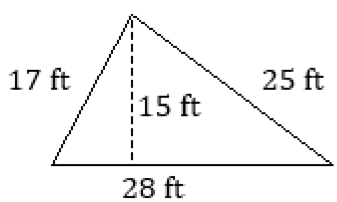
Area of a Triangle

$A = \frac{1}{2}bh$ or $A = bh \div 2$

As with a parallelogram, remember that the height must be perpendicular to the base.

Exercises

Find the area of each triangle.



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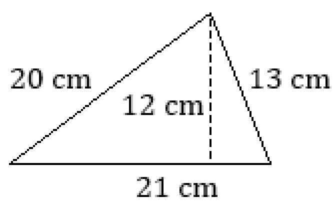

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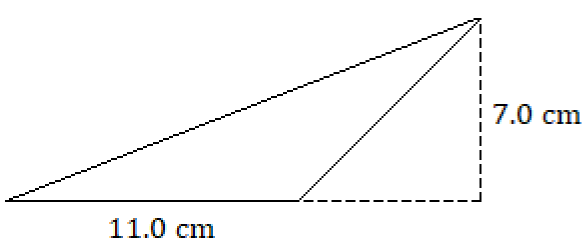

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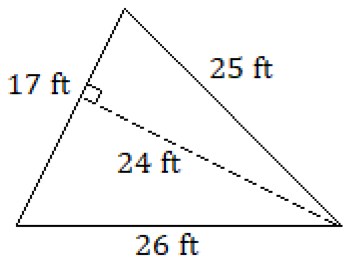
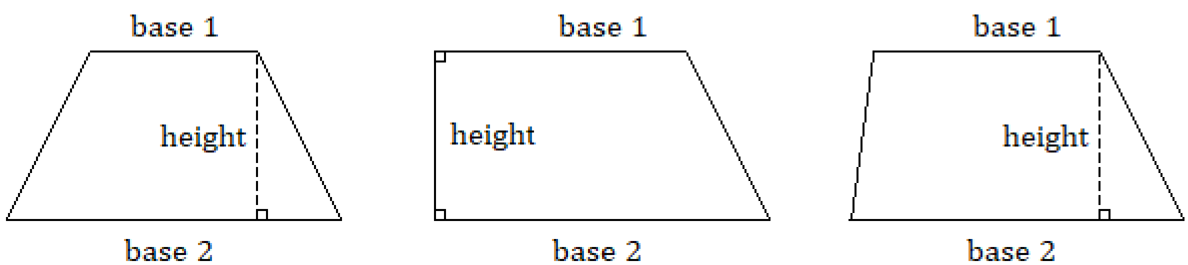
10. 

Area: Trapezoids

A somewhat less common quadrilateral is the ***trapezoid***, which has exactly one pair of parallel sides, which we call the bases. The first example shown below is called an isosceles trapezoid because, like an isosceles triangle, its two nonparallel sides have equal lengths.



There are a number of ways to show where the area formula comes from, but the explanations are better in video because they can be animated.[234]

Area of a Trapezoid

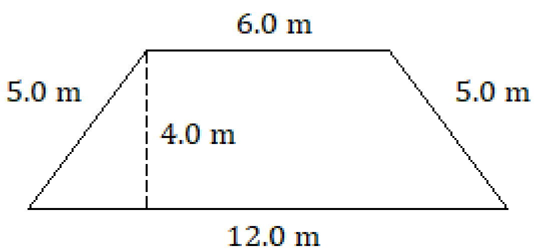
$A = \frac{1}{2}h(b_1 + b_2)$ or $A = (b_1 + b_2)h \div 2$

Don't be intimidated by the subscripts on b_1 and b_2; it's just a way to name two different measurements using the same letter for the variable. (Many people call the bases a and b instead; feel free to write it whichever way you prefer.) Whatever you call them, you just add the two bases, multiply by the height, and take half of that.

Exercises

11. The soaking tubs at Hot Lake Springs in La Grande, Oregon have benches formed by seven trapezoids. Each bench has an outer edge 33 inches long, an inner edge 23 inches long, and the edges are 11 inches apart. How much seating area do the benches provide? (Ignore the gaps between the boards.)

Find the area of each trapezoid.



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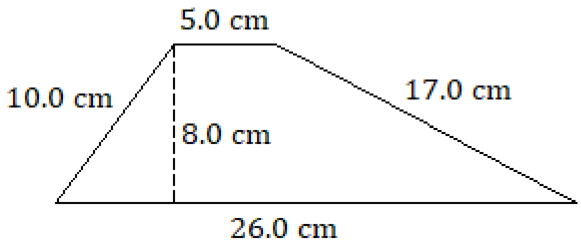

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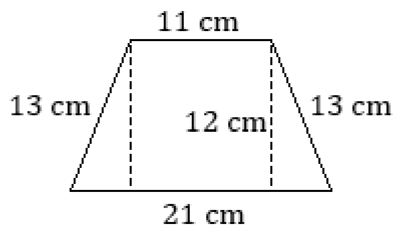
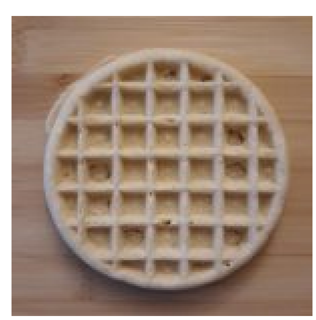


14.

Rocky the cat lounges at Hot Lake Springs.

Area: Circles



The area of a circle is π times the square of the radius: $A = \pi r^2$. The units are still square units, even though a circle is round. (Think of the squares on a round waffle.) Because we can't fit a whole number of squares—or an exact fraction of squares—inside the circle, the area of a circle will be an approximation.

If your calculator doesn't have a π key, use the approximation $\pi \approx 3.1416$.

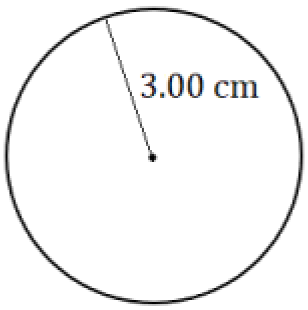
Area of a Circle

$A = \pi r^2$

Exercises

Find the area of each circle. Round each answer to the appropriate number of significant figures.

15.



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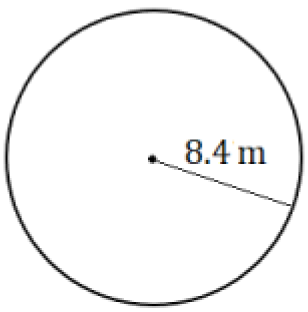

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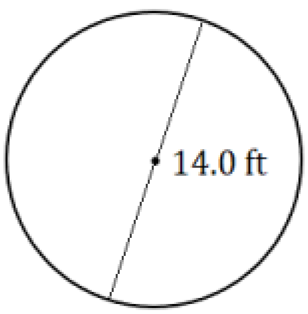
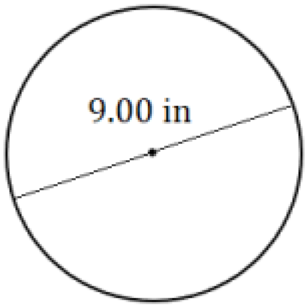


18. 

Exercise Answers

Notes

1. You might choose to use capital letters for the variables here because a lowercase letter "l" can easily be mistaken for a number "1".
2. https://youtu.be/yTnYRpcZA9c
3. https://youtu.be/WZtO3oERges
4. https://youtu.be/uLHc6Br2veg

[20]

Composite Figures

You may use a calculator in this module as needed.

Many objects have odd shapes made up of simpler shapes. A ***composite figure*** is a geometric figure which is formed by—or *composed* of—two or more basic geometric figures.

We will look at a handful of fairly simple examples, but this concept can of course be extended to much more complicated figures.

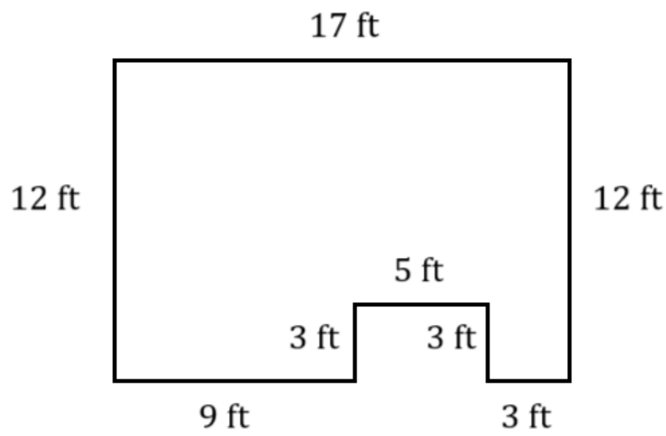
This sunburst design is composed of 276 tiles.

Composite Figures with Polygons

To find the area of a composite figure, it is generally a good idea to divide it into simpler shapes—rectangles, triangles, etc.—and either add or subtract their areas as necessary. You may need to figure out some unmarked dimensions during the process.

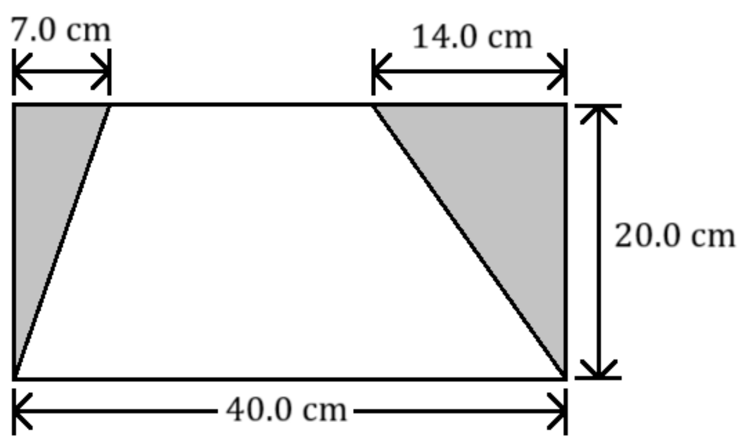
Exercises

A floor plan of a room is shown. The room is a 12-foot by 17-foot rectangle, with a 3 -foot by 5-foot rectangle cut out of the south side.



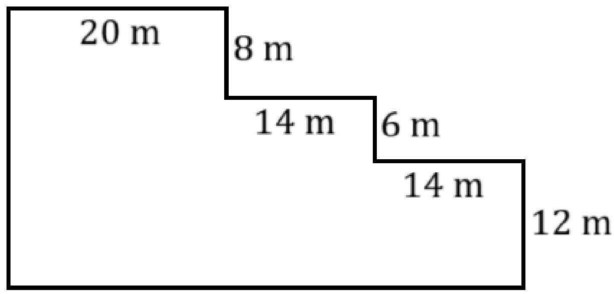
1. Determine the amount of molding required for the perimeter of the room.
2. Determine the amount of flooring required to cover the entire area.

A trapezoid is formed by removing two right triangles from a rectangle, as shown.



3. Determine the area of the trapezoid *without* using the trapezoid formula from Module 19.
4. Determine the area of the trapezoid using the trapezoid formula from Module 19. Does this agree with your previous answer?

A plan for an irregular parking lot is shown.



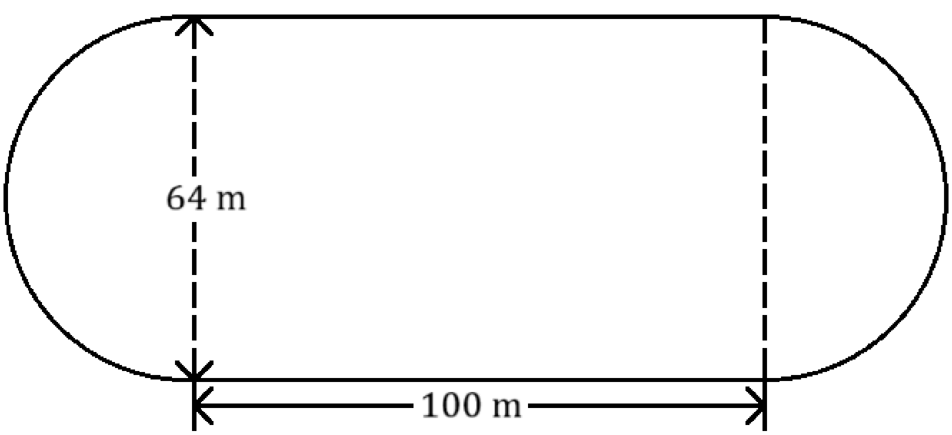
5. Calculate the perimeter.
6. Calculate the area.

Composite Figures with Circles

If a composite figure includes parts of circles, you'll need to approximate you answer because your calculation will have π in it.

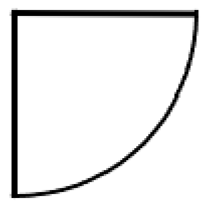
Exercises

A high school is building a track at its athletic fields. The track, which is formed by two straight sides and two semicircles as shown in the plans below, is supposed to have a total length of 400 meters.



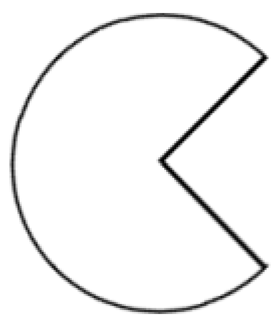
7. Determine the distance around the track. Will the track be the right length?
8. After the new track is built, landscapers need to lay sod on the field inside the track. What is the area of the field inside the track?

The radius of the quarter circle is 50.0 centimeters.



9. Calculate the area.
10. Calculate the perimeter.

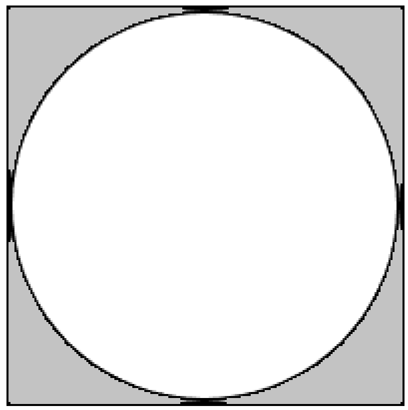
A quarter circle has been removed from a circle with a diameter of 7.0 feet.



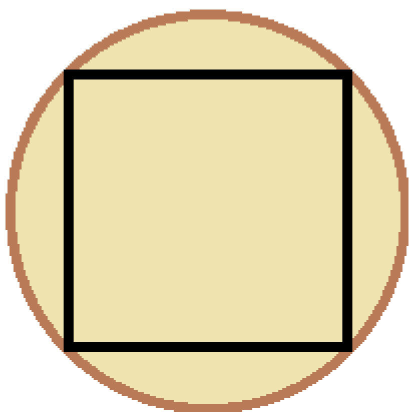
11. Calculate the area.
12. Calculate the perimeter.

If we need to determine a fraction or percent out of the whole, we may be able to solve the problem without knowing any actual measurements.

Exercises



13. Circular disks are being cut from squares of sheet metal, with the remainder around the corners being discarded. Assuming that the circles are made as large as possible, what percent of the sheet metal will be discarded?



14. A log with a circular cross-section is being planed to create a beam with a square cross-section. Assuming that the beam is made as large as possible, what percent of the log will be removed?

[Exercise Answers]

[21]

Converting Units of Area

You may use a calculator throughout this module.

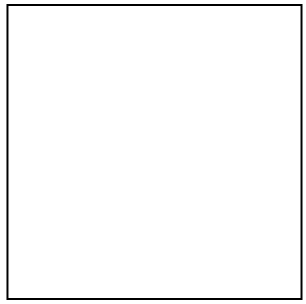
When we are converting between units of area, we need to be aware that square units behave differently than linear units. We'll begin by using our knowledge of linear units to investigate how area conversions are different.

U.S. System: Converting Measurements of Area

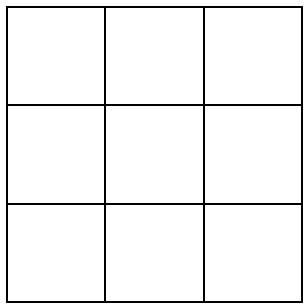
Exercises

1. A hallway's floor is 9 yards long and 2 yards wide. Convert each measurement to feet, then determine the area of the floor in square feet.
2. A hallway's floor has an area of 18 square yards. Determine the area of the floor in square feet.
3. A tabletop measures 24.0 inches by 42.0 inches. Convert each measurement to feet, then determine the area of the tabletop in square feet.
4. A tabletop has an area of $1,008$ square inches. Determine the area of the tabletop in square feet.

It may help to visualize the relationship between square yards and square feet. Consider a square yard; the area of a square with sides 1 yard long.



We know that 1 yard = 3 feet, so we can divide the square into three sections vertically and three sections horizontally to convert both dimensions of the square from yards to feet.



This forms a 3 by 3 grid, which shows us visually that 1 square yard equals 9 square feet! Instead of 1 to 3, the conversion ratio for the areas is 1 to 3^2, or 1 to 9.

Here's another way to think about it without a diagram: $1 \text{ yd} = 3 \text{ ft}$, so $(1 \text{ yd})^2 = (3 \text{ ft})^2$. To remove the parentheses, we must square the number *and* square the units: $(3 \text{ ft})^2 = 3^2 \text{ ft}^2 = 9 \text{ ft}^2$.

More generally, we need to **square** the linear conversion factors when converting units of area. If the linear units have a ratio of 1 to n, the square units will have a ratio of 1 to n^2.

Here are the conversion ratios for area in the U.S. system. As always, if you discover other conversion ratios that aren't provided here, it would be a good idea to write them down so you can use them as needed.

$1\text{ ft}^2 = 144\text{ in}^2$

$1\text{ yd}^2 = 9\text{ ft}^2$

$1\text{ yd}^2 = 1,296\text{ in}^2$

$1\text{ acre (ac)} = 4,840\text{ yd}^2$

$1\text{ ac} = 43,560\text{ ft}^2$

$1\text{ mi}^2 = 640\text{ ac}$

$1\text{ mi}^2 = 3,097,600\text{ yd}^2$

$1\text{ mi}^2 = 27,878,400\text{ ft}^2$

If you're curious, an acre is defined as the area of a 660 foot by 66 foot rectangle. (That's a furlong by a chain. Why? Because that's the amount of land that a medieval farmer with a team of eight oxen could plow in one day.)[1] An acre is defined as a unit of area; it would be wrong to say "acres squared" or put an exponent of 2 on the units.

Exercises

5. A rectangular sheet of fabric has an area of 6 square yards. Find its area in square inches.

6. A proposed site for an elementary school is 600 feet by 600 feet. Determine its area in acres, rounded to the nearest tenth.

Metric System: Converting Measurements of Area

Now let's take a look at converting area using the metric system.

Exercises

7. A hallway's floor is 9 meters long and 2 meters wide. Convert each measurement to centimeters, then determine the area of the floor in square centimeters.
8. A hallway's floor has an area of 18 square meters. Determine the area of the floor in square centimeters.
9. A sheet of A4 paper measures 210 mm by 297 mm. Convert each measurement to centimeters, then determine the area in square centimeters.
10. A sheet of A4 paper has an area of $62,370$ square millimeters. Determine the area in square centimeters.

As shown below, the conversions in the metric system are all powers of ten, which means they are all about moving the decimal point.

$$1 \text{ cm}^2 = 100 \text{ mm}^2$$

$$1 \text{ m}^2 = 10,000 \text{ cm}^2$$

$$1 \text{ m}^2 = 1,000,000 \text{ mm}^2$$

$$1 \text{ hectare (ha)} = 10,000 \text{ m}^2$$

$$1 \text{ km}^2 = 1,000,000 \text{ m}^2$$

$$1 \text{ km}^2 = 100 \text{ ha}$$

A hectare is defined as a square with sides 100 meters long. Dividing a square kilometer into ten rows and ten columns will make a 10 by 10 grid of 100 hectares. As with acres, it would be wrong to say "hectares squared" or put an exponent of 2 on the units.

Exercises

11. A rectangular sheet of fabric has an area of $60,000$ square centimeters. Find its area in square meters.

12. A proposed site for an elementary school is 200 meters by 200 meters. Find its area, in hectares.

Both Systems: Converting Measurements of Area

Converting between the U.S. and metric systems will involve messy decimal values. For example, because $1 \text{ in} = 2.54 \text{ cm}$, we can square both numbers and find that $(1 \text{ in})^2 = (2.54 \text{ cm})^2 = 6.4516 \text{ cm}^2$. The conversions are rounded to five significant figures in the table below.

$1 \text{ in}^2 = 6.4516 \text{ cm}^2$

$1 \text{ m}^2 \approx 10.764 \text{ ft}^2$

$1 \text{ m}^2 \approx 1.1960 \text{ yd}^2$

$1 \text{ mi}^2 \approx 2.5900 \text{ km}^2$

$1 \text{ ha} \approx 2.4711 \text{ ac}$ or $1 \text{ ac} \approx 0.40469 \text{ ha}$

Exercises

13. The area of Portland, Oregon is 145 mi^2. Convert this area to square kilometers.

14. How many hectares is a $5,\bar{0}00$ acre ranch?

15. A standard sheet of paper measures 8.50 inches by 11.00 inches. What is the area in square centimeters?

16. A football pitch (soccer field) is 100.0 meters long and 70.0 meters wide. What is its area in square feet?

Before the Denmark v Germany match, Rotterdam, Women's Euro 2017.

Areas of Similar Figures

Earlier in this module, it was stated that if the linear units have a ratio of 1 to n, the square units will have a ratio of 1 to n^2. This applies to similar figures as well.

If the linear dimensions of two similar figures have a ratio of 1 to n, then the areas will have a ratio of 1 to n^2.

