **integers**. The integers are the natural numbers, along with 0, and the negatives of the natural numbers. This set is often written as $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. We denote the integers with the symbol $\mathbb{Z}$.
- Notice that $\mathbb{N}$ is a proper subset of $\mathbb{Z}$, or, $\mathbb{N} \subset \mathbb{Z}$. All the ideas of this section apply to the natural numbers, while only some apply to all the integers.

Divisibility is when the integer n is divisible by m, if n can be written as m times another integer. Equivalently, there is no remainder when n is divided by m. There are many occasions when separating items into equal groups comes into play to ensure an equal distribution of whole items. For example, Francis, a preschool art teacher, has 15 students in one class. Francis has 225 sheets of construction paper and wants to provide each student with an equal number of pieces. To know if he will use all the construction paper, Francis is really asking if 225 can be evenly divided into 15 groups.

EXAMPLE 3.1

Determining if a Number Divides Another Number

Determine if 36 is divisible by 4.

Solution

We could divide 36 by 4 and see if there is a remainder, or we could see if we can write 36 as 4 times another integer. If we divide 36 by 4, we see $36 \div 4 = 9$ with no remainder. We see that 36 is divisible by 4. We can write 36 as 4 times another integer, $36 = 4 \times 9$. By the definition of divisibility, 36 is divisible by 4.

YOUR TURN 3.1

1. Determine if 54 is divisible by 9.

You can quickly check if a number is divisible by 2, 3, 4, 5, 6, 9, 10, and 12. Each has an easy-to-identify feature, or rule, that indicates the divisibility by those numbers, as shown in the following table.

Divisor	Rule
2	Last digit is even
3	Add the digits of the number together. If that sum is divisible by 3, then so is the original number
4	Look at only the last two digits. If this number is divisible by 4, so is the original number
5	Look at only the last digit. If it is a 5 or a 0, then the original number is divisible by 5
6	If the number passes the rule for divisibility by 2 and for 3, then the number is divisible by 6
9	Add the digits of the number together. If that number is divisible by 9, then so is the original number
10	Look at only the last digit. If it is a 0, then the original number is divisible by 10
12	If the number passes the rule for 3 and 4, the number is divisible by 12

EXAMPLE 3.2

Using Divisibility Rules

Using divisibility rules, determine if 245 is divisible by 5.

Solution

Since the last digit is a 5, the number 245 is divisible by 5 because the rule states if the last digit of the number is a 5 or a 0, then the original number is divisible by 5.

YOUR TURN 3.2

1. Using divisibility rules, determine if 45,730 is divisible by 5.

EXAMPLE 3.3

Using Divisibility Rules
Using divisibility rules, determine if 25,983 is divisible by 9.

Solution
The divisibility rule for 9 is when the digits of the number are added, the sum is divisible by 9. So, we calculate the sum of the digits.

$2 + 5 + 9 + 8 + 3 = 27$. Since 27 is divisible by 9, so is the original number 25,983.

YOUR TURN 3.3

1. Using divisibility rules, determine if 342,887 is divisible by 9.

EXAMPLE 3.4

Using Divisibility Rules
Can 298 coins be stacked into 6 stacks with an equal number of coins in each stack?

Solution
In order for the coins to be in equal-sized stacks, 298 would need to be divisible by 6. The divisibility rule for 6 is that the number passes the divisibility rules for both 2 and 3. Since the last digit is even, 298 is divisible by 2. To determine if 298 is divisible by 3, we first add the digits of the number: $2 + 9 + 8 = 19$. Since 19 is not divisible by 3, neither is 298. Because 298 is not divisible by 3, it is also not divisible by 6, which means they cannot be put into 6 equal stacks of coins.

YOUR TURN 3.4

1. Can 43,568 pieces of mail be separated into 6 bins with the same number of pieces of mail per bin?

EXAMPLE 3.5

Using Divisibility Rules
Using divisibility rules, determine if 4,259 is divisible by 10.

