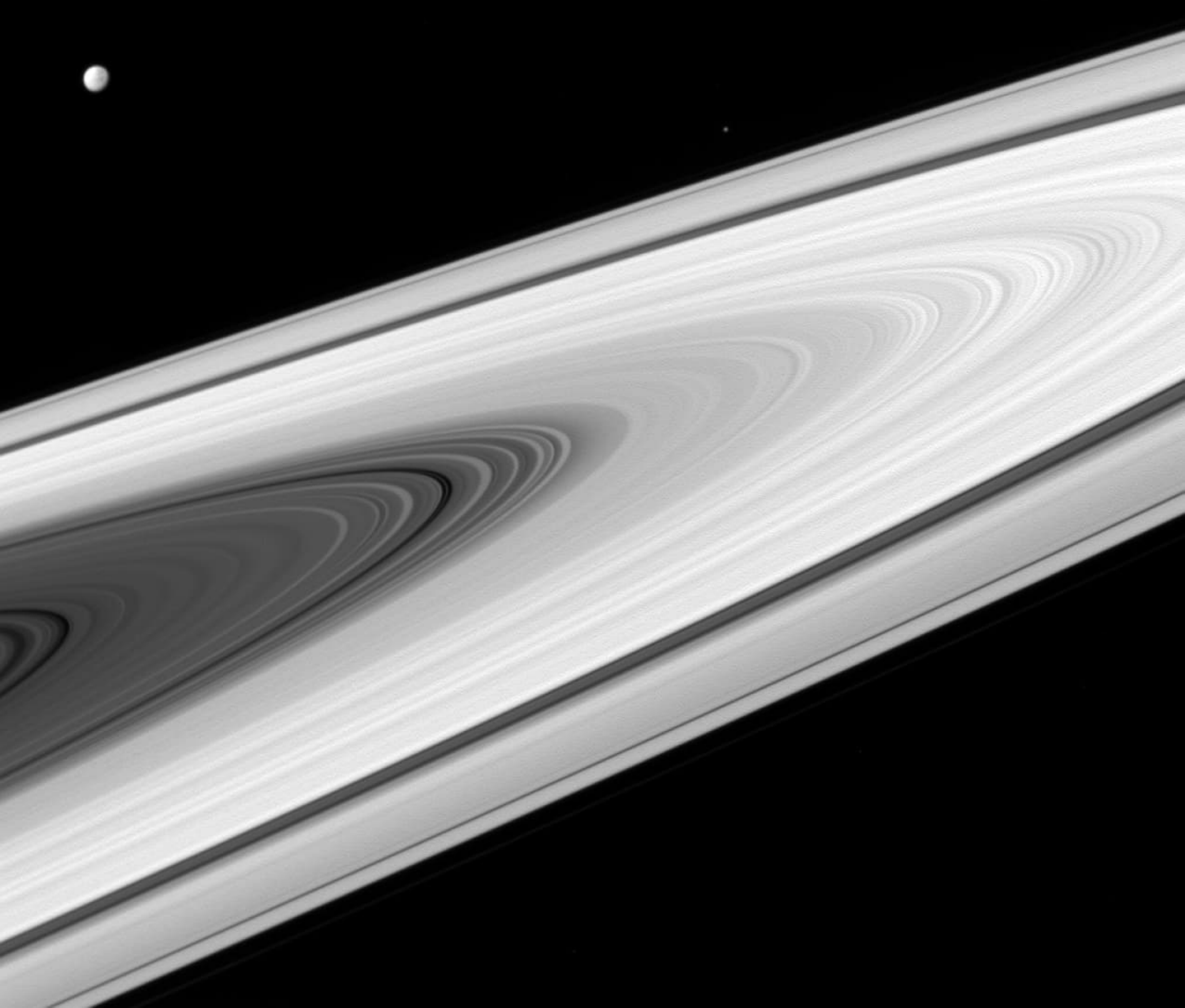
# University Physics I: Classical Mechanics

Julio Gea-Banacloche



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Cover image from NASA, https://www.nasa.gov/image-feature/jpl/not-really-starless-at-saturn

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# Contents

# Preface

**Students: if this is too long, at the very least read the last four paragraphs. Thank you!**

For many years Eric Mazur's *Principles and Practice of Physics* was the required textbook for University Physics I at the University of Arkansas. In writing this open-source replacement I have tried to preserve some of its best features, while at the same time condensing much of the presentation, and reworking several sections that did not quite fit the needs of our curriculum: primarily, the chapters on Thermodynamics, Waves, and Work. I have also skipped entirely the chapter on the "Principle of Relativity," and instead distributed its contents among other chapters: in particular, the Galilean reference frame transformations are now introduced at the very beginning of the book, as are the law of inertia and the concept of inertial reference frames.

Over the past few decades, there has been a trend to increase the size of introductory physics textbooks, by including more and more visual aids (pictures, diagrams, boxes…), as well as lengthier and more detailed explanations, perhaps in an attempt to reach as many students as possible, and maybe even to take the place of the instructor altogether. It seems to me that the result is rather the opposite: a massive (and expensive) tome that no student could reasonably be expected to read all the way through, at a time when "TL;DR" has become a popular acronym, and visual learning aids (videotaped lectures, demonstrations, and computer simulations) are freely available everywhere.

Our approach at the University of Arkansas, developed as a result of the work of Physics Education Research experts John and Gay Stewart, is based instead on two essential facts. First, that different students learn differently: some will learn best from a textbook, others will learn best from a lecture, and most will only really learn from a hands-on approach, by working out the answers to questions themselves. Second, just about everybody will benefit from repeated presentations of the material to be learned, in different environments and even from slightly different points of view.

In keeping with this, we start by requiring the students to read the textbook material before coming to lecture, and also take an online "reading quiz" where they can check their understanding of what they have read. Then, in the lecture, they will have an opportunity to see the material presented again, as a sort of executive summary delivered by, typically, a different instructor, who will also be able to answer any questions they might have about the book's presentation. Additionally, the instructor will directly test the students' understanding by means of conceptual questions asked of the whole class, which are to be answered using clickers. This prompts the students to think harder about the material, and encourages them to discuss it on the spot with their classmates.

Immediately after each lecture, the students will have a lab activity where they will be able to verify experimentally the concepts and principles to which they have just been introduced. Finally, every week they will have an "open response" homework assignment where they get to apply the principles, mathematically, to concrete problems. For both the labs and the homework, additional assistance is provided by a group of dedicated teaching assistants, who are often able to explain the material to the students in a way that better relates to their own experience.

In all this, the textbook is expected to play an important role, but certainly not to be the students' only (nor even, necessarily, the primary) source of understanding or knowledge. Its job is to start the learning process, and to stand by to provide a reference (among possibly several others) afterwards. To fulfill this role, perhaps the most essential requirement is that it should be *readable*, and hence concise enough for every reading assignment to be of manageable size. This (as well as a sensible organization, clear explanations, and a minimal assortment of worked-out exercises and end-of chapter problems) is what I have primarily tried to provide here.

A student who wants more information than provided in this textbook, or alternative explanations, or more worked-out examples, can certainly get these from many other sources: first, of course, the instructor and the teaching assistants, whose essential role as learning facilitators has to be recognized from the start. Then, there is a variety of alternative textbooks available: the best choice, probably, would still be Mazur's *Principles and Practice of Physics*, since it uses the same terminology and notation, introduces the material in almost the same sequence, and has tons of worked-out examples and self-quiz conceptual questions. That book is available on reserve in the Physics library, where it can be consulted by anybody. If a student feels the need for an alternative textbook that they can actually take home, one option is, of course, to actually buy Mazur's (which is what everybody had to do before); another option is to explore other open-source textbooks available online, which have been around longer than this one and benefit from more worked-out exercises and a more conventional presentation. One such book is *University Physics I* at openstax.org (https://openstax.org/details/books/university-physics-volume-1).

Finally, there are also a large number of other online resources, although I would advise the students to approach them with caution, since not all of them may be totally rigorous, and some may end up being more confusing than helpful. Some of my students have found the Khan Academy lectures and/or the "Flipping Physics" lectures helpful; I personally would recommend the lectures of Walter Lewin at MIT, if only for the wide array of cool demonstrations you can see there.

One last word, for the students who may have read this far, concerning the use of equations and "proofs" in this book. It is essential to the nature of physics to be able to cast its results in mathematical terms, and to use math to explain and predict new results; hence, equations and mathematical derivations are integral parts of any physics textbook. I have, however, tried to keep the math as simple as possible throughout, and I would not want a lengthy mathematical derivation to get in the way of your reading assignment. If you are reading the text and come upon several lines of math, skim them at first to see if they make sense, but if you get stuck do not spend too much time on them: move on to the bottom line, and keep reading from there. You can always ask your instructor or TA for help with the math later.

I would, however, encourage you to return, eventually, to any bit of math that you found challenging the first time around. Do try to go through all the algebra yourself! I have occasionally skipped intermediate steps, just to keep the math from overwhelming the text: but these are typically straightforward manipulations (multiplying or dividing both sides by something, moving something from one side of the equation to the other, multiplying out a parenthesis or, conversely, pulling out a common factor or denominator…). If you actually work out, on your own, all the missing steps, you will find it's a great way to improve your algebra skills. This is something that will make it much easier for you to deal with the homework and the exams later on this semester—and for the rest of your career as well.

**P.S. A few possibly useful features.**

The pdf version of this book is, of course searchable: you can use command-F on the Mac or control-F in Windows to search the text for any term (hint: try searching the textbook before hitting Google!!). Hopefully, this will make up a little for the lack of an actual index, which I just haven't found the time to compile yet.

The text is also pretty thoroughly hyperlinked: every equation number and figure number quoted is a link. Clicking on the equation number (for instance, the (6.30), in this sentence) will take you to where the equation was first displayed (in this case, in Chapter 6). Then you can use command-left arrow (on the Mac) or alt-left arrow (on Windows) to go back to the page you were reading before you clicked on the link. Command- (or alt-) left and right arrows will let you go back and forth between the two pages. (Note: these keyboard shortcuts only work in Adobe Acrobat, as far as I know.)

The same thing works if you click on the reference to a figure number (actually that takes you to the caption to the figure, so you may need to scroll up a bit to see the figure).

Finally, the entries in the table of contents are links to the respective sections as well (except for the preface, which for some reason does not work; oh, well… if you are reading this, at least you made it here!).

# Chapter 1

# Reference frames, displacement, and velocity

## 1.1 Introduction

Classical mechanics is the branch of physics that deals with the study of the motion of anything (roughly speaking) larger than an atom or a molecule. That is a lot of territory, and the methods and concepts of classical mechanics are at the foundation of any branch of science or engineering that is concerned with the motion of anything from a star to an amoeba—fluids, rocks, animals, planets, and any and all kinds of machines. Moreover, even though the accurate description of processes at the atomic level requires the (formally very different) methods of quantum mechanics, at least three of the basic concepts of classical mechanics that we are going to study this semester, namely, momentum, energy, and angular momentum, carry over into quantum mechanics as well, with the last two playing, in fact, an essential role.

