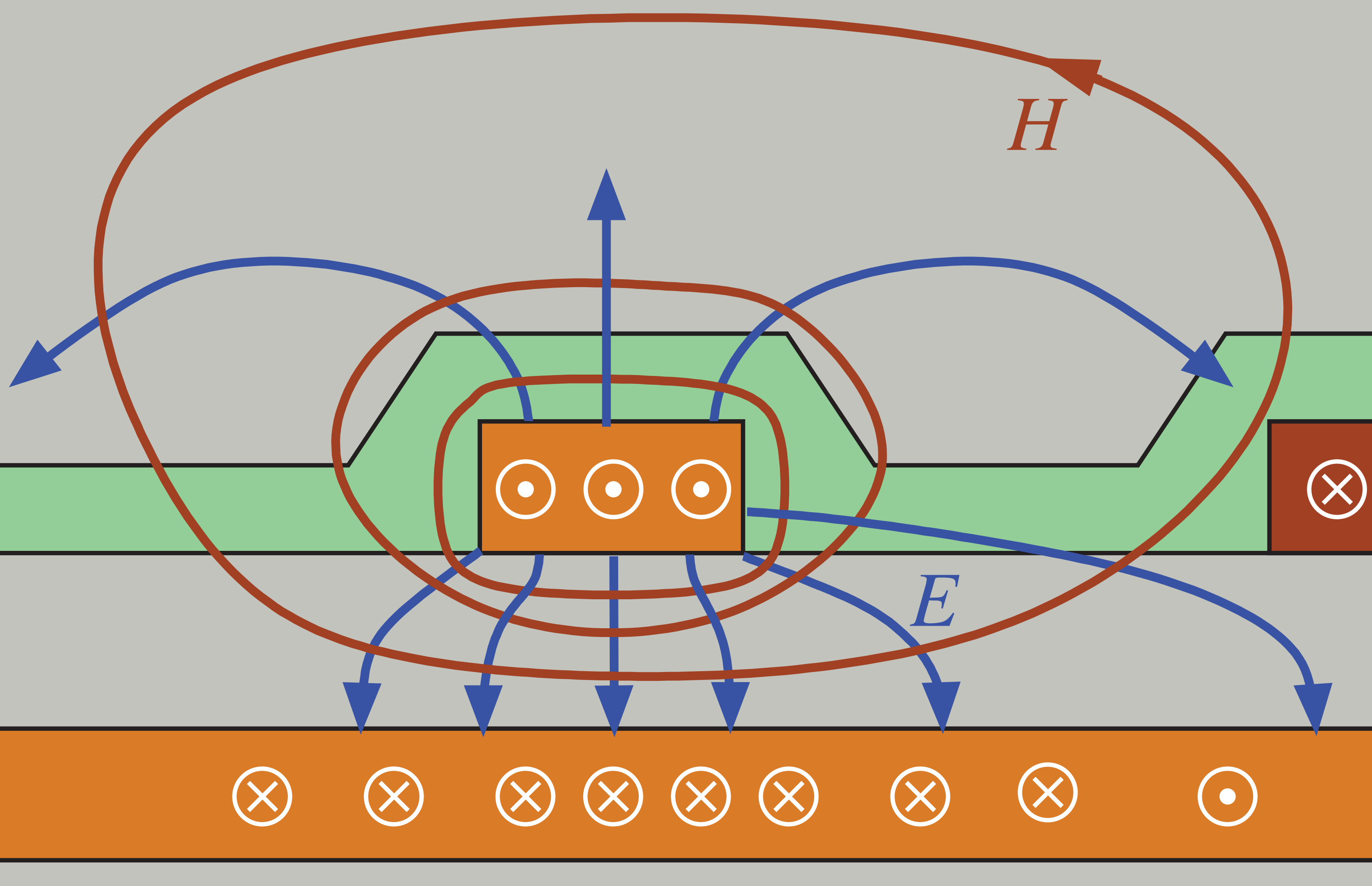
Volume 2

MICROWAVE AND RF DESIGN
TRANSMISSION LINES



Michael Steer

Third Edition

Microwave and RF Design
Transmission Lines

Volume 2

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To Conor and all my grandchildren

Preface

The book series *Microwave and RF Design* is a comprehensive treatment of radio frequency (RF) and microwave design with a modern "systems-first" approach. A strong emphasis on design permeates the series with extensive case studies and design examples. Design is oriented towards cellular communications and microstrip design so that lessons learned can be applied to real-world design tasks. The books in the Microwave and RF Design series are:

- Microwave and RF Design: Radio Systems, Volume 1
- Microwave and RF Design: Transmission Lines, Volume 2
- Microwave and RF Design: Networks, Volume 3
- Microwave and RF Design: Modules, Volume 4
- Microwave and RF Design: Amplifiers and Oscillators, Volume 5

The length and format of each is suitable for automatic printing and binding.

Rationale

The central philosophy behind this series's popular approach is that the student or practicing engineer will develop a full appreciation for RF and microwave engineering and gain the practical skills to perform system-level design decisions. Now more than ever companies need engineers with an ingrained appreciation of systems and armed with the skills to make system decisions. One of the greatest challenges facing RF and microwave engineering is the increasing level of abstraction needed to create innovative microwave and RF systems. This book series is organized in such a way that the reader comes to understand the impact that system-level decisions have on component and subsystem design. At the same time, the capabilities of technologies, components, and subsystems impact system design. The book series is meticulously crafted to intertwine these themes.

Audience

The book series was originally developed for three courses at North Carolina State University. One is a final-year undergraduate class, another an introductory graduate class, and the third an advanced graduate class. Books in the series are used as supplementary texts in two other classes. There are extensive case studies, examples, and end of chapter problems ranging from straight-forward to in-depth problems requiring hours to solve. A companion book, *Fundamentals of Microwave and RF Design*, is more suitable for an undergraduate class yet there is a direct linkage between the material in this book and the series which can then be used as a career-long reference text. I believe it is completely understandable for senior-level students where a microwave/RF engineering course is offered. The book series is a comprehensive RF and microwave text and reference, with detailed index, appendices, and cross-references throughout. Practicing engineers will find the book series a valuable systems primer, a refresher as needed, and a reference tool in the field. Additionally, it can serve as a valuable, accessible resource for those outside RF circuit engineering who need to understand how they can work with RF hardware engineers.

Organization

This book is a volume in a five volume series on RF and microwave design. The first volume in the series, *Microwave and RF Design: Radio Systems*, addresses radio systems mainly following the evolution of cellular radio. A central aspect of microwave engineering is distributed effects considered in the second volume of this book series, *Microwave and RF Design: Transmission Lines*. Here transmission lines are treated as supporting forward- and backward-traveling voltage and current waves and these are related to electromagnetic effects. The third volume, *Microwave and RF Design: Networks*, covers microwave network theory which is the theory that describes power flow and can be used with transmission line effects. Topics covered in *Microwave and RF Design: Modules*, focus on designing microwave circuits and systems using modules introducing a large number of different modules. Modules is just another term for a network but the implication is that is is packaged and often available off-the-shelf. Other topics that are important in system design using modules are considered including noise, distortion, and dynamic range. Most microwave and RF designers construct systems using modules developed by other engineers who specialize in developing the modules. Examples are filter and amplifier modules which once designed can be used in many different systems. Much of microwave design is about maximizing dynamic range, minimizing noise, and minimizing DC power consumption. The fifth volume in this series, *Microwave and RF Design: Amplifiers and Oscillators*, considers amplifier and oscillator design and develops the skills required to develop modules.

Volume 1: Microwave and RF Design: Radio Systems

The first book of the series covers RF systems. It describes system concepts and provides comprehensive knowledge of RF and microwave systems. The emphasis is on understanding how systems are crafted from many different technologies and concepts. The reader gains valuable insight into how different technologies can be traded off in meeting system requirements. I do not believe this systems presentation is available anywhere else in such a compact form.

Volume 2: Microwave and RF Design: Transmission Lines

This book begins with a chapter on transmission line theory and introduces the concepts of forward- and backward-traveling waves. Many examples are included of advanced techniques for analyzing and designing transmission line networks. This is followed by a chapter on planar transmission lines with microstrip lines primarily used in design examples. Design examples illustrate some of the less quantifiable design decisions that must be made. The next chapter describes frequency-dependent transmission line effects and describes the design choices that must be taken to avoid multimoding. The final chapter in this volume addresses coupled-lines. It is shown how to design coupled-line networks that exploit this distributed effect to realize novel circuit functionality and how to design networks that minimize negative effects. The modern treatment of transmission lines in this volume emphasizes planar circuit design and the practical aspects of designing around unwanted effects. Detailed design of a directional coupler is used to illustrate the use of coupled lines. Network equivalents of coupled lines are introduced as fundamental building blocks that are used later in the synthesis of coupled-line filters. The text, examples, and problems introduce the often hidden design requirements of designing to mitigate parasitic effects and unwanted modes of operation.

Volume 3: Microwave and RF Design: Networks

Volume 3 focuses on microwave networks with descriptions based on S parameters and $ABCD$ matrices, and the representation of reflection and transmission information on polar plots called Smith charts. Microwave measurement and calibration technology are examined. A sampling of the wide variety of microwave elements based on transmission lines is presented. It is shown how many of these have lumped-element equivalents and how lumped elements and transmission lines can be combined as a compromise between the high performance of transmission line structures and the compactness of lumped elements. This volume concludes with an in-depth treatment of matching for maximum power transfer. Both lumped-element and distributed-element matching are presented.

Volume 4: Microwave and RF Design: Modules

Volume 4 focuses on the design of systems based on microwave modules. The book considers the wide variety of RF modules including amplifiers, local oscillators, switches, circulators, isolators, phase detectors, frequency multipliers and dividers, phase-locked loops, and direct digital synthesizers. The use of modules has become increasingly important in RF and microwave engineering. A wide variety of passive and active modules are available and high-performance systems can be realized cost effectively and with stellar performance by using off-the-shelf modules interconnected using planar transmission lines. Module vendors are encouraged by the market to develop competitive modules that can be used in a wide variety of applications. The great majority of RF and microwave engineers either develop modules or use modules to realize RF systems. Systems must also be concerned with noise and distortion, including distortion that originates in supposedly linear elements. Something as simple as a termination can produce distortion called passive intermodulation distortion. Design techniques are presented for designing cascaded systems while managing noise and distortion. Filters are also modules and general filter theory is covered and the design of parallel coupled line filters is presented in detail. Filter design is presented as a mixture of art and science. This mix, and the thought processes involved, are emphasized through the design of a filter integrated throughout this chapter.

Volume 5: Microwave and RF Design: Amplifiers and Oscillators

The fifth volume presents the design of amplifiers and oscillators in a way that enables state-of-the-art designs to be developed. Detailed strategies for amplifiers and voltage-controlled oscillators are presented. Design of competitive microwave amplifiers and oscillators are particularly challenging as many trade-offs are required in design, and the design decisions cannot be reduced to a formulaic flow. Very detailed case studies are presented and while some may seem quite complicated, they parallel the level of sophistication required to develop competitive designs.

Case Studies

A key feature of this book series is the use of real world case studies of leading edge designs. Some of the case studies are designs done in my research group to demonstrate design techniques resulting in leading performance. The case studies and the persons responsible for helping to develop them are as follows.

1. Software defined radio transmitter.
2. High dynamic range down converter design. This case study was developed with Alan Victor.
3. Design of a third-order Chebyshev combline filter. This case study was developed with Wael Fathelbab.
4. Design of a bandstop filter. This case study was developed with Wael Fathelbab.
5. Tunable Resonator with a varactor diode stack. This case study was developed with Alan Victor.
6. Analysis of a 15 GHz Receiver. This case study was developed with Alan Victor.
7. Transceiver Architecture. This case study was developed with Alan Victor.
8. Narrowband linear amplifier design. This case study was developed with Dane Collins and National Instruments Corporation.
9. Wideband Amplifier Design. This case study was developed with Dane Collins and National Instruments Corporation.
10. Distributed biasing of differential amplifiers. This case study was developed with Wael Fathelbab.
11. Analysis of a distributed amplifier. This case study was developed with Ratan Bhatia, Jason Gerber, Tony Kwan, and Rowan Gilmore.
12. Design of a WiMAX power amplifier. This case study was developed with Dane Collins and National Instruments Corporation.
13. Reflection oscillator. This case study was developed with Dane Collins and National Instruments Corporation.
14. Design of a C-Band VCO. This case study was developed with Alan Victor.
15. Oscillator phase noise analysis. This case study was developed with Dane Collins and National Instruments Corporation.

Many of these case studies are available as captioned YouTube videos and qualified instructors can request higher resolution videos from the author.

Course Structures

Based on the adoption of the first and second editions at universities, several different university courses have been developed using various parts of what was originally one very large book. The book supports teaching two or three classes with courses varying by the selection of volumes and chapters. A standard microwave class following the format of earlier microwave texts can be taught using the second and third volumes. Such a course will benefit from the strong practical design flavor and modern treatment of measurement technology, Smith charts, and matching networks. Transmission line propagation and design is presented in the context of microstrip technology providing an immediately useful skill. The subtleties of multimoding are also presented in the context of microstrip lines. In such a class the first volume on microwave systems can be assigned for self-learning.

Another approach is to teach a course that focuses on transmission line effects including parallel coupled-line filters and module design. Such a class would focus on Volumes 2, 3 and 4. A filter design course would focus on using Volume 4 on module design. A course on amplifier and oscillator design would use Volume 5. This course is supported by a large number of case studies that present design concepts that would otherwise be difficult to put into the flow of the textbook.

Another option suited to an undergraduate or introductory graduate class is to teach a class that enables engineers to develop RF and microwave systems. This class uses portions of Volumes 2, 3 and 4. This class then omits detailed filter, amplifier, and oscillator design.

The fundamental philosophy behind the book series is that the broader impact of the material should be presented first. Systems should be discussed up front and not left as an afterthought for the final chapter of a textbook, the last lecture of the semester, or the last course of a curriculum.

The book series is written so that all electrical engineers can gain an appreciation of RF and microwave hardware engineering. The body of the text can be covered without strong reliance on this electromagnetic theory, but it is there for those who desire it for teaching or reader review. The book is rich with detailed information and also serves as a technical reference.

The Systems Engineer

Systems are developed beginning with fuzzy requirements for components and subsystems. Just as system requirements provide impetus to develop new base technologies, the development of new technologies provides new capabilities that drive innovation and new systems. The new capabilities may arise from developments made in support of other systems. Sometimes serendipity leads to the new capabilities. Creating innovative microwave and RF systems that address market needs or provide for new opportunities is the most exciting challenge in RF design. The engineers who can conceptualize and architect new RF systems are in great demand. This book began as an effort to train RF systems engineers and as an RF systems resource for practicing engineers. Many RF systems engineers began their careers when systems were simple. Today, appreciating a system requires higher levels of abstraction than in the past, but it also requires detailed knowledge or the ability to access detailed knowledge and expertise. So what makes a systems engineer? There is not a simple answer, but many partial answers. We know that system engineers have great technical confidence and broad appreciation for technologies. They are both broad in their knowledge of a large swath of technologies and also deep in knowledge of a few areas, sometimes called the "T" model. One book or course will not make a systems engineer. It is clear that there must be a diverse set of experiences. This book series fulfills the role of fostering both high-level abstraction of RF engineering and also detailed design skills to realize effective RF and microwave modules. My hope is that this book will provide the necessary background for the next generation of RF systems engineers by stressing system principles immediately, followed by core RF technologies. Core technologies are thereby covered within the context of the systems in which they are used.

Supplementary Materials

Supplementary materials available to qualified instructors adopting the book include PowerPoint slides and solutions to the end-of-chapter problems. Requests should be directed to the author. Access to downloads of the books, additional material and YouTube videos of many case studies are available at `https://www.lib.ncsu.edu/do/open-education`

Acknowledgments

Writing this book has been a large task and I am indebted to the many people who helped along the way. First I want to thank the more than 1200 electrical engineering graduate students who used drafts and the first two editions at NC State. I thank the many instructors and students who have provided feedback. I particularly thank Dr. Wael Fathelbab, a filter expert, who co-wrote an early version of the filter chapter. Professor Andreas Cangellaris helped in developing the early structure of the book. Many people have reviewed the book and provided suggestions. I thank input on the structure of the manuscript: Professors Mark Wharton and Nuno Carvalho of Universidade de Aveiro, Professors Ed Delp and Saul Gelfand of Purdue University, Professor Lynn Carpenter of Pennsylvania State University, Professor Grant Ellis of the Universiti Teknologi Petronas, Professor Islam Eshrah of Cairo University, Professor Mohammad Essaaidi and Dr. Otman Aghzout of Abdelmalek Essaadi Univeristy, Professor Jianguo Ma of Guangdong University of Technology, Dr. Jayesh Nath of Apple, Mr. Sony Rowland of the U.S. Navy, and Dr. Jonathan Wilkerson of Lawrence Livermore National Laboratories, Dr. Josh Wetherington of Vadum, Dr. Glen Garner of Vadum, and Mr. Justin Lowry who graduated from North Carolina State University.

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The book was produced using LaTeX and open access fonts, line art was drawn using xfig and inkscape, and images were edited in gimp. So thanks to the many volunteers who developed these packages.

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Contents

CHAPTER 1

Introduction to Distributed Microwave Circuits

1.1 Introduction

The universe has a speed limit, the finite speed of causality or c, the speed of light, which is embodied in Maxwell's equations and in the special theory of relativity. The result of this is that a cause, say a voltage, in one part of a circuit produces an effect, say a current, in another part of the circuit after a delay. When using circuits at low frequencies the impact of this delay is insignificant as the effect occurs long before there is even the slightest discernible change in the cause. However the delay is important at microwave and higher frequencies. Electromagnetic (EM) radiation travels 30 cm in a nanosecond in free space. Thus a circuit operating at 1 GHz that is 1 cm across will just see an appreciable affect due to the finite speed of the EM signal which, of course, is how the voltage and current "communicate." The impact will be more significant at higher frequencies and if the EM signal travels in a dielectric as a dielectric reduces the speed of light.

A microwave engineer must design circuits to manage this finite delay, but what is particularly interesting is that the effect gives rise to an enormous number of useful circuit elements that have no analog at lower frequencies. Whole circuits, and quite complex circuits at that, can be designed by exploiting distributed effects. This book explores the design of circuits with transmission lines and introduces effects, such as coupling from one transmission line to another, that can be exploited to build novel circuits.

This book is the second volume in a series on microwave and RF design. The first volume in the series addresses radio systems [1] mainly following the evolution of cellular radio. The third volume [2] covers microwave network theory which is the theory that describe power flow and can be used to describe transmission line effects. Topics covered in this volume include scattering parameters, Smith charts, and matching networks that enable maximum power transfer. The fourth volume [3] focuses on designing microwave circuits and systems using modules introducing a large number of different modules. Modules is just another term for a network but the implication is that is packaged and often available off-the-shelf. Other topics in this chapter that are important in system design using modules are considered including noise, distortion, and dynamic range. Most microwave and RF designers construct systems using modules developed by other engineers who specialize in developing the modules. Examples are filter and amplifier chip modules which once designed can be used in many different systems. Much of microwave design is about maximizing dynamic range, minimizing noise, and minimizing DC power consumption. The fifth volume in this series [4] considers amplifier and oscillator design and develops the skills required to develop modules.

The books in the Microwave and RF Design series are:

- Microwave and RF Design: Radio Systems
- Microwave and RF Design: Transmission Lines
- Microwave and RF Design: Networks
- Microwave and RF Design: Modules
- Microwave and RF Design: Amplifiers and Oscillators

1.2 Book Outline

The telegraphist's equations, used to analyze transmission lines, and their interpretations are presented in Chapter 2 where transmission line theory is presented. This leads to the analysis of terminated transmission lines with key results being how these can be used as circuit elements to realize narrow-band inductors and capacitors. The chapter concludes by developing circuit models of transmission lines that are suitable for use in circuit simulators and in designing circuits using transmission lines.

Chapters 2–6 describe transmission lines and the exploitation of transmission effects that are used in RF and microwave engineering. One of the aspects that distinguishes design at a few tens of megahertz and below from design above a few megahertz up to a terahertz is that at low frequencies design proceeds with the assumption that circuit elements and indeed whole circuits exist at a point and there is no time-of-flight, e.g. speed-of-light, limitation to a voltage and current at one point in a circuit instantaneously impacting the voltage and current anywhere else in the circuit. As the frequency of operation of a circuit increases, the effect of the finite speed of light becomes significant and distributed effects often need to be considered.

Chapter 3 focuses on a class of transmission lines called planar lines which are lines usually fabricated starting with a low loss dielectric substrate with metal sheets on both sides. The most important planar transmission line is the microstrip line shown in Figure 1-1. This is the most widely used

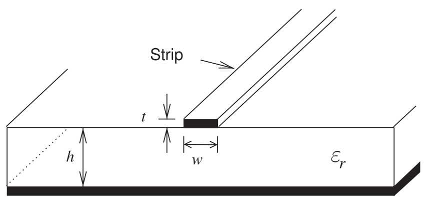


Figure 1-1: Microstrip transmission line.

interconnect at microwave frequencies. The strip of the line is formed by patterning the top conductor and etching away the unwanted metal. The defining distinction between a microstrip line, and any planar transmission line, and a trace on a printed circuit board is that at microwave frequencies attention must be given to providing a signal return path and maintaining as uniform an environment as possible for the electric and magnetic fields. Any discontinuity will introduce reflections of EM energy. This may actually be desired when implementing circuit elements but if not it is important that geometrical uniformity be maintained. Considering the microstrip line, see Figure 1-1, the width, w, of the strip and the thickness, h, of the substrate define the ratio of the voltage and current signals traveling along the microstrip line. This ratio is called the characteristic impedance of the line and it is critical for reliable signal transmission, i.e., good signal integrity, that the cross-sectional geometry be the same along the line as then the characteristic impedance of the line is constant.

While simple in concept, planar transmission lines needed to be invented. As well as conceptualizing a transmission line that can be realized by etching a planar metallic conductor on the printed circuit board, it is essential to provide the analytic tools that enable the propagation characteristics of the line to be calculated and enable structures such as couplers and filters to be synthesized using planar transmission lines. The origins of microstrip trace back to developments in the early 1940s, beginning with a coaxial line with a flat center conductor forming a rectangular coaxial line [5]. At that time printed circuit boards were being used for low-frequency circuits. In 1949 these came together when Barrett reasoned that the thick center conductor of the rectangular coaxial transmission line could be very thin with little effect on the properties of the line. This then meant that low-frequency printed circuit board techniques could be employed in microwave circuits and the transmission line system became known as stripline [5, 6].

The conceptual evolution of stripline is shown in Figure 1-2. The stripline configuration is developed by sandwiching a metallic strip between two metal-clad dielectric sheets. As initially envisioned, the strip could be stamped out or silk-screened using silver ink. Today it is most common to begin with a continuous metallic sheet bonded to one or both sides of a dielectric sheet. A pattern of an etch resistant material is then photolithographically defined on the sheet and the strip pattern appears after etching. The next advance in 1952 when Grieg and Engelmann removed one of stripline's ground planes thus becoming microstrip [7]. With an invention it is not sufficient to simply describe a structure, in this case understanding had to be provided as evidenced by presenting design equations for the electrical properties of the line. The essential electrical properties of a transmission line are its characteristic impedance, the ratio of the traveling voltage and current waves on the line, and its propagation constant, which relates to the speed of propagation of the voltage and current

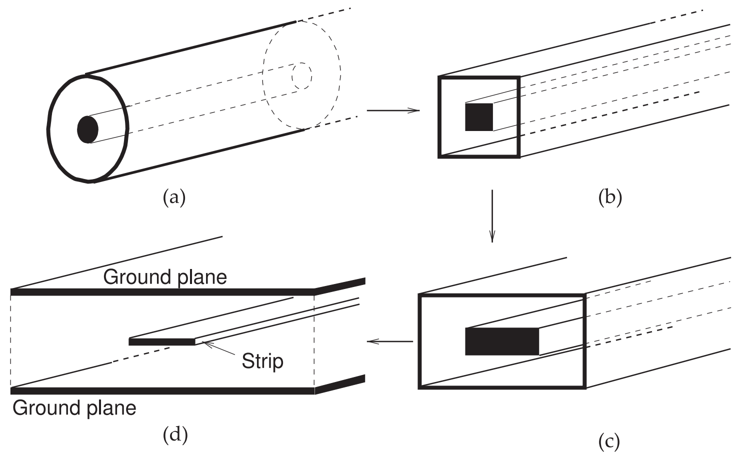


Figure 1-2: Evolution of the stripline transmission line: (a) coaxial line with a round center conductor; (b) square coaxial line with a square flat center conductor; (c) rectangular coaxial line with a flat center conductor; and (d) stripline.

waves on the lines.

Planar lines are not ideal transmission lines and undesired effects must be anticipated and design adjusted to reduce their occurrence or impact. These extraordinary effects are considered in Chapter 4. Understanding and anticipating these effects is really where the art of microwave design comes into play. Many of the effects occur when dimensions are too large but at the same time manufacturability improves when dimensions are large, so a design trade-off is required. Rarely is a design successful on the first pass. Thus it is important for a microwave engineer to interpret undesirable results and know what to modify in a design. It is difficult to do measurements at microwave frequencies as anything that is introduced in a measurement, e.g. a probe, will have significant impact on a circuit. Often a few indirect measurements, some analysis, and great intuition is required to correctly identify and hence resolve design issues.

Chapter 4 begins by considering frequency-dependent characteristics of planar lines and then moves on to developing analysis equations and design concepts for microstrip lines at high frequencies. As the transverse dimensions of a planar line approach a wavelength, say greater than a tenth of a wavelength, it is possible for the EM signal to travel down the line in two (or more field) orientations called modes and these travel at different velocities and corrupt a signal. Multimoding is nearly always undesirable so understanding the origins of multiple modes and designing to avoid multiple modes is an important consideration. However in some situations these modes is exploited, e.g. in realizing a small area inductor on an RFIC.

When two planar lines are close to each other the fields of one line can be in the vicinity of the other line and the lines are coupled. A portion of the energy of a signal traveling on one of the lines can appear on the other line. Many circuit elements can be developed by exploiting parallel coupled lines (PCLs). Chapter 5 begins by presenting the physics of coupling and then a generalization of the telegraphist's equations for coupled transmission lines. Design approaches and circuit models are developed and the circuit models enable the synthesis of microwave designs using parallel coupled lines Several PCL circuit elements are presented but many more will be described in other volumes of this book series.

The final chapter of this volume, Chapter 6, discusses rectangular waveguides. Instead of 'rectangular waveguide' just the term 'waveguide' is usually used with the understanding that reference is being made to a rectangular waveguide. This is a little confusing because really all transmission lines guide waves and so are waveguides. A rectangular waveguide is a very different type of transmission line that confines the EM fields inside a rectangular metal pipe. These are important elements starting at low microwave frequencies, 1 GHz and above, as they can carry very powerful signals with very little loss. They are almost always used in radar and similar high power applications. Waveguides are also very important at higher frequencies even when power levels are relatively low. They are often used at 15 GHz and above as they have very little loss while the loss of coaxial lines grows tremendously with frequency. With waveguides it is very difficult to use voltage and current concepts except over very narrow fractional bandwidths, a few percent at most. This means that it is very difficult to do circuit design using waveguides.

1.3 Chapter Outline

The chapter begins by considering distributed circuits in Section 1.4 and identifying the types of structures to be considered in this book. Then Section 1.5 revisits Maxwell's equations from a perspective different from the way it was (almost certainly) introduced to the reader. It is assumed that the reader has a knowledge of Maxwell's equations which are nearly always introduced following a treatment of the static field laws: Biot–Savart law, Ampere's circuital law, Gauss's law, Gauss's law for magnetism, and Faraday's law. The classic treatment follows the historical time line and then Maxwell's equations are introduced. It was almost as though Maxwell's equations were derived from the static field laws but that was never stated because they were not. Maxwell's equations represent a separate physical insight. Maxwell's equations cannot be derived from the static field laws, however the static field laws can be derived from Maxwell's equations. Maxwell had remarkable insight and interpreted the physical world in an amazing way and this places him among the greatest physicists of all time. He touched on relativity and quantum field theory in imagining that the electric and magnetic fields, and light, formed a two-component field where the spatial derivative of one component was related to the time derivative of the other. He imagined that this relationship imposed a cosmic speed limit. The components of Maxwell's field are the electric and magnetic fields but the insight was much more fundamental than that.

The next section, Section 1.6, demonstrates that all of the early static field laws can be derived from Maxwell's equation. Maxwell unified what seemed to be a number of unconnected observations but at the time many suspected that there was an underlying reality. The next few sections derive some important short-hand techniques that are useful in working with distributed circuits and in particular transmission lines. Section 1.7 describes how the EM fields interact with lossless materials. It is no secret that Maxwell's equations are fiendishly difficult to work with unless great simplifications are made. While tremendous computational EM analysis tools are available the design engineer needs insight. One of the critical pieces of insight that guides microwave engineers is to imagine a magnetic wall analogous to a conductor which is an electric wall. A magnetic wall is approximated at the interface of two dielectrics. The concepts of electric and magnetic walls analyzed in Section 1.8 provide an insight that is particularly useful to the microwave engineer is developed. These results will be used throughout the chapters in this book. Actual dielectrics and conductors are lossy so the effect of loss is considered in Section 1.9. Examples of how insight and fortuitous use of geometry, symmetries, and physical insight can be used in EM calculations are given in Section 1.9. Generally only particular structures with the requisite symmetries and simple geometry are used in design because if a structure is too difficult to analyze then the important intuitive insight is hard to acquire.

The final part of this chapter is an appendix on mathematical foundations with the required essentials of trigonometric expansions, mathematical identities, complex arithmetic, Butterworth and Chebyshev polynomials used in matching network and filter design, the properties of circles on a complex plane are used in dealing with complex numbers plotted on a polar plot, Kron's method used in network condensation and thus in simplifying designs, and the mathematics of random processes which are used in working with digitally modulated signals and with noise. Everything in this appendix is used somewhere in this book series.

1.4 Distributed Circuits

RF and microwave engineering has as its basis transmission line effects also known as distributed effects. As is well known, the voltage and current at one point, A, cannot instantaneously affect the voltage and current at another spatially separated point B. However if points A and B in a circuit are sufficiently close then a circuit can nearly always be treated as a lumped element circuit. What constitutes sufficiently close is related to the distance between A and B, d, compared to a wavelength, λ. If $d < \lambda/100$ then nearly always the circuit can be considered as being lumped. If $d \geq \lambda/10$ then distributed effects, limitations imposed by the finite time-of-flight, must be considered and the circuit is always regarded as being distributed. When d is between $\lambda/100$ and $\lambda/10$ then it is not clear whether the circuit must be regarded as distributed or can be regarded as lumped. It is of course much simpler to analyze and design with lumped-element circuits.

Distributed and transmission-line effects are synonymous. While a distributed circuit may be difficult to analyze, distributed effects can be exploited to realize a very large array of elements that usually have no equivalent at lower frequencies. For example, the fields from two adjacent transmission lines can overlap so that part of the signal (energy) from one of the lines appears on the other. This could be a problem in some situations but can be used to couple some of the power from one line onto another. Many novel circuit elements are based on this coupling effect.

Distributed effects and transmission line effects result from a signal, voltage or current, at one point in space not being able to instantaneously change the voltage and current at another point in space. To be more physical, voltage and current can be replaced by electric and magnetic field. The finite speed of light is encapsulated in Maxwell's equations which were developed between 1861 and 1862 [8, 9]. For microwave engineers Maxwell's equations are the most convenient and complete description of reality. They are the classical limit of a more fundamental theory called quantum electrodynamics. Maxwell's equations are important to the physical understanding of transmission line effects and distributed effects, but in reality are very difficult to manipulate and use in calculations. What are much more useful are the transmission line equations, or telegrapher's equations, developed by Heaviside in 1887 [10].

The telegrapher's equations use voltage and current which are related to the integral of electric field and integral of magnetic field respectively. The teleprapher's equations relate the time derivative of voltage (or current) to the spatial variation of current (or voltage). This is analogous to how Maxwell's equations relate the electric and magnetic fields. Heaviside also introduced the concept of phasors with the result that the four dimensions of Maxwell's equations, time and three spatial dimensions, are reduced to just one spatial dimension, along the length of a transmission line. Microwave designers, as with all circuit designers, is that they design based on lumped-element circuits that are naturally described by voltage and current. What is particularly important about the telegrapher's equations is that they enable direct relationship of transmission line circuits to lumped-element circuits. Never-the-less the telegraphist's equations are not always sufficient to understand distributed circuits so (sometimes reluctantly) microwave engineers must have working familiarity with Maxwell's equations to understand every situation that can be encountered. Putting this another way, microwave engineers want to express circuit quantities as voltage and currents rather than electric and magnetic fields but are always aware that microwave circuits are intricately connected to electric and magnetic fields and energy is stored in electric and magnetic fields. The telegraphist's equations provide a link between the voltage/current and EM field worlds and introduce concepts of traveling wave voltages and currents which further aid the connection.

1.5 Maxwell's Equations

In this and following sections, EM theory is presented in a form that aids in the understanding of distributed effects, such as propagation on transmission lines, coupling of transmission lines, and how transmission line effects can be used to realize components with unique functionality. While this is a review of material that most readers have previously learned, it is presented in a slightly different form than is usual. The treatment begins with Maxwell's equations and not the static field laws. This is not the way EM theory is initially presented.

Maxwell's equations are a remarkable insight and the early field laws can be derived from them. Most importantly, Maxwell's equations describe the propagation of an EM field. Maxwell's equations are presented in point form in Section 1.5.1 and in integral form in Section 1.5.5. From these, the early electric and magnetic field laws are derived. The effect of boundary conditions are introduced in Section 1.8 to arrive at implications for multimoding on transmission lines. Multimoding is almost always undesirable, and in designing transmission line structures so that multimoding is avoided, it is necessary to have rules that establish when multimoding can occur.

1.5.1 Point Form of Maxwell's Equations

The characteristics of EM fields are described by Maxwell's equations:

$$\nabla\times\bar{\mathcal{E}} = -\frac{\partial\bar{\mathcal{B}}}{\partial t} - \bar{\mathcal{J}}_m \qquad (1.1) \qquad\qquad \nabla\times\bar{\mathcal{H}} = \frac{\partial\bar{\mathcal{D}}}{\partial t} + \bar{\mathcal{J}} \qquad (1.3)$$

$$\nabla\cdot\bar{\mathcal{D}} = \rho_V \qquad (1.2) \qquad\qquad \nabla\cdot\bar{\mathcal{B}} = \rho_{mV}, \qquad (1.4)$$

where μ is called the permeability of the medium and ε is called the permittivity of the medium. (The left-hand side of Equation (1.1) is read as "curl E" and the left-hand side of Equation (1.2) as "div E.") They are the property of a medium and describe the ability to store magnetic energy and electric energy. The other quantities in Equations (1.1)–(1.4) are

- $\bar{\mathcal{E}}$, the **electric field**, with units of volts per meter (V/m), a time-varying vector
- $\bar{\mathcal{D}}$, the **electric flux** density, with units of coulombs per square meter (C/m^2)
- $\bar{\mathcal{H}}$, the three-dimensional **magnetic field**, with units of amperes per meter (A/m)
- $\bar{\mathcal{B}}$, the **magnetic flux** density, with units of teslas (T)
- $\bar{\mathcal{J}}$, the **electric current** density, with units of amperes per square meter (A/m^2)
- ρ_V, the **electric charge** density, with units of coulombs per cubic meter (C/m^3)
- ρ_{mV}, the magnetic charge density, with units of webers per cubic meter (Wb/m^3)
- $\bar{\mathcal{J}}_m$ is the magnetic current density, with units of webers per second per square meter ($\mathrm{Wb}\cdot\mathrm{s}^{-1}\cdot\mathrm{m}^{-2}$).

Magnetic charges do not exist, but their introduction in Maxwell's equations through the **magnetic charge** density, ρ_{mV}, and the **magnetic current** density, $\bar{\mathcal{J}}_m$, introduce an aesthetically appealing symmetry. Maxwell's equations are differential equations, and as with most differential equations, their solution is obtained with particular boundary conditions that here are imposed by conductors. Electric conductors (i.e., electric walls) support electric charges and hence electric current. By analogy, magnetic walls support magnetic charges and magnetic currents. Magnetic walls also provide boundary conditions to be used in the solution of Maxwell's equations. The notion of magnetic walls is important in RF and microwave engineering, as they are approximated by the boundary between two dielectrics of different permittivity. The greater the difference in permittivity, the more closely the boundary approximates a magnetic wall. As a result, the analysis of many structures with different dielectrics can be simplified, aiding in intuitive understanding.

The fields in Equations (1.1)–(1.4) are three-dimensional fields, e.g.,

$$\bar{\mathcal{E}} = \mathcal{E}_x\hat{\mathbf{x}} + \mathcal{E}_y\hat{\mathbf{y}} + \mathcal{E}_z\hat{\mathbf{z}}, \qquad (1.5)$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are the unit vectors (having a magnitude of 1) in the x, y and z directions, respectively. $\mathcal{E}_x$, $\mathcal{E}_y$, and $\mathcal{E}_z$ are the electric field components in the x, y, and z directions, respectively.

The symbols and units used with the various field quantities and some of the other symbols to be introduced soon are given in Table 1-1. $\bar{\mathcal{B}}$ and $\bar{\mathcal{H}}$, and $\bar{\mathcal{D}}$ and $\bar{\mathcal{E}}$ are related to each other by the properties of the medium embodied

Table 1-1: Quantities used in Maxwell's equations. Magnetic charge and current are introduced in establishing boundary conditions, especially at dielectric interfaces. The SI units of other quantities used in RF and microwave engineering are given in Table 2-1.

Symbol	SI unit	SI unit	Name and base units
E	volts per meter	V/m	electric field intensity base unit: $\mathrm{kg{\cdot}m{\cdot}s^{-3}{\cdot}A^{-1}}$
H	amps per meter	A/m	magnetic field intensity
D	coulombs per square meter	$\mathrm{C/m^2}$	$D = \varepsilon E$, electric flux density base unit: $\mathrm{A{\cdot}s{\cdot}m^{-2}}$
B	tesla, webers per square meter	T	$B = \mu H$, magnetic flux density base unit: $\mathrm{kg{\cdot}s^{-2}{\cdot}A^{-1}}$
I	amp	A	electric current
M	amps per meter	A/m	magnetization base unit: $\mathrm{A{\cdot}m^{-1}}$
q_e	coulomb	C	electric charge base unit: $\mathrm{A{\cdot}s}$
q_m	weber	Wb	magnetic charge base unit: $\mathrm{kg{\cdot}m^2{\cdot}s^{-2}{\cdot}A^{-1}}$
ψ_e	coulomb	C	electric flux base unit: $\mathrm{A{\cdot}s}$
ψ_m	weber	Wb	magnetic flux base unit: $\mathrm{kg{\cdot}m^2{\cdot}s^{-2}{\cdot}A^{-1}}$
ρ_V	coulombs per cubic meter	$\mathrm{C{\cdot}m^{-3}}$	charge density base unit: $\mathrm{A{\cdot}s{\cdot}m^{-3}}$
ρ_S	coulombs per square meter	$\mathrm{C/m^2}$	surface charge density base unit: $\mathrm{A{\cdot}s{\cdot}m^{-2}}$
ρ_{mV}	webers per cubic meter	$\mathrm{Wb/m^3}$	magnetic charge density base unit: $\mathrm{kg{\cdot}m^{-1}{\cdot}s^{-2}{\cdot}A^{-1}}$
ρ_{mS}	webers per square meter	$\mathrm{Wb/m^2}$	surface magnetic charge density base unit: $\mathrm{kg{\cdot}s^{-2}{\cdot}A^{-1}}$
J	amps per square meter	$\mathrm{A/m^2}$	electric current density
J_S	amps per meter	A/m	surface electric current density
J_m	webers per second per square meter	$\mathrm{Wb{\cdot}s^{-1}{\cdot}m^{-2}}$	magnetic current density base unit: $\mathrm{kg{\cdot}s^{-3}{\cdot}A^{-1}}$
J_{mS}	webers per second per meter	$\mathrm{Wb{\cdot}s^{-1}{\cdot}m^{-1}}$	surface magnetic current density base unit: $\mathrm{kg{\cdot}m^{-1}{\cdot}s^{-3}{\cdot}A^{-1}}$
S	square meters	$\mathrm{m^2}$	surface
V	cubic meters	$\mathrm{m^3}$	volume
ε	farads per meter	F/m	permittivity base unit: $\mathrm{kg^{-1}{\cdot}m^{-3}{\cdot}A^2{\cdot}s^4}$
μ	henry per meter	H/m	permeability base unit: $\mathrm{kg{\cdot}m{\cdot}s^{-2}{\cdot}A^{-2}}$
dS	square meter	$\mathrm{m^2}$	incremental area
$d\ell$	meter	m	incremental length
dV	cubic meter	$\mathrm{m^3}$	incremental volume
$\oint_C$			integral around a closed contour
$\oint_S$			integral over a closed surface
$\int_V$			volume integral

in μ and ε:

$$\bar{\mathcal{B}} = \mu\bar{\mathcal{H}} \quad (1.6) \qquad \bar{\mathcal{D}} = \varepsilon\bar{\mathcal{E}}. \quad (1.7)$$

The quantity μ is called the **permeability** and describes the ability to store **magnetic energy** in a region. The permeability in free space (or vacuum) is denoted μ_0 and the magnetic flux and magnetic field are related as (where $\mu_0 = 4\pi \times 10^{-7}$ H/m)

$$\bar{\mathcal{B}} = \mu_0\bar{\mathcal{H}}. \quad (1.8)$$

1.5.2 Moments and Polarization

In this subsection the response of materials to electric and magnetic fields are described. In a later section there will be a more specific discussion about the interaction of dielectrics and metals to electric fields. The main purpose here is to define electric and magnetic polarization vectors.

Material Response to an Applied Electric Field

The generation of electric moments is the atomic-level response of a material to an applied electric field. With an atom the center of negative charge, the center of the electron cloud, and the center of the positive charge, the nucleus, overlap. When an electric field is applied the centers of positive and negative charge separate and electrical energy additional to that stored in free space is stored in a manner very similar to potential energy storage in a stretched spring. The separation of charge (charge by distance) forms an electric dipole which has an electric dipole moment $\bar{p}$ having the SI units of $\mathrm{C \cdot m}$ or $\mathrm{A \cdot s \cdot m}$. The same occurs with solids but now the centers of positive and negative charge could be separated even without an external electric field and the material is said to be polarized. If there are n electric dipole moments $\bar{p}$ per unit volume (in SI per m^3), then the polarization density (or electric polarization or simply polarization) is

$$\bar{\mathcal{P}} = n\bar{p}. \quad (1.9)$$

which has the SI units of $\mathrm{C/m^2}$. $\bar{\mathcal{P}}$ has the same units and has the same effect as the electric flux density $\bar{\mathcal{D}}$. In a homogeneous, linear, isotropic material, $\bar{\mathcal{P}}$ is proportional to the applied electric field intensity $\bar{\mathcal{E}}$. A homogenous material has the same average properties everywhere and an isotropic material looks the same in all directions. Materials that are not isotropic are called anisotropic and most commonly these are crystals that have asymmetrical unit cells so that the separation of charge, the electric energy stored by this separation of charge (the strength of the spring) in response to an applied field depends on direction. As a result

$$\bar{\mathcal{P}} = \check{\chi}_e\varepsilon_0\bar{\mathcal{E}} \quad (1.10)$$

where $\check{\chi}_e$ is called the electric susceptibility of the material. The electric susceptibility has been written as a tensor, a 3×3 matrix that relates the three directions of $\bar{\mathcal{E}}$ to the three possible directions of $\bar{\mathcal{P}}$. For an amorphous solid or a crystal with high unit cell symmetry, the nine elements of $\check{\chi}_e$ are the same and so the tensor notation is dropped and χ_e is used as a scalar.

Material Response to an Applied Magnetic Field

All materials contain electrons each having a charge and a spin. The charges produce an electric field and the spin produces a magnetic field. The magnetic field produced by an electron is almost that which would be produced by a mechanically spinning charge, although there is no actual mechanical rotation. Spin is a quantum mechanical property. Still it is very hard to avoid using the rotating charge analogy. The spin of an electron points in a particular direction and in most materials these spins are randomly arranged and cancel each other out so that there is no net magnetic effect. Some materials can be magnetized by a large externally applied magnetic field and the spins are then permanently aligned even when the external magnetic field is removed. Sometimes the alignment is established by crystal geometry.

Consider a magnetic material without permanent magnetization. When a magnetic field is applied to these materials the electron spin vector tends to rotate and energy is stored. The ability to store magnetic energy above that which would be stored in vacuum is described by the concept of relative permeability.

The response of a material to an applied magnetic field is not directly analogous to the response to an applied electric field because of the fundamental difference in the source of magnetic moments. In most materials the magnetic moment of one electron is paired with an electron occupying the same orbital with a magnetic moment in the opposite direction so that there is no net magnetic moment. In magnetic materials there are some orbitals that do not have a pair of electrons so that there is a net magnetic moment. Additional energy is stored when the net magnetic moments are rotated away from their preferred long-term direction by an externally applied magnetic field $\bar{\mathcal{H}}$. This is like storing energy in a spring. It is as though there is an additional magnetic field called the magnetic polarization which has a particular density. The net magnetic polarization density is denoted $\bar{\mathcal{M}}$ which has the SI units of $\mathrm{A/m}$ and

$$\bar{\mathcal{M}} = \check{\chi}_m \bar{\mathcal{H}} \tag{1.11}$$

where $\check{\chi}_m$ is called the magnetic susceptibility of the material and in general can be a 3×3 matrix. The relative permeability is defined as

$$\check{\mu}_r = 1 + \check{\chi}_m. \tag{1.12}$$

Summary

A vacuum can store electric and magnetic energy. Materials can store additional energy and the propensity relative to vacuum is described by the relative permittivity and permeability, respectively, of the material.

1.5.3 Field Intensity and Flux Density

Permeability, μ, is generally a scalar, but in magnetic materials it can be a 3×3 matrix called a **dyadic** tensor with the relative permeability tensor being $\check{\mu}_r$. Then

$$\bar{\mathcal{B}} = \mu_0(\bar{\mathcal{H}} + \bar{\mathcal{M}}) \tag{1.13}$$

where $\bar{\mathcal{M}}$ is the magnetization of the material. Introducing phasors

$$\bar{B} = \mu_0(\bar{H} + \check{\chi}_m\bar{\mathcal{H}}) = \mu_o\check{\mu}_r\bar{H} \tag{1.14}$$

$$\begin{bmatrix} \mathcal{B}_x \\ \mathcal{B}_y \\ \mathcal{B}_z \end{bmatrix} = \check{\mu}_r \begin{bmatrix} \mathcal{H}_x \\ \mathcal{H}_y \\ \mathcal{H}_z \end{bmatrix} = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \begin{bmatrix} \mathcal{H}_x \\ \mathcal{H}_y \\ \mathcal{H}_z \end{bmatrix}. \tag{1.15}$$

The direction-dependent property indicated by this dyadic results from alignment of electron spins in a material. However, in most materials there is no alignment of spins and $\mu = \mu_0$. The **relative permeability**, μ_r, refers to the ratio of permeability of a material to its value in a vacuum:

$$\mu_r = \frac{\mu}{\mu_0}. \tag{1.16}$$

So $\mu_r > 1$ indicates that a material can store more magnetic energy than can a vacuum in a given volume.

The other material quantity is the **permittivity**, ε, which is denoted ε_0, and in a vacuum

$$\mathcal{D} = \varepsilon_0\mathcal{E}, \tag{1.17}$$

where ε_0 = 8.854×10^{-12} F·m^{-1}. The **relative permittivity**, ε_r, refers to the ratio of permeability of a material to its value in a vacuum:

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}. \tag{1.18}$$

So $\varepsilon_r > 1$ indicates that a material can store more electric energy in a volume than can be stored in a vacuum. In some calculations it is useful to introduce an **electric polarization**, P_e. Then

$$\mathcal{D} = \varepsilon\mathcal{E} = \varepsilon_0\mathcal{E} + \mathcal{P}_e, \tag{1.19}$$

and so the polarization vector is

$$\mathcal{P}_e = (\varepsilon - \varepsilon_0)\,\mathcal{E} = \varepsilon_0\chi_e\mathcal{E}, \tag{1.20}$$

where χ_e is called the **electric susceptibility**.

Some materials require a dyadic form of ε. This usually indicates a dependence on crystal symmetry, and the relative movement of charge centers in different directions when an E field is applied. Some commonly used microwave substrates, such as sapphire, have permittivities that are direction dependent rather than having full **dyadic permittivity**. A material in which the permittivity is a function of direction is called an anisotropic material or is said to have **dielectric anisotropy**. In an isotropic material the permittivity is the same in all directions; the material has **dielectric isotropy**. Most materials are isotropic.

Maxwell's original equations were put in the form of Equations (1.1)–(1.4) by the mathematician Oliver Heaviside from the less convenient form Maxwell originally used. The above equations are called the point form of Maxwell's equations, relating the field components to each other and to charge and current density at a point.

Maxwell's equations have three types of derivatives. First, there is the time derivative, $\partial/\partial t$. Then there are two spatial derivatives: $\nabla\times$, called curl, capturing the way a field circulates spatially (or the amount that it curls up on itself); and $\nabla\cdot$, called the div operator, describing the spreading out of a field (i.e., its divergence). **Curl** and **div** have different forms in different coordinate systems, and in the rectangular system can be expanded as ($\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$)

$$\nabla\times\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}} \tag{1.21}$$

and $$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{1.22}$$

Curl and div in cylindrical and spherical coordinates are given in Equations (1.134), (1.135), (1.143), and (1.144). Studying Equation (1.21) you will see that curl, $\nabla\times$, describes how much a field circles around the x, y, and z axes. That is, the curl describes how a field circulates on itself. So Equation (1.1) relates the amount an electric field circulates on itself to changes of the B field in time (and modified by the magnetic current). So a spatial derivative of electric fields is related to a time derivative of the magnetic field. Also in Equation (1.3), the spatial derivative of the magnetic field is related to the time derivative of the electric field (and modified by the electric current). These are the key elements that result in self-sustaining propagation.

Div, $\nabla\cdot$, describes how a field spreads out from a point. So the presence of net electric charge (say, on a conductor) will result in the electric field spreading out from a point (see Equation (1.2)). The magnetic field (Equation (1.4)) can never diverge from a point, which is a result of magnetic charges not existing. A magnetic wall approximation describes an open circuit and then effectively magnetic charges terminate the magnetic field. What actually happens in free space or on a transmission line depends on boundary conditions, and in the case of transmission lines, the dimensions involved.

How fast a field varies with time, $\partial\bar{\mathcal{B}}/\partial t$ and $\partial\bar{\mathcal{D}}/\partial t$, depends on frequency. The more interesting property is how fast a field can change spatially, $\nabla\times\bar{\mathcal{E}}$ and $\nabla\times\bar{\mathcal{H}}$—this depends on wavelength relative to geometry. So if the cross-sectional dimensions of a transmission line are much less than a wavelength (say, less than $\lambda/4$), then it will be impossible for the fields to curl up on themselves and so there will be only one or, in some cases, no solutions to Maxwell's equations.

EXAMPLE 1.1 Energy Storage

Consider a material with a relative permittivity of 65 and a relative permeability of 1000. There is a static electric field E of 1 kV/m. How much energy is stored in the E field in a 10 cm^3 volume of the material?