Solution
The divisibility rule for 10 is that the last digit of the number is 0. Since the last digit of 4,259 is not 0, then 4,259 is not divisible by 10.

YOUR TURN 3.5

1. Using divisibility rules, determine if 87,762 is divisible by 10.

EXAMPLE 3.6

Using Divisibility Rules
Using divisibility rules, determine if 936,276 is divisible by 4.

Solution
The divisibility rule for 4 is to check the last two digits of the number. If the number formed by the last two digits of the original number is divisible by 4, then so is the original number. The last two digits make the number 76 and 76 is

divisible by 4, since $76 = 4 \times 19$. Since 76 is divisible by 4, so is 936,276.

YOUR TURN 3.6

1. Using divisibility rules, determine if 43,568 is divisible by 4.

VIDEO

Divisibility Rules (https://openstax.org/r/Divisibility_Rules)

Prime and Composite Numbers

Sometimes, a natural number has only two unique divisors, 1 and itself. For instance, 7 and 19 are **prime**. In other words, there is no way to divide a prime number into groups with an equal number of things, unless there is only one group, or those groups have one item per group. Other natural numbers have more than two unique divisors, such as 4, or 26. These numbers are called **composite**. The number 1 is special; it is neither prime nor composite.

To determine if a number is prime or composite, you have to determine if the number has any divisors other than 1 and itself. The divisibility rules are useful here, and can quickly show you if a number has a divisor on that list.

However, if none of those divide the number, you still have to check all other possible prime divisors. What are the prime numbers that are possibly divisors of the number you are checking? You need only check the prime numbers up to the square root of the number in question. For instance, if you want to know if 2,117 is prime, you need to determine if any primes up to the square root of 2,117 (which is 46.0 when rounded to one decimal place) divide 2,117. If any of those primes are divisors of the number in question, then the number is composite. If none of those primes work, then the number is itself prime.

We can check divisibility with whatever tool we wish. Divisibility rules are quick for some prime divisors (2 and 5 come to mind) but aren't quick for other values (like 11). In place of divisibility rules, we could just use a calculator. If the prime number divides the number evenly (that is, there is no decimal or fractional part), then the number is divisible by that prime. Table 3.1 is a quick list of the prime numbers up to 50. There are 15 prime numbers less than 50.

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47

Table 3.1 Prime Numbers Less than 50

EXAMPLE 3.7

Determining If a Number Is Prime or Composite

Determine if 2,117 is prime or composite.

Solution

The square root of 2,117 is 46.0 (rounded to one decimal place). So, we need to check if 2,117 is divisible by any prime up to 46.

Step 1: First we'll use the rules of divisibility we learned earlier:

- We can tell 2,117 is not divisible by 2, as the last digit isn't even.
- 2,117 is not divisible by 5 (the last digit isn't 0 or 5).
- Add the digits of 2,117 to get 11, which is not divisible by 3. So, 2,117 is also not divisible by 3.

Step 2: Now we repeat the process for all the primes up to 46.

Using a calculator, we find that 2,117 divided by the prime numbers 7, 11, 13, 17, 19, and 23 results in a remainder, a decimal part. So, we know that 2,117 is not divisible by these prime numbers. (You should check these results yourself.)

Moving on, we check the next prime: 29. Using the calculator to divide 2,117 by 29 results in 73. Since there is no decimal part, 2,117 is divisible by 29.

This means that 2,117 is not a prime number, but rather, a composite number. Writing 2,117 as the product of 29 and another natural number, $2{,}117 = 29 \times 73$.

YOUR TURN 3.7

1. Determine if 1,429 is prime or composite.

EXAMPLE 3.8

Determining if a Number Is Prime or Composite
Determine if 423 is prime or composite.

Solution
The square root of 423 is 20.57 (rounded to two decimal places). So, we need to check if 423 is divisible by any prime up to 20.

Step 1: Check 2. We can tell 423 is not divisible by 2, as the last digit isn't even.

Step 2: Check 5. It is not divisible by 5 (the last digit isn't 0 or 5).