### 1.1.1 Particles in classical mechanics

In the study of motion, the most basic starting point is the concept of the *position* of an object. Clearly, if we want to describe accurately the position of a macroscopic object such as a car, we may need a lot of information, including the precise shape of the car, whether it is turned this way or that way, and so on; however, if all we want to know is how far the car is from Fort Smith or Fayetteville, we do not need any of that: we can just treat the car as a dot, or mathematical point, on the map—which is the way your GPS screen will show it, anyway. When we do this, we say that are describing the car (or whatever the macroscopic object may be) as a **particle**.

In classical mechanics, an “ideal” particle is an object with no appreciable size—a mathematical point. In one dimension (that is to say, along a straight line), its position can be specified just by giving a single number, the distance from some reference point, as we shall see in a moment (in three dimensions, of course, three numbers are required). In terms of energy (which is perhaps the most important concept in all of physics, and which we will introduce properly in due course), an ideal particle has only one kind of energy, what we will later call *translational kinetic energy*; it cannot have, for instance, rotational kinetic energy (since it has “no shape” for practical purposes), or any form of internal energy (elastic, thermal, etc.), since we assume it is too small to have any internal structure in the first place.

The reason this is a useful concept is not just that we can often treat extended objects as particles in an approximate way (like the car in the example above), but also, and most importantly, that if we want to be more precise in our calculations, *we can always treat an extended object (mathematically) as a collection of “particles.”* The physical properties of the object, such as its energy, momentum, rotational inertia, and so forth, can then be obtained by adding up the corresponding quantities for all the particles making up the object. Not only that, but the interactions between two extended objects can also be calculated by adding up the interactions between all the particles making up the two objects. This is how, once we know the form of the gravitational force between two particles (which is fairly simple, as we will see in Chapter 10), we can use that to calculate the force of gravity between a planet and its satellites, which can be fairly complicated in detail, depending, for instance, on the relative orientation of the planet and the satellite.

The mathematical tool we use to calculate these “sums” is *calculus*—specifically, integration—and you will see many examples of this... in your calculus courses. Calculus I is only a corequisite for this course, so we will not make a lot of use of it here, and in any case you would need multidimensional integrals, which are an even more advanced subject, to do these kinds of calculations. But it may be good for you to keep these ideas on the back of your mind. Calculus was, in fact, invented by Sir Isaac Newton precisely for this purpose, and the developments of physics and mathematics have been closely linked together ever since.

Anyway, back to particles, the plan for this semester is as follows: we will start our description of motion by treating every object (even fairly large ones, such as cars) as a “particle,” because we will only be concerned at first with its translational motion and the corresponding energy. Then we will progressively make things more complex: by considering systems of two or more particles, we will start to deal with the *internal energy* of a system. Then we will move to the study of *rigid bodies*, which are another important idealization: extended objects whose parts all move together as the object undergoes a translation or a rotation. This will allow us to introduce the concept of rotational kinetic energy. Eventually we will consider *wave motion*, where different parts of an extended object (or “medium”) move relative to each other. So, you see, there is a logical progression here, with most parts of the course building on top of the previous ones, and energy as one of the main connecting themes.

### 1.1.2 Aside: the atomic perspective

As an aside, it should perhaps be mentioned that the building up of classical mechanics around this concept of ideal particles had nothing to do, initially, with any belief in "atoms," or an atomic theory of matter. Indeed, for most 18th and 19th century physicists, matter was supposed to be a continuous medium, and its (mental) division into particles was just a mathematical convenience.

The atomic hypothesis became increasingly more plausible as the 19th century wore on, and by the 1920's, when quantum mechanics came along, physicists had to face a surprising development: matter, it turned out, was indeed made up of "elementary particles," but these particles could *not*, in fact, be themselves described by the laws of classical mechanics. One could not, for instance, attribute to them simultaneously well-defined positions and velocities. Yet, in spite of this, most of the conclusions of classical mechanics remain valid for macroscopic objects, because, most of the time, it is OK to (formally) "break up" extended objects into chunks that are small enough to be treated as particles, but large enough that one does not need quantum mechanics to describe their behavior.

Quantum properties were first found to manifest themselves at the macroscopic level when dealing with thermal energy, because at one point it really became necessary to figure out where and how the energy was stored at the truly microscopic (atomic) level. Thus, after centuries of successes, classical mechanics met its first failure with the so-called *problem of the specific heats*, and a completely new physical theory—quantum mechanics—had to be developed in order to deal with the newly-discovered atomic world. But all this, as they say, is another story, and for our very brief dealings with thermal physics—the last chapter in this book—we will just take specific heats as given, that is to say, something you measure (or look up in a table), rather than something you try to calculate from theory.

## 1.2 Position, displacement, velocity

*Kinematics* is the part of mechanics that deals with the mathematical description of motion, leaving aside the question of what causes an object to move in a certain way. Kinematics, therefore, does not include such things as forces or energy, which fall instead under the heading of dynamics. It may be said, then, that kinematics by itself is not true physics, but only applied mathematics; yet it is still an essential part of classical mechanics, and its most natural starting point. This chapter (and parts of the next one) will introduce the basic concepts and methods of kinematics in one dimension.

### 1.2.1 Position

As stated in the previous section, we are initially interested only in describing the motion of a "particle," which can be thought of as a mathematical point in space. (Later on we will see that, even for an extended object or system, it is often useful to consider the motion of a specific point that we call the system's *center of mass*.) A point in three dimensions can be located by giving three numbers, known as its *Cartesian coordinates* (or, more simply, its *coordinates*). In two dimensions, this works as shown in Figure 1.1 below. As you can see, the coordinates of a point just tell us how to find it by first moving a certain distance $x$, from a previously-agreed origin, along a horizontal (or $x$) axis, and then a certain distance $y$ along a vertical (or $y$) axis. (Or, of course, you could equally well first move vertically and then horizontally.)

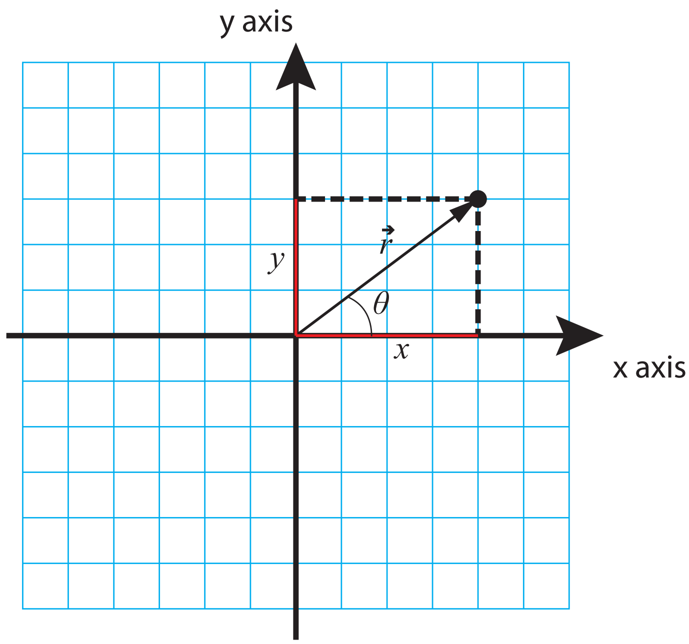



**Figure 1.1:** The position vector, $\vec{r}$, of a point, and its $x$ and $y$ components (the point's coordinates).

The quantities $x$ and $y$ are taken to be positive or negative depending on what side of the origin the point is on. Typically, we will always start by choosing a *positive direction* for each axis, as the direction along which the algebraic value of the corresponding coordinate increases. This is often chosen to be to the right for the horizontal axis, and upwards for the vertical axis, but there is nothing that says we cannot choose a different convention if it turns out to be more convenient. In Figure 1.1, the arrows on the axes denote the positive direction for each. Going by the grid, the coordinates of the point shown are $x = 4$ units, $y = 3$ units.

In two or three dimensions (and even, in a sense, in one dimension), the coordinates of a point can be interpreted as the *components of a vector* that we call the point's **position vector**, and denote by $\vec{r}$ (sometimes boldface letters are used for vectors, instead of an arrow on top; in that case, the position vector would be denoted by **r**). A **vector** is a mathematical object, with specific geometric and algebraic properties, that physicists use to represent a quantity that has both a magnitude and a direction. The *magnitude* of the position vector in Fig. 1.1 is just the length of the arrow, which is to say, 5 length units (by the Pythagorean theorem, the length of $\vec{r}$, which we will often write using absolute value bars as $|\vec{r}|$, is equal to $\sqrt{x^2+y^2}$); this is just the straight-line distance of the point to the origin. The *direction* of $\vec{r}$, on the other hand, can be specified in a number of ways; a common convention is to give the value of the angle that it makes with the positive $x$ axis, which I have denoted in the figure as $\theta$ (in this case, you can verify that $\theta = \tan^{-1}(y/x) = 36.9^\circ$). In three dimensions, two angles would be needed to completely specify the direction of $\vec{r}$.

As you can see, giving the magnitude and direction of $\vec{r}$ is a way to locate the point that is completely equivalent to giving its coordinates $x$ and $y$. By the same token, the coordinates $x$ and $y$ are a way to specify the vector $\vec{r}$ that is completely equivalent to giving its magnitude and direction. As I stated above, we call $x$ and $y$ the components (or sometimes, to be more specific, the Cartesian components) of the vector $\vec{r}$. In a sense all the vectors that will be introduced later on this semester will derive their geometric and algebraic properties from the position vector $\vec{r}$, so once you know how to deal with one vector, you can deal with them all. The geometric properties (by which I mean, how to relate a vector's components to its magnitude and direction) I have just covered, and will come back to later on in this chapter, and again in Chapter 8; the algebraic properties (how to add vectors and multiply them by ordinary numbers, which are called *scalars* in this context) I will introduce along the way.

For the first few chapters this semester, we are going to be primarily concerned with motion in one dimension (that is to say, along a straight line, backwards or forwards), in which case all we need to locate a point is one number, its $x$ (or $y$, or $z$) coordinate; we do not then need to worry particularly about vector algebra. Alternatively, we can simply say that a vector in one dimension is essentially the same as its only component, which is just a positive or negative number (the magnitude of the number being the magnitude of the vector, and its sign indicating its direction), and has the algebraic properties that follow naturally from that.

The description of the motion that we are aiming for is to find a *function of time*, which we denote by $x(t)$, that gives us the point's position (that is to say, the value of $x$) for any value of the time parameter, $t$. (See Eq. (1.10), below, for an example.) Remember that $x$ stands for a number that can be positive or negative (depending on the side of the origin the point is on), and has dimensions of length, so when giving a numerical value for it you must always include the appropriate units (meters, centimeters, miles…). Similarly, $t$ stands for the time elapsed since some more or less arbitrary "origin of time," or time zero. Normally $t$ should always be positive, but in special cases it may make sense to consider negative times (think of how we count years: "AD" would correspond to "positive" and "BC" would correspond to negative—the difference being that there is actually no year zero!). Anyway, $t$ also is a number with dimensions, and must be reported with its appropriate units: seconds, minutes, hours, etc.