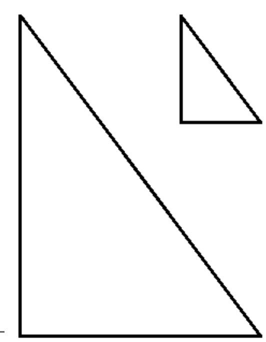
This is true for circles, similar triangles, similar rectangles, similar hexagons, you name it. We'll verify this in the following exercises.

Exercises

A personal pizza has a 7-inch diameter. A medium pizza has a diameter twice that of a personal pizza.

17. Determine the area of the medium pizza.
18. Determine the area of the personal pizza.
19. What is the ratio of the areas of the two pizzas?

A right triangle has legs 3 cm and 4 cm long. A larger right triangle has legs triple these dimensions.



20. Determine the area of the larger triangle.
21. Determine the area of the smaller triangle.
22. What is the ratio of the areas of the two triangles?

Exercise Answers

Notes

1. Source: https://en.wikipedia.org/wiki/Acre

[22]

Surface Area of Common Solids

You may use a calculator throughout this module.

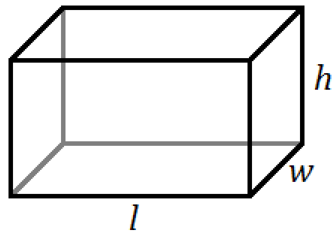
We will now turn our attention from two-dimensional figures to three-dimensional figures, which we often call ***solids***, even if they are hollow inside.

In this module, we will look the surface areas of some common solids. (We will look at volume in a later module.) Surface area is what it sounds like: it's the sum of the areas of all of the outer surfaces of the solid. When you are struggling to wrap a present because your sheet of wrapping paper isn't quite big enough, you are dealing with surface area.

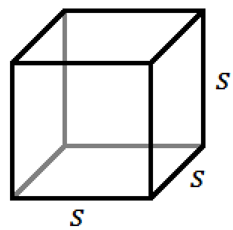
There are two different kinds of surface area that are important: the ***lateral surface area*** (LSA) and ***total surface area*** (TSA).

To visualize the difference between LSA and TSA, consider a can of food. The lateral surface area would be used to measure the size of the paper label around the can. The total surface area would be used to measure the amount of sheet metal needed to make the can. In other words, the total surface area includes the top and bottom, whereas the lateral surface area does not.

Surface Area: Rectangular Solids



A ***rectangular solid*** looks like a rectangular box. It has three pairs of equally sized rectangles on the front and back, on the left and right, and on the top and bottom.



A ***cube*** is a special rectangular solid with equally-sized squares for all six faces.

The lateral surface area is the combined total area of the four vertical faces of the solid, but not the top and bottom. If you were painting the four walls of a room, you would be thinking about the lateral surface area.

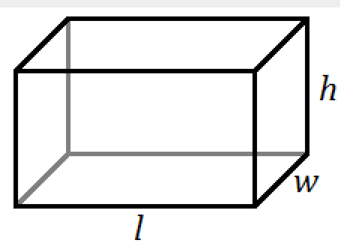
The total surface area is the combined total area of all six faces of the solid. If you were painting the four walls, the floor, and the ceiling of a room, you would be thinking about the total surface area.

For a rectangular solid with length l, width w, and height h...

$$LSA = 2lh + 2wh$$

$$TSA = 2lh + 2wh + 2lw$$

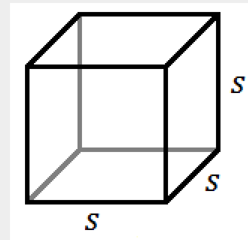
$$TSA = LSA + 2lw$$



For a cube with side length s...

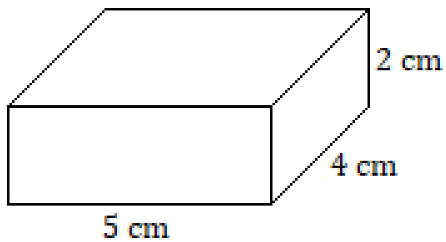
$LSA = 4s^2$

$TSA = 6s^2$



Note: These dimensions are sometimes called base, depth, and height.

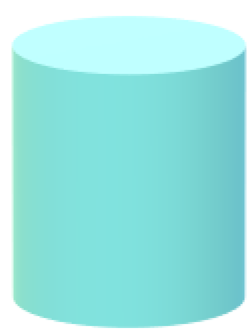
Exercises



1. Find the lateral surface area of this rectangular solid.
2. Find the total surface area of this rectangular solid.

Surface Area: Cylinders

As mentioned earlier in this module, the lateral surface area of a soup can is the paper label, which is a rectangle. Therefore, the lateral surface area of a ***cylinder*** is a rectangle; its width is equal to the circumference of the circle, $2\pi r$, and its height is the height of the cylinder.



Since a cylinder has equal-sized circles at the top and bottom, its total surface area is equal to the lateral surface area plus twice the area of one of the circles.

For a cylinder with radius r and height h...

$LSA = 2\pi rh$

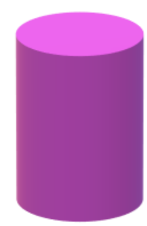
$TSA = 2\pi rh + 2\pi r^2$

$TSA = LSA + 2\pi r^2$

Be aware that if you are given the diameter of the cylinder, you will need to cut it in half before using these formulas.

Exercises

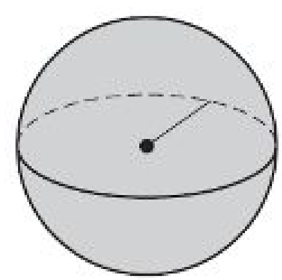
A cylinder has a diameter of $10.0\ \mathrm{cm}$ and a height of $15.0\ \mathrm{cm}$.



3. Find the lateral surface area.

4. Find the total surface area.

Surface Area: Spheres



The final solid of this module is the ***sphere***, which can be thought of as a circle in three dimensions: every point on the surface of a sphere is the same distance from the center. Because of this, a sphere has only one important measurement: its radius. Of course, its diameter could be important also, but the idea is that a sphere doesn't have different dimensions such as length, width, and height. A sphere has the same radius (or diameter) in every direction.

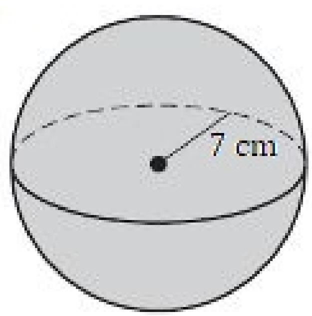
We would need to use calculus to derive the formula for the surface area of a sphere, so we'll just assume it's true and get on with the business at hand. Notice that, because a sphere doesn't have top or bottom faces, we don't need to worry about finding the lateral surface area. The only surface area is the total surface area.

For a sphere with radius r or diameter d...

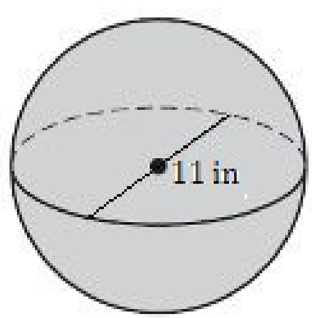
$SA = 4\pi r^2$ or $SA = \pi d^2$

Coincidentally, the surface area of a sphere is 4 times the area of the cross-sectional circle at the sphere's widest part. You may find it interesting to try to visualize this, or head to the kitchen for a demonstration: if you cut an orange into four quarters, the peel on one of those quarter oranges has the same area as the circle formed by the first cut.

Exercises



5. Find the surface area of this sphere.



6. Find the surface area of this sphere.

Exercise Answers

[23]

Area of Regular Polygons

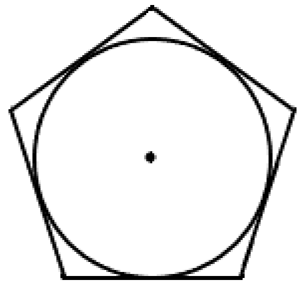
You may use a calculator throughout this module.

"The Pentagon" by David B. Gleason is licensed under CC BY-SA 2.0

The Pentagon building spans 28.7 acres ($116,000\ \mathrm{m}^2$), and includes an additional 5.1 acres ($21,000\ \mathrm{m}^2$) as a central courtyard.[1] (That's roughly 25 American football fields… or 23 Canadian football fields.) In this module, we will focus on calculating the area of regular polygons such as this one.

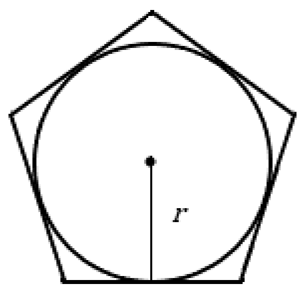
A ***regular polygon*** has all sides of equal length and all angles of equal measure. Because of this symmetry, a circle can be ***inscribed***—drawn inside the polygon touching each side at one point—or ***circumscribed***—drawn outside the polygon intersecting each vertex. We'll focus on the inscribed circle first.

Using the Distance from the Center to an Edge



inscribed circle

Let's call the radius of the inscribed circle lowercase r; this is the distance from the center of the polygon perpendicular to one of the sides. (The inner radius is more commonly called the *apothem* and labeled a, but we are trying to keep the jargon to a minimum in this textbook.)



We can derive a formula for the area of a regular polygon by dividing the polygon into equally-sized triangles and combining the areas of those triangles. If the length of each side of is s, each triangle's area is its base times its height divided by 2, or $s \cdot r \div 2$. If the polygon has n sides, then there are n of these triangles, and the total combined area is $n \cdot s \cdot r \div 2$.

Area of a Regular Polygon (with a radius drawn to the center of one side)

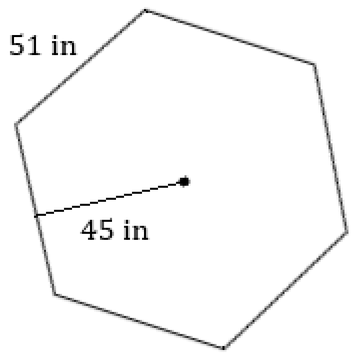
For a regular polygon with n sides of length s, and inscribed (inner) radius r,

$A = nsr \div 2$

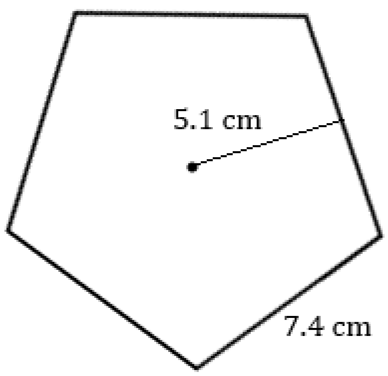
Note: This formula is more commonly written as one-half the apothem times the perimeter: $A = \frac{1}{2}ap$. The apothem is what we're calling r, and the perimeter is $n \cdot s$, so $A = \frac{1}{2}ap$ and $A = nsr \div 2$ are equivalent formulas.

Exercises

1. Calculate the area of this regular hexagon.



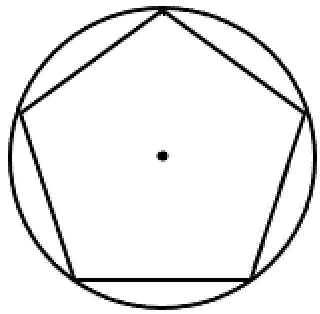
2. Calculate the area of this regular pentagon.



3. A stop sign has a height of 30 inches, and each edge measures 12.5 inches. Find the area of the sign.

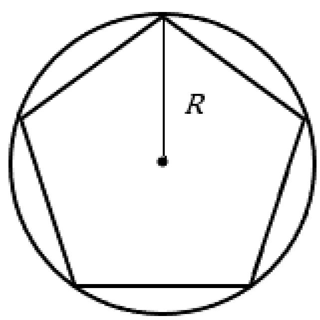
Using the Distance from the Center to a Vertex

Okay, but what if we know the distance from the center to a vertex (a corner of the polygon) instead of the distance from the center to an edge? We'll need to imagine a circumscribed circle.



circumscribed circle

Let's call the radius of the circumscribed circle capital R; this is the distance from the center of the polygon to one of the vertices (corners).



Area of a Regular Polygon (with a radius drawn to a vertex)

For a regular polygon with n sides of length s, and circumscribed (outer) radius R,

$$A = 0.25ns\sqrt{4R^2 - s^2}$$

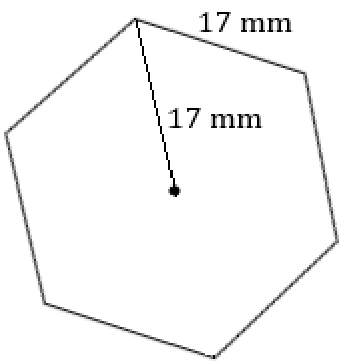
or

$$A = ns\sqrt{4R^2 - s^2} \div 4$$

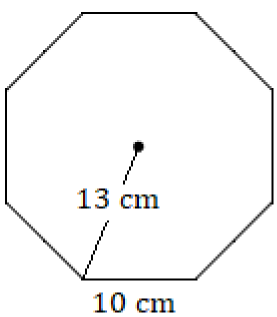
Your author created this formula because every other version of it uses trigonometry, which we haven't covered yet. [2]

Exercises

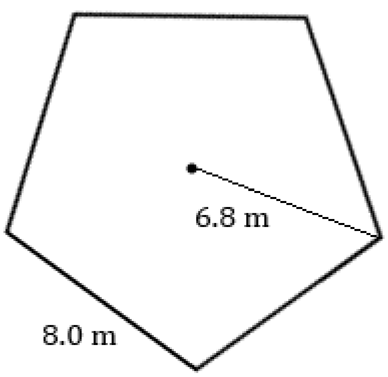
4. Calculate the area of this regular hexagon.



5. Calculate the area of this regular octagon.



6. Calculate the area of this regular pentagon.

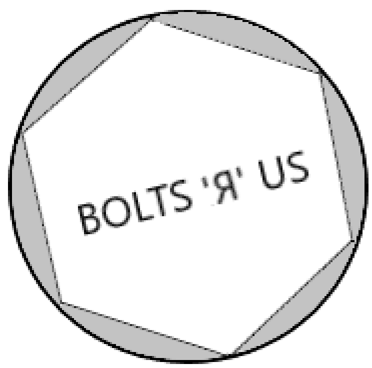


Fun fact: In a regular hexagon, the radius to a vertex is always equal to the side length. If you divide the hexagon into six equilateral triangles, you'll see why.

Composite Figures with Regular Polygons

As you know, a ***composite figure*** is a geometric figure which is formed by joining two or more basic geometric figures. Let's look at a composite figure formed by a circle and a regular polygon.

Exercises



7. The hexagonal head of a bolt fits snugly into a circular cap with a circular hole with inside diameter 46 mm as shown in this diagram. Opposite sides of the bolt head are 40 mm apart. Find the total empty area in the hole around the edges of the bolt head.

Exercise Answers

Notes

1. Source: https://en.wikipedia.org/wiki/The_Pentagon
2. This formula is derived from dividing the polygon into *n* equally-sized triangles and combining the areas of those triangles; it includes a square root because it involves the Pythagorean theorem.

[24]

Volume of Common Solids

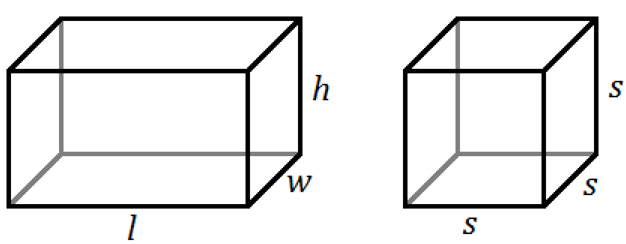
You may use a calculator throughout this module.

Note: Some of the diagrams in this module have dimensions with only one significant figure, but we would lose a lot of information if we rounded the results to only one sig fig. Therefore, we will not necessarily follow the accuracy-based rounding rules in the answer key for this module.

Wanda Ortiz of The Iron Maidens turns up the volume. Photo by Alejandro Páez on flickr.

The surface area of a solid is the sum of the areas of all its faces; therefore, surface area is two-dimensional and measured in square units. The ***volume*** is the amount of space inside the solid. Volume is three-dimensional, measured in cubic units. You can imagine the volume as the number of cubes required to completely fill up the solid.

Volume: Rectangular Solids



Volume of a Rectangular Solid

For a rectangular solid with length l, width w, and height h:

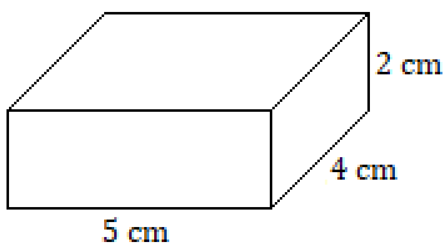
$V = lwh$

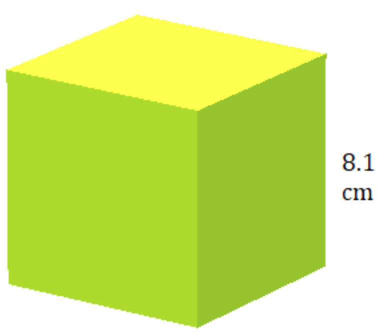
For a cube with side length s:

$V = s^3$

Exercises

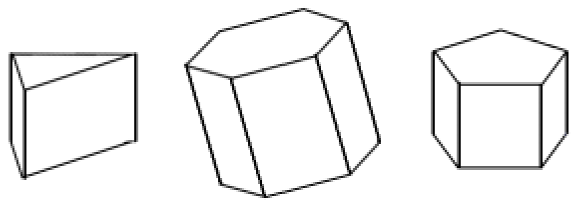
Find the volume of each solid.

1. 

2. 

Volume: Rectangular Prisms

A solid with two equal sized polygons as its bases and rectangular lateral faces is called a right-angle prism. Some examples are shown below. We will refer to them simply as ***prisms*** in this textbook. (We will not be working with oblique prisms, which have parallelograms for the lateral faces.)



If you know the area of one of the bases, multiplying it by the height gives you the volume of the prism. In the formula below, we are using a capital B to represent the *area* of the base.

Volume of a Prism

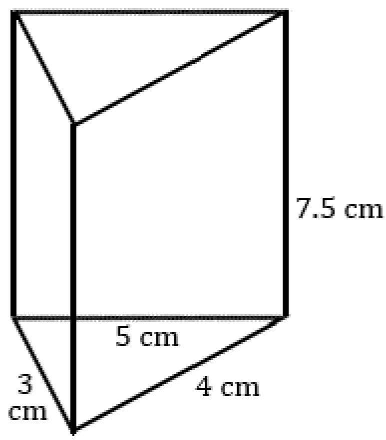
For a prism with base area B and height h:

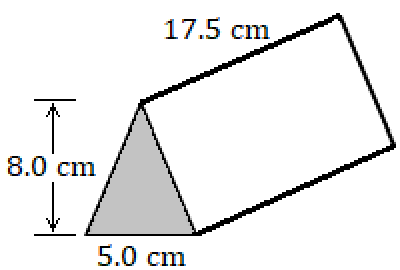
$V = Bh$

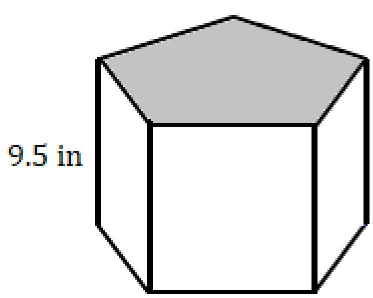
If the prism is lying on its side, the "height" will look like a length. No matter how the prism is oriented, the height is the dimension that is perpendicular to the planes of the two parallel bases.

Exercises

Find the volume of each prism.

3. 

4. 

5. 

The area of the pentagon is 55 square inches.

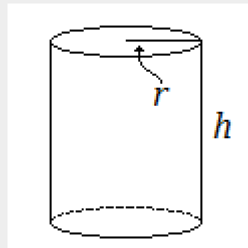
Volume: Cylinders

A cylinder can be thought of as a prism with bases that are circles, rather than polygons. Just as with a prism, the volume is the area of the base multiplied by the height.

Volume of a Cylinder

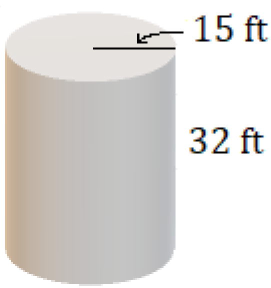
For a cylinder with radius r (or diameter d) and height h:

$V = \pi r^2 h$ or $V = \pi d^2 h \div 4$

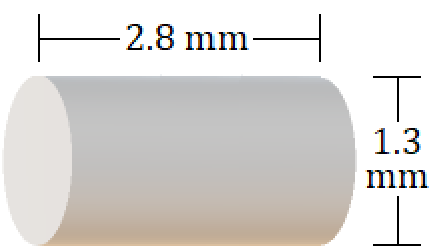


Exercises

Find the volume of each cylinder.



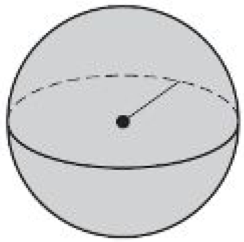
6.



7.

Volume: Spheres

As with surface area, we would need to use calculus to derive the formula for the volume of a **sphere**. Just believe it. ¯_(ツ)_/¯



Volume of a Sphere

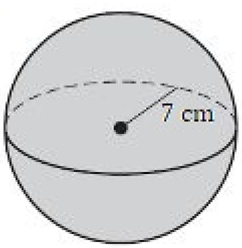
For a sphere with radius r or diameter d:

$V = 4\pi r^3 \div 3$ or $V = \pi d^3 \div 6$

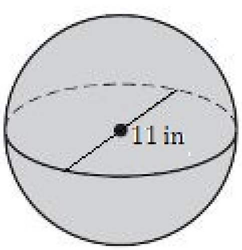
Exercises

Find the volume of each sphere.

8.