Step 3: Check 3. To check if 423 is divisible by 3, we use the divisibility rule for 3. When we take the sum of the digits of 423, the result is 9. Since 9 is divisible by 3, so is 423.

Since 423 is divisible by 3, then 423 is a composite number. Writing 423 as the product of 3 and another natural number, $423 = 3 \times 141$.

YOUR TURN 3.8

1. Determine if 859 is prime or composite.

EXAMPLE 3.9

Determining if a Number Is Prime or Composite
Determine if 1,034 is prime or composite

Solution
A quick inspection of 1,034 shows it is divisible by 2 since the last digit is even, and so 1,034 is a composite number.

YOUR TURN 3.9

1. Determine if 5,067,322 is prime or composite.

EXAMPLE 3.10

Determining if a Number Is Prime or Composite
Determine if 2,917 is prime or composite.

Solution

The square root of 2,917 is 50.01 (rounded to two decimal places). So, we need to check if 2,917 is divisible by any prime up to 50.

Step 1: Check 2. We can tell 2,917 is not divisible by 2, as the last digit isn't even.

Step 2: Check 5. It is it divisible by 5 (the last digit isn't 0 or 5).

Step 3: Check 3. Using the divisibility rule for 3, we take the sum of the digits of 2,917, which is 19. Since 19 is not divisible by 3, neither is 2,917.

Step 4: Check the rest of the primes up to 50 using a calculator. When 2,917 is divided by every prime number up to 50, the result has a decimal part.

Since no prime up to 50 divides 2,917, it is a prime number.

YOUR TURN 3.10

1. Determine if 1,477 is prime.

WHO KNEW?

ILLEGAL PRIMES

Large primes are a hot commodity. Using two very large primes (some have more than 22 million digits!) is necessary for secure encryption. Anyone who has a new prime that is large enough can use that prime to create a new encryption. Of course, whoever discovers a large prime could sell it to a security company. These primes are so useful for encryption, it is necessary to protect that intellectual property. In fact, at least one prime number was declared illegal.

VIDEO

Illegal Prime Number (https://openstax.org/r/Illegal_Prime_Number)

PEOPLE IN MATHEMATICS

Sophie Germain

Figure 3.3 Sophie Germain (credit: “Sophie Germain at 14 years,” Illustration from histoire du socialisme, approx. 1880, Wikimedia Commons, public domain)

Born into a wealthy French family in 1776, Sophie Germain discovered and fell in love with mathematics by browsing her father’s books. Clandestine study, hard and tenacious work, and a mathematical mindset did not lead to college, however, as she was not allowed to attend. She did manage, through friends, to obtain problem sets and submit brilliant solutions under the name Monsieur LaBlanc. One of her great interests was number theory, which is the study of properties of integers. One of her theorems, titled “Sophie Germain’s Theorem,” partially solved one of the great mathematical mysteries, Fermat’s Last Theorem. From this she discovered what are now known as Sophie Germain Primes. A Sophie Germain Prime is a prime number that can be written in the form $2p + 1$, where p is a prime number. For instance, 23 is prime: $2(23) + 1 = 47$, which is prime, so 47 is a Sophie Germain Prime. It should be noted that $2p + 1$, where p is a prime number, may or may not be prime (check for $p = 7$!).

Finding the Prime Factorization of Composite Numbers

Before we can start with prime factorization, let’s remind ourselves what it means to factor a number. We factor a number by identifying two (or more) numbers that, when multiplied, result in the original number. For instance, 3 and 24, when multiplied, give 72. So, 72 can be factored into 3×24. Notice that we could have factored the 72 differently, say as $72 = 6 \times 12$, or $72 = 2 \times 36$, or even as $72 = 3 \times 4 \times 6$.

Finding the **prime factorization** of a composite number means writing the number as the product of all of its prime factors. For example, $80 = 2 \times 2 \times 2 \times 2 \times 5$. Notice that all the numbers being multiplied on the right-hand side are prime numbers. Sometimes prime numbers repeat themselves in the factorization. When prime factors do repeat, we may write the prime factorization using **exponents**, as in $80 = 2^4 \times 5$. In that equation, the 2 is raised to the 4th power. The 4 is the exponent, and the 2 is the **base**. More generally, in the exponent notation a^b, the number represented by a is the base, and the number represented by b is the exponent.