Solution:

Energy stored in a static electric field $= \int_V \bar{D} \cdot \bar{E} \cdot dv$:

$$D = \varepsilon E \text{ and field is constant (uniform)}$$

$$\text{Energy stored } = \varepsilon E^2 \times (\text{volume}) = 65\varepsilon_0 \cdot \left(10^3\right)^2 \cdot \left(10 \cdot 10^{-6}\right) = 5.75 \cdot 10^{-9}\text{ J} = 5.75\text{ nJ}.$$

The complete calculation using SI units is

$$\begin{aligned}\text{Energy stored } &= 65(8.854 \cdot 10^{-12} \cdot \text{F}\cdot\text{m}^{-1})\left(10^3 \cdot \text{V}\cdot\text{m}^{-1}\right)^2 \cdot \left(10 \cdot 10^{-6} \cdot \text{m}^3\right)\\ &= 65(8.854 \cdot 10^{-12} \cdot \text{kg}^{-1}\cdot\text{m}^{-3}\cdot\text{A}^2\cdot\text{s}^4)\cdot\left(10^3 \cdot \text{kg}\cdot\text{m}^2\cdot\text{A}^{-1}\cdot\text{s}^{-3}\cdot\text{m}^{-1}\right)^2\cdot\left(10^{-5} \cdot \text{m}^3\right)\\ &= 65\cdot 8.854\cdot 10^{-12}\cdot 10^6\cdot 10^{-5}\cdot\text{kg}\cdot\text{m}^2\cdot\text{s}^{-2} = 5.75 \cdot 10^{-9} \cdot \text{kg}\cdot\text{m}^2\cdot\text{s}^{-2}\\ &= 5.75 \cdot 10^{-9}\text{ J} = 5.75\text{ nJ}.\end{aligned}$$

EXAMPLE 1.2 Polarization Vector

A time-varying electric field in the x direction has a strength of 100 V/m and a frequency of 10 GHz. The medium has a relative permittivity of 10. What is the polarization vector? Express this vector in the time domain?

Solution:

$\bar{\mathcal{D}} = \varepsilon\bar{\mathcal{E}} = \varepsilon_0\bar{\mathcal{E}} + \bar{\mathcal{P}}_e$.
The polarization vector is

$$\bar{\mathcal{P}}_e = (\varepsilon - \varepsilon_0)\,\bar{\mathcal{E}} = (10-1)\,\varepsilon_0\bar{\mathcal{E}} = 9\varepsilon_0\bar{\mathcal{E}}.$$

Now $\bar{\mathcal{E}} = 100\cos\left(2 \cdot \pi \cdot 10^{10} t\right)\hat{\mathbf{x}}$
and $\varepsilon_0 = 8 \cdot 85 \cdot 10^{12}$, so

$$\begin{aligned}\bar{\mathcal{P}}_e &= 9\left(8.85 \times 10^{-12}\right) \cdot (100)\cos\left(6.28 \times 10^{10} t\right)\hat{\mathbf{x}}\\ &= 79.7 \times 10^{-10} \cdot \cos\left(6.28 \times 10^{10} t\right)\hat{\mathbf{x}}\\ &= 7.79\cos\left(6.28 \times 10^{10} t\right)\hat{\mathbf{x}}\text{ nC/m}^2.\end{aligned} \tag{1.23}$$

1.5.4 Maxwell's Equations in Phasor Form

Phasors reduce the dimensionality of Maxwell's equations by replacing a time derivative by a complex scalar. The phasor form is used with a cosinusoidally varying quantity so, for example, an x-directed electric field in rectangular coordinates,

$$\bar{\mathcal{E}} = |E_x|\cos(\omega t + \phi)\hat{x}, \tag{1.24}$$

is represented as the phasor

$$E_x = |E_x|e^{\jmath\phi}\hat{x}. \tag{1.25}$$

The full three-dimensional electric field is

$$\bar{\mathcal{E}} = |E_x|\cos(\omega t + \phi_x)\hat{\mathbf{x}} + |E_y|\cos(\omega t + \phi_y)\hat{\mathbf{y}} + |E_z|\cos(\omega t + \phi_z)\hat{\mathbf{z}} \tag{1.26}$$

and has the phasor form

$$\bar{E} = |E_x|e^{\jmath\phi_x}\hat{\mathbf{x}} + |E_y|e^{\jmath\phi_y}\hat{\mathbf{y}} + |E_z|e^{\jmath\phi_z}\hat{\mathbf{z}}. \tag{1.27}$$

To express Maxwell's equations in phasor form, the full complex form of Equation (1.26) must be considered. Using Equation (1.94), Equation (1.26) becomes

$$\begin{aligned}\bar{\mathcal{E}} =& \frac{1}{2}\left(|E_x|e^{\jmath\phi_x}\hat{\mathbf{x}} + |E_y|e^{\jmath\phi_y}\hat{\mathbf{y}} + |E_z|e^{\jmath\phi_z}\hat{\mathbf{z}}\right)e^{\jmath\omega t}\\ &+ \frac{1}{2}\left(|E_x|e^{-\jmath\phi_x}\hat{\mathbf{x}} + |E_y|e^{-\jmath\phi_y}\hat{\mathbf{y}} + |E_z|e^{-\jmath\phi_z}\hat{\mathbf{z}}\right)e^{-\jmath\omega t}\end{aligned} \tag{1.28}$$

with the time derivative

$$\begin{aligned}\frac{\partial \mathbf{E}}{\partial t} =& \frac{1}{2}\jmath\omega\left(|E_x|e^{\jmath\phi_x}\hat{\mathbf{x}} + |E_y|e^{\jmath\phi_y}\hat{\mathbf{y}} + |E_z|e^{\jmath\phi_z}\hat{\mathbf{z}}\right)e^{\jmath\omega t}\\ &- \frac{1}{2}\jmath\omega\left(|E_x|e^{-\jmath\phi_x}\hat{\mathbf{x}} + |E_y|e^{-\jmath\phi_y}\hat{\mathbf{y}} + |E_z|e^{-\jmath\phi_z}\hat{\mathbf{z}}\right)e^{-\jmath\omega t}.\end{aligned} \tag{1.29}$$

Equation (1.29) has the phasor form (from the $e^{\jmath\omega t}$ component)

$$\frac{2}{e^{\jmath\omega t}}\left\{\begin{array}{ll} e^{\jmath\omega t} & \\ \text{component of} & \dfrac{\partial\bar{\mathcal{E}}}{\partial t}\end{array}\right\} = \jmath\omega\left(|E_x|e^{\jmath\phi_x}\hat{\mathbf{x}} + |E_y|e^{\jmath\phi_y}\hat{\mathbf{y}} + |E_z|e^{\jmath\phi_z}\hat{\mathbf{z}}\right). \tag{1.30}$$

Noting this, **Maxwell's equations in phasor form**, from Equations (1.1)–(1.4), are

$$\nabla\times\bar{E} = -\jmath\omega\bar{B} - \bar{J}_m \tag{1.31}$$

$$\nabla\times\bar{H} = \jmath\omega\bar{D} + \bar{J} \tag{1.32}$$

$$\nabla\cdot\bar{D} = \rho_V \tag{1.33}$$

$$\nabla\cdot\bar{B} = \rho_{mV}. \tag{1.34}$$

1.5.5 Integral Form of Maxwell's Equations

It is sometimes more convenient to use the integral forms of Maxwell's equations, and Equations (1.1)–(1.4) become

$$\oint_s \nabla\times\bar{\mathcal{E}}\ \cdot d\mathbf{s} = \oint_C \bar{\mathcal{E}}\cdot d\ell = -\oint_s \frac{\partial\bar{\mathcal{B}}}{\partial t}\cdot d\mathbf{s} - \oint_s \bar{\mathcal{J}}_m\cdot d\mathbf{s} \tag{1.35}$$

$$\oint_s \nabla\times\bar{\mathcal{H}}\ \cdot d\mathbf{s} = \oint_C \bar{\mathcal{H}}\cdot d\ell = \int_s \frac{\partial\bar{\mathcal{D}}}{\partial t}\cdot d\mathbf{s} + \bar{\mathcal{I}} \tag{1.36}$$

$$\int_v \nabla\cdot\bar{\mathcal{D}}\,dv = \oint_s \bar{\mathcal{D}}\cdot d\mathbf{s} = \int_V \rho_v\,dv = Q_{\text{enclosed}} \tag{1.37}$$

$$\int_v \nabla\cdot\bar{\mathcal{B}}\,dv = \oint_s \bar{\mathcal{B}}\cdot d\mathbf{s} = 0, \tag{1.38}$$

where Q_{enclosed} is the total charge in the volume enclosed by the surface, S. The subscript S on the integral indicates a surface integral and the circle on the integral sign indicates that the integral is over a closed surface. Two mathematical identities were used in developing Equations (1.35)–(1.38). The first identity is **Stoke's theorem**, which states that for any vector field $\mathbf{X}$,

$$\oint_\ell \mathbf{X} \cdot d\ell \equiv \oint_s (\nabla \times \mathbf{X}) \cdot d\mathbf{s}. \tag{1.39}$$

So the contour integral of X around a closed contour, C (with incremental length vector $d\ell$), is the integral of $\nabla \times X$ over the surface enclosed by the contour and $d\mathbf{s}$ is the incremental area multiplied by a unit vector normal to the surface. This identity is used in Equations (1.35) and (1.36). The **divergence theorem** is the other identity and is used in Equations (1.37) and (1.38). For any vector field $\mathbf{X}$, the divergence theorem states that

$$\oint_s \mathbf{X} \cdot d\mathbf{s} = \int_v \nabla \cdot \mathbf{X}\, dv. \tag{1.40}$$

That is, the volume integral of $\nabla \cdot \mathbf{X}$ is equal to the closed surface integral of $\mathbf{X}$.

Using phasors the integral-based forms of Maxwell's equations become

$$\oint_C \bar{E} \cdot d\ell = -\jmath\omega \oint_s \bar{B} \cdot d\mathbf{s} - \oint_s \bar{J}_m \cdot d\mathbf{s} \tag{1.41}$$

$$\oint_C \bar{H} \cdot d\ell = \jmath\omega \int_s \bar{D} \cdot d\mathbf{s} + \bar{I} \tag{1.42}$$

$$\oint_s \bar{D} \cdot d\mathbf{s} = \int_V \rho_v\, dv = Q_{\text{enclosed}} \tag{1.43}$$

$$\oint_s \bar{B} \cdot d\mathbf{s} = 0, \tag{1.44}$$

1.6 Electric and Magnetic Field Laws

Before Maxwell's equations were postulated, several laws of electromagnetics were known. These are the Biot–Savart law, Ampere's circuital law, Gauss's law, Gauss's law for magnetism, and Faraday's law. The laws are for specific circumstances and mostly for static fields. In this section they are derived from Maxwell's equations.

1.6.1 Ampere's Circuital Law

Ampere's circuital law, often called just Ampere's law, relates direct current and the static magnetic field $\bar{H}$. The relationship is based on Figure 1-3. In the static situation, $\partial\bar{D}/\partial t = 0$, and so one of Maxwell's equations, Equation (1.36), reduces to

$$\oint_\ell \bar{H} \cdot d\ell = I_{\text{enclosed}}. \tag{1.45}$$

This is Ampere's circuital law.

1.6.2 Biot–Savart Law

The Biot–Savart law relates current to static magnetic field. With $\partial\mathbf{D}/\partial t = 0$, Equation (1.3) becomes

$$\nabla \times \bar{H} = \bar{J}. \tag{1.46}$$

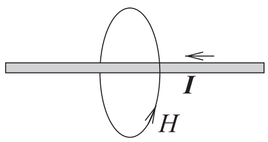


Figure 1-3: Diagram illustrating Ampere's law. Ampere's law relates the current on a wire to the magnetic field around it.

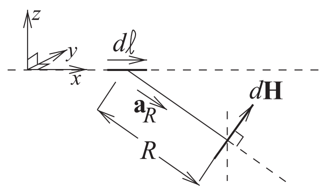


Figure 1-4: Diagram illustrating the Biot–Savart law. The Biot–Savart law relates current to magnetic field.

In Cartesian coordinates,

$$\nabla \times \bar{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{\mathbf{z}}, \quad (1.47)$$

and if the current is confined to a thin wire as shown in Figure 1-4, $\nabla \times \bar{H}$ reduces to an operation along the wire, but also with vector information. So the first step in the development of the Biot–Savart law, Equation (1.46), is simplified to (ignoring vector information for now)

$$\frac{\partial H}{\partial \ell} = \frac{I}{4\pi R^2}. \quad (1.48)$$

With vector information and moving the infinitesimal length of wire, $d\ell$, from the left side of the equation to the right, the final form of the Biot–Savart law is obtained:

$$d\bar{H} = \frac{I d\ell \times \hat{\mathbf{a}}_R}{4\pi R^2}, \quad (1.49)$$

which has the units of amps per meter in the SI system. In Equation (1.49), I is current, $d\ell$ is a length increment of a filament of current I, $\hat{\mathbf{a}}_R$ is the unit vector in the direction from the current filament to the magnetic field, and R is the distance between the filament and the magnetic field increment. So Equation (1.49) is saying that a filament of current produces a magnetic field at a point. The total static magnetic field from a current on a wire or surface can be found by modeling the wire or surface as a number of current filaments, and the total magnetic field at a point is obtained by summing up the contributions from each filament.

1.6.3 Gauss's Law

The third law is Gauss's law, which relates the static electric flux density vector, $\mathbf{D}$, to charge. With reference to Figure 1-5, Gauss's law in integral form is

$$\oint_s \bar{D} = \int_v \rho_v . dv = Q_{\text{enclosed}}, \quad (1.50)$$

which states that the integral of the electric flux density vector, $\bar{D}$, is equal to the total charge enclosed by the surface, Q_{enclosed}. This is just the integral form of the third Maxwell equation (Equation (1.37)).

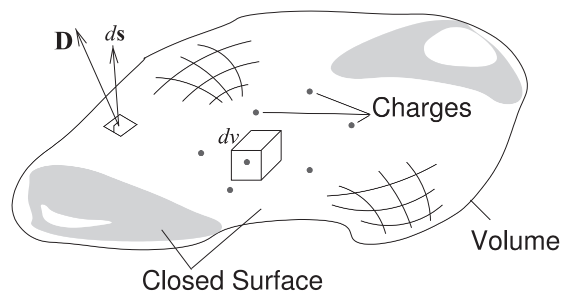


Figure 1-5: Diagram illustrating Gauss's law. Charges are distributed in the volume enclosed by the closed surface. An incremental area is described by the vector $d\mathbf{s}$ which is normal to the surface and whose magnitude is the area of the incremental area.

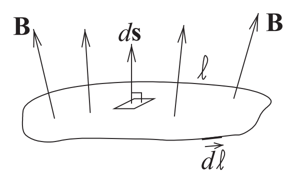


Figure 1-6: Diagram illustrating Faraday's law. The contour ℓ encloses the surface.

There is a similar law called Coulomb's law which says that the magnitude of the force between two stationary point charges is directly proportional to the product of the magnitudes of charges and inversely proportional to the square of the distance between them. This law can be derived from Gauss's law but is less general. Coulomb's law applies only to stationary charges whereas with Gauss's law the charges can be moving.

Gauss's law has also been used for magnetic field where it is called Gauss's law for magnetism. In integral form Gauss's law for magnetism is

$$\oint_s \bar{B} = 0, \tag{1.51}$$

where $\bar{B}$, is the static magnetic flux density. This law is the same as saying that there are no magnetic charges. "Gauss's law for magnetism" is a name that is not universally used and many people simply say that there are no magnetic charges.

1.6.4 Faraday's Law

Faraday's law relates a time-varying magnetic field to an induced voltage drop, V, which is now understood to be $\oint_\ell E \cdot d\ell$, that is, the closed contour integral of the electric field,

$$V = \oint_\ell \bar{\mathcal{E}} \cdot d\ell = -\oint_s \frac{\partial \bar{\mathcal{B}}}{\partial t} \cdot d\mathbf{s}, \tag{1.52}$$

and this has the units of volts in the SI unit system. This is just the first of Maxwell's equations in integral form (see Equation (1.35)). The operation described in Equation (1.52) is illustrated in Figure 1-6.

1.7 Electromagnetic Fields in Dielectrics and Metals

The discussion in this section follows the description by Max Born [11] (who was instrumental in the development of quantum mechanics). Extensions are made to relates quantum descriptions to effects important to microwave engineering. In particular the descriptions here relate phenomena at the discrete particle level to continuum properties such as permittivity, ϵ.

1.7.1 Electromagnetic Fields in a Dielectric

A dielectric has no electrons that are free to move through a material under the influence of an electric field. When an EM field is incident on a dielectric the electric field moves charge centers. For a crystal these charge centers are at the scale of the lattice, and for composite materials like plastics the centers are at the molecular scale. With some materials, which we know as materials having high permittivity, the charge centers are normally separated but by less than the intra-atom spacing. With all dielectric materials, the major impact of an applied electric field is movement of the charge centers, alternately moving them apart and moving them together as the applied electric field alternates. Now being a dielectric, the charges cannot move freely through the material and the energy that is transferred to the charges is stored as electric potential energy in stretched bonds or, sometimes, as a distorted lattice (such as with the piezoelectric effect). This energy storage is much like storing mechanical energy in a spring. Now the charges move and thus excite an electric field of their own. However the charges move sluggishly and so the phase of the EM signal they produce is out of phase with respect to an externally applied alternating EM field. The charge centers are moving the fastest at the peak of the applied sinusoidal field and the net effect is a $90°$ phase lag.

Another phenomenon is that the combined effect of the reradiated fields from the moving charge centers produces an EM wave with the same frequency as the applied field but with a smaller phase velocity (smaller because the phase velocity is averaged over many paths). The oscillating charge centers radiate in all directions (which is called scattering) and so some of the EM energy will be scattered in the direction from which the applied EM wave came.

An ideal dielectric has no loss. That is, there is no dielectric relaxation loss associated with heating as the charge centers move, and there is no conductivity due to moving free charges in the dielectric.

1.7.2 Refractive Index

Dielectrics were first characterized by their refractive index long before the concept of EM waves and permittivity were developed. The refractive index, n, of a medium is defined as the ratio of the speed of light (i.e., of an EM wave) in a vacuum, c, of an EM wave to the phase velocity, v_p, of the wave in the medium:

$$n = \frac{c}{v_p} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \frac{\sqrt{\epsilon\mu}}{1}. \tag{1.53}$$

For a loss-free medium $\quad n = \sqrt{\varepsilon_r \mu_r}. \tag{1.54}$

For a lossy medium the complex index of refraction is

$$\tilde{n} = n + \jmath\kappa, \tag{1.55}$$

where n is called the refractive index and is directly related to the phase velocity, v_p, and κ is called the extinction coefficient, which describes loss when the EM wave propagates through the material.

Conversion between refractive index and permittivity is as follows [12, 13]; the complex permittivity

$$\varepsilon = \epsilon_1 + \jmath\epsilon_2 = (n + \jmath\kappa)^2, \tag{1.56}$$

where $\quad \varepsilon_1 = n^2 - \kappa^2 \quad \text{and} \quad \varepsilon_2 = 2n\kappa.$. (1.57)

The components of the complex index of refraction are then

$$n = \sqrt{\frac{\sqrt{\varepsilon_1^2 + \varepsilon_2^2} + \varepsilon_1}{2}} \quad \text{and} \quad \kappa = \sqrt{\frac{\sqrt{\varepsilon_1^2 + \varepsilon_2^2} - \epsilon_1}{2}}. \tag{1.58}$$

The permittivity is just the square of the (complex) refractive index in a nonmagnetic medium ($\mu_r = 1$). The refractive index is used with optics and the permittivity is used when working with Maxwell's equations and with electronics.

The refractive index, and thus permittivity, of a dielectric can vary significantly between microwave and optical frequencies, a result of how quickly different types of charge centers can move. For water at 20°C and at the standard optical wavelength of 589 nm (the yellow doublet sodium D-line), $n = 1.333$ ($\epsilon_r = 1.78$) [14]. At 1 GHz $n = 8.94$ ($\epsilon_r = 80$) [15].

1.7.3 Electromagnetic Fields in a Metal

The discussion here refers to the behavior of EM fields in metals at frequencies from DC to 10 THz. Above these frequencies the EM wave changes direction much faster than electrons can move. Metals have a large number of free charges that can move through a metal under the influence of an electric field. On transmission lines the energy is contained in the EM fields between metal guides, and electric and magnetic fields are present at the surface of the metal. The main effect of the EM fields at the surface of the metal is to accelerate the free electrons at the surface, with the electrons accelerating in the direction opposite to that of the electric field. While some of the EM energy propagates into the metal, the overwhelming effect is transfer of energy from the EM photons to kinetic energy of the free electrons. There is also some transfer of energy to the bound electrons, however, this effect is smaller as the bound electrons are shielded by the sea of free electrons. Electrons have mass and accelerate relatively slowly. Even when the electric field is reversed, the electrons will continue moving considerably in the same direction as the electric field before reversing. The moving and accelerating electrons also produce electric fields of their own that in turn influence the movement of other electrons. Electrons are sluggish and their position is almost 180° out of phase with the applied E field. The net effect is that the electric field produced by the electrons almost completely cancels

Table 1-2: Electromagnetic properties of metals at optical frequencies. (n and κ are measured, ε_1 and ε_2 are derived using Equation (1.57).

Metal	Photon energy (eV)	Wavelength, λ (µm)	Frequency f (THz)	n	κ	ε_1	ε_2
Gold	0.1	12.398	24.197	8.17	82.83	-6794	1353
Copper	0.1	12.398	24.197	29.69	71.57	-4240	4250
Aluminum	0.1	12.398	24.197	98.595	203.7	-31 663	40 168

the incoming electric field. It is therefore almost meaningless to talk about the (group or phase) velocity of the EM fields in the metal. The overriding effect is transfer of energy to moving electrons that eventually make the lattice move and transfer their energy to the lattice, producing heat.

The accelerating electrons radiate electric field in all directions and in turn this field accelerates other electrons. There is also a damping force as the free electrons collide with atoms. So there is a rapidly diminishing component of the EM field that travels through the metal at the speed of light. Rather than energy being stored in the electric field through the movement of charge centers as in dielectrics, electric energy in the forward-propagating field is almost entirely lost in the scattering and collision processes. So at microwave frequencies the effective permittivity of a metal is almost entirely imaginary (corresponding to loss), and this imaginary component of the relative permittivity is 8–10 orders of magnitude greater than the real component, which is believed to be 1 [16]. The real component of the relative permittivity is 1 because the charge centers are masked by the sea of free electrons. It is not possible to directly measure the real part of the permittivity of metals at microwave frequencies. An ideal conductor has infinite conductivity, so there are no EM fields inside the metal and net charges are confined to the surface of the conductor. At optical frequencies metals are known to have a negative permittivity (see Table 1-2).

1.8 Electric and Magnetic Walls

The boundaries imposed by conductors and the interface between dielectrics define the mode structure (i.e., orientation of the fields) supported by a transmission line. Understanding the fields at the interfaces provides the intuition required to understand the operational frequency limits of distributed structures such as transmission lines.

1.8.1 Electric Wall

An electric wall is formed by an ideal conductor, as shown in Figure 1-7, where there are electric charges at the surface of the conductor. Consider that the conductor in Figure 1-7(a) carries a current of density J and charge of density ρ_V. As the conductivity of the conductor goes to infinity, the conductor becomes a perfect conductor and forms an electric wall, as shown in Figure 1-7(b). The current and charge now occupy the very skin of the conductor at the interface to the region above which there is either free space or a dielectric. The behavior of the fields at the interface can be understood

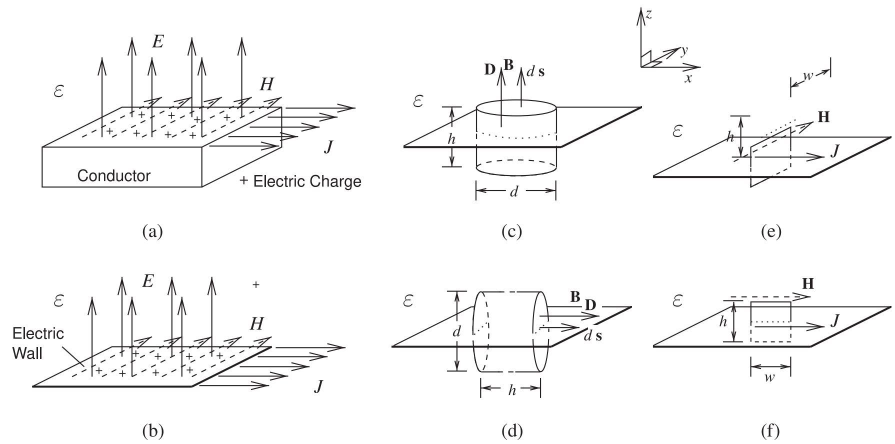


Figure 1-7: Properties of an electric wall: (a) the electrical field is perpendicular to a conductor and the magnetic field is parallel to it and is orthogonal to the current; (b) a conductor can be approximated by an electric wall; (c) cylinder used in deriving the normal electric field; (d) cylinder used in showing that the tangential electric field is zero; (e) contour used in deriving the tangential magnetic field in the y direction; and (f) contour used in deriving the z- and x-directed magnetic fields (they are zero).

by considering the integration surfaces and volumes shown in Figure 1-7(c and d). The cylinder in Figure 1-7(c) with height h and diameter d enables Gauss's law (Equation (1.50)) to be applied. As $h \longrightarrow 0$ (h approaching zero), Equation (1.50) yields

$$\oint_s \bar{\mathcal{D}}.d\mathbf{s} = D_z \tfrac{1}{4}\pi d^2 = \int_v \rho_v .dv = Q_{\text{enclosed}} = \rho_S \tfrac{1}{4}\pi d^2, \tag{1.59}$$

where ρ_S is the surface charge density (in SI units of $\mathrm{C/m^2}$) and so

$$D_z = \rho_S. \tag{1.60}$$

Now considering the cylinder in Figure 1-7(d), again as $h \longrightarrow 0$,

$$\oint_s \bar{\mathcal{D}}.d\mathbf{s} = D_x \tfrac{1}{4}\pi d^2 = \int_v \rho_v .dv = Q_{\text{enclosed}}$$
$$= \rho_S \pi d h \longrightarrow 0. \tag{1.61}$$

That is, $$D_x = 0. \tag{1.62}$$

Changing the orientation of the cylinder, it can be seen that

$$D_y = 0. \tag{1.63}$$

So the electric field is normal to the surface of the conductor.

The properties of the magnetic field at the electric wall can be deduced from Faraday's circuital law and the Biot–Savart law (Equation (1.49)). Consider the rectangular contour shown in Figure 1-7(e) with height h and width w. With $h \longrightarrow 0$, only the y-directed component of the magnetic field will contribute to the static magnetic field integral of Faraday's law, so from

Equation (1.52),

$$\oint_{\ell} \bar{H} \cdot d\ell = H_y w = I_{\text{enclosed}} = J_S w, \tag{1.64}$$

where J_S is the surface current density (with SI units of A/m) and

$$H_y = J_S. \tag{1.65}$$

For the field oriented in the x direction, the rectangular contour shown in Figure 1-7(f) can be used, but now no current is enclosed by the contour, so that with $h \longrightarrow 0$,

$$\oint_{\ell} \bar{H} \cdot d\ell = H_x w = I_{\text{enclosed}} = 0 \tag{1.66}$$

and $$H_x = 0. \tag{1.67}$$

H_z has not yet been considered. Again, using the rectangle in Figure 1-7(f) and with $w \longrightarrow 0$,

$$\oint_{\ell} \mathbf{H} \cdot d\ell = H_z h = I_{\text{enclosed}} = 0 \tag{1.68}$$

and $$H_z = 0. \tag{1.69}$$

This could also have been derived from the Biot–Savart law (Equation (1.49)). Immediately above the electric wall $R \longrightarrow 0$ and only the filament of current immediately below the magnetic field contributes. Thus a_R is normal to the surface so that **H** will be perpendicular to the current, but in the plane of the current.

Gathering the results together, the EM fields immediately above the electric wall have the following properties:

The E field is perpendicular to the electric wall.	The H field is parallel to the electric wall.
$D_{\text{normal}} = \rho_S$, the surface charge density.	$H_{\text{parallel}} = J_S$, the surface current density.

1.8.2 Magnetic Wall

A true magnetic wall does not exist, but it is approximated by the interface between two dielectrics (as shown in Figure 1-8) when the permittivity of the dielectrics differ so that $\varepsilon_2 \gg \varepsilon_1$. Considering the top dielectric region, the magnetic wall model can then be introduced at the interface between the two dielectrics, complete with magnetic charges and magnetic current density, M. The situation is analogous to the electric wall situation with the roles of the electric and magnetic fields interchanged. The EM fields immediately above the magnetic wall have the following properties:

The H field is perpendicular to the magnetic wall.	The E field is parallel to the magnetic wall.
$H_{\text{normal}} = \rho_{mS}$, the surface magnetic charge density.	$E_{\text{parallel}} = J_{mS}$, the surface magnetic current density.

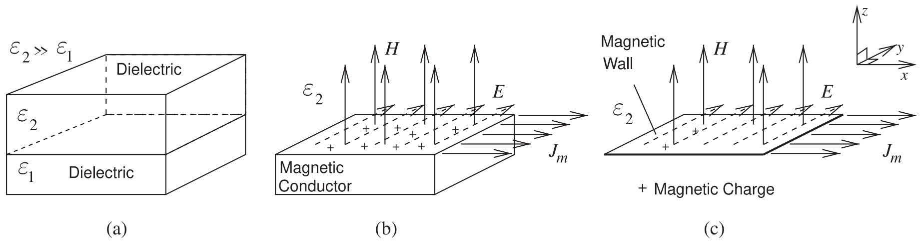


Figure 1-8: Properties of the EM field at a magnetic wall: (a) the interface of two dielectrics of contrasting permittivities approximates a magnetic wall with the magnetic field perpendicular to the interface and the electric field parallel to it; (b) the dielectric with lower permittivity can be approximated as a magnetic conductor; and (c) the magnetic conductor forms a magnetic wall at the interface with the material with higher permittivity.

Table 1-3: Properties of electric and magnetic walls.

	Electric field	Magnetic field
Electric wall	Normal	Parallel
Magnetic wall	Parallel	Normal

What constitutes a good magnetic wall is subject to debate, as the contrast between dielectrics is not as great as the contrast between the conductivity of a good conductor and that of a good dielectric. However, it is an essential concept in understanding the coupling of signals and approximating transmission lines to yield intuitive understanding.

The summary that is used in the text to understand multimoding is given in Table 1-3.

1.9 Fields in Lossy Mediums

Lossy mediums result in power loss of EM fields. In RF and microwave systems materials are ideally either perfect conductors or perfect dielectrics. In practice, conductors and dielectrics have finite conductivity and dielectrics have another type of loss called dielectric relaxation loss. Dielectric relaxation loss is due to heating that can result from the movement of charge centers in a material.

1.9.1 Lossy Dielectrics

In a lossy dielectric there can be both current flow and relaxation loss. Current flow results from applied electric field and is described by the **conductivity**, σ, of the material. **Relaxation loss** results from polarization of the material and the transfer of energy to the lattice of the material as the electric field changes direction. This is also referred to as **dielectric damping**. The polarization vector describes the additional energy storage capacity of a material (above that of vacuum) in response to an applied electric field. The capacity is described by the electric susceptibility, χ_e, in Equation (1.20). In phasor form, loss is captured by a complex χ_e or a **complex permittivity**:

$$\varepsilon = \varepsilon' - \jmath\varepsilon'' = \varepsilon(1 + \chi_e). \tag{1.70}$$

Both current and relaxation respond to an applied electric field. In phasor form, Maxwell's curl equation for a magnetic field, Equation (1.32), becomes

$$\begin{aligned}\nabla \times \bar{H} &= \jmath\omega\bar{D} + \bar{J} = \jmath\omega\varepsilon\bar{E} + \sigma\bar{E} = \jmath\omega(\varepsilon' - \jmath\varepsilon'')\bar{E} + \sigma\bar{E} \\ &= \jmath\omega\varepsilon'\bar{E} + (\sigma + \omega\varepsilon'')\bar{E} \\ &= \jmath\omega\left(\varepsilon' - \jmath\varepsilon'' - \jmath\frac{\sigma}{\omega}\right)\bar{E}. \end{aligned} \tag{1.71}$$

In the above, $\omega\varepsilon'$ describes the ability to store electric energy, $\omega\varepsilon''$ describes dielectric damping loss, and σ describes conductivity loss. At a single frequency, damping and conductivity losses are indistinguishable and so the term $(\omega\varepsilon'' + \sigma)$ is taken as the effective conductivity. The effective (complex) relative permittivity is

$$\varepsilon_e = \left(\varepsilon' - \jmath\varepsilon'' - \jmath\frac{\sigma}{\omega}\right)\frac{1}{\varepsilon_0}. \tag{1.72}$$

Another quantity that describes loss is the **loss tangent**, $\tan\delta$, which is the ratio of power lost to energy stored:

$$\tan\delta = \frac{\omega\varepsilon'' + \sigma}{\omega\varepsilon'}. \tag{1.73}$$

For many dielectric materials used with EM structures, $\tan\delta$ is approximately independent of frequency up to 100s of gigahertz, so it is commonly used to characterize the dielectric loss of a material. Most exceptions are semiconductors such as silicon which can have appreciable **conductive loss**. The frequency independence of $\tan\delta$ indicates that conductivity loss is negligible.

EXAMPLE 1.3 Electromagnetic Wave Propagation in a Lossy Dielectric

Consider a plane EM wave propagating in a medium with permittivity ε and permeability μ. The complex permittivity measured at two frequencies is experimentally characterized by the relative permittivity $(\varepsilon/\varepsilon_0)$ of the real, $\Re\{\varepsilon/\varepsilon_0\}$, and imaginary, $\Im\{\varepsilon/\varepsilon_0\}$, parts as follows:

Measured relative permittivity:

Frequency	Real part	Imaginary part
1 GHz	1.9	−0.0490
2 GHz	1.9	−0.0314

Measured relative permeability:

Frequency	Real part	Imaginary part
1 GHz	1.9	−0.001
2 GHz	1.9	−0.001

Since there is an imaginary part of the dielectric constant, there could be loss due to either dielectric damping or finite material conductivity, or both. When characterizing materials, it is only possible to distinguish the contribution to the measured imaginary part of the permittivity by considering the experimentally derived imaginary permittivity at two different frequencies.

(a) Determine the dielectric loss tangent at 1 GHz.
(b) Determine the relative dielectric damping factor at 1 GHz (the part of the permittivity due to dielectric damping).
(c) What is the conductivity of the dielectric?
(d) What is the magnetic conductivity of the medium?

Solution:

(a) At 1 GHz, $\varepsilon = \varepsilon_r \varepsilon_0$, and the measured relative permittivity is

$$\varepsilon_r = 1.9 - \jmath 0.049 = \varepsilon'_{r,e} - \jmath \varepsilon''_{r,e}, \tag{1.74}$$

where $\varepsilon'_{r,e}$ and $\varepsilon''_{r,e}$ are the effective (measured) values of ε'_r and ε''_r obtained from measurement. Now $\varepsilon'_e = \varepsilon'$, but the imaginary permittivity includes the effect of dielectric conductivity, σ, as well as of dielectric damping, which is described by ε'' alone. That is,

$$\varepsilon''_e = \varepsilon'' + \frac{\sigma}{\omega} \cdot \varepsilon_r = \varepsilon''_{r,e} = 0.0490 \quad \text{and} \quad \tan\delta = 0.0490/1.9 = 0.0258.$$

(b) The imaginary relative permittivity is found by noting that $\varepsilon''_{r,e} = \varepsilon''_r + \sigma/\omega$ and $\omega\varepsilon''_r + \sigma = \omega\varepsilon''_{r,e}$. Let $\omega_1 = 2\pi\,(1\text{ GHz})$. Then

$$\begin{aligned}
\omega_2 &= 2\omega_1 \\
\omega_1 \varepsilon''_{r,e}\,(1\text{ GHz}) &= \omega_1\varepsilon''_r + \sigma = \omega_1 \cdot 0.0490 \quad \text{(A)} \\
\omega_2 \varepsilon''_{r,e}\,(2\text{ GHz}) &= \omega_2\varepsilon''_r + \sigma = \omega_2 \cdot 0.0314 \\
2\omega_1 \varepsilon''_e + \sigma &= 2\omega_1\,(0.0314) \qquad\qquad \text{(B)} \\
\text{(B)} - \text{(A)} \rightarrow \omega_1\varepsilon''_r &= \omega_1\,[0.0628 - 0.0490] \\
\varepsilon''_r &= 0.0138 \;\; \text{(at all frequencies)}\,.
\end{aligned}$$

(c) Recall that $\varepsilon''_{r,e} = \varepsilon''_r + \sigma/\omega$.
At 1 GHz, $\omega = 2\pi \times 10^9$, $\varepsilon''_{r,e} = 0.0490$, $\varepsilon''_r = 0.0138$, so

$$\sigma = \omega\left(\varepsilon''_{r,e} - \varepsilon''_r\right) = 2 \cdot \pi \cdot 10^9\,(0.0490 - 0.0138) = 0.221 \cdot 10^9\ \text{S/m} = 221\ \text{MS/m}.$$

(d) Magnetic conductivity is nonexistent, thus magnetic conductivity is zero.

1.9.2 Lossy Conductors

Perfect conductors would have infinite conductivity, but since the conductivity of real conductors is finite, EM fields penetrate into the interior of a conductor. Still the energy stored in the current is much greater than the (dielectric) electric energy storage capability, $\sigma \gg \omega\varepsilon'$, and the losses from conductivity are much higher than dielectric loss, $\sigma \gg \omega\varepsilon''$, so that the effective relative permittivity from Equation (1.72) is

$$\varepsilon_e = \left(\varepsilon' - \jmath\varepsilon'' - \jmath\frac{\sigma}{\omega}\right)\frac{1}{\varepsilon_0} \approx \left(-\jmath\frac{\sigma}{\omega}\right)\frac{1}{\varepsilon_0} \approx \left(\frac{\sigma}{\jmath\omega}\right)\frac{1}{\varepsilon_0}. \tag{1.75}$$

For most conductors the permeability is μ_0 and the working permittivity is

$$\varepsilon = \varepsilon_e\varepsilon_0 = \frac{\sigma}{\jmath\omega}. \tag{1.76}$$

The propagation constant of the field in a conductor is

$$\gamma = \alpha + \jmath\beta \approx \jmath\omega\sqrt{\mu\varepsilon} \approx \jmath\omega\sqrt{\mu}\sqrt{\frac{\sigma}{\jmath\omega}} \approx (1+\jmath)\sqrt{\frac{\omega\mu\sigma}{2}}. \tag{1.77}$$

So $\alpha = \sqrt{\omega\mu\sigma/2}$. The EM field in the conductor reduces in amplitude as $\mathrm{e}^{-\alpha x}$ after a distance x. Thus the field reduces to $1/\mathrm{e}$ of its value after a distance called the skin depth, δ_s:

$$\delta_s = \sqrt{\frac{2}{\omega\mu\sigma}}. \tag{1.78}$$

If the amplitude of the electric field at the surface of a conductor is $A(0)$, after a distance δ_s the amplitude of the electric field will have the amplitude

$$\begin{aligned} A(\delta_s) &= \mathrm{e}^{-\alpha x} = \mathrm{e}^{-\left(\sqrt{\omega\mu\sigma/2}\sqrt{2/\omega\mu\sigma}\right)} \\ &= \mathrm{e}^{-1}. \end{aligned} \tag{1.79}$$

Table 1-4 lists the skin depth and phase velocities of conductors commonly used in microwave circuits. The resistivity values here are those of a single crystal. The phase velocity, v_p, is calculated from the propagation constant β. From Equation (1.77), $\beta = \sqrt{\omega\mu\sigma/2}$.

Now $v_p = \omega/\beta$, so that
$$v_p = \frac{\omega}{\beta} = \frac{\omega\sqrt{2}}{\sqrt{\omega\mu\sigma}} = \sqrt{\frac{2\omega}{\mu\sigma}}. \tag{1.80}$$

This can be simplified further if the relative permeability of the material is 1, as then $\mu = \mu_0 = 4\pi 10^{-7}$ H/m. If SI units are used for f and σ:

$$v_p = \sqrt{\frac{2\omega}{\mu_0\sigma}} = \sqrt{\frac{4\pi f}{4\pi 10^{-7}\sigma}} = \sqrt{\frac{10^7 f}{\sigma}}. \tag{1.81}$$

From Table 1-4 note that the skin depth at gigahertz frequencies is close to the thickness of microstrip lines which are often only a few microns thick. Also the speed of an EM wave in a conductor is very slow. The speed of EM in copper (ideal), for example, is 0.004% of the speed of light in a vacuum.

Table 1-4: Skin depth and effective phase velocity at 1 GHz of several conductors commonly used in microwave circuits.

Material	Electrical conductivity, σ (MS.m^{-1})	Skin depth, δ at 1 GHz (µm)	Phase velocity, v_p at 1 GHz (km/s)
1 GHz			
Aluminum (crystal)	37.7	2.59	16.3
Aluminum (2× resistivity)	18.9	3.66	23.0
Copper (single crystal)	59.6	2.06	13.0
Copper (2× resistivity)	29.8	2.92	18.3
Gold (single crystal)	45.2	2.37	14.9
Gold (2× resistivity)	22.6	3.35	21.0
Silver (single crystal)	63.0	2.01	12.6
Silver (2× resistivity)	31.5	2.84	17.8
Titanium (single crystal)	0.238	32.6	200
Titanium (2× resistivity)	0.119	46.1	290

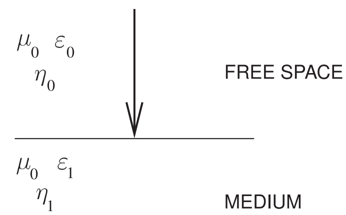


Figure 1-9: Two regions used in Example 1.4.

EXAMPLE 1.4 Electromagnetic Calculations

This example provides a review of basic EM theory. In Figure 1-9, a plane wave in free space is normally incident on a lossless medium occupying a half space with a dielectric constant of 16. Calculate the electric field reflection coefficient at the interface referred to the medium.

Solution:

Since it is not explicitly stated, assume $\mu_1 = \mu_0$. (This should be done in general since μ_0 is the default value for μ, as most materials have unity relative permeability.) The characteristic impedance of a medium is $\eta = \sqrt{\mu/\varepsilon}$. Now η is used for characteristic impedance (also called wave impedance) of a TEM propagating wave in a medium. It is used instead of Z_0, which is generally reserved for the characteristic impedance of transmission lines. The electric field reflection coefficient is

$$\Gamma^E = \frac{\eta_1 - \eta_0}{\eta_1 + \eta_0} = \frac{\eta_1/\eta_0 - 1}{\eta_1/\eta_0 + 1}$$

$$\frac{\eta_1}{\eta_0} = \sqrt{\frac{\mu_0}{\varepsilon_1}} \cdot \sqrt{\frac{\varepsilon_0}{\mu_0}} = \sqrt{\frac{\varepsilon_0}{\varepsilon_1}} = \sqrt{\frac{1}{16}} = \frac{1}{4},$$

so

$$\Gamma^E = \frac{1/4 - 1}{1/4 + 1} = \frac{1-4}{1+4} = -\frac{3}{5} = -0.6.$$

EXAMPLE 1.5 Intuitive Solution of Electromagnetic Problems

This example provides an intuitive way of solving an EM problem. A plane wave in free space is normally incident on a lossless medium occupying a half space with a dielectric constant of 16. What is the magnetic field reflection coefficient?

Solution:

There are two ways to answer this question. An intuitive approach will be used first.

Intuitively $|\Gamma^H| = |\Gamma^E|$,.

With $\bar{\mathcal{E}}$ and $\bar{\mathcal{H}}$ being vectors, the Poynting vector $\bar{\mathcal{E}} \times \bar{\mathcal{H}}$ points in the direction of propagation for a plane wave, since the reflected wave is in the opposite direction to the incident wave, thus

$$\text{sign}\left(\Gamma^H\right) = -\text{sign}\left(\Gamma^E\right).$$

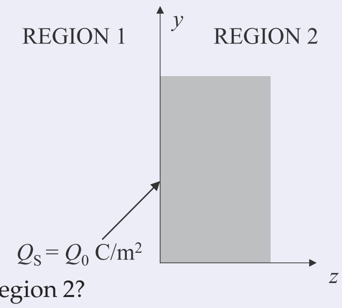
Therefore $\Gamma^H = -\Gamma^E = +0.6$. ($\Gamma^E$ was calculated in Example 1.4.)

Now the second approach. The formula for the magnetic field reflection coefficient is

$$\Gamma_{\text{H}} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} = \frac{1 - \eta_2/\eta_1}{1 + \eta_2/\eta_1} = \frac{1 - 1/4}{1 + 1/4} = \frac{3}{5} = 0.6.$$

EXAMPLE 1.6 Wave Impedance and Propagation Constant

A 4 GHz time-varying EM field is traveling in the $+z$ direction in Region 1 and is normally incident on another material in Region 2. The boundary between the two regions is in the $z = 0$ plane. The permittivity of Region 1 is $\varepsilon_1 = \varepsilon_0$ and that of Region 2 is $\varepsilon_2 = (4 - \jmath 0.4)\varepsilon_0$. For both regions, $\mu_1 = \mu_2 = \mu_0$. The phasor of the forward-traveling electric field (i.e., the incident field) is $\mathbf{E}^+ = 100\,\hat{y}$, and the phase is normalized with respect to $z = 0$.



What are the wave impedance and propagation constant of Region 2?

Solution:

The wave impedance is

$$\begin{aligned}\eta &= \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{\mu_r}{\varepsilon_r}} = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}} = 377\sqrt{\frac{1}{(4-\jmath 0.4)}}\ \Omega \\ &= 187.8 + \jmath 9.4\ \Omega. \end{aligned} \tag{1.82}$$

The propagation constant is

$$\begin{aligned}\gamma &= \jmath\omega\sqrt{\mu\varepsilon} = \jmath\omega\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r} \\ &= \jmath 4\cdot 2\pi\cdot 10^9\cdot\sqrt{4\pi\cdot 10^{-7}\cdot 8.854\cdot 10^{-12}}\sqrt{4-\jmath 0.4} = (8.4 + \jmath 167.9)\ \mathrm{m}^{-1}. \end{aligned} \tag{1.83}$$

1.10 Summary

This introductory chapter provided the background for the following chapters on distributed effects. A review of Maxwell's equations and their implications was presented. The notable change in the way these were presented here to the first time they are introduced in electromagnetics courses is the first time the static electric and magnetic field laws are presented and then Maxwell's equations are covered almost as though there was a continuous derivation. However Maxwell's equations cannot be derived from the static laws even with some additional insight. Instead Maxwell's equations embed the quantum field theory that there are two inseparable component fields which we call electric and magnetic fields, and the fundamental relativity effect of the finite speed of light which results from relating the spatial derivative of each of the field components to the time derivative of the other component. The insight places Maxwell among the ten greatest physicists of all time.

The finite speed for electromagnetic radiation is at the core of microwave engineering. At microwave frequencies the finite speed at which information can move means that microwave circuit design is quite different from low frequency circuit design, and the distributed and electromagnetic coupling effects lead to a large number of microwave circuit components that provide a very rich design space.

1.11 References

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1.12 Exercises

1. The plane wave electric field of a radar pulse propagating in free space in the z direction is given by (with $\omega_0 \gg \pi/(2\tau)$)

$$\bar{\mathcal{E}}(t) = \begin{cases} A\cos\left(\frac{\pi t}{2\tau}\right)\cos(\omega_0 t)\hat{\mathbf{x}} & -\tau \le t \le \tau \\ 0 & \text{otherwise,} \end{cases}$$

   (a) Sketch $\bar{\mathcal{E}}$(t) over the interval $-1.5\tau \le t \le 1.5\tau$.
   (b) What is, $\bar{\mathcal{H}}(t)$.
   (c) What is the Poynting vector, $\bar{\mathcal{S}}(t)$.
   (d) Determine the total energy density in the pulse (in units of J/m^2).

2. Consider a material with a relative permittivity of 72, a relative permeability of 1, and a static electric field (E) of 1 kV/m. How much energy is stored in the E field in a $10\ \text{cm}^3$ volume of the material? [Parallels Example 1.1]

3. A time-varying electric field in the x direction has a strength of 1 kV/m and a frequency of 1 GHz. The medium has a relative permittivity of 70. What is the time-domain polarization vector? [Parallels Example 1.2]

4. A medium has a dielectric constant of 20. What is the index of refraction?

5. Consider a plane EM wave propagating in a medium with a permittivity ε and permeability μ. The complex permittivity measured at two frequencies is characterized by the relative permittivity $(\varepsilon/\varepsilon_0)$ of the real $\Re\{\varepsilon/\varepsilon_0\}$ and imaginary parts $\Im\{\varepsilon/\varepsilon_0\}$ as follows:

   Measured relative permittivity:

Frequency	Real part	Imaginary part
1 GHz	3.8	-0.05
10 GHz	4.0	-0.03

   Measured relative permeability:

Frequency	Real Part	Imaginary Part
1 GHz	0.999	-0.001
10 GHz	0.998	-0.001

   Since there is an imaginary part of the dielectric constant there could be either dielectric damping or material conductivity, or both. [Parallels Example 1.3]

   (a) Determine the dielectric loss tangent at 10 GHz.
   (b) Determine the relative dielectric damping factor at 10 GHz (the part of the permittivity due to dielectric damping).
   (c) What is the conductivity of the dielectric at 10 GHz?

6. A medium has a relative permittivity of 13 and supports a 5.6 GHz EM signal. By default, if not specified otherwise, a medium is lossless and will have a relative permeability of 1.

   (a) Calculate the characteristic impedance of an EM plane wave.
   (b) Calculate the propagation constant of the medium.

7. A plane wave in free space is normally incident on a lossless medium occupying a half space with a dielectric constant of 12. [Parallels Example 1.4]

   (a) Calculate the electric field reflection coefficient referred to the interface medium.
   (b) What is the magnetic field reflection coefficient?

8. Water, or more specifically tap water or sea water, has a complex dielectric constant resulting from two effects: conductivity resulting from dissolved ions in the water leading to charge carriers that can conduct current under the influence of an electric field, and dielectric loss resulting from rotation or bending of the water molecules themselves under the influence of an electric field. The rotation or bending of the water molecules results in motion of the water molecules and thus heat. The relative permittivity is

$$\varepsilon_{wr} = \varepsilon'_{wr} - \jmath\varepsilon''_{wr} \tag{1.84}$$

   and the real and imaginary components are

$$\varepsilon'_{wr} = \varepsilon_{w00} + \frac{\varepsilon_{w0} - \varepsilon_{w00}}{1 + (2\pi f \tau_w)^2} \tag{1.85}$$

$$\varepsilon''_{wr} = \frac{2\pi f \tau_w(\varepsilon_{w0} - \varepsilon_{w00})}{1 + (2\pi f \tau_w)^2} + \frac{\sigma_i}{2\pi\varepsilon_0 f}, \tag{1.86}$$

   where f is frequency, $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m is the permittivity magnitude, and $\varepsilon_{w00} = 4.9$ is the high-frequency limit of ε_r and is known to be independent of salinity and is also that of pure deionized water. The other quantities are τ_w, the relaxation time of water; ε_{w0}, the static relative dielectric constant of water (which for pure water at 0°C is equal to 87.13); and σ_i, the ionic conductivity of the water. At 5°C and for tap water with a salinity of 0.03 (in parts per thousand by weight) [17, 18], ε_{w0} = 85.9, τ_w = 14.6 ps, and σ_i = 0.00548 S/m.

   (a) Calculate and then plot the real and imaginary dielectric constant of water over the frequency range from DC to 40 GHz for tap water at 5°C. You should see two distinct regions. Identify the part of the imaginary dielectric constant graph that results from ionic conductivity and the part of the graph that relates to dielectric relaxation.
   (b) Consider a plane wave propagating in water. Determine the attenuation constant and the propagation constant of the wave at 500 MHz.

(c) Consider a plane wave propagating in water. Determine the attenuation constant and the propagation constant of the wave at 2.45 GHz.

(d) Consider a plane wave propagating in water. Determine the attenuation constant and the propagation constant of the wave at 22 GHz.

(e) Again considering the plane wave in water, calculate the loss factor over a distance of 1 cm at 2.45 GHz. Express your answer in terms of decibels.

(f) Consider a plane wave making normal incidence from above into a bucket of tap water that is 1 m deep so that you do not need to consider the reflection at the bottom of the bucket. Calculate the reflection coefficient of the incident plane wave at 2.45 GHz referred to the air immediately above the surface of the water.