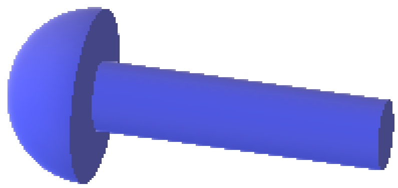
9.



Volume: Composite Solids

Of course, not every three-dimensional object is a prism, cylinder, or sphere. A composite solid is made up of two or more simpler solids. As with two-dimensional composite figures, breaking the figure into recognizable solids is a good first step.

Exercises



10. A rivet is formed by topping a cylinder with a hemisphere. The width of the cylindrical part (the rivet pin) is 1.6 cm and the length is 7.0 cm. The width of the hemisphere-shaped top (the rivet head) is 3.2 cm. Find the rivet's volume.

Casper relaxes in the shadow of the propane tank.
Photo by David Dodge on flickr.

A 250-gallon propane tank is roughly in the shape of a cylinder with a hemisphere on each end. The length of the cylindrical part is 6.0 feet long, and the cross-sectional diameter of the tank is 2.5 feet.

11. Calculate the volume of the tank in cubic feet.

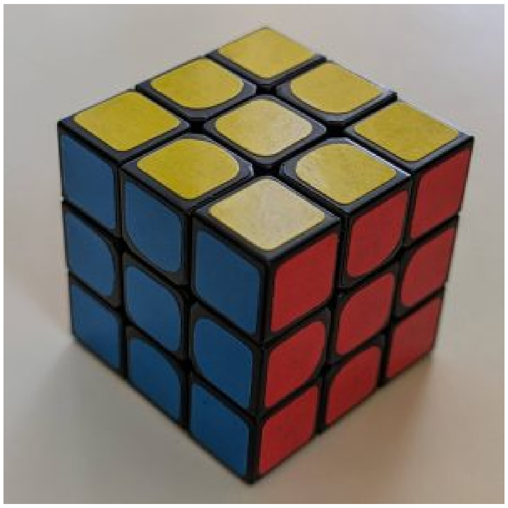
12. Verify that the tank can hold 250 gallons of liquid propane.

Exercise Answers

[25]

Converting Units of Volume

You may use a calculator throughout this module.



Just as we saw with area, converting between units of volume requires us to be careful because cubic units behave differently than linear units.

Quantities of mulch, dirt, or gravel are often measured by the cubic yard. How many cubic feet are in one cubic yard?

1 yard = 3 feet, so we can divide the length into three sections, the width into three sections, and the height into three sections to convert all three dimensions of the cube from yards to feet. This forms a 3 by 3 by 3 cube, which shows us that 1 cubic yard equals 27 cubic feet. The linear conversion ratio of 1 to 3 means that that the conversion ratio for the volumes is 1 to 3^3, or 1 to 27.

Here's another way to think about it without a diagram: $1 \text{ yd} = 3 \text{ ft}$, so $(1 \text{ yd})^3 = (3 \text{ ft})^3$. To remove the parentheses, we must cube the number *and* cube the units: $(3 \text{ ft})^3 = 3^3 \text{ ft}^3 = 27 \text{ ft}^3$.

More generally, we need to **cube** the linear conversion factors when converting units of volume. If the linear units have a ratio of 1 to n, the cubic units will have a ratio of 1 to n^3.

U.S. System: Converting Measurements of Volume

$$1 \text{ ft}^3 = 1,728 \text{ in}^3$$

$$1 \text{ yd}^3 = 27 \text{ ft}^3$$

$$1 \text{ yd}^3 = 46,656 \text{ in}^3$$

Exercises

1. True story: A friend at the National Guard base gave us three long wooden crates to use as raised planting beds. (The crates probably carried some kind of weapons or ammunition, but our friend wouldn't say.) Henry, who was taking geometry in high school, was asked to measure the crates and figure out how much soil we needed. The inside dimensions of each crate were 112 inches long, 14 inches wide, and 14 inches deep. We wanted to fill them most of the way full with soil, leaving about 4 inches empty at the top. How many cubic yards of soil did we need to order from the supplier?

2. True story, continued: We decided to check our answer and did a rough estimate by rounding each dimension to the nearest foot, then figuring out the volume from there. Did this give the same result?

This is one of the three crates.

We can convert between units of volume and liquid capacity. As you might expect, the numbers are messy in the U.S. system.

$$1 \text{ fl oz} \approx 1.8047 \text{ in}^3$$
$$1 \text{ ft}^3 \approx 7.4805 \text{ gal}$$

Exercises

3. A circular wading pool has a diameter of roughly 5 feet and a depth of 6 inches. How many gallons of water are required to fill it about 80% of the way full?
4. A standard U.S. soda pop can has a diameter of $2\frac{1}{2}$ inches and a height of $4\frac{3}{4}$ inches. Verify that the can is able to hold 12 fluid ounces of liquid.
5. A water jug that is roughly cube-shaped is advertised with dimensions 10.51 inches deep, 10.51 inches wide, and 10.51 inches high. How many gallons is its capacity?

Tiger the cat couldn't jump onto the storage container, so I made him a staircase. He didn't understand the concept.

Metric System: Converting Measurements of Volume

$$1\ \text{cm}^3 = 1\ \text{cc} = 1\ \text{mL}$$
$$1\ \text{cm}^3 = 1,000\ \text{mm}^3$$
$$1\ \text{m}^3 = 1,000,000\ \text{cm}^3$$
$$1\ \text{L} = 1,000\ \text{cm}^3$$
$$1\ \text{m}^3 = 1,000\ \text{L}$$

It's no surprise that the metric conversion ratios for volume are all powers of 10 ... but they are actually powers of $1,000$, or 10^3, because the linear conversions get cubed when we go three-dimensional.

Exercises

6. The circular soaking tubs at Hot Lake Springs in La Grande, Oregon have an inner diameter of 2.00 meters, and the water has a depth of 0.75 meters. What is the approximate volume of water, in liters, in one of the tubs?

7. A 1.5 L carton of orange juice has a square base 9.5 centimeters on each side, and the height of the rectangular section is 19 centimeters. (Ignore the triangular section at the top.) Verify that the carton holds 1.5 liters of juice.

8. A 1.89 L carton of POG juice has a square base 9.5 centimeters on each side, and the height of the rectangular section is 19.5 centimeters. This is not enough volume to hold 1.89 liters of juice unless the sides of the carton bulge out. If you unfold the triangular top section to make a rectangular column and squeeze the sides of the carton so they aren't bulging out, what height will the juice rise to?

Both Systems: Converting Measurements of Volume

Converting volumes between the U.S. and metric systems will of course involve messy decimal values. For example, because $1 \text{ in} = 2.54 \text{ cm}$, we can cube both numbers and find that $1 \text{ in}^3 = (2.54 \text{ cm})^3 = 16.387064 \text{ cm}^3$. The conversions in the table below are rounded to five significant figures.

$$1 \text{ in}^3 \approx 16.387 \text{ cm}^3$$
$$1 \text{ m}^3 \approx 35.315 \text{ ft}^3$$
$$1 \text{ m}^3 \approx 1.3080 \text{ yd}^3$$

Exercises

9. A large dumpster in Iceland has a volume of 15 cubic meters. Convert this to cubic yards.

10. Suppose you need to know the volume of an Icelandic dumpster in cubic feet. Convert $15\ \text{m}^3$ to ft^3.

Hvaða rusl!

11. A "two yard" dumpster at a campground in Washington has a volume of 2 cubic yards. Convert this to cubic meters.

What rubbish!

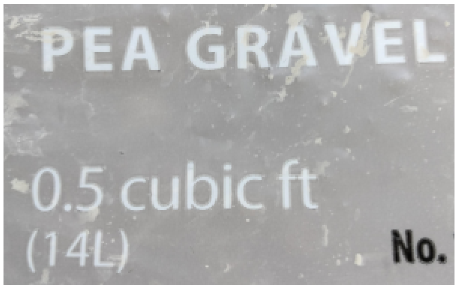
12. The engine of a 1964 Corvette has 8 cylinders with a bore (diameter) of 4.00 inches and a stroke (height) of 3.25 inches. Find the total displacement (volume) of the cylinders in this engine, rounded to the nearest cubic inch.[1]

13. Convert the displacement of the Corvette engine into cubic centimeters.

14. Convert the displacement of the Corvette engine into liters.

Photo by Chevrolet via Wikimedia Commons

15. A hardware store sells pea gravel in $0.5\ \text{ft}^3$ bags. The bags are also marked $14\ \text{L}$. Verify that these two measures are equivalent.



Density

The density of a material is its weight per volume such as pounds per cubic foot, or mass per volume such as grams per cubic centimeter. Multiplying the volume of an object by its density will give its weight or mass.

Exercises

16. A 0.5 ft^3 bag of pea gravel weighs approximately 50 pounds. Determine the density of pea gravel in pounds per cubic foot.

17. The standard size of a gold bar in the U.S. Federal Reserve is 7 inches by $3\frac{5}{8}$ inches by $1\frac{3}{4}$ inches.[2] The density of gold is 0.698 pounds per cubic inch. How much does one gold bar weigh?

18. A cylindrical iron bar has a diameter of 3.0 centimeters and a length of 20.0 centimeters. The density of iron is 7.87 grams per cubic centimeter. What is the bar's mass, in kilograms?

Volumes of Similar Solids

Earlier in this module, it was stated that if the linear units have a ratio of 1 to n, the cubic units will have a ratio of 1 to n^3. This applies to similar solids as well.

> If the linear dimensions of two similar solids have a ratio of 1 to n, then the volumes will have a ratio of 1 to n^3.

We'll verify this in the following exercises.

Exercises

A table tennis (ping pong) ball has a diameter of 4 centimeters. A wiffle® ball has a diameter twice that of a table tennis ball.

19. Determine the volume of the wiffle® ball.

20. Determine the volume of the table tennis ball.

21. What is the ratio of the volumes of the two balls?

Rectangular solid A has dimensions 3 inches by 4 inches by 5 inches. Rectangular solid B has dimensions triple those of A's.

22. Determine the volume of the larger solid, B.

23. Determine the volume of the smaller solid, A.

24. What is the ratio of the volumes of the two solids?

Exercise Answers

Notes

1. The formula is often written as $V = \frac{\pi}{4} \cdot b^2 \cdot s \cdot c$, where c is the number of cylinders.
2. Source: https://www.usmint.gov/about/mint-tours-facilities/fort-knox

[26]

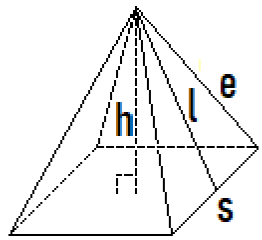
Pyramids and Cones

You may use a calculator throughout this module.

Pyramids

The pyramid complex at Giza, Egypt. Photo by Ricardo Liberato on Wikipedia

A **pyramid** is a geometric solid with a polygon base and triangular faces with a common vertex (called the *apex* of the pyramid). Pyramids are named according to the shape of their bases. The most common pyramids have a square or another regular polygon for a base, making all of the faces identical isosceles triangles. The height, h, is the distance from the apex straight down to the center of the base. Two other measures used with pyramids are the edge length e, the sides of the triangular faces, and the slant height l, the height of the triangular faces.



Volume of a Pyramid

In general, the volume of a pyramid with base of area B and height h is

$V = \frac{1}{3}Bh$ or $V = Bh \div 3$

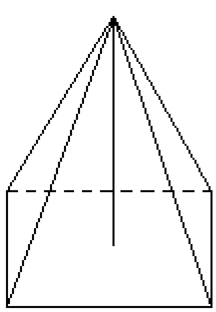
If the base is a square with side length s, the volume is

$V = \frac{1}{3}s^2h$ or $V = s^2h \div 3$

Interestingly, the volume of a pyramid is $\frac{1}{3}$ the volume of a prism with the same base and height.

Exercises

1. A pyramid has a square base with sides 16.0 centimeters long, and a height of 15.0 centimeters. Find the volume of the pyramid.



2. The Great Pyramid at Giza has a height of 137 meters and a square base with sides $23\overline{0}$ meters long.[1] Find the volume of the pyramid.

The lateral surface area (LSA) of a pyramid is found by adding the area of each triangular face.

Lateral Surface Area of a Pyramid

If the base of a pyramid is a regular polygon with n sides each of length s, and the slant height is l, then

$LSA = \frac{1}{2}nsl$ or $LSA = nsl \div 2$

If the base is a square, then

$LSA = 2sl$

The total surface area (TSA) is of course found by adding the area of the base B to the lateral surface area. If the base is a regular polygon, you will need to use the techniques we studied in Module 23.

Total Surface Area of a Pyramid

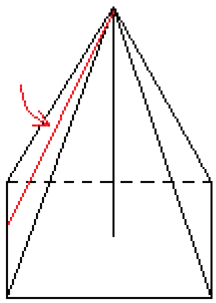
$TSA = LSA + B$

If the base is a square, then

$TSA = 2sl + s^2$

Exercises

3. A pyramid has a square base with sides 16.0 centimeters long, and a slant height of 17.0 centimeters. Find the lateral surface area and total surface area of the pyramid.



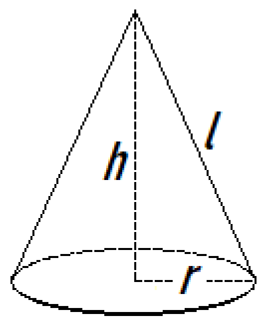
4. The Great Pyramid at Giza has a slant height of 179 meters and a square base with sides $23\overline{0}$ meters long. Find the lateral surface area of the pyramid.

Cones

A large cone-shaped pile of grain in Eastern Oregon.

A cone is like a pyramid with a circular base. You may be able to determine the height h of a cone (the altitude from the apex, perpendicular to the base), or the slant height l (which is the length from the apex to the edge of the circular base).

Note that the height, radius, and slant height form a right triangle with the slant height as the hypotenuse. We can use the Pythagorean theorem to determine the following equivalences.



The slant height l, height h, and radius r of a cone are related as follows:

$l = \sqrt{r^2 + h^2}$

$h = \sqrt{l^2 - r^2}$

$r = \sqrt{l^2 - h^2}$

Just as the volume of a pyramid is $\frac{1}{3}$ the volume of a prism with the same base and height, the volume of a cone is $\frac{1}{3}$ the volume of a cylinder with the same base and height.

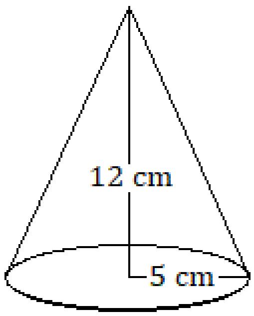
Volume of a Cone

The volume of a cone with a base radius r and height h is

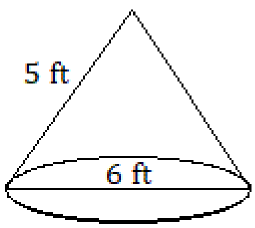
$V = \frac{1}{3}\pi r^2 h$ or $V = \pi r^2 h \div 3$

Exercises

5. The base of a cone has a radius of 5.0 centimeters, and the vertical height of the cone is 12.0 centimeters. Find the volume of the cone.



6. The base of a cone has a diameter of 6.0 feet, and the slant height of the cone is 5.0 feet. Find the volume of the cone.

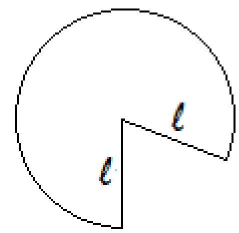


For the surface area of a cone, we have the following formulas.

Surface Area of a Cone

$LSA = \pi r l$

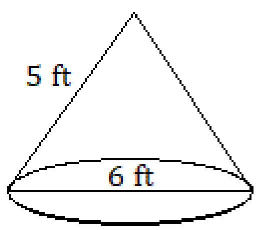
$TSA = LSA + \pi r^2 = \pi r l + \pi r^2$



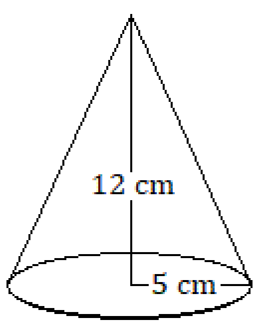
It's hard to explain the LSA formula in words, but here goes. The lateral surface of a cone, when flattened out, is a circle with radius l that is missing a wedge. The circumference of this partial circle, because it matched the circumference of the circular base, is $2\pi r$. The circumference of the entire circle with radius l would be $2\pi l$, so the part we have is just a fraction of the entire circle. To be precise, the fraction is $\frac{2\pi r}{2\pi l}$, which reduces to $\frac{r}{l}$. The area of the entire circle with radius l would be πl^2. Because the partial circle is the fraction $\frac{r}{l}$ of the entire circle, the area of the partial circle is $\pi l^2 \cdot \frac{r}{l} = \pi r l$.

Exercises

7. The base of a cone has a diameter of 6.0 feet, and the slant height of the cone is 5.0 feet. Find the lateral surface area and total surface area of the cone.



8. The base of a cone has a radius of 5.0 centimeters, and the vertical height of the cone is 12.0 centimeters. Find the lateral surface area and total surface area of the cone.



Exercise Answers

Notes

1. https://en.wikipedia.org/wiki/Great_Pyramid_of_Giza

[27]

Percents Part 3

Photo by Karim Manjra on Unsplash

You may use a calculator throughout this module.

There is one more situation involving percents that often trips people up: working backwards from the result of a percent change to find the original value.

$$\text{Amount} = \text{Rate} \cdot \text{Base}$$

$$A = R \cdot B$$

Finding the Base After a Percent Increase

Suppose a 12% tax is added to a price; what percent of the original is the new amount?

Well, the original number is 100% of itself, so the new amount must be $100\% + 12\% = 112\%$ of the original.

As a proportion, $\frac{A}{B} = \frac{112}{100}$.

As an equation, $A = 1.12 \cdot B$.

A very common error is to find 12% of the new amount and subtract that from the new amount. However, this doesn't give the correct result. Instead, we must ***divide*** the new amount by 112%.

> If an unknown number is increased by a percent, add that percent to 100% and use that result for the rate.
>
> $$\frac{\text{new amount}}{\text{unknown base}} = \frac{100\text{+percent}}{100}$$

Exercises

1. A sales tax of 8% is added to the selling price of a lawn tractor, making the total price $\$1,402.92$. What is the selling price of the lawn tractor without tax?
2. Clackamas Community College's enrollment in Fall 2023 was $17,605$ students, which was an increase of 18.465% from Fall 2022. What was the enrollment in Fall 2022?
3. While your author and his children were stuck overnight at the Seattle airport, they bought three 20-ounce bottles of Sprite for $\$11.86$, but the prices weren't shown and they weren't given a receipt. Suspecting that the sales tax rate was 10.1% from previous experience, your author was able to figure out the cost of one Sprite. Can you?

Finding the Base After a Percent Decrease

Suppose a 12% discount is applied to a price; what percent of the original is the new amount?

As above, the original number is 100% of itself, so the new amount must be $100\% - 12\% = 88\%$ of the original.

As a proportion, $\frac{A}{B} = \frac{88}{100}$.

As an equation, $A = 0.88 \cdot B$.

As above, the most common error people make is finding 12% of the new amount and adding that to the new amount, but this doesn't give the correct result. Instead, we must ***divide*** the new amount by 88%.

> If an unknown number is **decreased** by a percent, subtract that percent from 100% and use that result for the rate.
>
> $$\frac{\text{new amount}}{\text{unknown base}} = \frac{100-\text{percent}}{100}$$

Exercises

4. A city department's budget was cut by 5.00% this year. If this year's budget is $\$3.04$ million, what was last year's budget?

5. The estimated population of San Francisco in July 2022 was $808,400$ people, which was a decrease of 7.50% from April 2020. What was the estimated population in April 2020? (Round to the nearest hundred people.)[1]

6. An educational website claims that by purchasing access for $\$5$, you'll save 69% off the standard price. What is the standard price?

Exercise Answers

Notes

1. Source: https://www.census.gov/quickfacts/fact/table/sanfranciscocitycalifornia/PST045222

[28]

Mean, Median, Mode

You may use a calculator throughout this module.

We often describe data using a ***measure of central tendency***. This is a number that we use to describe the typical data value. In this module, we will look at three measures of central tendency: the ***mean***, the ***median***, and the ***mode***. Each of these has pros and cons, depending on the particular data set.

Mean

The mean of a set of data is what we commonly call the average: add up all of the numbers and then divide by how many numbers there were.

Exercises

1. The table below shows the amount of time, rounded to the nearest half minute, it took Marty to complete the Friday crossword puzzle in the New York Times. Calculate the mean completion time for these thirteen puzzles.

Oct 6, 2023	10.0 min
Oct 13, 2023	13.0 min
Oct 20, 2023	11.0 min
Oct 27, 2023	9.0 min
Nov 3, 2023	8.5 min
Nov 10, 2023	9.5 min
Nov 17, 2023	11.0 min
Nov 24, 2023	12.0 min
Dec 1, 2023	11.5 min
Dec 8, 2023	9.5 min
Dec 15, 2023	11.0 min
Dec 22, 2023	11.0 min
Dec 29, 2023	7.0 min

2. The table below shows the average price of a gallon of regular unleaded gasoline in the Seattle metro area for ten weeks in late 2023.[1] Compute the mean price over this time period.

Oct 23, 2023	\$4.81
Oct 30, 2023	\$4.70
Nov 6, 2023	\$4.63
Nov 13, 2023	\$4.57
Nov 20, 2023	\$4.49
Nov 27, 2023	\$4.45
Dec 4, 2023	\$4.39
Dec 11, 2023	\$4.34
Dec 18, 2023	\$4.28
Dec 25, 2023	\$4.22

Median

The median is the middle number in a set of data; it has an equal number of data values below it as above it. The numbers must be arranged in order, usually smallest to largest but largest to smallest would also work. Then we can count in from both ends of the list and find the median in the middle.

If there are an odd number of data values, there will be one number in the middle, which is the median.