One has to wonder if finding the prime factorization could result in different factorizations. The **Fundamental Theorem of Arithmetic** tells us that there is only one prime factorization for a given natural number.

Fundamental Theorem of Arithmetic

Every natural number, other than 1, can be expressed in exactly one way, apart from the arrangement, as a product of primes.

The process of finding the prime factorization of a number is iterative, which means we do a step, then repeat it until we cannot do the step any longer. The step we use is to identify one prime factor of the number, then write the number as the prime factor times another factor. We repeat this step on the other, newly found, factor. This step is repeated until no more primes can be factored from the remaining factor. This is easier to see and explain with an example.

EXAMPLE 3.11

Finding the Prime Factorization

Find the prime factorization of 140.

Solution

Step 1: Identify a prime number that divides 140. Since 140 is even (the last digit is even), 2 divides 140. We then factor the 2 out of the 140, giving us $140 = 2 \times 70$.

Step 2: With the other factor, 70, find a prime factor of 70. Since 70 is also even, 2 divides 70. We factor the 2 out of the 70 and the factorization is now $140 = 2 \times 2 \times 35 = 2^2 \times 35$.

Notice that we expressed the two factors of 2 as 2^2.

Step 3: Look to the remaining factor, 35. The last digit of 35 is 5, so 5 is a factor of 35. We factor the 5 out of the 35. The factorization is now $140 = 2^2 \times 5 \times 7$.

Step 4: Look to the remaining factor, 7. Since 7 is prime, the process is complete.

The prime factorization of 140 is $2^2 \times 5 \times 7$.

YOUR TURN 3.11

1. Find the prime factorization of 90.

Factor Trees

A useful tool for helping with prime factorization is a **factor tree**. To create a factor tree for the natural number n (where n is not 1), perform the following steps:

Step 1: If n is prime, you're done. If n is composite, continue to the next step.

Step 2: Identify two divisors of n, call them a and b.

Step 3: Write the number n down, and draw two branches extending down (or to the right) of the number n.

Step 4: Label the end of one branch a, the other as b. See Figure 3.4.

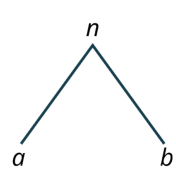


Figure 3.4

Step 5: If a and b are prime, the tree is complete. When a number at this step is a prime number, we refer to it as a leaf of the tree diagram.

Step 6: If either a or b are composite, repeat Steps 2 through 4 for a and b.

Step 7: The process stops when the leaves are all prime.

Step 8: The prime factorization is then the product of all the leaves.

This is best seen in an example.

EXAMPLE 3.12

Finding the Prime Factorization

Find the prime factorization of 66.

Solution

Since 66 is even, 2 is a factor.

Step 1: Factor out the 2. The factorization is $66 = 2 \times 33$. The factor tree is shown in Figure 3.5.

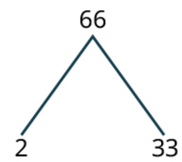


Figure 3.5

Since 2 is a factor, that branch is done, and 2 is a leaf.

Step 2: The 33, though, is divisible by 3, and is the product of 3 and 11. We attach that to the factor tree (Figure 3.6).

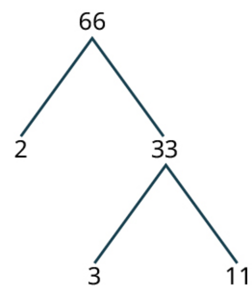


Figure 3.6

Since the 2, 3 and 11 are all prime, the factor tree is done.

The prime factorization of 66 is the product of the leaves, so $66 = 2 \times 3 \times 11$. The factorization is complete.