**Figure 1.2:** A possible position vs. time graph for an object moving in one dimension.

We will be often interested in plotting the position of an object as a function of time—that is to say, the graph of the function $x(t)$. This may, in principle, have any shape, as you can see in Figure 1.2 above. In the lab, you will have a chance to use a position sensor that will automatically generate graphs like that for you on the computer, for any moving object that you aim the position sensor at. It is, therefore, important that you learn how to "read" such graphs. For example, Figure 1.2 shows an object that starts, at the time $t = 0$, a distance 0.2 m away and to the right of the origin (so $x(0) = 0.2\,\text{m}$), then moves in the negative direction to $x = -0.15\,\text{m}$, which it reaches at $t = 0.5\,\text{s}$; then turns back and moves in the opposite direction until it reaches the point $x = 0.1\,\text{m}$, turns again, and so on. Physically, this could be tracking the damped oscillations of a system such as an object attached to a spring and sliding over a surface that exerts a friction force on it (see Example 11.5.1).

### 1.2.2 Displacement

In one dimension, the **displacement** of an object over a given time interval is a quantity that we denote as $\Delta x$, and equals the difference between the object's initial and final positions (in one dimension, we will often call the "position coordinate" simply the "position," for short):

$$\Delta x = x_f - x_i \tag{1.1}$$

Here the subscript $i$ denotes the object's position at the beginning of the time interval considered, and the subscript $f$ its position at the end of the interval. The symbol $\Delta$ will consistently be used throughout this book to denote a *change* in the quantity following the symbol, meaning the difference between its initial value and its final value. The time interval itself will be written as $\Delta t$ and can be expressed as

$$\Delta t = t_f - t_i \tag{1.2}$$

where again $t_i$ and $t_f$ are the initial and final values of the time parameter (imagine, for instance, that you are reading time in seconds on a digital clock, and you are interested in the change in the object's position between second 130 and second 132: then $t_i = 130$ s, $t_2 = 132$ s, and $\Delta t = 2$ s).

You can practice reading off displacements from Figure 1.2. The displacement between $t_i = 0.5\,\mathrm{s}$ and $t_f = 1\,\mathrm{s}$, for instance, is $0.25\,\mathrm{m}$ ($x_i = -0.15\,\mathrm{m}$, $x_f = 0.1\,\mathrm{m}$). On the other hand, between $t_i = 1\,\mathrm{s}$ and $t_f = 1.3\,\mathrm{s}$, the displacement is $\Delta x = 0 - 0.1 = -0.1\,\mathrm{m}$.

Notice two important things about the displacement. First, it can be positive or negative. Positive means the object moved, overall, in the positive direction; negative means it moved, overall, in the negative direction. Second, even when it is positive, the displacement does not always equal the distance traveled by the object (distance, of course, is always defined as a positive quantity), because if the object "doubles back" on its tracks for some distance, that distance does not count towards the overall displacement. For instance, looking again at Figure 1.2 , in between the times $t_i = 0.5\,\mathrm{s}$ and $t_f = 1.5\,\mathrm{s}$ the object moved first $0.25\,\mathrm{m}$ in the positive direction, and then $0.15\,\mathrm{m}$ in the negative direction, for a total distance traveled of $0.4\,\mathrm{m}$; however, the total displacement was just $0.1\,\mathrm{m}$.

In spite of these quirks, the total displacement is, mathematically, a useful quantity, because often we will have a way (that is to say, an equation) to calculate $\Delta x$ for a given interval, and then we can rewrite Eq. (1.1) so that it reads

$$x_f = x_i + \Delta x \tag{1.3}$$

That is to say, if we know where the object started, and we have a way to calculate $\Delta x$, we can easily figure out where it ended up. You will see examples of this sort of calculation in the homework later on.

**Extension to two dimensions**

In two dimensions, we write the displacement as the vector

$$\Delta \vec{r} = \vec{r}_f - \vec{r}_i \tag{1.4}$$

The components of this vector are just the differences in the position coordinates of the two points involved; that is, $(\Delta \vec{r})_x$ (a subscript $x, y$, etc., is a standard way to represent the $x, y \ldots$ component of a vector) is equal to $x_f - x_i$, and similarly $(\Delta \vec{r})_y = y_f - y_i$.

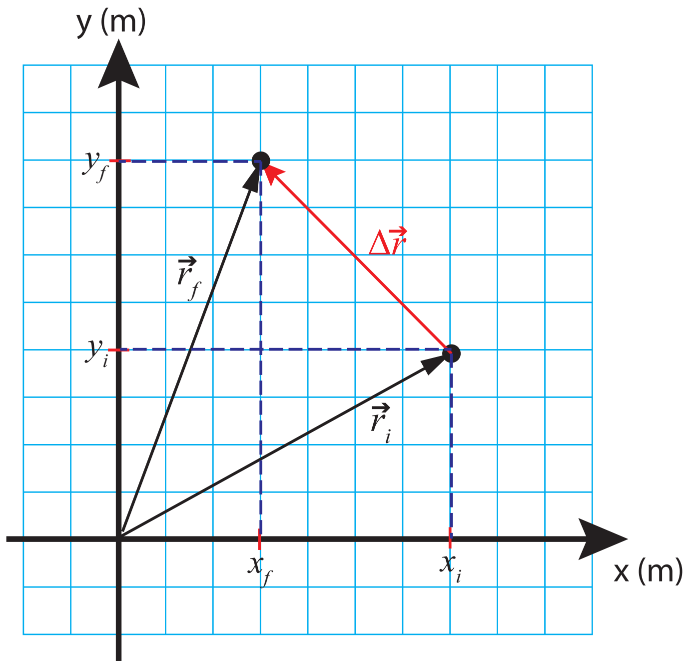



**Figure 1.3:** The displacement vector for a particle that was initially at a point with position vector $\vec{r}_i$ and ended up at a point with position vector $\vec{r}_f$ is the *difference* of the position vectors.

Figure 1.3 shows how this makes sense. The $x$ component of $\Delta\vec{r}$ in the figure is $\Delta x = 3 - 7 = -4$ m; the $y$ component is $\Delta y = 8 - 4 = 4$ m. This basically shows you how to subtract (and, by extension, add, since $\vec{r}_f = \vec{r}_i + \Delta\vec{r}$) vectors: you just subtract (or add) the corresponding components. Note how, by the Pythagorean theorem, the length (or magnitude) of the displacement vector, $|\Delta\vec{r}| = \sqrt{(x_f - x_i)^2 + (y_f - y_i)^2}$, equals the straight-line distance between the initial point and the final point, just as in one dimension; of course, the particle could have actually followed a very different path from the initial to the final point, and therefore traveled a different distance.

### 1.2.3 Velocity

#### Average velocity

If you drive from Fayetteville to Fort Smith in 50 minutes, your average speed for the trip is calculated by dividing the distance of 59.2 mi by the time interval:

$$\text{average speed} = \frac{\text{distance}}{\Delta t} = \frac{59.2\,\text{mi}}{50\,\text{min}} = \frac{59.2\,\text{mi}}{50\,\text{min}} \times \frac{60\,\text{min}}{1\,\text{hr}} = 71.0\,\text{mph} \tag{1.5}$$

(this equation, incidentally, also shows you how to convert units, and how you will be expected to work with significant figures this semester: the rule of thumb is, keep four significant figures in all intermediate calculations, and report three in the final result).

The way we define *average velocity* is similar to average speed, but with one important difference: we use the *displacement*, instead of the distance. So, the average velocity $v_{av}$ of an object, moving along a straight line, over a time interval $\Delta t$ is

$$v_{av} = \frac{\Delta x}{\Delta t} \tag{1.6}$$

This definition has all the advantages and the quirks of the displacement itself. On the one hand, it automatically comes with a sign (the same sign as the displacement, since $\Delta t$ will always be positive), which tells us in what direction we have been traveling. On the other hand, it may not be an accurate estimate of our average *speed*, if we doubled back at all. In the most extreme case, for a roundtrip (leave Fayetteville and return to Fayetteville), the average velocity would be zero, since $x_f = x_i$ and therefore $\Delta x = 0$.

It is clear that this concept is not going to be very useful in general, if the object we are tracking has a chance to double back in the time interval $\Delta t$. A way to prevent this from happening, and also getting a more meaningful estimate of the object's speed at any instant, is to make the time interval very small. This leads to a new concept, that of *instantaneous velocity*.

**Instantaneous velocity**

We define the instantaneous velocity of an object (a "particle"), at the time $t = t_i$, as the mathematical limit

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \tag{1.7}$$

The meaning of this is the following. Suppose we compute the ratio $\Delta x/\Delta t$ over successively smaller time intervals $\Delta t$ (all of them starting at the same time $t_i$). For instance, we can start by making $t_f = t_i + 1$ s, then try $t_f = t_i + 0.5$ s, then $t_f = t_i + 0.1$ s, and so on. Naturally, as the time interval becomes smaller, the corresponding displacement will also become smaller—the particle has less and less time to move away from its initial position, $x_i$. The hope is that the successive ratios $\Delta x/\Delta t$ will *converge* to a definite value: that is to say, that at some point we will start getting very similar values, and that beyond a certain point making $\Delta t$ any smaller will not change any of the significant digits of the result that we care about. This limit value is the *instantaneous velocity* of the object at the time $t_i$.

When you think about it, there is something almost a bit self-contradictory about the concept of instantaneous velocity. You cannot (in practice) determine the velocity of an object if all you are given is a literal instant. You cannot even tell if the object is moving, if all you have is one instant! Motion requires more than one instant, the passage of time. In fact, all the "instantaneous" velocities that we can measure, with any instrument, are always really average velocities, only the average is taken over very short time intervals. Nevertheless, the fact is that for any reasonably well-behaved position function $x(t)$, the limit in Eq. (1.7) is *mathematically* well-defined, and it equals what we call, in calculus, the *derivative* of the function $x(t)$:

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \tag{1.8}$$

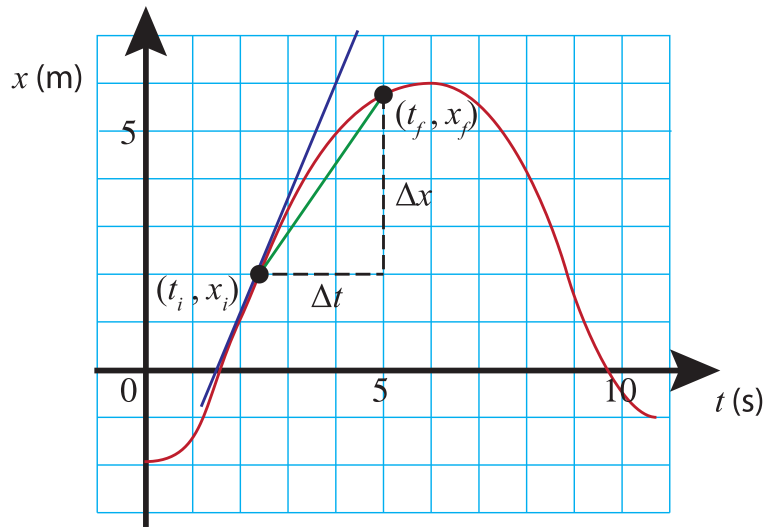



**Figure 1.4:** The slope of the green segment is the average velocity for the time interval $\Delta t$ shown. As $\Delta t$ becomes smaller, this approaches the slope of the tangent at the point $(t_i, x_i)$

In fact, there is a nice geometric interpretation for this quantity: namely, it is the slope of a line tangent to the $x$-vs-$t$ curve at the point $(t_i, x_i)$. As Figure 1.4 shows, the average velocity $\Delta x/\Delta t$ is the slope (rise over run) of a line segment drawn from the point $(t_i, x_i)$ to the point $(t_f, x_f)$ (the green line in the figure). As we make the time interval smaller, by bringing $t_f$ closer to $t_i$ (and hence, also, $x_f$ closer to $x_i$), the slope of this segment will approach the slope of the tangent line at $(t_i, x_i)$ (the blue line), and this will be, by the definition (1.7), the instantaneous velocity at that point.