If there are an even number of data values, there will be two numbers in the middle. The mean of these two numbers is the median.

Exercises

3. Here are Marty's Friday crossword puzzle completion times again, in minutes, listed in order from fastest to slowest. What is the median completion time? $7.0, 8.5, 9.0, 9.5, 9.5, 10.0, 11.0, 11.0, 11.0, 11.0, 11.5, 12.0, 13.0$

4. Here are the Seattle gas prices again, listed in order from lowest to highest. What is the median price?$\$4.22$, $\$4.28$, $\$4.34$, $\$4.39$, $\$4.45$, $\$4.49$, $\$4.57$, $\$4.63$, $\$4.70$, $\$4.81$

The five houses on a block have these property values: $\$250,000$; $\$300,000$; $\$320,000$; $\$190,000$; $\$220,000$.

5. Find the mean property value.

6. Find the median property value.

A new house is built on the block, making the property values $\$250,000$; $\$300,000$; $\$320,000$; $\$190,000$; $\$220,000$; $\$750,000$.

7. Find the mean property value.

8. Find the median property value.

9. Which of these measures appears to give a more accurate representation of the typical house on the block?

The mean is better to work with when we do more complicated statistical analysis, but it is sensitive to extreme values; in other words, one very large or very small number can have a significant effect on the mean. The median is not sensitive to extreme values, which can make it a better measure to use when describing data that has one or two numbers very different from the remainder of the data.

Mode

The mode is the value that appears **most** frequently in the data set. On the game show *Family Feud*, the goal is to guess the mode: the most popular answer.

If no numbers are repeated, then the data set has no mode. If there are two values that are tied for most frequently occurring, then they are both considered a mode and the data set is called ***bimodal***. If there are more than two values tied for the lead, we usually say that there is no mode.[2] (It's like in sports: there is usually one MVP, but occasionally there are two co-MVPs. Having three or more MVPs would start to get ridiculous.)

Exercises

10. Here are Marty's Friday crossword puzzle completion times one last time, in minutes. What is the mode of the completion times? 7.0, 8.5, 9.0, 9.5, 9.5, 10.0, 11.0, 11.0, 11.0, 11.0, 11.5, 12.0, 13.0

11. Here are the Seattle gas prices one last time. What is the mode of the prices? \$4.22, \$4.28, \$4.34, \$4.39, \$4.45, \$4.49, \$4.57, \$4.63, \$4.70, \$4.81

12. One hundred cell phone owners are asked which carrier they use. What is the mode of the data?

AT&T Mobility	Verizon Wireless	T-Mobile US	Dish Wireless	U.S. Cellular
43	29	24	2	2

13. Fifty people are asked what their favorite type of Girl Scout cookie is. What is the mode?

S'Mores	Samoas	Tagalongs	Trefoils	Thin Mints
4	16	5	9	16

Let's put it all together and find the mean, median, and mode of some data sets. Sportsball!

Exercises

From 2001-2019, these are the numbers of games won by the New England Patriots each NFL season.[3]
11, 9, 14, 14, 10, 12, 16, 11, 10, 14, 13, 12, 12, 12, 12, 14, 13, 11, 12.

14. Find the mean number of games won from 2001 to 2019.

15. Find the median number of games won from 2001 to 2019.

16. Find the mode of the number of games won from 2001 to 2019.

17. Do any of these measures appear to be misleading, or do they all represent the data fairly well?

From 2001-2019, these are the numbers of games won by the Buffalo Bills each NFL season.[4]
3, 8, 6, 9, 5, 7, 7, 7, 6, 4, 6, 6, 6, 9, 8, 7, 9, 6, 10.

18. Find the mean number of games won from 2001 to 2019.

19. Find the median number of games won from 2001 to 2019.

20. Find the mode of the number of games won from 2001 to 2019.

21. Do any of these measures appear to be misleading, or do they all represent the data fairly well?

Some sets of data may not be easy to describe with one measure of central tendency.

Exercises

Thirteen clementines are weighed. Their masses, in grams, are
$82, 90, 90, 92, 93, 94, 94, 102, 107, 107, 108, 109, 109$.

22. Determine the mean. Does the mean appear to represent the mass of a typical clementine?

23. Determine the median. Does the median appear to represent the mass of a typical clementine?

24. Determine the mode. Does the mode appear to represent the mass of a typical clementine?

Suppose that the 108-gram clementine is a tiny bit heavier and the masses are actually
$82, 90, 90, 92, 93, 94, 94, 102, 107, 107, 109, 109, 109$.

25. Determine the new mean. Is the new mean different from the original mean?

26. Determine the new median. Is the new median different from the original median?

27. Determine the new mode. Is the new mode different from the original mode? Does it represent the mass of a typical clementine?

Clementine the cat weighs more than 109 grams.

Exercise Answers

Notes

1. Source: https://www.eia.gov/petroleum/gasdiesel/
2. The concepts of trimodal and multimodal data exist, but we aren't going to consider anything beyond bimodal in this textbook.
3. Source: https://www.pro-football-reference.com/teams/nwe/index.htm
4. Source: https://www.pro-football-reference.com/teams/buf/index.htm

[29]

Probability

Probability is the likelihood that some event occurs. If the event occurs, we call that a ***favorable outcome***. The set of all possible events (or outcomes) is called the ***sample space*** of the event.

We will limit our focus to ***independent*** events, which do not influence each other. For example, if we roll a 5 on one die, that does not affect the probability of rolling a 5 on the other die. (We will not be studying *dependent* events, which do influence each other.)

Theoretical Probability

If we are working with something simple like dice, cards, or coin flips where we know all of the possible outcomes, we can calculate the ***theoretical probability*** of an event occurring. To do this, we divide the number of ways the event can occur by the total number of possible outcomes. We may choose to write the probability as a fraction, a decimal, or a percent depending on what form seems most useful.

Theoretical probability of an event:

$$P(\text{event}) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

Suppose two six-sided dice, numbered 1 through 6, are rolled. There are $6 \cdot 6 = 36$ possible outcomes in the sample space. If we are playing a game where we take the sum of the dice, the only possible outcomes are 2 through 12. However, as the following table shows, these outcomes are not all equally likely. For example, there are two different ways to roll a 3, but only one way to roll a 2.

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Exercises

Two six-sided dice numbered 1 through 6 are rolled. Find the probability of each event occurring.

1. The sum of the dice is 7.
2. The sum of the dice is 11.
3. The sum of the dice is 7 or 11.
4. The sum of the dice is greater than 1.
5. The sum of the dice is 13.

Some things to notice...

If an event is impossible, its probability is 0% or 0.

If an event is certain to happen, its probability is 100% or 1.

If it will be tedious to count up all of the favorable outcomes, it may be easier to count up the unfavorable outcomes and subtract from the total.

Exercises

Two six-sided dice numbered 1 through 6 are rolled. Find the probability of each event occurring.

6. The sum of the dice is 5.

7. The sum of the dice is not 5.

8. The sum of the dice is greater than 9.

9. The sum of the dice is 9 or lower.

The set of outcomes in which an event does not occur is called the ***complement*** of the event. The event "the sum is not *5*" is the complement of "the sum is *5*". Two complements *complete* the sample space.

If the probability of an event happening is p, the probability of the complement is $1 - p$.

Exercises

A bowl of 60 Tootsie Rolls Fruit Chews contains the following: 15 cherry, 14 lemon, 13 lime, 11 orange, 7 vanilla.

10. If one Tootsie Roll is randomly selected from the bowl, what is the probability that it is cherry?

11. If one Tootsie Roll is randomly selected from the bowl, what is the probability that it is not cherry?

12. What is the probability that a randomly selected Tootsie Roll is either orange or vanilla?

13. What is the probability that a randomly selected Tootsie Roll is not orange or vanilla?

Here's where we try to condense the basics of genetic crosses into one paragraph.

Each parent gives one allele to their child. The allele for brown eyes is B, and the allele for blue eyes is b. If two parents both have genotype Bb, the table below (which biologists call a Punnett square) shows that there are four equally-likely outcomes: BB, Bb, Bb, bb. The allele for brown eyes, B, is dominant over the gene for blue eyes, b, which means that if a child has any B alleles, they will have brown eyes. The only genotype for which the child will have blue eyes is bb.

	B	b
B	BB	Bb
b	Bb	bb

Exercises

Two parents have genotypes Bb and Bb. (B = brown, b = blue)

14. What is the probability that their child will have blue eyes?

15. What is the probability that their child will have brown eyes?

Now suppose that one parent has genotype Bb but the other parent has genotype bb. The Punnett square will look like this.

	B	b
b	Bb	bb
b	Bb	bb

Exercises

Two parents have genotypes *Bb* and *bb*. (*B* = brown, *b* = blue)

16. What is the probability that their child will have blue eyes?

17. What is the probability that their child will have brown eyes?

Empirical Probability

The previous methods work when we know the total number of outcomes and we can assume that they are all equally likely. (The dice aren't loaded, for example.) However, life is usually more complicated than a game of dice or a bowl of Tootsie Rolls. In many situations, we have to observe what has happened in the past and use that data to predict what might happen in the future. If someone predicts that an Alaska Airlines flight has an 85.4% of arriving on time[1], that is of course based on Alaska's past rate of success and this number will vary from month to month. When we calculate the probability this way, by observation, we call it an ***empirical probability***.

Empirical probability of an event:

$$P(\text{event}) = \frac{\text{number of favorable observations}}{\text{total number of observations}}$$

Although the wording may seem complicated, we are still just thinking about $\frac{\text{part}}{\text{whole}}$.

Exercises

A photocopier makes 250 copies, but 8 of them are unacceptable because they have toner smeared on them.

18. What is the empirical probability that a copy will be unacceptable?

19. What is the empirical probability that a copy will be acceptable?

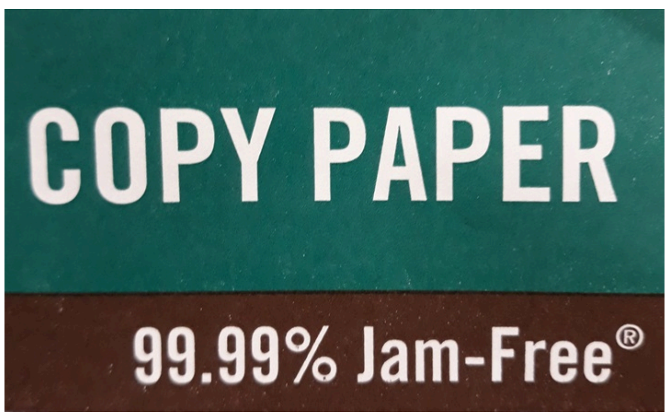
20. Out of the next $1,000$ copies, how many should we expect to be acceptable?

An auditor examined 200 tax returns and found errors on 44 of them.

21. What percent of the tax returns contained errors?

22. How many of the next $1,000$ tax returns should we expect to contain errors?

23. What is the probability that a given tax return, chosen at random, will contain errors?



Probability of More Than One Event

It was mentioned earlier in this module that ***independent events*** have no influence on each other. Some examples:

- Rolling two dice are independent events because the result of the first die does not affect the probability of what will happen with the second die.

- If we flip a coin ten times, each flip is independent of the previous flip because the coin doesn't remember how it landed before. The probability of heads or tails remains $\frac{1}{2}$ for each flip.

- Drawing marbles out of a bag are independent events only if we put the first marble back in the bag before drawing a second marble. If we draw two marbles at once, or we draw a second marble without replacing the first marble, these are *dependent events*, which we are not studying in this course.

- Drawing two cards from a deck of 52 cards are independent events only if we put the first card back in the deck before drawing a second card. If we draw a second card without replacing the first card, these are *dependent events*; the probabilities change because there are only 51 cards available on the second draw.

If two events are independent, then the probability of both events happening can be found by multiplying the probability of each event happening separately.

If A and B are independent events, then $P(A \text{ and } B) = P(A) \cdot P(B)$.

Note: This can be extended to three or more events. Just multiply all of the probabilities together.

Exercises

An auditor examined 200 tax returns and found errors on 44 of them.

24. What is the probability that the next two tax returns both contain errors?

25. What is the probability that the next three tax returns all contain errors?

26. What is the probability that the next tax return contains errors but the one after it does not?

27. What is the probability that the next tax return does not contain errors but the one after it does?

28. What is the probability that neither of the next two tax returns contain errors?

29. What is the probability that none of the next three tax returns contain errors?

30. What is the probability that at least one of the next three tax returns contain errors? (This one is tricky!)

Exercise Answers

Notes

1. Source (PDF file, see page 6): https://www.transportation.gov/sites/dot.gov/files/2024-01/December%202023%20ATCR.pdf

[30]

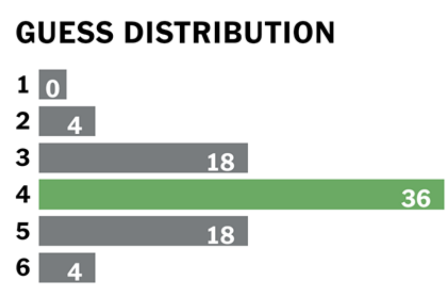
Standard Deviation

You probably won't need a calculator for this module.

This topic requires a leap of faith. It is one of the rare times when this textbook will say "don't worry about *why* it's true; just accept it." 😂

Normal Distributions & Standard Deviation

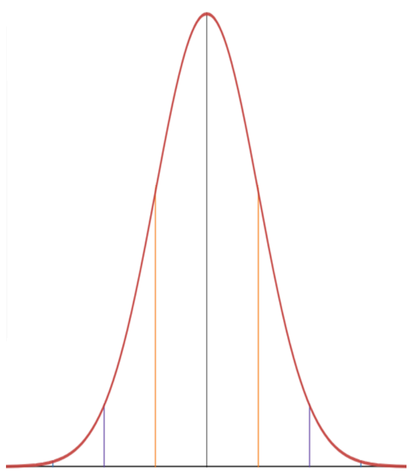
A ***normal distribution***, often referred to as a ***bell curve***, is symmetrical on the left and right, with the mean, median, and mode being the value in the center. There are lots of data values near the center, then fewer and fewer as the values get further from the center. A normal distribution describes the data in many real-world situations.



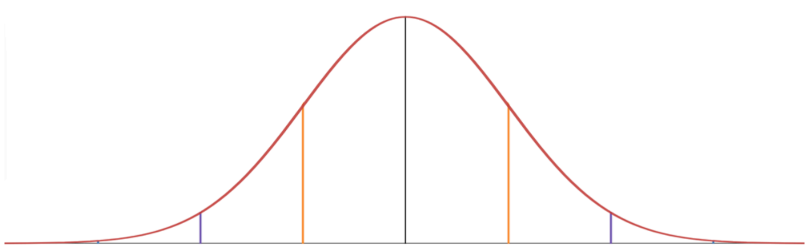
This normal distribution of Wordle scores is formatted horizontally.

One of the best ways to demonstrate the normal distribution is to drop balls through a board of evenly spaced pegs, as shown here. (The Plinko game on *The Price Is Right* is a well known example of this.) Each time a ball hits a peg, it has a fifty-fifty chance of going left or right. For most balls, the number of lefts and rights are roughly equal, and the ball lands near the center. Only a few balls have an extremely lop-sided number of lefts and rights, so there are not many balls at either end. As you can see, the distribution is not perfect, but it is approximated by the normal curve drawn on the glass.

The ***standard deviation*** is a measure of the spread of the data: data with lots of results close to the mean has a smaller standard deviation, and data with results spaced further from the mean has a larger standard deviation. (In this textbook, you will be given the value of the standard deviation of the data and will never need to calculate it.) The standard deviation is a measuring stick for a particular set of data.

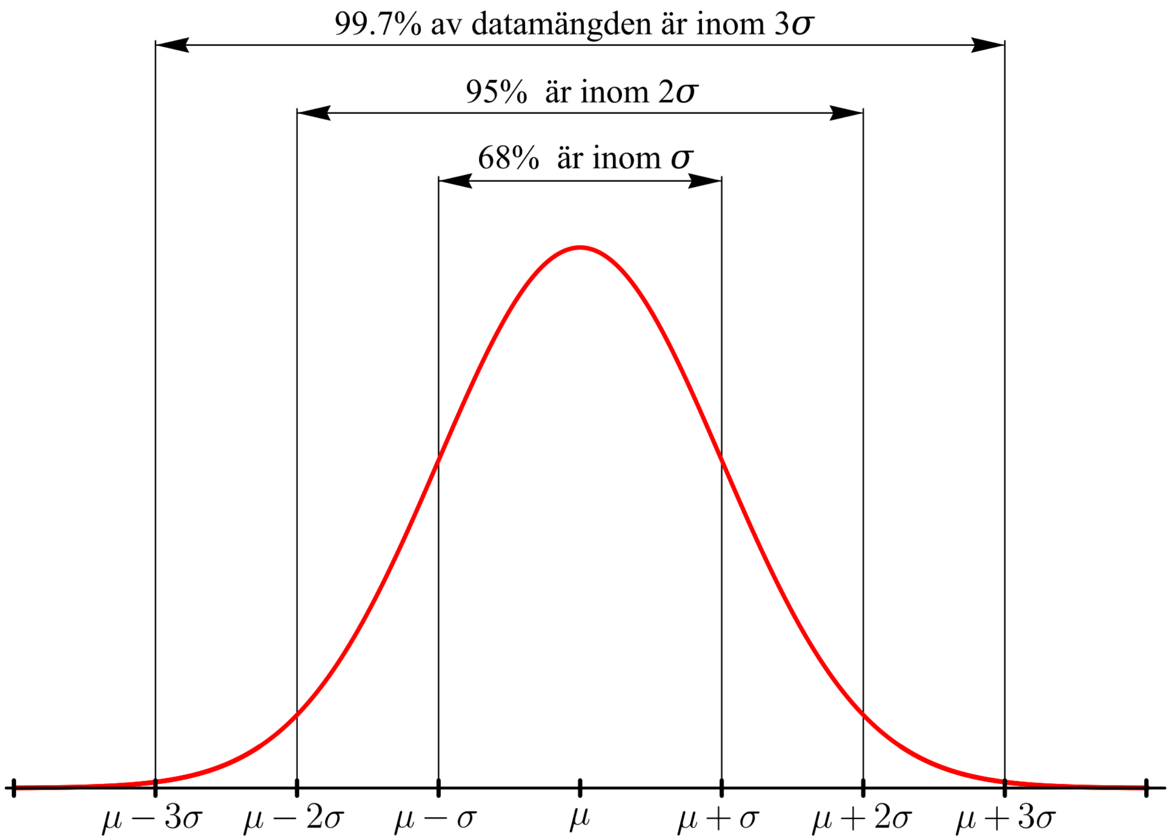


a distribution with a small standard deviation



a distribution with a large standard deviation

The 68-95-99.7 Rule



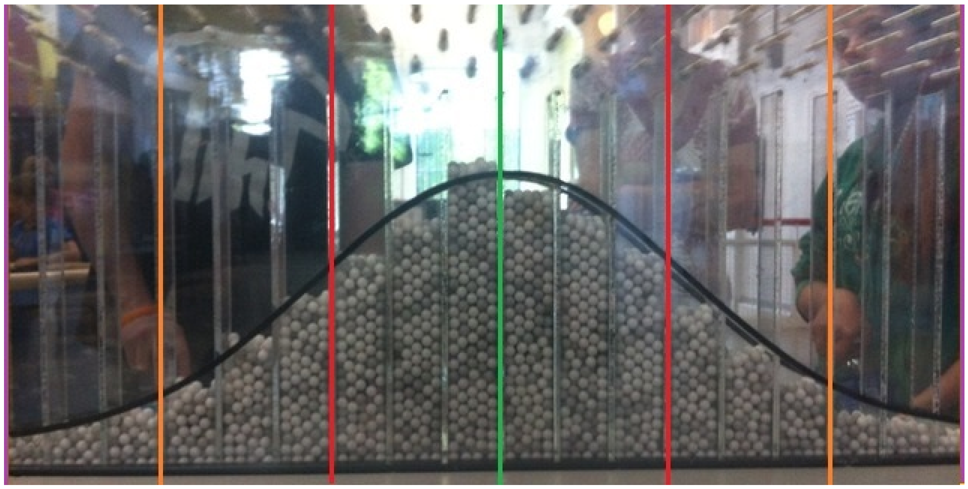
The 68-95-99.7 rule, in Swedish. Image credit: Svjo, hosted at Wikimedia Commons.

The 68-95-99.7 rule: In a normal distribution, approximately…

- 68% of the numbers are within 1 standard deviation above or below the mean
- 95% of the numbers are within 2 standard deviations above or below the mean
- 99.7% of the numbers are within 3 standard deviations above or below the mean

This is an *empirical* rule because it is based on observation of how the world works, rather than being based on a formula.[1]

Returning to the ball-dropping experiment, let's assume that the standard deviation is three columns wide.[2] In the picture below, the **green** line marks the center of the distribution.



First, the two **red** lines are each three columns away from the center, which is **one standard deviation** above and below the **center**, so about **68%** of the balls will land between the red lines.

Next, the two **orange** lines are another three columns farther away from the center, which is six columns or **two standard deviations** above and below the **center**, so about **95%** of the balls will land between the orange lines.

And finally, the two **purple** lines are another three columns farther away from the center, which is nine columns or **three standard deviations** above and below the **center**, so about **99.7%** of the balls will land between the purple lines. We can expect that 997 out of $1,000$ balls will land between the purple lines, leaving only 3 out of $1,000$ landing beyond the purple lines on either end.

Okay, that was a lot of information. For our purposes, the following restatement of the 68-95-99.7 rule may be more practical.

The 68-95-99.7 rule: In a normal distribution with mean μ and standard deviation σ...