YOUR TURN 3.12

1. Find the prime factorization of 85.

VIDEO

Using a Factor Tree to Find the Prime Factorization (https://openstax.org/r/Prime_Factorization)

EXAMPLE 3.13

Finding the Prime Factorization

Find the prime factorization of 135.

Solution

The number 135 is divisible by 3, and so 3 is a factor of 135.

Step 1: Factor out the 3. The factorization is $135 = 3 \times 45$. Using the factor tree (Figure 3.7),

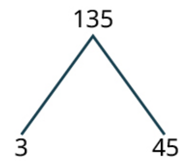


Figure 3.7

45 is also divisible by 3.

Step 2: Factor out a 3 from 45. The other factor is 15. The factor tree is shown in Figure 3.8.

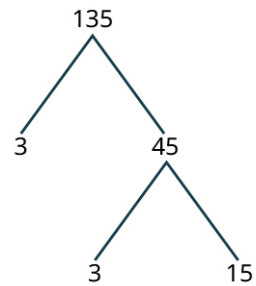


Figure 3.8

15 is also divisible by 3.

Step 3: The factors of 15 are 3 and 5. The factor tree is shown in Figure 3.9.

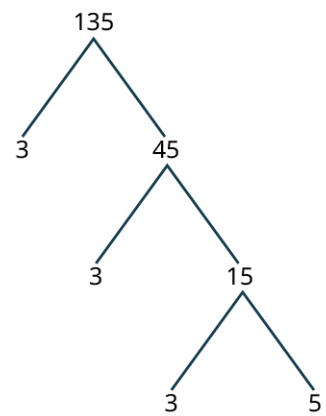


Figure 3.9

All the leaves are prime, so the process is complete. The prime factorization of 135 is $3^3 \times 5$.

YOUR TURN 3.13

1. Find the prime factorization of 280.

EXAMPLE 3.14

Identifying Prime Factors

How many different prime factors does 10,241 have?

Solution

To know how many different prime factors 10,241 has, we need the prime factorization of 10,241.

Step 1: Use divisibility rules, to see that the number 10,241 is not divisible by 2 or by 3 (the sum of the digits is 8), or by 5. However, it is divisible by 7.

Step 2: After factoring the 7, the factorization is $10,241 = 7 \times 1463$; 1,463 is also divisible by 7.

Step 3: After factoring the 7, the factorization is $10,241 = 7 \times 7 \times 209 = 7^2 \times 209$. The number 209 is not divisible by 7.

Step 4: Check the next prime number: 11; 11 does divide 1,463.

Step 5: After factoring the 11, the factorization is $10,241 = 7^2 \times 11 \times 19$.

Since 19 is prime, the prime factorization of 10,241 is complete. We see that 10,241 has three different prime factors: 7, 11, and 19.

YOUR TURN 3.14

1. Find the number of different prime factors of 180.

VIDEO

Finding the Prime Factorization of 168 (https://openstax.org/r/Prime_Factorization_of_168)

TECH CHECK

Using Wolfram Alpha to Find Prime Factorizations

The Wolfram Alpha website (https://openstax.org/r/wolframalpha) is a powerful resource available for free to use. It is designed using AI so that it understands natural language requests. For instance, typing the question "What is the prime factorization of 543,390?" gets a rather quick answer of $2 \times 3 \times 5 \times 59 \times 307$. So, if you want to find the prime factorization of a number, you can simply ask Wolfram Alpha to find the prime factorization of the number.

Finding the Greatest Common Divisor

Two numbers often have more than one divisor in common (all pairs of natural numbers have the common divisor 1). When listing the common divisors, it's often the case that the largest is of interest. This divisor is called the **greatest common divisor** and is denoted **GCD**. It is also sometimes referred to as the greatest common factor (GCF).

For instance, 6 is the greatest common divisor of 12 and 18. We can see this by listing all the divisors of each number and, by inspection, select the largest value that shows up in each list.

The divisors of 12	1, 2, 3, 4, 6, 12
The divisors of 18	1, 2, 3, 6, 9

It is easy to see that 6 is the largest value that appears in both lists.