This geometric interpretation makes it easy to get a qualitative feeling, from the position-vs-time graph, for when the particle is moving more or less fast. A large slope means a steep rise or fall, and that is when the velocity will be largest—in magnitude. A steep rise means a large positive velocity, whereas a steep drop means a large negative velocity, by which I mean a velocity that is given by a negative number which is large in absolute value. In the future, to simplify sentences like this one, I will just use the word "speed" to refer to the magnitude (that is to say, the absolute value) of the instantaneous velocity. Thus, speed (like distance) is always a positive number, by definition, whereas velocity can be positive or negative; and a steep slope (positive or negative) means the speed is large there.

Conversely, looking at the sample $x$-vs-$t$ graphs in this chapter, you may notice that there are times when the tangent is horizontal, meaning it has zero slope, and so the instantaneous velocity at those times is zero (for instance, at the time $t = 1.0\,\mathrm{s}$ in Figure 1.2). This makes sense when you think of what the particle is actually doing at those special times: it is just changing direction, so its velocity is going, for instance, from positive to negative. The way this happens is, it slows down, down... the velocity gets smaller and smaller, and then, for just an instant (literally, a mathematical point in time), it becomes zero before, the next instant, going negative.

We will be coming back to this "reading of graphs" in the lab and the homework, as well as in the next chapter, when we introduce the concept of acceleration.

**Motion with constant velocity**

If the instantaneous velocity of an object never changes, it means that it is always moving in the same direction with the same speed. In that case, the instantaneous velocity and the average velocity coincide, and that means we can write $v = \Delta x/\Delta t$ (where the size of the interval $\Delta t$ could now be anything), and rewrite this equation in the form

$$\Delta x = v\,\Delta t \tag{1.9}$$

which is the same as

$$x_f - x_i = v\,(t_f - t_i)$$

Now suppose we keep $t_i$ constant (that is, we fix the initial instant) but allow the time $t_f$ to change, so we will just write $t$ for an arbitrary value of $t_f$, and $x$ for the corresponding value of $x_f$. We end up with the equation

$$x - x_i = v\,(t - t_i)$$

which we can also write as

$$x(t) = x_i + v\,(t - t_i) \tag{1.10}$$

after some rearranging, and where the notation $x(t)$ has been introduced to emphasize that we want to think of $x$ as a function of $t$. This is, not surprisingly, the equation of a straight line—a "curve" which is its own tangent and always has the same slope.

(Please make sure that you are not confused by the notation in Eq. (1.10). The parentheses around the $t$ on the left-hand side mean that we are considering the position $x$ as a function of $t$. On the other hand, the parentheses around the quantity $t - t_i$ on the right-hand side mean that we are multiplying this quantity by $v$, which is a constant here. This distinction will be particularly important when we introduce the function $v(t)$ next.)

Either one of equations (1.9) or (1.11) can be used to solve problems involving motion with constant velocity, and again you will see examples of this in the homework.

**Motion with changing velocity**

If the velocity changes with time, obtaining an expression for the position of the object as a function of time may be a nontrivial task. In the next chapter we will study an important special case, namely, when the velocity changes at a constant rate (constant acceleration).

For the most general case, a graphical method that is sometimes useful is the following. Suppose that we know the function $v(t)$, and we graph it, as in Figure 1.5 below. Then the area under the curve in between any two instants, say $t_i$ and $t_f$, is equal to the total displacement of the object over that time interval.

The idea involved is known in calculus as *integration*, and it goes as follows. Suppose that I break

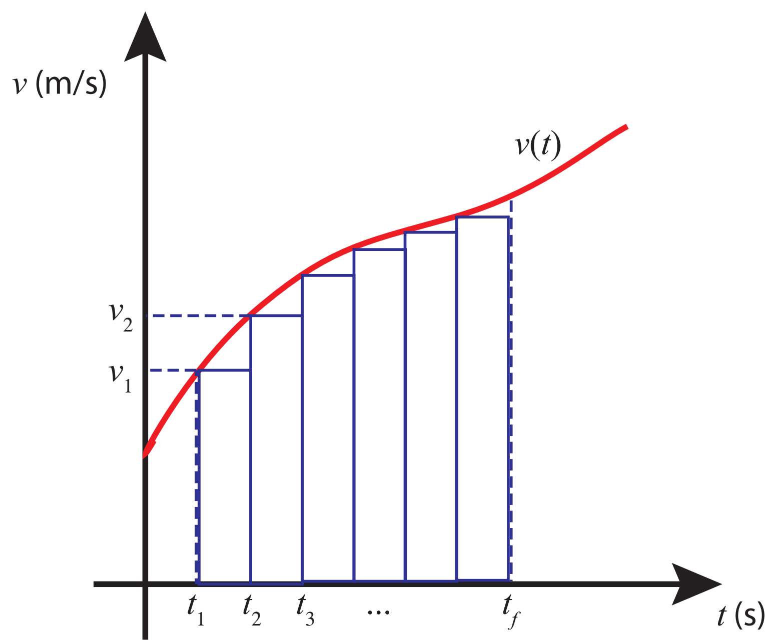



**Figure 1.5:** How to get the displacement from the area under the $v$-vs-$t$ curve.

down the interval from $t_i$ to $t_f$ into equally spaced subintervals, beginning at the time $t_i$ (which I am, equivalently, going to call $t_1$, that is, $t_1 \equiv t_i$, so I have now $t_1, t_2, t_3, \ldots t_f$). Now suppose I treat the object's motion over each subinterval as if it were motion with constant velocity, the velocity being that at the beginning of the subinterval. This, of course, is only an approximation, since the velocity is constantly changing; but, if you look at Figure 1.5, you can convince yourself that it will become a better and better approximation as I increase the number of intermediate points and the rectangles shown in the figure become narrower and narrower. In this approximation, the displacement during the first subinterval would be

$$\Delta x_1 = v_1(t_2 - t_1) \tag{1.11}$$

where $v_1 = v(t_1)$; similarly, $\Delta x_2 = v_2(t_3 - t_2)$, and so on.

However, Eq. (1.11) is just the area of the first rectangle shown under the curve in Figure 1.5 (the base of the rectangle has "length" $t_2 - t_1$, and its height is $v_1$). Similarly for the second rectangle, and so on. So the sum $\Delta x_1 + \Delta x_2 + \ldots$ is both an approximation to the area under the $v$-vs-$t$ curve, and an approximation to the total displacement $\Delta t$. As the subdivision becomes finer and finer, and the rectangles narrower and narrower (and more numerous), both approximations become more and more accurate. In the limit of "infinitely many," infinitely narrow rectangles, you get both the total displacement and the area under the curve exactly, and they are both equal to each other. Mathematically, we would write

$$\Delta x = \int_{t_i}^{t_f} v(t)\, dt \tag{1.12}$$

where the stylized "S" (for "sum") on the right-hand side is the symbol of the operation known as *integration* in calculus. This is essentially the inverse of the process know as differentiation, by which we got the velocity function from the position function, back in Eq. (1.8).

This graphical method to obtain the displacement from the velocity function is sometimes useful, if you can estimate the area under the $v$-vs-$t$ graph reliably. An important point to keep in mind is that rectangles under the horizontal axis (corresponding to negative velocities) have to be added as having negative area (since the corresponding displacement is negative); see example 1.5.1 at the end of this chapter.

**Extension to two dimensions**

In two (or more) dimensions, you define the average velocity vector as a vector $\vec{v}_{av}$ whose components are $v_{av,x} = \Delta x/\Delta t$, $v_{av,y} = \Delta y/\Delta t$, and so on (where $\Delta x, \Delta y, \ldots$ are the corresponding components of the displacement vector $\Delta\vec{r}$). This can be written equivalently as the single vector equation

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} \tag{1.13}$$

This tells you how to multiply (or divide) a vector by an ordinary number: you just multiply (or divide) each component by that number. Note that, if the number in question is positive, this operation does not change the direction of the vector at all, it just *scales* it up or down (which is why ordinary numbers, in this context, are called *scalars*). If the scalar is negative, the vector's direction is flipped as a result of the multiplication. Since $\Delta t$ in the definition of velocity is always positive, it follows that the average velocity vector always points in the same direction as the displacement, which makes sense.

To get the instantaneous velocity, you just take the limit of the expression (1.13) as $\Delta t \to 0$, for each component separately. The resulting vector $\vec{v}$ has components $v_x = \lim_{\Delta t \to 0} \Delta x/\Delta t$, etc., which can also be written as $v_x = dx/dt, v_y = dy/dt, \ldots$.

All the results derived above hold for each spatial dimension and its corresponding velocity component. For instance, the graphical method shown in Figure 1.5 can always be used to get $\Delta x$ if the function $v_x(t)$ is known, or equivalently to get $\Delta y$ if you know $v_y(t)$, and so on.

Introducing the velocity vector at this point does cause a little bit of a notational difficulty. For quantities like $x$ and $\Delta x$, it is pretty obvious that they are the $x$ components of the vectors $\vec{r}$ and $\Delta\vec{r}$, respectively; however, the quantity that we have so far been calling simply $v$ should more properly be denoted as $v_x$ (or $v_y$ if the motion is along the $y$ axis). In fact, there is a convention that if you use the symbol for a vector without the arrow on top or any $x, y, \ldots$ subscripts, you must mean the *magnitude* of the vector. In this book, however, I have decided *not* to follow that convention, at least not until we get to Chapter 8 (and even then I will use it only for forces). This is because we will spend most of our time dealing with motion in only one dimension, and it makes the notation unnecessarily cumbersome to keep having to write the $x$ or $y$ subscripts on every component of every vector, when you really only have one dimension to worry about in the first place. So $v$ will, throughout, refer to the relevant component of the velocity vector, to be inferred from the context, until we get to Chapter 8 and actually need to deal with both a $v_x$ and a $v_y$ explicitly.

Finally, notice that the magnitude of the velocity vector, $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$, is equal to the *instantaneous speed*, since, as $\Delta t \to 0$, the magnitude of the displacement vector, $|\Delta\vec{r}|$, becomes the actual distance traveled by the object in the time interval $\Delta t$.

## 1.3 Reference frame changes and relative motion

Everything up to this point assumes that we are using a fixed, previously agreed upon **reference frame**. Basically, this is just an origin and a set of axes along which to measure our coordinates, as shown in Figure 1.

There are, however, a number of situations in physics that call for the use of different reference frames, and, more importantly, that require us to *convert* various physical quantities from one reference frame to another. For instance, imagine you are on a boat on a river, rowing downstream. You are moving with a certain velocity relative to the water around you, but the water itself is flowing with a different velocity relative to the shore, and your actual velocity relative to the shore is the sum of those two quantities. Ships generally have to do this kind of calculation all the time, as do airplanes: the "airspeed" is the speed of a plane relative to the air around it, but that air is usually moving at a substantial speed relative to the earth.