(g) Again consider the situation in (f), but this time with the plane wave incident on the air–water interface from the water side, calculate the reflection coefficient of the incident wave at 2.45 GHz referred to the water immediately below the surface of the water.

(h) If the bucket in (f) and (g) is 0.5 cm deep you will need to consider the reflection at the bottom of the bucket. Consider that the bottom of the bucket is a perfect conductor. Draw the bounce diagram for calculating the reflection of the plane wave in air. Determine the total reflection coefficient of the plane wave (referred to the air immediately above the surface of the water) by using the bounce diagram.

(i) Repeat (h) using the formula for multiple reflections.

9. A 4 GHz time-varying EM field is traveling in the $+z$ direction in Region 1 and is incident on another material in Region 2. The permittivity of Region 1 is $\varepsilon_1 = \varepsilon_0$ and that of Region 2 is $\varepsilon_2 = (4-\jmath 0.04)\varepsilon_0$. For both regions $\mu_1 = \mu_2 = \mu_0$. [Parallels Example 1.6]

   (a) What is the characteristic impedance (or wave impedance) in Region 2?

   (b) What is the propagation constant in Region 1?

10. A 4 GHz time-varying EM field is traveling in the $+z$ direction in Region 1 and is normally incident on another material in Region 2. The boundary between the two regions is in the $z = 0$ plane. The permittivity of Region 1 is $\varepsilon_1 = \varepsilon_0$, and that of Region 2 is $\varepsilon_2 = 4\varepsilon_0$. For both regions, $\mu_1 = \mu_2 = \mu_0$. The phasor of the forward-traveling electric field (i.e., the incident field) is ﬃ$E^+ = 100\ \hat{\mathbf{y}}$ V/m and the phase is normalized with respect to $z = 0$. $Q_0 = 0$. [Parallels Example 1.6]

   (a) What is the wave impedance of Region 1?

   (b) What is the wave impedance of Region 2?

   (c) What is the electric field reflection coefficient at the boundary?

   (d) What is the magnetic field reflection coefficient at the boundary?

   (e) What is the electric field transmission coefficient at the boundary?

   (f) What is the power transmitted into Region 2?

   (g) What is the power reflected from the boundary back into Region 1?

1.12.1 Exercises By Section

†challenging, ‡very challenging

§1.5 $1^{\dagger}, 2^{\dagger}, 3^{\dagger}$ §1.7 $4, 5^{\dagger}$ §1.9 $6, 7^{\dagger}, 8^{\ddagger}, 9^{\dagger}, 10^{\dagger}$

1.12.2 Answers to Selected Exercises

4 4.47 8(c) $\gamma = 49 + \jmath 468\,\mathrm{m}^{-1}$ 9 $0.84 + \jmath 168\,\mathrm{m}^{-1}$

Appendix

1.A Mathematical Foundations

1.A.1 Angles of a Triangle

The law of cosines relates the angles and lengths of the sides of the triangle in Figure 1-10(a):

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma) \tag{1.87}$$

$$\cos(\alpha) = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos(\beta) = \frac{a^2 + c^2 - b^2}{2ac}, \quad \cos(\gamma) = \frac{a^2 + b^2 - c^2}{2ab}. \tag{1.88}$$

Referring to the right-angle triangle in Figure 1-10(b):

$$\sin(x) = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos(x) = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan(x) = \frac{\text{opposite.}}{\text{adjacent}} \tag{1.89}$$

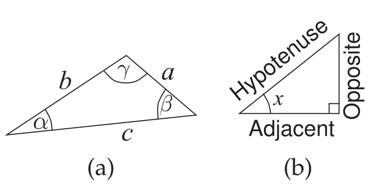


Figure 1-10: Triangles: (a) triangle with sides having lengths a, b, and c, and angles α, β, and γ; and (b) right-angle triangle.

1.A.2 Trigonometric Identities

$$\text{if } \sin(\theta) = x \text{ then } \theta = \arcsin(x) \qquad \text{if } \cos(\theta) = x \text{ then } \theta = \arccos(x) \tag{1.90}$$
$$\text{if } \tan(\theta) = x \text{ then } \theta = \arctan(x) \tag{1.91}$$

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x \tag{1.92}$$
$$\csc(-x) = -\csc x \qquad \sec(-x) = \sec x \qquad \cot(-x) = -\cot x \tag{1.93}$$
$$\cos(x) = \tfrac{1}{2}\left(e^{\jmath x} + e^{-\jmath x}\right) \qquad \sin(x) = \tfrac{1}{2}\jmath\left(e^{-\jmath x} - e^{\jmath x}\right) \tag{1.94}$$
$$\sin x = 1/\csc x \qquad \cos x = 1/\sec x \qquad \tan x = 1/\cot x \tag{1.95}$$
$$\tan x = \sin x/\cos x \tag{1.96}$$
$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x \tag{1.97}$$
$$\sin\left(x - \tfrac{\pi}{2}\right) = -\cos x \qquad \cos\left(x - \tfrac{\pi}{2}\right) = \sin x \qquad \tan\left(x - \tfrac{\pi}{2}\right) = -\cot x \tag{1.98}$$
$$\sin\left(x + \tfrac{\pi}{2}\right) = \cos x \qquad \cos\left(x + \tfrac{\pi}{2}\right) = -\sin x \qquad \tan\left(x + \tfrac{\pi}{2}\right) = -\cot x \tag{1.99}$$
$$\sin\left(\tfrac{\pi}{2} - x\right) = \cos x \qquad \cos\left(\tfrac{\pi}{2} - x\right) = \sin x \qquad \tan\left(\tfrac{\pi}{2} - x\right) = \cot x \tag{1.100}$$
$$\csc\left(x - \tfrac{\pi}{2}\right) = -\sec x \qquad \sec\left(x - \tfrac{\pi}{2}\right) = \csc x \qquad \cot\left(x - \tfrac{\pi}{2}\right) = -\tan x \tag{1.101}$$
$$\csc\left(x + \tfrac{\pi}{2}\right) = \sec x \qquad \sec\left(x + \tfrac{\pi}{2}\right) = -\csc x \qquad \cot\left(x + \tfrac{\pi}{2}\right) = -\tan x \tag{1.102}$$
$$\csc\left(\tfrac{\pi}{2} - x\right) = \sec x \qquad \sec\left(\tfrac{\pi}{2} - x\right) = \csc x \qquad \cot\left(\tfrac{\pi}{2} - x\right) = \tan x \tag{1.103}$$
$$\sin(2x) = 2\sin x \cos y \qquad \cos(2x) = \cos^2(x) - \sin^2(x) \tag{1.104}$$
$$\tan(2x) = \frac{2\tan x}{1 - \tan^2 x} \qquad \cot(2x) = \tfrac{1}{2}[\cot(x) - \tan(x)] \tag{1.105}$$
$$\sin^2 x = \tfrac{1}{2}\left[1 - \cos(2x)\right] \qquad \cos^2 x = \tfrac{1}{2}\left[1 + \cos(2x)\right] \qquad \tan^2 x = \frac{1 - \cos(2x)}{1 + \cos(2x)} \tag{1.106}$$
$$\tan(x + y) = \frac{\tan(x) + \tan(y)}{\tan(x) - \tan(y)} \qquad \tan(x - y) = \frac{\tan(x) - \tan(y)}{\tan(x) + \tan(y)} \tag{1.107}$$
$$\sin x + \sin y = 2\sin\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right) \qquad \sin x - \sin y = 2\cos\left(\tfrac{x+y}{2}\right)\sin\left(\tfrac{x-y}{2}\right) \tag{1.108}$$
$$\cos x + \cos y = 2\cos\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right) \qquad \cos x - \cos y = -2\cos\left(\tfrac{x+y}{2}\right)\sin\left(\tfrac{x-y}{2}\right) \tag{1.109}$$
$$\sin x \sin y = \tfrac{1}{2}\left[\cos(x - y) - \cos(x + y)\right] \qquad \sin x \cos y = \tfrac{1}{2}\left[\sin(x + y) + \sin(x - y)\right] \tag{1.110}$$
$$\cos x \cos y = \tfrac{1}{2}\left[\cos(x - y) + \cos(x + y)\right] \qquad \cos x \sin y = \tfrac{1}{2}\left[\sin(x + y) - \sin(x - y)\right] \tag{1.111}$$

1.A.3 Trigonometric Derivatives

$$\begin{array}{lll} \dfrac{d}{dx}\sin(x) = \cos(x) & \dfrac{d}{dx}\cos(x) = -\sin(x) & \dfrac{d}{dx}\tan(x) = \sec^2(x) \\ \dfrac{d}{dx}\csc(x) = -\csc(x)\cot(x) & \dfrac{d}{dx}\sec(x) = \sec(x)\tan(x) & \dfrac{d}{dx}\cot(x) = -\csc^2(x) \end{array} \tag{1.112}$$

1.A.4 Complex Numbers and Phasors

A complex number z is an ordered pair of two real numbers x and y, and a complex number has a special meaning when working in the frequency domain. The rectangular form of the complex number is $z = x + \jmath y$, where $\jmath = \sqrt{-1}$ is an imaginary number.[1] The polar form of the complex number $z = r\angle\varphi$ relates directly to the amplitude and phase of a cosinusoid. In determining phase, reference is made to a cosinusoid of the form $u(t) = r\cos(\omega t + \varphi)$, as this has a value of r when the radian frequency $\omega = 0$ (i.e., at DC). The cosinusoid is shown as the solid line in Figure 1-11(a), where r is the amplitude and ϕ is the phase of the cosinusoid. The cosinusoid can be presented as the complex valued function

$$f(t) = z e^{\jmath\omega t}, \qquad \text{where} \qquad z = r e^{\jmath\phi} \tag{1.113}$$

[1] Outside electrical engineering $\imath$ (i without the dot) is used for $\sqrt{-1}$, but this is replaced by $\jmath$ (j without the dot) in electrical engineering to avoid confusion with the usage of i for current.

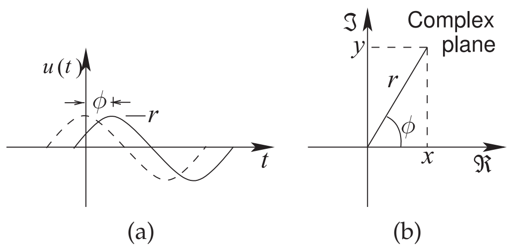


Figure 1-11: Sinusoidal signal representations: (a) waveform referenced to the zero-phase cosinusoid shown as the dashed waveform; and (b) representation as a phasor on the complex plane where Re is the real part and Im is the imaginary part.

is called the phasor of $u(t)$. The phasor contains the amplitude and phase of the cosinusoid, and given the frequency of the cosinusoid, the original time-domain waveform can be reconstructed. The time-domain form of the complex function is

$$u(t) = \Re\{f(t)\} = \Re\left\{z e^{\jmath\omega t}\right\} = \Re\left\{r e^{\jmath\phi} e^{\jmath\omega t}\right\} = \Re\left\{r e^{[\jmath(\omega t+\phi)]}\right\} = r\cos(\omega t + \phi). \tag{1.114}$$

$\Re\{\}$ indicates that the real part of a complex number is taken, and similarly $\Im\{\}$ indicates that the imaginary part of a complex number is taken. The complex number can be represented in rectangular or magnitude-phase form so that

$$z = x + \jmath y = r\angle\phi = r e^{\jmath\phi} = r(\cos\phi + \jmath\sin\phi) \tag{1.115}$$

with the relationship shown in Figure 1-11(b). Note that in Equation (1.115), ϕ must be expressed in radians and anywhere it is used in calculations. Also

$$x = r\cos(\phi) \quad \text{and} \quad y = r\sin(\phi). \tag{1.116}$$

Basic calculations using the complex numbers

$$w = a + \jmath b = q\angle\theta \quad \text{and} \quad z = c + \jmath d = r\angle\phi \tag{1.117}$$

(with a, b, c, d , θ, ϕ, r, and q being real numbers) are

- $\jmath \cdot \jmath = \jmath^2 = -1$
- Addition: $w + z = (a + \jmath b) + (c + \jmath d) = (a + c) + \jmath(b + d)$
- Subtraction: $w - z = (a + \jmath b) - (c + \jmath d) = (a - c) + \jmath(b - d)$
- Multiplication: $w \cdot z = (a + \jmath b) \cdot (c + \jmath d) = (ac - bd) + \jmath(bc + ad)$
- Magnitude: $|w| = \sqrt{a^2 + b^2} = q$; $|z| = \sqrt{c^2 + d^2} = r$
- Angle: $\theta = \arctan(b/a) = \arccos(a/q) = \arcsin(b/q)$ (Correction may be required to get the phase in the right quadrant. This requires that the signs of a and b be examined.)
- Multiplication (alternative): $w \cdot z = qr\angle(\theta + \phi)$
- Division: $w/z = \dfrac{a + \jmath b}{c + \jmath d} = \dfrac{(a + \jmath b)(c - \jmath d)}{(c + \jmath d)(c - \jmath d)} = \left(\dfrac{ac + bd}{c^2 + d^2}\right) + \jmath\left(\dfrac{bc - ad}{c^2 + d^2}\right)$
- Division (alternative): $w/z = q/r\angle(\theta - \phi)$
- Square root: $\sqrt{w} = \sqrt{q}\angle(\theta/2 + m\pi), \quad m = 0, 1 \quad (\text{i.e. } \sqrt{w} = \sqrt{q}\angle(\theta/2) \text{ and } \sqrt{q}\angle(\theta/2 + \pi))$
- nth root: $\sqrt[n]{w} = \sqrt[n]{q}\angle\left(\theta/n + 2\pi\dfrac{m}{n}\right), \quad m = 0, 1, \ldots (n - 1)$
- Negative: $-w = -q\angle\theta = q\angle(\theta + \pi)$
- Complex conjugate: $w^* = a - \jmath b = q\angle(-\theta)$
- Squared magnitude: $|z|^2 = z \cdot z^*$
- Conjugate addition: $(w + z)^* = w^* + z^*$
- Conjugate power: $(z^n)^* = (z^*)^n$
- Conjugate multiplication: $(wz)^* = w^* z^*$
- Conjugate division: $(w/z)^* = w^*/z^*$
- Conjugate exponential: $\exp(z^*) = (\exp z)^*$

- Conjugate logarithm: $\log(z^*) = (\log z)^*$
- Conjugate conjugate: $(w^*)^* = w$
- Complex inverse: $z^{-1} = z^*/|z|^2$
- Complex magnitude: $|z^*| = |z|$

EXAMPLE 1.7 Complex numbers

Consider the complex numbers $w = -0.5 - \jmath 1.6$ and $z = 5 - \jmath 3$, and the waveform $v(t) = 2.3\cos(2\pi 10t - 1.2)$.

- In polar form, $w = 1.676\angle(-1.874)$, where 1.874 is in radians; alternatively $w = 1.676\angle(-107.4)^\circ$.
- In polar form, $z = 5.831\angle(-0.54)$.
- $2w = -1 - \jmath 3.2 = 3.353\angle(-1.874)$.
- $z \cdot w = w \cdot z = w \cdot z = (1.676 \cdot 5.831)\angle(-1.874 - 0.54) = 9.774\angle(-2.414) = -7.3 - \jmath 6.5$.
- $\sqrt{z} = 2.327 - \jmath 0.645$.
- Alternatively $\sqrt{z} = \sqrt{5.831}\angle(-0.54/2) = 2.415\angle(-0.27)$.
- $w/z = (1.676/5.831)\angle[-1.874 - (-0.54)] = 0.287\angle{-1.333} = 0.068 - \jmath 6.5$.
- The radian frequency of $v(t)$ is 20π rads/s.
- The frequency of $v(t)$ is 10 Hz.
- The phasor of $v(t)$ is $v = 2.3\angle(-1.2) = 2.3\angle(-68.755^\circ) = 0.833 - \jmath 2.144$.
- If w is a phasor and its frequency is 1 GHz, the waveform equivalent of w is $w(t) = 1.676\cos(2\pi \cdot 10^9 t - 1.874)$ and the phase is -1.874 radians or -107.35° or 252.65°.

1.A.5 Vector Operators

Vector Multiplication

There are two types of operations that multiply vectors. $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are vectors with

$$\mathbf{A} = a_i\mathbf{i} + a_j\mathbf{j} + a_k\mathbf{k}, \tag{1.118}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, are orthogonal vectors such as the unit vectors of the Cartesian coordinate system, that is, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ in the x, y and z directions, respectively.

Dot Product

$\mathbf{A} \cdot \mathbf{B}$ is called the dot product, the scalar product or the inner product. It is read as “a dot b.”

$\mathbf{A} \cdot \mathbf{B} = a_ib_i + a_jb_j + \ldots + a_kb_k$.

Commutative property: $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$.

Distributive property: $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$.

Bilinear property (r is a scalar): $\mathbf{A} \cdot (r\mathbf{B} + \mathbf{C}) = r(\mathbf{A} \cdot \mathbf{B}) + (\mathbf{A} \cdot \mathbf{C})$.

Scalar multiplication property (r and s are scalars): $(r\mathbf{A}) \cdot (s\mathbf{B}) = (rs)(\mathbf{A} \cdot \mathbf{B})$.

Orthogonal property: Two nonzero vectors $\mathbf{A}$ and $\mathbf{B}$ are orthogonal if and only if $\mathbf{A} \cdot \mathbf{B} = 0$.

Cross Product $\mathbf{A} \times \mathbf{B}$ is called the cross product. It is read as "a cross b." Now the vector components of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ need to be more tightly specified as orthogonal vectors, typically having unit amplitude, that have the property

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}, \quad \text{and} \quad \mathbf{i} \times \mathbf{i} = 0 = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} \tag{1.119}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}. \tag{1.120}$$

The unit vectors of the Cartesian coordinate system have this property with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, corresponding to the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ in the x, y and z directions, respectively.

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_i\mathbf{i} + a_j\mathbf{j} + a_k\mathbf{k}) \times (b_i\mathbf{i} + b_j\mathbf{j} + b_k\mathbf{k}) \\ &= a_ib_i\mathbf{i} \times \mathbf{i} + a_ib_j\mathbf{i} \times \mathbf{j} + a_ib_k\mathbf{i} \times \mathbf{k} + a_jb_i\mathbf{j} \times \mathbf{i} + a_jb_j\mathbf{j} \times \mathbf{j} + a_jb_k\mathbf{j} \times \mathbf{k} + \\ &\quad a_kb_i\mathbf{k} \times \mathbf{i} + a_kb_j\mathbf{k} \times \mathbf{j} + a_kb_k\mathbf{k} \times \mathbf{k} \\ &= (a_jb_k - a_kb_j)\mathbf{i} + (a_kb_i - a_ib_k)\mathbf{j} + (a_ib_j - a_jb_i)\mathbf{k}.\end{aligned} \tag{1.121}$$

Anticommutative property: $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

Distributive property: $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$.

Bilinear property (r is a scalar): $\mathbf{A} \times (r\mathbf{B} + \mathbf{C}) = r(\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$.

Scalar multiplication property (r and s are scalars): $(r\mathbf{A}) \times (s\mathbf{B}) = (rs)(\mathbf{A} \times \mathbf{B})$.

Lagrange's formula

$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{B} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.

Del (∇) Vector Operator

Operations in Cartesian, Cylindrical, and Spherical Coordinate Systems

This section defines the del, or the nabla ∇, a calculus operator that operates on vector quantities and also on a scalar to create a vector. The various del operators in the Cartesian (rectangular), spherical, and cylindrical coordinate systems are presented. Here a scalar is denoted by f and a vector by $\mathbf{F}$:

- grad, ∇, is the gradient of a scalar with direction. ∇f is read as "grad f."
- div, ∇, is a measure of how a vector is spreading out from a point. $\nabla \cdot \mathbf{F}$ is read as "div F."
- curl, $\nabla\times$, is a measure of the rotation of a vector. $\nabla \times \mathbf{F}$ is read as "curl F."

Cartesian (rectangular) coordinates (x, y, z):
$\hat{x}$ is the unit vector in the x direction and F_x is the component of the field in the x-direction. Similarly for the other components.

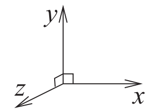


Figure 1-12

$$\mathbf{F} = F_x\hat{x} + F_y\hat{y} + F_z\hat{z} \tag{1.122}$$

$$\text{grad} \quad \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \tag{1.123}$$

$$\text{div} \quad \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{1.124}$$

$$\text{curl} \quad \nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{z}} \tag{1.125}$$

$$\nabla^2\mathbf{F} = \frac{\partial^2\mathbf{F}}{\partial x^2} + \frac{\partial^2\mathbf{F}}{\partial y^2} + \frac{\partial^2\mathbf{F}}{\partial z^2} \tag{1.126}$$

$$\text{differential length, } d\ell = dx\ \hat{\mathbf{x}} + dy\ \hat{\mathbf{y}} + dz\ \hat{\mathbf{z}} \tag{1.127}$$

$$\text{differential normal area, } d\mathbf{s} = dy\ dz\ \hat{\mathbf{x}} + dx\ dz\ \hat{\mathbf{y}} + dx\ dy\ \hat{\mathbf{z}} \tag{1.128}$$

$$\text{differential volume, } dv = dx\ dy\ dz. \tag{1.129}$$

Cylindrical coordinates (ρ, ϕ, z):
The cylindrical coordinates are ρ for radius, ϕ for angle, and z for height. These are related to the Cartesian coordinates by

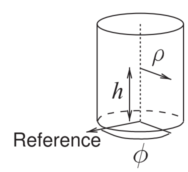


Figure 1-13

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z \tag{1.130}$$

$$\rho = \sqrt{x^2 + y^2}, \quad \phi = \text{atan2}(y, x), \quad z = z \tag{1.131}$$

$$\text{vector} \quad \mathbf{F} = F_\rho\hat{\boldsymbol{\rho}} + F_\phi\hat{\boldsymbol{\phi}} + F_z\hat{\mathbf{z}} \tag{1.132}$$

$$\text{grad} \quad \nabla f = \frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \tag{1.133}$$

$$\text{div} \quad \nabla \cdot \mathbf{F} = \frac{1}{\rho}\frac{\partial \rho F_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \tag{1.134}$$

$$\text{curl} \quad \nabla \times \mathbf{F} = \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left(\frac{\partial(\rho F_\phi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi}\right)\hat{\mathbf{z}} \tag{1.135}$$

$$\text{differential length, } d\ell = d\rho\ \hat{\boldsymbol{\rho}} + \rho d\phi\ \hat{\boldsymbol{\phi}} + dz\ \hat{\mathbf{z}} \tag{1.136}$$

$$\text{differential normal area, } d\mathbf{s} = \rho d\phi\ dz\ \hat{\boldsymbol{\rho}} + d\rho\ dz\ \hat{\boldsymbol{\phi}} + \rho d\rho\ d\phi\ \hat{\mathbf{z}} \tag{1.137}$$

$$\text{differential volume, } dv = \rho d\rho\ d\phi\ dz. \tag{1.138}$$

Spherical coordinates (ρ, θ, ϕ):
The spherical coordinates are ρ for radius, θ for angle of the horizontal projection, and ϕ for the elevation angle. These are related to the Cartesian coordinates by

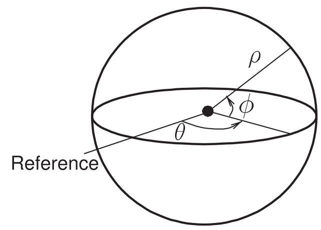


Figure 1-14

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta \tag{1.139}$$

$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \arccos(z/r), \quad \phi = \operatorname{atan2}(y,x) \tag{1.140}$$

$$\text{vector } \mathbf{F} = F_r\hat{\mathbf{r}} + F_\theta\hat{\boldsymbol{\theta}} + F_\phi\hat{\boldsymbol{\phi}} \tag{1.141}$$

$$\text{grad } \nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{1.142}$$

$$\text{div } \nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(F_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi} \tag{1.143}$$

$$\begin{aligned}\text{curl } \nabla\times\mathbf{F} = &\frac{1}{r\sin\theta}\left(\frac{\partial(F_\phi\sin\theta)}{\partial\theta} - \frac{\partial F_\theta}{\partial\phi}\right)\hat{\mathbf{r}} \\ &+ \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial(rF_\phi)}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}}\end{aligned} \tag{1.144}$$

$$\text{differential length } d\ell = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\,\hat{\boldsymbol{\phi}} \tag{1.145}$$

$$\text{differential area } d\mathbf{s} = r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}} + r\sin\theta\,dr\,d\phi\,\hat{\boldsymbol{\theta}} + r\,dr\,d\theta\,\hat{\boldsymbol{\phi}} \tag{1.146}$$

$$\text{differential volume } dv = r^2\sin\theta\,dr\,d\theta\,d\phi. \tag{1.147}$$

Identities

Identities for the del, ∇, operator:

$$\text{div grad } f = \nabla\cdot(\nabla f) = \nabla^2 f = \triangle f(\text{Laplacian}) \tag{1.148}$$

$$\triangle(fg) = f\triangle g + 2\nabla f\cdot\nabla g + g\triangle f \tag{1.149}$$

$$\text{curl grad } f = \nabla\times(\nabla f) = 0 \tag{1.150}$$

$$\text{div curl } \mathbf{F} = \nabla\cdot(\nabla\times\mathbf{F}) = 0 \tag{1.151}$$

$$\text{curl curl } \mathbf{F} = \nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot F) - \nabla^2 F \tag{1.152}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})\,. \tag{1.153}$$

1.A.6 Hyperbolic Functions and Complex Numbers

$$\cosh(x) = \tfrac{1}{2}\left(e^x + e^{-x}\right) = \cos(\jmath x) \tag{1.154}$$
$$\cosh(\jmath x) = \tfrac{1}{2}\left(e^{\jmath x} + e^{-\jmath x}\right) = \cos(x) \tag{1.155}$$
$$\cosh(-x) = \cosh(x) \tag{1.156}$$

$$\sinh(x) = \tfrac{1}{2}\left(e^x - e^{-x}\right) = -\jmath\sin(\jmath x) \tag{1.157}$$
$$\sinh(\jmath x) = \tfrac{1}{2}\left(e^{\jmath x} - e^{-\jmath x}\right) = \jmath\sin(x) \tag{1.158}$$
$$\sinh(-x) = -\sinh(x) \tag{1.159}$$

$$\cosh^2(x) - \sinh^2(x) = 1 \tag{1.160}$$

$$\tanh(x) = \sinh(x)/\cosh(x) \quad (1.161)$$
$$\tanh(x) = -\jmath\tan(\jmath x) \quad (1.162)$$
$$\tanh(\jmath x) = \jmath\tan(x) \quad (1.163)$$
$$\tanh(-x) = -\tanh(x) \quad (1.164)$$

$$\sin(\jmath x) = \jmath\sinh(x) \quad (1.165)$$
$$\cos(\jmath x) = \cosh(x) \quad (1.166)$$
$$\tan(\jmath x) = \jmath\tanh(x) \quad (1.167)$$

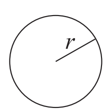
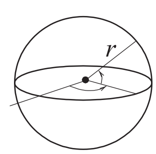
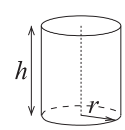
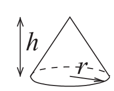
$$\mathrm{e}^{x} = \cosh(x) + \sinh(x) \qquad \mathrm{e}^{-x} = \cosh(x) - \sinh(x) \quad (1.168)$$
$$\mathrm{e}^{\jmath x} = \cos(x) + \jmath\sin(x) \qquad \mathrm{e}^{-\jmath x} = \cos(x) - \jmath\sin(x) \quad (1.169)$$

$$\sin(x + \jmath y) = \sin(x)\cosh(y) + \jmath\cos(x)\sinh(y) \quad (1.170)$$
$$\cos(x + \jmath y) = \cos(x)\cosh(y) - \jmath\sin(x)\sinh(y) \quad (1.171)$$

$$\mathrm{e}^{\gamma} = \mathrm{e}^{(\alpha+\jmath\beta)} = \mathrm{e}^{\alpha}\mathrm{e}^{\jmath\beta}, \text{ where } \gamma = \alpha + \jmath\beta \quad (1.172)$$

$$\cot(x) = \tfrac{1}{2}\left[\cot(x/2) - \tan(x/2)\right]. \quad (1.173)$$

1.A.7 Volumes and Areas

	Circle	Sphere	Cylinder	Cone
	r = radius 	r = radius 	r = radius h = height 	r = radius of base h = height 
Area	πr^2	$4\pi r^2$	$2\pi r h$ (cylinder) $2\pi r^2$ (base & top)	
Volume		$4\pi r^3/3$	$\pi r^2 h$	$\pi r^2 h/3$

1.A.8 Series Expansions

Square root:

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(1-2n)(n!)^2(4^n)} x^n$$
$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \ldots \text{ for } -1 < x \leq 1. \quad (1.174)$$

Exponential function (the natural number e raised to a power):

$$\mathrm{e}^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \text{ for all } x. \quad (1.175)$$

Natural logarithm:

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \text{ for } -1 \le x < 1 \tag{1.176}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \text{ for } -1 < x \le 1. \tag{1.177}$$

Binomial series:

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \quad \text{for all } |x| < 1 \text{ and all complex } \alpha \tag{1.178}$$

where n is an integer and the binomial coefficient is

$$\binom{\alpha}{n} = \prod_{k=1}^{n} \frac{\alpha - k + 1}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}. \tag{1.179}$$

If α is replaced by an integer N the binomial coefficient is

$$\binom{N}{n} = \frac{N!}{(N-n)!n!}. \tag{1.180}$$

Infinite geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1. \tag{1.181}$$

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n \quad \text{for } |x| < 1. \tag{1.182}$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} \quad \text{for } |x| < 1. \tag{1.183}$$

1.A.9 Trigonometric Series Expansions

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \text{ for all } x \tag{1.184}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \text{ for all } x \tag{1.185}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \text{ for } |x| < \frac{\pi}{2} \tag{1.186}$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \text{ for all } x \tag{1.187}$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \text{ for all } x \tag{1.188}$$

$$\tanh x = \sum_{n=1}^{\infty} \frac{B_{2n} 4^n (4^n - 1)}{(2n)!} x^{2n-1} = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots \text{ for } |x| < \frac{\pi}{2} \tag{1.189}$$

$$\text{arsinh}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \text{ for } |x| \leq 1 \tag{1.190}$$

$$\text{artanh}(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \text{ for } |x| < 1. \tag{1.191}$$

1.A.10 Special Polynomials

Several polynomials have special properties of use in the design of filters and matching networks. Their arguments are put in terms of frequency and a transfer function of a two-port is expressed in terms on one of the special polynomials or its inverse.

Butterworth Polynomial

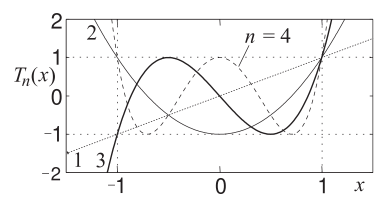
The Butterworth polynomials have the general form

$$B_n(x) = \begin{cases} \prod_{k=1}^{n/2} \left\{ x^2 - 2x\cos\left[\frac{(2k+n-1)\pi}{2n}\right] + 1 \right\} & \text{for } n \text{ even} \\ (x+1)\prod_{k=1}^{(n-1)/2} \left\{ x^2 - 2x\cos\left[\frac{(2k+n-1)\pi}{2n}\right] + 1 \right\} & \text{for } n \text{ odd} \end{cases} \tag{1.192}$$

These of course do not look like polynomials but their expansions are, see Table 1-5.

Chebyshev Polynomial

The nth-order Chebyshev polynomial of the first kind, $T_n(x)$, (often just called the Chebyshev polynomial) has the special property that the maximum absolute value of $T_n(x)$ for $-1 \leq x \leq 1$ is 1. This can be seen in the figure to the right where the first four Tchebyshev polynomials are plotted.



When x is frequency this means that the frequency response in a bandwidth has equal ripples across the passband and outside (both above and below) the value of $T_n(x)$ rapidly diverges (the magnitude becomes monotonically larger). For a filter or matching network the reflection coefficient is expressed in terms of $T_n(x)$ and so the skirts of the transmission response are steep. Normalization of x and scaling of the Chebyshev polynomials is used to scale the response in frequency and amplitude. The Chebyshev polynomial is obtained using the recursion formula

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad \text{with} \quad T_1(x) = x \quad \text{and} \quad T_2(x) = 2x^2 - 1. \tag{1.193}$$

For example, with $n = 3$,

$$T_3(x) = 2xT_{3-1}(x) - T_{3-2}(x) = 2x(2x^2 - 1) - x = 4x^3 - 2x - x = 4x^3 - 3x. \tag{1.194}$$

Table 1-5: Butterworth polynomials.

n	Butterworth polynomial $B_n(x)$
1	$x + 1$
2	$x^2 + 1.4142x + 1$
3	$(x+1)(x^2 + x + 1)$
4	$(x^2 + 0.7564x + 1)(x^2 + 1.8478x + 1)$
5	$(x+1)(x^2 + 0.1680x + 1)(x^2 + 1.6180x + 1)$
6	$(x^2 + 0.5176x + 1)(x^2 + 1.4142x + 1)(x^2 + 1.9319x + 1)$
7	$(x+1)(x^2 + 0.4450x + 1)(x^2 + 1.2470x + 1)(x^2 + 1.8019x + 1)$
8	$(x^2 + 0.3902x + 1)(x^2 + 1.1111x + 1)(x^2 + 1.6629x + 1)(x^2 + 1.9616x + 1)$

Several useful expansions of Chebyshev polynomials are used in microwave engineering to relate the frequency response of a function or circuit to the Chebyshev characteristic. Generally the frequency is referred to the electrical length θ (which for a transmission line θ is proportional to frequency). Some of the more useful expansions used in the text follow:

$$T_n(\cos\theta) = \cos(n\theta) \tag{1.195}$$

and note that when $\theta = \pi/2$, $\cos(n\theta) = 0$ when is odd and $\cos(n\theta) = \pm 1$ when is even. From this identity two more useful identities are derived:

$$T_n(x) = \cos\left(n\cos^{-1}x\right) \quad \text{for } |x| \leq 1, \quad \text{and} \quad T_n(x) = \cosh\left(n\cosh^{-1}x\right) \quad \text{for } |x| \geq 1. \tag{1.196}$$

A normalized argument $\cos\theta/\cos\theta_m$ is often used to define a passband as being between θ_m and $(\pi - \theta_m)$. That is that we can relate the Chebyshev response to frequency and define a frequency bandwidth. Then a useful Chebyshev polynomial expansion is

$$T_n\left(\cos\theta/\cos\theta_m\right) = T_n(\sec\theta_m\cos\theta) = \cos\left\{n\left[\cos^{-1}\left(\cos\theta/\cos\theta_m\right)\right]\right\}. \tag{1.197}$$

From Section 1.A.2 note that $\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)]$, $\cos\theta\cos(n\theta) = \frac{1}{2}\cos[(n+1)\theta]\cos[(n-1)\theta]$, and $\sec(\theta_m) = 1/\cos\theta_m$. Then (after ingenious manipulation) Equation (1.193) becomes

$$\begin{aligned}
T_1(\cos\theta/\cos\theta_m) &= \sec\theta_m\cos\theta\\
T_2(\cos\theta/\cos\theta_m) &= \sec^2\theta_m[\cos(2\theta)+1]-1 = \sec^2\theta_m\cos(2\theta)+(\sec^2\theta_m-1)\\
T_3(\cos\theta/\cos\theta_m) &= \sec^3\theta_m[\cos(3\theta)+3\cos\theta]-3\sec\theta_m\cos\theta\\
&= \sec^3\theta_m\cos(3\theta)+3(\sec^3\theta_m-\sec\theta_m)\cos\theta\\
T_4(\cos\theta/\cos\theta_m) &= \sec^4\theta_m[\cos(4\theta)+4\cos(2\theta)+3]-4\sec^2\theta_m[\cos(2\theta)+1]+1\\
&= \sec^4\theta_m\cos(4\theta)+4(\sec^4-\sec^2\theta_m)\cos(2\theta)+(3\sec^4\theta_m-4\sec^2\theta_m+1)\\
K_5(\cos\theta/\cos\theta_m) &= \sec^5\theta_m[\cos(5\theta)+5\cos(3\theta)+7\cos\theta]-\sec^3\theta_m[5\cos(3\theta)+11\cos\theta]\\
&\quad+4\sec\theta_m\cos\theta\\
&= \sec^5\theta_m\cos(5\theta)+5(\sec^5\theta_m-\sec^3\theta_m)\cos(3\theta)\\
&\quad+(7\sec^5\theta_m-11\sec^3\theta_m+4\sec\theta_m)\cos\theta.
\end{aligned} \tag{1.198}$$

1.A.11 Matrix Operations

A, **B**, and **C** are square matrices and V is a vector.
Determinant:

$$|\mathbf{A}| = \det(\mathbf{A}). \tag{1.199}$$

Transpose:

$$(\mathbf{ABC})^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} \tag{1.200}$$

$$\left(\mathbf{ABC}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{CB}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}. \tag{1.201}$$

Inverse:

$$(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \tag{1.202}$$

$$\left(\mathbf{ABC}^{-1}\right)^{-1} = \left(\mathbf{C}^{-1}\right)^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} = \mathbf{CB}^{-1}\mathbf{A}^{-1} \tag{1.203}$$

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{1}. \tag{1.204}$$

Two row square matrices and vectors

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{1.205}$$

Multiplication:

$$\mathbf{AV} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} (a_{11}v_1 + a_{12}v_2) \\ (a_{21}v_1 + a_{22}v_2) \end{bmatrix} \tag{1.206}$$

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{21} + a_{12}b_{22}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) \end{bmatrix}. \tag{1.207}$$

Determinant:

$$|\mathbf{A}| = \det(\mathbf{A}) = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{1}{ad - bc}. \tag{1.208}$$

Transpose:

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \tag{1.209}$$

Inverse:

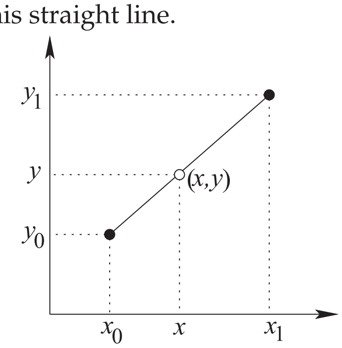
$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} d & -b \\ -c & d \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & d \end{bmatrix}. \tag{1.210}$$

1.A.12 Interpolation

Linear Interpolation

Linear interpolation can be used to extract data from tables. In the table below there are two known points, (x_0, y_0) and (x_1, y_1), and the linear interpolant is the straight line between them. The unknown point (x, y) is found by locating it on this straight line.

Variable	
Independent	Dependent
x_0	y_0
x	y
x_1	y_1



Thus

$$\frac{y - y_0}{y_1 - y_0} = \frac{x - x_0}{x_1 - x_0} \tag{1.211}$$

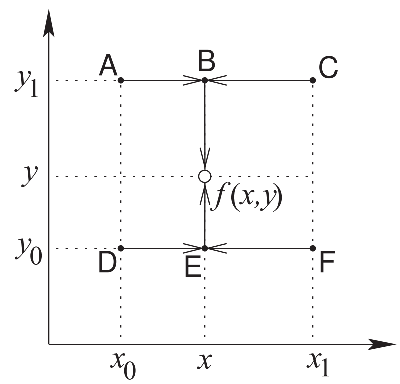
so that, given x,

$$y = y_0 + (x - x_0)\frac{y_1 - y_0}{x_1 - x_0}. \tag{1.212}$$

Bilinear Interpolation

Bilinear interpolation is a two-dimensional generalization of linear interpolation. In the table below, $f(\mathsf{D}) = f(x_0, y_0)$, $f(\mathsf{A}) = f(x_0, y_1)$, $f(\mathsf{F}) = f(x_1, y_0)$, and $f(\mathsf{C}) = f(x_1, y_1)$ are known and $f(x, y)$ is to be found. The bilinear interpolation technique is illustrated in the figure.

	Variable	
Independent	Independent	Dependent
x_0	y_0	f_0
x	y	f
x_1	y_1	f_1



First, find the value of the function at point B from the values at Point A and Point C. Thus, from Equation (1.212),

$$f(\mathsf{B}) = f(x, y_1) = f(\mathsf{A}) + (x - x_0)\frac{f(\mathsf{C}) - f(\mathsf{A})}{x_1 - x_0}. \tag{1.213}$$

Similarly

$$f(\mathsf{E}) = f(x, y_0) = f(\mathsf{D}) + (x - x_0)\frac{f(\mathsf{F}) - f(\mathsf{D})}{x_1 - x_0}. \tag{1.214}$$

Linear interpolation between Point B and Point E yields

$$f(x, y) = f(\mathsf{E}) + (y - y_0)\frac{f(\mathsf{B}) - f(\mathsf{E})}{y_1 - y_0}. \tag{1.215}$$

Combining these, the function obtained from bilinear interpolation is

$$\begin{aligned} f(x, y) = &\frac{f(x_0, y_0)}{(x_1 - x_0)(y_1 - y_0)}(x_1 - x)(y_1 - y) + \frac{f(x_1, y_0)}{(x_1 - x_0)(y_1 - y_0)}(x - x_0)(y_1 - y) \\ &+ \frac{f(x_0, y_1)}{(x_1 - x_0)(y_1 - y_0)}(x_1 - x)(y - y_0) + \frac{f(x_1, y_1)}{(x_1 - x_0)(y_1 - y_0)}(x - x_0)(y - y_0). \end{aligned} \tag{1.216}$$

When interpolating from a table, choosing the points closest to the final point will yield greatest accuracy.

1.A.13 Circles on the Complex Plane

A common situation that occurs when working with complex numbers involves equating the magnitude of a complex number being equal to a real number. As will be shown, this defines a circle in the complex plane. Consider the relation

$$|S - c| = r, \tag{1.217}$$

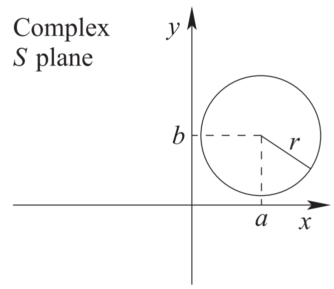


Figure 1-15: A circle in the S plane defined by the relation $|S - c| = r$, where $c = a + \jmath b$ and $S = x + \jmath y$.

where $S = x + \jmath y$ and $c = a + \jmath b$ are complex numbers and r is a real number. Substituting for S and c in Equation (1.217) yields

$$|(x - a) + \jmath(y - b)| = r, \tag{1.218}$$

that is,
$$[(x - a) + \jmath(y - b)]\,[(x - a) - \jmath(y - b)] = r^2 \tag{1.219}$$
$$(x - a)^2 + (y - b)^2 = r^2. \tag{1.220}$$

The above is a general equation for a circle in the x-y plane with center (a, b) and radius r (see Figure 1-15).

1.A.14 Bilinear Transform

The bilinear transformation maps a circle in the complex plane onto another circle in the complex plane [19]. The bilinear transformation of a complex variable z is

$$w = \frac{Az + B}{Cz + 1}. \tag{1.221}$$

where A, B, and C are real numbers. In RF and microwave usage, z and w could be reflection coefficients or impedances. Rearranging Equation (1.221) results in

$$w = \frac{A}{C} + \frac{B - \frac{A}{C}}{Cz + 1}, \tag{1.222}$$

which can be rearranged again as
$$\frac{w - \frac{A}{C}}{B - \frac{A}{C}} = \frac{1}{Cz + 1}. \tag{1.223}$$

The complexity of the derivation is reduced by introducing the intermediate variables W and Z defined as

$$W = \frac{w - \frac{A}{C}}{B - \frac{A}{C}} \tag{1.224}$$

and
$$Z = Cz + 1, \tag{1.225}$$

and noting that, from Equation (1.223),
$$W = 1/Z. \tag{1.226}$$

If, in Equation (1.225), the locus of z describes a circle, then the locus of Z will also be a circle since C is a constant. If the center of the circle in Z space is C_Z and the distance from any point on the circle to the center is the radius R_Z, then

$$|Z - C_Z| = R_Z. \tag{1.227}$$

Considering Equation (1.225), then in z space, the center of the circle is

$$C_z = \frac{C_Z}{C} - \frac{1}{C} \tag{1.228}$$

and its radius is

$$R_z = R_Z/C. \tag{1.229}$$

Removing the magnitude signs in Equation (1.227) leads to

$$(Z - C_Z)(Z - C_Z)^* = (Z - C_Z)(Z^* - C_Z^*) = R_Z^2, \tag{1.230}$$

or

$$ZZ^* - C_Z Z^* - C_Z^* Z + (|C_Z|^2 - R_Z^2) = 0, \tag{1.231}$$

which is the general equation for a circle in the complex plane.

Now, the circle in the Z plane will be related to what happens in the W plane. Substituting Equation (1.226) into Equation (1.231) results in

$$\frac{1}{WW^*} - C_Z\frac{1}{W^*} - C_Z^*\frac{1}{W} + (|C_Z|^2 - R_Z^2) = 0, \tag{1.232}$$

and rearranging,

$$WW^* - \frac{C_Z}{(|C_Z|^2 - R_Z^2)}W - \frac{C_Z^*}{(|C_Z|^2 - R_Z^2)}W^* + \frac{1}{(|C_Z|^2 - R_Z^2)} = 0. \tag{1.233}$$

This is in the same form as Equation (1.231), so Equation (1.233) describes a circle in W space with center

$$C_W = \frac{C_Z^*}{(|C_Z|^2 - R_Z^2)} \tag{1.234}$$

and radius

$$R_W = \left|\frac{R_Z}{|C_Z|^2 - R_Z^2}\right|. \tag{1.235}$$

Now the locus of w will also be a circle. From Equation (1.224),

$$w = W\left(B - \frac{A}{C}\right) + \frac{A}{C} \tag{1.236}$$

so that W is scaled and a constant added. The center of the circle in w space is

$$C_w = C_W\left(B - \frac{A}{C}\right) + \frac{A}{C} = \frac{C_Z^*\,(B - A/C)}{(|C_Z|^2 - R_Z^2)} + \frac{A}{C} \tag{1.237}$$

and the radius is

$$R_w = R_W\left|B - \frac{A}{C}\right| = \left|\frac{(B - A/C)}{|C_Z|^2 - R_Z^2}\right|. \tag{1.238}$$

Substituting Equations (1.228) and (1.229) in the above relates the centers and radii in the w and z spaces:

$$C_w = \frac{(CC_z + 1)^*(B - A/C)}{\left(|CC_z + 1|^2 - |CR_z|^2\right)} + \frac{A}{C} \tag{1.239}$$

$$R_w = \left|\frac{(B - A/C)CR_z}{|CC_z + 1|^2 - |CR_z|^2}\right| = \left|\frac{(BC - A)R_z}{|CC_z + 1|^2 - |CR_z|^2}\right|. \tag{1.240}$$

Thus the bilinear transform, Equation (1.221), maps all points on a circle in the z plane to points on a circle in the w plane.

1.A.15 Quadratic and Cubic Equations

Quadratic Equation

The general form of the quadratic equation in x is (a, b, and c can be complex)

$$ax^2 + bx + c = 0, \tag{1.241}$$

where $a \neq 0$There are two solutions, called roots, given by the quadratic formula, [20]

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{1.242}$$

and $\pm$ indicates two possible roots,

$$x_+ = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_- = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{1.243}$$

Even if a, b, and c are real, the roots may be complex and they could be degenerate, that is, the same. The factors of the quadratic equation come from the roots, that is,

$$ax^2 + bx + c = a\,(x - x_+)\,(x - x_-)\,. \tag{1.244}$$

Cubic Equation

The general form of the cubic equation in x is (a, b, c and d can be complex)

$$ax^3 + bx^2 + cx + d = 0, \tag{1.245}$$

where $a \neq 0$There are three solutions, called roots, given by the cubic formula, [20]

$$x_k = \frac{1}{3a}\left(b + u_k + \frac{\Delta_0}{u_k C}\right), \quad k \in \{1, 2, 3\} \tag{1.246}$$

where the u_ks are the cube roots of one: $u_1 = 1, u_2 = \frac{1}{2}\left(-1 + \jmath\sqrt{3}\right), u_3 = \frac{1}{2}\left(-1 - \jmath\sqrt{3}\right)$, and

$$C = \left[\tfrac{1}{2}\left(\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}\right)\right]^{1/3} \tag{1.247}$$

with $\Delta_0 = b^2 - 3ac$ and $\Delta_1 = 2b^3 - 9abc + 27a^2d$. In Equation (1.247) any choice of the square or cube roots can be made as the effect is simply to exchange x_1, x_2 and x_3. Note that $ax^3 + bx^2 + cx + d = (x - x_1)(x - x_2)(x - x_3)$.

1.A.16 Kron's Method: Network Condensation

Kron's method [21–23] is also called network condensation and is used in developing simpler equivalent networks of large linear circuits with algebraic y parameters, that is, for resistive networks, or for general linear circuits in frequency-domain analysis where the y parameters are complex numbers. Network condensation is a numerical approach particular to circuits. It is of use in reduced-order modeling of linear circuits and in filter design.

The indefinite nodal admittance formulation of a network with four external terminals (see Figure 1-16) is

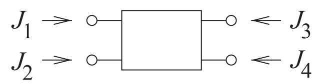


Figure 1-16: A four-terminal network.

$$\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1,k} & y_{1,k+1} \\ y_{21} & y_{22} & \cdots & y_{2,k} & y_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_{k1} & y_{k2} & \cdots & y_{k,k} & y_{k,k+1} \\ y_{k+1,1} & y_{k+1,2} & \cdots & y_{k+1,k} & y_{k+1,k+1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ \vdots \\ v_k \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \tag{1.248}$$

as there are no external current sources at the internal terminals $5, 6, \cdots, k+1$. Since the $(k+1)$th current source is zero, the $(k+1)$th row and column of $\mathbf{Y}$ can be eliminated, yielding

$$\begin{bmatrix} y'_{11} & y'_{12} & \cdots & y'_{1,k} \\ y'_{21} & y'_{22} & \cdots & y'_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ y'_{k1} & y'_{k2} & \cdots & y'_{k,k} \end{bmatrix} \begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \\ v'_4 \\ v'_5 \\ \vdots \\ v'_k \end{bmatrix} = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{1.249}$$

$$\text{where} \qquad y'_{ij} = y_{ij} - y_{k+1,j}\frac{y_{i,k+1}}{y_{k+1,k+1}} \quad \text{and} \quad v'_i = v_i - v_{k+1}. \tag{1.250}$$

It is of no concern that the terminal voltages have been altered, as they have all had the same quantity added to them. This has the same effect as choosing another reference terminal. The reference terminal is arbitrary until it is connected to the full circuit. The process can be continued until the network equations are reduced to that of a four-terminal element, that is

$$\begin{bmatrix} y''_{11} & y''_{12} & y''_{13} & y''_{14} \\ y''_{21} & y''_{22} & y''_{23} & y''_{24} \\ y''_{31} & y''_{32} & y''_{33} & y''_{34} \\ y''_{41} & y''_{42} & y''_{43} & y''_{44} \end{bmatrix} \begin{bmatrix} v''_1 \\ v''_2 \\ v''_3 \\ v''_4 \end{bmatrix} = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{bmatrix}. \tag{1.251}$$

The y-parameter matrix in Equation (1.251) is that of the four-terminal element shown in Figure 1-16.

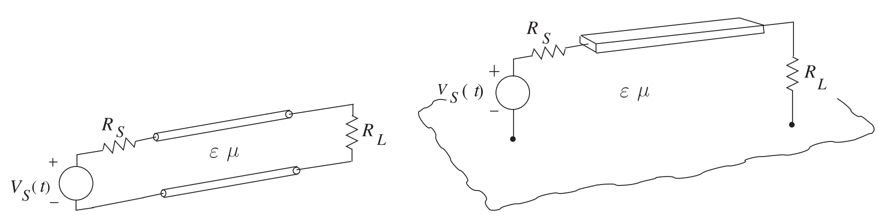
CHAPTER 2

Transmission Lines

2.1 Introduction

A transmission line stores electric and magnetic energy, and for an alternating signal at a position on the line, energy is converted from one form to the other as time progresses. As such a line has a circuit form that combines inductors, Ls (for the magnetic energy), capacitors, Cs (for the electric energy), and resistors, Rs (modeling losses), whose values depend on the line geometry and material properties.