EXAMPLE 3.15

Finding the Greatest Common Divisor Using Lists

Find the greatest common divisor of 1,400 and 250 by listing all divisors of each number.

Solution

We create a list of all the divisors of 1,400 and of 250, and choose the largest one.

The divisors of 1,400 are

1, 2, 4, 5, 7, 8, 10, 14, 20, 25, 28, 35, 40, 50, 56, 70, 100, 140, 175, 200, 280, 350, 700, 1,400.

The divisors of 250 are

1, 2, 5, 10, 25, 50, 125, 250.

The largest value that appears on both lists is 50, so the greatest common divisor of 1,400 and 250 is 50.

YOUR TURN 3.15

1. Find the greatest common divisor of 270 and 99 by listing all divisors of each number.

Listing all the divisors of the numbers in the set will always work, but for some relatively small numbers, the set of all divisors can become pretty big, and finding them all can be a chore. Another approach to finding the greatest common divisor is to use the prime factorization of the numbers. To do so, use the following steps:

Step 1: Find the prime factorization of the numbers.

Step 2: Identify the prime factors that appear in every number's prime factorization. These are called the common prime factors.

Step 3: Identify the smallest exponent of each prime factor identified in Step 2 in the prime factorizations.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3. The result is the greatest common divisor.

EXAMPLE 3.16

Finding the Greatest Common Divisor Using Prime Factorization
Find the greatest common divisor of 1,400 and 250 by using their prime factorizations.

Solution
Step 1: Find the prime factorizations of the numbers.

The prime factorization of 1,400 is $2^3 \times 5^2 \times 7$.

The prime factorization of 250 is $250 = 2 \times 5^3$.

Step 2: Identify the prime factors that appear in every number's prime factorization.

The common prime factors are 2 and 5.

Step 3: Identify the smallest exponent of each prime identified in Step 2 in the prime factorizations.

The exponent of common prime factor 2 in the prime factorization of 1,400 is 3, and in the prime factorization of 250 is 1. The smallest of those exponents is 1.

The exponent of the common prime factor 5 in the prime factorization of 1,400 is 2 and is in the prime factorization of 250 is 3. The smallest of these exponents is 2.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

This gives $2^1 \times 5^2 = 50$. The greatest common divisor of 1,400 and 250 is $2 \times 5^2 = 50$.

Notice that the answer matches the one we found in Example 3.15.

YOUR TURN 3.16

1. Using prime factorization, determine the greatest common divisor of 36 and 128.

EXAMPLE 3.17

Finding the Greatest Common Divisor Using Prime Factorization
Find the greatest common divisor of 600 and 784 by using their prime factorizations.

Solution

Step 1: Find the prime factorizations of the numbers.

The prime factorization of 600 is $2^3 \times 3 \times 5^2$.

The prime factorization of 784 is $2^4 \times 7^2$.

Step 2: Identify the prime factors that appear in every number's prime factorization.

There is only one common prime factor, 2.

Step 3: Identify the smallest exponent of each prime from identified in Step 2 in the prime factorizations.

The exponent of 2 in the prime factorization of 600 is 3. The exponent of 2 in the prime factorization of 784 is 4. So, the smallest exponent of 2 is 3.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

This gives $2^3 = 8$. The greatest common divisor of 600 and 784 is 8.

YOUR TURN 3.17

1. Using prime factorization, determine the greatest common divisor of 120 and 200.

PEOPLE IN MATHEMATICS

Srinivasa Ramanujan

Figure 3.10 Srinivasa Ramanujan (credit: Srinivasa Ramanujan, photo by Konrad Jacobs/Oberwolfach Photo Collection/public domain)

Ramanujan was born in southern India in 1887, during British rule. He was a self-taught mathematician, who, while in high school, began working through a two-volume text of mathematical theorems and results. His work included examination of the distribution of primes. He eventually came to the attention of British mathematician, G.H. Hardy. During one visit, Hardy mentioned to Ramanujan that his taxicab number was 1,729, remarking that 1,729 appeared to be a rather dull number. To which Ramanujan responded, "It is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."