- 68% of the numbers are between $\mu - \sigma$ and $\mu + \sigma$
- 95% of the numbers are between $\mu - 2\sigma$ and $\mu + 2\sigma$
- 99.7% of the numbers are between $\mu - 3\sigma$ and $\mu + 3\sigma$

Exercises

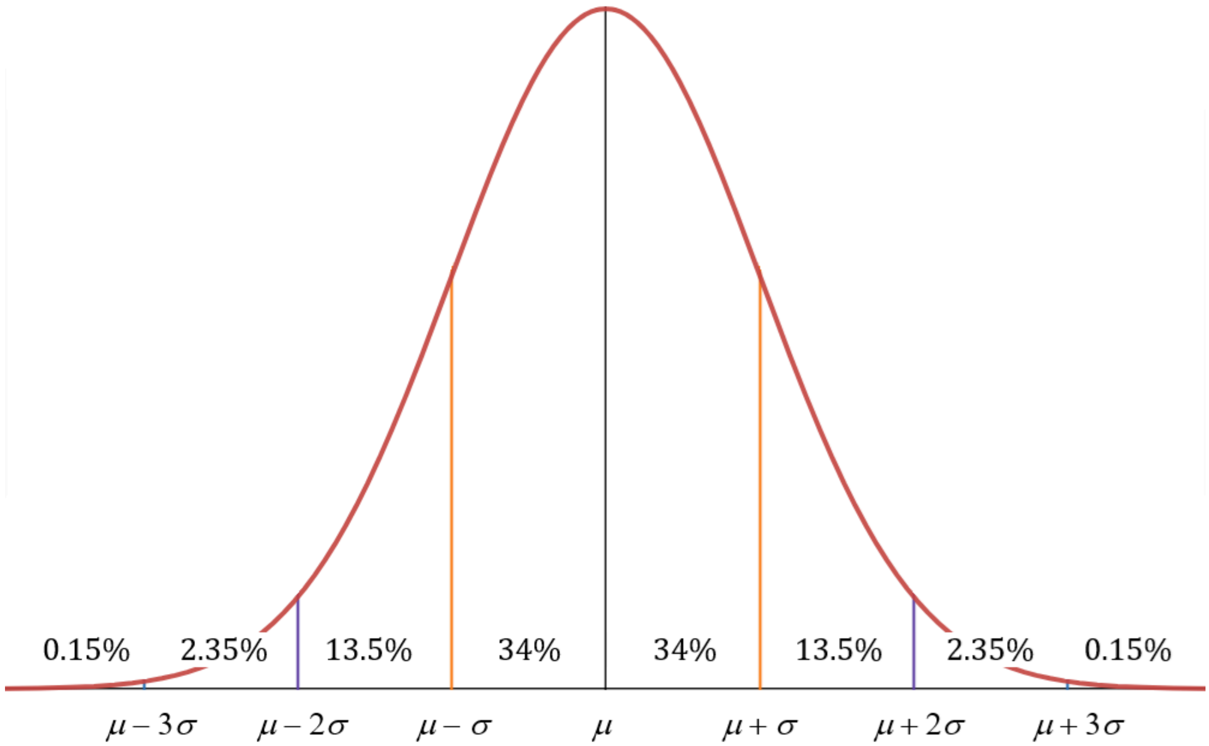
The heights of U.S. females are normally distributed. The average height is around 63.5 inches (5 ft 3.5 in) and the standard deviation is 3 inches. Use the 68-95-99.7 rule to fill in the blanks.

1. About 68% of the women should be between _______ and _______ inches tall.
2. About 95% of the women should be between _______ and _______ inches tall.
3. About 99.7% of the women should be between _______ and _______ inches tall.

The heights of U.S. males are normally distributed. The average height is around 69.5 inches (5 ft 9.5 in) and the standard deviation is 3 inches. Use the 68-95-99.7 rule to fill in the blanks.

4. About 68% of the men should be between _______ and _______ inches tall.
5. About 95% of the men should be between _______ and _______ inches tall.
6. About 99.7% of the men should be between _______ and _______ inches tall.

This graph provides another way to think about the distribution of the data.



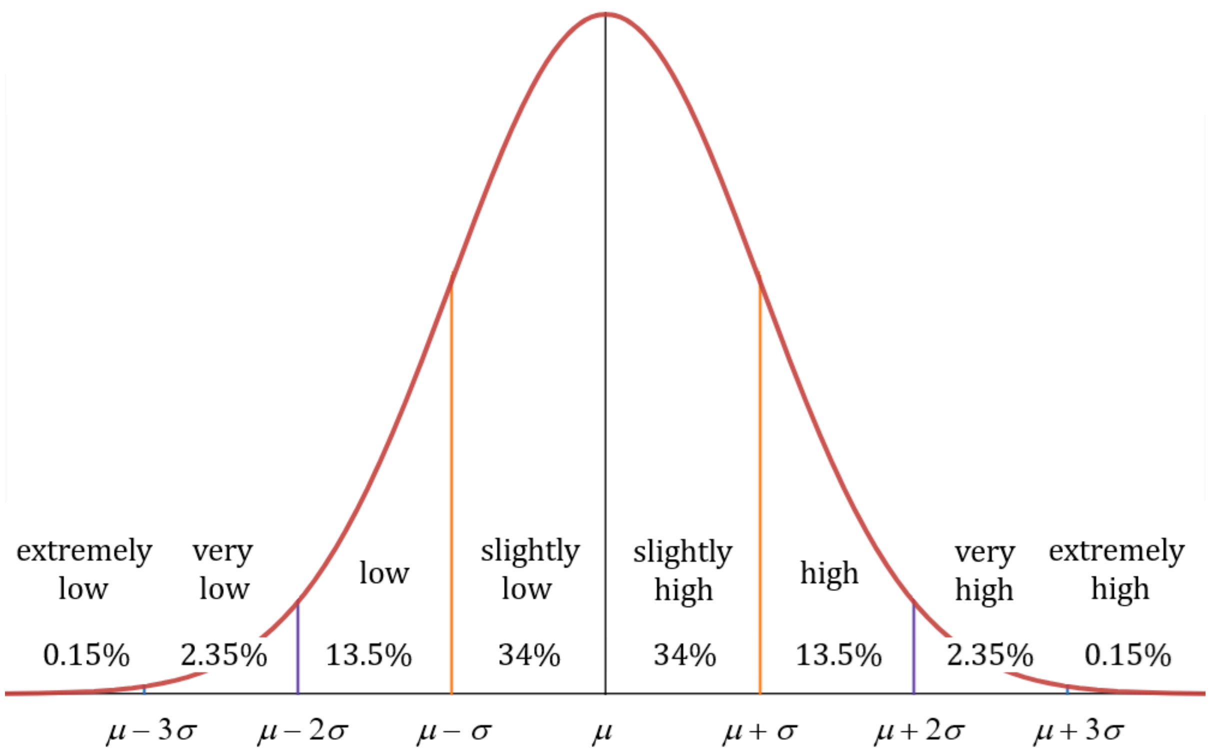
Because 68% of the data are within one standard deviation of the mean, we have 34% of the data slightly below the mean and 34% slightly above. Moving outwards one more standard deviation in each direction, we have another 13.5% below the mean and another 13.5% above the mean, encompassing a total of 95% of the data. Moving outwards one more standard deviation, we have another 2.35% far below the mean and another 2.35% far above the mean, bringing the total up to 99.7% of the data. This leaves only 0.15% of the data more than three standard deviations below the mean and 0.15% of the data more than three standard deviations above the mean.

Exercises

Around 16% of U.S. males in their forties weigh less than 160 lb and 16% weigh more than 230 lb. Assume a normal distribution.[3]

7. What percent of U.S. males weigh between 160 lb and 230 lb?
8. What is the average weight? (Hint: think about symmetry.)
9. What is the standard deviation? (Hint: You have to work backwards to figure this out, but the math isn't complicated.)
10. Based on the empirical rule, about 95% of the men should weigh between _______ and _______ pounds.

This version of the graph can help us group the data into general categories. This is not official terminology, but hopefully it gets the point across.



Looking at the middle 68% of the data, 34% could be considered "slightly low" and 34% could be considered "slightly high". Moving outwards, we have another

13.5% that could be considered "low" and another 13.5% could be considered "high". Moving outwards again, we have another 2.35% that could be considered "very low" and another 2.35% that could be considered "very high". Finally, 0.15% of the data could be considered "extremely low" and 0.15% could be considered "extremely high".

If you are asked only one question about the empirical rule instead of three in a row (68%, 95%, 99.7%), you will most likely be asked about the 95%. This is related to the "95% confidence interval" that is often mentioned in relation to statistics. For example, the margin of error for a poll is usually close to two standard deviations.[4]

Let's finish up by comparing the performance of three NFL teams at the beginning of this century.

Exercises

The numbers of regular-season games won by the New England Patriots[5] each NFL season from 2001-2019: $11, 9, 14, 14, 10, 12, 16, 11, 10, 14, 13, 12, 12, 12, 12, 14, 13, 11, 12$.

The mean number of wins is 12.2, and a spreadsheet tells us that the standard deviation is 1.7 wins.

11. There is a 95% chance of the Patriots winning between _______ and _______ games in a season.

12. In 2020, the Patriots won 7 games. Could you have predicted that based on the data? How many standard deviations from the mean is this number of wins?

The numbers of regular-season games won by the Buffalo Bills[6] each NFL season from 2001-2019: $3, 8, 6, 9, 5, 7, 7, 7, 6, 4, 6, 6, 6, 9, 8, 7, 9, 6, 10$.

The mean number of wins is 6.8, and a spreadsheet tells us that the standard deviation is 1.7 wins.

13. There is a 95% chance of the Bills winning between _______ and _______ games in a season.

14. In 2020, the Bills won 13 games. Could you have predicted that based on the data? How many standard deviations from the mean is this number of wins?

The numbers of regular-season games won by the Denver Broncos[7] each NFL season from 2001-2019: $8, 9, 10, 10, 13, 9, 7, 8, 8, 4, 8, 13, 13, 12, 12, 9, 5, 6, 7$.

The mean number of wins is 9.1, and a spreadsheet tells us that the standard deviation is 2.6 wins.

15. There is a 95% chance of the Broncos winning between _______ and _______ games in a season.

16. In 2020, the Broncos won 5 games. Could you have predicted that based on the data? How many standard deviations from the mean is this number of wins?

Exercise Answers

Notes

1. Well, there is a formula, $y = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, but it was discovered after the fact.
2. I eyeballed it and it seemed like a reasonable assumption.
3. Source (PDF file): https://www2.census.gov/library/publications/2010/compendia/statab/130ed/tables/11s0205.pdf
4. Source: https://en.wikipedia.org/wiki/Standard_deviation
5. Source: https://www.pro-football-reference.com/teams/nwe/index.htm
6. Source: https://www.pro-football-reference.com/teams/buf/index.htm
7. Source: https://www.pro-football-reference.com/teams/den/index.htm

[31]

Right Triangle Trigonometry

Bob Brown and Morgan Chase

You may use a calculator throughout this module.

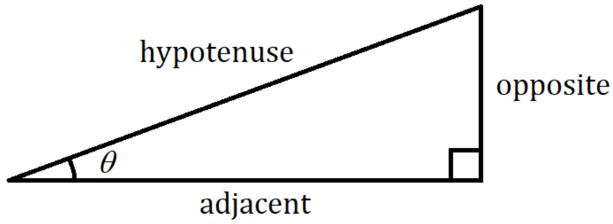
Photo by Georgios Karamanis on flickr.

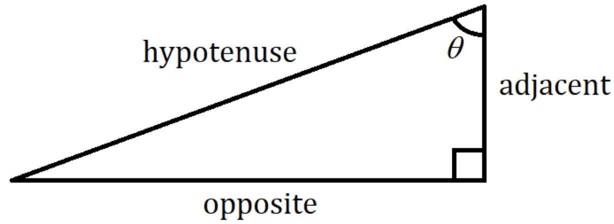
The word "trigonometry" essentially means the measurement of triangles. In this module, we will focus on right triangles. If we know some information about a right triangle, such as the measure of one side and one angle, we can use trigonometry to determine the measure of another side.

Sine, Cosine, Tangent

Consider one of the acute angles in a right triangle.

- The *sine* of the angle is the ratio of the opposite side to the hypotenuse.
- The *cosine* of the angle is the ratio of the adjacent side to the hypotenuse.
- The *tangent* of the angle is the ratio of the opposite side to the adjacent side.



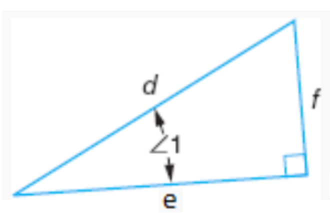


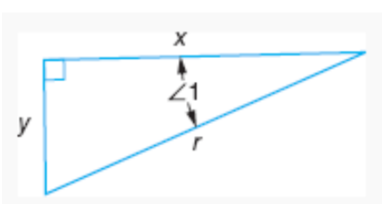
SOHCAHTOA

$$\text{sine} = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\text{cosine} = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\text{tangent} = \frac{\text{opposite}}{\text{adjacent}}$$

Exercises

In relation to $\angle 1$, identify which side of the triangle is the adjacent side, the opposite side, and the hypotenuse.

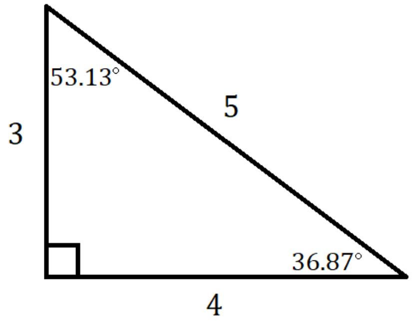
1. 

2. 

Now let's look at a specific example, a right triangle with sides of length 3, 4, and 5. Measuring the angles with a protractor will show us that the smaller acute angle is approximately 37° and the larger acute angle is 53°; to the nearest hundredth of a degree, these values are 36.87° and 53.13°.

Exercises

Use the diagram to determine each of the following, first as a fraction and then as a decimal (rounded to four significant figures, if necessary).



3. $\sin 36.87°$

4. $\cos 36.87°$

5. $\tan 36.87°$

6. $\sin 53.13°$

7. $\cos 53.13°$

8. $\tan 53.13°$

Note that we would have gotten the same results for any right triangle whose sides are in the ratio 3:4:5. If the sides had been 6, 8, 10, the angles measures would still be $36.87°$ and $53.13°$. If the sides had been 30, 40, 50, the angles measures would still be $36.87°$ and $53.13°$.

Or to look at it another way, any right triangle with angles $36.87°$ and $53.13°$ will have these relationships: the shorter leg is $\frac{3}{5}$ the hypotenuse, the longer leg is $\frac{4}{5}$ the hypotenuse, and the shorter leg is $\frac{3}{4}$ the longer leg. Your scientific calculator knows this and is programmed with all the trigonometric values you need.

Exercises

Use your calculator to determine each of the following as a decimal (rounded to four significant figures, if necessary). Be sure that your calculator is in degree mode, not radian mode, before you begin.

9. $\sin 36.87°$

10. $\cos 36.87°$

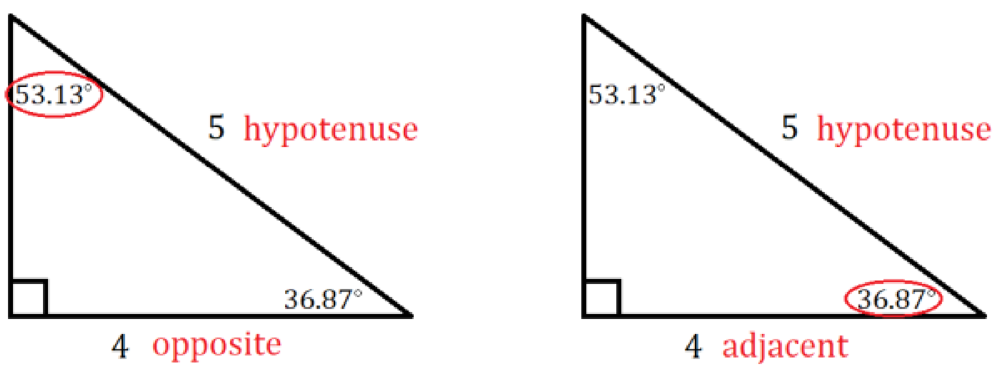
11. $\tan 36.87°$

12. $\sin 53.13°$

13. $\cos 53.13°$

14. $\tan 53.13°$

You may notice some repeated answers here, which makes sense because, for example, the side opposite the $53.13°$ angle is adjacent to the $36.87°$ angle, which means that $\sin 53.13°$ must be equal to $\cos 36.87°$.

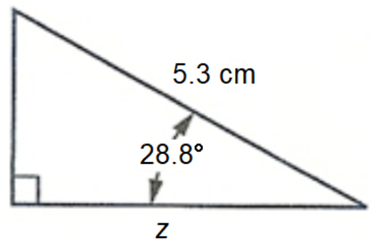


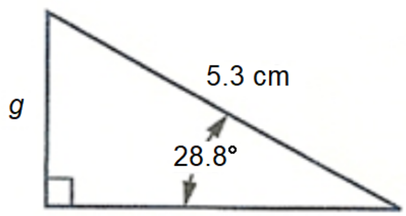
Trigonometry Applications

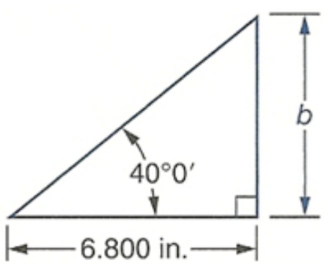
The sine, cosine, and tangent ratios apply to any right triangle, not just a 3-4-5 triangle, which is what makes trigonometry so useful. If we know the measure of one side and one acute angle in a right triangle, we can use SOHCAHTOA to find the length of another side of the triangle. For each of the following exercises, we'll need to decide whether the parts we're dealing with will require us to use sine, cosine, or tangent.

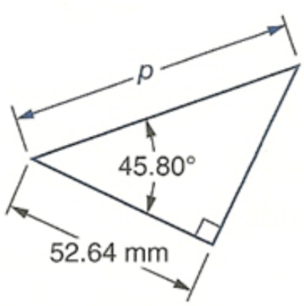
Exercises

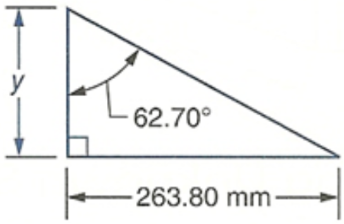
Determine the length of the labeled unknown side of each right triangle. Round each answer appropriately.

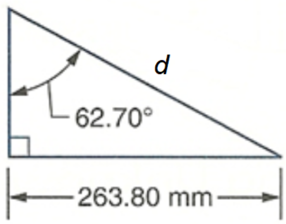
15. 

16. 

17. 

18. 

19. 

20. 

21. A guy wire is attached to a utility pole. The wire is anchored to the ground 15 feet from the base of the pole, forming a $56°$ angle with the level ground. What is the length of the wire, to the nearest foot?

22. A 19-foot ladder is leaning against a wall, forming a $70°$ angle with the level ground. What is the vertical height of the place where the ladder contacts the wall?

23. During intermission of a hockey game, one lucky fan gets to shoot a puck from the blue line at the goal, 64 feet away, for the chance to win a team jersey. Actually, there is a sheet of fabric blocking the entire goal except for a 6-inch wide hole in the bottom center. If the fan shoots the puck hard enough, aiming directly at the center of the hole, they win. If their aim is off by $1°$, will they still win the jersey? (If it helps, a hockey puck is 3 inches in diameter.)

Inverse Sine, Inverse Cosine, Inverse Tangent

Okay, we now know how to find the length of a side if we know an angle and another side. But what if we know the lengths of the sides and want to determine a missing angle? Rather than carefully making a scale drawing and using a protractor, we can use a calculator.

Your calculator's SIN, COS, and TAN keys take an angle measure for the input and give a ratio (in decimal form) as the output. Your calculator's SIN^{-1}, COS^{-1}, and TAN^{-1} keys allow you to work backwards; they take a ratio for the input and give an angle measure for the output.

Exercises

Use your calculator to determine each of the following angle measures (rounded to the nearest hundredth of a degree, if necessary).

24. $\sin^{-1}\left(\frac{3}{5}\right)$

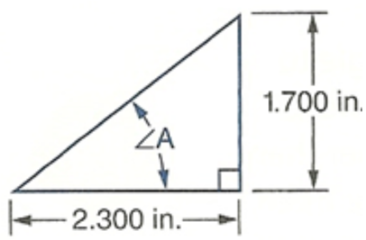
25. $\cos^{-1}\left(\frac{1}{2}\right)$

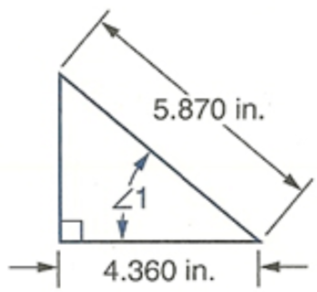
26. $\tan^{-1}\left(\frac{4}{3}\right)$

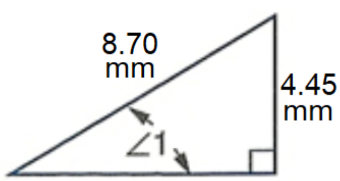
Inverse Trigonometry Applications

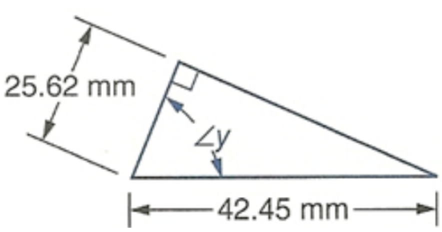
Exercises

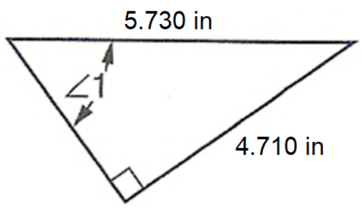
Determine the measure of the marked angle. Round each answer to the nearest hundredth of a degree.