The transmission lines considered in this chapter are restricted to just two parallel conductors, as shown in Figure 2-1, with the distance between the two wires (i.e., in the transverse direction) being substantially smaller than the wavelengths of the signals on the line. The correct physical interpretation is that the conductors of a transmission line confine and guide an EM field. The EM field contains the energy of the signal and not the current on the line.



(a) Two-wire line

(b) Strip-above-ground line

Figure 2-1: Two conductor transmission lines.

However, with electrically small transverse dimensions, a two conductor line may be satisfactorily analyzed on the basis of voltages and currents.

The earliest understanding of signal transmission led to telegraphy over distances. The critical theoretical step that enabled transmission over more than short distances was the development of an understanding of signal transmission on lines using phasor analysis [1]. This frequency-domain analysis is still the best way to understand transmission lines. This transmission line theory with modern developments is presented in Section 2.2 and useful formulas and concepts developed in Section 2.3 for lossless transmission lines. Section 2.4 presents several configurations of lossless lines that are particularly useful in microwave circuit design and used in many places in this book series. Section 2.5 repeats the analysis of lossless lines but now for lossy lines. Reflections and transmission of power at multiple interacting interfaces is considered in Section 2.6. Important network models for transmission lines are developed in Section Section 2.7. The last section, Section 2.8, presents summary formulas for many two-conductor transmission lines.

2.1.1 When Must a Line be Considered a Transmission Line

The key determinant of whether a transmission line can be considered as an invisible connection between two points is whether the signal anywhere along the interconnect has the same value at a particular instant. If the value of the signal (say, voltage) varies along the line (at an instant), then it may be necessary to consider transmission line effects. A typical criterion used is that if the length of the interconnect is less than $1/20$th of the wavelength of the highest-frequency component of a signal, then transmission line effects can be safely ignored and the circuit can be modeled as a single RLC circuit [2]. The actual threshold used—$\lambda/20$, $\lambda/10$, or $\lambda/5$—is based on experience and the particular application. For example, an interconnect carrying a digital signal clocking at 4 GHz has an appreciable frequency component at 20 GHz. Then the interconnect reaches the $\lambda/10$ threshold when it is 4.5 mm long. Thus it takes a finite time for the variation of a voltage at one end of the interconnect to impact the voltage at the other end. The ultimate limit is determined by the speed of light, c, but this is reduced by the relative permittivity and permeability of the material in which the fields exist. The relative permittivity and permeability describe the effect of excess potential energy storage in the material, above that of vacuum, in a manner that is analogous to storing mechanical energy in a spring.

2.1.2 Movement of a Signal on a Transmission Line

A **coaxial line** (Figure 2-2(a)) is the quintessential transmission line as it is one of the few transmission line structures that can be described exactly from first principles when there is no loss. This is because the cross section of a coaxial line matches the cylindrical coordinate system, an orthogonal coordinate system with boundary conditions that can be expressed in terms of just one coordinate (the radius). It is known how to solve Maxwell's equations,which are just differential equations, in an orthogonal coordinate system. This is done for a coaxial line in Section 2.9.

When a positive voltage pulse is applied to the center conductor of the

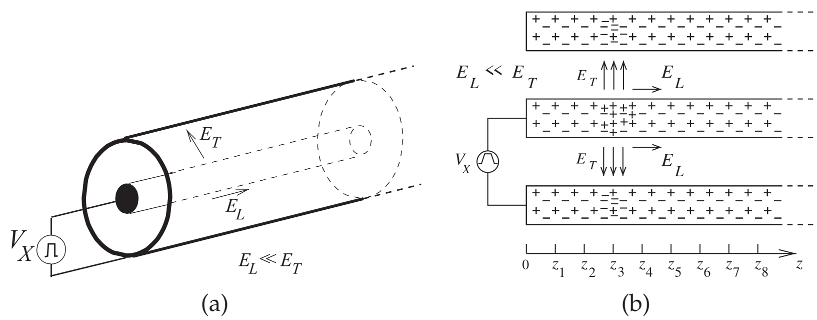


Figure 2-2: A coaxial transmission line: (a) three-dimensional view; (b) the line with pulsed voltage source showing the electric fields at an instant in time as a voltage pulse travels down the line. $E_L \ll E_T$.

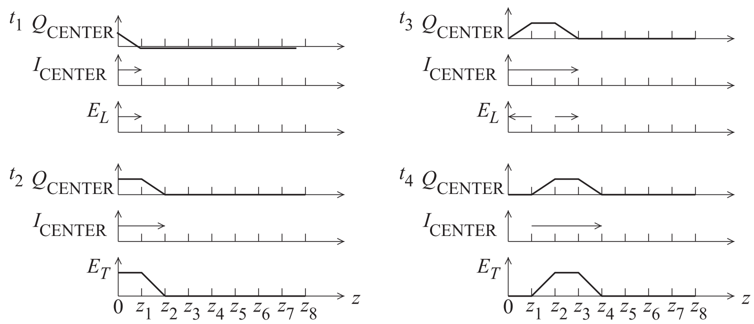


Figure 2-3: Fields, currents, and charges on the coaxial transmission line of Figure 2-2 at times $t_4 > t_3 > t_2 > t_1$. Q_{CENTER} is the net free charge on the center conductor. I_{CENTER} is the current on the center conductor.

coaxial line, as shown in Figure 2-2(a), an electric field results that is directed from the center conductor to the outer conductor. Referring to Figure 2-2(b), the component of the field that is directed along the shortest path from the center conductor to the outer conductor is denoted E_T, and the subscript T denotes the transverse component of the field with the transverse plane being perpendicular to the direction of propagation along the line. Figure 2-2(b) shows the fields in the structure after the pulse has started moving along the line. This is shown in another view in Figure 2-3 at four different times. The transverse voltage, V_T, is given by E_T integrated along a path between the inner and outer conductors: $V_T \approx E_T(a - b)$. This is a good measure, provided that the transverse dimensions are small compared to a wavelength (otherwise the integral is then path dependent). The voltage pulse exciting the line has a trapezoidal shape and Figure 2-3 shows the charge, Q_{CENTER}, and current, I_{CENTER}, on the center conductor and the transverse field, E_T. The voltage between the inner and outer conductor has the same form as E_T. As indicated by Maxwell's equations, a change in time of the electric field results in a spatial change in the magnetic field and hence current. As a result there is a variation of the current in time and this results in a spatial change of the electric field. The chasing from a time variation to a spatial variation and then back to a time variation causes the pulse to move down the line.

The pulse moves down the line at the **group velocity**, which for a lossless coaxial line is the same as the **phase velocity**, v_p.[1] This is determined by the physical properties of the region between the conductors. The permittivity,

[1] The phase velocity is the apparent velocity of a point of constant phase on a sinewave and is almost frequency independent for a low-loss coaxial line of small transverse dimensions (less than $\lambda/10$).

ε, describes energy storage associated with the electric field, E, and the energy storage associated with the magnetic field, H, is described by the permeability, μ. (Both ε and μ are properties of the medium—the material.) It has been determined that[2]

$$v_p = 1/\sqrt{\mu\varepsilon}. \tag{2.1}$$

In a vacuum $\varepsilon = \varepsilon_0$, the free-space permittivity, and $\mu = \mu_0$, the free-space permeability. These are physical constants and have the values

Permittivity of free space: $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m.
Permeability of free space: $\mu_0 = 4\pi \times 10^{-7}$ H/m.

One conclusion here is that EM energy can be stored in **free space**, i.e. in a vacuum. In free space $v_p = c = 1/\sqrt{\mu_0\varepsilon_0} = 3 \times 10^8$ m/s. The free-space wavelengths, $\lambda_0 = c/f$, at several frequencies, f, are

f	100 MHz	1 GHz	10 GHz
λ_0	3 m	30 cm	3.0 cm

Commonly λ_0 is used to indicate the wavelength in free space and λ_g, the guide wavelength, is used to denote the wavelength on a transmission line. It is convenient to use the relative permittivity (or the less commonly used term **dielectric constant**), ε_r, defined as

$$\varepsilon_r = \varepsilon/\varepsilon_0. \tag{2.2}$$

Similarly the relative permeability is

$$\mu_r = \mu/\mu_0, \tag{2.3}$$

and most materials have $\mu_r = 1$. The permittivity, permeability, and conductivity of materials used in RF and microwave circuits are given in Appendix 2.A.

EXAMPLE 2.1 Transmission Line Wavelength

A length of coaxial line is filled with a dielectric having a relative permittivity of 20 and is designed to be 1/4 wavelength long at a frequency, f, of 1.850 GHz.

(a) What is the free-space wavelength?
(b) What is the wavelength of the signal in the dielectric-filled coaxial line?
(c) How long is the line?

Solution:

(a) $\lambda_0 = c/f = 3 \times 10^8/1.85 \times 10^9 = 0.162$ m $= 16.2$ cm.
(b) Note that for a dielectric-filled line with $\mu_r = 1$, $\lambda = v_p/f = c/(\sqrt{\varepsilon_r} f) = \lambda_0/\sqrt{\varepsilon_r}$, so $\lambda = \lambda_0/\sqrt{\varepsilon_r} = 16.2$ cm$/\sqrt{20} = 3.62$ cm.
(c) $\lambda_g/4 = 3.62$ cm$/4 = 9.05$ mm.

[2] This is derived from Maxwell's equations and describes the situation where the fields have a very simple structure, in general where the fields are solely directed in the transverse plane.

2.2 Transmission Line Theory

This section develops the theory of signal propagation on transmission lines. The first section, Section 2.2.1, makes the argument that a circuit with resistors, inductors, and capacitors is a good model for a transmission line. The complete development of transmission line theory is presented in Section 2.2.2, and Section 2.2.3 relates the RLGC transmission line model to the properties of a medium. The dimensions of some of the quantities that appear in transmission line theory are discussed in Section 2.2.4. Section 2.2.5 summarizes the important parameters of a lossless line and then two particularly important lines, coaxial lines and microstrip lines, are considered in Sections 2.2.6 and 2.2.7.

2.2.1 Transmission Line RLGC Model

Regardless of the actual structure, a segment of uniform transmission line (i.e., a line with constant cross section along its length) as shown in Figure 2-4(a) can be modeled by the circuit shown in Figure 2-4(b) with

Resistance along the line	$= R$	
Inductance along the line	$= L$	all specified
Conductance shunting the line	$= G$	per unit length.
Capacitance shunting the line	$= C$	

Thus R, L, G, and C are also referred to as resistance, inductance, conductance, and capacitance per unit length. (Sometimes p.u.l. is used as shorthand for **per unit length**.) In the metric system, ohms per meter (Ω/m), henries per meter (H/m), siemens per meter (S/m) and farads per meter (F/m), respectively, are used. The values of R, L, G, and C are affected by the geometry of the transmission line and by the electrical properties of the dielectrics and conductors. C describes the ability to store electrical energy and is mostly due to the properties of the dielectric. G describes loss in the dielectric which derives from conduction in the dielectric and from dielectric relaxation. Most microwave substrates have negligible conductivity so dielectric relaxation loss dominates. Dielectric relaxation loss results from the movement of charge centers which result in distortion of the dielectric lattice (if a crystal) or molecular structure. The periodic variation of the E field transfers energy from the EM field to mechanical vibrations. R is due to ohmic loss in the metal more than anything else. L describes the ability to store magnetic energy and is mostly a function of geometry, as most materials used with transmission lines have $\mu_r = 1$ (so no more magnetic energy is stored than in a vacuum).

For most lines the effects due to L and C dominate because of the relatively low series resistance and shunt conductance. The propagation characteristics of the line are described by its loss-free, or lossless, equivalent line, although in practice some information about R or G is necessary to determine power

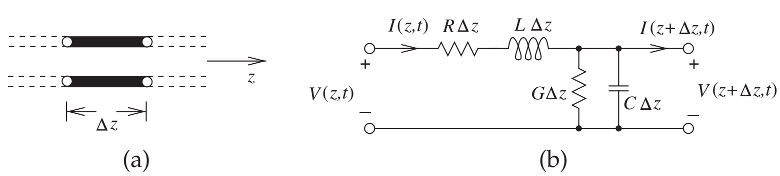


Figure 2-4: Transmission line segment: (a) of length Δz; and (b) lumped-element model.

losses. The lossless concept is just a useful and good approximation.

2.2.2 Derivation of Transmission Line Properties

In this section the differential equations governing the propagation of signals on a transmission line are derived. These are coupled first-order differential equations and are akin to Maxwell's Equations in one dimension. Solution of the differential equations describes how signals propagate, and leads to the extraction of a few parameters that describe transmission line properties.

Applying **Kirchoff's laws** applied to the model in Figure 2-4(b) and taking the limit as $\Delta z \to 0$ the **transmission line equations** are

$$\frac{\partial v(z,t)}{\partial z} = -Ri(z,t) - L\frac{\partial i(z,t)}{\partial t} \tag{2.4}$$

$$\frac{\partial i(z,t)}{\partial z} = -Gv(z,t) - C\frac{\partial v(z,t)}{\partial t}. \tag{2.5}$$

$V(z)$ is a phasor and $v(z,t) = \Re\{V(z)e^{\jmath\omega t}\}$. $\Re\{w\}$ denotes the real part of w, a complex number.

In sinusoidal steady-state using cosine-based phasors these become

$$\frac{dV(z)}{dz} = -(R+\jmath\omega L)I(z) \quad (2.6) \quad \text{and} \quad \frac{dI(z)}{dz} = -(G+\jmath\omega C)V(z). \quad (2.7)$$

Eliminating $I(z)$ in the above yields the wave equation for $V(z)$:

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0. \quad (2.8) \quad \text{Similarly} \quad \frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0, \quad (2.9)$$

where the propagation constant is

$$\gamma = \alpha + \jmath\beta = \sqrt{(R+\jmath\omega L)(G+\jmath\omega C)}, \tag{2.10}$$

with SI units of m^{-1} and where α is the attenuation coefficient and has units of nepers per meter (Np/m), and β is the phase-change coefficient, or phase constant, and has units of radians per meter (expressed as rad/m or radians/m). Nepers and radians are dimensionless units, but serve as prompts for what is being referred to.

Equations (2.8) and (2.9) are second-order differential equations that have solutions of the form

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \quad (2.11) \quad \text{and} \quad I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}. \quad (2.12)$$

The physical interpretation of these solutions is that $V^+(z) = V_0^+ e^{-\gamma z}$ and $I^+(z) = I_0^+ e^{-\gamma z}$ are forward-traveling waves (moving in the $+z$ direction) and $V^-(z) = V_0^- e^{\gamma z}$ and $I^-(z) = I_0^- e^{\gamma z}$ are backward-traveling waves (moving in the $-z$ direction). $V(z)$, V_0^+, V_0^-, $I(z)$, I_0^+ and I_0^- are all phasors. Substituting Equation (2.11) in Equation (2.6) results in

$$I(z) = \frac{\gamma}{R+\jmath\omega L}\left[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}\right]. \tag{2.13}$$

Then from Equations (2.13) and (2.12)

$$I_0^+ = \frac{\gamma}{R+\jmath\omega L}V_0^+ \quad \text{and} \quad I_0^- = \frac{\gamma}{R+\jmath\omega L}(-V_o^-). \tag{2.14}$$

The characteristic impedance is defined as

$$Z_0 = \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-} = \frac{R+\jmath\omega L}{\gamma} = \sqrt{\frac{R+j\omega L}{G+j\omega C}}, \tag{2.15}$$

with the SI unit of ohms (Ω). Equations (2.11) and (2.12) can be rewritten as

$$V(z) = V_0^+ \mathrm{e}^{-\gamma z} + V_0^- \mathrm{e}^{\gamma z} \quad (2.16) \quad \text{and} \quad I(z) = \frac{V_0^+}{Z_0}\mathrm{e}^{-\gamma z} - \frac{V_0^-}{Z_0}\mathrm{e}^{\gamma z}. \quad (2.17)$$

Converting back to the time domain:

$$v(z,t) = \left|V_0^+\right| \cos(\omega t - \beta z + \varphi^+)\mathrm{e}^{-\alpha z} + \left|V_0^-\right| \cos(\omega t + \beta z + \varphi^-)\mathrm{e}^{\alpha z}, \quad (2.18)$$

where φ^+ and φ^- are phases of the forward- and backward-traveling waves, respectively. The phasors of the traveling voltage waves are

$$V_0^+(z) = \left|V_0^+\right| \mathrm{e}^{\jmath\varphi^+} \mathrm{e}^{-\jmath\beta z} \quad \text{and} \quad V_0^-(z) = \left|V_0^-\right| \mathrm{e}^{\jmath\varphi^-} \mathrm{e}^{\jmath\beta z}. \quad (2.19)$$

The following quantities are defined:

$$\begin{aligned}
\text{Characteristic impedance:}\quad & Z_0 = \sqrt{(R + \jmath\omega L)/(G + \jmath\omega C)} && (2.20)\\
\text{Propagation constant:}\quad & \gamma = \sqrt{(R + \jmath\omega L)(G + \jmath\omega C)} && (2.21)\\
\text{Attenuation constant:}\quad & \alpha = \Re\{\gamma\} && (2.22)\\
\text{Phase constant:}\quad & \beta = \Im\{\gamma\} && (2.23)\\
\text{Wavenumber:}\quad & k = -\jmath\gamma && (2.24)\\
\text{Phase velocity:}\quad & v_p = \omega/\beta && (2.25)\\
\text{Wavelength:}\quad & \lambda = \frac{2\pi}{|\gamma|} = \frac{2\pi}{|k|}, && (2.26)
\end{aligned}$$

where $\omega = 2\pi f$ is the radian frequency and f is the frequency with the SI units of hertz (Hz). The wavenumber k as defined here is used in electromagnetics and where wave propagation is concerned. Considering one of the traveling waves, the phase velocity refers to the apparent velocity of which a point of constant phase on the sinewave appears to move.

For low-loss materials, $\alpha \ll \beta$, and so $\beta \approx |k|$, then the following approximations are valid:

$$\begin{aligned}
\text{Characteristic impedance:}\quad & Z_0 \approx \sqrt{L/C} && (2.27)\\
\text{Phase constant:}\quad & \beta \approx \omega\sqrt{LC} && (2.28)\\
\text{Wavenumber:}\quad & k \approx \beta && (2.29)\\
\text{Phase velocity:}\quad & v_p = \omega/\beta && (2.30)\\
\text{Wavelength:}\quad & \lambda \approx \frac{2\pi}{\beta} = v_p/f. && (2.31)
\end{aligned}$$

The important result here is that a voltage wave (and a current wave) can be defined on a transmission line. One more parameter needs to be introduced: the group velocity,

$$v_g = \frac{\partial\omega}{\partial\beta}. \quad (2.32)$$

The group velocity is the velocity of a modulated waveform's envelope and describes how fast information propagates. It is the velocity at which the energy (i.e. information) in the waveform moves. Thus group velocity can never be more than the speed of light in a vacuum, c. Phase velocity, however, can be more than c. If the speed at which information moves varies with frequency, then a signal such as a pulse will spread out. Such a line is said to have dispersion. For a lossless, dispersionless line, the group and phase velocity are the same. If the phase velocity is frequency independent, then β is linearly proportional to ω.

Electrical length is used in designs with transmission lines prior to establishing the physical length of the line. The electrical length of a line is expressed either as a fraction of a wavelength or in degrees (or radians), where a wavelength corresponds to 360° (or 2π radians). So if β is the phase constant of a signal on the line and ℓ is its physical length, the electrical length of the line in radians is $\beta\ell$.

EXAMPLE 2.2 Physical and Electrical Length

A transmission line is 10 cm long and at the operating frequency the phase constant β is 30 rad/m. What is the electrical length of the line?

Solution:

The physical length of the line is ℓ = 10 cm = 0.1 m. Then the electrical length of the line is $\ell_e = \beta\ell = (30\ \text{rad/m}) \times 0.1$ m = 3 radians. The electrical length can also be expressed in terms of wavelength noting that 360° corresponds to 2π radians, which also corresponds to λ. Thus $\ell_e = (3\ \text{radians}) = 3 \times 360/(2\pi)$ = 171.9° or $\ell_e = 3/(2\pi)\ \lambda = 0.477\ \lambda$.

EXAMPLE 2.3 *RLGC* Parameters

A transmission line has the $RLGC$ parameters $R = 100\ \Omega/\text{m}$, $L = 80\ \text{nH/m}$, $G = 1.6\ \text{S/m}$, and $C = 200\ \text{pF/m}$. Consider a traveling wave at 2 GHz on the line.

(a) What is the attenuation constant?
(b) What is the phase constant?
(c) What is the phase velocity?
(d) What is the characteristic impedance of the line?
(e) What is the group velocity?

Solution:

(a) α: $\gamma = \alpha + \jmath\beta = \sqrt{(R + \jmath\omega L)(G + \jmath\omega C)}$; $\omega = 12.57 \cdot 10^9$ rad/s

$$\gamma = \sqrt{(100 + \jmath\omega \cdot 80 \cdot 10^{-9})(1.6 + \jmath\omega 200 \times 10^{-12})} = (17.94 + \jmath 51.85)\ \text{m}^{-1}$$
$$\alpha = \Re\{\gamma\} = 17.94\ \text{Np/m}$$

(b) Phase constant: $\beta = \Im\{\gamma\} = 51.85$ rad/m

(c) Phase velocity:
$$v_p = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = \frac{12.57 \times 10^9 \text{rad} \cdot \text{s}^{-1}}{51.85\ \text{rad} \cdot \text{m}^{-1}} = 2.42 \times 10^8\ \text{m/s}$$

(d) $Z_0 = (R + \jmath\omega L)/\gamma = (100 + \jmath\omega \cdot 80 \cdot 10^{-9})/(17.94 + \jmath 51.85) = (17.9 + \jmath 4.3)\ \Omega$
Note also that $Z_0 = \sqrt{(R + \jmath\omega L)/(G + j\omega C)}$, which yields the same answer.

(e) Group velocity:
$$v_g = \left.\frac{\partial\omega}{\partial\beta}\right|_{f=2\ \text{GHz}}$$

Numerical derivatives will be used, thus $v_g = \Delta\omega/\Delta\beta$. Now β is already known at 2 GHz. At 1.9 GHz, $\gamma = 17.884 + \jmath 49.397\ \text{m}^{-1}$, and so $\beta = 49.397$ rad/m.

$$v_g = \frac{2\pi(2\ \text{GHz} - 1.9\ \text{GHz})}{\beta(2\ \text{GHz}) - \beta(1.9\ \text{GHz})} = \frac{2\pi(2 - 1.9)10^9\ \text{Hz}}{(51.85 - 49.397)\ \text{m}^{-1}} = 2.563 \times 10^8\ \text{m/s}.$$

(Note that Hz = s^{-1}. Note that $v_g \neq v_p$, and so the transmission line has dispersion.)

2.2.3 Relationship of RLGC parameters to Permittivity and Permeability of a Medium

In the previous section the telegrapher's equations for a transmission line modeled as subsections of *RLGC* elements was derived. Very good accuracy is obtained if the length of a subsection is no more than one-twentieth of a wavelength. The full transmission line model is the cascade of many transmission line subsections (e.g. see Figure 2-5). In this section the $RLGC$ parameters are related to the physical parameters of permittivity and permeability. The development does not go into much detail, as the derivation is involved and can only be derived analytically for a few regular transmission line structures. If you are curious, the development is done for a parallel plate waveguide and a rectangular waveguide in Chapter 6 and for a coaxial line in Section 2.9.

The main parameters describing propagation on a transmission line are Z_0 and γ, and these depend on the permeability and permittivity of the medium and the spatial variation of the E and H fields resulting from the geometry of the conductors. Only a few geometries permit analytic solution of the fields so in general a numerical field solution is required and Z_0 and γ derived. Very often equations are curve fit to the numerical solutions but the structure of the equations have a theoretical foundation. It is found that there can be a number of possible field solutions (not unusual for the solution of differential equations) each of which is called a mode. Modes that have all the fields in the transverse plane (perpendicular to the propagation direction) is called a transverse EM (TEM) mode and these modes exist at DC. The other modes are only possible above a cut-off frequency. It is found that the propagation constant has the form

$$\gamma^2 = -\left(k^2 - k_c^2\right), \tag{2.33}$$

where k_c is the cutoff wavenumber and for a particular line usually has a different value for each mode. The wavenumber of a homogeneous line is

$$k = \jmath\omega\sqrt{\mu\varepsilon}. \tag{2.34}$$

A mode can only exist and describe a propagating signal when $\beta = \Im\{\gamma\}$ is not zero which requires that γ^2 be negative. Thus a mode can support a propagating mode only if the wavenumber is greater than the cut-off wavenumber, i.e. when $k > k_c$. A TEM mode has $k_c = 0$ so a signal, other than DC can always propagate on the line in a TEM mode. A homogeneous line has just one type of medium supporting the EM fields. An example of a homogeneous line is a teflon-filled coaxial line. A non-homogeneous line has two or or more dielectric mediums, such as air and a dielectric. For non-homogeneous lines the concept of an equivalent homogeneous line with an effective permittivity ε_{eff} and effective permeability μ_{eff} is used. Then

$$k = \jmath\omega\sqrt{\mu_{\text{eff}}\varepsilon_{\text{eff}}} \tag{2.35}$$

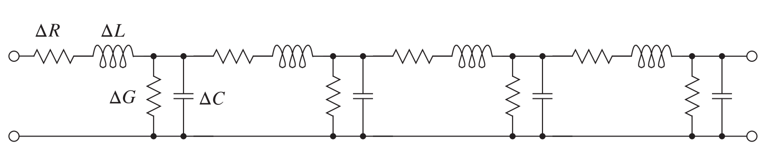


Figure 2-5: $RLGC$ model of a transmission line.

Loss is incorporated in the imaginary parts of ε and μ for TEM modes. As before, for any mode,

$$\text{Attenuation constant:} \quad \alpha = \Re\{\gamma\} \tag{2.36}$$

$$\text{Phase constant:} \quad \beta = \Im\{\gamma\} \tag{2.37}$$

$$\text{Phase velocity:} \quad v_p = \omega/\beta \tag{2.38}$$

$$\text{Wavelength:} \quad \lambda = \frac{v_p}{f} = \frac{2\pi}{\beta}, \tag{2.39}$$

Comparing γ in Equations (2.10) and (2.35), an equivalence can be developed between the lumped-element form of transmission line propagation and the propagation of an EM wave in a medium. Specifically,

$$-\omega^2 \mu_{\text{eff}} \varepsilon_{\text{eff}} = (R + \jmath\omega L)(G + \jmath\omega C). \tag{2.40}$$

Lossless Medium

If the medium is lossless (μ and ε are real and $R = 0 = G$), then

$$\mu_{\text{eff}} \varepsilon_{\text{eff}} = LC. \tag{2.41}$$

When the medium is free space (a vacuum), then a subscript zero is used. Free space is also lossless, so the following results hold:

$$\alpha_0 = 0 \quad \text{and} \quad \beta_0 = -\jmath\gamma = \omega\sqrt{\mu_0 \varepsilon_0} = \omega\sqrt{LC}. \tag{2.42}$$

If frequency is specified in gigahertz (indicated by f_{GHz}), in free space

$$\beta_0 = 20.958 f_{\text{GHz}} \quad \text{units of rads/m.} \tag{2.43}$$

At 1 GHz, $\beta_0 = 20.958$ rad/m and $\lambda_0 = 29.98$ cm (use $\lambda_0 \approx 30$ cm at 1 GHz as a reference). In a lossless medium with **effective relative permeability** $\mu_e = \mu_{\text{eff}}/\mu_0 = 1$ and **effective relative permittivity** $\varepsilon_e = \varepsilon_{\text{eff}}/\varepsilon_0$,

$$\beta = \sqrt{\varepsilon_e}\beta_0. \tag{2.44}$$

Z_0 depends strongly on the spatial variation of the fields. When there is no variation in the plane transverse to the direction of propagation (i.e. for plane wave propagation)

$$Z_0 = \sqrt{\mu/\varepsilon}. \tag{2.45}$$

However, if there is variation of the fields

$$Z_0 = \kappa\sqrt{\mu_{\text{eff}}/\varepsilon_{\text{eff}}} = \kappa\sqrt{\mu_0/\varepsilon_0}\sqrt{\mu_e/\varepsilon_e}, \tag{2.46}$$

where κ captures the effect of geometric variation of the fields. Spatial variation of the fields stores additional energy in the E and H fields, affecting γ as well as Z_0.

If the boundary conditions on a transmission line are such that a required spatial variation of the fields cannot be supported, then the signal cannot propagate. The critical frequency at which $k = \jmath\omega\sqrt{\mu\varepsilon} = k_c$ is called the cutoff frequency, f_c. Signals cannot propagate on the line if the frequency is below f_c. For the rest of this chapter the only lines considered are those for which $k_c = 0$, (and $f_c = 0$), i.e. TEM modes on transmission lines. That is, the lines carry DC signals.

Low Loss Medium

Transmission line loss is due to the resistance of conductors, which is described by R, and loss in the dielectric described by G. For most dielectrics there is very little conductive loss and loss is due almost entirely to dielectric relaxation. (The most common exception is when the dielectric is silicon as there can be appreciable conduction in silicon.) In dielectric relaxation an electric field causes charge centers to move and these cause the lattice to vibrate which we know is heat so energy is transferred from the electric field to heat. The energy lost is proportional to frequency and so G is directly proportional to frequency and consequently, provided there is no conduction in the dielectric, the **loss tangent** of the transmission line is thus defined as:

$$\tan\delta = G/\omega C = \text{independent of frequency.} \tag{2.47}$$

One consequence of Equation (2.47), and noting that C is approximately independent of frequency, is that if G is known at one frequency then its value at another frequency can be quickly determined.

The loss tangent of the dielectric medium and the loss tangent of the transmission line may not be the same as the EM fields may not be confined just to the medium, e.g. fields are in air and in a dielectric.

2.2.4 Dimensions of γ, α, and β

The SI unit of γ are inverse meters (m^{-1}) and the attenuation constant, α, and the phase constant, β, have, strictly speaking, the same units. However, the convention is to introduce the dimensionless quantities Neper and radian to convey additional information. Thus the attenuation constant α has the units of Nepers per meter (Np/m) and the phase constant β has the units radians per meter (rad/m). The unit Neper comes from the name of the e (= 2.7182818284590452354. . .) symbol (written in upright font and not italics since it is a constant), which is called the Neper. The Neper is used in calculating transmission line signal levels, as in Equations (2.11) and (2.12). The attenuation and phase constants are often separated and then the attenuation constant describes the decrease in signal amplitude as the signal travels down a transmission line. So when $\alpha\ell$ = 1 Np, where ℓ is the length of the line, the signal has decreased to $1/\mathrm{e}$ of its original value, and the power drops to $1/\mathrm{e}^2$ of its original value. The decrease in signal level represents loss and the units of decibels per meter (dB/m) are used with 1 Np = $20\log\mathrm{e}$ = 8.6858896381 dB. So expressing α as 1 Np/m is the same as saying that the attenuation loss is 8.6859 dB/m. To convert from dB to Np multiply by 0.1151. Thus α = x dB/m = $x \times 0.1151$ Np/m.

The name for e derives from John **Napier**, who developed the theory of logarithms [3]. e is sometimes called Euler's constant.

In engineering $\log x \equiv \log_{10} x$ and $\ln x \equiv \log_{\mathrm{e}} x$.

EXAMPLE 2.4 Transmission Line Characteristics

A line has an attenuation of 10 dB/m and a phase constant of 50 radians/m at 2 GHz.

(a) What is the complex propagation constant of the transmission line?
(b) If the capacitance of the line is 100 pF/m and the conductive loss is zero (i.e., $G = 0$), what is the characteristic impedance of the line?

Solution:

(a) $\alpha|_{\mathrm{Np}} = 0.1151 \times \alpha|_{\mathrm{dB}} =$ 0.1151× (10 dB/m) = 1.151 Np/m, $\beta = 50$ rad/m
Propagation constant, $\gamma = \alpha + \jmath\beta = (1.151 + \jmath 50)\ \mathrm{m}^{-1}$

(b) $\gamma = \sqrt{(R + \jmath\omega L)\,(G + \jmath\omega C)}$, and $Z_0 = \sqrt{(R + \jmath\omega L)/(G + \jmath\omega C)}$,
therefore $Z_0 = \gamma/(G + \jmath\omega C)$; $\omega = 2\pi \cdot 2 \times 10^9\ s^{-1}$; $G = 0$; $C = 100 \times 10^{-12}$ F,
so $Z_0 = 39.8 - \jmath 0.916\ \Omega$.

2.2.5 Lossless Transmission Line

If the conductor and dielectric are ideal (i.e., lossless), then $R = 0 = G$ and the equations for the transmission line characteristics simplify. The transmission line parameters from Equations (2.15) and (2.21)–(2.26) are then

$$Z_0 = \sqrt{\tfrac{L}{C}} \qquad (2.48) \qquad\qquad \alpha = 0 \qquad (2.49)$$

$$\beta = \omega\sqrt{LC} \qquad (2.50) \qquad\qquad v_p = 1/\sqrt{LC} \qquad (2.51)$$

$$\lambda_g = \frac{2\pi}{\omega\sqrt{LC}} = \frac{v_p}{f}. \qquad (2.52)$$

Note that there is a distinction between a transmission line and an RLC circuit. When a transmission line is referred to as having an impedance of 50 Ω, this is referring to the line having a characteristic impedance of 50 Ω, the line cannot be replaced by a 50 Ω resistor.

A line cannot be replaced by a lumped element except as follows:

1. When calculating the forward voltage wave of a line that is infinitely long (or there are no reflections from the load). Then the line can be replaced by an impedance equal to the characteristic impedance of the line. The total voltage is then only the forward-traveling component.
2. The characteristic impedance and load impedance are used to calculate the input impedance of the terminated line at a particular frequency.

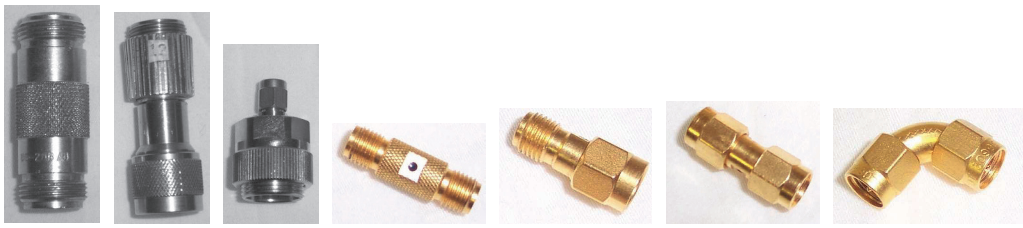
2.2.6 Coaxial Line

The analytic calculation of the characteristic impedance of a transmission line from geometry is not always possible except for a few regular geometries (matching orthogonal coordinate systems). For a coaxial line, the electric fields extend in a radial direction from the center conductor to the outer conductor. So it is possible to calculate the voltage by integrating this E field from the center to the outer conductor. The magnetic field is circular, centered on the center conductor, so the current on the conductor can be calculated as the closed integral of the magnetic field. (Here the field lines and the conductor boundaries correspond to the cylindrical coordinate system.) Solving for the fields in the region between the center and outer conductors yields the following formula for the characteristic impedance of a coaxial line (the derivation is presented in Section 2.9):

$$Z_0 = 138\sqrt{\frac{\mu_r}{\varepsilon_r}}\log\left(\frac{b}{a}\right)\,\Omega = 60\sqrt{\frac{\mu_r}{\varepsilon_r}}\ln\left(\frac{b}{a}\right)\,\Omega, \qquad (2.53)$$

where ε_r is the relative permittivity of the medium between the center and outer conductors, b is the inner diameter of the outer conductor, and a is the outer diameter of the inner conductor. With a higher ε, more energy is stored in the electric field and the capacitance per unit length of the line, C, increases. As the relative permittivity of the line increases, the characteristic impedance of the line reduces. Equation (2.53) is an exact formulation for the characteristic impedance of a coaxial line. Such a formula can only be approximated for nearly every other line.

Most coaxial cables have Z_0 of 50 Ω, but different ratios of b and a yield special properties of the coaxial line. When the ratio is 1.65 for an air-filled line, $Z_0 = 30\ \Omega$ and the line has **maximum power-carrying** capability. The



(a) (b) (c) (d) (e) (f) (g)

Figure 2-6: Coaxial line adaptors: (a) N-type female-to-female (N(f)-to-N(f)); (b) APC-7 to N-type male (APC-7-to-N(m)); (c) APC-7 to SMA-type male (APC-7-to-SMA(m)); SMA adapters: (d) SMA-type female-to-female (SMA(f)-to-SMA(f)); (e) SMA-type male-to-female (SMA(m)-to-SMA(f)); (f) SMA-type male-to-male (SMA(m)-to-SMA(m)); and (g) SMA elbow. N, APC-7 and SMA are known as connector series.

ratio for maximum voltage breakdown is 2.7, corresponding to $Z_0 = 60\ \Omega$ for an air-filled line. The characteristic impedance for minimum attenuation in an air-filled line is $77\ \Omega$, with a diameter ratio of 3.59. This optimum ratio of outer-to-inner conductors for **minimum loss** is independent of the dielectric and typical dielectrics have negligible loss with line loss due almost entirely due to the conductors. If the dielectric filling the coaxial line is polyethylene (which is most common) with $\varepsilon_r = 2.3$, the characteristic impedance of the minimum loss line is $50.6\ \Omega$.

The velocity of propagation in a lossless coaxial line having uniform medium is the same as that for a plane wave in the medium, i.e. $v_g = c/\sqrt{\varepsilon_r}$. There is one caveat. This is true for all transmission line structures supporting the minimum variation of the fields corresponding to a TEM mode. Higher-order modes, with greater spatial variations of the fields will travel more slowly (i.e. $v_g < c/\sqrt{\varepsilon_r}$), will be considered in Chapter 4. The diameter of the outer conductor and the type of internal supports for the internal conductor determines the frequency range of coaxial components. At high frequencies, and hence short wavelengths, large internal dimensions of a large diameter cable can support more than one spatial variation of the EM fields, i.e. more than one mode. This is undesirable because two modes will travel at different speeds and a propagating signal divides energy between the two modes and, since the modes have different group velocities, the signal will become garbled. Thus the transverse dimensions of the cable determine its upper frequency limit for reliable signal transmission.

There are various coaxial lines with different diameters and different levels of attention given to the uniformity of the lines. Each type of line is called a series and it is necessary to convert between series using adaptors. Various adaptors are shown in Figure 2-6. Primarily these have different cost, loss, and uniformity of Z_0. It is often necessary to convert between series and also to convert between the sexes (male and female) of connectors. Note that a plug (or jack) could be either female or male.

The different construction of connectors can be seen in Figure 2-7. With the APC-7 connector shown in Figure 2-7(a) the inside diameter of the outer conductor is 7 mm. The unique feature of this connector is that it is sexless, with the interface plate being spring-loaded. That is, an APC-7 connector can always be connected to another APC-7 connector. These are precision

(c) APC-7 (a) N(f) (b) SMA(m) (d) SMA attenuator (e) N-type inline filter

Figure 2-7: Various coaxial 50 Ω transmission line connectors: (a) APC-7 coaxial connector (7 mm outer conductor diameter); (b) female N-type (N(f)) coaxial connector (7 mm); (c) male SMA-type (SMA(m)) coaxial connector (3.5 mm); (d) DC to 26 GHz, 2-W SMA precision fixed attenuator; and (e) N-type inline 500 MHz lowpass filter. ((d) and (e) copyright 2012 Scientific Components Corporation d/b/a Mini-Circuits, used with permission [4]).

(c)

(a) (b) (d) (e)

Figure 2-8: Coaxial lines: (a) SMA cables (from the top): flexible cable type I, type II, type III, and semirigid cable; (b) coaxial cables showing layers; (c) semirigid coaxial cable bender; (d) bender with line; and (e) hand-formable cable supporting tight radius with low return and insertion loss. ((b) Copyright Megaphase LLC, used with permission [5].) ((e) Copyright 2012 Scientific Components Corporation d/b/a Mini-Circuits, used with permission [4].)

connectors used in some microwave measurements. The N-type connector in Figure 2-7(b) and the SMA connector in Figure 2-7(c) are more common day-to-day connectors. Different views of these connectors are shown in Figures 2-7(d and e). Unlike an APC-7 connector, an N-type (or SMA) male connector, for example, can only be connected to a female N-type connector. There are a large number of different types or series of connectors for high-power applications, different frequency ranges, distortion levels, and cost.

There are also many types of coaxial cables, as shown in Figure 2-8(a). These are cables with SMA connectors (with 3.5 mm outer conductor diameter). Microwave cables without connectors are shown in Figure 2-8(b). These cables range in cost, flexibility, and the number of times they can be reliably flexed or bent. The semirigid cable shown at the bottom of Figure 2-8(a) must be bent using a bending tool, as shown in Figure 2-8(c), and in

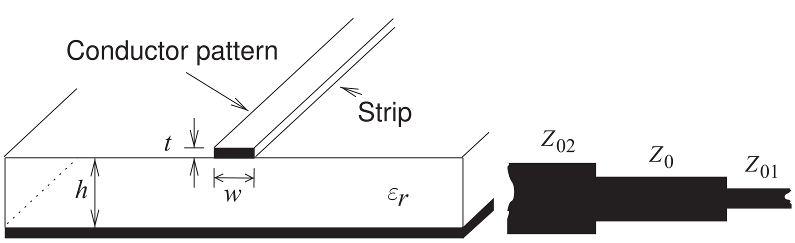
Figure 2-9: Torque wrench used in making repeatable coaxial connections. Copyright MegaPhase LLC, used with permission [5].

use in Figure 2-8(d). The controlled bending radius ensures minimal change in the characteristic impedance and propagation constant of the cable. Semirigid cables can only be bent once however. The highest precision bend is realized using an elbow bend, shown in Figure 2-6(g). Various flexible cables have different responses to bending, with higher precision (and more expensive) cables having the least impact on characteristic impedance and phase variations as cables are flexed.

Precision measurements require that connectors be repeatably attached using a torque wrench, as shown in Figure 2-9. A torque wrench can be used in a conventional manner, but once the prescribed torque is obtained the wrench breaks just above the head. This ensures that two connectors are repeatably joined with the same force and so the contact resistance and quality of the connection is repeatable.

2.2.7 Microstrip Line

A microstrip line is shown in Figure 2-10(a). This is a commonly used transmission line, as it can be cheaply fabricated using printed circuit board techniques. This line consists of a metal-backed substrate of relative permittivity ε_r on top of which is a metal strip. Above that is air. The width of the strip determines the characteristic impedance of the line. The characteristic impedance of microstrip lines having various strip widths is shown in Figure 2-11 for several substrate permittivities. So the wider the strip and the higher the substrate permittivity, the lower the characteristic impedance of the line. The EM fields are partly in air and partly in the dielectric and an effective permittivity must be used when calculating the electrical length of the line. The results of field simulations of the effective permittivity of lines of various widths and with various substrate permittivities are shown in Figure 2-12, where it can be seen that the effective relative permittivity, ε_e, increases for wide strips. This is because more of the EM field is in the substrate. Microstrip transmission line structures are often drawn showing just the layout of the strip, as shown in Figure 2-10(b), where



(a) Microstrip (b) Layout of microstrip lines

Figure 2-10: Microstrip transmission line. The layout (or top) view is commonly used with circuit designs using microstrip. This is the pattern of the strip where (b) shows three lines of different width.

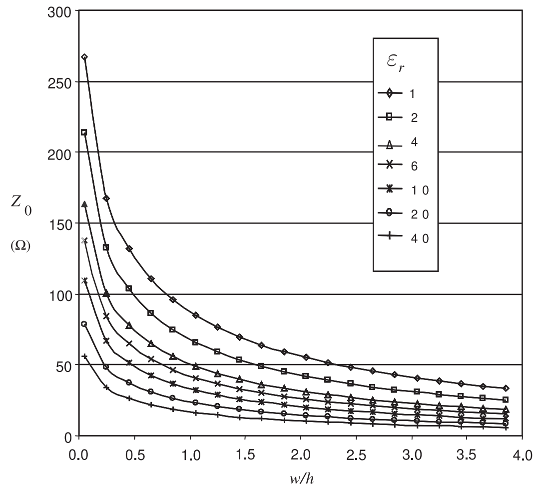


Figure 2-11: Dependence of Z_0 of a microstrip line at 1 GHz for various ε_r and aspect (w/h) ratios. Calculated using EM simulation.

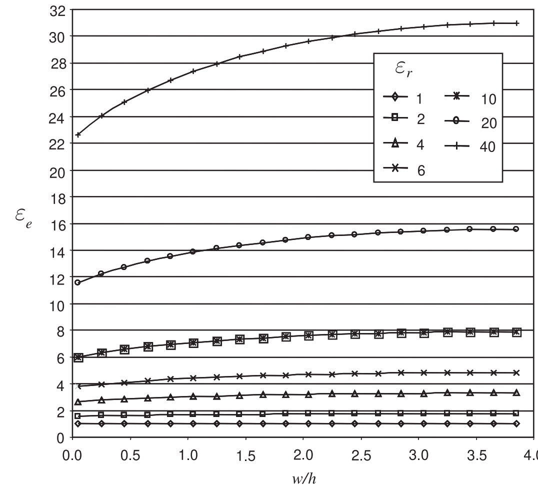


Figure 2-12: Dependence of effective relative permittivity ε_e of a microstrip line at 1 GHz for various permittivities and aspect ratios (w/h).

the three lines have different characteristic impedances. The next chapter presents detailed analyses of microstrip and other planar transmission lines.

2.2.8 Summary

The important takeaway from this section is that a signal moves on a transmission line as forward- and backward-traveling waves. The energy transferred is in the traveling waves. The total voltage and current at a point on the line is the sum of the traveling voltage and traveling current waves, respectively, but the total voltage/current view is not sufficient to describe how a transmission line works. Transmission line theory is expressed in terms of traveling voltage and current waves and these are akin to a one-dimensional form of Maxwell's equations. An argument was developed that posits a model of a line as cascaded sections of RLGC circuit and these circuit elements can be loosely related to the material properties of the medium in which the transmission line is embedded. There are a modification factors due to the actual orientation of the electric and magnetic fields and developing these requires detailed field analysis of the actual transmission line geometry. This can be done for a coaxial line but a microstrip line requires numerical analysis.

2.3 The Lossless Terminated Line

Microwave engineers want to work with total voltage and current when possible and the art of design synthesis usually requires relating the total voltage and current world of a lumped element circuit to the traveling voltage world of transmission lines. This section develops the important abstractions that enable the total voltage and current view of the world to be used with transmission lines. The first step in this process is in Section 2.3.1 where total voltages and currents are related to forward- and backward-

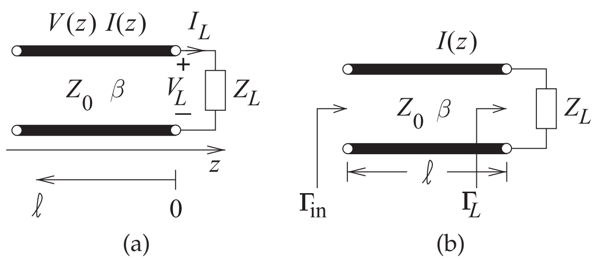


Figure 2-13: A terminated transmission line.

traveling voltages and currents. Insight into traveling waves and reflections is presented in Section 2.3.2. Important abstractions are presented first for the input reflection coefficient of a terminated lossless line in Section 2.3.3 and then for the input impedance of the line in Section 2.3.4. The last section, Section 2.3.5, presents a view of the total voltage on the transmission line and describes the voltage standing wave concept.

2.3.1 Total Voltage and Current on the Line

Consider the terminated line shown in Figure 2-13(a). Assume an incident or forward-traveling wave, with traveling voltage $V_0^+ \mathrm{e}^{-\jmath\beta z}$ and current $I_0^+ \mathrm{e}^{-\jmath\beta z}$ propagating toward the load Z_L at $z = 0$. The characteristic impedance of the transmission line is the ratio of the voltage and current traveling waves so that

$$\frac{V_0^+(z)}{I_0^+(z)} = \frac{V_0^+ \mathrm{e}^{-\jmath\beta z}}{I_0^+ \mathrm{e}^{-\jmath\beta z}} = \frac{V_0^+(0)}{I_0^+(0)} = \frac{V_0^+}{I_0^+} = Z_0. \tag{2.54}$$

The reflected wave has a similar relationship (but note the sign change):

$$\frac{V_0^- \mathrm{e}^{\jmath\beta z}}{-I_0^- \mathrm{e}^{\jmath\beta z}} = \frac{V_0^-}{-I_0^-} = Z_0. \tag{2.55}$$

The load Z_L imposes an additional constraint on the relationship of the total voltage and current at $z = 0$:

$$\frac{V_L}{I_L} = \frac{V(z=0)}{I(z=0)} = Z_L. \tag{2.56}$$

When $Z_L \neq Z_0$ there must be a reflected wave with appropriate amplitude to satisfy the above equations. Now the total voltage

$$V(z) = V_0^+ \mathrm{e}^{-\jmath\beta z} + V_0^- \mathrm{e}^{\jmath\beta z}, \tag{2.57}$$

and the total current, $I(z)$, is related to the traveling current waves by

$$I(z) = \frac{V_0^+}{Z_0} \mathrm{e}^{-\jmath\beta z} - \frac{V_0^-}{Z_0} \mathrm{e}^{\jmath\beta z} = I_0^+ \mathrm{e}^{-\jmath\beta z} + I_0^- \mathrm{e}^{\jmath\beta z}. \tag{2.58}$$

Thus at the termination of the line $(z = 0)$,

$$\frac{V(0)}{I(0)} = Z_L = Z_0 \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}.$$

This can be rearranged as the ratio of the reflected voltage to the incident voltage:

$$\frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}.$$

This ratio is defined as the voltage **reflection coefficient** at the load,

$$\Gamma_L = \Gamma_L^V = \frac{V_0^-(0)}{V_0^+(0)} = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}. \tag{2.59}$$

That is, at the load

$$V_0^- = \Gamma_L V_0^+. \tag{2.60}$$

The relationship of the traveling waves on the line can also be described using the transmission coefficient T (this is the capital Greek letter tau which looks the same as the English letter 'T'.)

The voltage transmission coefficient from a port at position z to a port at position 0 is (for the transmission line)

$$T = T^V = \frac{V_0^+(\text{at end of line})}{V_0^+(\text{at start of line})} = \frac{V_0^+(0)}{V_0^+(z)} = \frac{V_0^+}{V_0^+ \mathrm{e}^{-\jmath\beta z}} = \mathrm{e}^{\jmath\beta z}. \tag{2.61}$$

The relationship in Equation (2.59) can be rewritten so that the input load impedance can be obtained from the reflection coefficient:

$$Z_L = Z_0 \frac{1 + \Gamma^V}{1 - \Gamma^V}. \tag{2.62}$$

Similarly, the current reflection coefficient can be written as

$$\Gamma^I = \frac{I_0^-}{I_0^+} = \frac{-Z_L + Z_0}{Z_L + Z_0} = -\Gamma^V. \tag{2.63}$$

The voltage reflection coefficient is used most of the time, so the reflection coefficient, Γ, on its own refers to the voltage reflection coefficient, $\Gamma^V = \Gamma$.

There are several special cases that are noteworthy. The most important of these is the case when there is no reflected wave and $\Gamma = 0$. To obtain $\Gamma = 0$, the value of load impedance, Z_L, is equal to Z_0, the characteristic impedance of the transmission line as seen in Equation (2.59).

The total voltage and current waves on the line can be written as

$$V(z) = V_0^+[\mathrm{e}^{-\jmath\beta z} + \Gamma \mathrm{e}^{\jmath\beta z}] \tag{2.64}$$

$$I(z) = \frac{V_0^+}{Z_0}[\mathrm{e}^{-\jmath\beta z} - \Gamma \mathrm{e}^{\jmath\beta z}]. \tag{2.65}$$

From Equations (2.64) and (2.65) it can be seen that the total voltage and current on the line consist of superpositions of incident and reflected waves.