The way we deal with all these situations is by introducing two reference frames, which here I am going to call A and B. One of them, say A, is "at rest" relative to the earth, and the other one is "at rest" relative to something else—which means, really, moving along with that something else. (For instance, a reference frame at rest "relative to the river" would be a frame that's moving along with the river water, like a piece of driftwood that you could measure your progress relative to.)

In any case, graphically, this will look as in Figure 1.6, which I have drawn for the two-dimensional case because I think it makes it easier to visualize what's going on:

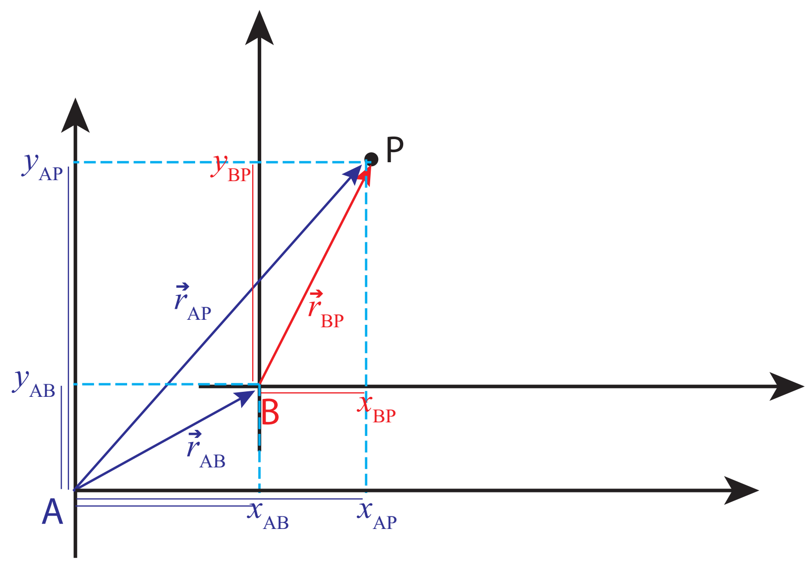



**Figure 1.6:** Position vectors and coordinates of a point $P$ in two different reference frames, A and B.

In the reference frame A, the point $P$ has position coordinates $(x_{AP}, y_{AP})$. Likewise, in the reference frame B, its coordinates are $(x_{BP}, y_{BP})$. As you can see, the notation chosen is such that every coordinate in A will have an "A" as a first subscript, while the second subscript indicates the object to which it refers, and similarly for coordinates in B.

The coordinates $(x_{AB}, y_{AB})$ are special: they are the coordinates, in the reference frame A, of the *origin* of reference frame B. This is enough to fully locate the frame B in A, as long as the frames are not rotated relative to each other.

The thin colored lines I have drawn along the axes in Figure 1.6 are intended to make it clear that the following equations hold:

$$
\begin{aligned}
x_{AP} &= x_{AB} + x_{BP} \\
y_{AP} &= y_{AB} + y_{BP}
\end{aligned}
\tag{1.14}
$$

Although the figure is drawn for the easy case where all these quantities are positive, you should be able to convince yourself that Eqs. (1.14) hold also when one or more of the coordinates have negative values.

All these coordinates are also the components of the respective position vectors, shown in the figure and color-coded by reference frame (so, for instance, $\vec{r}_{AP}$ is the position vector of $P$ in the frame A), so the equations (1.14) can be written more compactly as the single vector equation

$$\vec{r}_{AP} = \vec{r}_{AB} + \vec{r}_{BP} \tag{1.15}$$

From all this you can see how to add vectors: algebraically, you just add their components separately, as in Eqs. (1.14); graphically, you draw them so the tip of one vector coincides with the tail of the other (we call this "tip-to-tail"), and then draw the sum vector from the tail of the first one to the tip of the other one. (In general, to get two arbitrary vectors tip-to-tail you may need to displace one of them; this is OK provided you do not change its orientation, that is, provided you only displace it, not rotate it. We'll see how this works in a moment with velocities, and later on with forces.)

Of course, I showed you already how to *subtract* vectors with Fig. 1.3: again, algebraically, you just subtract the corresponding coordinates, whereas graphically you draw them with a common origin, and then draw the vector from the tip of the vector you are subtracting to the tip of the other one. If you read the previous paragraph again, you can see that Fig. 1.3 can equally well be used to show that $\Delta\vec{r} = \vec{r}_f - \vec{r}_i$, as to show that $\vec{r}_f = \vec{r}_i + \Delta\vec{r}$.

In a similar way, you can see graphically from Fig. 1.6 (or algebraically from Eq. (1.15)) that the position vector of $P$ in the frame B is given by $\vec{r}_{BP} = \vec{r}_{AP} - \vec{r}_{AB}$. The last term in this expression can be written in a different way, as follows. If I follow the convention I have introduced above, the quantity $x_{BA}$ (with the order of the subscripts reversed) would be the $x$ coordinate of the origin of frame A in frame B, and algebraically that would be equal to $-x_{AB}$, and similarly $y_{BA} = -y_{AB}$. Hence the vector equality $\vec{r}_{AB} = -\vec{r}_{BA}$ holds. Then,

$$\vec{r}_{BP} = \vec{r}_{AP} - \vec{r}_{AB} = \vec{r}_{AP} + \vec{r}_{BA} \tag{1.16}$$

This is, in a way, the "inverse" of Eq. (1.15): it tells us how to get the position of $P$ in the frame B if we know its position in the frame A.

Let me show next you how all this extends to displacements and velocities. Suppose the point $P$ indicates the position of a particle at the time $t$. Over a time interval $\Delta t$, both the position of the particle and the relative position of the two reference frames may change. We can add yet another subscript, $i$ or $f$, (for initial and final) to the coordinates, and write, for example,

$$\begin{aligned} x_{AP,i} &= x_{AB,i} + x_{BP,i} \\ x_{AP,f} &= x_{AB,f} + x_{BP,f} \end{aligned} \tag{1.17}$$

Subtracting these equations gives us the corresponding displacements:

$$\Delta x_{AP} = \Delta x_{AB} + \Delta x_{BP} \tag{1.18}$$

Dividing Eq. (1.18) by $\Delta t$ we get the average velocities[1], and then taking the limit $\Delta t \to 0$ we get the instantaneous velocities. This applies in the same way to the $y$ coordinates, and the result is the vector equation

$$\vec{v}_{AP} = \vec{v}_{BP} + \vec{v}_{AB} \tag{1.19}$$

I have rearranged the terms on the right-hand side to (hopefully) make it easier to visualize what's going on. In words: the velocity of the particle $P$ relative to (or *measured in*) frame A is equal to the (vector) sum of the velocity of the particle as measured in frame B, plus the velocity of frame B relative to frame A.

The result (1.19) is just what we would have expected from the examples I mentioned at the beginning of this section, like rowing in a river or an airplane flying in the wind. For instance, for the airplane $\vec{v}_{BP}$ could be its "airspeed" (only it has to be a vector, so it would be more like its "airvelocity": that is, its velocity relative to the air around it), and $\vec{v}_{AB}$ would be the velocity of the air relative to the earth (the wind velocity, at the location of the airplane). In other words, A represents the earth frame of reference and B the air, or wind, frame of reference. Then, $\vec{v}_{AP}$ would be the "true" velocity of the airplane relative to the earth. You can see how it would be important to add these quantities as vectors, in general, by considering what happens when you fly in a cross wind, or try to row across a river, as in Figure 1.7 below.

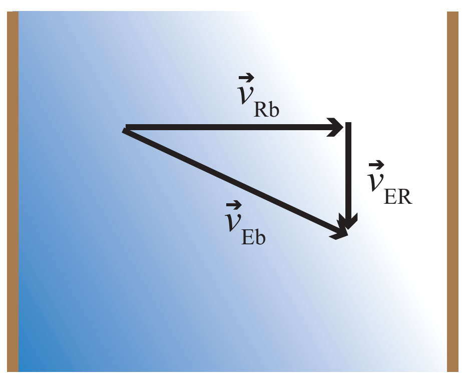



**Figure 1.7:** Rowing across a river. If you head "straight across" the river (with velocity vector $\vec{v}_{Rb}$ in the moving frame of the river, which is flowing with velocity $\vec{v}_{ER}$ in the frame of the earth), your actual velocity relative to the shore will be the vector $\vec{v}_{Eb}$. This is an instance of Eq. (1.19), with frame A being E (the earth), frame B being R (the river), and "b" (for "boat") standing for the point P we are tracking.

As you can see from this couple of examples, Equation (1.19) is often useful as it is written, but

[1]We have made a very natural assumption, that the time interval $\Delta t$ is the same for observers tracking the particle's motion in frames A and B, respectively (where each observer is understood to be moving along with his or her frame). This, however, turns out to be *not* true when any of the velocities involved is close to the speed of light, and so the simple addition of velocities formula (1.19) does not hold in Einstein's relativity theory.
(This is actually the first bit of real physics I have told you about in this book, so far; unfortunately, you will have no use for it this semester!)

sometimes the information we have is given to us in a different way: for instance, we could be given the velocity of the object in frame A ($\vec{v}_{AP}$), and the velocity of frame B as seen in frame A ($\vec{v}_{AB}$), and told to calculate the velocity of the object as seen in frame B. This can be easily accomplished if we note that the vector $\vec{v}_{AB}$ is equal to $-\vec{v}_{BA}$; that is to say, the velocity of frame B as seen from frame A is just the opposite of the velocity of frame A as seen from frame B. Hence, Eq. (1.19) can be rewritten as

$$\vec{v}_{AP} = \vec{v}_{BP} - \vec{v}_{BA} \tag{1.20}$$

For most of the next few chapters we are going to be considering only motion in one dimension, and so we will write Eq. (1.19) (or (1.20)) without the vector symbols, and it will be understood that $v$ refers to the component of the vector $\vec{v}$ along the coordinate axis of interest.

A quantity that will be particularly important later on is the *relative velocity* of two objects, which we could label 1 and 2. The velocity of object 2 relative to object 1 is, by definition, the velocity which an observer moving along with 1 would measure for object 2. So it is just a simple frame change: let the earth frame be frame E and the frame moving with object 1 be frame 1, then the velocity we want is $v_{12}$ ("velocity of object 2 in frame 1"). If we make the change A$\rightarrow$ 1, B$\rightarrow E$, and P$\rightarrow$ 2 in Eq. (1.20), we get

$$v_{12} = v_{E2} - v_{E1} \tag{1.21}$$

In other words, the velocity of 2 relative to 1 is just the velocity of 2 minus the velocity of 1. This is again a familiar effect: if you are driving down the highway at 50 miles per hour, and the car in front of you is driving at 55, then its velocity relative to you is 5 mph, which is the rate at which it is moving away from you (in the forward direction, assumed to be the positive one).