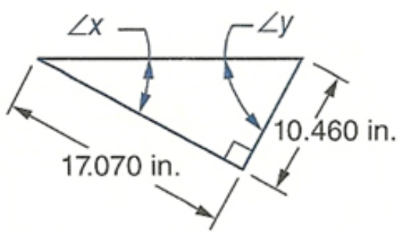
27. 

28. 

29. 

30. 

31. 

32. 

33. A rule of thumb for ladder safety is that the vertical height where the ladder touches the wall should be four times the horizontal distance from the base of the ladder to the wall.[1] To the nearest degree, what is the angle between the ladder and the floor in this scenario?

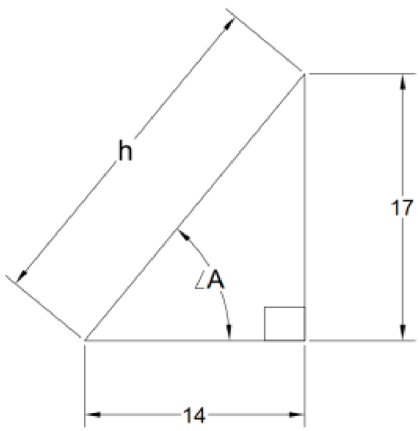
34. The entrance to a public building is 2 feet higher than the courtyard in front of the entrance. A ramp is being constructed, continuous with no switchbacks or level platforms, with a surface length of 25 feet. ADA regulations[2] limit the angle of elevation to a maximum of approximately $4.75°$. Does the ramp's design meet ADA standards?

Photo by [UNDP in Europe and Central Asia]() on [flickr]().

Using Multiple Methods

In Module 18, we used the Pythagorean Theorem, $a^2 + b^2 = c^2$, to determine the length of an unknown side in a right triangle. We can use the Pythagorean Theorem in conjunction with trigonometry to solve a problem in more than one way or to double-check our results.

Exercises



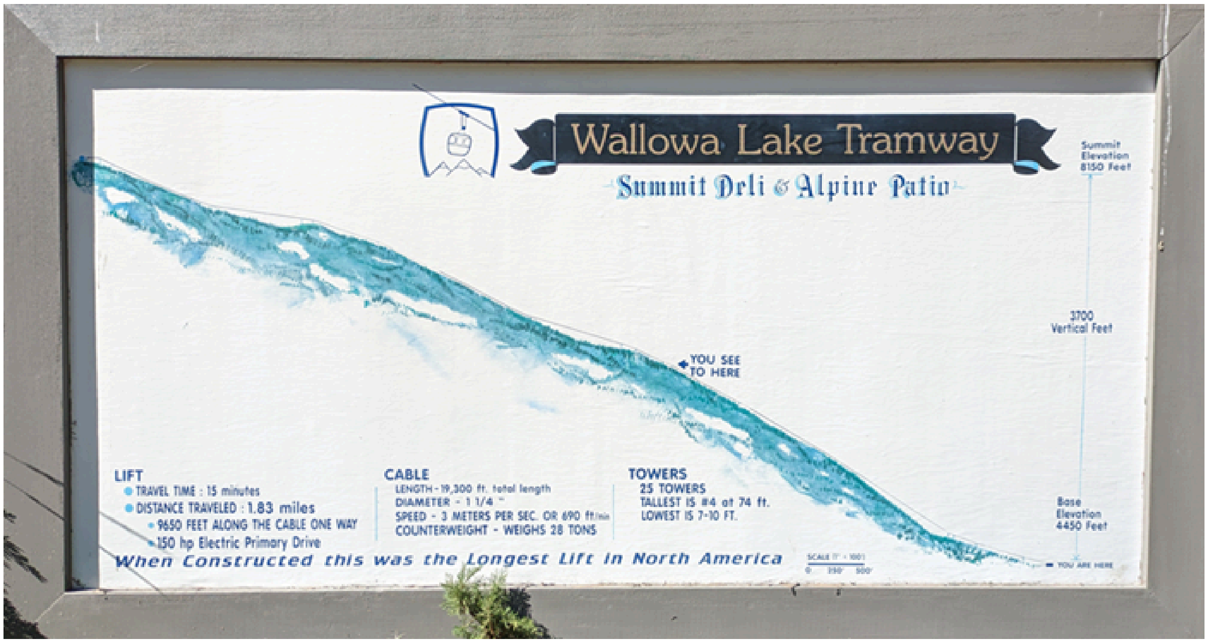
35. First, use inverse tangent to determine the value of A. Round to the nearest tenth of a degree.

36. Next, use sine with $\angle A$ to determine the value of h. Round to the nearest tenth.

37. Next, use cosine with $\angle A$ to determine the value of h. Round to the nearest tenth.

38. Finally, use the Pythagorean Theorem to determine the value of h. Round to the nearest tenth.

39. Compare your answers for Exercises 36-38. Are they the same or are they different?



40. The Wallowa Lake Tramway in Eastern Oregon carries passengers along a $9,650$ ft cable, with a vertical rise of $3,700$ ft. Although the cable is not straight along its entire length, let's assume that it is. To the nearest degree, what is the angle of elevation of the cable?

41. What is the approximate horizontal run covered by the tramway cable?

Exercise Answers

Notes

1. Source: https://www.americanladderinstitute.org/page/Ladders101
2. This is implied at https://www.ada-compliance.com/ada-compliance/ada-ramp but not directly stated. We'll verify this angle measure in the next module.

[32]

Slope

Bob Brown and Morgan Chase

You may use a calculator throughout this module.

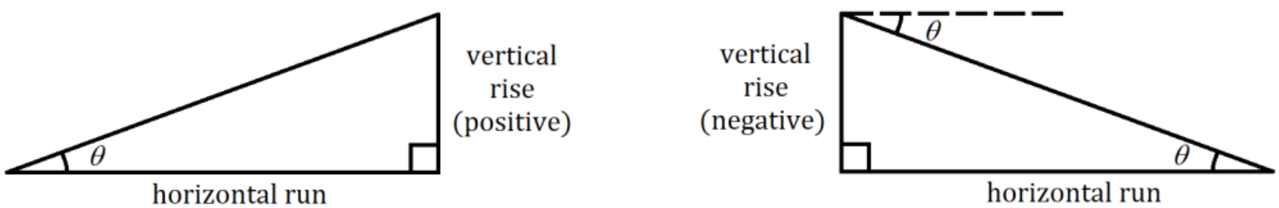
The ***slope*** of a surface is a measure of its steepness. In some cases, such as a walkway or ramp or street, a shallow slope is safer than a steep slope. In other cases, such as a roof, a steep slope may be preferred because it allows rainwater or accumulated snow to move off the roof surface more easily than a shallow slope.[1]

This section of Kearney Street in San Francisco is too steep for a sidewalk and has stairs instead. Photo by Marcus Lenk on Unsplash.

We will first look at slope in terms of vertical and horizontal distances, and we will then look at slope in terms of angles. Come on, let's hit the slopes!

Slope as a Ratio

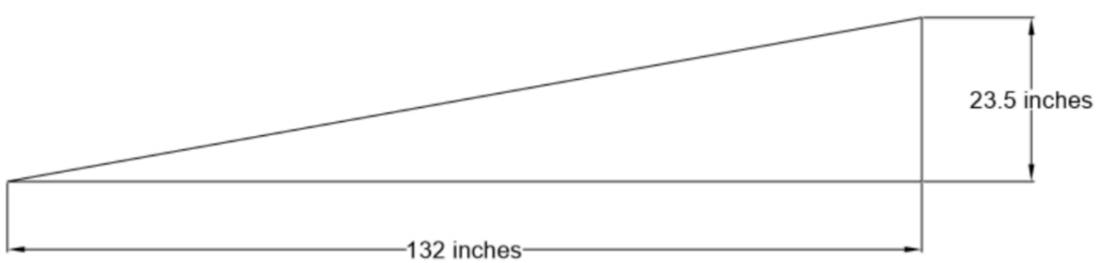
Slope is defined as the ratio of the vertical rise to the horizontal run.



$$\text{slope} = \frac{\text{vertical rise}}{\text{horizontal run}}$$

- A line that is increasing in height has a positive slope.
- A line that is decreasing in height has a negative slope.[2]
- The slope of a horizontal line is 0; it is not increasing or decreasing.

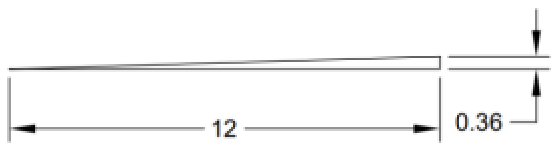
A slope may be expressed as a ratio, a decimal, or a percent grade. For example, consider this loading ramp with a vertical rise of 23.5 inches and a horizontal run of 132 inches.



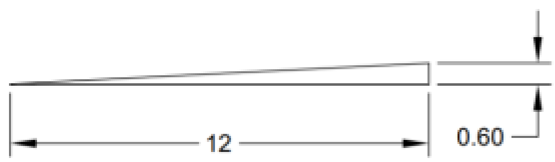
As a ratio, the slope is 23.5 : 132. Dividing the rise by the run gives a decimal value of approximately 0.18. Moving the decimal point two places to the right gives a grade of approximately 18%.

Exercises

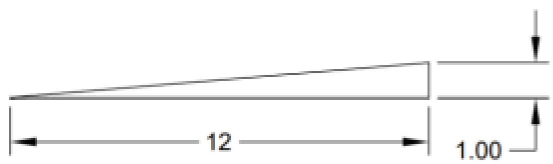
Express each slope in three ways: as a ratio, a decimal, and a percent grade.

1. 

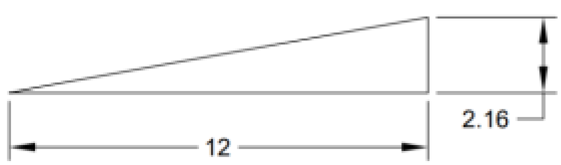
(This is the preferred maximum slope for sidewalks.)

2. 

(This is the maximum allowed slope for sidewalks.)[3]

3. 

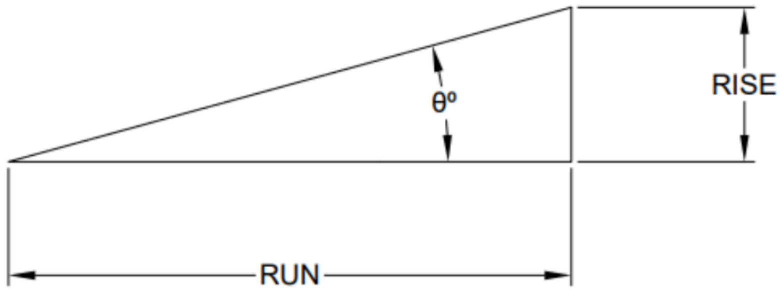
(This is the maximum allowed slope for ADA-accessible ramps.)[4]

4. 

(This is the preferred maximum slope for open pit mines.)

Slope as an Angle

The steepness of a line may also be described by its angle of elevation above the horizontal (or its angle of depression below the horizontal).



Notice that the vertical rise is the side *opposite* the angle, and the horizontal run is the side *adjacent* to the angle. Therefore, trigonometry tells us that $\text{tangent} = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{rise}}{\text{run}}$. The tangent of the angle is equal to the slope.

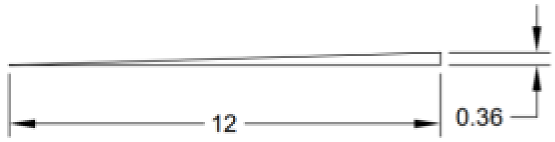
$$\tan\,\theta = \frac{\text{rise}}{\text{run}}$$

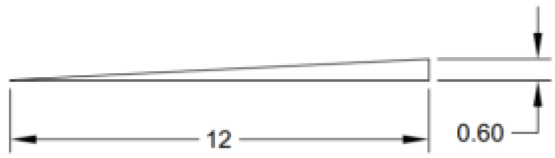
Or, thinking about it in reverse, the inverse tangent of the slope is the angle.

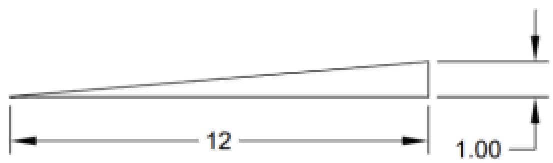
$$\tan^{-1}\left(\frac{\text{rise}}{\text{run}}\right) = \theta$$

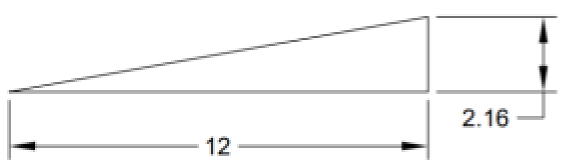
Exercises

Determine the angle of elevation for each slope. Round to the nearest hundredth of a degree.

5. 

6. 

7. 

8. 

For proper drainage, the ground around a building should slope downwards, away from the building.

9. The minimum downward grade of the ground is 1%. Assuming this grade, if a point on the ground is 50 feet horizontally from the base of the house, how much lower is the ground at that point?

10. The preferred minimum downward grade of the ground is 2%. Assuming this grade, if a point on the ground is 50 feet horizontally from the base of the house, how much lower is the ground at that point?

11. The maximum acceptable downward grade of the ground is 10%. Assuming this grade, if a point on the ground is 50 feet horizontally from the base of the house, how much lower is the ground at that point?

---

12. A motion sensor needs to be installed on the outside of a warehouse door 12 feet above the ground. The sensor should go off if anyone approaching the warehouse gets within 20 feet of the door. What angle from vertical should

the sensor point away from the building to detect someone at the appropriate distance from the warehouse? Round your answer to the nearest degree.

13. A ramp is being constructed to the entrance to a public building that is 2.5 feet higher than the level courtyard in front of the entrance. Assuming that the ramp will be continuous with no switchbacks or level platforms,[5] what is the minimum horizontal distance required for the ramp so that it will comply with ADA regulations?

Photo by [Forest Service Pacific Northwest Region] on [flickr].

[Exercise Answers]

Notes

1. Source: https://www.nachi.org/roof-slope-pitch.htm
2. We won't worry about negative slopes in this textbook, because we can always express a negative slope using a word like "decrease", "decline", or "depression" in combination with a positive number.
3. Source: https://www.ada-compliance.com/ada-compliance/403-walking-surfaces
4. Source: https://www.ada-compliance.com/ada-compliance/405-ramps
5. If you were curious, this is the maximum rise allowed for this situation; a rise of more than 2.5 feet would require a switchback or a level platform.

[33]

Non-Right Triangle Trigonometry

You may use a calculator throughout this module.

Your author among the California Redwoods.

Basic trigonometry applies to right triangles, but there are two useful formulas that can be used with non-right triangles; these are known as the Law of Sines and the Law of Cosines.

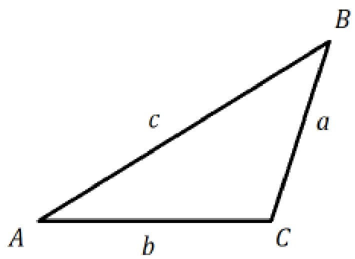
The Law of Sines allows you to use a proportion to solve for a missing value in a triangle.

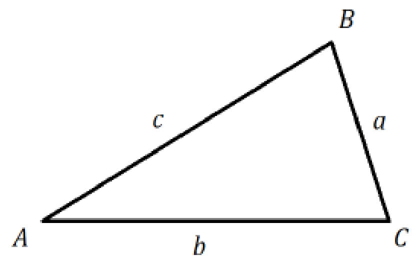
The Law of Cosines is essentially the Pythagorean Theorem with an extra twist that makes it work with any kind of triangle.

These formulas have applications in many fields, most obviously in surveying and forestry where distances cannot be measured directly.

Law of Sines

Suppose we have a triangle with its vertices labeled A, B, C, and the sides opposite each vertex labeled a, b, and c.



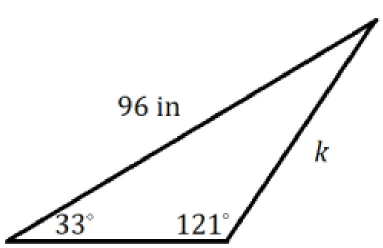


The Law of Sines is useful when you are dealing with two sides and two angles; if you know three of those values, you can use the formula to figure out the fourth value. Depending on which sides and angles you know, you'll use one of these three versions:

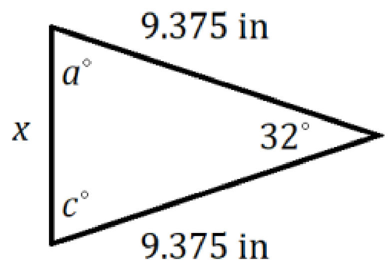
$$\frac{\sin\ A}{a}=\frac{\sin\ B}{b} \qquad \frac{\sin\ B}{b}=\frac{\sin\ C}{c} \qquad \frac{\sin\ C}{c}=\frac{\sin\ A}{a}$$

Exercises

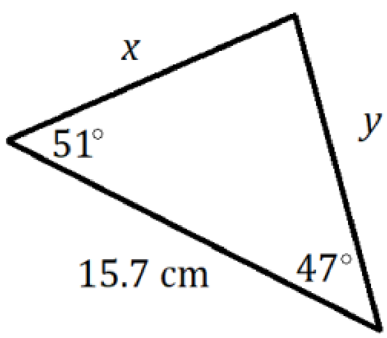
Determine the unknown value(s) in each triangle.



1.

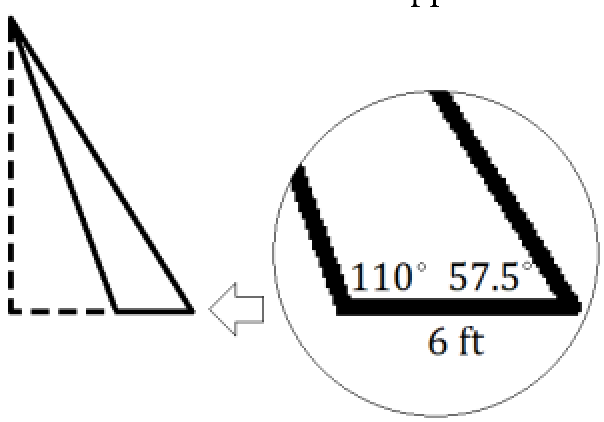


2.



3.

4. A utility pole is supported by two guy wires as shown in the figure below. The shorter wire makes a $110°$ angle with the sidewalk, the longer wire makes a $57.5°$ angle with the sidewalk, and the wires meet the sidewalk a distance of 6 feet from each other. Determine the approximate length of each wire.

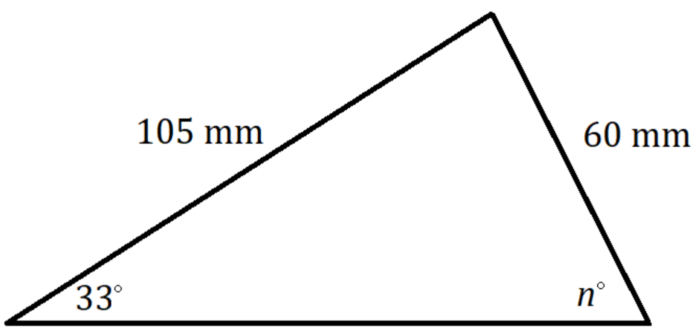


When we need to determine an angle measure, we sometimes have a situation with two possible results: the angle could be acute or obtuse. (For this ambiguous situation to arise, we must know the lengths of two sides and the measure of an acute angle that is *not* between those two sides.) The inverse sine function on a calculator will always be programmed to give an acute angle measure for the result. If it is clear that the angle should be obtuse, simply subtract the calculator's result from 180 degrees.

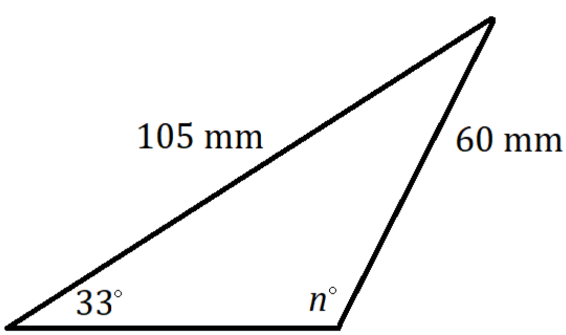
Exercises

Determine the unknown angle measure in each triangle.

5. Assume that n represents an acute angle.

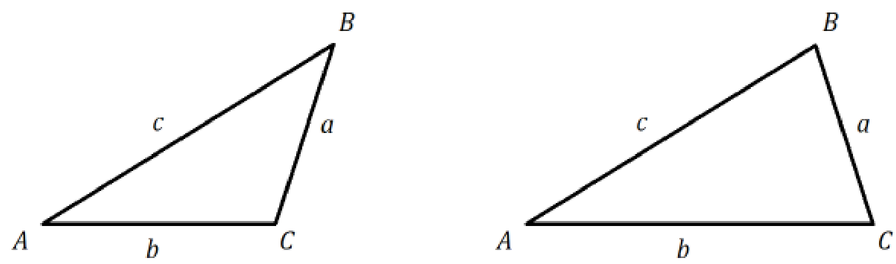


6. Assume that n represents an obtuse angle.



Law of Cosines

Again, suppose we have a triangle with its vertices labeled A, B, C, and the sides opposite each vertex labeled a, b, and c.



The Law of Cosines is useful when you are dealing with three sides and one angle; if you know three of those values, you can use the formula to figure out the fourth value.