EXAMPLE 2.5 Forward- and Backward-Traveling Waves at an Open Circuit

A lossless transmission line is terminated in an open circuit. What is the relationship between the forward- and backward-traveling voltage waves at the end of the line?

Solution:

At the end of the line the total current is zero, so that $I^+ + I^- = 0$ and so

$$I^- = -I^+. \tag{2.66}$$

The forward- and backward traveling voltages and currents are related to the characteristic impedance by

$$Z_0 = V^+/I^+ = -V^-/I^-, \tag{2.67}$$

Note the change in sign, as a result of the direction of propagation changing but the positive reference for current is in the same direction. Substituting for I^- at the termination,

$$V^+ = -V^- I^+/I^- = -V^- I^+/(-I^+) = V^-. \tag{2.68}$$

Thus the total voltage at the end of the line, V_{TOTAL}, is $V^+ + V^- = 2V^+$. Note that the total voltage at the end of the line is twice the incident (forward-traveling) voltage.

EXAMPLE 2.6 Current Reflection Coefficient

A load consists of a shunt connection of a capacitor of 10 pF and a resistor of 60 Ω. The load terminates a lossless 50 Ω transmission line. The operating frequency is 5 GHz.

(a) What is the impedance of the load?
(b) What is the normalized impedance of the load (normalized to Z_0 of the line)?
(c) What is the reflection coefficient of the load?
(d) What is the current reflection coefficient of the load?

Solution:

(a) $C = 10 \cdot 10^{-12}$ F; $R = 60\ \Omega$; $f = 5 \cdot 10^9$ Hz; $\omega = 2\pi f$; $Z_0 = 50\ \Omega$

$$Z_L = R||C = (1/R + \jmath\omega C)^{-1} = 0.168 - \jmath 3.174\ \Omega.$$

(b) $z_L = Z_L/Z_0 = 3.368 \cdot 10^{-3} - \jmath 0.063$.
(c) This is the voltage reflection coefficient. $\Gamma_L = (z_L - 1)/(z_L + 1) = -0.985 - \jmath 0.126 = 0.993\angle 187.3^\circ$.
(d) $\Gamma_L^I = -\Gamma_L = 0.985 + \jmath 0.126 = 0.993\angle(187.3 - 180)^\circ = 0.993\angle 7.3^\circ$.

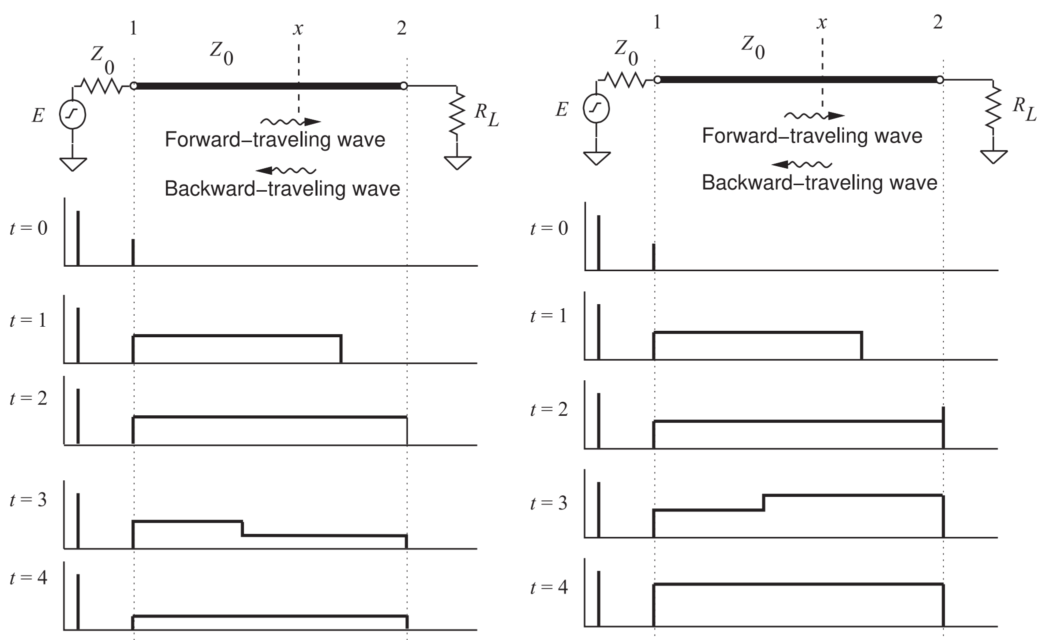
2.3.2 Forward- and Backward-Traveling Pulses

Reflections at the end of a line produce a backward-traveling signal. Forward- and backward-traveling pulses are shown in Figure 2-14(a) for the situation where the resistance at the end of the line is lower than the characteristic impedance of the line ($Z_L < Z_0$). The voltage source is a step voltage that is zero for time $t < 0$. At time $t = 0$, the step is applied to the line and it begins traveling down the line, as shown at time $t = 1$. This voltage step moving from left to right is called the forward-traveling voltage wave.

At time $t = 2$, the leading edge of the step reaches the load, and as the load has lower resistance than the characteristic impedance of the line, the total voltage across the load drops below the level of the forward-traveling voltage step. The reflected wave is called the backward-traveling wave and it must be negative, as it adds to the forward-traveling wave to yield the total voltage. Thus the voltage reflection coefficient, Γ, is negative and the total voltage on the line, which is all that can be directly observed, drops. A reflected, smaller, and opposite step signal travels in the backward direction and adds to the forward-traveling step to produce the waveform shown at $t = 3$. The impedance of the source matches the transmission line impedance so that the reflection at the source is zero. The signal on the line at time $t = 4$, the time for round-trip propagation on the line, therefore remains at the lower value. The easiest way to remember the polarity of the reflected pulse is to consider the situation with a short-circuit at the load. Then the total voltage on the line at the load must be zero. The only way this can occur when a signal is incident is if the reflected signal is equal in magnitude but opposite in sign, in this case $\Gamma = -1$. So whenever $|Z_L| < |Z_0|$, the reflected pulse will tend to subtract from the incident pulse.

The opposite situation occurs when the resistance at the load is higher than the characteristic impedance of the line (Figure 2-14(b)). In this case the reflected pulse has the same polarity as the incident signal. Again, to remember this, think of the open-circuited case. The voltage across the load doubles, as the reflected pulse has the same sign as well as magnitude as that of the incident signal, in this case $\Gamma = +1$. This is required so that the total current is zero.

A more illustrative situation is shown in Figure 2-15, where a more



(a) $R_L < Z_0$ (b) $R_L > Z_0$

Figure 2-14: Reflection of a voltage pulse from a load: (a) when the resistance of the load, R_L is lower than the characteristic impedance of the line, Z_0; and (b) when R_L is greater than Z_0.

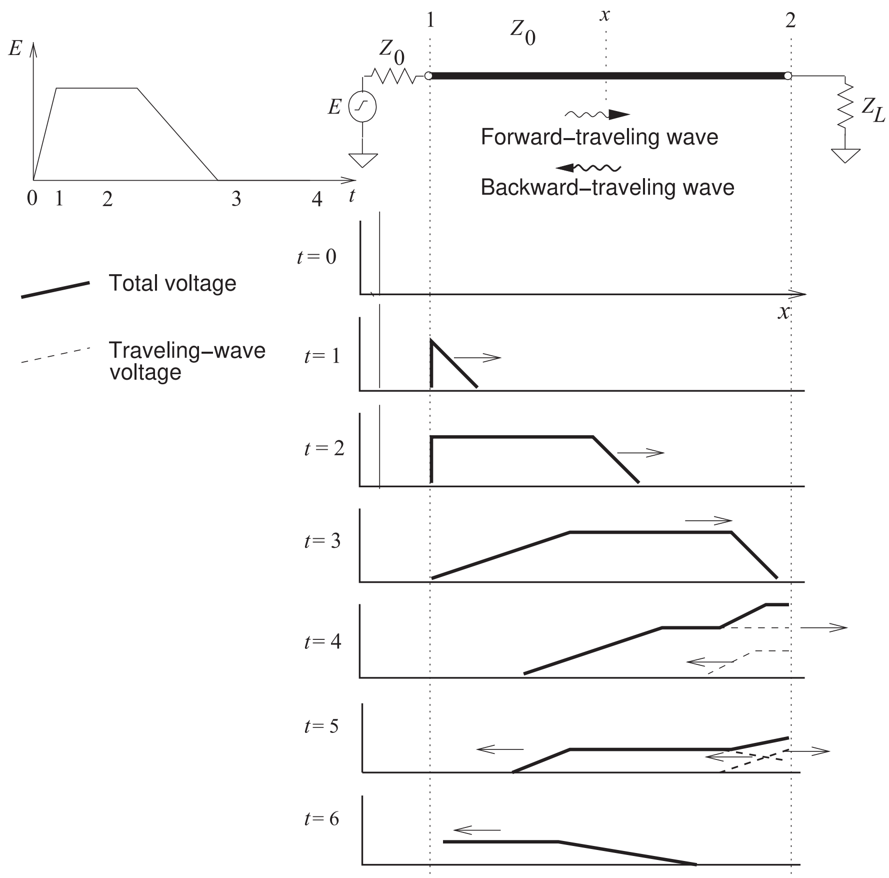


Figure 2-15: Reflection of a pulse on an interconnect showing forward- and backward-traveling pulses. $Z_L > Z_0$.

complicated signal is incident on a load that has a resistance higher than that of the characteristic impedance of the line. The peaking of the voltage that results at the load is typically the design objective in many long digital interconnects, as less overall signal energy needs to be transmitted down the line, or equivalently a lower current drive capability of the source is required to achieve first incidence switching. This is at the price of having reflected signals on the interconnects, but these are dissipated through a combination of line loss and absorption of the reflected signal at the driver.

2.3.3 Input Reflection Coefficient of a Lossless Line

The reflection coefficient looking into a line varies with position along the line as the forward- and backward-traveling waves change in relative phase. Referring to Figure 2-16, at a distance ℓ from the load (i.e., $z = -\ell$), the input

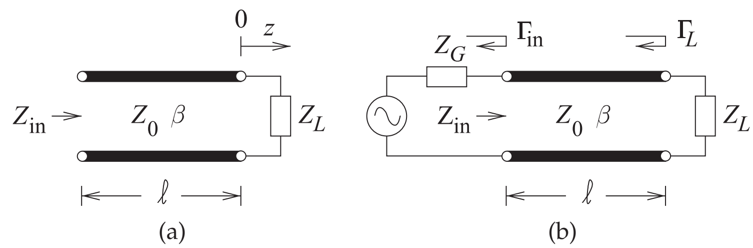


Figure 2-16: Terminated transmission line: (a) a transmission line terminated in a load impedance, Z_L, with an input impedance of Z_{in}; and (b) a transmission line with source impedance Z_G and load Z_L.

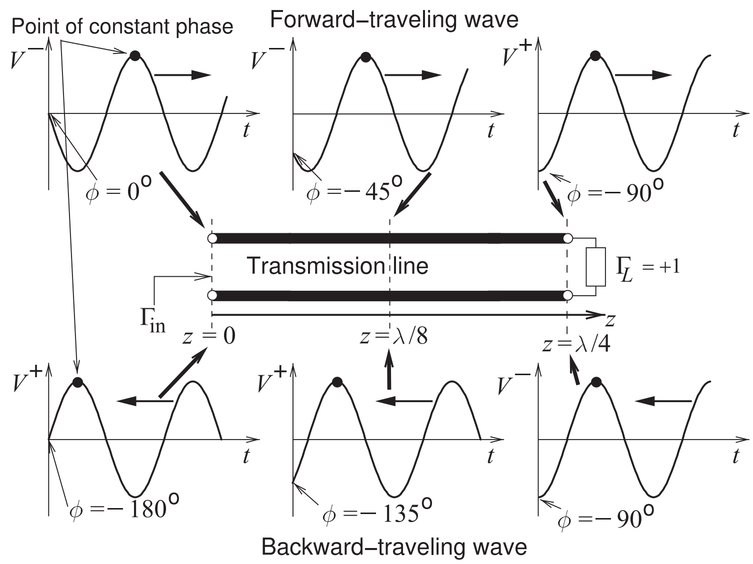


Figure 2-17: The forward-traveling wave $v^+(t,z) = |V^+|\cos(\omega t - \beta z) = |V^+|\cos(\omega t + \phi(z))$ and the backward-traveling wave $v^-(t,z) = |V^+|\cos(\omega t + \beta z) = |V^+|\cos[\omega t + \phi(z)]$. The phase, ϕ, of the forward-traveling wave becomes increasingly negative along the line as z increases, and when reflected the phase ϕ of the backward-traveling wave becomes increasingly negative as the wave moves away from the load (i.e. as z decreases).

reflection looking into a terminated lossless line is

$$\Gamma_{in}|_{z=-\ell} = \frac{V^-(z=-\ell)}{V^+(z=-\ell)} = \frac{V^-(z=0)e^{-\jmath\beta\ell}}{V^+(z=0)e^{+\jmath\beta\ell}} = \frac{V^-(z=0)}{V^+(z=0)}\frac{e^{-\jmath\beta\ell}}{e^{+\jmath\beta\ell}} = \Gamma_L e^{-\jmath 2\beta\ell} \tag{2.69}$$

Note that Γ_{in} has the same magnitude as Γ_L but rotates in the clockwise direction (becomes increasingly negative) at twice the rate of increase of the electrical length $\beta\ell$.

It is important to graphical concepts introduced later that there be a full appreciation for the angle of Γ_{in} becoming increasingly negative at twice the rate at which the electrical length of the line increases. Figure 2-17 is a way of visualizing this. The transmission line here is $\lambda/4$ long with an electrical length of $90°$ and is terminated in a load with reflection coefficient $\Gamma_L = +1$. At position $z = 0$ the forward-traveling voltage wave is $v^+(t,0) = |V^+|\cos(\omega t)$, and this then propagates down the line in the $+z$ direction. The forward-traveling voltage at point $z = \lambda/8$ at $t = 0$ will be the same as the voltage at $z = 0$ at a time one-eighth of a period in the past. The voltage at $z = \lambda/8$ is $v^+(t,\lambda/8) = |V^+|\cos(\omega t - 2\pi/8)$, i.e. there is a phase rotation of $-45°$. Then at $z = \lambda/4$, $v^+(t,\lambda/4) = |V^+|\cos(\omega t - 2\pi/4)$, i.e. at time $t = 0$ there is a phase rotation of $-90°$ relative to $v^+(0,0)$, and this is the negative of the electrical length of the line. The voltage wave reflects at the load and becomes a backward-traveling wave. Here $\Gamma_L = +1$ and so, at the load, the phase of the backward- and forward-traveling waves are the same. The backward-traveling wave continues to travel in the $-z$ direction and its phase at $t = 0$ becomes increasingly negative as z gets closer to the input of the line. The phase of the backward-traveling wave at $z = 0$ is rotated $-90°$ with respect to the backward-traveling wave at the load, and has rotated $-180°$ relative to the forward-traveling wave at $z = 0$. For a lossless line, in general, the angle of $\Gamma_{\text{in}} = $ [phase of $V^-(z = 0)$ relative to the phase of $V^+(z = 0)$] + (the phase of Γ_L) $= -2$(electrical length of the line) + (the phase of Γ_L).

2.3.4 Input Impedance of a Lossless Line

The impedance looking into a lossless line varies with position, as the forward- and backward-traveling waves combine to yield position-dependent total voltage and current. At a distance ℓ from the load (i.e., $z = -\ell$), the input impedance seen looking toward the load is

$$Z_{\text{in}}|_{z=-\ell} = \frac{V(z=-\ell)}{I(z=-\ell)} = Z_0 \frac{1 + |\Gamma|\, e^{\jmath(\Theta - 2\beta\ell)}}{1 - |\Gamma|\, e^{\jmath(\Theta - 2\beta\ell)}} = Z_0 \frac{1 + \Gamma_L e^{\jmath(-2\beta\ell)}}{1 - \Gamma_L e^{\jmath(-2\beta\ell)}}. \quad (2.70)$$

Another form is obtained by substituting Equation (2.59) in Equation (2.70):

$$\begin{aligned} Z_{\text{in}} &= Z_0 \frac{(Z_L + Z_0)e^{\jmath\beta\ell} + (Z_L - Z_0)e^{-\jmath\beta\ell}}{(Z_L + Z_0)e^{\jmath\beta\ell} - (Z_L - Z_0)e^{-\jmath\beta\ell}} = Z_0 \frac{Z_L \cos(\beta\ell) + \jmath Z_0 \cos(\beta\ell)}{Z_0 \cos(\beta\ell) + \jmath Z_L \cos(\beta\ell)} \\ &= Z_0 \frac{Z_L + \jmath Z_0 \tan\beta\ell}{Z_0 + \jmath Z_L \tan\beta\ell}. \end{aligned} \quad (2.71)$$

This is the **lossless telegrapher's equation**. The electrical length, $\beta\ell$, is in radians when used in calculations.

2.3.5 Standing Waves and Voltage Standing Wave Ratio

The total voltage on a terminated line is the sum of forward- and backward-traveling waves. This sum produces what is called a standing wave. Figure 2-18 shows the total and traveling waveforms on a line terminated in a reactance and evaluated at times equal to multiples of an eighth of a period. Here the traveling waves have the same amplitude indicating that the termination of the line is reactive, $|\Gamma| = 1$. The interesting property here is that the total voltage appears as a standing wave with fixed points called nodes where the total voltage is always zero. This is more easily seen in Figure 2-19(a), where the total voltage is overlaid for many times. If the termination has resistance, then the magnitude of the backward-traveling wave will be less than that of the forward-traveling wave and the overlaid total voltage is as shown in Figure 2-19(b). This is still a standing wave, but the minima are now not zero. The envelope of this standing wave is shown in Figure 2-19(c), where there is a maximum amplitude V_{max} and a minimum amplitude V_{min}.

Now this situation will be examined mathematically to relate the standing wave to the reflection coefficient. If $\Gamma = 0$, then the magnitude of the total voltage on the line, $|V(z)|$, is equal to $\left|V_0^+\right|$ anywhere on the line. For this reason, such a line is said to be "flat." If there is reflection the magnitude of the total voltage on the line is not constant (see Figure 2-19(b)). Thus from Equations (2.64) and (2.65):

$$|V(z)| = \left|V_0^+\right| \left|1 + \Gamma e^{2\jmath\beta z}\right| = \left|V_0^+\right| \left|1 + \Gamma e^{-2\jmath\beta\ell}\right|, \quad (2.72)$$

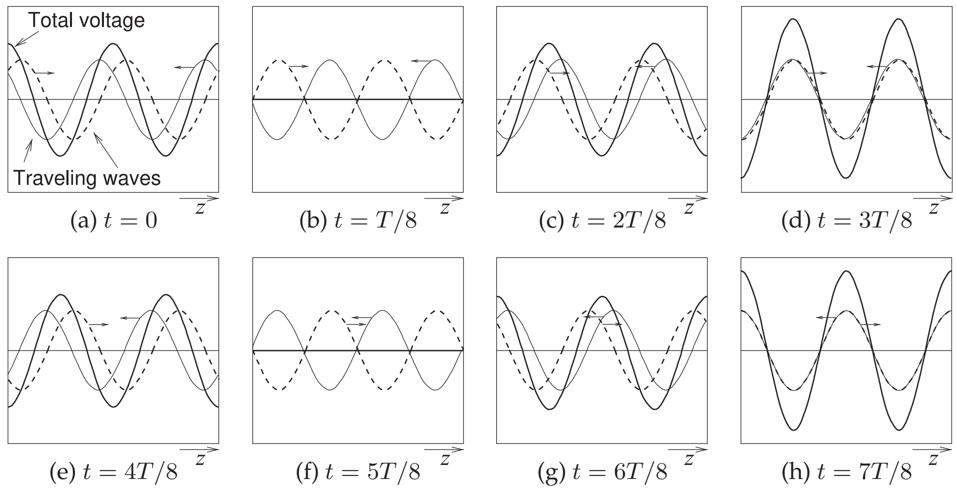


Figure 2-18: Evolution of a standing wave with a reactive load as the sum of forward- and backward-traveling waves (to the right and left, respectively) of equal amplitude evaluated at times t equal to eighths of the period T. At $t = T/8$ and $t = 5T/8$ the total voltage everywhere on the line is zero.

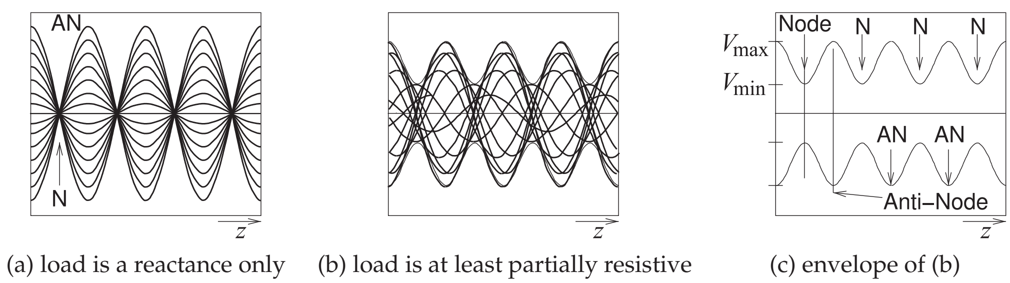


Figure 2-19: Standing waves as an overlay of waveforms at many times: (a) when the forward- and backward-traveling waves have the same amplitude; (b) when the waves have different amplitudes; and (c) the envelope of the standing wave. N is a node (a minimum) and AN is an antinode (a maximum). Nodes, N, are separated by $\lambda/2$. Antinodes, AN, are separated by $\lambda/2$.

where $z = -\ell$ is the positive distance measured from the load at $z = 0$ toward the generator. Or, setting $\Gamma = |\Gamma|e^{\jmath\Theta}$,

$$|V(z)| = \left|V_0^+\right|\left|1 + |\Gamma|e^{\jmath(\Theta - 2\beta\ell)}\right|, \tag{2.73}$$

where Θ is the phase of the reflection coefficient ($\Gamma = |\Gamma|e^{\jmath\Theta}$) at the load. This result shows that the voltage magnitude oscillates with position z along the line. The maximum value occurs when $e^{\jmath(\Theta - 2\beta\ell)} = 1$ and is given by

$$V_{\rm max} = \left|V_0^+\right|(1 + |\Gamma|). \tag{2.74}$$

Similarly the minimum value of the total voltage magnitude occurs when the phase term is $e^{\jmath(\Theta-2\beta l)} = -1$, and is given by

$$V_{\min} = \left|V_0^+\right|(1-|\Gamma|). \tag{2.75}$$

A mismatch can be defined by the **voltage standing wave ratio** (**VSWR**):

$$\mathrm{VSWR} = \frac{V_{\max}}{V_{\min}} = \frac{(1+|\Gamma|)}{(1-|\Gamma|)}. \tag{2.76}$$ Also $$|\Gamma| = \frac{\mathrm{VSWR}-1}{\mathrm{VSWR}+1}. \tag{2.77}$$

Notice that in general Γ is complex, but VSWR is necessarily always real and $1 \leq \mathrm{VSWR} \leq \infty$. For the matched condition, $\Gamma \;= 0$ and VSWR $= 1$, and the closer VSWR is to 1, the closer the load is to being matched to the line and the more power is delivered to the load. The magnitude of the reflection coefficient on a line with a short-circuit or open-circuit load is 1, and in both cases the VSWR is infinite.

To determine the position of the standing wave maximum, $\ell_{\max}$, consider Equation (2.73) and note that at the maximum

$$\Theta - 2\beta\ell_{\max} = 2n\pi, \quad n = 0, 1, 2, \ldots\ . \tag{2.78}$$

Here Θ is the angle of the reflection coefficient at the load:

$$\Theta - 2n\pi = 2\frac{2\pi}{\lambda_g}\ell_{\max}\ . \tag{2.79}$$

Thus the position of the voltage maxima, $l_{\max}$, normalized to wavelength is

$$\frac{\ell_{\max}}{\lambda_g} = \frac{1}{2}\left(\frac{\Theta}{2\pi} - n\right), \quad n = 0, -1, -2, \ldots. \tag{2.80}$$

Similarly the position of the voltage minima is (using Equation (2.73)):

$$\Theta - 2\beta\ell_{\min} = (2n+1)\pi. \tag{2.81}$$

After rearranging the terms,

$$\frac{\ell_{\min}}{\lambda_g} = \frac{1}{2}\left(\frac{\Theta}{2\pi} - n + \frac{1}{2}\right), \quad n = 0, -1, -2, \ldots. \tag{2.82}$$

Summarizing from Equations (2.80) and (2.82):

1. The distance between two successive maxima is $\lambda_g/2$.
2. The distance between two successive minima is $\lambda_g/2$.
3. The distance between a maximum and an adjacent minimum is $\lambda_g/4$.
4. From the measured VSWR the magnitude of the reflection coefficient $|\Gamma|$ can be found. From the measured $\ell_{\max}$ the angle Θ of Γ can be found. Then from Γ the load impedance can be found.

In a similar manner to that above, the magnitude of the total current on the line is

$$|I(\ell)| = \frac{\left|V_0^+\right|}{Z_0}\left|1 - |\Gamma|e^{\jmath(\Theta-2\beta\ell)}\right|. \tag{2.83}$$

Hence the standing wave current is maximum where the standing-wave voltage amplitude is minimum, and minimum where the standing-wave voltage amplitude is maximum.

Z_{in} in Equation 2.71 is a periodic function of length with period $\lambda/2$ and it varies between $Z_{\max}$ and $Z_{\min}$, where

$$Z_{\max} = \frac{V_{\max}}{I_{\min}} = Z_0 \times \mathrm{VSWR} \quad \text{and} \quad Z_{\min} = \frac{V_{\min}}{I_{\max}} = \frac{Z_0}{\mathrm{VSWR}}. \tag{2.84}$$

EXAMPLE 2.7 Standing Wave Ratio

In Example 2.6 the load consisted of a capacitor of 10 pF in shunt with a resistor of 60 Ω. The load terminated a lossless 50 Ω transmission line. The operating frequency is 5 GHz.

(a) What is the SWR?
(b) What is the current standing wave ratio (ISWR)? (When SWR is used on its own it is assumed to refer to VSWR.)

Solution:

(a) From Example 2.6 $\Gamma_L = 0.993\angle 187.3^\circ$ and so

$$\text{VSWR} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = \frac{1+0.993}{1-0.993} = 285.$$

(b) ISWR = VSWR = 285.

EXAMPLE 2.8 Standing Waves

A load has an impedance $Z_L = 45 + \jmath 75\ \Omega$ and the system reference impedance, Z_0, is 100 Ω.

(a) What is the reflection coefficient?
(b) What is the current reflection coefficient?
(c) What is the SWR?
(d) What is the ISWR?
(e) The power available from a source with a 100 Ω Thevenin equivalent impedance is 1 mW. The source is connected directly to the load, Z_L. Use the reflection coefficient to calculate the power delivered to Z_L.
(f) What is the total power absorbed by the Thevenin equivalent source impedance?
(g) Discuss the effect on power flow of inserting a lossless 100 Ω transmission line between the source and the load.

Solution:

(a) The voltage reflection coefficient is

$$\begin{aligned}\Gamma_L &= (Z_L - Z_0)/(Z_L + Z_0) = (45 + \jmath 75 - 100)/(45 + \jmath 75 + 100) \\ &= (93.0\angle(2.204\ \text{rads}))/(163.2\angle(0.4773\ \text{rads})) \\ &= 0.570\angle(1.726\ \text{rads}) = 0.570\angle 98.9^\circ = -0.0881 + \jmath 0.563 = \Gamma^V. \qquad (2.85)\end{aligned}$$

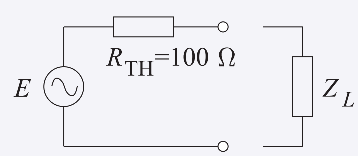
(b) The current reflection coefficient is

$$\Gamma^I = -\Gamma^V = 0.0881 - \jmath 0.563 = 0.570\angle(98.9^\circ - 180^\circ) = 0.570\angle 81.1^\circ. \qquad (2.86)$$

(c) The SWR is the VSWR, so

$$\text{SWR} = \text{VSWR} = \frac{V_{\max}}{V_{\min}} = \frac{1+|\Gamma^V|}{1-|\Gamma^V|} = \frac{1+0.570}{1-0.570} = 3.65. \qquad (2.87)$$

(d) The current SWR is ISWR = VSWR.
(e) To determine the reflection coefficient of the load, begin by developing the Thevenin equivalent circuit of the load. The power available from the source is $P_A = 1\ \text{mW}$, so the Thevenin equivalent circuit is



The power reflected by the load is

$$P_R = P_A \left|\Gamma_L^2\right| = 1\ \text{mW} \cdot (0.570)^2 = 0.325\ \text{mW}$$

and the power delivered to the load is

$$P_D = P_A \left(1 - \left|\Gamma_L^2\right|\right) = 0.675\ \text{mW}.$$

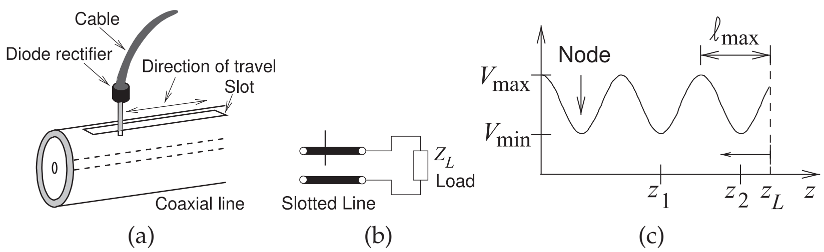


Figure 2-20: Measurement of standing waves: (a) coaxial slotted line; (b) schematic of slotted line; (c) measured standing wave.

(f) It is tempting to think that the power dissipated in R_{TH} is just P_R. However, this is not correct. Instead, the current in R_{TH} must be determined and then the power dissipated in R_{TH} found. Let the current through $R_{\text{TH}cdot}$ be I, and this is composed of forward- and backward-traveling components:

$$I = I^+ + I^- = (1 + \Gamma_I)I^+,$$

where I^+ is the forward-traveling current wave. Thus

$$P_A = \tfrac{1}{2}|I^+|^2 R_{\text{TH}} = \tfrac{1}{2}|I^+|^2 \times 100 = 1 \text{ mW} = 10^{-3} \text{ W},$$

so $I^+ = 4.47$ mA, and

$$I = (1 + \Gamma_I)I^+ = (1 + 0.0881 - \jmath 0.563) \times 4.47 \times 10^{-3} \text{ A}, \qquad |I| = 5.48 \text{ mA}.$$

The power dissipated in R_{TH} is

$$P_{\text{TH}} = \tfrac{1}{2}|I|^2 R_{\text{TH}} = \tfrac{1}{2}(5.48 \times 10^{-3})^2 R_{\text{TH}} = 1.50 \text{ mW}. \tag{2.88}$$

The circuit is that shown in part (e) and so the current in R_{TH} is the same as the current in Z_L. Thus the power delivered to the load Z_L is due to the real part of Z_L:

$$P_D = \tfrac{1}{2}\left|I^2\right| \Re(Z_L) = \tfrac{1}{2}(5.48 \times 10^{-3})^2 \times 45 = 0.676 \text{ mW} \tag{2.89}$$

(g) Inserting a transmission line with the same characteristic impedance as the Thevenin equivalent impedance will have no effect on power flow.

VSWR Measurement

The measurement of standing waves can be used to calculate the impedance of a load. The device that does this measurement, called a slotted line, is shown in Figure 2-20(a). A probe is inserted a small distance into the transmission line to measure the electric field. The RF electric field produces an RF voltage on the probe that is rectified by the diode detector. The DC voltage at the output of the detector is proportional to the total voltage on the line. The probe can be moved along the line and the ratio of V_{max} to V_{min} determined. This is just the VSWR. To find the complex load impedance it is also necessary to determine the position of the node of the standing wave. From the measured VSWR the magnitude of the reflection coefficient $|\Gamma|$ can be found. From the measured ℓ_{max} the angle Θ of Γ can be found. From Γ the load impedance can be found. This is demonstrated in the next example.

EXAMPLE 2.9 Slotted Line Measurement of Impedance

A slotted line is used to determine the properties of the standing wave on a terminated 50 Ω line see Figure 2-19(c). $V_{\max} = 5$ V and $V_{\min} = 2$ V, and the first minimum is 2 cm from the load. The guide wavelength is 10 cm. What is the load impedance Z_L?

Solution:

Now VSWR = $V_{\max}/V_{\min} = 5/2 = 2.5$. So from Equation (2.77)

$$|\Gamma| = |\Gamma_L| = \frac{\text{VSWR} - 1}{\text{VSWR} + 1} = \frac{2.5 - 1}{2.5 + 1} = 0.428. \tag{2.90}$$

Equation (2.82) and the position of the first node can be used to determine the angle of Γ_L. For the first node (minimum), $n = 0$ and

$$\frac{\ell_{\min}}{\lambda_g} = \tfrac{1}{2}\left(\frac{\Theta}{2\pi} + \frac{1}{2}\right). \tag{2.91}$$

Rearranging,

$$\Theta = 2\pi\left(2\frac{\ell_{\min}}{\lambda_g} - \frac{1}{2}\right) \text{ radians.} \tag{2.92}$$

Now $\ell_{\min} = 2$ cm and $\lambda_g = 10$ cm. So, in degrees,

$$\Theta = 360\left(2\frac{\ell_{\min}}{\lambda_g} - \frac{1}{2}\right) = 360\left(2\frac{2}{10} - \frac{1}{2}\right) = -36^\circ. \tag{2.93}$$

Thus $\Gamma_L = 0.428\angle(-36^\circ) = 0.3463 - \jmath 0.2516$, so the load impedance is (where $Z_0 = 50\ \Omega$)

$$Z_L = Z_0\left(\frac{1 + \Gamma_L}{1 - \Gamma_L}\right) = 83.2 - \jmath 51.3\ \Omega. \tag{2.94}$$

2.3.6 Summary

This section related the physics of traveling voltage and current waves on lossless transmission lines to the total voltage and current view. First the input reflection coefficient of a terminated lossless line was developed and from this the input impedance, which is the ratio of total voltage and total current, derived. At any point along a line the amplitude of total voltage varies sinusoidally, tracing out a standing wave pattern along the line and yielding the VSWR metric which is the ratio of the maximum amplitude of the total voltage to the minimum amplitude of that voltage. This is an important metric that is often used to provide an indication of how good a match, i.e. how small the reflection is, with a VSWR $= 1$ indicating no reflection and a VSWR $= \infty$ indicating total reflection, i.e. a reflection coefficient magnitude of 1.

2.4 Special Lossless Line Configurations

The lossless transmission line configurations considered in this section are used as circuit elements in RF designs and are used elsewhere in this book series. The first element considered in Section 2.4.1 is a short length of short-circuited line which looks like an inductor. The element considered in Section 2.4.2 is a short length of open-circuited line which looks like a capacitor. Then lengths of short-circuited and open-circuited lines, called stubs, used nearly always as shunt elements to introduce an admittance in a circuit, are described in Sections 2.4.3 and 2.4.4. Another type of element, described in Section 2.4.5, is a short length of line with either high or low characteristic impedance realizing a small series inductor or capacitor respectively. The final element described in Section 2.4.6 is a quarter-wave transformer, a quarter-wavelength long line with a particular characteristic impedance which is used in two ways. It can be used to provide maximum power transfer from a source to a load resistance, and it can invert an impedance, e.g. making a capacitor terminating the line look like an inductor.

2.4.1 Short Length of Short-Circuited Line

A transmission line terminated in a short circuit ($Z_L = 0$) has the input impedance (using Equation (2.71))

$$Z_{\text{in}} = \jmath Z_0 \tan(\beta\ell). \tag{2.95}$$

So a short length of line, $\ell < \lambda_g/4$, looks like an inductor with inductance L_s,

$$Z_0 \tan(\beta\ell) = \omega L_s, \quad \text{and so} \quad L_s = \frac{Z_0}{\omega}\tan\frac{2\pi\ell}{\lambda_g}. \tag{2.96}$$

From Equation (2.96) it can be seen that for a given ℓ, L_s is proportional to Z_0. Hence, for larger values of L_s, sections of transmission line of high characteristic impedance are needed. So microstrip lines with narrow strips can be used to realize inductors in planar microstrip circuits.

2.4.2 Short Length of Open-Circuited Line

An open-circuited line has $Z_L = \infty$ and so (using Equation (2.71))

$$Z_{\text{in}} = -\jmath\frac{Z_0}{\tan\beta\ell}\,. \tag{2.97}$$

For lengths ℓ such that $\ell < \lambda/4$, an open-circuited segment of line realizes a capacitor C_0 for which

$$\frac{1}{\omega C_0} = \frac{Z_0}{\tan\beta\ell} \quad \text{and so} \quad C_0 = \frac{1}{Z_0}\frac{\tan(\beta\ell)}{\omega}. \tag{2.98}$$

From the above relationship, it can be seen that C_0 is inversely proportional to Z_0. Hence, for larger values of C_0, sections of transmission line with low characteristic impedance need to be used.

2.4.3 Short-Circuited Stub

A stub is a section of open-circuited or short-circuited transmission line and is used as a series or shunt element in a microwave circuit. There are several representations. A shorted stub is shown in Figure 2-21(a) as a transmission line with characteristic impedance Z_{01} that is short circuited. The input impedance of the line is Z_1. If the line is lossless, the usual assumption, then Z_{01} will be real and Z_1 will be imaginary. Stubs are commonly used in microwave circuits and generally all stubs in a network have the same length, such as $\lambda/4$ long or $\lambda/8$ long. Which it is is specified in the design. Realistically they do not need to have the same length but there are some special properties for certain lengths, as will become clearer. A cleaner way to indicate a shorted stub is shown in Figure 2-21(b), where the value of the stub is as indicated. The absence of a 0 subscript (which would indicate

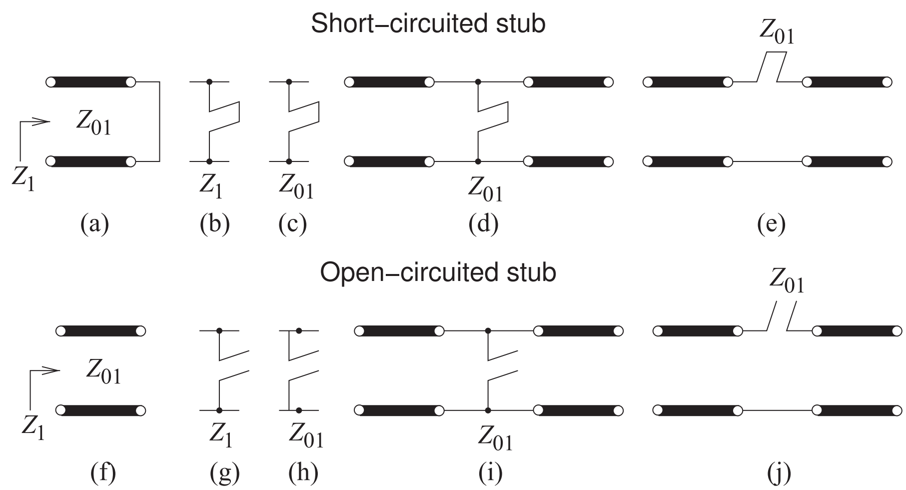


Figure 2-21: Transmission line stubs: (a)–(e) short-circuited stubs; and (f)–(j) open-circuited stubs.

characteristic impedance) means that this is the reactive input impedance of the stub. If a 0 subscript is used, as in Figure 2-21(c), the characteristic impedance of the stub is indicated. If a numerical value is given then an imaginary impedance indicates that the input impedance is being specified, whereas a real impedance indicates the characteristic impedance of the stub. The shorted stub is shown as a shunt element in Figure 2-21(d) and as a series element in Figure 2-21(e). However in nearly all transmission line technologies, including microstrip, only shunt stubs can be realized. The open-circuited stubs with annotations are shown in Figures 2-21(f–j) with similar assignments of meaning. The length of a stub is often indicated by its resonant frequency, f_r. This is the frequency at which the stub is $\lambda/4$ long.

The shorted stub in Figure 2-21(a) has the input impedance (from Equation (2.71))

$$Z_1 = \jmath Z_{01} \tan \beta\ell, \tag{2.99}$$

where ℓ is the physical length of the line. Since the stub is $\lambda/4$ long at f_r, then at frequency f, the input impedance of the stub is

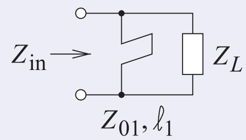
$$Z_1 = \jmath Z_{01} \tan\left(\frac{\pi}{2}\frac{f}{f_r}\right). \tag{2.100}$$

A special situation, and the most commonly used in design, is when the operating frequency is around one-half of the resonant frequency (i.e., $f \approx \frac{1}{2}f_r$). Then the stub is one-eighth of a wavelength long and the argument of the tangent function in Equation (2.100) is approximately $\pi/4$ and Z_1 becomes

$$Z_1 \approx \jmath Z_{01} \tan\left(\frac{\pi}{4}\right) = \jmath Z_{01}, \tag{2.101}$$

thus realizing an inductance at $f = f_r/2$ with a reactance equal to the characteristic impedance of the line.

EXAMPLE 2.10 Short-Circuited Stub



Develop the electrical design of the shunt stub shown with a load of impedance $Z_L = 75 + \jmath 15\ \Omega$ so that the total impedance of the load and stub is real.

Solution:

The short-circuited stub has characteristic impedance Z_{01} and length ℓ_1. Choose $Z_{01} = 75\ \Omega$ (generally this must be between 15 Ω and 100 Ω for most transmission line technologies). The stub needs to be designed so that the susceptances of the stub and load sum to zero. The admittance of the load $Y_L = 1/Z_L = 0.01282 - \jmath 0.002564$ S. The required admittance of the stub is $Y_{\text{STUB}} = \jmath 0.002564$ S so, using Equation (2.99),

$$Z_{\text{STUB}} = 1/Y_{\text{STUB}} = \jmath Z_{01} \tan \beta\ell_1 = -\jmath 390\ \Omega.$$

Therefore, the electrical length of the stub is

$$\beta\ell_1 = \arctan(-\jmath 390/\jmath 75) = -1.381 + n\pi \text{ radians}, n = 0, 1, 2, \ldots. \tag{2.102}$$

The first positive angle is taken so the stub has the shortest length. So

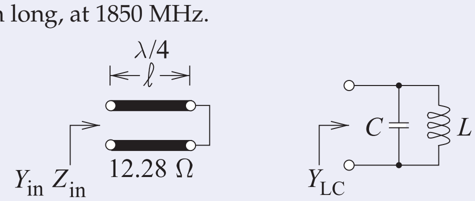
$$\beta\ell_1 = 1.761 \text{ radians} = 100.9^\circ. \tag{2.103}$$

The complete electrical design of the stub is that it is a shunt short-circuited stub with a characteristic impedance of 75 Ω and with an electrical length of 100.9°. The combined impedance of the stub and load is $Z_X = 1/(\Re\{Y_L\}) = 1/0.01282 = 78.00\ \Omega$.

EXAMPLE 2.11 Model of a Resonant Shorted Transmission Line

This example presents an analytic approach to developing the equivalent circuit of a shorted stub at resonance. Consider a shorted transmission line with characteristic impedance $Z_0 = 12.28\ \Omega$ and resonant, i.e. a quarter-wavelength long, at 1850 MHz.

Solution:



At the first resonant frequency, $f_r = \omega_r/(2\pi)$, the transmission line presents an open circuit and the appropriate circuit model is shown on the right.

The strategy here is to develop the LC equivalent circuit by equating the derivatives of the resonator and the LC circuit. This is in addition to equating the input admittances of the two circuits at the resonant frequency $f_r = \omega_r/(2\pi)$. That is, at frequency f_r

$$Y_{\text{in}}(f_r) = 0 = Y_{\text{LC}}(f_r) = \jmath\left[\omega_r C - 1/(\omega_r L)\right] \quad \text{and so} \quad \omega_r^2 = 1/(LC). \tag{2.104}$$

The input impedance of the line at frequency $f = \omega/(2\pi)$ is $Z_{\text{in}}(\omega) = \jmath Z_0 \tan(\beta\ell)$, and so its input admittance is

$$Y_{\text{in}}(\omega) = \frac{-\jmath}{Z_0} \cot(\beta\ell). \tag{2.105}$$

The derivative of the transmission line admittance is (using Equation (1.112))

$$\frac{\partial Y_{\text{in}}}{\partial \omega} = \frac{\partial \beta\ell}{\partial \omega}\frac{\partial Y_{\text{in}}}{\partial \beta\ell} = \frac{\partial \beta\ell}{\partial \omega}\left(\frac{-\jmath}{Z_0}\right)\left[-\csc^2(\beta\ell)\right]. \tag{2.106}$$

Now β is proportional to ω for a lossless dispersionless line so

$$\frac{\partial Y_{\text{in}}}{\partial \omega} = \frac{\beta\ell}{\omega}\frac{\jmath}{Z_0}\csc^2(\beta\ell).$$

Since the line is approximately $\lambda/4$ long near the resonant frequency, for $f \approx f_r$, $\beta\ell \approx \pi/2$, $\csc(\beta\ell) \approx 1$ (which is a good approximation since at f_r, $[\partial \csc(\beta\ell)]/(\partial\omega) = 0$), and

$$\left.\frac{\partial Y_{\text{in}}}{\partial \omega}\right|_{\omega_r} = \frac{\jmath\beta\ell}{\omega_r Z_0}. \tag{2.107}$$

The input admittance of the parallel LC circuit is

$$Y_{\text{LC}} = \jmath\left(\omega C - \frac{1}{\omega L}\right) \quad \text{and} \quad \frac{\partial Y_{\text{LC}}}{\partial \omega} = \jmath\left(C + \frac{1}{\omega^2 L}\right). \tag{2.108}$$

At and near resonance $\omega^2 \approx 1/(LC)$ and so

$$\left.\frac{\partial Y_{\text{LC}}}{\partial \omega}\right|_{\omega_r} = \jmath 2C. \tag{2.109}$$

Equating the derivatives of Y_{in}, Equation (2.107), and of Y_{LC}, Equation (2.109), yields

$$\frac{-\jmath\beta\ell}{\omega Z_0} = \jmath 2C. \tag{2.110}$$

Thus (since $\beta\ell \approx \pi/2$)

$$C = \frac{\pi}{4\omega_r Z_0} = \frac{\pi}{4 \cdot 2\pi \cdot 1850 \cdot 10^6 \cdot 12.28} = 5.502 \cdot 10^{-12}\text{F} = 5.502\ \text{pF} \tag{2.111}$$

and, since $\omega_r^2 = 1/(LC)$,

$$L = \frac{1}{\omega_r^2 C} = \frac{1}{(2\pi \cdot 1850 \cdot 10^6)^2 \cdot 5.502 \cdot 10^{-12}} = 1.345 \cdot 10^{-9}\ \text{H} = 1.345\ \text{nH}. \tag{2.112}$$

2.4.4 Open-Circuited Stub

An open-circuited transmission line is commonly used as a circuit element called an open stub shown in Figure 2-21(f–j). From Equation (2.71) and noting that $Z_L = \infty$, the open stub input impedance is

$$Z_1 = -\jmath Z_{01}\frac{1}{\tan\beta\ell}. \tag{2.113}$$

With the stub one-quarter wavelength long at the frequency f_r, the input impedance at f_r is a short circuit and the stub is said to be resonant at f_r. Then at a frequency f, the input impedance of the stub is

$$Z_1 = -\jmath Z_{01}\tan^{-1}\left(\frac{\pi}{2}\frac{f}{f_r}\right). \tag{2.114}$$

When $f = \frac{1}{2}f_r$ the stub is one-eighth wavelength long and

$$Z_1 = -\jmath Z_{01}\frac{1}{\tan\left(\frac{\pi}{4}\right)} = -\jmath Z_{01}. \tag{2.115}$$

So a $\lambda/8$ long open-circuited stub ($\lambda/4$ at f_r, $f = \frac{1}{2}f_r$) realizes a capacitance with a reactance equal to the characteristic impedance of the line.

If the length of a stub can be changed then the stub can be used as a tuning element. A common microstrip tuning technique is shown in Figure 2-22, where bonding to different pads enables a variable length stub to be realized.

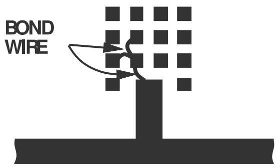
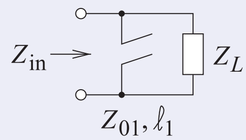


Figure 2-22: Open-circuited stub with variable length realized using wire bonding from the fixed stub to one of the bond pads. The bond pads are on the same layer as the strip metal layer and bonding to them extends the length of the open-circuited stub.

EXAMPLE 2.12 Open-Circuited Stub



Develop the electrical design of the open-circuit stub shown with a load of impedance $Z_L = 75 + \jmath 15\ \Omega$ so that the total impedance of the load and stub is real.

Solution:

The open-circuited stub has characteristic impedance Z_{01} and length ℓ_1. A good choice is to choose Z_{01} around the impedance level of the load as long as it can be realized; so choose $Z_{01} = 75\ \Omega$. The stub needs to be designed so that the susceptances of the stub and load sum to zero. The admittance of the load $Y_L = 1/Z_L = 0.01282 - \jmath 0.002564$ S. The required admittance of the stub is $Y_{\text{STUB}} = \jmath 0.002564$ S, so, using Equation (2.113),

$$Z_{\text{STUB}} = 1/Y_{\text{STUB}} = -\jmath Z_{01}/\tan\beta\ell_1 = -\jmath 390\ \Omega.$$

Therefore, the electrical length of the stub is

$$\beta\ell_1 = \arctan(\jmath 75/\jmath 390) = 0.1900 + n\pi \text{ radians}, n = 0, 1, 2, \ldots. \tag{2.116}$$

The first positive angle is taken so the stub has the shortest length. Then

$$\beta\ell_1 = 0.1900 \text{ radians} = 10.89^\circ. \tag{2.117}$$

The complete electrical design of the stub is that it is a shunt open-circuited stub with a characteristic impedance of 75 Ω and an electrical length of 10.89°. The combined impedance of the stub and load is $Z_X = 1/(\Re\{Y_L\}) = 1/0.01282 = 78.00\ \Omega$.

2.4.5 Electrically Short Lossless Line

Consider the input impedance, Z_{in}, of an electrically short line (i.e., $\beta\ell$ is small) (see Figure 2-23). Using Equation (2.71),

$$Z_{\text{in}} \approx \frac{Z_L + \jmath Z_0(\beta\ell)}{1 + \jmath(Z_L/Z_0)(\beta\ell)} \approx [Z_L + \jmath Z_0(\beta\ell)]\left[1 - \jmath\frac{Z_L}{Z_0}(\beta\ell)\right]. \tag{2.118}$$

Since $Z_0\beta = \sqrt{L/C}(\omega\sqrt{LC}) = \omega L$ and $\beta/Z_0 = (\omega\sqrt{LC})/\sqrt{L/C}) = \omega C$ (where L and C are the inductance and capacitance per unit length of the line), Equation (2.118) can be written as

$$Z_{\text{in}} \approx Z_L\left[1 + (\beta\ell)^2\right] + \jmath\left[\omega(L\ell) - Z_L^2\omega(C\ell)\right]. \tag{2.119}$$

Since $\beta\ell$ is small, $(\beta\ell)^2$ is very small, and so the $(\beta\ell)^2$ term can be ignored. Then the input impedance of an electrically short line terminated in impedance Z_L is

$$Z_{\text{in}} \approx Z_L + \jmath\left[\omega(L\ell) - Z_L^2\omega(C\ell)\right]. \tag{2.120}$$

Some special cases of this result will be considered in the following examples.

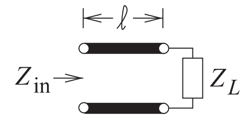
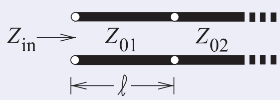


Figure 2-23: An electrically short line.