It is important to realize that all these velocities are *real* velocities, each in its own reference frame. Something may be said to be truly moving at some velocity in one reference frame, and just as truly moving with a different velocity in a different reference frame. I will have a lot more to say about this in the next chapter, but in the meantime you can reflect on the fact that, if a car moving at 55 mph collides with another one moving at 50 mph in the same direction, the damage will be basically the same as if the first car had been moving at 5 mph and the second one had been at rest. For practical purposes, where you are concerned, another car's velocity relative to yours *is* that car's "real" velocity.

**Resources**

A good app for practicing how to add vectors (and how to break them up into components, magnitude and direction, etc.) may be found here:
`https://phet.colorado.edu/en/simulation/vector-addition`.

Perhaps the most dramatic demonstration of how Eq. (1.19) works in the real world is in this episode of *Mythbusters*: `https://www.youtube.com/watch?v=BLuI118nhzc`. (If this link does not work, do a search for "Mythbusters cancel momentum.") They shoot a ball from the bed of a truck, with a velocity (relative to the truck) of 60 mph backwards, while the truck is moving forward at 60 mph. I think the result is worth watching. (Do not be distracted by their talk about momentum. We will get there, in time.)

A very old, but also very good, educational video about different frames of reference is this one: https://www.youtube.com/watch?v=sS17fCom0Ns. You should try to watch at least part of it. Many things will be relevant to later parts of the course, including projectile motion, and the whole discussion of relative motion coming up next, in Chapter 2.

## 1.4 In summary

1. To describe the motion of an object in one dimension we treat it as a mathematical point, and consider its *position coordinate*, $x$ (often shortened to just the *position*), as a function of time: $x(t)$.

2. Numerically, the position coordinate is the distance to a chosen origin, with a positive or negative sign depending on which side of the origin the point is. For every problem, when we introduce a coordinate axis we need to specify a positive direction. Starting from the origin in that direction, the position coordinate is positive and increasing, whereas going from the origin in the opposite direction (negative direction) it becomes increasingly negative.

3. The displacement of an object over a time interval from an initial time $t_i$ to a final time $t_f$ is the quantity $\Delta x = x_f - x_i$, where $x_f$ is the position of the object at the final time (or, the final position), and $x_i$ the position at the initial time (or initial position).

4. The average velocity of an object over the time interval from $t_i$ to $t_f$ is defined as $v_{av} = \Delta x/\Delta t$, where $\Delta t = t_f - t_i$.

5. The *instantaneous velocity* (often just called the *velocity*) of an object at the time $t$ is the limit value of the quantity $\Delta x/\Delta t$, calculated for successively shorter time intervals $\Delta t$, all with the same initial time $t_i = t$. This is, mathematically, the definition of the derivative of the function $x(t)$ at the time $t$, which we express as $v = dx/dt$.

6. Graphically, the instantaneous velocity of the object at the time $t$ is the slope of the tangent line to the $x$-vs-$t$ graph at the time $t$.

7. The instantaneous velocity of an object is a positive or negative quantity depending on whether the object, at that instant, is moving in the positive or the negative direction.

8. For an object moving with *constant* velocity $v$, the position function is given by [Eq. (1.10)]:

$$x(t) = x_i + v\,(t - t_i)$$

where $t_i$ is an arbitrarily chosen initial time and $x_i$ the position at that time. This can also be written in the form given by Eq. (1.9). The argument ($t$) on the left-hand side of (1.10) is optional, and $t_i$ is often set equal to zero, giving just $x = x_i + vt$. This, however, is not quite as generally applicable as the result (1.9) or (1.10).

9. For an object moving with changing velocity, the total displacement in between times $t_i$ and $t_f$ is equal to the total area under the $v$-vs-$t$ curve in between those times; areas below the horizontal ($t$) axis must be treated as negative.

10. In two or more dimensions one introduces, for every point in space, a *position vector* whose components are just the Cartesian coordinates of that point; then the displacement vector is defined as $\Delta\vec{r} = \vec{r}_f - \vec{r}_i$, the average velocity vector is $\vec{v}_{av} = \Delta\vec{r}/\Delta t$, and the instantaneous velocity vector is the limit of this as $\Delta t$ goes to zero. Vectors are added by adding their components separately; to multiply a vector by an ordinary number, or *scalar*, we just multiply each component by that number.

11. When tracking the motion of an object, "P", in two different reference frames, A and B, the position vectors are related by $\vec{r}_{AP} = \vec{r}_{AB} + \vec{r}_{BP}$, and likewise the velocity vectors: $\vec{v}_{AP} = \vec{v}_{AB} + \vec{v}_{BP}$. Here, the first subscript tells you in which reference frame you are measuring, and the second subscript what it is that you are looking at; $\vec{r}_{AB}$ is the position vector of the origin of frame B as seen in frame A, and $\vec{v}_{AB}$ its velocity.

## 1.5 Examples

### 1.5.1 Motion with (piecewise) constant velocity

You leave your house on your bicycle to go visit a friend. At your normal speed of 9 mph, you know it takes you 6 minutes to get there. This time, though, when you have traveled half the distance you realize you forgot a book at home that you were going to return to your friend, so you turn around and pedal at twice your normal speed, get back home, grab the book, and start off again for your friend's house at 18 mph (imagine you are really fit to pull this off!)
(a) How far away from you does your friend live?
(b) What is the total distance you travel on this trip?
(c) How long did the whole trip take?
(d) Draw a position versus time and a velocity versus time graph for the whole trip. Use SI units for both graphs. Neglect the time it takes you to stop and turn around, and also the time it takes you to run into your house and grab the book (in other words, assume those changes in your direction of motion happen instantly).
(e) Show explicitly, using your $v$-vs-$t$ graph, that the graphical method of Figure 1.5 gives you the total displacement for your trip.

**Solution**

I am going to work out this problem using both miles and SI units, the first because it seems most natural, and the second because we are asked to use SI units for part (d), so we might as well use them from the start. In general, you should use SI units whenever you can. If you are unsure of what to do in a specific problem, ask your instructor!

(a) We are told that at 9 miles per hour it would take 6 minutes to get there, so let us use

$$\Delta x = v\Delta t \tag{1.22}$$

with $v = 9\,\text{mph}$ and $\Delta t = 6\,\text{min}$. We have to either convert the hours to minutes, or vice-versa. Again, in this case it seems easiest to realize that 6 min equals 1/10 of an hour, so:

$$\Delta x = \left(9\,\frac{\text{miles}}{\text{hr}}\right) \times 0.1\,\text{hr} = 0.9\,\text{miles}. \tag{1.23}$$

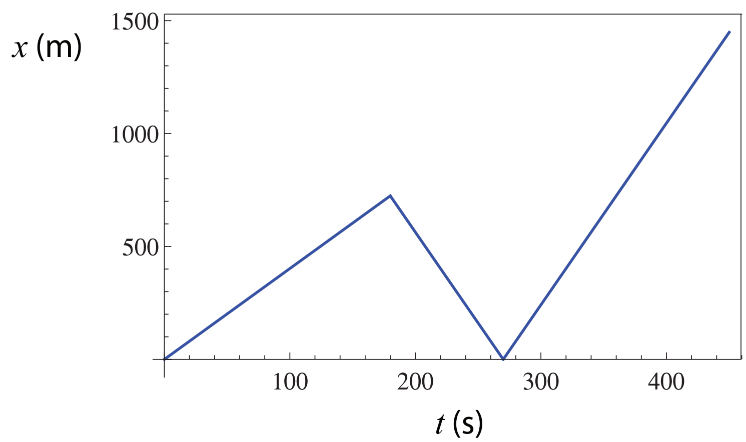
In SI units, $9\,\text{mph} = 4.023\,\text{m/s}$, and $6\,\text{min} = 360\,\text{s}$, so $\Delta x = 1448\,\text{m}$.

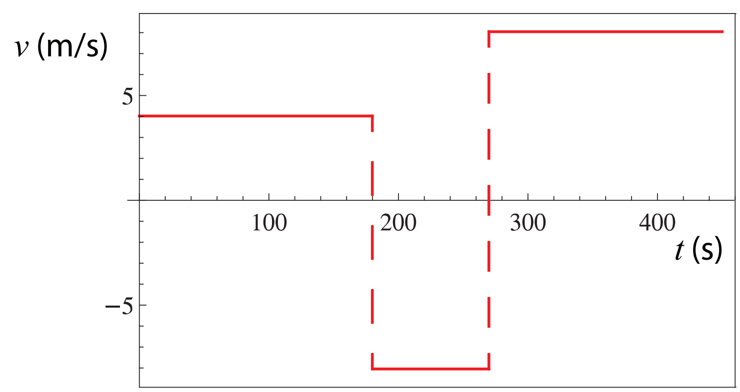
(b) This is just a matter of keeping track of the distance traveled in the various parts of the trip. You start by riding half the distance to your friend's house, which is to say, 0.45 miles, and then you ride that again back home, so that's 0.9 miles, and then you're back where you started, so you still have to go the 0.9 miles to your friend's house. So overall, you ride for 1.8 miles, or 2897 m.

(c) The whole trip consists, as detailed above, of 0.45 miles at 9 mph, and the rest, which is 1.35 miles, at 18 mph. Applying $\Delta t = \Delta x/v$ to each of these intervals, we get a total time of

$$
\begin{aligned}
\Delta t &= \frac{0.45\,\text{miles}}{9\,\text{mph}} + \frac{1.35\,\text{miles}}{18\,\text{mph}} = 0.125\,\text{hours} \\
&= 0.125 \times 60\,\text{min} = 7.5\,\text{min} \\
&= 7.5 \times 60\,\text{s} = 450\,\text{s}
\end{aligned}
\tag{1.24}
$$

(d) The graphs are shown below. Details on how to get them follow.




• First interval: from $t = 0$ to $t = 180\,\text{s}$ (3 min, which is what it would take to cover half the distance to your friend's house at 9 mph). The velocity is a constant $v = 4.023\,\text{m/s}$. For the position graph, use Eq. (1.10) with $x_i = 0$, $t_i = 0$ and $v = 4.023\,\text{m/s}$.

• Second interval: from $t = 180\,\text{s}$ to $t = 270\,\text{s}$ (it takes you half of 3 min, which is to say 90 s, to cover the same distance as above at twice the speed). The velocity is a constant $v = -8.046\,\text{m/s}$

(twice what it was earlier, but in the opposite direction). For the position graph, use Eq. (1.10) with $x_i = 724\,\text{m}$ (this is half of the distance to your friend's house, and the starting position for this interval), $t_i = 180\,\text{s}$ and $v = -8.046\,\text{m/s}$.

• Third interval: from $t = 270\,\text{s}$ to $t = 450\,\text{s}$. The velocity is a constant $v = 8.046\,\text{m/s}$ (same speed as just before, but in the opposite direction). For the position graph, use Eq. (1.10) with $x_i = 0\,\text{m}$ (you start back at your house), $t_i = 270\,\text{s}$ and $v = 8.046\,\text{m/s}$.

If you are familiar with the software package *Mathematica*, the position graph was produced using the command
Plot[If[t<180, 4.023 t, If[t<270, 4.023*180-8.046 (t-180), 8.046 (t-270)]], {t, 0, 450}]
and the velocity graph was produced using
Plot[If[t<180, 4.023 , If[t<270, -8.046, 8.046]], {t, 0, 450}]
(and then connecting the horizontal lines by hand, which is not necessary, but helps to visualize what's going on).