If you're trying to determine one of the side lengths, use this version.

$$c = \sqrt{a^2 + b^2 - 2ab\ \cos\ C}$$

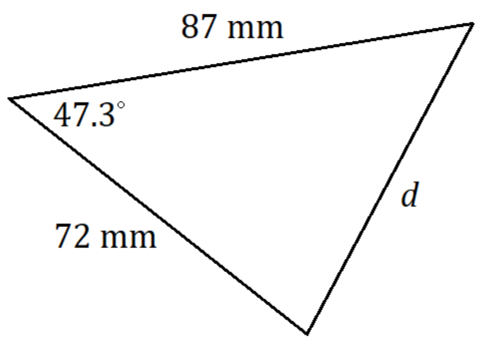
If you're trying to determine an angle measure, use this version.

$$C = \cos^{-1}\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

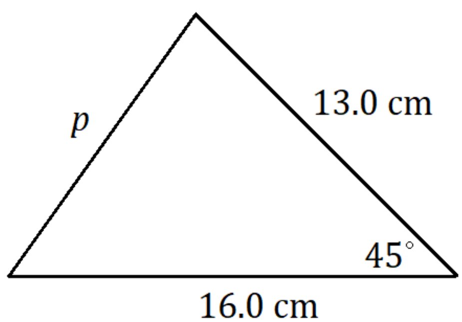
In either case, the side you name c must be opposite the angle you name C.

Exercises

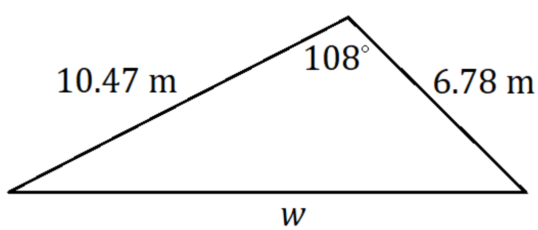
Determine the unknown value in each triangle.



7.

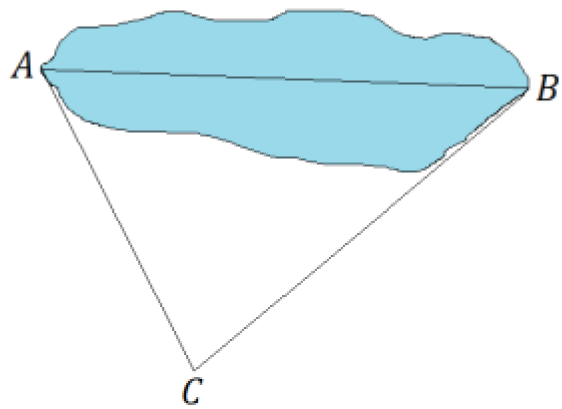


8.

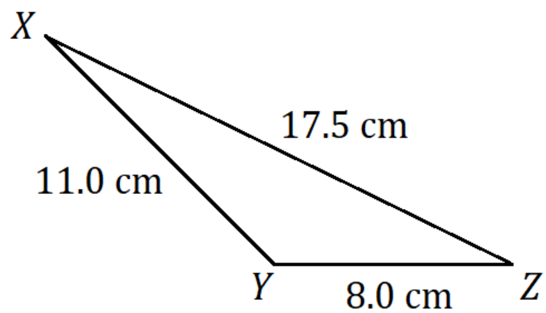


9.

10. Amateur surveyors have determined that the distance from C to A is 53 meters, the distance from C to B is 75 meters, and the measure of $\angle C$ is $77°$, as shown in the figure below. What is the length of the pond?



11. Determine the measure of each angle.



Exercise Answers

[34]

Radian Measure

You may use a calculator as needed in this module.

In everyday life, we measure angles in degrees. However, degrees are an arbitrary unit of measure; why is a right angle 90 degrees and a full circle 360 degrees?[1]

Another unit that is used in surveying is the *gradian*; a right angle is equal to 100 gradians. Using gradians makes some calculations easier, but this is still an arbitrary unit of measure.

In this module, we will focus on a third unit of angle measure known as ***radians***.

Radian Measure

To explain radian measure, we can visualize a circle. Next, let's measure a length along the circumference of a circle that is equal to the radius. Finally, let's draw an angle with its vertex at the center of the circle, intersecting the circle at the endpoints of this arc. The measure of this angle is defined to be 1 radian.

In fewer words, a radian is the measure of the central angle that forms an arc equal to the circle's radius.

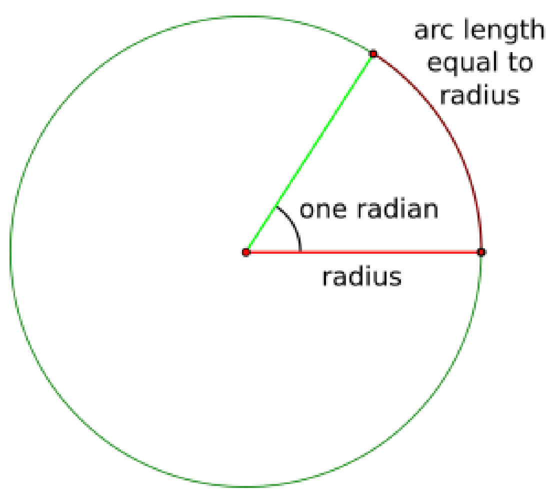
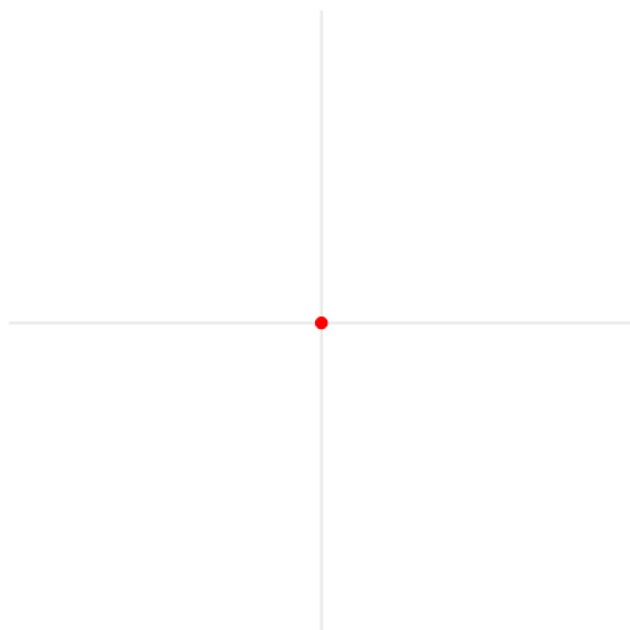


Image by Adrignola on Wikimedia Commons.

Because the circumference of a circle is 2π times the radius, there are 2π radians in one full circle, and therefore 2π radians are equivalent to $360°$.

$$2\pi \text{ rad} = 360°$$

$$\pi \text{ rad} = 180°$$



Animation by Lucas Vieira on Wikimedia Commons.

We can convert between degrees and radians using the conversion factor $\pi \text{ rad} = 180°$.

Exercises

Convert each degree measure into radian measure. Leave your answers in terms of π.

1. $30°$
2. $45°$
3. $60°$

4. $90°$
5. $120°$
6. $225°$
7. $240°$
8. $270°$

Convert each radian measure into degree measure.

9. $\frac{\pi}{12}$
10. $\frac{3\pi}{4}$
11. $\frac{5\pi}{6}$
12. $\frac{7\pi}{4}$
13. 1

[Exercise Answers](#)

Notes

1. See https://en.wikipedia.org/wiki/Degree_(angle)#History for some theories.

Exercise Answers

Module 1: Order of Operations

1. 7
2. 13
3. 7
4. 13
5. 2
6. 8
7. 18
8. 6
9. 25
10. 49
11. 80
12. 31
13. 28
14. 67
15. 22
16. 4

17. 160
18. 19
19. 2
20. 12
21. 40
22. 200
23. 2
24. 14
25. $9 \cdot 2 + 30 = 48^{\circ}\mathrm{F}$
26. $(72 - 30) \div 2 = 21^{\circ}\mathrm{C}$

Module 2: Negative Numbers

1. 5
2. 5
3. -15
4. -22
5. 4
6. -4
7. -9
8. 9
9. $18^{\circ}\mathrm{F}$
10. 3
11. -200

12. 3

13. -3

14. -7

15. -7

16. 7

17. 7

18. 3

19. -3

20. 55

21. 55

22. $11,123$ ft

23. 543 ft

24. -12

25. -40

26. 18

27. 21

28. 4

29. -8

30. 16

31. -32

32. -7

33. -4

34. 9

35. 0

36. 0

37. undefined

38. 19

39. -73

40. 1

41. -6

42. -8

43. 40

Module 3: Decimals

If you've seen Modules 5 & 6, don't worry about accuracy or precision on these exercises.

1. 90.23

2. 7.056

3. 16.55

4. 184.015

5. 8.28

6. 15.756

7. $4,147$

8. 414.7

9. 41.47

10. 4.147

11. $65,625$

12. 65.625

13. 6.5625

14. $\$656.25$

15. 243.5

16. $2,435$

17. $243,500$

18. 24.35

19. $6,000$

20. $6,380$

21. 0.71

22. 0.715

23. $\$3.67$ per month

24. 7.5 miles per hour

Module 4: Fractions

1. $\frac{11}{30}$

2. $\frac{19}{30}$

3. $\frac{12}{15}$

4. $\frac{8}{12}$

5. 7

6. 1

7. 0

8. undefined
9. $\frac{3}{4}$
10. $\frac{5}{3}$
11. 2
12. $\frac{1}{2}$
13. $\frac{5}{12}$
14. 1
15. at least 45 questions
16. 16
17. $\frac{3}{5}$
18. 6 scoops
19. A requires $\frac{1}{12}$ cup more than B
20. $\frac{1}{2}$ of the pizza
21. $\frac{2}{3}$ more
22. $\frac{5}{8}$ inches combined
23. $\frac{1}{8}$ inches difference
24. $\frac{7}{12}$ combined
25. $\frac{1}{12}$ more
26. 2.75
27. 0.35
28. $0.\overline{5}$ or $0.555...$
29. $1.\overline{63}$ or $1.636363...$

30. $11\frac{1}{2}$

31. $4\frac{2}{3}$

32. $\frac{11}{5}$

33. $\frac{20}{3}$

34. $10\frac{3}{8}$

35. $4\frac{7}{8}$

36. $8\frac{1}{6}$

37. $1\frac{5}{6}$ cup

Module 5: Accuracy and Significant Figures

1. exact value
2. approximation
3. exact value
4. approximation
5. exact value
6. approximation
7. three significant figures
8. four significant figures
9. five significant figures
10. two significant figures
11. three significant figures
12. four significant figures
13. two significant figures; the actual value could be anywhere between $28,500$ and

$29,500$

14. three significant figures; the actual value could be anywhere between $28,950$ and $29,050$
15. four significant figures; the actual value could be anywhere between $28,995$ and $29,005$
16. five significant figures; the actual value could be anywhere between $28,999.5$ and $29,000.5$
17. $51,800$
18. $51,840$
19. 4.3
20. 4.28
21. $14,000$
22. $14,\bar{0}00$
23. 2.6
24. 2.60
25. $29,000$ ft
26. $29,\bar{0}00$ ft
27. $29,030$ ft
28. $29,032$ ft
29. $29,031.7$ ft
30. 107
31. 640
32. 14.4
33. 12

34. $\$23$

Module 6: Precision and GPE

1. thousands
2. hundreds
3. tens
4. thousandths
5. ten thousandths
6. hundred thousandths
7. $82,000$
8. $82,\bar{0}00$
9. $82,0\bar{0}0$
10. 0.6
11. 0.60
12. 0.600
13. 39.3 lb
14. 39 lb
15. $\$9,800$
16. $\$8\bar{0}0$
17. thousands place; the nearest $1,000$ people
18. ± 500 people
19. hundred thousandths place; the nearest 0.00001 in
20. ± 0.000005 in

21. hundredths place; the nearest 0.01 mil
22. ±0.005 mil
23. $30\overline{0}$ miles
24. ones place; the nearest 1 mile
25. ±0.5 mi
26. ones place; the nearest 1 minute
27. ±0.5 min
28. two sig figs
29. three sig figs
30. 55 mi/hr
31. When dividing, we must round the result based on the accuracy; i.e., the number of significant figures.

Module 7: Formulas

1. $\$2.46$
2. $\$4.14$
3. $\$17.80$
4. $\$24.80$
5. $\$3.80$
6. 6 representatives
7. 10 representatives
8. 52 representatives
9. 8 electoral votes

10. 12 electoral votes
11. 54 electoral votes
12. $\approx 115^{\circ}$F; the official record high in the city was 116°F.
13. 37°C
14. -0.4°F
15. 200°C
16. 90 mm Hg
17. around 107 mm Hg
18. 70 in
19. 74 in
20. yes
21. no; too large
22. no; too small
23. yes

Module 8: Perimeter and Circumference

1. 70 ft
2. 58 cm
3. 28 cm
4. 104 ft
5. 130 ft
6. 56 ft
7. 24 in

8. 20 cm
9. 28.3 in
10. 18.8 cm
11. 44.0 ft
12. 53 m

Module 9: Percents Part 1

1. 47%
2. 53%
3. $\frac{71}{100}$
4. $\frac{1.3}{100} = \frac{13}{1000}$
5. $\frac{0.04}{100} = \frac{1}{2500}$
6. $\frac{106}{100} = \frac{53}{50}$
7. 0.71
8. 0.013
9. 0.0004
10. 1.06
11. 23%
12. 7%
13. 8.5%
14. 250%
15. 28%
16. 12.5%

17. 31.5

18. 22.5

19. 67.5

20. 100

21. 38.6

22. 2.25

23. $\$9.35$

24. $\$119.32$

Module 10: Ratios, Rates, Proportions

1. $\frac{3}{8}$

2. $\frac{105 \text{ mi}}{2 \text{ hr}}$

3. $\frac{52.5 \text{ mi}}{1 \text{ hr}}$ or 52.5 miles per hour

4. $\approx \$0.335/\text{oz}$, or $33.5¢/\text{oz}$

Carmella Creeper parties like it’s 1995.

5. $\approx \$0.281/\text{oz}$, or $28.1¢/\text{oz}$

6. $\approx \$0.281/\text{oz}$, or $28.1¢/\text{oz}$

7. $\frac{3}{4} = \frac{3}{4}$; true

8. $\frac{2}{3} \neq \frac{4}{5}$; false

9. $168 = 168$; true

10. $200 \neq 240$; false

11. $70 \neq 60$; false

12. $20 = 20$; true

13. $x = 12$

14. $n = 5$

15. $k = 4$

16. $w = 10$

17. $x = 10.4$

18. $m = 2.0$

19. 256 miles

20. 20 hours

21. ≈ 50 miles (rounding to one sig fig seems like a good idea here)

22. 190 pixels wide

Module 11: Scientific Notation

1. Earth's mass is larger because it's a 25-digit number and Mars' mass is a 24 -digit number, but it might take a lot of work counting the zeros to be sure.

2. Earth's mass is about ten times larger, because the power of 10 is 1 higher than that of Mars.

3. A chlorine atom's radius is larger because it has 9 zeros before the significant digits begin, but a hydrogen atom's radius has 10 zeros before the significant digits begin. As above, counting the zeros is a pain in the neck.

4. The chlorine atom has a larger radius because its power of 10 is 1 higher than that of the hydrogen atom. (Remember that -10 is larger than -11 because -10 is farther to the right on a number line.)

5. $\$100,000$; 1.00000×10^5

6. $\$999,999$; 9.99999×10^5

7. 1.234×10^3

8. 1.02×10^7

9. 8.70×10^{-4}

10. 7.32×10^{-2}

11. $35,000$

12. $90,120,000$

13. 0.00825

14. 0.000014

15. 8×10^7

16. 3.5×10^{13}

17. 6×10^{-5}

18. 4.8×10^5

19. $1,260$ people per square mile

20. 250 people per square mile

21. the proton's mass is roughly $1,830$ or 1.83×10^3 times larger

22. 335.9×10^6; 3.359×10^8

23. 8.020×10^9; 8.020×10^9

24. 33.9×10^{12}; 3.39×10^{13}

25. $\approx \$101,000$ per person

Module 12: Percents Part 2 and Error Analysis

1. 93% or 93.3%
2. 37.5%
3. $\$2,500$
4. 720
5. 93% or 93.3%
6. 37.5%
7. $\$2,500$
8. 720
9. 44.8% or 45%
10. 50 grams of added sugars is the recommended daily intake for a $2,000$ calorie diet.
11. 56% increase
12. 10.1% sales tax
13. 36% decrease
14. 2.7% decrease
15. $0.1875 \div 25 \approx 0.75\%$
16. $0.13 \div 10.8 \approx 1.2\%$
17. 4.806 g; 5.194 g
18. 3.88%
19. 5.443 g; 5.897 g

20. 4.00%

Module 13: The US Measurement System

We generally won't worry about significant figures in these answers; we'll probably say "2 miles" even if "2.000 miles" is technically correct.

1. 54 in
2. 54 ft
3. 36 in
4. $1,760$ yd
5. $14\frac{2}{3}$ ft or 14 ft 8 in
6. 15 yd
7. 2 mi
8. 30 yd
9. 40 oz
10. $2,400$ lb
11. 18.75 lb
12. $32,000$ oz
13. 48 fl oz
14. 7 pt
15. 8 pt
16. 5 c
17. 1.25 gal
18. 64 fl oz

19. 3 lb 7 oz
20. 7 c 3 fl oz
21. 15 ft 2 in
22. 4 t 500 lb
23. 20 lb or 20 lb 0 oz combined
24. 3 lb 2 oz heavier
25. 9 ft 1 in combined
26. 1 ft 5 in longer

Module 14: The Metric System

1. 5 m
2. 28 cm
3. 3.8 km
4. 1.6 m
5. 160 cm
6. 3 mm
7. 536 cm
8. $5,360$ mm
9. $1,609$ m
10. $160,900$ cm
11. 0.297 m
12. 29.7 cm
13. 0.828 km

14. 82.8 dam
15. 100 g
16. 80 kg
17. 500 mg
18. $2,000$ kg
19. $2,270$ g
20. $2,270,000$ mg
21. $6,500$ cg
22. $65,000$ mg
23. 0.065 kg
24. 9.5 cg
25. 0.095 g
26. 50 L
27. 30 mL
28. 0.5 L
29. 10.5 dL
30. $1,750$ mL
31. 0.25 L
32. they are equal in size
33. about 11 to 1
34. 4 bottles; this is easier if you know that a 500-milliliter bottle of Mexican Coke is called a *medio litro*.

Doce litros de Coca-Cola

Module 15: Converting Between Systems

We may round some of these answers to three significant figures even if the given number has fewer than three sig figs. On the other hand, you'll see that some of these answers include a critique of a manufacturer's decision to round numbers a certain way.

1. 31 mi
2. 183 cm
3. 164.0 ft
4. yes, $4 \text{ in} = 101.6 \text{ mm}$ according to the conversion, but is it really 4.000 in to begin with? Rounding the result to 100 mm or 102 mm seems reasonable.
5. 20 in converts to around 50.8 cm, and 50.0 cm converts to around 19.7 in. It looks like somebody used the conversion $1 \text{ in} = 2.5 \text{ cm}$, which is fine if you're estimating but not if you're going to report a number to three sig figs.
6. not exactly but they're pretty close; the error is around 0.3%.
7. yes; using either conversion gives a result of 294.8 kg, rounded to the nearest tenth. However, it doesn't make sense for this to be accurate to four sig figs. It would be best to round to 295 or perhaps even $3\bar{0}0$ kg.
8. not exactly; using 1 kg ≈ 2.205 lb gives a result of $1,473.9$ kg, and using 1 lb

≈ 0.4536 kg gives a result of $1,474.2$ kg. Again, the accuracy here doesn't make sense, especially when you consider that the numbers on the box don't agree with each other: $294.8 \cdot 5 \neq 1,474.1$.

9. 11 lb
10. $\approx 230 \text{ g}$
11. $\approx 110 \text{ g}$
12. $\approx 1.8 \text{ oz}$
13. about $\$1.09$
14. about $\$0.99$
15. $\approx 1.3 \text{ gal}$
16. 355 mL
17. 3.4 fl oz
18. no matter which conversion you use, the result should round to 22.7 liters.
19. a bit less than 600 km
20. 11 km/L

Module 16: Other Conversions

We will generally round these answers to three significant figures; your answer may be slightly different depending on which conversion ratio you used.

1. $525,600$ min; if you're familiar with the musical *Rent*, then you already knew the answer.
2. this is roughly 31.7 years, which is indeed possible
3. 37.6 km/hr
4. 23.3 mi/hr

5. $1,770$ mi/hr
6. 29.5 mi in 1 min
7. 20.3 min
8. 0.17 mi/gal
9. 5.8 gal/mi
10. 171 gal in 1 min
11. the capacity increased by a factor of 14.4
12. 4 times greater
13. $1,200$ megawatts per home
14. 1 watt per gallon
15. $2,500$ times more powerful
16. 0.4 ms, 0.04 ms, 0.004 ms; 400 µs, 40 µs, 4 µs
17. the ratio of the wavelengths of red and infrared is 7 to 100;
    the ratio of the wavelengths of infrared and red is around 14 to 1
18. this is equivalent to $2,500$ chest x-rays

Module 17: Angles

1. right angle
2. obtuse angle
3. reflexive angle
4. straight angle
5. acute angle
6. $a = 127°$; $b = 53°$; $c = 127°$

7. $27°$
8. $97°$
9. $23°$ each
10. $45°$ each
11. $60°$ each
12. $A = 61°$; $B = 80°$; $C = 39°$
13. $18.9111°$
14. $155.6808°$
15. $34.1924°$
16. $29°58'30''$
17. $31°8'15''$
18. $76°20'48.1''$

Module 18: Triangles

1. right isosceles triangle
2. obtuse scalene triangle
3. acute equilateral triangle (yes, an equilateral triangle will always be acute)
4. $w = 35$ ft
5. $x = 8$ cm; $y = 10.5$ cm
6. $d = 268$ ft
7. $n = 55$ cm
8. this is a right triangle, because $5^2 + 12^2 = 13^2$.
9. this is not a right triangle, because $8^2 + 17^2 \neq 19^2$.