EXAMPLE 2.13 Capacitive Transmission Line Segment

This example demonstrates that a predominantly capacitive behavior can be obtained from a short segment of transmission line, the Z_{01} line here, of low characteristic impedance. Consider the transmission line system shown below with lines having the characteristic impedances, Z_{01} and Z_{02}, $Z_{02} \gg Z_{01}$.



The value of $Z_{\rm in}$ is (treating Z_{02} as the load)

$$Z_{\rm in} = Z_{01} \frac{Z_{02} + \jmath Z_{01} \tan \beta\ell}{Z_{01} + \jmath Z_{02} \tan \beta\ell}. \tag{2.121}$$

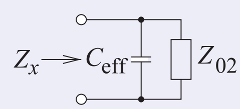
Now $(1 + \jmath x)^{-1} \approx 1 - \jmath x - x^2$. Thus for a short line (and so dropping the $\tan^2(\beta\ell)$ term)

$$Z_{\rm in} \approx Z_{02} - \jmath \frac{Z_{02}^2}{Z_{01}} \tan(\beta\ell) + \jmath Z_{01} \tan(\beta\ell) = Z_{02} + \jmath Z_{01} \tan(\beta\ell) \left[1 - \frac{Z_{02}^2}{Z_{01}^2}\right]. \tag{2.122}$$

For $Z_{02} \gg Z_{01}$ and for a short line, $\tan(\beta\ell) \approx \beta\ell$, and this becomes

$$Z_{\rm in} \approx Z_{02} - \jmath \frac{Z_{02}^2}{Z_{01}} \tan(\beta\ell) \approx Z_{02} - \jmath \frac{Z_{02}^2}{Z_{01}} \beta\ell, \tag{2.123}$$

which is capacitive. Now consider the circuit to the right where an effective capacitance $C_{\rm eff}$ is in shunt with a load Z_{02}. This has the input impedance



$$Z_x = \left(\jmath\omega C_{\rm eff} + \frac{1}{Z_{02}}\right)^{-1} = \frac{Z_{02}}{1 + \jmath\omega C_{\rm eff} Z_{02}} = Z_{02}[1 - \jmath\omega C_{\rm eff} Z_{02} - (\jmath\omega C_{\rm eff} Z_{02})^2 + \ldots . \tag{2.124}$$

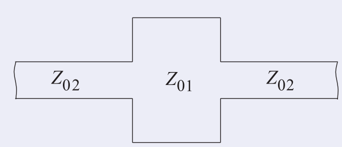
For $\omega C_{\rm eff} Z_{02} \ll 1$ (i.e. an electrically short line)

$$Z_x \approx Z_{02} - \jmath\omega C_{\rm eff} Z_{02}^2 \tag{2.125}$$

Equating Equations (2.123) and (2.125), the effective value of the shunt capacitor realized by the short length of low-impedance line, the Z_{01} line, is

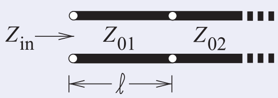
$$C_{\rm eff} = \frac{1}{\omega Z_{02}^2} \frac{Z_{02}^2 \beta\ell}{Z_{01}} = \frac{\beta}{\omega} \frac{\ell}{Z_{01}}. \tag{2.126}$$

Thus a shunt capacitor can be realized approximately by a low-impedance line embedded between two high-impedance lines. The microstrip layout of this is shown in the figure on the right. Recall that a wide microstrip line has a low characteristic impedance.



EXAMPLE 2.14 Inductive Transmission Line Segment

This example demonstrates that a (predominantly) inductive behavior can be obtained from a segment of transmission line. Consider the transmission line system shown below with lines having two different characteristic impedances, Z_{01} and Z_{02}, $Z_{02} \ll Z_{01}$.



The value of Z_{in} is (using Equation (2.71))

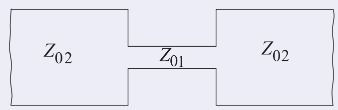
$$Z_{\text{in}} = Z_{01} \frac{Z_{02} + \jmath Z_{01} \tan \beta\ell}{Z_{01} + \jmath Z_{02} \tan \beta\ell}, \tag{2.127}$$

which for a short line can be expressed as

$$Z_{\text{in}} \approx Z_{02} \left[1 + \tan(\beta l)\right] + \jmath Z_{01} \tan(\beta\ell) . \tag{2.128}$$

Note that $\jmath Z_{01} \tan(\beta\ell)$ is the dominant part for $\ell < \lambda/8$ and $Z_{02} \ll Z_{01}$.

Thus a microstrip realization of a series inductor is a high-impedance line embedded between two low-impedance lines. A top view of such a configuration in microstrip is shown in the figure. A narrow microstrip line has high characteristic impedance.



The previous two examples showed how a shunt capacitance or series inductance can be realized using short sections of line, the Z_{01} line here, with low or high characteristic impedance respectively. This enables realization of some lumped element circuits in microstrip form. A lumped element lowpass filter is shown in Figure 2-24(a) and this can be realized using wide and narrow microstrip lines, as shown in Figure 2-24(b).

2.4.6 Quarter-Wave Transformer

Figure 2-25(a) shows a resistive load R_L and a section of transmission line with length $\ell = \lambda_g/4$ (hence the name quarter-wave transformer). The input

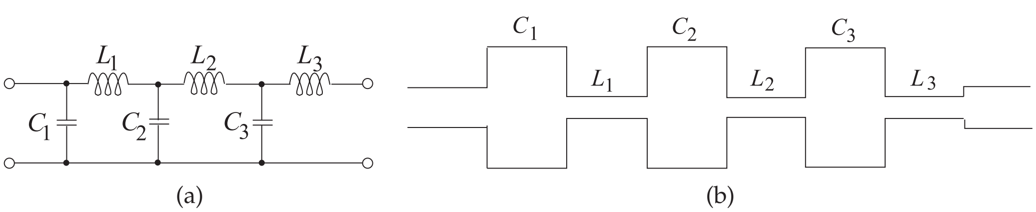


Figure 2-24: A lowpass filter: (a) in the form of an LC ladder network; and (b) realized using microstrip lines.

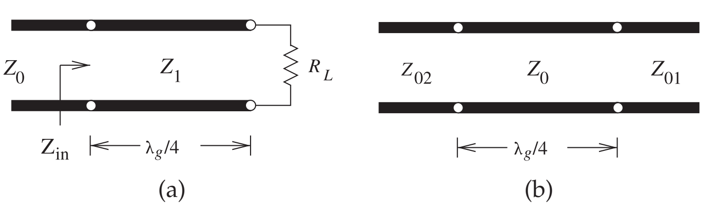


Figure 2-25: The quarter-wave transformer line: (a) transforming a load; and (b) interfacing two lines.

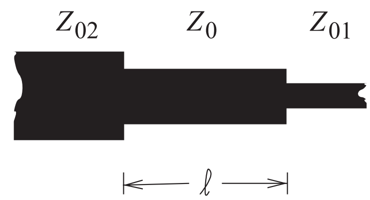


Figure 2-26: Layout of a microstrip quarter-wave transformer.

impedance of the line is

$$Z_{\text{in}} = Z_1 \frac{R_L + \jmath Z_1 \tan(\beta\ell)}{Z_1 + \jmath R_L \tan(\beta\ell)} = Z_1 \frac{R_L + \jmath Z_1 \infty}{Z_1 + \jmath R_L \infty} = \frac{Z_1^2}{R_L} \, . \tag{2.129}$$

The input impedance is matched to the transmission line Z_0 if

$$Z_{\text{in}} = Z_0^* = Z_0, \tag{2.130}$$

since here the characteristic impedance is real. Thus

$$Z_1 = \sqrt{Z_0 R_L} \tag{2.131}$$

and so the one-quarter wavelength long line acts as an ideal impedance transformer.

Another example of the quarter-wave transformer is shown in Figure 2-25(b). The input impedance looking into the quarter-wave transformer (from the left) is given by

$$Z_{\text{in}} = Z_0 \frac{Z_{01} + \jmath Z_0 \tan(\beta\ell)}{Z_0 + \jmath Z_{01} \tan(\beta\ell)} = Z_0 \frac{Z_{01} + \jmath Z_0 \infty}{Z_0 + \jmath Z_{01} \infty} = \frac{Z_0^2}{Z_{01}}. \tag{2.132}$$

Hence a section of transmission line of length $\ell = \lambda_g/4 + n\lambda_g/2$, where $n = 0, 1, 2, \ldots$, can be used to match lines having different impedances, Z_{01} and Z_{02}, by constructing the line so that its characteristic impedance is

$$Z_0 = \sqrt{Z_{01} Z_{02}}. \tag{2.133}$$

Note that for a design center frequency f_0, the matching section provides a perfect match only at the center frequency and at frequencies where $\ell = \lambda_g/4 + n\lambda_g/2$.

The layout of a microstrip quarter-wave transformer is shown in Figure 2-26, where $\ell = \lambda_g/4$ and the characteristic impedance of the transformer, Z_0, is the geometric mean of the impedances on either side, that is, $Z_0 = \sqrt{Z_{01} Z_{02}}$.

A quarter-wave transformer has an interesting property that is widely used. Examine the final result in Equation (2.132), which is repeated here:

$$Z_{\text{in}} = \frac{Z_0^2}{Z_{01}} \, . \tag{2.134}$$

Equation (2.134) indicates that a one-quarter wavelength long line is an impedance inverter presenting, at Port 1, the inverse of the impedance presented at port 2, Z_{01}. This result also applies to complex impedances replacing Z_{01}. This impedance inversion is scaled by the square of the characteristic impedance of the line. This inversion holds in the reverse direction as well.

2.4.7 Summary

The lossless transmission line configurations considered in this section are those most commonly used in microwave circuit design. It is important to note that the stub line is almost always used in shunt configuration to provide an admittance in a circuit. Most transmission line technologies, including coaxial lines and microstrip, only permit shunt stubs. The quarter-wave transformer is a particularly interesting element enabling maximum power transfer from a source to a load that may be different. An interesting feature that is widely exploited is that the quarter-wave transformer inverts an impedance. For example, turning a small resistance into a large resistance, or even turning a small capacitor into a large inductance. These transformations are valid over a moderate bandwidth.

2.5 The Lossy Terminated Line

Previously Section 2.3 presented abstractions that enabled a total voltage and current view to be used with lossless transmission lines. A similar development is presented here for lossy lines. Important abstractions are presented first for the input reflection coefficient of a terminated lossy line in Section 2.5.1 and then for the input impedance of first a long lossy line in Section 2.5.2 and then for a finite length line in Section 2.5.3. Section 2.5.4 presents a simple approximation for the attenuation on a line if it is low loss. Power flow on a lossy line is considered in Section 2.5.5 and then the impact of dispersion on signal integrity considered in Section 2.5.6. The final section, Section 2.5.7, describes a technique for the design of a finite bandwidth dispersion-less line.

2.5.1 Input Reflection Coefficient of a Lossy Line

Γ_{in} of a lossy line can be developed by replacing $\jmath\beta$ in Section 2.3.3 by γ. Referring to Figure 2-16, at a distance ℓ from the load (i.e., $z = -\ell$), the input reflection looking into a lossy line toward the load is

$$\begin{aligned}\Gamma_{\text{in}}|_{z=-\ell} &= \frac{V^-(z=-\ell)}{V^+(z=-\ell)} = \frac{V^-(z=0)\mathrm{e}^{-\gamma\ell}}{V^+(z=0)\mathrm{e}^{+\gamma\ell}} = \frac{V^-(z=0)}{V^+(z=0)}\frac{\mathrm{e}^{-\gamma\ell}}{\mathrm{e}^{+\gamma\ell}} \\ &= \Gamma_L\mathrm{e}^{-2\gamma\ell} = \Gamma_L\mathrm{e}^{-2\alpha\ell}\mathrm{e}^{-2\jmath\beta\ell} \end{aligned} \tag{2.135}$$

As the line becomes longer the magnitude of Γ_{in}decreases exponentially, approaching zero, because of attenuation described by the $\mathrm{e}^{-2\alpha\ell}$ term.

2.5.2 Input Impedance of a Long Lossy Line

Figure 2-27(a) shows an infinitely long line with characteristic impedance Z_0. The input impedance Z_{in} of the line is the ratio of the total voltage V_1 to the total current I_1 at the input of the line:

$$Z_{\text{in}} = \frac{V_1}{I_1}. \tag{2.136}$$

If the line is infinitely long or sufficiently lossy there is negligible reflected wave and thus the total voltage and current are just the forward-traveling

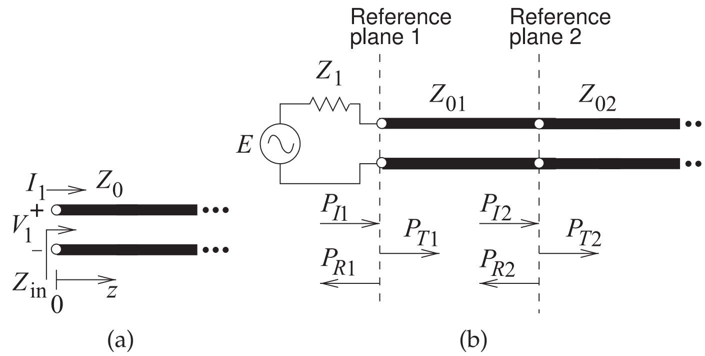


Figure 2-27: Transmission line networks: (a) an infinitely long line; and (b) with a finite-length line of characteristic impedance Z_{01} and an infinitely long transmission line of characteristic impedance Z_{02}.

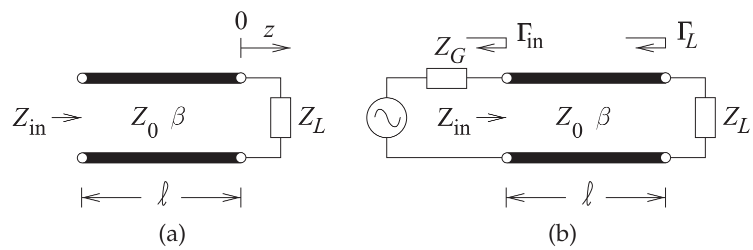


Figure 2-28: Terminated transmission line: (a) a transmission line terminated in a load impedance, Z_L, with an input impedance of Z_{in}; and (b) a transmission line with source impedance Z_G and load Z_L.

voltage and current and

$$Z_{in} = \frac{V_1}{I_1} = \frac{V^+(0)}{I^+(0)} = Z_0. \tag{2.137}$$

The infinitely long line is approximated by a very long, slightly lossy, cable. It will not matter how the line is terminated (e.g., in a resistor, open circuit, or short circuit), there will be a negligible backward-traveling wave, and the input impedance of the cable will be its characteristic impedance.

2.5.3 Input Impedance of a Lossy Line

The impedance looking into the line varies with position, as the forward- and backward-traveling waves combine to yield position-dependent total voltage and current. At a distance ℓ from the load (i.e., $z = -\ell$), the input impedance seen looking toward the load is

$$Z_{in}|_{z=-\ell} = \frac{V(z=-\ell)}{I(z=-\ell)} = Z_0 \frac{1 + |\Gamma|\, e^{(\jmath\Theta - 2\gamma\ell)}}{1 - |\Gamma|\, e^{(\jmath\Theta - 2\gamma\ell)}} = Z_0 \frac{1 + \Gamma_L e^{-2\gamma\ell}}{1 - \Gamma_L e^{-2\gamma\ell}}. \tag{2.138}$$

Another form comes from substituting Equation (2.59) in Equation (2.138):

$$\begin{aligned} Z_{in} &= Z_0 \frac{(Z_L + Z_0)e^{\gamma\ell} + (Z_L - Z_0)e^{-\gamma\ell}}{(Z_L + Z_0)e^{\gamma\ell} - (Z_L - Z_0)e^{-\gamma\ell}} = Z_0 \frac{Z_L \cosh(\gamma\ell) + \jmath Z_0 \cosh(\gamma\ell)}{Z_0 \cosh(\gamma\ell) + \jmath Z_L \cosh(\gamma\ell)} \\ &= Z_0 \frac{Z_L + Z_0 \tanh \gamma\ell}{Z_0 + Z_L \tanh \gamma\ell}. \end{aligned} \tag{2.139}$$

This equation is also known as the **lossy telegrapher's equation**.

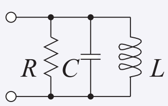
Note that Z_{in} is a quasi-periodic function of ℓ and approaches Z_0 for a long lossy line (i.e. provided that there is attenuation and the line's γ has a real part as then $\tanh \gamma\ell$ goes to one).

EXAMPLE 2.15 Transmission Line Resonator

A shorted line is used as a resonator. The first resonance is a parallel resonance at 1 GHz.

(a) Draw the lumped-element equivalent circuit of the resonator.
(b) What is the impedance looking into the line at resonance?
(c) What is the electrical length of the resonator?
(d) If the resonator is $\lambda_g/4$ longer, what is the input impedance of the resonator now?

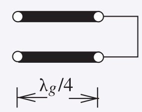
Solution:

(a) 

LC is an open circuit at resonance.
If the line is lossless, $R = 0$.

(b) $Z_{\text{in}} = \infty$ (for a lossless line).

(c) From Equation (2.71) and with $Z_L = 0$, $Z_{\text{in}} = \jmath Z_0 \tan(\beta\ell)$. $Z_{\text{in}} = \infty$ when $\tan(\beta\ell) = \infty$, i.e. when $\beta\ell = \pi/2 = \lambda_g/4 = 90°$.



(d) $Z_{\text{in}} = 0\ \Omega$.

2.5.4 Attenuation on a Low-Loss Line

Recall that γ, the propagation constant, is given by

$$\gamma = \sqrt{(R + \jmath\omega L)(G + \jmath\omega C)}. \tag{2.140}$$

This can be written as

$$\gamma = \jmath\omega\sqrt{LC}\sqrt{\left(1 + \frac{R}{\jmath\omega L}\right)\left(1 + \frac{G}{\jmath\omega C}\right)}. \tag{2.141}$$

With a low-loss line, $R \ll \omega L$ and $G \ll \omega C$, and so, using a Taylor series approximation (see Equation (1.174)),

$$\left(1 + \frac{R}{\jmath\omega L}\right)^{1/2} \approx 1 + \tfrac{1}{2}\frac{R}{\jmath\omega L} \tag{2.142}$$

and

$$\left(1 + \frac{G}{\jmath\omega C}\right)^{1/2} \approx 1 + \tfrac{1}{2}\frac{G}{\jmath\omega C}, \tag{2.143}$$

thus

$$\gamma \approx \tfrac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right) + \jmath\omega\sqrt{LC}. \qquad g \tag{2.144}$$

Hence for low-loss lines (in Np/m if SI units are used),

$$\alpha \approx \tfrac{1}{2}\left(\frac{R}{Z_0} + GZ_0\right) \quad (2.145) \qquad \text{and} \qquad \beta \approx \omega\sqrt{LC}\,. \quad (2.146)$$

What Equation (2.145) indicates is that for low-loss lines the attenuation constant, α, consists of dielectric- and conductor-related parts; that is,

$$\alpha = \alpha_d + \alpha_c \quad (2.147) \qquad \alpha_d \approx GZ_0/2 \quad (2.148) \qquad \text{and} \qquad \alpha_c \approx R/(2Z_0) \quad (2.149)$$

where α_d is the attenuation contributed by dielectric loss and is called dielectric attenuation, and α_d is the attenuation contributed by the conductor loss and is called ohmic or conductive attenuation.

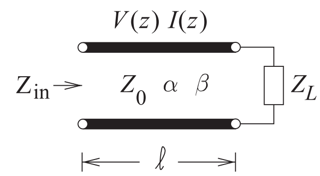


Figure 2-29: A low-loss transmission line. The propagation constant $\gamma = \alpha + \jmath\beta$.

2.5.5 Power Flow on a Terminated Lossy Line

In this section a lossy transmission line with low loss is considered so that $R \ll \omega L$ and $G \ll \omega C$, and the characteristic impedance is $Z_0 \approx \sqrt{L/C}$. Figure 2-29 is a lossy transmission line and the total voltage and current at any point on the line are given by

$$V(z) = V_0^+ \left[e^{-\gamma z} + \Gamma e^{\gamma z}\right] \quad \text{and} \quad I(z) = \frac{V_0^+}{Z_0}\left[e^{-\gamma z} - \Gamma e^{\gamma z}\right]. \tag{2.150}$$

Section 2.5.3 derived the lossy telegrapher's equation:

$$Z_{in} = Z_0 \frac{Z_L + Z_0 \tanh \gamma\ell}{Z_0 + Z_L \tanh \gamma\ell}. \tag{2.151}$$

For a lossy transmission line not all of the power applied at the input will be delivered to the load as power will be lost on the line due to attenuation. The power delivered to the load (which is at position $z = 0$) is

$$P_L = \frac{1}{2}\mathcal{R}\{V(0)I^*(0)\} = \frac{|V_0^+|^2}{2Z_0}(1- \mid \Gamma_L \mid^2), \tag{2.152}$$

where Γ_L is the reflection coefficient of the load. Similarly the power at the input of the line (at position $z = -\ell$) is

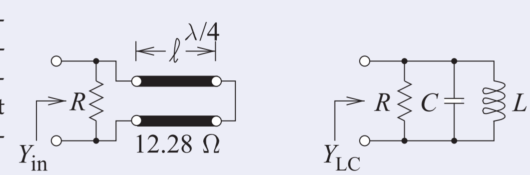
$$P_{in} = \frac{1}{2}\Re\{V(-\ell)I^*(-\ell)\} = \frac{|V_0^+|^2}{2Z_0}\left[1- \mid \Gamma_L \mid^2 e^{-4\alpha\ell}\right] e^{2\alpha\ell} \tag{2.153}$$

and the power lost in the line is

$$P_{loss} = P_{in} - P_L = \frac{|V_0^+|^2}{2Z_0}\left[\left(e^{2\alpha\ell} - 1\right) + \mid \Gamma_L \mid^2 \left(1 - e^{-2\alpha\ell}\right)\right]. \tag{2.154}$$

EXAMPLE 2.16 Q of a Transmission Line Resonator Revisited

Energy is stored on a transmission line and this must be considered in deriving the Q of a transmission line resonator. Consider the $\lambda/4$-long shorted transmission line examined in Examples 2.5 and 2.11. This line has a characteristic impedance of 12.28 Ω, it is resonant at 1850 MHz, and a 10 kΩ resistor, R, is at the input as shown in the left figure.



Solution:

At resonance

$$Q = 2\pi \frac{\text{Peak energy stored}}{\text{Energy dissipated per cycle}} = \omega_r \frac{\text{Peak energy stored}}{\text{Power loss in the resistor}} \tag{2.155}$$

$$Q = 1/(\text{Fractional bandwidth}), \tag{2.156}$$

where ω_r is the radian frequency at resonance. So there are two fundamental ways that the Q can be determined: determining the peak energy stored and the power loss, and finding the fractional bandwidth. However, using the fractional bandwidth is only an approximate method, as Q is a measure of energy stored and energy dissipated.

For an RLC lumped-element resonator

$$Q = \begin{cases} \dfrac{1}{R}\sqrt{\dfrac{L}{C}} = \dfrac{\omega_r L}{R} & \text{for a series } RLC \text{ circuit} \\ R\sqrt{\dfrac{C}{L}} = \dfrac{\omega_r C}{G} & \text{for a shunt } RLC \text{ circuit.} \end{cases} \tag{2.157}$$

The narrowband lumped-element equivalent circuit of the resonator developed in Example 2.5, with C = 5.503 pF and L = 1.345 pH can be used. Thus

$$Q = R\sqrt{\frac{C}{L}} = 10 \cdot 10^3 \sqrt{\frac{5.503 \cdot 10^{-12}}{1.345 \cdot 10^{-9}}} = 640.$$

It is important to note that the input impedance of the transmission line at the resonant frequency cannot be used to find the Q of the transmission line resonator, as it does not convey any information about the energy stored. However, the narrowband model of the resonator at resonance does capture the energy storage information and so can be used to calculate the Q of the resonator, as was done here.

2.5.6 Lossy Transmission Line Dispersion

On a lossy line, phase velocity, group velocity, and attenuation constant are frequency dependent and so a lossy line is, in general, dispersive. That is, different frequency components of a signal travel at different speeds, and the phase velocity, v_p, and group velocity, v_g, are functions of frequency. As a result, the signal will spread out in time and, if the line is long enough, it will be difficult to extract the original information.

In the previous section it was seen, in Equation (2.146), that for a TEM line $\omega/\beta = v_p = v_g$ is approximately frequency independent for a low-loss line. This is true for most two-conductor lines (lines that support TEM modes), but not for all transmission line structures that guide EM waves (a rectangular waveguide is an example of where v_g and v_p differ significantly). Also, the conductive component of the attenuation constant, α_c in Equation (2.149), is approximately frequency independent. However, the dielectric component, α_d in Equation (2.148), is frequency dependent even for a low-loss line. This is because G is mostly due to energy loss when a material lattice or molecular components are distorted by the E field. So there is more loss and G increases linearly as the frequency increases. (Conductivity of the dielectric also affects G, but this is usually a much smaller effect except for a silicon substrate.) If the transmission line has moderate loss, all of the propagation parameters will be frequency dependent and the line is dispersive.

2.5.7 Design of a Dispersionless Lossy Line

Over a moderate bandwidth a lossy line can be designed to be dispersionless (i.e. $v_g \approx v_p \approx$ constant). The parameters that are important in describing signal propagation on a transmission line are the propagation constant, γ, and the characteristic impedance, Z_0. When considering dispersion it is more appropriate to examine α and $v_p \approx \omega/\beta$ (for a low loss line), as these parameters are generally frequency dependent for a lossy line. For a line to be dispersionless α, ω/β, and Z_0 should be independent of frequency.

For any low-loss line the propagation constant is

$$\gamma = \sqrt{(R + \jmath\omega L)(G + \jmath\omega C)} = \jmath\omega\sqrt{LC}\left[\left(1 + \frac{R}{\jmath\omega L}\right)\left(1 + \frac{G}{\jmath\omega C}\right)\right]^{1/2}. \tag{2.158}$$

If the line is designed so that

$$R/L = G/C, \tag{2.159}$$

then

$$\gamma = \alpha + \jmath\beta = \jmath\omega\sqrt{LC}\left(1 + \frac{R}{\jmath\omega L}\right) = R\sqrt{\frac{C}{L}} + \jmath\omega\sqrt{LC}.$$

From this

$$\alpha = R\sqrt{\frac{C}{L}}, \quad \beta = \omega\sqrt{LC} \text{ and } v_p = \omega/\beta = 1/\sqrt{LC}. \tag{2.160}$$

The analysis is completed by considering the characteristic impedance

$$Z_0 = \sqrt{\frac{R + \jmath\omega L}{G + \jmath\omega C}} = \sqrt{\frac{L}{C}}\sqrt{\frac{R/L + \jmath\omega}{G/C + \jmath\omega}}, \tag{2.161}$$

and, referring to Equation (2.159), note that the last square root term is 1, so

$$Z_0 = \sqrt{L/C}, \tag{2.162}$$

which is frequency independent. The important characteristics describing signal propagation are now frequency independent and the line is dispersionless. In practice a lossy dispersionless line can only be approximated over a small bandwidth since G is linearly dependent on frequency because of the frequency dependence of dielectric relaxation loss.

2.5.8 Summary

An earlier section developed the input reflection coefficient and input impedance of a lossless line. This section did the same thing but for a lossy line. Short-hand derivations were presented for the attenuation of a low loss line in terms of the per-unit-length resistance and characteristic impedance of the line. Dispersion on a line was discussed and a design technique presented for the design of a dispersionless line. This design is valid over a narrow or moderate bandwidth and this bandwidth limitation applies to all transmission line-based designs.

2.6 Reflections at Interfaces

Transmission lines transfer power from one point to another and there are often many interfaces at which power is reflected and transferred. This section is focused on developing an intuitive understanding of reflection and transmission of power at interfaces and over a transmission line. Section 2.6.1 presents an understanding of power flow and introduces the return loss concept applying this analysis to a single interface in Section 2.6.3. Microwave engineering is greatly concerned with maximum power transfer and the analysis that supports design choices, the maximum power transfer theorem, is revisited in Section 2.6.2. Section 2.6.4 presents bounce diagram analysis which is concerned with reflection and transmission at multiple interfaces. This can be a confusing analysis but it provides the essential understanding for the development of intuition for how power flows, and understanding of situations where power flow may be disrupted. The final section, Section 2.6.5, presents the theory of small reflections, derived from bounce diagram analysis, which is used in several places in this book series in the synthesis of transmission line networks.

2.6.1 Power Flow and Return Loss

Now consider the power flow on the lossless line in Figure 2-13. The incident (forward-traveling) wave has the power

$$\begin{aligned} P^+ &= \frac{1}{2}\Re\Big[V^+\left(I^+\right)^*\Big] = \frac{1}{2}\Re\left[V^+\left(\frac{V^+}{Z_0}\right)^*\right] \\ &= \frac{1}{2}\Re\left\{\left(V_0^+\,\mathrm{e}^{-\jmath\beta z}\right)\left(\frac{V_0^+\,\mathrm{e}^{-\jmath\beta z}}{Z_0}\right)^*\right\} = \frac{1}{2}\frac{\left|V_0^+\right|^2}{Z_0}, \end{aligned} \tag{2.163}$$

and the reflected wave has the power (using Equation (2.60))

$$P^- = \frac{1}{2}\frac{\left|V_0^-\right|^2}{Z_0} = \frac{|\Gamma|^2}{2}\frac{\left|V_0^+\right|^2}{Z_0}. \tag{2.164}$$

Considering conservation of power, the power delivered to the load, P_L, is the difference of the forward- and backward-traveling powers:

$$P_L = \frac{1}{2}\Re\{V_L I_L^*\} = P^+ - P^- = P^+\Big(1 - |\Gamma|^2\Big). \tag{2.165}$$

Noteworthy cases are when there is an open circuit, a short circuit, or a purely reactive load at the end of a transmission line. These have $|\Gamma| = 1$. Thus all power is reflected back to the source and $P_L = 0$.

The power that is absorbed by the load appears as a loss as far as the incident and reflected waves are concerned. To describe this, the concept of return loss (RL) is introduced and defined as

$$\mathrm{RL} = -20\log|\Gamma|\ \mathrm{dB}\,. \tag{2.166}$$

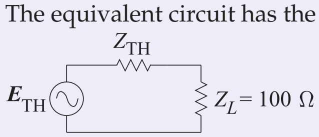
RL indicates the available power not delivered to the load. A matched load ($\Gamma = 0$) has $\mathrm{RL} = \infty$ dB, and a total reflection ($|\Gamma| = 1$) has $\mathrm{RL} = 0$ dB.

EXAMPLE 2.17 Transmission Line Model at the Load

A transmission line with a characteristic impedance of 75 Ω supports a forward-traveling wave with a power of 1 μW. The line is terminated in a resistance of 100 Ω. Draw the lumped-element equivalent circuit at the interface between the transmission line and the load.

Solution:

The equivalent circuit has the form

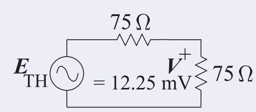


where E_{TH} is the Thevenin equivalent generator and Z_{TH} is the Thevenin equivalent generator impedance.

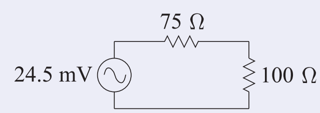
The amplitude of the forward-traveling voltage wave is obtained by calculating the power in the forward-traveling wave:

$$P^+ = \tfrac{1}{2}(V^+)^2/Z_0 = (V^+)^2/150 = 1\ \mu\text{W} = 10^{-6}\ \text{W}.$$

So $V^+ = \sqrt{150 \cdot 10^{-6}} = 12.25$ mV. Note that V^+ is not E_{TH}. To calculate E_{TH}, consider the circuit to the right that results in maximum power transfer.



So $E_{\mathrm{TH}} = 2V^+ = 24.5$ mV. Since the line has a characteristic impedance of 75 Ω, then $Z_{\mathrm{TH}} = 75\ \Omega$. So the lumped-element equivalent circuit at the load is



2.6.2 Maximum Power Transfer Theorem

Many transmission line calculations can be solved using the concepts of maximum available power, incident power, and reflected power. At microwave frequencies the output impedance of sources, for example, cannot be ignored as can often be done at low frequencies. Thus the output of an RF source is defined in terms of maximum available power.

Consider the source shown in Figure 2-30 with an impedance $Z_S = R_S + \jmath X_S$ driving a load $Z_L = R_L + \jmath X_L$. The aim here is to find Z_L for maximum delivery of power to the load. The phasor current in the load is

$$I = \frac{E}{Z_S + Z_L} \tag{2.167}$$

and thus the average power in the load is

$$P_L = \tfrac{1}{2}|I|^2 R_L = \tfrac{1}{2}\left(\frac{|E|}{|Z_S + Z_L|}\right)^2 R_L = \frac{\tfrac{1}{2}|E|^2 R_L}{(R_S + R_L)^2 + (X_S + X_L)^2}. \tag{2.168}$$

The load required for maximum power transfer is obtained by first considering a fixed value of R_L and then finding the value of X_L required to minimize

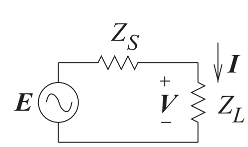


Figure 2-30: A source terminated in a load. Maximum power transfer occurs when $Z_L = Z_S^*$.

Equation (2.168). Since X_L only affects the denominator in Equation (2.168), it is clear that the denominator is minimized by making $X_L = -X_S$. So now Equation (2.168) reduces to

$$P_L = \frac{\frac{1}{2}|E|^2 R_L}{(R_S + R_L)^2} \quad (2.169) \quad \text{and} \quad P_L = \frac{\frac{1}{2}|E|^2}{(R_S^2/R_L + 2R_S + R_L)}. \quad (2.170)$$

Now the power delivered to the load is maximized by minimizing the denominator in Equation (2.170), which will occur when the derivative of the denominator in Equation (2.170) is zero. That is, when

$$\frac{d}{dR_L}\left(R_S^2/R_L + 2R_S + R_L\right)^2 = \frac{-R_S^2}{R_L^2} + 1 = 0. \quad (2.171)$$

The derivative is zero when

$$R_S^2/R_L^2 = 1, \quad \text{that is, when} \quad R_L = \pm R_S. \quad (2.172)$$

When the derivative is zero, the power transferred to the load is either the minimum or maximum power that can be delivered to the load. Clearly $R_L = -R_S$ is a nonsensical solution as the load and source resistance should be positive. In addition, the situations at the extremes need to be check, that is, when R_L is very small, $R_L \to 0$, and when R_L is very large, $R_L \to \infty$. As $R_L \to 0$, Equation (2.169) becomes $P_L \to 0$, and as $R_L \to \infty$, Equation (2.169) becomes $P_L \to 0$. Clearly there will not be negative power dissipated in R_L, so when $R_L = R_S$ the maximum power, $P_L|_{\max}$, is dissipated in the load. Equation (2.169) becomes

$$P_L|_{\max} = \frac{1}{2}\frac{|E|^2 R_L}{(R_S + R_S)^2} = \frac{1}{8}\frac{|E|^2}{R_S}. \quad (2.173)$$

This is called the maximum available power, or simply the **available power** of the source. So the conditions for maximum power transfer are $R_L = R_S$ and $X_L = -X_S$; that is, the maximum power transfer from source to load requires

$$Z_L = Z_S^*. \quad (2.174)$$

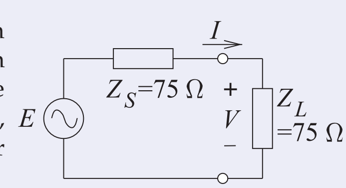
Also note that if the source impedance in Figure 2-30 is resistive, $V = \frac{1}{2}E$ at maximum power transfer.

EXAMPLE 2.18 Available Power

A 75 Ω source with an available power of 1 W is terminated in a short circuit. What is the power dissipated in the Thevenin equivalent resistance of the source?

Solution:

At RF, "75 Ω source" refers to a source with a Thevenin equivalent resistance of 75 Ω which here is Z_S. The condition for maximum power transfer, and when the full available power is delivered to the load, is when the load impedance, Z_L is the complex conjugate of $Z_S = 75\ \Omega$. The circuit for maximum power transfer is as shown to the right.



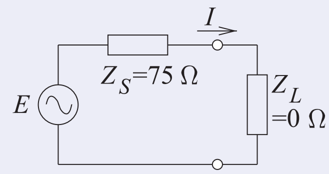
To solve the problem we must determine E as this will not change even though the current through Z_S, I, will depend on the load impedance. The available power is

$P_A = \frac{1}{2}|I|^2\Re(Z_S) = \frac{1}{2}|I|^2 \times 75 = 1$ W and so $I = \sqrt{2 \times 1/75} = 0.1633$ A.

From this the voltage across the load can be determine:

$V = IZ_L = 0.1633 \times 75 = 12.25$ V and from symmetry $E = 2V = 24.50$ V.

Now the condition with a short circuit, $Z_L = 0$, can be determined. Note that I and V will change but E will be fixed. Considering the circuit to the right, now



$I = E/(Z_S + Z_L) = 24.50/(75 + 0) = 0.3267$ A.

The power dissipated in Z_S is

$P_S = \frac{1}{2}I^2\Re(Z_S) = 0.3267 \times 75 = 4.002$ W.

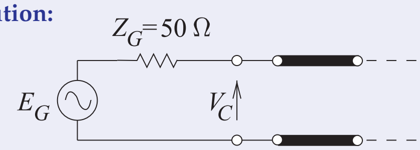
Some comments on this result can be made. With a short circuit the current that flows in the circuit doubles compared to the maximum power transfer condition. Since the power dissipated in Z_S is proportion to the square of the current the power dissipated in the Thevenin equivalent resistance increases by a factor of 4 (P_S should be exactly 4 W). The ideal voltage source E can provide unlimited power. However no matter what the value of Z_L, the power dissipated in Z_L can never be more than the available power.

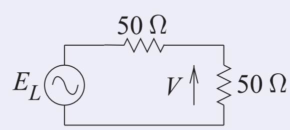
EXAMPLE 2.19 Forward-Traveling Wave Calculations

A transmission line is driven by a generator with a maximum available power of 20 dBm and a Thevenin equivalent impedance of 50 Ω. The transmission line has a characteristic impedance of 50 Ω.

(a) What is the Thevenin equivalent generator voltage?
(b) What is the magnitude of the forward-traveling voltage wave on the line? Assume that the line is infinitely long.
(c) What is the power of the forward-traveling voltage wave?

Solution:





(a) Maximum available power is delivered to the load when it is conjugately matched to the generator impedance (see right diagram). Then $P_{\rm LOAD} = \frac{1}{2}V^2/R$ (V is peak voltage) (i.e., $P_{\rm LOAD} = 20$ dBm $= 0.1$ W) and

$$V = \sqrt{2P_{\rm LOAD} \cdot R} = \sqrt{2 \times 0.1 \times 50}\ {\rm V_{peak}} = 3.16\ {\rm V}, \qquad E_G = 2V = 6.32\ {\rm V}.$$

(b) This is just V, so $V^+ = 3.16$ V. (c) $P^+ = 20$ dBm.

2.6.3 A Single Interface

Many transmission line calculations are most conveniently addressed using the maximum available power concept. This is particularly so when there are multiple transmission lines. The concept is that the maximum available power from a source is incident on a load and if there is a mismatch, power is reflected and the rest of the power is transmitted. The circuits used to illustrate these calculations are shown in Figure 2-31. The available power of the source is $P_{\rm av}$, which is incident on the reference plane shown in Figure 2-31(b) where it is called the incident power P_I. At the reference plane power

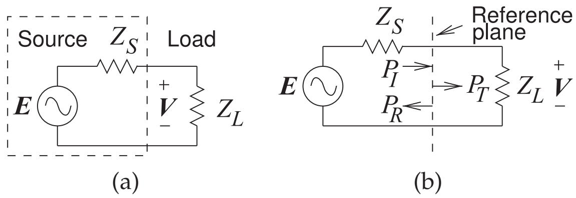


Figure 2-31: Calculations using incident and reflected power: (a) source terminated in a load; and (b) reference plane illustrating the use of incident and reflected power at a load.

is reflected, P_R, and power is transmitted, P_T. Using the reflection coefficient normalized to the impedance of the source (i.e., Z_S)

$$P_R = |\Gamma|^2 P_I, \qquad (2.175) \qquad \text{where} \quad \Gamma = \frac{Z_S - Z_L}{Z_S + Z_L} \qquad (2.176)$$

and the transmitted power is

$$P_T = \left(1 - |\Gamma|^2\right) P_I. \qquad (2.177)$$

EXAMPLE 2.20 Transmission Line Power Calculations

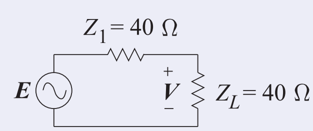
The transmission line shown in Figure 2-27(b) consists of a source with Thevenin impedance $Z_1 = 40\ \Omega$ and source $E = 5$ V (peak), connected to a $\lambda/4$ long line of characteristic impedance $Z_{01} = 40\ \Omega$, which in turn is connected to an infinitely long line of characteristic impedance $Z_{02} = 100\ \Omega$. The transmission lines are lossless. Two reference planes are shown in Figure 2-27(b). At reference plane 1 the incident power is P_{I1} (the maximum available power from the source), the reflected power is P_{R1}, and the transmitted power is P_{T1}. P_{I2} (the maximum available power from Z_{01}), P_{R2}, and P_{T2} are similar quantities at Reference Plane 2. P_{I1}, P_{R1}, P_{T1}, P_{I2}, P_{R2}, and P_{T2} are steady-state quantities.

(a) What is P_{I1}?
(b) What is P_{T2}?

Solution: Since the infinitely long line does not have a backward-traveling wave this problem reduces to a single transmission line interface problem.

First develop some expectations. This will be a sanity check during the problem. P_{I1} and P_{I2} are maximum available powers and since the Z_{01} line (this is a short way of talking about the line with a characteristic impedance Z_{01}) is lossless they should be equal: $P_{I1} = P_{I2}$. P_{T1} and P_{T2} are the total powers delivered to the right of the respective interfaces. Again since the Z_{01} line is lossless $P_{T1} = P_{T2}$, Also $P_{T1} \leq P_{I1}$ as power P_{R1} is reflected from the interface. Similarly $P_{T2} \leq P_{I2}$.

(a) P_{I1} is the available power from the generator. Since the Thevenin impedance of the generator is 40 Ω, P_{I1} is the power that would be delivered to a matched load (the maximum available power). An equivalent problem is shown to the right where $V = \frac{1}{2}E = 2.5\ \mathrm{V_{peak}}$. So

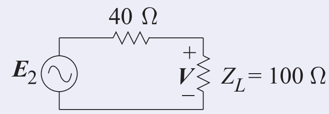


$$P_{I1} = \text{power in } Z_L = \tfrac{1}{2}(V)^2 \frac{1}{Z_L} = \tfrac{1}{2}(2.5)^2 \frac{1}{40} = 0.07813\ \mathrm{W} = 78.13\ \mathrm{mW}. \qquad (2.178)$$

Note that the $\frac{1}{2}$ occurs because peak voltage is used in RF calculations.

(b) Now the problem becomes interesting and there are many ways to solve it. One of the key observations is that the first transmission line has the same characteristic impedance as the Thevenin equivalent impedance of the generator, $Z_{01} = Z_1$, and so can be ignored where appropriate. This observation will be used in this example. One way to proceed is to directly calculate P_{T2}, and a second approach is to calculate the incident and reflected powers at reference plane 2 and then to determine P_{T2}.

(i) First approach: looking to the left from reference plane 2, the circuit can be modeled as an equivalent circuit having a Thevenin equivalent resistance of 40 Ω and a Thevenin equivalent voltage that has an available power of 78.13 mW. So in the circuit to the right, E_2 is just E or 5 V.



The load is 100 Ω as the second transmission line is infinitely long. A reasonable question to ask is why E_2 is not 2.5 V, as this would be the voltage across Z_L = 40 Ω in part (a). However, 2.5 V is the voltage of the forward-traveling voltage wave on the first transmission line with characteristic impedance Z_{01} = 40 Ω. It is not the Thevenin equivalent voltage of the source. The voltage across the load is

$$V = E_2 \frac{100}{40 + 100} = E\frac{100}{140} = 3.57\ \mathrm{V_{peak}}. \tag{2.179}$$

The power transmitted at reference plane 2 is also the power delivered to the load:

$$P_{T2} = P_L = \tfrac{1}{2}(V)^2 \frac{1}{Z_L} = \frac{1}{2}(3.57)^2 \frac{1}{100} = 0.0638\ \mathrm{W} = 63.8\ \mathrm{mW}. \tag{2.180}$$

A quick check is that this is less than P_{I1}, as it should be.

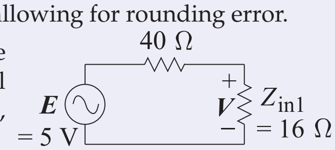
(ii) Second approach: This time P_{R2}, the reflected power at reference plane 2, will be calculated. The incident power at plane 2, P_{I2}, is just P_{I1}. P_{I2} is the maximum available power at reference plane 2 and not necessarily the power that is incident there. In general, to calculate P_{I2} the Thevenin equivalent source looking to the left from reference plane 2 would need to be calculated. However since here $Z_{01} = Z_1$, $P_{I2} = P_{I1} = 78.13$ mW.
P_{R2} can be calculated from the voltage reflection coefficient at reference plane 2:

$$\Gamma_2 = \frac{Z_L - Z_{01}}{Z_L + Z_{01}} = \frac{100 - 40}{100 + 40} = 0.429 \tag{2.181}$$

$$P_{R2} = \Gamma_2^2 P_{I2} = 0.429^2 \times 78\ \mathrm{mW} = 14.36\ \mathrm{mW}. \tag{2.182}$$

So $P_{T2} = P_{I2} - P_{R2} = 78.13\ \mathrm{mW} - 14.36\ \mathrm{mW} = 63.7$ mW, which is the same as the transmitted power calculated in the first approach, allowing for rounding error.

(iii) A third approach is to calculate the input impedance looking to the right from reference plane 1, call this $Z_{\text{in}1}$. Using the lossless telegrapher's equation, Equation 2.71, $Z_{\text{in}1}$ is calculated to be 16 Ω.



The voltage across the 16 Ω resistor is $16/(40 + 16)(5\ \mathrm{V})$ = 1.429 V. So the power dissipated in $Z_{\text{in}1}$ is

$$P = \tfrac{1}{2}(1.429\ \mathrm{V})^2 \frac{1}{16} = 0.0638\ \mathrm{W} = 63.8\ \mathrm{mW},$$

as before. All of this power must be transmitted to the infinitely long line, i.e. it is P_{T2}, as the system is otherwise lossless.

2.6.4 Bounce Diagram

A bounce diagram is a graphical representation of reflections at the interfaces between networks. A microwave network is shown in Figure 2-32 where there are two reference planes at boundaries between different parts of the network. At each boundary there are reflections and transmissions of what could be viewed as small packets of sinusoidal signals. While the bounce diagram yields a steady-state result such as the input impedance, the thought experiment is that small sinusoidal packets bounce around the transmission-line network. Bounce diagrams are used to explore the impact of multiple reflections in a network leading to understanding, and from that to design decisions. Bounce diagrams are best illustrated through an example.

Reflection and Transmission Coefficients at a Boundary

Bounce diagram analysis requires that the reflection and transmission coefficients at a boundary be referenced to a common impedance. Figure 2-33(a) shows the interface of two transmission lines of characteristic impedance Z_{01} and Z_{02} at a reference plane. The problem is determining the reflection coefficient, Γ, and transmission coefficient, T, at the boundary referencing them to the same impedance. Here they will be referenced to Z_{01}. The reflection coefficient as seen from the Z_{01} line referenced to Z_{01} is

$$\Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}. \tag{2.183}$$

Now the transmission coefficient referenced to Z_{01} is

$$T = {}^{Z01}V_2^+ / V_1^+, \tag{2.184}$$

where ${}^{Z01}V_2^+$ is the forward-traveling voltage wave on the Z_{02} line (traveling to the right) referenced to Z_{01}. It is not the actual forward-traveling voltage on the Z_{02} line. $V_1^+ = {}^{Z01}V_1^+$ is the traveling voltage wave on the Z_{01} line that is incident at the boundary. It is obtained using the circuit in Figure 2-33(b). Figure 2-33(b) shows the maximum power transfer condition so that

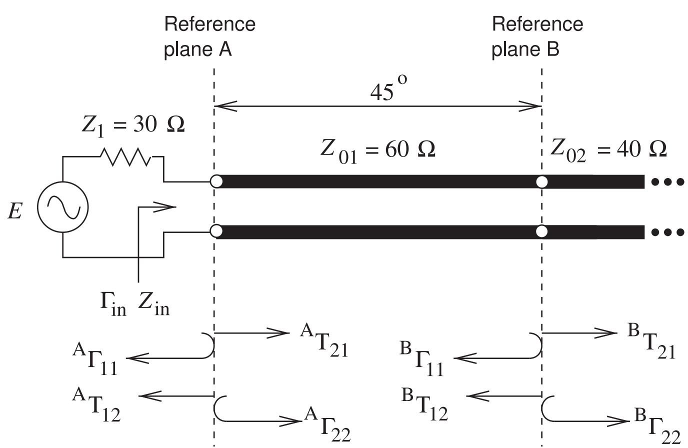


Figure 2-32: Transmission line network with a finite-length line of characteristic impedance Z_{01} and an infinitely long transmission line of characteristic impedance Z_{02}. ${}^A\Gamma_{1i}$ is the ith discrete reflection of an incident wave at the left-hand side of reference plane A. ${}^AT_{1i}$ is the transmission at reference plane A for the same event. ${}^A\Gamma_{2i}$ is the ith discrete reflection of an incident wave at the right-hand side of reference plane A.

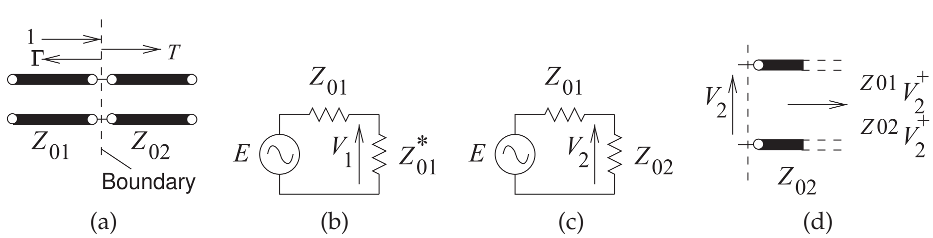


Figure 2-33: Reflection (Γ) and transmission (T) at the boundary between two transmission lines of characteristic impedance Z_{01} and Z_{02}.

the forward-traveling wave on the Z_{01} line at the left of the boundary is

$$V_1^+ = V_1 = E\frac{Z_{01}}{Z_{01} + Z_{01}^*} = E\frac{Z_{01}}{2\Re(Z_{01})}. \tag{2.185}$$

(For real impedances $V_1^+ = \frac{1}{2}E$.)