The graphs could also have been produced using the free plotting software package Gnuplot (available here: `http://www.gnuplot.info/download.html`) with the following commands:
gnuplot> set dummy t
gnuplot> f(t) = t<180 ? 4.023*t : t<270 ? 4.023*180-8.046*(t-180) : 8.046*(t-270)
gnuplot> plot [0:450] f(t)
The first line sets the default independent variable to $t$ (instead of $x$, which is what Gnuplot expects). The second line defines the piecewise function using the *ternary operator* (? :) borrowed from the C programming language. The third line plots the function over the range indicated.

(e) For this we need to find the area under the $v$-vs-$t$ graph we just plotted. Basically, we have three rectangles: the first one has base 180 units (s) and height 4 units (m/s), so its area is $4 \times 180 = 720$ (m). The second rectangle has base 90 units and height $-8$ (negative, because it is below the horizontal axis!), so its area is $-720$. The last one has base 180 units again (from 270 to 450) and height 8, so its area is $8 \times 180 = 1440$. So the total area "under" the $v$-vs-$t$ curve is

$$720 - 720 + 1440 = 1440\,\text{meters}$$

which is (approximately) your total displacement, that is, the 9 miles to your friend's house. (Of course, we would have obtained a more accurate result if we had used the more accurate values for the "heights" of 4.023, $-8.046$, and 8.046, but if all we have to go by is the graph, such accuracy is pretty much impossible.)

### 1.5.2 Addition of velocities, relative motion

This example was inspired by the "race on a moving sidewalk" demo at
http://physics.bu.edu/~duffy/classroom.html.
Please go take a look at it!

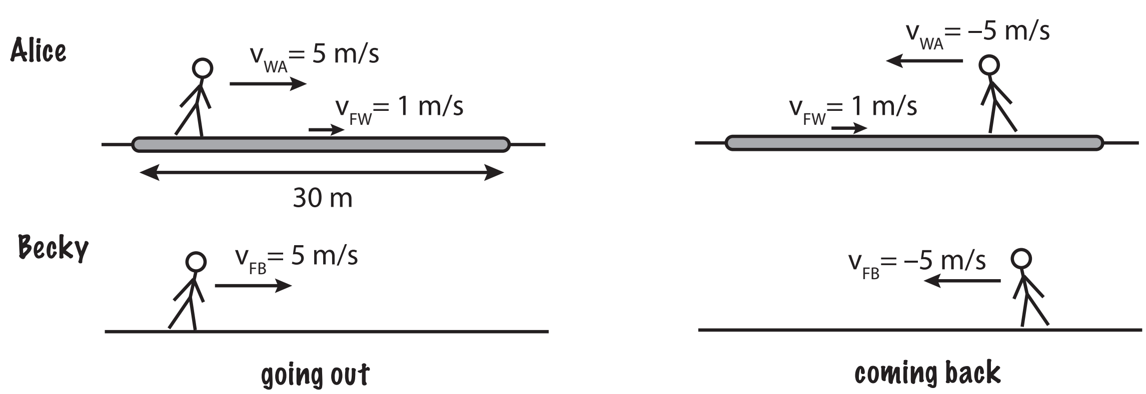
Two girls, Ann and Becky (yes, A and B) decide to have a race while they wait for a plane at a nearly-deserted airport. Ann will run on the moving walkway, to the end of it (which is 30 m away) and back, whereas Becky will run alongside her on the (non-moving) floor, also 30 m out and back. The walkway moves at 1 m/s, and the girls both run at the same constant speed of 5 m/s *relative to the surface they are standing on*.

(a) Relative to the (non-moving) floor, what is Ann's velocity for the first leg of her race, when she is moving in the same direction as the walkway (take that to be the positive direction)? What is her velocity for the return leg?
(b) How long does it take each of the girls to complete their race?
(c) When both girls are running in the positive direction, what is Becky's velocity relative to Ann? (That is, how fast does Ann see Becky move, and in what direction?)
(d) When Ann turns around and starts running in the negative direction, but Becky is still running in the positive direction, what is Becky's velocity relative to Ann?
(e) What is the total distance Ann runs *in the moving walkway's frame of reference*?

**Solution**

I am going to solve this in the format that you will be required to use this semester for most of the homework and exam problems. I will not be able to do this for every single example, but you should! Please follow this carefully.

To begin with, you must draw a sketch of the situation described in the problem, detailed enough to include all the relevant information you are given. Here is mine:



Note that I have drawn one picture for each half of the race, and that all the information given in the text of the problem is there. The figure makes it clear also the notation I will be using for each of the girls' velocities, and to see at a glance what is happening.

You should next state what kind of problem this is and what basic result (theorem, principle, or equation(s)) you are going to use to solve it. For this problem, you could say:

"This is a relative motion/reference frame transformation problem. I will use Eq. (1.19)

$$\vec{v}_{AP} = \vec{v}_{BP} + \vec{v}_{AB}$$

as well as the basic equation for motion with constant velocity:"

$$\Delta x = v \Delta t$$

After that, solve each part in turn, and make sure to show all your work!

Part (a): Let $F$ stand for the floor frame of reference, and $W$ the walkway frame. In the notation of Section 1.3, we have $v_{FW} = 1\,\text{m/s}$. For the first leg of her race, we are told that Ann's velocity *relative to the walkway* is $5\,\text{m/s}$, so $v_{WA} = 5\,\text{m/s}$. Then, by Eq. (1.19) (with the following change of indices: $A \to F$, $B \to W$, and $P \to A$),

$$v_{FA} = v_{FW} + v_{WA} = 1\,\frac{\text{m}}{\text{s}} + 5\,\frac{\text{m}}{\text{s}} = 6\,\frac{\text{m}}{\text{s}} \tag{1.25}$$

(when you see an equation like this, full of subscripts, it is a good practice to read it out, mentally, to yourself: "Ann's velocity relative to the floor equals the velocity of the walkway relative to the floor plus Ann's velocity relative to the walkway." Then take a moment to see if it makes sense! Here is a place where the picture can be really helpful.)

For the return leg, use the same formula, but note that now her velocity relative to the walkway is *negative*, $v_{WA} = -5\,\text{m/s}$, since she is moving in the opposite direction:

$$v_{FA} = v_{FW} + v_{WA} = 1\,\frac{\text{m}}{\text{s}} - 5\,\frac{\text{m}}{\text{s}} = -4\,\frac{\text{m}}{\text{s}} \tag{1.26}$$

Part (b): Relative to the floor reference frame, we have just seen that Ann first covers 30 m at a speed of 6 m/s, and then the same 30 m at a speed of 4 m/s, so her total time is

$$\Delta t_A = \frac{30\text{m}}{6\text{m/s}} + \frac{30\text{m}}{4\text{m/s}} = 5\,\text{s} + 7.5\,\text{s} = 12.5\,\text{s} \tag{1.27}$$

whereas Becky just runs 30 m at 5 m/s both ways, so it takes her 6 s either way, for a total of 12 s, which means she wins the race.

Part (c): The quantity we want is written, in the notation of Section 1.3, $v_{AB}$ ("velocity of Becky relative to Ann"). To calculate this, we just need to know the velocities of both girls in some frame of reference (the same for both!), then subtract Ann's velocity from Becky's (this is what Eq. (1.21) is saying). In this case, if we just choose the floor's reference frame, we have $v_{FA} = 6\,\text{m/s}$ and $v_{FB} = 5\,\text{m/s}$, so

$$v_{AB} = v_{FB} - v_{FA} = 5\,\frac{\text{m}}{\text{s}} - 6\,\frac{\text{m}}{\text{s}} = -1\,\frac{\text{m}}{\text{s}} \tag{1.28}$$

The negative sign makes sense: Ann sees Becky *falling behind her*, so relative to her Becky is moving *backwards*, which is to say, in the direction we have identified as negative.

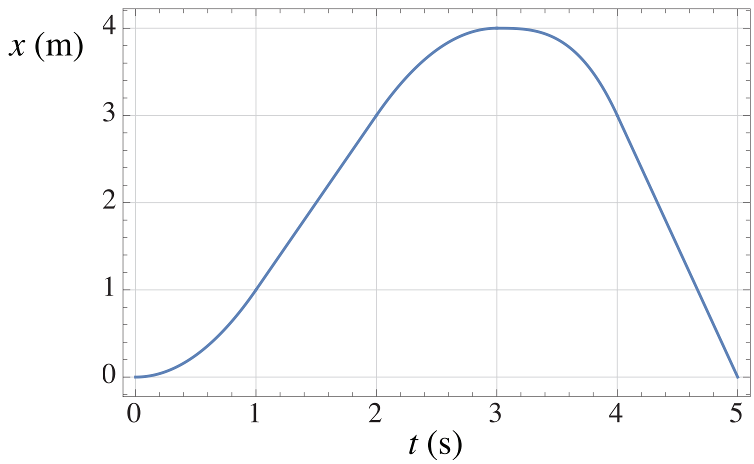
Part (d): Again we use the same equation, and Becky's velocity is still the same, but now Ann's velocity is $v_{FA} = -4\,\text{m/s}$ (note the negative sign!), so

$$v_{AB} = v_{FB} - v_{FA} = 5\,\frac{\text{m}}{\text{s}} - \left(-4\,\frac{\text{m}}{\text{s}}\right) = 9\,\frac{\text{m}}{\text{s}} \tag{1.29}$$

Part (e): You may find this a bit surprising, but if you think about it the explanation for why Ann lost the race, despite her running at the same speed as Becky *relative to the surface she was standing on*, has to be that she actually ran a longer distance on that surface! Since she was running for a total of 12.5 s at a constant *speed* (not velocity!) of 5 m/s in the walkway frame, then in that frame she ran a distance $d = |v|\Delta t = 5 \times 12.5 = 62.5\,\text{m}$. That is the total length of walkway that she actually stepped on.

## 1.6 Problems

### Problem 1




The above figure is the position (in meters) versus time (in seconds) graph of an object in motion. Only the segments between $t = 1\,\mathrm{s}$ and $t = 2\,\mathrm{s}$, and between $t = 4\,\mathrm{s}$ and $t = 5\,\mathrm{s}$, are straight lines. The peak of the curve is at $t = 3\,\mathrm{s}$, $x = 4\,\mathrm{m}$.

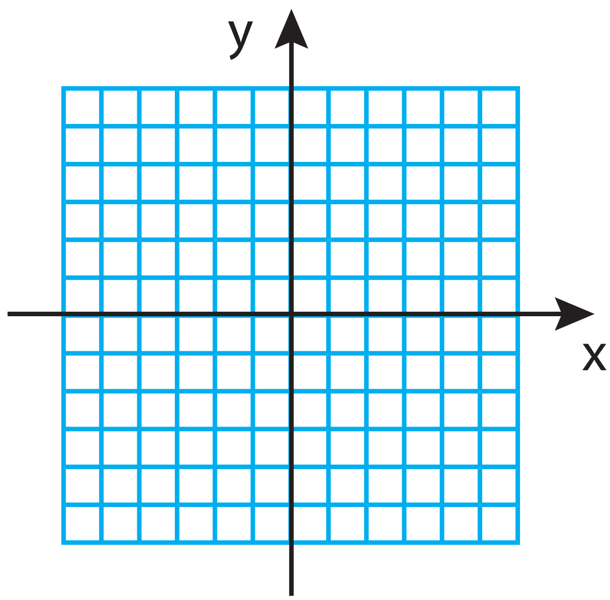
Answer the following questions, and provide a brief justification for your answer in every case.