10. 7.07
11. 17.20
12. 30.71
13. 10 ft
14. 15 ft
15. 12.3 cm
16. 1.8 cm

Module 19: Area of Polygons and Circles

We may occasionally include extra sig figs in these answers so you can be sure that your answer matches ours.

1. 20 cm^2
2. 16 cm^2
3. 4.86 m^2
4. 12.25 ft^2
5. 120 in^2
6. 360 m^2
7. 210 ft^2
8. 126 cm^2
9. 38.5 cm^2
10. 204 ft^2
11. $2,160 \text{ in}^2$
12. 36 m^2

13. 124 cm^2

14. 192 cm^2

15. 28.3 cm^2

16. 220 m^2

17. 154 ft^2

18. 63.6 in^2

Module 20: Composite Figures

1. 64 ft

2. 189 ft^2

3. 590 cm^2; the area of the rectangle is 800 cm^2 and the areas of the triangles are 70 cm^2 and 140 cm^2.

4. 590 cm^2; hey, that's what we got for #3!

5. 148 m

6. 940 m^2

7. Based on the stated measurements, the distance around the track will be 401 meters, which appears to be 1 meter too long. In real life, precision would be very important here, and you might ask for the measurements to be given to the nearest tenth of a meter.

8. around $9,620 \text{ m}^2$

9. $1,960 \text{ cm}^2$

10. 178.5 cm

11. 29 ft^2

12. 47 ft

13. around 21.5% (Hint: Make up an easy number for the side of the square, like 2 or 10.)

14. around 36.3% (Hint: The diagonals of the square are equal to the circle's diameter.)

Module 21: Converting Units of Area

We may occasionally include extra sig figs in these answers so that you can be sure that your answer matches ours.

1. 162 ft^2
2. 162 ft^2
3. 7 ft^2
4. 7 ft^2
5. $7,776 \text{ in}^2$
6. 8.3 ac
7. $180,000 \text{ cm}^2$
8. $180,000 \text{ cm}^2$
9. 623.7 cm^2
10. 623.7 cm^2
11. 6 m^2
12. 4 ha
13. 376 km^2
14. $2,\overline{0}00 \text{ ha}$, rounded to two sig figs
15. 603 cm^2
16. $75,300 \text{ ft}^2$, rounded to three sig figs

17. 154 in^2

18. 38.5 in^2

19. 4 to 1; if we double the linear measurement, we get four times the area.

20. 54 cm^2

21. 6 cm^2

22. 9 to 1; if we triple the linear measurement, we get nine times the area.

Module 22: Surface Area of Common Solids

1. 36 cm^2

2. 76 cm^2

3. 471 cm^2

4. 628 cm^2

5. 616 cm^2

6. 380 in^2

Module 23: Area of Regular Polygons

All answers have been given to two or three significant figures.

1. $6,900 \text{ in}^2$

2. 94 cm^2

3. 750 in^2

4. 751 mm^2

5. 480 cm^2

6. 110 m^2

7. 280 mm^2 (the area of the circle $\approx 1,660\text{ mm}^2$ and the area of the hexagon is $1,380\text{ mm}^2$)

Module 24: Volume of Common Solids

1. 40 cm^3
2. 531 cm^3
3. 45 cm^3
4. 350 cm^3
5. 520 cm^3
6. $22,600\text{ ft}^3$
7. 3.7 mm^3
8. $1,440\text{ cm}^3$
9. 697 in^3 or $7\overline{0}0\text{ in}^3$
10. 22.7 cm^3 (the cylinder's volume $\approx 14.1\text{ cm}^3$ and the hemisphere's volume $\approx 8.6\text{ cm}^3$.)
11. 37.6 ft^3 (the cylinder's volume $\approx 29.45\text{ ft}^3$ and the two hemispheres' combined volume $\approx 8.18\text{ ft}^3$)
12. $37.6\text{ ft}^3 \approx 282\text{ gal}$, which is more than 250 gal.

Module 25: Converting Units of Volume

1. the result is very close to 1 cubic yard:
   $(112\text{ in} \cdot 14\text{ in} \cdot 10\text{ in}) \cdot 3\text{ crates} = 47,040\text{ in}^3 \approx 1.01\text{ yd}^3$
2. this estimate is also 1 cubic yard: $(9\text{ ft} \cdot 1\text{ ft} \cdot 1\text{ ft}) \cdot 3\text{ crates} = 27\text{ ft}^3 = 1\text{ yd}^3$
3. around 60 gallons

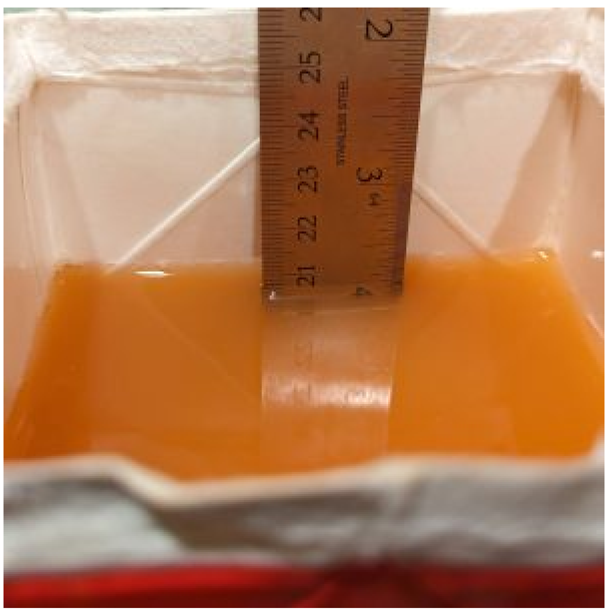
4. yes, the can is able to hold 12 fluid ounces; the can's volume is roughly $23.3 \text{ in}^3 \approx 12.9 \text{ fl oz}$.

5. 5 gallons

6. around $2,300$ to $2,400$ liters; a calculator says $2,356$ liters which should technically be rounded up to $2,400$ liters, but it would be reasonable to round down to $2,300$ liters instead if you considered the volume of the benches and the fact that the sides might slope inwards near to bottom of the tub.

7. the rectangular section of the carton has a volume of 1.7 liters, which is larger than the required 1.5 L.

8. ≈ 21 cm high



Independent verification from my kitchen.

9. 19.6 yd^3

10. $53\overline{0} \text{ ft}^3$

11. 1.53 m^3

12. 327 in^3

13. $5,350 \text{ cm}^3$

14. 5.35 L

15. $0.5 \text{ ft}^3 \approx 14.16 \text{ L}$; assuming that it's 0.50 ft^3, these measures are equivalent to two sig figs.

16. 100 lb/ft^3

17. 31 lb; now you can laugh whenever you see someone in a heist movie load a cheap duffel bag with gold bars and carry it out of the vault.

18. 1.1 kg

19. 268 cm^3

20. 33.5 cm^3

21. 8 to 1

22. $1,620 \text{ in}^3$

23. 60 in^3

24. 27 to 1

Module 26: Pyramids and Cones

1. $1,280 \text{ cm}^3$

2. $2,420,000 \text{ m}^3$

3. 544 cm^2; $80\overline{0} \text{ cm}^2$

4. $82,300 \text{ m}^2$

5. 310 cm^3 (if we had greater accuracy, the result would be 314.16 because it's 100 times π.)

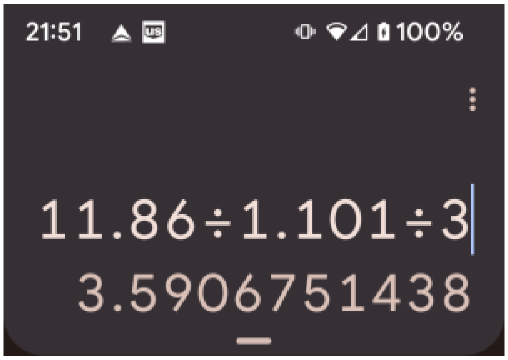
6. 38 ft^3

7. 47 ft^2; 75 ft^2

8. $2\overline{0}0 \text{ cm}^2$; 280 cm^2

Module 27: Percents Part 3

1. $\$1,299.00$
2. $14,861$ students; notice that if the percent had fewer than five sig figs, we wouldn't have been able to get an answer that was accurate to the nearest whole number.
3. Yes, you can! Each bottle cost $\$3.59$.



Real time screenshot of my phone's calculator.

4. $\$3.20$ million
5. $873,900$ people
6. $\$16$; the percent has only two sig figs, so it doesn't make sense to assume that the price was $\$16.13$. They probably rounded the percent from 68.75% to make the numbers in the advertisement seems less complicated.

Module 28: Mean, Median, Mode

1. 10.3 min
2. $\$4.488$
3. 11.0 min (because it is the seventh value in the list of thirteen)
4. $\$4.475$
5. $\$256,000$
6. $\$250,000$

7. $\$338,000$

8. $\$275,000$

9. the median is more representative because the mean is higher than five of the six home values.

10. 11.0 min (because it appears four times in the list)

11. no mode (there are no repeated values)

12. AT&T Mobility

13. Samoas and Thin Mints

14. 12.2 games

15. 12 games

16. 12 games

17. they all represent the data fairly well; 12 wins represents a typical Patriots season.

18. 6.8 games

19. 7 games

20. 6 games

21. they all represent the data fairly well; 6 or 7 wins represents a typical Bills season.

22. 98.2 grams; the mean doesn't seem to represent a typical clementine because there is a group of smaller ones (from 82 to 94 grams) and a group of larger ones (from 102 to 109 grams) with none in the middle.

23. 94 grams; for the same reason, the median doesn't represent a typical clementine, but you could say it helps split the clementines into a lighter group and a heavier group.

24. no mode; too many values appear twice.

25. 98.3 grams; this is a small increase over the previous mean.

26. 94 grams; the median does not change when one of the highest numbers increases.

27. 109 grams; you might say it represents the mass of a typical large clementine, but it doesn't represent the entire group.

Module 29: Probability

1. $\frac{6}{36} = \frac{1}{6}$

2. $\frac{2}{36} = \frac{1}{18}$

3. $\frac{8}{36} = \frac{2}{9}$

4. $\frac{36}{36} = 1$

5. $\frac{0}{36} = 0$

6. $\frac{4}{36} = \frac{1}{9}$

7. $\frac{32}{36} = \frac{8}{9}$

8. $\frac{6}{36} = \frac{1}{6}$

9. $\frac{30}{36} = \frac{5}{6}$

10. $\frac{15}{60} = \frac{1}{4} = 0.25 = 25\%$

11. $\frac{45}{60} = \frac{3}{4} = 0.75 = 75\%$

12. $\frac{18}{60} = \frac{3}{10} = 0.3 = 30\%$

13. $\frac{42}{60} = \frac{7}{10} = 0.7 = 70\%$

14. $\frac{1}{4} = 25\%$

15. $\frac{3}{4} = 75\%$

16. $\frac{2}{4} = 50\%$

17. $\frac{2}{4} = 50\%$

18. $\frac{8}{250} = 3.2\%$

19. $\frac{242}{250} = 96.8\%$

20. we should expect 968 copies to be acceptable

21. $\frac{44}{200} = 22\%$

22. we should expect 220 tax returns to have errors

23. $22\% = 0.22$

24. $(0.22)^2 \approx 4.8\%$

25. $(0.22)^3 \approx 1.1\%$

26. $0.22 \cdot 0.78 \approx 17.2\%$

27. $0.78 \cdot 0.22 \approx 17.2\%$

28. $(0.78)^2 \approx 60.8\%$

29. $(0.78)^3 \approx 47.5\%$

30. $1 - (0.78)^3 \approx 52.5\%$

Module 30: Standard Deviation

1. 60.5; 66.5

2. 57.5; 69.5

3. 54.5; 72.5

4. 66.5; 72.5

5. 63.5; 75.5

6. 60.5; 78.5

7. 68% because $100\% - (16\% + 16\%) = 68\%$

8. 195 lb because this is halfway between 160 and 230 lb

9. 35 lb because $195 - 35$ lb and $195 + 35$ lb encompasses 68% of the data

10. 125; 265

11. 8.8; 15.6

12. You would not have predicted this from the data because it is more than two standard deviations below the mean, so there would be a roughly 2.5% chance of this happening randomly. In fact, $(12.2 - 7) \div 1.7$ is slightly larger than 3, so this is more than three standard deviations below the mean, making it even more unlikely. (You might have predicted that the Patriots would get worse when Tom Brady left them for Tampa Bay, but you wouldn't have predicted only 7 wins based on the previous nineteen years of data.)

13. 3.4; 10.2

14. You would not predict this from the data because it is more than two standard deviations above the mean, so there would be a roughly 2.5% chance of this happening randomly. In fact, $(13 - 6.8) \div 1.7 \approx 3.6$, so this is more than three standard deviations above the mean, making it even more unlikely. This increased win total is partly due to external forces (i.e., the Patriots becoming weaker and losing two games to the Bills) but even 11 wins would have been a bold prediction, let alone 13.

15. 3.9; 14.3

16. The trouble with making predictions about the Broncos is that their standard deviation is so large. You could choose any number between 4 and 14 wins and be within the 95% interval. $(9.1 - 5) \div 2.6 \approx 1.6$, so this is around 1.6 standard deviations below the mean, which makes it not very unusual. Whereas the Patriots and Bills are more consistent, the Broncos' win totals fluctuate quite a bit and are therefore more unpredictable.

Module 31: Right Triangle Trigonometry

1. the adjacent side is *e*, the opposite side is *f*, and the hypotenuse is *d*.

2. the adjacent side is x, the opposite side is y, and the hypotenuse is r.
3. $\frac{3}{5} = 0.6$
4. $\frac{4}{5} = 0.8$
5. $\frac{3}{4} = 0.75$
6. $\frac{4}{5} = 0.8$
7. $\frac{3}{5} = 0.6$
8. $\frac{4}{3} \approx 1.333$
9. 0.6000
10. 0.8000
11. 0.7500
12. 0.8000
13. 0.6000
14. 1.333
15. $z \approx 4.6 \text{ cm}$
16. $g \approx 2.6 \text{ cm}$
17. $b \approx 5.706 \text{ in}$
18. $p \approx 75.51 \text{ mm}$
19. $y \approx 136.18 \text{ mm}$
20. $d \approx 296.87 \text{ mm}$
21. the wire is approximately 27 ft long
22. $\approx 17.85 \text{ ft}$, which is roughly $17 \text{ ft}, 10 \text{ in}$
23. No; $64 \cdot \tan 1^\circ \approx 1.1 \text{ ft}$, so the puck will hit the fabric over 1 foot away from the center of the hole. (The person in the photo was given a bunch of pucks

and 30 seconds to score, but he scored on his first shot. Boston Bruins at Vancouver Canucks, February 24, 2024.)

24. $36.87°$

25. $60°$

26. $53.13°$

27. $\angle A \approx 36.47°$

28. $\angle 1 \approx 42.03°$

29. $\angle 1 \approx 30.76°$

30. $\angle y \approx 52.88°$

31. $\angle 1 \approx 55.28°$

32. $\angle x \approx 31.50°$; $\angle y \approx 58.50°$

33. $\tan^{-1}\left(\frac{4}{1}\right) \approx 76°$ angle of elevation

34. Yes; $\sin^{-1}\left(\frac{2}{25}\right) \approx 4.59°$, which is less than $4.75°$.

35. $\tan^{-1}\left(\frac{17}{14}\right) \approx 50.5°$

36. $17 \div \sin\ 51° \approx 22.0$

37. $14 \div \cos\ 51° \approx 22.0$

38. $\sqrt{14^2 + 17^2} \approx 22.0$

39. All three answers are the same rounded to three significant figures. This is true because we rounded $\angle A$ to the nearest tenth; if we had rounded it to $51°$ instead of $50.5°$, we would have decreased the accuracy of #36 & #37 to only two sig figs and the three results all would have been slightly different.

40. $\approx 23°$

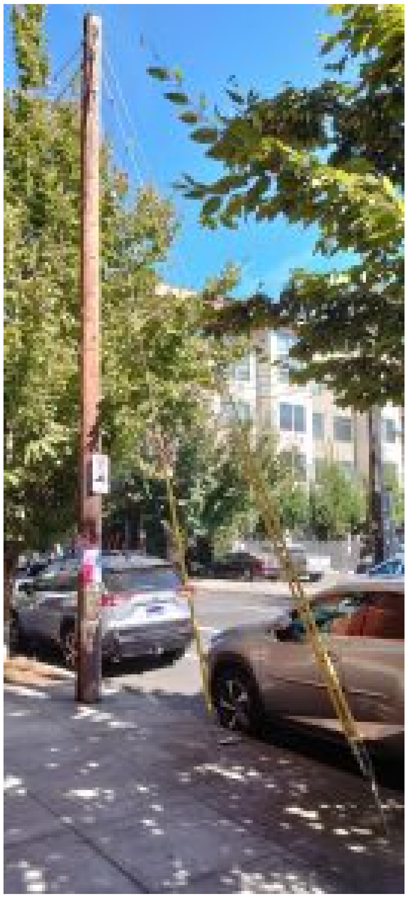
41. $\approx 8,900$ ft

Module 32: Slope

1. 0.36:12, 0.03, 3%
2. 0.60:12, 0.05, 5%
3. 1.00:12, 0.0833, 8.33%
4. 2.16:12, 0.18, 18%
5. $1.72°$
6. $2.86°$
7. $4.76°$
8. $10.20°$
9. 0.5 ft
10. 1 ft
11. 5 ft
12. $\tan^{-1}(20/12) \approx 59°$ from vertical. (this would be an angle of depression of $31°$.)
13. 30 ft; the proportion $\frac{1}{12} = \frac{2.5}{x}$ gives a result of exactly 30 ft.
    Using the result from #7, the equation $\tan(4.76°) = \frac{2.5}{x}$ gives a result very close to 30.0 ft.

Module 33: Non-Right Triangle Trigonometry

1. $k \approx 61 \text{ in}$
2. $a = c = 74°$; $x \approx 5.168 \text{ in}$
3. $x \approx 11.6 \text{ cm}$; $y \approx 12.3 \text{ cm}$
4. the shorter wire is roughly 23.5 feet; the longer wire is roughly 26 feet.



Exercise 4 is based on actual events!

5. $n \approx 72.4°$

6. $n \approx 107.6°$

7. $d \approx 65 \text{ mm}$

8. $p \approx 11.4 \text{ cm}$

9. $w \approx 14.12 \text{ m}$

10. the distance across the pond is roughly 81.5 meters.

11. $X \approx 19.4°$; $Y \approx 133.5°$; $Z \approx 27.1°$

Module 34: Radian Measure

1. $\frac{\pi}{6}$

2. $\frac{\pi}{4}$
3. $\frac{\pi}{3}$
4. $\frac{\pi}{2}$
5. $\frac{2\pi}{3}$
6. $\frac{5\pi}{4}$
7. $\frac{4\pi}{3}$
8. $\frac{3\pi}{2}$
9. $15°$
10. $135°$
11. $150°$
12. $315°$
13. $\approx 57.3°$

Accessibility

This section would normally belong in the front matter, but that would cause inconsistencies between the page numbering of the two PDF versions because I didn't include this section in the print textbook. Please don't assume that its position at the back of the book indicates that it is merely an afterthought.

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- All content can be navigated using a keyboard.
- There is an option to increase font size.
- Images have alt tags.
- Tables have header rows and columns when appropriate.
- Links are anchored to descriptive text rather than the word "here".
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Known accessibility issues and areas for improvement

While your author tried to ensure that this resource is as accessible and usable as possible, he worked on this entire project essentially by himself and surely made mistakes. He did his best but has a rudimentary knowledge of html and can't always make things function the way he'd prefer. Any issues that are identify will be listed below.

List of Known Accessibility Issues

Location	Issue	Timeline	Explanation
Web version	Kurzweil 3000 text to speech can't read the math text correctly	Unknown	Math expressions are rendered using QuickLaTeX integration with Pressbooks, which is not fully accessible to screen reader software. (For some examples, it interprets some things correctly, such as square meters, but not others, such as square centimeters. It also reads out the underlying QuickLaTeX coding, which means it says "backslash" quite a bit and reads 6 squared, 6^2, as "6 circumflex 2".) It may not be possible to fix this, but I have reached out for help.
PDF versions	Kurzweil 3000 text to speech can't read the math text correctly	Unknown	Math expressions are rendered using QuickLaTeX integration with Pressbooks, which is not fully accessible to screen reader software. (The results are different in many ways from what happens in the web version. It reads 6 squared, 6^2, as "sixty-two". It frequently ignores symbols. It reads the starting page number in Spanish and "Exercises" in French.) It may not be possible to fix this, but I have reached out for help.
Modules 17 and 19	Poor contrast with blue link on gray background	Unknown	I tried a few things but I'm not sure whether the link color to a footnote can be adjusted because they are programmed differently from other links. (This is minor compared to the screenreader issues mentioned above.)
Module 30	Three tables have an empty cell in the upper left corner	N/A	This is intentional.
Throughout	Links open in new tab	March 31	Be aware that this is the default. It will take time to either change this behavior sitewide or add "new tab" to each link.

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- The location of the problem by providing a web address or page description
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using that can help us diagnose and solve your issue: e.g., Windows 11, Google Chrome Version 122.0.6261.112 (Official Build) (64-bit), NVDA screen reader

Email the author at morganc[at]clackamas.edu or morganchase64[at]gmail.com to report an issue. Apologies for the non-clickable email addresses, but I am nervous about receiving spam if I make these addresses readable by malicious bots.

This statement was last updated on March 12, 2024.