The next parameter to determine is ${}^{Z01}V_2^+$. This begins by determining the total voltage, V_2, at the boundary using the circuit in Figure 2-33(c):

$$V_2 = E\frac{Z_{02}}{Z_{01} + Z_{02}}. \tag{2.186}$$

Now consider that the Z_{02} line is infinitely long, then it is clear that $V_2 = {}^{Z02}V_2^+$. This is the actual forward-traveling voltage on the Z_{02} line. Now the problem is how to change the reference impedance of V_2^+. This is done by noting that the forward-traveling power wave is independent of the reference impedance, therefore the forward-traveling power wave on the Z_{02} line traveling to the right, away from the reference plane, is

$$P_{Z02} = \frac{1}{2}\left[\frac{\left({}^{Z02}V_2^+\right)^2}{\Re(Z_{02})}\right] = \frac{1}{2}\left[\frac{\left({}^{Z01}V_2^+\right)^2}{\Re(Z_{01})}\right]. \tag{2.187}$$

So the desired forward-traveling voltage wave is (since ${}^{Z02}V_2^+ = V_2$)

$${}^{Z01}V_2^+ = \sqrt{\frac{\Re(Z_{01})}{\Re(Z_{02})}}V_2 = E\sqrt{\frac{\Re(Z_{01})}{\Re(Z_{02})}}\left(\frac{Z_{02}}{Z_{01} + Z_{02}}\right). \tag{2.188}$$

So the transmission coefficient referenced to Z_{01} is

$$\begin{aligned} T = \frac{{}^{Z01}V_2^+}{V_1^+} &= E\sqrt{\frac{\Re(Z_{01})}{\Re(Z_{02})}}\left(\frac{Z_{02}}{Z_{01} + Z_{02}}\right)\left(\frac{2\Re(Z_{01})}{Z_{01}}\frac{1}{E}\right) \\ &= \sqrt{\frac{\Re(Z_{01})}{\Re(Z_{02})}}\left[\frac{\Re(Z_{01})}{Z_{01}}\right]\left(\frac{2Z_{02}}{Z_{01} + Z_{02}}\right). \end{aligned} \tag{2.189}$$

An alternative way of determining T is to consider power conservation at the boundary. Then

$$|T|^2 = 1 - |\Gamma|^2, \quad \text{that is} \quad |T| = \pm\sqrt{1 - |\Gamma|^2}. \tag{2.190}$$

This can be used if T is expected to be real, which it will be if Z_{01} and Z_{02} are real. The positive sign should be taken. The more general formula, Equation (2.189), can be used with complex impedances.

EXAMPLE 2.21 Bounce Diagram

In the transmission line system in Figure 2-32 the electrical length of the Z_{01} line (i.e., the line with characteristic impedance Z_{01}) is $45°$ and the Z_{02} line is infinitely long. The aim here is to find the input impedance, Z_{in}. This will be arrived at in two ways. The first technique uses a bounce diagram approach and emphasizes reflection and transmission at the interface planes. The second approach uses the telegrapher's equation.

(a) What are the reflection and transmission parameters at reference plane A?

${}^{A}\Gamma_{11}$ is the reflection coefficient of signals incident on Reference Plane A from the left. ${}^{A}T_{21}$ is the transmission coefficient at the plane for signals from the left. ${}^{A}\Gamma_{22}$ and ${}^{A}T_{12}$ are the corresponding parameters for scattering from signals coming from the right to the plane. So, normalizing to Z_1,

$$ {}^{A}\Gamma_{11} = \frac{Z_{01} - Z_1}{Z_{01} + Z_1} = 0.333; \qquad {}^{A}\Gamma_{22} = -{}^{A}\Gamma_{11} = -0.333. \tag{2.191} $$

Using Equation (2.189),

$$ {}^{A}T_{21} = \sqrt{\frac{Z_1}{Z_{01}}}\left(\frac{2Z_{01}}{Z_1 + Z_{01}}\right) = \sqrt{\frac{30}{60}}\left(\frac{2\cdot 60}{30+60}\right) = 0.943. \tag{2.192} $$

Similarly, ${}^{A}T_{12} = 0.943$, which is due to reciprocity. As an additional sanity check (this can only be done if Z_{01} and Z_{02} are real)

$$ {}^{A}T_{21} = \sqrt{1 - {}^{A}\Gamma_{11}^2} = 0.943, \qquad {}^{A}T_{12} = \sqrt{1 - {}^{A}\Gamma_{22}^2} = 0.943. \tag{2.193} $$

(b) What are the scattering parameters Γ_2 and T_2 at reference plane B?
The reference is Z_0, and so the reflection coefficient going from the Z_{01} line to the Z_{02} line is

$$ {}^{B}\Gamma_{11} = \frac{Z_{02} - Z_0}{Z_{02} + Z_0}. \tag{2.194} $$

The question now is what system reference impedance to use? Should it be Z_1, Z_{01}, or even Z_{02}? The problem could be solved using any of these, but the simplest procedure is to use the same reference impedance throughout, and since the eventual aim is to calculate the overall input reflection coefficient, the appropriate choice is $Z_0 = Z_1$. Note, however, that the actual voltage levels on the lines are not being calculated (which would need to be referenced to the characteristic impedance of the lines being considered), but instead a traveling wave referenced to a universal system impedance. So the scattering parameters at reference plane B referenced to the impedance Z_1 are

$$ {}^{B}\Gamma_{11} = \frac{Z_{02} - Z_1}{Z_{02} + Z_1} = 0.143; \qquad {}^{B}\Gamma_{22} = -{}^{B}\Gamma_{11} = -0.143 \tag{2.195} $$

$$ {}^{B}T_{21} = \sqrt{1 - {}^{A}\Gamma_{11}^2} = 0.990; \qquad {}^{B}T_{12} = \sqrt{1 - {}^{B}\Gamma_{22}^2} = 0.990. \tag{2.196} $$

The second transmission line is infinitely long and so no signal from the line will be incident at reference plane B from the right.

(c) What is the transmission coefficient of the Z_{01} transmission line referenced to Z_{01}?

T_{01} is the ratio of the forward-traveling wave at the end of the line to its value at the start of the line. Using a reference impedance of Z_{01}, the magnitude of the transmission coefficient is one and it rotates by the line's electrical length $\Theta_1 = 45°$ or 0.785 radians:

$$T_{01} = \mathrm{e}^{-\jmath\Theta_1} = \exp(-\jmath 0.785) = 0.707 - \jmath 0.707. \tag{2.197}$$

(d) Draw the bounce diagram of the transmission line network.

The bounce diagram is shown in Figure 2-34.

(e) What is Γ_{in} and hence what is Z_{in}?

Γ_{in} is the steady-state input reflection coefficient and is obtained by adding all of the signals going to the left from reference plane A in Figure 2-34. So

$$\begin{aligned}\Gamma_{\text{in}} =& {}^A\Gamma_{11} + {}^AT_{12}{}^AT_{21}T_{01}^2\,{}^B\Gamma_{11} \\ &+ {}^AT_{12}{}^AT_{21}T_{01}^4\,{}^B\Gamma_{11}^2\,{}^A\Gamma_{22} + {}^AT_{12}{}^AT_{21}T_{01}^6\,{}^B\Gamma_{11}^3\,{}^A\Gamma_{22}^2 \cdots \\ =& {}^A\Gamma_{11} + {}^AT_{12}{}^AT_{21}T_{01}^2\,{}^B\Gamma_{11}\left[1 + x + x^2 + \cdots\right],\end{aligned} \tag{2.198}$$

where $x = T_{01}^2\,{}^B\Gamma_{11}\,{}^A\Gamma_{22}$. Now $1/(1-x) = 1 + x + x^2 + \cdots$, and thus

$$\begin{aligned}\Gamma_{\text{in}} &= {}^A\Gamma_{11} + \frac{{}^AT_{12}{}^AT_{21}T_{01}^2\,{}^B\Gamma_{11}}{1 - T_{01}{}^2\,{}^B\Gamma_{11}\,{}^A\Gamma_{22}} \\ &= 0.333 + \frac{0.943 \times 0.943 \times (0.707 - \jmath 0.707)^2 \times (-0.2)}{1 - (0.707 - \jmath 0.707)^2 \times (-0.2) \times (-0.333)} \\ &= 0.345 - \jmath 0.177.\end{aligned} \tag{2.199}$$

Γ_{in} is the reflection at reference plane A and is referenced to $Z_1 = 30\ \Omega$. So the input impedance is

$$Z_{\text{in}} = Z_1\left(\frac{1+\Gamma_{\text{in}}}{1-\Gamma_{\text{in}}}\right) = 30\left(\frac{1 + 0.345 - \jmath 0.177}{1 - 0.345 + \jmath 0.177}\right) = 55.39 + \jmath 23.08\ \Omega. \tag{2.200}$$

(f) Use the lossless telegrapher's equation, Equation (2.71), to find Z_{in}.

The infinitely long line presents an impedance Z_{02} to the 60 Ω transmission line. So the input impedance looking into the 60 Ω line at reference plane A is, using the lossless telegrapher's equation,

$$Z_{\text{in}} = Z_{01}\left(\frac{Z_{02} + \jmath Z_{01}\tan\beta l}{Z_{01} + \jmath Z_{02}\tan\beta l}\right),$$

where the electrical length βl is 45° or $\pi/4$ radians. So

$$Z_{\text{in}} = 60\left(\frac{40 + \jmath 60\tan(\pi/4)}{60 + \jmath 40\tan(\pi/4)}\right) = 55.39 + \jmath 23.08\ \Omega,$$

equivalent to the result obtained using the bounce diagram method (see Equation (2.200)).

The bounce diagram technique aids in physical understanding, however using the telegrapher's equation is a less error prone approach to solving transmission line problems.

2.6.5 Theory of Small Reflections

If the discontinuity at the boundary A in Figure 2-32 is small (i.e. Z_1 is close to Z_{01} instead of the 30 Ω/60 Ω discontinuity shown) then a useful approximation is obtained by noting that ${}^A\Gamma_{11}$ and ${}^A\Gamma_{22}$ are small and ${}^AT_{21} \approx {}^AT_{12} \approx 1$ so that in Equation (2.198) $x \approx 0$. Then Equation (2.198) becomes, using Equation (2.197), [6]

$$\Gamma_{\text{in}} \approx {}^A\Gamma_{11} + {}^AT_{12}{}^AT_{21}T_{01}^2\,{}^B\Gamma_{11} \approx {}^A\Gamma_{11} + {}^B\Gamma_{11}\mathrm{e}^{-2\jmath\theta_1}. \tag{2.201}$$

That is, if the discontinuity at reference plane A is small then the input reflection coefficient is dominated by the initial reflection at A, from Z_1 to Z_{01}, and the initial reflection at B, from Z_{01} to Z_{02}, rotated by twice the electrical length, θ_1, of the line.

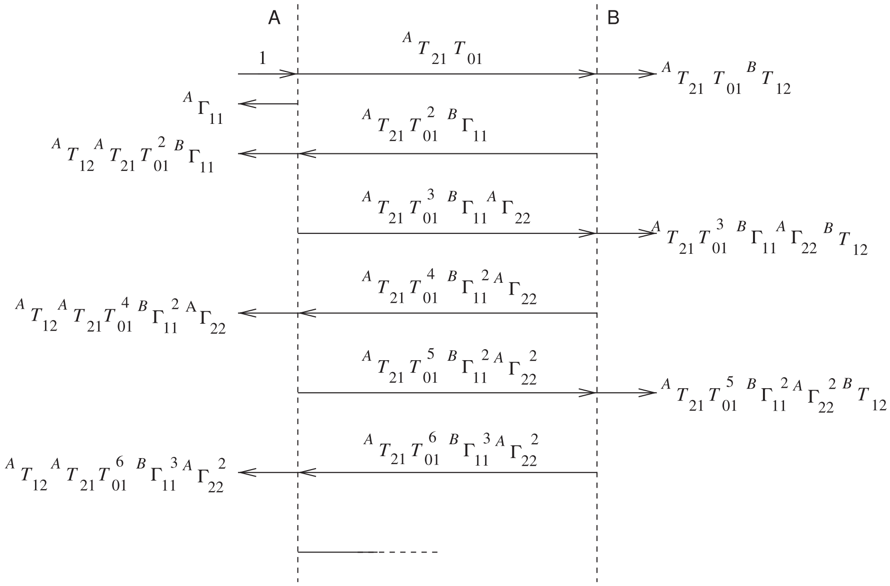


Figure 2-34: Bounce diagram of the transmission line network in Figure 2-32.

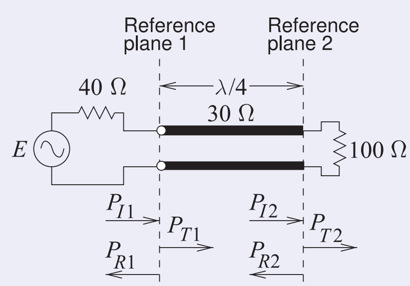
EXAMPLE 2.22 Advanced Power Calculations

This example is similar to Example 2.20. Again, the transmission line network of Figure 2-27 is considered, but now the characteristic impedance of the first transmission line is not the same as the generator impedance and so the simplification used in the previous example can no longer be used. Now the generator has a Thevenin impedance $Z_1 = 40\ \Omega$ and source $E = 5\ \mathrm{V_{peak}}$, connected to a one-quarter wavelength long line of characteristic impedance $Z_{01} = 30\ \Omega$ that in turn is connected to an infinitely long line of characteristic impedance $Z_{02} = 100\ \Omega$. The transmission lines are lossless. Two reference planes are shown in Figure 2-27. At reference plane 1 the incident power is P_{I1} (the maximum available power from the source), the reflected power is P_{R1} and the transmitted power is P_{T1}. P_{I2} (the maximum available power from Z_{01}), P_{R2}, and P_{T2} are similar quantities at reference plane 2. P_{I1}, P_{R1}, P_{T1}, P_{I2}, P_{R2}, and P_{T2} are steady-state quantities.

(a) What is P_{I1}?
(b) What is P_{T2}?
(c) Determine P_{T1}, P_{I2}, P_{R1}, and P_{R2}.

Solution:

One of the first things to note is that the infinitely long $100\ \Omega$ transmission line is indistinguishable from a $100\ \Omega$ resistor, so the reduced form of the problem is as shown to the right.



(a) P_{I1} was calculated in Example 2.20 in Equation (2.178):

$$P_{I1} = 78.13\ \mathrm{mW}. \quad (2.202)$$

P_{I1} is the available power from the source and this is the power that would be delivered to a load that is conjugately matched to the Thevenin equivalent source impedance.

(b) The problem here is finding P_{T2}. Recall that the powers here are steady-state quantities so that multiple reflections of, say, a pulse are not being considered. Since the system is lossless, the power delivered by the generator must be the power delivered to the infinitely long transmission line Z_{02} (i.e., P_{T2}). The telegrapher's equation can be used to calculate the input impedance, Z_{in}, of the two transmission line system; that is, the input impedance of Z_{01} from the generator end. However, a simpler way to find this impedance is to realize that the Z_{01} line is a $\lambda/4$ transformer so that

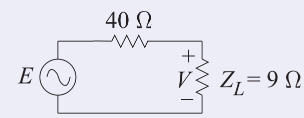
$$Z_{01} = 30\ \Omega = \sqrt{Z_{\text{in}} Z_{02}} = \sqrt{100 Z_{\text{in}}}, \tag{2.203}$$

and so

$$Z_{\text{in}} = 9\ \Omega. \tag{2.204}$$

The equivalent circuit is as shown to the right, where E is the original generator voltage of 5 V and

$$V = \frac{9}{40+9} 5 = 0.9184\ \text{V}. \tag{2.205}$$



The power delivered by the generator to the 9 Ω load is

$$P_{T2} = \tfrac{1}{2} V^2/9 = 0.04686\ \text{W} = 46.86\ \text{mW}. \tag{2.206}$$

(c) The power transmitted into the system at reference plane 1, P_{T1}, is the same as the power transmitted to the 100 Ω load, as the first transmission line is lossless; that is,

$$P_{T1} = P_{T2} = 46.86\ \text{mW}. \tag{2.207}$$

Also

$$P_{R1} = P_{I1} - P_{T1} = (78.13 - 46.86)\ \text{mW} = 31.27\ \text{mW}. \tag{2.208}$$

The two remaining quantities to determine are P_{I2} and P_{R2}. There can be a number of interpretations of what these should be, but one thing that is certain is that

$$P_{T2} = P_{I2} - P_{R2} = 46.86\ \text{mW}. \tag{2.209}$$

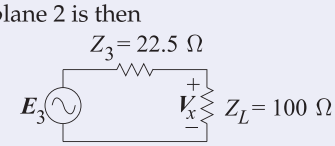
One interpretation that will be followed here is based on the equivalent circuit at reference plane 2. Let Z_{out} be the impedance looking to the left from reference plane 2. Again, using the property of a one-quarter wavelength long transmission line,

$$Z_{01} = 30 = \sqrt{Z_{\text{out}} \times 40} \tag{2.210}$$

and so

$$Z_{\text{out}} = Z_{01}^2/40 = 30^2/40 = 22.5\ \Omega. \tag{2.211}$$

The equivalent circuit at reference plane 2 is then



Now determine V_x that results in a power P_{T2} being delivered to the 100 Ω load, so

$$P_{T2} = \tfrac{1}{2} V_x^2/100 = 0.04686\ \text{W} \tag{2.212}$$

and

$$V_x = 3.06\ \text{V}. \tag{2.213}$$

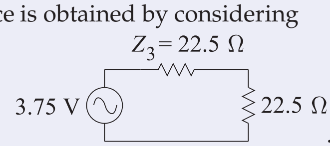
From the circuit above,

$$V_x = \frac{100}{100 + 22.5} E_3 \tag{2.214}$$

and so

$$E_3 = 3.75\ \text{V}. \tag{2.215}$$

The available power from this source is obtained by considering



The available power at reference plane 2 is

$$P_{I2} = \frac{1}{2} \frac{(E_3/2)^2}{22.5} = \frac{1.875^2}{2 \times 22.5} = 0.07813\ \text{W} = 78.13\ \text{mW}. \tag{2.216}$$

From Equation (2.209),

$$P_{R2} = P_{I2} - P_{T1} = (78.13 - 46.86)\ \text{mW} = 31.27\ \text{mW}. \tag{2.217}$$

Note that P_{R2} in Equation (2.217) is the same as P_{R1} in Equation (2.208), as expected.

2.7 Models of Transmission Lines

Models of transmissions lines enable lines to be incorporated in simulators and related to lumped-element circuits. Section 2.7.1 presents models used in circuit simulators such as Spice. Synthesis of microwave circuits often requires that a lumped-element design be realized using an equivalent transmission line network. The ABCD parameter technique described in Section 2.7.2 is the main means for relating these networks and thus synthesizing transmission line-based designs. This method is used many times in designs in this book series.

2.7.1 Circuit Models of Transmission Lines

Circuit models of transmission lines are required if they are to be used in a circuit simulator. RF and microwave engineering uses two types of simulators. Spice-like simulators use lumped-element transmission line models in which an $RLGC$ model of a short segment of line is replicated for the length of the line. If the ground plane is treated as a universal ground, then the model of a segment of length Δz is as shown in Figure 2-35(a). In this segment $r = R\Delta z$, $l = L\Delta z$, $g = G\Delta z$, and $c = C\Delta z$, where R, L, G, and C are the per unit length parameters of the line. Cascading the segments to get the length of the line yields the complete lumped-element model of the line, as shown in Figure 2-35(b). This model is adequate if there is a well-defined ground plane. Otherwise the more accurate form of the segment model shown in Figure 2-35(c) is used. The L and C of a two-conductor transmission line model relate to the fields between the conductors and not to the conductors themselves. So assigning them to just one conductor is not accurate. The capacitor is already shared between the two conductors in the original model, Figure 2-35(a), but the inductor was assigned to just one. The change in the distributed ground model of Figure 2-35(c) is that inductance

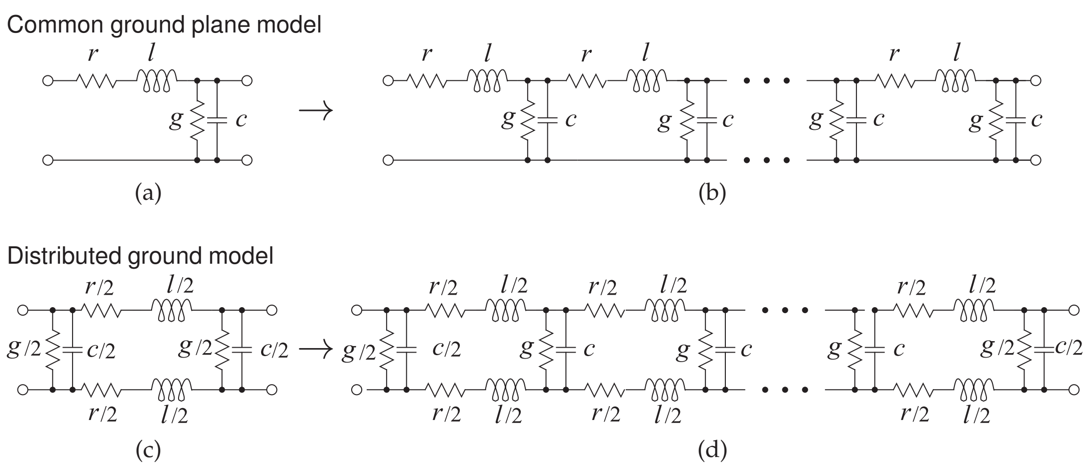


Figure 2-35: Lumped-element transmission line models: (a) model of a short segment (e.g. $\lambda/20$ long); (b) complete model of a transmission line; (c) more accurate model of a segment ; and (d) complete accurate model of a transmission line.

is assigned to each of the conductors.

One of the problems with the lumped-element model is that it is not possible to handle G correctly. G is largely due to dielectric relaxation loss and so is linearly dependent on frequency. Fortunately it is very small for RF and microwave substrates and so it can nearly always be ignored. An exception is a silicon integrated circuit where there is appreciable dielectric loss. Parasitic extraction software is used in silicon integrated circuit design to extract models similar to that in Figure 2-35(d) but account for coupling with other circuits. Especially in digital circuits, it is sometimes difficult to identify a ground plane (a current return path) except for critical interconnect networks such as clock and power distribution circuits.

EXAMPLE 2.23 Transmission Line Resonator

Communication filters are often constructed using several shorted transmission line resonators that are coupled to each other. Consider a coaxial line that is short-circuited at one end. The permittivity filling the coaxial line has $\varepsilon = 20$ and the resonator is to be designed to resonate at a center frequency, f_0, of 1850 MHz when it is $\lambda/4$ long.

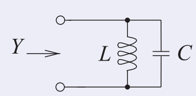
(a) What is the wavelength in the dielectric-filled coaxial line?
(b) What is the form of the equivalent circuit (in terms of inductors and capacitors) of the one-quarter wavelength long resonator if the coaxial line is lossless? Note that the input impedance of a shorted $\lambda/4$-long line is infinite.
(c) What is the physical length of the resonator?

Solution:

The first thing to realize with this example is that the first resonance will occur when the length of the resonator is one-quarter wavelength ($\lambda/4$) long. Resonance generally means that the impedance is either an open or a short circuit and there is energy stored. When the shorted line is $\lambda/4$ long, the input impedance is an open circuit and also energy is stored.

(a) $\lambda_g = \lambda_0/\sqrt{\varepsilon_r} = 16.2\ \text{cm}/\sqrt{20} = 3.62$ cm. (Here $\lambda_0 = 30$ cm at 1 GHz was used. Thus at 1850 MHz $\lambda_0 = (30\ \text{cm})/1.85 = 16.2$ cm.)

(b)



$Y = 0$ at resonance

$$Y = Y_L + Y_C = \frac{1}{j\omega L} + j\omega C = \frac{\omega^{-1}}{jL} + j\omega C.$$

(c) $\ell = (0.0362\ \text{m})/4 = 9.05$ mm.

EXAMPLE 2.24 Resonator Model

This example builds on Example 2.23. RF communication filters are often constructed using several shorted transmission line resonators that are coupled to each other. Consider a coaxial line that is short-circuited at one end. The dielectric filling the coaxial line has a $\varepsilon = 20$ and the resonator is to be designed to resonate at a center frequency $f_0 = 1850$ MHz.

(a) What is the wavelength in the dielectric-filled coaxial line?
(b) What is the form of the equivalent circuit (in terms of inductors and capacitors) of the one-quarter wavelength long resonator if the coaxial line is lossless?
(c) What is the physical length of the resonator?
(d) Determine the derivative with respect to frequency of the admittance of the LC equivalent circuit developed in (b). Determine an analytic expression for the derivative at the resonant frequency.

(e) If the diameter of the inner conductor of the coaxial line is 2 mm and the inside diameter of the outer conductor is 5 mm, what is the characteristic impedance of the coaxial line?

(f) Calculate the input admittance of the dielectric-filled coaxial line at $0.99f_0$, f_0, and $1.01f_0$. Determine the numerical frequency derivative of the line admittance at f_0.

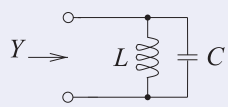
(g) Derive the numeric values of the equivalent circuit of the resonator at resonance.

Solution:

The first thing to realize is that the first resonance will occur when the length of the resonator is one-quarter wavelength ($\lambda/4$) long. Resonance generally means that the impedance is either an open or a short circuit and there is energy stored. When the shorted line is $\lambda/4$ long, the input impedance will be an open circuit. When the line is of zero length energy is not stored, so a zero-length line is not a resonator.

(a) $\lambda_g = \lambda_0/\sqrt{\varepsilon_r} = 16.2\ \text{cm}/\sqrt{20} = 3.62$ cm.

(b)



$$Y = Y_L + Y_C = \frac{1}{\jmath\omega L} + \jmath\omega C = \frac{\omega^{-1}}{\jmath L} + \jmath\omega C.$$

$Y = 0$ at resonance.

(c) ℓ = (0.0362 m)/4 = 9.05 mm.

(d) From (b),

$$\frac{\partial Y}{\partial \omega} = -\frac{\omega^{-2}}{\jmath L} + \jmath C = \frac{\jmath}{\omega^2 L} + \jmath C = \jmath\left(\frac{1}{\omega^2 L} + C\right). \tag{2.218}$$

(e) From Equation (2.53), $Z_0 = 12.28\ \Omega$.

(f) $Z_L = 0\ \Omega$, $\ell = (0.0362\ \text{m})/4 = 9.05$ mm, $\beta = \beta_0\sqrt{\varepsilon_r}$, where β_0 is the phase constant of free space. From Equations (2.43) and (2.44),

$$\beta = 20.958 \times f_0|_{\text{GHz}} \times \sqrt{20} = 173.4\ \text{rad/m}.$$

At $0.99f_0$ and using Equation (2.71), $Z_{\text{in}} = \jmath 781.7\ \Omega;\ Y_{\text{in}} = -\jmath 0.001279$ S.
At $f_0,\ Z_{\text{in}} = -\jmath\infty\ \Omega;\ Y_{\text{in}} = 0$ S. At $1.01f_0,\ Z_{\text{in}} = -\jmath 781.7\ \Omega;\ Y_{\text{in}} = \jmath 0.001279$ S.
So the derivative of the input admittance is

$$\frac{\partial Y}{\partial \omega} \approx \frac{\Delta Y_{\text{in}}}{\Delta \omega} = \frac{\jmath(0.001279 + 0.001279)}{2\pi f_0(1.01 - 0.99)} = \jmath 1.101 \cdot 10^{-11}\ \text{S} \cdot \text{s} = \jmath 1.101 \cdot 10^{-11}\ \text{F}. \tag{2.219}$$

Note the SI unit equivalence S · s = F obtained by examining Table 2-1.

(g) The LC circuit resonates at $f_0 = 1.85$ GHz when $1/(\jmath\omega_0 L) = -\jmath\omega_0 C$; $\omega_0 = 2\pi f_0$. Thus

$$LC = \omega_0^{-2} = 0.7401 \times 10^{-20}\ \text{s}^2. \tag{2.220}$$

Equating Equations (2.218) and (2.219):

$$\frac{1}{\omega_0^2 L} + C = 1.101 \times 10^{-11}\ \text{F} \qquad \text{and so} \quad C = 1.101 \times 10^{-11} - \frac{1}{\omega_0^2 L}\ \text{F}. \tag{2.221}$$

Substituting this in Equation (2.220) (noting the SI unit equivalence $\text{S}^{-1} \cdot \text{s}^{-1}$ = H:

$$L\left(1.101 \times 10^{-11} - \frac{1}{\omega_0^2 L}\right) = 0.7401 \times 10^{-20}\ \text{H}$$

$$1.101 \times 10^{-11} L = 1.4802 \times 10^{-20}\ \text{H} \quad \text{and} \quad L = 1.345 \times 10^{-9}\ \text{H} = 1.345\ \text{nH}. \tag{2.222}$$

From Equation (2.221),

$$C = 0.7401 \times 10^{-20}/L = 5.503 \times 10^{-12}\ \text{F} = 5.503\ \text{pF}. \tag{2.223}$$

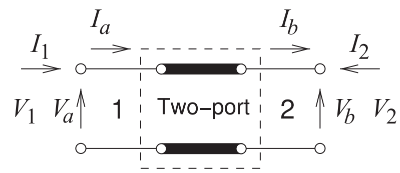
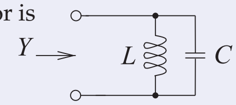


Figure 2-36: Two-port network of a transmission line with cascadable voltage and current definitions.

So the equivalent circuit of the resonator is



with L = 1.345 nH and C = 5.503 pF.
This equivalent circuit is valid for a range (say 5%–10%) of frequencies around 1.85 GHz. The broadband equivalence was obtained here by matching both the admittance and the derivative of the admittance.

2.7.2 ABCD (Chain) Parameters of a Transmission Line

ABCD parameters, also called chain parameters, are used in RF and microwave engineering to electrically equate two different structures such as a lumped-element network and a transmission line structure. Often the electrical design is undertaken using lumped elements and then these are converted to equivalent transmission line structures with the equivalence achieved using *ABCD* parameters.

Figure 2-36, shows the voltages and currents of a two-port in cascadable form so that the input variables can be written in terms of the output variables:

$$\begin{bmatrix} V_a \\ I_a \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_b \\ I_b \end{bmatrix}. \tag{2.224}$$

Note that the current at port 2 is in the opposite direction to the usual definition of the two-port current. So in terms of the port voltage and current at port 1, (V_1 and I_1), and the port voltage and current at port 2, (V_2 and I_2),

$$V_a = V_1, \quad I_a = I_1, \quad V_b = V_2, \quad \text{and} \quad I_b = -I_2. \tag{2.225}$$

In terms of traveling-wave voltages, the total voltage and current on a line of length ℓ, propagation constant γ, and characteristic impedance $Z_0 = 1/Y_0$ at position z are

$$V(z) = V_0^+ \mathrm{e}^{-\jmath\gamma(z-\ell)} + V_0^- \mathrm{e}^{\jmath\gamma(z-\ell)} \tag{2.226}$$

$$I(z) = Y_0 V_0^+ \mathrm{e}^{-\jmath\gamma(z-\ell)} - Y_0 V_0^- \mathrm{e}^{\jmath\gamma(z-\ell)}. \tag{2.227}$$

where V_0^+ and V_0^- are the traveling wave voltage phasors at $z = \ell$, (port 2), i.e.

$$V_0^+ = \tfrac{1}{2}\left(V_2 + Z_0 I_2\right) \quad \text{and} \quad V_0^- = \tfrac{1}{2}\left(V_2 - Z_0 I_2\right). \tag{2.228}$$

Substituting these into Equations (2.226) and (2.227),

$$V(z) = \tfrac{1}{2}\left(V_2 + Z_0 I_2\right) \mathrm{e}^{-\jmath\gamma(z-\ell)} + \tfrac{1}{2}\left(V_2 - Z_0 I_2\right) \mathrm{e}^{\jmath\gamma(z-\ell)} \tag{2.229}$$

$$I(z) = \tfrac{1}{2}\left(Y_0 V_2 + I_2\right) \mathrm{e}^{-\jmath\gamma(z-\ell)} - \tfrac{1}{2}\left(Y_0 V_2 - I_2\right) \mathrm{e}^{\jmath\gamma(z-\ell)}. \tag{2.230}$$

At Port 1 ($z = 0$)

$$V_1 = V(0) = \tfrac{1}{2}\left(V_2 + Z_0 I_2\right) \mathrm{e}^{\jmath\gamma\ell} + \tfrac{1}{2}\left(V_2 - Z_0 I_2\right) \mathrm{e}^{-\jmath\gamma\ell} \tag{2.231}$$

$$I_1 = I(0) = \tfrac{1}{2}\left(Y_0 V_2 + I_2\right) \mathrm{e}^{\jmath\gamma\ell} - \tfrac{1}{2}\left(Y_0 V_2 - I_2\right) \mathrm{e}^{-\jmath\gamma\ell}. \tag{2.232}$$

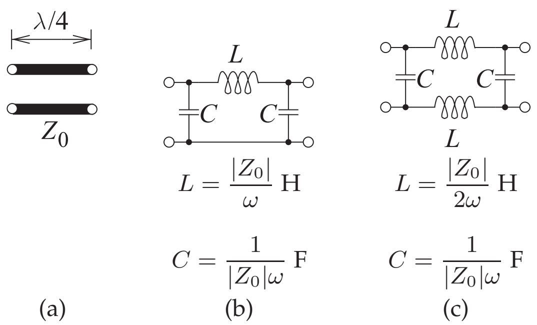


Figure 2-37: Lumped equivalent circuit of a one-quarter wavelength long line of characteristic impedance Z_0 which is the same as that of an impedance inverter (of Z_0 Ω): (a) one-quarter wavelength long line segment; (b) lumped-element equivalent circuit; and (c) alternative lumped-element model.

Using the trigonometric identities in Section 1.A.2 (and noting that $I_2 = -I_b$),

$$V_a = V_1 = \cosh(\gamma\ell)V_2 - Z_0 \sinh(\gamma\ell)I_2 = \cosh(\gamma\ell)V_b + Z_0 \sinh(\gamma\ell)I_b \quad (2.233)$$

$$I_a = I_1 = Y_0 \sinh(\gamma\ell)V_2 - \cosh(\gamma\ell)I_2 = Y_0 \sinh(\gamma\ell)V_b + \cosh(\gamma\ell)I_b. \quad (2.234)$$

That is, the *ABCD* parameters of a lossy line are

$$\begin{array}{ll} A = \cosh(\gamma\ell) & B = Z_0 \sinh(\gamma\ell) \\ C = Y_0 \sinh(\gamma\ell) & D = \cosh(\gamma\ell) \end{array} \quad (2.235)$$

and the *ABCD* parameters of a lossless transmission line, where $\gamma = \jmath\beta$, are

$$\begin{array}{ll} A = \cos(\beta\ell) & B = \jmath Z_0 \sin(\beta\ell) \\ C = \jmath Y_0 \sin(\beta\ell) & D = \cos(\beta\ell) \end{array}. \quad (2.236)$$

EXAMPLE 2.25 Lumped Element Model of a Quarter-Wavelength Long Line

Develop the lumped-element model of a one-quarter wavelength long lossless transmission line having characteristic impedance Z_0.

Solution:

The model is developed by equating *ABCD* parameters. The *ABCD* parameters of a one-quarter wavelength long lossless line (with $\beta\ell = \pi/2$) are

$$A = \cos(\beta\ell) = 0 \quad (2.237) \qquad C = \jmath Y_0 \sin(\beta\ell) = \jmath/Z_0 \quad (2.239)$$

$$B = \jmath Z_0 \sin(\beta\ell) = \jmath Z_0 \quad (2.238) \qquad D = \cos(\beta\ell) = 0. \quad (2.240)$$

The *ABCD* parameters of a Pi network (see Table 2-1 of [7]) are

$$A = 1 + y_2/y_3 \quad (2.241) \qquad C = y_1 + y_2 + y_1 y_2/y_3 \quad (2.243)$$

$$B = 1/y_3 \quad (2.242) \qquad D = 1 + y_1/y_3. \quad (2.244)$$

Equating Equations (2.238) and (2.242), $$y_3 = 1/(\jmath Z_0), \quad (2.245)$$

and with Equations (2.237), (2.240), (2.241), and (2.244),

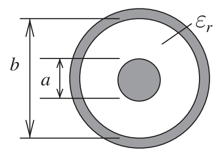
$$y_1 = y_2 = -y_3 = -1/(\jmath Z_0). \quad (2.246)$$

The lumped equivalent circuit of the quarter-wave transformer in Figure 2-37(a) is shown in Figure 2-37(b). An alternative lumped-element model is shown in Figure 2-37(c). The lumped-element model of the 50 Ω line at 400 MHz is shown in Figure 2-37(b) with L = 19.89 nH and C = 7.968 pF. This is a simpler and more accurate model of a $\lambda/4$ long line than a model with multiple short line segments as in Figure 2-35.

2.8 Two-Conductor Transmission Lines

Nearly every two-conductor transmission line supports EM fields at frequencies down to DC. As a result, such lines can usually be characterized at DC and the properties of the lines will be essentially the same at RF frequencies. In the following, the characteristic impedances of the most common types of two-conductor lines are presented. For all of these lines the phase constant is the same as it would be in the medium without conductors (i.e., $\beta = \omega\sqrt{\mu\varepsilon}$). The characteristic impedances are modifications of the free-space wave impedance $\eta_0 = \sqrt{\mu_0/\varepsilon_0} = 120\pi\ \Omega = 376.73\ \Omega$. The properties of many other transmission lines are given in [8].

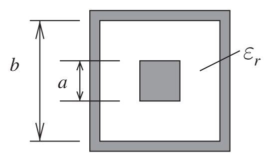
Coaxial line
Figure 2-38



$$Z_0 = \frac{\eta_0}{2\pi}\sqrt{\frac{\mu_r}{\varepsilon_r}}\ln\left(\frac{b}{a}\right) = 60\sqrt{\frac{\mu_r}{\varepsilon_r}}\ln\left(\frac{b}{a}\right) \tag{2.247}$$

Exact. See derivation in Section 2.9.

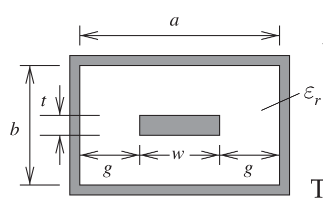
Square coaxial line
Figure 2-39



$$Z_0 \approx \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}\left[4\left(\frac{2a}{b-a}+0.558\right)\right]^{-1} \tag{2.248}$$

Typical accuracy: <1% for $b/a \le 4$. Reference [9].

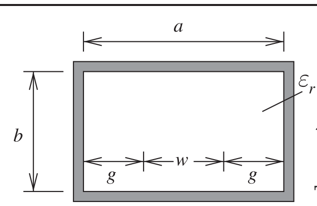
Rectangular coaxial line
Figure 2-40



$$Z_0 = \frac{\eta_0}{4}\sqrt{\frac{\mu_r}{\varepsilon_r}}\left(\frac{w}{b-t}+\frac{1}{\pi}\left\{\frac{b}{b-t}\ln\left(\frac{2b-t}{t}\right) + \ln\left[\frac{t(2b-t)}{(b-t)^2}\right]\right\}\frac{\ln\left[1+\coth(\pi g/b)\right]}{\ln 2}\right)^{-1} \tag{2.249}$$

Typical accuracy: 1%. References [10–12].

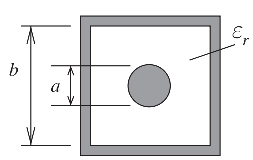
Rectangular coaxial line, thin strip $t \to 0$
Figure 2-41



$$Z_0 = \frac{\eta_0}{4}\sqrt{\frac{\mu_r}{\varepsilon_r}}\left\{\frac{w}{b}+\frac{2}{\pi}\ln\left[1+\coth\left(\frac{\pi g}{b}\right)\right]\right\}^{-1} \tag{2.250}$$

Typical accuracy: 1%. Reference [10].

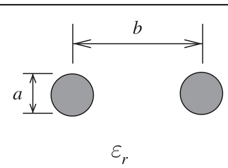
Square coax with circular inner conductor
Figure 2-42



$$Z_0 = \frac{\eta_0}{2\pi}\sqrt{\frac{\mu_r}{\varepsilon_r}}\ln\left(\frac{1.0787b}{a}\right) \tag{2.251}$$

Typical accuracy: 1.5%. References [8, 13–15].

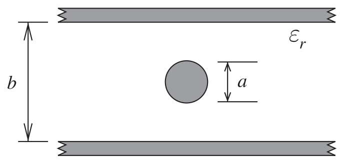
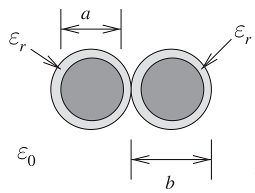
Parallel wires
Figure 2-43



$$Z_0 = \frac{\eta_0}{\pi}\sqrt{\frac{\mu_r}{\varepsilon_r}}\operatorname{arccosh}\left(\frac{b}{a}\right) \tag{2.252}$$

Typical accuracy: 1%. References [16, 17].

Characteristics of two-conductor transmission lines.

Slab line **Figure 2-44**		$$Z_0 = 15\sqrt{\frac{\mu_r}{\varepsilon_r}}\ln\left[1 + 1.314g + \sqrt{(1.314g)^2 + 2g}\right]$$ where $g = (b/a)^4 - 1$ (2.253) Typical accuracy: 0.5%. Reference [18].
Twisted pair with T twists per unit length **Figure 2-45**		$$Z_0 = \frac{\eta_0}{\pi}\sqrt{\frac{\mu_r}{\varepsilon_e}}\text{arccosh}\left(\frac{b}{a}\right) \quad (2.254)$$ $$\varepsilon_e = 1 + q(\varepsilon_r - 1),\ q = 0.25 + 0.0004\Theta^2$$ $$T = \frac{\tan\Theta}{\pi D} \ \text{ so } \ \Theta = \arctan(T\pi D)$$ Typical accuracy: 1%. Reference [19].

Characteristics of two-conductor transmission lines.

2.9 Coaxial Line

In this section the characteristic impedance of a coaxial line is calculated. The technique assumes that the dynamic electric and magnetic fields will be the same as they would be at DC. This physical insight enables the problem to be solved relatively simply using the static field laws. Later the dynamic situation is considered when the fields on a realistic coaxial line are considered.

2.9.1 Characteristic Impedance of a Coaxial Line

In the following the per unit length parameters—R, L, G, and C—of a coaxial line are obtained. From these the characteristic impedance and propagation constant of the coaxial line are derived. The development is an example of how these parameters can be developed for any transmission line. The key is using a coordinate system that matches the structure being considered. Then Maxwell's equations can be expressed very simply, usually as one-dimensional ordinary differential equations. Very often the static electric and magnetic field laws can be used that make the development even simpler. If an orthogonal coordinate system does not coincide with the dimensions of the line, then sometimes it is possible to use a mathematical technique called conformal mapping to map a nonconforming shape such as a microstrip line into a shape that matches an orthogonal coordinate system [20]. Once the transmission line parameters are derived the result is mapped back to the original structure. Fortunately most transmission line structures conform to an orthogonal coordinate system.

The cross section of a coaxial line is shown in Figure 2-46(a) and it has two concentric conductors with the inner conductor having an outer radius of a and the inside of the outer conductor with a radius b. The fields between the conductors can be solved by expressing the field relations in cylindrical coordinates. The following is a simplification of the treatment in [21], which can be applied to a broader range of coaxial cables than just the circularly symmetrical conventional coaxial cable in Figure 2-46. The simplest possible field distribution is shown in Figure 2-46(b). The first parameter to be developed is C. For this, the charge on one of the conductors is related to the voltage between the conductors. Consider a section of line of length Δz, then using Gauss's law, Equation (1.50), the electric flux is related to the total charge enclosed. So at radius r,

$$\oint_s \bar{D} = Q_{\text{enclosed}}$$
$$2\pi r D_r(r)\Delta z = \rho_\ell \Delta z \quad \text{and} \quad 2\pi r\varepsilon E_r(r)\Delta z = \rho_\ell \varepsilon \Delta z, \tag{2.255}$$

where ρ_ℓ is the charge on the inner conductor per unit length (in the z direction). So the radial electric field is

$$\bar{E} = E_r(r)\hat{\mathbf{r}} = \frac{\rho_\ell}{2\pi r\varepsilon}\hat{\mathbf{r}} \tag{2.256}$$

and $\mathbf{r}$ is the unit vector in the radial direction. The voltage between the conductors is

$$V = \int_a^b E_r.dr = \int_a^b \frac{\rho_\ell}{2\pi r\varepsilon}.dr = \frac{\rho_\ell}{2\pi\varepsilon}\ln\left(\frac{b}{a}\right). \tag{2.257}$$

Now, the total capacitance of a section of line is

$$C_{\text{TOTAL}} = C\Delta z = \frac{Q}{V} = \frac{\rho_\ell \Delta z}{V}, \tag{2.258}$$

and so the capacitance per unit length of line is (with SI units of F/m)

$$C = \frac{2\pi\varepsilon}{\ln(b/a)}. \tag{2.259}$$

The conductance of the line is found beginning with the current density in the dielectric, which is

$$J = \sigma E + \jmath\omega\varepsilon E. \tag{2.260}$$

Integrating this over the volume of the dielectric,

$$I = GV + \jmath\omega CV. \tag{2.261}$$

Using a similar development to that which led to Equation (2.259), the conductance per unit length is found to be (with the SI units of S/m)

$$G = \frac{2\pi\sigma}{\ln(b/a)}. \tag{2.262}$$

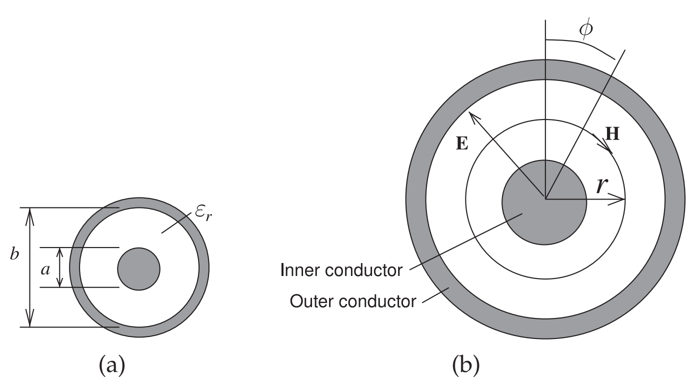


Figure 2-46: Coaxial line: (a) with inner an conductor of radius a and an outer conductor with inside radius b; and (b) with cylindrical coordinates used in calculation.

The development of the expressions for L and R is more involved, as there will be current in the interior of the inner conductor. To simplify matters the high-frequency limit will be considered where the skin effect is fully established so that the currents on the inner and outer conductors are confined to a thin layer near the surfaces of the conductors at radius a for the inner conductor and at radius b for the outer conductor. Then the magnetic field will be confined to the region between the conductors. Using Ampere's circuital law, Equation (1.45),

$$\oint_{\ell} \bar{H} \cdot d\ell = \oint_{\ell} H_{\phi}\hat{\boldsymbol{\phi}} \cdot d\ell = I_{\text{enclosed}} = I, \tag{2.263}$$

where the closed line integral is on a circle of constant radius. Noting that H_r is only a function of r, then for $a < r < b$,

$$I = 2\pi r H_r(r) \quad \text{and} \quad B_r = \mu H_r = \frac{\mu I}{2\pi r}. \tag{2.264}$$

This enables the line inductance to be calculated as the total magnetic flux (obtained by integrating over the cross section of the coaxial line between radii a and b) per unit current. So the inductance of the line per unit length is

$$L = \int_a^b \left(\frac{\mu}{2\pi r}\right) dr = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right), \tag{2.265}$$

which has SI units of H/m.

The resistance of the lines is calculated using the surface resistance of the conductors, R_s. So the resistance of the line per unit length is

$$R = \frac{R_s}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right), \tag{2.266}$$

which has the SI units of Ω/m.

Then the lossless characteristic impedance of the coaxial line is

$$Z_0 = \sqrt{\frac{L}{C}} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}} \ln\left(\frac{b}{a}\right), \tag{2.267}$$

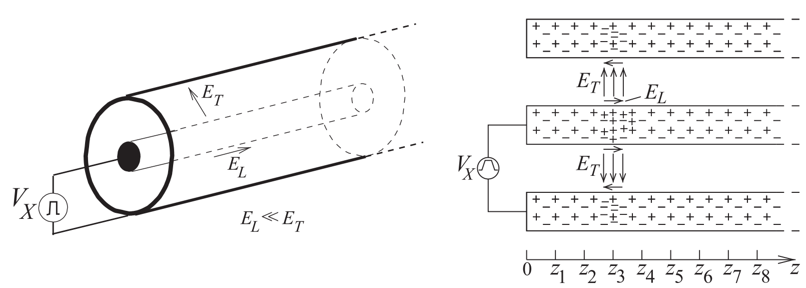
which has units of Ωs if all the quantities in the expression are in SI units. A complex characteristic impedance is derived using $Z_0 = \sqrt{(R + \jmath\omega L)/(G + \jmath\omega C)}$, but if the loss is small, this will be very close to the lossless characteristic impedance. Also

$$Z_0 = \sqrt{\frac{\mu_r}{\varepsilon_r}} \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}} \ln\left(\frac{b}{a}\right) = 60\sqrt{\frac{\mu_r}{\varepsilon_r}} \ln\left(\frac{b}{a}\right). \tag{2.268}$$

Note that the 60 is exact and not an approximation.

For actual coaxial lines the dielectric has negligible loss and the resistance, R, is small. Using the results developed in Section 2.5.4 for a low-loss line, the attenuation coefficient for a low-loss coaxial line is, from Equation (2.149),

$$\alpha \approx \alpha_c = \frac{R}{2Z_0} = \frac{R_s}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right)\frac{2\pi}{\ln(b/a)}\sqrt{\frac{\varepsilon}{\mu}} = \left(\frac{1}{a} + \frac{1}{b}\right)\frac{R_s}{\ln(b/a)}\sqrt{\frac{\varepsilon}{\mu}} \tag{2.269}$$



(a) (b)

Figure 2-47: A coaxial transmission line: (a) three-dimensional view; (b) the line with pulsed voltage source showing the electric fields at an instant in time as a voltage pulse travels down the line.

which is the attenuation due to the conductor, that is, due to ohmic or conductor loss. From Equation (2.269) it is seen that for a given characteristic impedance, and hence a fixed ratio b/a, the attenuation coefficient and thus loss, is minimized when the line has a large cross section (i.e., large a and b). With a fixed a, attenuation is minimized with b/a = 3.59 independent of μ and ε [17]. For an air-filled coaxial line with $\mu_r = 1 = \varepsilon_r$, the characteristic impedance of this minimum-loss line is 76.7 Ω. For a typically Teflon-filled coaxial line $\varepsilon_r = 2.1$ and the characteristic impedance of a minimum loss line, from Equation (2.268), is 52.9 Ω. The above result is when the conductivity of the inner and outer conductors is the same and the internal inductance of the conductors is negligible. If the internal inductance is not negligible, or the conductors have different conductivity, which happens when the conductors are not solid, then the optimum characteristic impedance could be higher or lower [17]. The choice of 50 Ω for the characteristic impedance is a good compromise. Other optimum line dimensions can be derived for maximizing breakdown voltage and optimizing power transfer [22].

The upper operating frequency and dimensions of coaxial lines are chosen to ensure single-mode propagation [23–25]. A useful approximation to the cutoff frequency of the first higher-order mode is [23]

$$\lambda_c \approx \frac{\pi(a+b)}{2}, \tag{2.270}$$

which is the circumference of a circle halfway between the inner and outer conductors. Since b is several times a, then the radius of the outer conductor largely determines the upper operating frequency. Minimizing loss given a fixed cutoff frequency of the first higher-order mode yields another optimum value of b/a for minimum loss (see [22]).

2.9.2 Electromagnetic Fields on a Realistic Coaxial Line

In this section a realistic coaxial line is considered with conductors having a small amount of loss. This expands on the discussion in Section 2.1.2. When a positive voltage pulse is applied to the center conductor of the coaxial line, as shown in Figure 2-47(a), an electric field results that is essentially directed from the center conductor to the outer conductor. A much smaller component of the electric field will also be directed along the line. These fields are shown in Figure 2-47(b), where the direction of the electric field is the direction in which a positive charge would move if it was released into the field. The component of the field that is directed along the shortest path from the center conductor to the outer conductor (in the transverse plane)

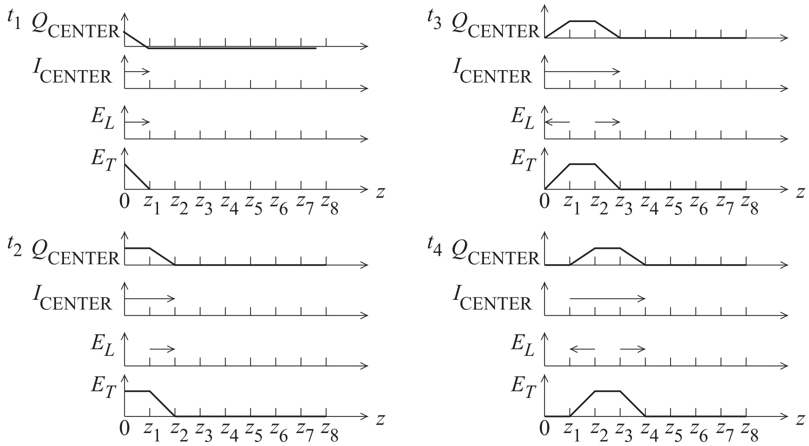


Figure 2-48: Fields, currents, and charges on the coaxial transmission line of Figure 2-2 at times $t_4 > t_3 > t_2 > t_1$. Q_{CENTER} is the net free charge on the center conductor. I_{CENTER} is the current on the center conductor.

is denoted E_T, and the component directed along the line is denoted E_L. (The subscripts T and L denote transverse and longitudinal components, respectively.) Thus, while $E_L \ll E_T$, it is necessary to accelerate electrons on the conductors and so give rise to current flow, and hence the movement of the pulse along the line. The electrons do not accelerate indefinitely, as they collide with atoms and scatter so that there is a net velocity of free electrons along the inner and outer conductors.

Snapshots of a pulse traveling along a line are shown in Figure 2-48 at four different times.

2.10 Summary

In this chapter a classical treatment of transmission lines was presented. Transmission lines are distributed elements and form the basis of microwave circuits. A distinguishing feature is that they support forward- and backward-traveling waves and they can be used to implement circuit functions. RF and microwave engineers are, of course, electrical engineers who learned how to design circuits using lumped elements. Distributed circuits can be made functionally equivalent to lumped-element circuits, at least over a narrow frequency range. The most important technique for establishing this is equating the *ABCD* parameters of the distributed circuit and those of a lumped-element circuit. As will be seen, this technique will be exploited many times in the rest of this book. Distributed circuits enable an RF designer to realize functions that often have superior performance to their lumped-element analogs. At least at microwave frequencies, the loss of lumped elements can be significantly higher than those of distributed elements. Superior performance can be obtained by exploiting the intrinsic behavior of distributed structures. Sometimes functionality can only be conceived of using transmission-line structures, then, often, a low-frequency lumped-element equivalent can be developed. Exploiting the functionality of distributed structures requires a solid understanding of the behavior, modeling, and circuits that can be realized using transmission lines. The technology covered in this chapter provides the theoretical underpinning for all of the other topics covered in this book.

The most important of the formulas presented in this chapter are listed here. Reflection coefficients are referenced to an impedance Z_0, the load impedance is Z_L, and a line has a characteristic impedance Z_0, physical length ℓ, and propagation constant γ (or electrical length in radians of $\beta\ell$ where ℓ is the physical length of the line.

Reflection coefficient of a load impedance Z_L:

$$\Gamma = \Gamma^V = \frac{Z_L - Z_{\rm REF}}{Z_L + Z_{\rm REF}} \quad (2.59)$$

Load impedance in terms of reflection coefficient Γ:

$$Z_L = Z_{\rm REF}\frac{1+\Gamma}{1-\Gamma} \quad (2.62)$$

Input reflection coefficient of a lossless line of length ℓ

$$\Gamma_{\rm in} = \Gamma_L \mathrm{e}^{-\jmath 2\beta\ell} \quad (2.69)$$

Input reflection coefficient of a lossy line of length ℓ

$$\Gamma_{\rm in} = \Gamma_L \mathrm{e}^{-2\gamma\ell} = \Gamma_L \mathrm{e}^{-2\alpha\ell}\mathrm{e}^{-2\jmath\beta\ell} \quad (2.135)$$

Input impedance of a lossless line

$$Z_{\rm in} = Z_0\frac{Z_L + \jmath Z_0 \tan\beta\ell}{Z_0 + \jmath Z_L \tan\beta\ell} \quad (2.71)$$

Input impedance of a lossy line

$$Z_{\rm in} = Z_0\frac{Z_L + Z_0 \tanh\gamma\ell}{Z_0 + Z_L \tanh\gamma\ell} \quad (2.151)$$

Reflection coefficient in terms of VSWR

$$|\Gamma| = \frac{\mathrm{VSWR}-1}{\mathrm{VSWR}+1} \quad (2.77)$$

VSWR in terms of reflection coefficient

$$\mathrm{VSWR} = \frac{(1+|\Gamma|)}{(1-|\Gamma|)} \quad (2.76)$$

2.11 References

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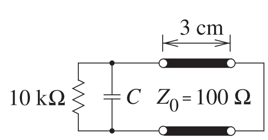
2.12 Exercises

1. A coaxial line is short-circuited at one end and is filled with a dielectric with a relative permittivity of 64. [Parallels Example 2.1]
   (a) What is the free-space wavelength at 18 GHz?
   (b) What is the wavelength in the dielectric-filled coaxial line at 18 GHz?
   (c) The first resonance of the coaxial resonator is at 18 GHz. What is the physical length of the resonator?
2. A transmission line has the following $RLGC$ parameters: $R = 100\ \Omega/\text{m}$, $L = 85\ \text{nH/m}$, $G = 1\ \text{S/m}$, and $C = 150\ \text{pF/m}$. Consider a traveling wave on the transmission line with a frequency of 1 GHz. [Parallels Example 2.3]
   (a) What is the attenuation constant?
   (b) What is the phase constant?
   (c) What is the phase velocity?
   (d) What is the characteristic impedance of the line?
   (e) What is the group velocity?
3. A transmission line has the per-unit length parameters $L = 85\ \text{nH/m}$, $G = 1\ \text{S/m}$, and $C = 150\ \text{pF/m}$. Use a frequency of 1 GHz. [Parallels Example 2.3]
   (a) What is the phase velocity if $R = 0\ \Omega/\text{m}$?
   (b) What is the group velocity if $R = 0\ \Omega/\text{m}$?
   (c) If $R = 10\ \text{k}\Omega/\text{m}$ what is the phase velocity?
   (d) If $R = 10\ \text{k}\Omega/\text{m}$ what is the group velocity?
4. A line is 10 cm long and at the operating frequency the phase constant β is 40 rad/m. What is the electrical length of the line? [Parallels Example 2.2]
5. A dielectric-filled lossless transmission line carrying a 1 GHz signal has the parameters $L = 80\ \text{nH/m}$ and $C = 200\ \text{pF/m}$. When the dielectric is replaced by air the line's capacitance is $C_{\text{air}} = 50\ \text{pF/m}$. What is the relative permittivity of the dielectric?
6. A coaxial transmission line is filled with lossy dielectric material with a relative permittivity of $5 - \jmath 0.2$. If the line is air-filled it would have a characteristic impedance of $100\ \Omega$. What is the input impedance of the line if it is 1 km long? Use reasonable approximations. [Hint: Does the termination matter?]
7. A transmission line has the per unit length parameters $R = 2\ \Omega/\text{cm}$, L=100 nH/m, $G = 1\ \text{mS/m}$, $C = 200\ \text{pF/m}$.
   (a) What is the propagation constant of the line at 5 GHz?
   (b) What is the characteristic impedance of the line at 5 GHz?
   (c) Plot the magnitude of the characteristic impedance versus frequency from 100 MHz to 10 GHz.

8. A line is 20 cm long and at 1 GHz the phase constant β is 20 rad/m. What is the electrical length of the line in degrees?
9. What is the electrical length of a line that is a quarter of a wavelength long,
   (a) in degrees?
   (b) in radians?
10. A lossless transmission line has an inductance of 8 nH/cm and a capacitance of 40 pF/cm.
    (a) What is the characteristic impedance of the line?
    (b) What is the phase velocity on the line at 1 GHz?
11. A 50 Ω coaxial airline is a coaxial line without a dielectric (i.e., it is air-filled) and with thin dielectric discs supporting the inner conductor have negligible effect. If the air is replaced by a dielectric having a relative permittivity of 20, what is the characteristic impedance of the dielectric-filled line?
12. A transmission line has an attenuation of 2 dB/m and a phase constant of 25 radians/m at 2 GHz. [Parallels Example 2.4]
    (a) What is the complex propagation constant of the transmission line?
    (b) If the capacitance of the line is 50 pF·m^{-1} and $G = 0$, what is the characteristic impedance of the line?
13. A very low-loss microstrip transmission line has the following per unit length parameters: $R = 2\ \Omega$/m, $L = 80$ nH/m, $C = 200$ pF/m, and $G = 1$ µS/m.
    (a) What is the characteristic impedance of the line if loss is ignored?
    (b) What is the attenuation constant due to conductor loss?
    (c) What is the attenuation constant due to dielectric loss?
14. A lossless transmission line carrying a 1 GHz signal has the following per unit length parameters: $L = 80$ nH/m, $C = 200$ pF/m.
    (a) What is the attenuation constant?
    (b) What is the phase constant?
    (c) What is the phase velocity?
    (d) What is its characteristic impedance?
15. A transmission line has a characteristic impedance Z_0 and is terminated in a load with a reflection coefficient of $0.8\angle 45°$. A forward-traveling voltage wave on the line has a power of 1 dBm.
    (a) How much power is reflected by the load?
    (b) What is the power delivered to the load?
16. A transmission line has an attenuation of 0.2 dB/cm and a phase constant of 50 radians/m at 1 GHz.
    (a) What is the complex propagation constant of the transmission line?
    (b) If the capacitance of the line is 100 pF/m and $G = 0$, what is the complex characteristic impedance of the line?
    (c) If the line is driven by a source modeled as an ideal voltage and a series impedance, what is the impedance of the source for maximum transfer of power to the transmission line?
    (d) If 1 W is delivered (i.e. in the forward-traveling wave) to the transmission line by the generator, what is the power in the forward-traveling wave on the line at 2 m from the generator?
17. The transmission line shown in Figure 2-27 consists of a source with Thevenin impedance $Z_1 = 40\ \Omega$ and source $E = 5$ V (peak) connected to a $\lambda/4$ long line of characteristic impedance $Z_{01} = 50\ \Omega$, which in turn is connected to an infinitely long line of characteristic impedance $Z_{02} = 100\ \Omega$. The transmission lines are lossless. Two reference planes are shown in Figure 2-27. At reference plane 1 the incident power is P_{I1}, the reflected power is P_{R1}, and the transmitted power is P_{T1}. P_{I2}, P_{R2}, and (P_{T2}) are similar quantities at reference plane 2. [Parallels Examples 2.20 and 2.22]
    (a) What is P_{I1}?
    (b) What is P_{T2}?
18. A lossless, 10 cm-long, 75 Ω transmission line is driven by a 1 GHz generator with a Thevenin equivalent impedance of 50 Ω. The maximum power that can be delivered to a load attached to the generator is 2 W. The line is terminated in a load that has a complex reflection coefficient (referred to 50 Ω) of $0.65 + \jmath 0.65$. The effective relative permittivity, ε_e, of the non-magnetic transmission line is 2.0.
    (a) Calculate the forward-traveling voltage wave (at the generator end of the transmission line). Ignore reflections from the load at the end of the 75 Ω line.
    (b) What is the load impedance?
    (c) What is the wavelength of the forward-traveling voltage wave?
    (d) What is the VSWR on the line?
    (e) What is line's propagation constant?
    (f) What is the input reflection coefficient (at the generator end) of the line?
    (g) What is the power delivered to the load?

19. The first resonance of a lossless non-magnetic open-circuited transmission line is at 30 GHz. The effective relative permittivity of the line is 12.
    (a) What is the resonator's input impedance?
    (b) Draw its LC equivalent circuit.
    (c) What is its electrical length?
    (d) What is its physical length?
20. An open-circuited transmission line is used as a resonator. What is the electrical length of the line at its first resonance?
21. The second resonance of an open-circuited transmission line is used as a resonator.
    (a) What is its input impedance?
    (b) What is its electrical length?
22. A lossless transmission line is driven by a 1 GHz generator having a Thevenin equivalent impedance of 50 Ω. The transmission line is lossless, has a characteristic impedance of 75 Ω, and is infinitely long. The maximum power that can be delivered to a load attached to the generator is 2 W.
    (a) What is the total (phasor) voltage at the input to the transmission line?
    (b) What is the magnitude of the forward-traveling voltage wave at the generator side of the line?
    (c) What is the magnitude of the forward-traveling current wave at the generator side of the line?
23. A transmission line is terminated in a short circuit. What is the ratio of the forward- and backward-traveling voltage waves at the termination? [Parallels Example 2.5]
24. A 50 Ω transmission line is terminated in a 40 Ω load. What is the ratio of the forward- to the backward-traveling voltage waves at the termination? [Parallels Example 2.5]
25. A 50 Ω transmission line is terminated in an open circuit. What is the ratio of the forward- to the backward-traveling voltage waves at the termination? [Parallels Example 2.5]
26. The resonator below is constructed from a 3.0 cm length of 100 Ω air-filled coaxial line, shorted at one end and terminated with a capacitor at the other end.



    (a) What is the lowest resonant frequency of this circuit without the capacitor (ignore the 10 kΩ resistor)?
    (b) What is the capacitor value to achieve the lowest-order resonance at 6.0 GHz (ignore the 10 kΩ resistor)?
    (c) Assume that loss is introduced by placing a 10 kΩ resistor in parallel with the capacitor. What is the Q of the circuit?
    (d) Approximately what is the bandwidth of the circuit?
27. A 50 Ω transmission line is terminated in a load that results in a reflection coefficient of $0.5 + \jmath 0.5$.
    (a) What is the load impedance?
    (b) What is the VSWR on the line?
    (c) What is the input impedance if the line is one-half wavelength long?
28. Communication filters are often constructed using several shorted transmission line resonators that are coupled by passive elements such as capacitors. Consider a coaxial line that is short-circuited at one end. The dielectric filling the coaxial line has a relative permittivity of 64 and the resonator is to be designed to resonate at a center frequency, f_0, of 800 MHz. [Parallels Example 2.24]
    (a) What is the wavelength in the dielectric-filled coaxial line?
    (b) What is the form of the equivalent circuit (in terms of inductors and capacitors) of the one-quarter wavelength long resonator if the coaxial line is lossless?
    (c) What is the length of the resonator?
    (d) If the diameter of the inner conductor of the coaxial line is 2 mm and the inside diameter of the outer conductor is 5 mm, what is the characteristic impedance of the coaxial line?
    (e) Calculate the input admittance of the dielectric-filled coaxial line at $0.99 f_0$, f_0, and $1.01 f_0$. Determine the numerical derivative of the line admittance at f_0.
    (f) Derive the values of the equivalent circuit of the resonator at the resonant frequency and derive the equivalent circuit of the resonator. Hint: Match the derivative expression derived in (e) with the actual derivative derived in Example 2.24.
29. Develop an analytic formula relating a reflection coefficient (Γ_1) in one reference system (Z_{01}) to a reflection coefficient (Γ_2) in another reference system (Z_{02}).
30. A line has a characteristic impedance Z_0 and is terminated in a load with a reflection coefficient of 0.8. A forward-traveling voltage wave on the line has a power of 1 W.
    (a) How much power is reflected by the load?
    (b) What is the power delivered to the load?

31. A load consists of a shunt connection of a capacitor of 10 pF and a resistor of 25 Ω. The load terminates a lossless 50 Ω transmission line. The operating frequency is 1 GHz. [Parallels Example 2.6]
    (a) What is the impedance of the load?
    (b) What is the normalized impedance of the load (normalized to the characteristic impedance of the line)?
    (c) What is the reflection coefficient of the load?
    (d) What is the current reflection coefficient of the load?
    (e) What is the standing wave ratio (SWR)?
    (f) What is the current standing wave ratio (ISWR)?
32. A 50 Ω air-filled transmission line is connected between a 40 GHz source with a Thevenin equivalent impedance of 50 Ω and a load. The SWR on the line is 3.5.
    (a) What is the magnitude of the reflection coefficient, Γ_L, at the load?
    (b) What is the phase constant, β, of the line?
    (c) If the first minimum of the standing wave voltage on the transmission line is 2 mm from the load, determine the electrical distance (in degrees) of the SWR minimum from the load.
    (d) Determine the angle of Γ_L at the load.
    (e) What is Γ_L in magnitude-phase form?
    (f) What is Γ_L in rectangular form?
    (g) Determine the load impedance, Z_L.
33. A load consists of a resistor of 100 Ω in parallel with a 5 pF capacitor with an electrical signal at 2 GHz.
    (a) What is the load impedance?
    (b) What is the reflection coefficient in a 50 Ω reference system?
    (c) What is the SWR on a 50 Ω transmission line connected to the load?
34. An amplifier is connected to a load by a transmission line matched to the amplifier. If the SWR on the line is 1.5, what percentage of the available amplifier power is absorbed by the load?
35. An output amplifier can tolerate a mismatch with a maximum SWR of 2.0. The amplifier is characterized by a Thevenin equivalent circuit with an impedance of 50 Ω and is connected directly to an antenna characterized by a load resistance R_L. Determine the tolerance limits on R_L so that the amplifier does not self-destruct.
36. A load has a reflection coefficient of 0.5 when referred to 50 Ω. The load is placed at the end of a 100 Ω-transmission line.
    (a) What is the complex ratio of the forward-traveling wave to the backward-traveling wave on the $100\ \Omega$ line at the load end of the line?
    (b) What is the VSWR on the $100\ \Omega$ line?
37. A load has a reflection coefficient of 0.5 when referred to $50\ \Omega$. The load is at the end of a line with a $50\ \Omega$ characteristic impedance.
    (a) If the line has an electrical length of $45°$, what is the reflection coefficient calculated at the input of the line?
    (b) What is the VSWR on the $50\ \Omega$ line?
38. A 100 Ω resistor in parallel with a 5 pF capacitor terminates a 100 Ω transmission line. Calculate the SWR on the line at 2 GHz.
39. A lossless 50 Ω transmission line has a 50 Ω generator at one end and is terminated in 100 Ω. What is the VSWR on the line?
40. A lossless 75 Ω line is driven by a 75 Ω generator. The line is terminated in a load that with a reflection coefficient (referred to 50 Ω) of $0.5 + \jmath 0.5$. What is the VSWR on the line?
41. A load with a 20 pF capacitor in parallel with a 50 Ω resistor terminates a 25 Ω line. The operating frequency is 5 GHz. [Parallels Example 2.7]
    (a) What is the VSWR?
    (b) What is ISWR?
42. A load $Z_L = 55 - \jmath 55\ \Omega$ and the system reference impedance, Z_0, is 50 Ω. [Parallels Example 2.8]
    (a) What is the load reflection coefficient Γ_L?
    (b) What is the current reflection coefficient?
    (c) What is the VSWR on the line?
    (d) What is the ISWR on the line?
    (e) Now consider a source connected directly to the load. The source has a Thevenin equivalent impedance $Z_G = 60\ \Omega$ and an available power of 1 W. Use Γ_L to find the power delivered to Z_L.
    (f) What is the total power absorbed by Z_G?
43. A slotted line, as shown in Figure 2-19(c), is used to characterize a 50 Ω line terminated in a load Z_L. $V_{\max} = 1$ V and $V_{\min} = 0.1$ V, and the first minimum is 5 cm from the load. The guide wavelength is 30 cm. What is Z_L? [Parallels Example 2.9]
44. A shorted coaxial line is used as a resonator. The first resonance is determined to be a parallel resonance and is at 1 GHz.
    (a) Draw the lumped-element equivalent circuit of the resonator.
    (b) What is the electrical length of the resonator?

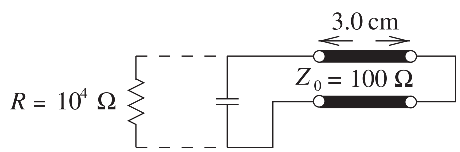
(c) What is the impedance looking into the line at resonance?
(d) If the resonator is $\lambda/4$ longer, what is the impedance of the resonator now?

45. A load of 100 Ω is to be matched to a transmission line with a characteristic impedance of 50 Ω. Use a quarter-wave transformer. What is the characteristic impedance of the quarter-wave transformer?

46. Determine the characteristic impedance of a quarter-wave transformer used to match a load of 50 Ω to a generator with a Thevenin equivalent impedance of 75 Ω.

47. A transmission line is to be inserted between a 5 Ω line and a 50 Ω load so that there is maximum power transfer to the 50 Ω load at 20 GHz.
(a) How long is the inserted line in terms of wavelengths at 20 GHz?
(b) What is the characteristic impedance of the line at 20 GHz?

48. The resonator below is constructed from a 3.0 cm length of 100 Ω air-filled coaxial line shorted at one end and terminated with a capacitor at the other end:



(a) What is the lowest resonant frequency of this circuit without the capacitor (ignore the resistor)?
(b) What is the capacitor value required to achieve resonance at 6.0 GHz?
(c) Assume that loss is introduced by placing a 10 kΩ resistor in parallel with the capacitor. What is the Q of the circuit?
(d) What is the bandwidth of the circuit?

49. A coaxial transmission line is filled with lossy material with a relative permittivity of $5 - \jmath 0.2$. If the line is air-filled it would have a characteristic impedance of 100 Ω.
(a) What is the characteristic impedance of the dielectric-filled line?
(b) What is the propagation constant at 500 MHz?
(c) What is the input impedance of the line if it has an electrical length of 280° and is terminated in a 35 Ω resistor?

50. A coaxial line is filled with a very slightly lossy material with a relative permittivity of 5. The line would have a characteristic impedance of 100 Ω if it was air-filled.
(a) What is the characteristic impedance of the dielectric-filled line?
(b) What is the propagation constant at 500 MHz? Use the fact that the velocity of an EM wave in a lossless air-filled line is the same as that of free-space propagation in air.
(c) What is the input impedance of the line if it has an electrical length of 90° and it is terminated in a 35 Ω resistor?
(d) What is the input impedance of the line if it has an electrical length of 180° and is terminated in an impedance $\jmath 35$ Ω?
(e) What is the input impedance of the line if it is 1 km long? Use reasonable approximations.

51. A lossy transmission line with a characteristic impedance of $60 - \jmath 2$ Ω is driven by a generator with a Thevenin equivalent impedance Z_g. If the line is infinitely long, what is Z_g for maximum power transfer from the generator to the line?

52. A 25 Ω-transmission line is driven by a generator with an available power of 23 dBm and a Thevenin equivalent impedance of 60 Ω. [Parallels Example 2.19]
(a) What is the Thevenin equivalent generator voltage?
(b) What is the magnitude of the forward-traveling voltage wave on the line? Assume the line is infinitely long.
(c) What is the power of the forward-traveling voltage wave?

53. A open-circuited coaxial line is used as a resonator. The first resonance is a series resonance at 2 GHz. [Parallels Example 2.15]
(a) Draw the lumped-element equivalent circuit of the resonator.
(b) What is the resonator's electrical length?
(c) What is the impedance looking into the line at resonance?
(d) If the resonator is $3\lambda_g/4$ longer, what is the input impedance of the resonator?

54. The forward-traveling wave on a 60 Ω line has a power of 2 mW. The line is terminated in a resistance of 50 Ω. How much power is delivered to the 50 Ω load.

55. The forward-traveling wave on a 40 Ω line has a power of 2 mW. The line is terminated in a resistance of 60 Ω. How much power is in the backward traveling wave?

56. The forward-traveling wave on a 60 Ω line has a power of 2 mW. The line is terminated in a resistance of 50 Ω. Draw the lumped-element equiv-

alent circuit at the interface of the line and the load. [Parallels Example 2.17]

57. A source is connected to a load by a one wavelength long transmission line having a loss of 1.5 dB. The source reflection coefficient (referred to the transmission line) is 0.2 and the load reflection coefficient is 0.5.
    (a) What is the transmission coefficient?
    (b) Draw the bounce diagram using the transmission and reflection coefficients. Determine the overall effective transmission coefficient from the source to the load. Calculate the power delivered to the load from a source with an available power of 600 mW.
58. Consider a coaxial line that is short-circuited at one end. The dielectric filling the line has $\varepsilon_r = 20$ and the line has its lowest frequency resonance at 2.4 GHz. [Parallels Example 2.23]
    (a) What is the guide wavelength?
    (b) Draw the resonator's equivalent circuit.
    (c) What is the resonator's physical length?
59. Consider a lossless coaxial line that is open-circuited at one end and is used as a resonator that is resonant at $f_0 = 2.4$ GHz. The line's dielectric has $\varepsilon_r = 81$. [Parallels Example 2.24]
    (a) What is the wavelength in the line?
    (b) Draw the lumped-element equivalent circuit of a $\lambda_g/4$ long resonator?
    (c) What is the resonator's physical length?
    (d) What is the derivative with respect to frequency of the admittance of the LC equivalent circuit developed in (b).
    (e) If the diameter of the inner conductor of the line is 1 mm and the inside diameter of the outer conductor is 3 mm, what is the characteristic impedance of the line?
    (f) Determine the numerical frequency derivative of the line admittance at f_0.
    (g) Derive the values of the equivalent circuit of the resonator at resonance.
60. Develop the lumped-element model of a half wavelength long line having characteristic impedance Z_0. [Parallels Example 2.25]
61. The diameter of the inner conductor of a coaxial line is 2 mm and the interior diameter of the outer conductor 8 mm. The coaxial line is filled with polyimide which has a relative permittivity of 3.2.
    (a) What is the characteristic impedance of the line?
    (b) Describe the conditions by which a non-TEM mode can be supported. Refer to two different families of higher-order modes.
    (c) For the coaxial line here, at what frequency will a second propagating mode be first supported?

2.12.1 Exercises By Section

†challenging, ‡very challenging

§2.1	1
§2.2	2†, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14†, 15
§2.3	16†, 17, 18‡, 19‡, 20, 21, 22, 23, 24, 25, 26†, 27‡, 28‡, 29, 30†, 31†, 32‡, 33‡, 34†, 35†, 36†, 37, 38, 39, 40, 41, 42, 43
§2.4	44†, 45, 46, 47
§2.5	48‡, 49†, 50†, 51, 52‡, 53
§2.6	54, 55, 56, 57†
§2.7	58, 59†, 60
§2.9	61

2.12.2 Answers to Selected Exercises

11	$11.2\,\Omega$
12	$0.23 + \jmath 25\ \mathrm{m}^{-1}$
17(b)	74.0 mW
28(c)	1.17 cm
28(e)	$\jmath 4.55 \cdot 10^{-11}\,\mathrm{S}\cdot\mathrm{s}$
32(f)	$0.544 + \jmath 0.116$
33(f)	41.45
35	$25\,\Omega \leq Z_L \leq 100\,\Omega$
46	$61.2\,\Omega$
57	354.2 mW

Appendix

2.A Physical Constants and Material Properties

2.A.1 SI Units

The main SI units used in RF and microwave engineering are given in Table 2-1. Symbols for SI units[3] (from the French name Système International d'Unités) are written in upright roman font. (Source: U.S. National Institute of Standards and Technology (2006) [26], and 2002 CODATA recommended values of constants [27].)

The International Organization for Standardization (ISO) maintains the International System of Quantities (ISQ) defines the quantities that are measured in SI units [28]. These are set out in the ISO 8000 standard (which is jointly published with the International Electrotechnical Commission (IEC) as the IEC 8000).

- The fundamental units of the SI system are meter, kilogram, second, candela, mole, and Kelvin.
- The unit of length is spelled meter in the United States and metre in other countries.
- The unit designations, such as m for meter, is called a symbol and not an abbreviation.
- Symbols for units are written in lowercase unless the symbol is derived from the name of a person. For example, the symbol for the unit of force is N as it is named after Isaac Newton. An exception is the use of L for liter to avoid possible confusion with l, which looks like the numeral one and the letter i.
- A space separates a value from the symbol for the unit (e.g., 5.6 kg). There is an exception for degrees, with the symbol $^\circ$. For example, 45 degrees is written 45°.

SI Unit Combinations

When SI units are multiplied a center dot is used. For example, newton meters is written N·m. When a unit is derived from the ratio of symbols then either a solidus (/) or a negative exponent is used; the symbol for velocity (meters per second) is either m/s or $\mathrm{m{\cdot}s^{-1}}$. The use of multiple solidi for a combination symbol is confusing and must be avoided. So the symbol for acceleration is $\mathrm{m{\cdot}s^{-2}}$ or m/s^2 and not m/s/s. Another example is the thermal conductivity of aluminum at room temperature which is $k = 237\ \mathrm{kW \cdot m^{-1} \cdot K^{-1}}$ and not k = 237 kW/m/K or 237 kW/m·K. However, 237 kW /(mK) is sometimes used.

Consider calculation of the thermal resistance of a rod of cross-sectional area A and length ℓ:

$$R_{\mathrm{TH}} = \frac{\ell}{kA}. \tag{2.271}$$

If $A = 0.3\ \mathrm{cm}^2$ and $\ell = 2$ mm, the thermal resistance is

$$R_{\mathrm{TH}} = \frac{(2\ \mathrm{mm})}{(237\ \mathrm{kW{\cdot}m^{-1}{\cdot}K^{-1}}) \cdot (0.3\ \mathrm{cm}^2)} = \frac{(2 \cdot 10^{-3}\ \mathrm{m})}{237 \cdot (10^3 \cdot \mathrm{W{\cdot}m^{-1}{\cdot}K^{-1}}) \cdot 0.3 \cdot (10^{-2} \cdot \mathrm{m})^2} \tag{2.272}$$

$$= \frac{2 \cdot 10^{-3}}{237 \cdot 10^3 \cdot 0.3 \cdot 10^{-4}} \cdot \frac{\mathrm{m}}{\mathrm{W{\cdot}m^{-1}{\cdot}K^{-1}{\cdot}m^2}} \tag{2.273}$$

$$= 2.813 \cdot 10^{-4}\ \mathrm{K{\cdot}W^{-1}} = 281.3\ \mu\mathrm{K/W}. \tag{2.274}$$

This would be an error-prone calculation if the thermal conductivity was taken as 237 kW/m/K.

The use of SI units initially means that calculations can be undertaken without the tedium of tracing units through calculations. This requires that the SI unit of the final result be known and assigned. Repeating the above calculation for the thermal resistance of a rod using Equation (2.271), first express the

[3] The older metric systems used different fundamental units; for example, the mks metric system used meter, kilogram, and second as fundamental units; the cgs metric system used centimeter, gram, and second as fundamental units.

Table 2-1: Main SI units used in RF and microwave engineering. The SI units used in electromagnetics are given in Table 1-1.

SI unit	Name	Usage	In terms of fundamental units
A	ampere	current (abbreviated as amp)	Fundamental unit
cd	candela	luminous intensity	Fundamental unit
C	coulomb	charge	$\mathrm{A{\cdot}s}$
F	farad	capacitance	$\mathrm{kg^{-1}{\cdot}m^{-2}{\cdot}A^{-2}{\cdot}s^{4}}$
g	gram	weight	=kg/1000
H	henry	inductance	$\mathrm{kg{\cdot}m^{2}{\cdot}A^{-2}{\cdot}s^{-2}}$
J	joule	unit of energy	$\mathrm{kg{\cdot}m^{2}{\cdot}s^{-2}}$
K	kelvin	thermodynamic temperature	Fundamental unit
kg	kilogram	SI fundamental unit	Fundamental unit
m	meter	length	Fundamental unit
mol	mole	amount of substance	Fundamental unit
N	newton	unit of force	$\mathrm{kg{\cdot}m{\cdot}s^{-2}}$
Ω	ohm	resistance	$\mathrm{kg{\cdot}m^{2}{\cdot}A^{-2}{\cdot}s^{-3}}$
Pa	pascal	pressure	$\mathrm{kg{\cdot}m^{-1}{\cdot}s^{-2}}$
s	second	time	Fundamental unit
S	siemen	admittance	$\mathrm{kg^{-1}{\cdot}m^{-2}{\cdot}A^{2}{\cdot}s^{3}}$
V	volt	voltage	$\mathrm{kg{\cdot}m^{2}{\cdot}A^{-1}{\cdot}s^{-3}}$
W	watt	power	$J \cdot s^{-1}$

quantities in SI units: $\ell = 2 \cdot 10^{-2}$ m; $k = 2.37 \cdot 10^{5}$ W/(m·K); and $A = 0.3 \cdot 10^{-4}$ m^2, then

$$R_{\mathrm{TH}} = \frac{2}{2.37 \cdot 10^{5} \cdot 0.3 \cdot 10^{-4}} = 2.813 \cdot 10^{-4}. \tag{2.275}$$

The SI unit of R_{TH} is K/W, so that $R_{\mathrm{TH}} = 2.813 \cdot 10^{-4}\ \mathrm{K{\cdot}W^{-1}} = 281.3\ \mu\mathrm{K/W}$.

SI Prefixes

A prefix before a unit indicates a multiple of a unit (e.g., 1 pA is 10^{-12} amps). (Source: 2015 ISO/IEC 8000 [28].) In 2009 new definitions of the prefixes for bits and bytes were adopted [28] removing the confusion over the earlier use of quantities such as kilobit to represent either 1,000 bits or 1,024 bits. Now kilobit (kbit) always means 1,000 bits and a new term kibibit (Kibit) means 1,024 bit. Also the now obsolete usage of kbps is replaced by kbit/s (kilobit per second). The prefix k stands for kilo (i.e. 1,000) and Ki is the symbol for the binary prefix kibi- (i.e. 1,024). (Note that "K" is sometimes used as an abbreviation for 1,024 but this is non-standard.) The symbol for byte (= 8 bits) is "B".

Table 2-2: SI prefixes.

SI Prefixes			SI Prefixes			Prefixes for bits and bytes		
Symbol	Factor	Name	Symbol	Factor	Name	Name		
10^{-24}	y	yocto	10^{1}	da	deca	kilobit	kbit	1000 bit
10^{-21}	z	zepto	10^{2}	h	hecto	megabit	Mbit	1000 kbit
10^{-18}	a	atto	10^{3}	k	kilo	gigabit	Gbit	1000 Mbit
10^{-15}	f	femto	10^{6}	M	mega	terabit	Tbit	1000 Gbit
10^{-12}	p	pico	10^{9}	G	giga	kibibit	Kibit	1024 bit
10^{-9}	n	nano	10^{12}	T	tera	mebibit	Mibit	1024 Kibit
10^{-6}	μ	micro	10^{15}	P	peta	gibibit	Gibit	1024 Mibit
10^{-3}	m	milli	10^{18}	E	exa	tebibit	Tibit	1024 Gibit
10^{-2}	c	centi	10^{21}	Z	zetta	kilobyte	kB	1000 B
10^{-1}	d	deci	10^{24}	Y	yotta	kibibyte	KiB	1024 B

Physical and Mathematical Constants

Physical and mathematical constants in SI units. Source: U.S. National Institute of Standards and Technology (2006) [26], and 2002 CODATA recommended values of constants [27].

Table 2-3: Physical and mathematical constants.

Parameter	Value	Description
c	$299792458\ \mathrm{m{\cdot}s^{-1}}$	Speed of light in a vacuum (free space)
e	1.6021765310^{-19} C	Elementary charge (negative of an electron's charge)
e	2.718281828459045	Natural log base
γ	0.577215664901532	Euler's ratio
ϕ	1.618033988749894	Golden ratio
ε_0	$8.854187817 \times 10^{-12}\ \mathrm{F{\cdot}m^{-1}}$	Permittivity of a vacuum (free space)
h	$6.6260693 \times 10^{-34}\ \mathrm{J{\cdot}s}$	Planck constant (alt. $\hbar = h/(2\pi)$)
k	$1.3806505 \times 10^{-23}\ \mathrm{J{\cdot}K^{-1}}$	the Boltzmann constant
m_e	$9.1093826 \times 10^{-31}$ kg	Electron mass
μ_0	$12.566370614 \times 10^{-7}\ \mathrm{N{\cdot}A^{-2}}$	Permeability of free space = $4\pi \times 10^{-7}\ \mathrm{N{\cdot}A^{-2}}$
π	3.14159265358979323846264	Pi, ratio of circumference to diameter of a circle
	101325 Pa	Standard atmosphere (pressure)
η	$376.730313461\ \Omega$	Characteristic impedance of vacuum (free space)
	6.241509×10^{18} eV	1 joule of energy in terms of electron-volts
	1.602176×10^{-19} J	1 eV of energy in joules (the energy required to move a charge e through a potential of 1 V)

Accuracy and Precision

Precision is a description of statistical variability or random error while accuracy includes systematic errors in a measurement or calculation (and these are not random) combined with statistical variability. For example, consider the calculation of the resistance of a uniform length of metal using dimensions (length ℓ, width w, thickness t, which are known accurately) and resistivity ρ where the ρ may not be known accurately. The resistance $R = \rho\ell/(wt)$.

The resistance calculation would not be accurate if the resistivity is not known accurately. However the calculation would be precise if the resistivity of the metal is known to be fixed (and not statisically variable). If the resistivity of the metal varies from place to place, i.e. it is not homogeneous, then there is statistical variability of the resistivity depending on the section of metal chosen and so both the precision and the accuracy of the resistance calculation would be poor. You can have a precise answer that is not accurate, but you cannot have an accurate result that is not precise.

The number of significant digits in a calculation or measurement implies the accuracy and/or precision of a number. Sometimes the accuracy is explicitly stated, for example 4.01 ± 0.02 m, but more commonly in engineering the number of significant digits implies the accuracy and/or precision. Error is taken to be one half of the last significant digit. So 1200 m implies an accuracy of 0.5 m. While 1.2 km would imply an accuracy of 0.05 km or 20 m. Usually 4 is enough but there are departures from this.

One common situation is when we are talk about the center frequency of a carrier in a communication system. That is because we can precisely set the frequency of a carrier and can put communication bands very close together. For example, if we have frequency bands that are 25 kHz wide on a carriers near 900 MHz. We would need to indicate the frequency of the carrier to a fraction of a kilohertz. So there is a significant difference between specifying a carrier as being at 1.000000 GHz or at 1.000001 GHz. Sometimes however it is not important to specify frequency so accurately. In the great majority of situations four digits of accuracy is enough, in any case it is very difficult to fabricate something with better than 0.1% accuracy.

When using decibels, a logarithmic scale, two digits after the decimal point is the usually sufficient and required. One digit after the decimal point is not enough accuracy. For example, $0\ \mathrm{dB} = 10^0 = 1.000$ and $0.1\ \mathrm{dB} = 10^{0.01} = 1.023$. So if there is only one digit after the decimal point is used then the accuracy implied is 1%. Now $0.01\ \mathrm{dB} = 10^{0.001} = 1.0023$ and so with two digits after the decimal point the implied accuracy is 0.1% which is about four digits of precision. With four digits after the decimal digit, $0.0001\ \mathrm{dB} = 10^{0.00001} = 1.000023$. which is the same as claiming around 6 digits of accuracy. Achieving this is highly unlikely except in circumstances involving frequency or when circuits are tuned after fabrication.

There are a few exceptions to the accuracy implied by the number of digits used with decibels. When a number is written as 0 dB or 10 dB, for example, with no digits after the decimal point, then the implication is that it is exactly 0 dB (= 1.00000000) or 10 dB (= 10.0000000). Another exception is a factor of 2 which in decibels is 3.01 dB. It is common to write this as 3 dB and when one sees 3 dB then most people automatically recognize this as a factor of exactly 2. Similarly for 6 dB the implication is that it refers to a factor of exactly 4.

Using more significant digits in a result than is justified can leave an impression of a lack of understanding, that the result was simply a matter of plugging numbers into a calculation rather than understanding what was being done. When an engineer presents results, the observer, usually another engineer, wants to develop trust for the engineering process. Engineering is pragmatic and abstractions are made, you always want to create an impression that you are in control and know what you are doing. Sure interim results can have many significant digits of accuracy but final results should have reasonable accuracy.

Standard Temperatures

Table 2-4: Temperature constants.

Description	Value	In Terms of Fundamental Units
Absolute zero temperature	0 K	Fundamental unit $= -273.15$°C
Room temperature	290–298 K	19–25°C, generally used as an imprecise measurement implying that properties are unchanged over a few degrees variation.
Standard temperature	290 K	In microwave engineering [29], different in other disciplines.
Available noise of a resistor at room temperature		-174 dBm/Hz. (e.g., in 2 Hz bandwidth the available noise of a resistor at room temperature is -171 dBm). (At 290 K the available noise power is -173.97 dBm/Hz, at 293 K it is -173.93 dBm/Hz, at 298 K it is -173.86 dBm/Hz.)

2.A.2 Greek Alphabet and Additional Characters

Name	Uppercase	Lowercase
alpha	A	α
beta	B	β
gamma	Γ	γ
delta	Δ	δ
epsilon	E	ϵ
zeta	Z	ζ
eta	H	η
theta	Θ	θ
iota	I	ι
kappa	K	κ
lambda	Λ	λ
mu	M	μ

Name	Uppercase	Lowercase
nu	N	ν
xi	Ξ	ξ
omicron	O	o
pi	Π	π
rho	P	ρ
sigma	Σ	σ
tau	T	τ
upsilon	Υ	υ
phi	Φ	ϕ
chi	X	χ
psi	Ψ	ψ
omega	Ω	ω

Additional characters	
nabla	∇
cross	$\times$
times	$\times$
varepsilon	ε
varphi	φ
varpi	ϖ
varrho	ϱ
varsigma	ς
vartheta	ϑ
aleph	$\aleph$

2.A.3 Conductors, Dielectrics, and Magnetic Materials

Electrical and thermal properties of RF and microwave materials are given in the tables below. A parameter listed as a range indicates that the parameter depend on the formulation of the alloy. $\perp$ indicates the property in the direction perpendicular to the crystal axis. // indicates the property in the direction parallel to the crystal axis.

Material data from several sources including the *Standard Reference Data* database of the U.S. National Institute of Standards and Technology [26], the CODATA databases of the International Council for Science, Committee on Data for Science and Technology [27], and references [30–33]. Electrical and especially thermal properties are functions of temperature; properties at temperatures other than 300 K should be researched.

Table 2-5: Relative permeability of metals.

Material	Relative permeability μ_r	Material	Relative permeability μ_r
Aluminum	1.00000065	Nickel	50–600
Cobalt	60	Palladium	1.000 8
Copper	0.999 994	Permalloy 45	2 500
Ferrite (NiZn)	16–640	Platinum	1.000 265
Gold	0.999 998	Silver	0.99999981
Iron	5 000–6 000	Steel	100–40 000
Lead	0.999 983	Superconductors	0
Magnesium	1.000 006 93	Supermalloy	100 000
Manganese	1.000 125	Tungsten	1.000 068
Mumetal	20 000–1000 000	Wood (dry)	0.99999942

Some calculations require the use of volumetric heat capacity, c_v obtained from

$$c_v = c_p \rho \tag{2.276}$$

but ensure the use of SI units, i.e. convert the denisty ρ to $\mathrm{kg \cdot m^{-3}}$.

The electrical resistivities listed in Table 2-8 for single-element metals are those of single-crystal metals. The resistivity of the best fabricated metal with multiple crystal grains tends to be up to 5% above that of a single-crystal. Poorly fabricated metals can have a resistivity twice as high.

Table 2-6: Electrical properties of dielectrics and nonconductors.

Material	Resistivity (MΩ·m ρ, at 300 K)	Relative permittivity (ε_r at 1 GHz)	Loss tangent ($\tan\delta$ at 1 GHz)
Air (dry, sea level)	4×10^7	1.000 5	0.000
Alumina			
99.5%	$> 10^6$	9.8	0.000 1–0.000 2
96%	$> 10^6$	9.0	0.000 6
85%	$> 10^6$	8.5	0.001 5
Aluminum nitride	10^6	8.9	0.001
Bakelite	1–100	4.74	0.022
Beryllium oxide (toxic)	$> 10^8$	6.7	0.004
Diamond	10^5–10^{10}	5.68	$< 0.000\,1$
Ferrite (MnZn)	0.1–10 Ω·m	13–16	0.000 4
Ferrite (NiZn)	0.1–12.4	13–16	0.000 4
FR-4 circuit board	8×10^5	4.3–4.5	0.01
GaAs	1.0	12.85	0.000 6
InP	Up to 0.001	12.4	0.001
Glass	2×10^8	4–7	0.002
Mica	2×10^5	5.4	0.000 6
Mylar	10^{10}	3.2	0.005
Paper, white	3.5×10^6	3	0.008
Polyethylene	$> 10^7$	2.26	0.000 2
Polyimide	10^{10}	3.2	0.005
Polypropylene	$> 10^7$	2.25	0.000 3
Quartz (fused)	7.5×10^{11}	3.8	0.000 75
Sapphire			
//	$> 10^6$	11.6	0.000 04–0.000 07
$\perp$	$> 10^6$	9.4	0.000 04–0.000 07
Polycrystalline	$> 10^6$	10.13	0.000 04–0.000 07
Silicon (undoped)			
Low resistivity (used in CMOS)	50 μΩ·m	11.68	0.005
High resistivity	300 mΩ·m	11.68	0.005
Carbide (SiC)	100	10.8	0.002
Dioxide (SiO_2)	5.8×10^7	3.7–4.1	0.001
Nitride (Si_3N_4)	10^7	7.5	0.001
Poly	0.1–10 kΩ·m	11.7	0.005
Teflon (PTFE)	10^{10}	2.1	0.000 3
Vacuum	∞	1	0
Water			
Distilled	182	80	0.1
Ice (273 K)	1	4.2	0.05
Wood (dry oak)	3×10^{11}	1.5–4	0.01
Zirconia (variable)	10^4	28	0.000 9

Table 2-7: Thermal properties of dielectrics and nonconductors.

Material	Thermal conductivity, k $(\mathrm{W \cdot m^{-1} \cdot K^{-1}})$ (at 300 K)	Specific heat capacity, c_p $(\mathrm{kJ \cdot kg^{-1} \cdot K^{-1}})$ (at 25°C)	Density, ρ $(\mathrm{g \cdot cm^{-3}})$ (at 25°C)	Speed of sound, c_s $(\mathrm{m \cdot s^{-1}})$ (at 25°C)
Air (dry, sea level)	0.026	1.005	0.0018	343
Alumina (Al_2O_3)				9900–10520
100%	30	0.78	3.8	
99.5%	26.9–30	0.78	3.8	
96%	24.7	0.78	3.8	
85%	16	0.92	3.5	
Aluminum nitride	285	0.74	3.28	11000
Bakelite (wood filler)	0.2–1.4	1.38	1.25–1.36	
Beryllium oxide (toxic)	64–210	1.75 (@0°C)	1.85–2.85	
Diamond	1 000–2 000	0.52–0.63	3.50–3.53	12000
Ferrite (MnZn)	3.5–5	0.7–0.8	4.9	
Ferrite (NiZn)	3.5–5	0.75	4.5	
FR-4 circuit board	0.16–0.3	0.6	1.3–1.8	
Graphite	25–470	0.71–0.83	1.3–2.27	1200
GaAs	50–59	0.37	5.32	4730
InP	68	0.31	4.81	
Glass	0.8–1.2	0.5–0.84	2.0–8.0	3950–5640
Mica	260–750	0.5	0.72	
Mylar (polyethylene-terephthalate)	0.08	1.19	1.4	1900–2430
Paper (white bond)	40–90	1.4	0.72	
Polyethylene	0.42–0.51	2.3–2.9	2.30	1900–2430
Polyimide	0.12	1.09–1.15	1.43	
Polypropylene	0.35–0.40	1.7–2.0	0.855	2740
Quartz (fused)	1.30–1.44	0.67–0.74	2.2	5800
Sapphire				11100
//	35	0.74–0.78	4.05	
$\perp$	32	0.74–0.78	4.05	
Polycrystalline	31–33	0.74–0.78	3.97–4.05	
Silicon (undoped)				
low resistivity (CMOS)	149	0.705	2.34	8433
high resistivity	149	0.705	2.34	8433
carbide (SiC)	350–490	0.75	2.55	13060
dioxide (SiO_2)	1.4	1.0	2.27–2.63	
nitride (Si_3N_4)	28	0.711	3.44	11000
polysilicon	12.5–157	0.71–0.75	2.2–2.3	
Teflon (PTFE)	0.20–0.25	0.97	2.1–2.2	1400
Vacuum	0	0	0	
Water				
Distilled	580	4.18	0.997	1480
Ice (at 273 K)	2.22	2.05	0.917	4000
Wood (dry oak)	170	2	0.6–0.9	3960
Zirconia (variable)	1.7–2.2	0.40–0.50	5.6–6.1	

Table 2-8: Thermal and electrical properties of conductors.

Conductors						
Material	Electrical resistivity, ρ ($\mathrm{n\Omega \cdot m}$) (at 20 °C)	Thermal conductivity, k ($\mathrm{W \cdot m^{-1} \cdot K^{-1}}$) (at 300 K)	Specific heat capacity, c_p ($\mathrm{kJ \cdot kg^{-1} \cdot K^{-1}}$) (at 25°C)	Density, ρ ($\mathrm{g \cdot cm^{-3}}$) (at 25°C)	Thermal coefficient of resistance ($\mathrm{K^{-1}}$) (at 20°C)	Speed of sound, c_s ($\mathrm{m \cdot s^{-1}}$) (at 20°C)
Aluminum	26.50	237	0.897	2.70	0.004308	6420
Brass	Variable	120	0.38	8.4–8.7	0.0015	3500–4700
Bronze	Variable	110	0.38	7.4–8.9		
Chromium	125	93.9	0.450	7.15		5490
Constantan	500	19.5	0.39	8.9	± 0.00003	
Copper	16.78	401	0.39	8.94	0.004041	3560–4700
Gold	22.14	318	0.129	19.30	0.003715	3240
Graphite						
// c-axis	1 200	1 950	0.71	2.09–2.23	-0.0002	
$\perp$ c-axis	41 000	5.7	0.71	2.09–2.23	-0.0002	
Iridium	47.1	147	0.131	22.6		
Iron (cast,hard)	96.1	80.2	0.449	7.87	0.005671	5600–5900
Lead	208	35.3	0.127	11.3		1160–2200
Manganin	430–480	22	0.406	8.4	± 0.000015	
Mercury	961	8.34	0.139	13.53	0.0089	
Nickel	69.3	90.9	0.445	8.90	0.0058–0.0064	5600
NiChrome	1 100	11.3	0.432	8.40	0.00017	
Palladium	105.4	71.8	0.244	12.0		3070
Platinum	105	71.6	0.133	21.5	0.0037–0.0038	3300
Silver	15.87	429	0.235	10.49	0.003819	3600–3650
Solder						
tin-lead Pb, Sn 50% Pb	17.2	34	0.167	8.89		
lead-free 77.2% Sn, 2.8% Ag, 20% In	170	53.5	0.23	7.25		
Steel, stainless	720	16	0.483	7.48–8.00		5740–5790
Steel, carbon (standard)	208	46	0.49	7.85	0.003–0.006	4880–5050
Tantalum	133	57.5	0.14	16.69	0.0038	
Tin	115	66.8	0.227	7.27		3300
Titanium	4 200	21.9	0.522	4.51		6070–6100
Tungsten	52.8	173	0.132	19.3	0.004403	5200
Zinc	59.0	116	0.388	7.14	0.0037–0.0038	4200

CHAPTER 3

Planar Transmission Lines

3.1 Introduction

The majority of transmission lines used in high-speed digital, RF, and microwave circuits are planar, as these can be defined using masks, photoresist, and etching of metal sheets. The most common planar transmission line is the microstrip line shown in Figure 3-1 and in cross section in Figure 3-2. This cross section is typical of what would be found on a semiconductor or **printed wiring board** (**PWB**), which is also called a **printed circuit board** (**PCB**). Current flows in both the top and bottom conductor, but in opposite directions. The physics is such that if there is a signal current on the top conductor, there must be a return signal current, which will tend to be as close to the signal current as possible to minimize stored energy. The provision of a signal return path close to the signal path is important in maintaining the integrity of (i.e., a predictable signal waveform on) an interconnect.

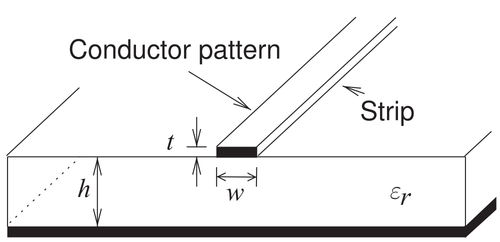


Figure 3-1: Microstrip transmission line.