(a) At what time(s) is the object's velocity equal to zero?
(b) For what range(s) of times is the object moving with constant velocity?
(c) What is the object's position coordinate at $t = 1\,\mathrm{s}$?
(d) What is the displacement of the object between $t = 1\,\mathrm{s}$ and $t = 4\,\mathrm{s}$?
(e) What is the *distance* traveled between $t = 1\,\mathrm{s}$ and $t = 4\,\mathrm{s}$?
(f) What is the instantaneous velocity of the object at $t = 1.5\,\mathrm{s}$?
(g) What is its average velocity between $t = 1\,\mathrm{s}$ and $t = 3\,\mathrm{s}$?

### Problem 2

A particle is initially at $x_i = 3\,\mathrm{m}$, $y_i = -5\,\mathrm{m}$, and after a while it is found at the coordinates $x_f = -4\,\mathrm{m}$, $y_f = 2\,\mathrm{m}$.

(a) On the grid below (next page), draw the initial and final position vectors, and the displacement vector.
(b) What are the components of the displacement vector?
(c) What are the magnitude and direction of the displacement vector? (You can specify the direction by the angle it makes with either the positive $x$ or the positive $y$ axis.)




**Problem 3**

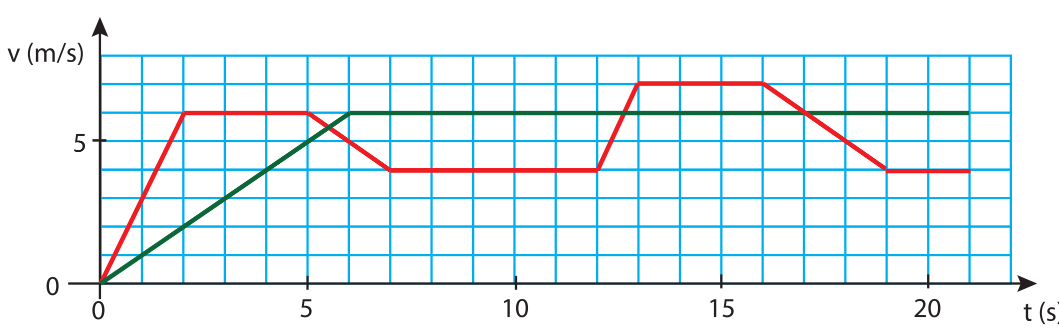
Marshall Dillon is riding at 30 mph after the robber of the Dodge City bank, who has a head start of 15 minutes, but whose horse can only make 25 mph on a good day. How long does it take for Dillon to catch up with the bad guy, and how far from Dodge City are they when this happens? (Assume the road is straight, for simplicity.)

**Problem 4**

The picture below shows the velocity versus time graph of the first 21 seconds of a race between two friends, "Red" and "Green."

(a) Who is ahead at $t = 10\,\text{s}$, and by how much?

(b) Who passes the 100 m marker first?



**Problem 5**

You are trying to pass a truck on the highway. The truck is driving at 55 mph, so you speed up to 60 mph and move over to the left lane. If the truck is 17 m long, and your car is 3 m long

(a) how long does it take you to pass the truck completely?
(b) How far (along the highway) have you traveled in that time?
Note: to answer part (a) look at the problem from the perspective of the truck driver. How far are you going relative to him, and how far would it take you to cover 20 m at that speed?

**Problem 6**

Suppose the position function of a particle moving in one dimension is given by

$$x(t) = 5 + 3t + 2t^2 - 0.5t^3 \tag{1.30}$$

where the coefficients are such that the result will be in meters if you enter the time in seconds. What is the particle's velocity at $t = 2\,\text{s}$? There are two ways you can do this:

- If you know calculus, calculate the derivative of (1.30) and evaluate it at $t = 2\,\text{s}$.
- If you do *not* yet know how to take derivatives, calculate the limit in the definition (1.8). That is to say, calculate $\Delta x/\Delta t$ with $t_i = 2\,\text{s}$ and $\Delta t$ equal, first, to 0.1 s, then to 0.01 s, and then to 0.001 s. You will need to keep more than the usual 4 decimals in the intermediate calculations if you want an accurate result, but you should still report only 3 significant digits in the final result.

**Problem 7**

Suppose you are rowing across a river, as in Figure 1.7. Your speed is 2 miles per hour relative to the current, which is moving at a leisurely 1 mile per hour. If the river is 10 m wide,

(a) How far downstream do you end up?
(b) To row straight across you would need to have an upstream velocity component (relative to the current). How large would that be?
(c) If your rowing speed is still only 2 miles per hour, how long does it take you to row across the river now?

# Chapter 2

# Acceleration

## 2.1 The law of inertia

There is something funny about motion with constant velocity: it is *indistinguishable from rest.* Of course, you can usually tell whether you are moving *relative to something else.* But if you are enjoying a smooth airplane ride, without looking out the window, you have no idea how fast you are moving, or even, indeed (if the flight is exceptionally smooth) whether you are moving at all. I am actually writing this on an airplane. The flight screen informs me that I am moving at 480 mph relative to the ground, but I do not feel anything like that: just a gentle rocking up and down and sideways that gives me no clue as to what my forward velocity is.

If I were to drop something, I know from experience that it would fall on a straight line—relative to me, that is. If it falls from my hand it will land at my feet, just as if we were all at rest. But we are *not* at rest. In the half second or so it takes for the object to fall, the airplane has moved forward 111 meters relative to the ground. Yet the (hypothetical) object I drop does not land 300 feet behind me! It moves forward with me as it falls, even though I am not touching it. *It keeps its initial forward velocity,* even though it is no longer in contact with me or anything connected to the airplane.

(At this point you might think that the object is still in contact with the air inside the plane, which is moving with the plane, and conjecture that maybe it is the air inside the plane that "pushes forward" on the object as it falls and keeps it from moving backwards. This is not necessarily a dumb idea, but a moment's reflection will convince you that it is impossible. We are all familiar with the way air pushes on things moving through it, and we know that the force an object experiences depends on its mass and its shape, so if that was what was happening, dropping objects of different masses and shapes I would see them falling in all kind of different ways—as I would, in fact, if I were dropping things from rest outdoors in a strong wind. But that is not what we experience on an airplane at all. The air, in fact, has no effect on the forward motion of the falling object. It does not push it in any way, because it is moving at the same velocity. This, in fact, reinforces our previous conclusion: the object keeps its forward velocity while it is falling, in the absence of any external influence.)

This remarkable observation is one of the most fundamental principles of physics (yes, we have started to learn physics now!), which we call **the law of inertia**. It can be stated as follows: in the absence of any external influence (or *force*) acting on it, an object at rest will stay at rest, while an object that is already moving with some velocity will keep that same velocity (speed and direction of motion)—at least until it is, in fact, acted upon by some force.

Please let that sink in for a moment, before we start backtracking, which we have to do now on several accounts. First, I have used repeatedly the term "force," but I have not defined it properly. Or have I? What if I just said that forces are precisely any "external influences" that may cause a change in the velocity of an object? That will work, I think, until it is time to explore the concept in more detail, a few chapters from now.

Next, I need to draw your attention to the fact that the object I (hypothetically) dropped did not actually keep its *total* initial velocity: it only kept its initial *forward* velocity. In the downward direction, it was speeding up from the moment it left my hand, as would any other falling object (and as we shall see later in this chapter). But this actually makes sense in a certain way: there was no forward force, so the forward velocity remained constant; there was, however, a vertical force acting all along (the force of gravity), and so the object did speed up in that direction. This observation is, in fact, telling us something profound about the world's geometry: namely, that forces and velocities are *vectors*, and laws such as the law of inertia will typically apply to the vector as a whole, as well as to each component separately (that is to say, each dimension of space). This anticipates, in fact, the way we will deal, later on, with motion in two or more dimensions; but we do not need to worry about that for a few chapters still.

Finally, it is worth spending a moment reflecting on how radically the law of inertia seems to contradict our intuition about the way the world works. What it seems to be telling us is that, if we throw or push an object, it should continue to move forever with the same speed and in the same direction with which it set out—something that we know is certainly not true. But what's happening in "real life" is that, just because we have left something alone, it doesn't mean the world has left it alone. After we lose contact with the object, all sorts of other forces will continue to act on it. A ball we throw, for instance, will experience air resistance or drag (the same effect I was worrying about in that paragraph in parenthesis in the previous page), and that will slow it down. An object sliding on a surface will experience friction, and that will slow it down too. Perhaps the closest thing to the law of inertia in action that you may get to see is a hockey puck sliding on the ice: it is remarkable (perhaps even a bit frightening) to see how little it slows down, but even so the ice does a exert a (very small) frictional force that would bring the puck to a stop eventually.

This is why, historically, the law of inertia was not discovered until people started developing an appreciation for frictional forces, and the way they are constantly acting all around us to oppose the relative motion of any objects trying to slide past each other.

This mention of relative motion, in a way, brings us full circle. Yes, relative motion is certainly detectable, and for objects in contact it actually results in the occurrence of forces of the frictional, or drag, variety. But *absolute* (that is, without reference to anything external) motion with constant velocity is fundamentally undetectable. And in view of the law of inertia, it makes sense: if no force is required to keep me moving with constant velocity, it follows that as long as I am moving with constant velocity I should not be feeling any net force acting on me; nor would any other detection apparatus I might be carrying with me.

What we do feel in our bodies, and what we can detect with our inertial navigation systems (now you may start to guess why they are called "inertial"), is a *change* in our velocity, which is to say, our *acceleration* (to be defined properly in a moment). We rely, ultimately, on the law of inertia to detect accelerations: if my plane is shaking up and down, because of turbulence (as, in fact, it is right now!), the water in my cup may not stay put. Or, rather, the water may try to stay put (really, to keep moving, at any moment, with whatever velocity it has at that moment), but if the cup, which is connected to my hand which is connected, ultimately, to this bouncy plane, moves suddenly out from under it, not all of the water's parts will be able to adjust their velocities to the new velocity of the cup in time to prevent a spill.

This is the next very interesting fact about the physical world that we are about to discover: forces cause accelerations, or changes in velocity, but they do so in different degrees for different objects; and, moreover, the ultimate change in velocity *takes time*. The first part of this statement has to do with the concept of *inertial mass*, to be introduced in the next chapter; the second part we are going to explore right now, after a brief detour to define *inertial reference frames*.

### 2.1.1 Inertial reference frames

The example I just gave you of what happens when a plane in flight experiences turbulence points to an important phenomenon, namely, that there may be times where the law of inertia may not *seem* to apply in a certain reference frame. By this I mean that an object that I left at rest, like the water in my cup, may suddenly start to move—relative to the reference frame coordinates—even though nothing and nobody is acting on it. More dramatically still, if a car comes to a sudden stop, the passengers may be "projected forward"—they were initially at rest relative to the car frame, but now they find themselves moving forward (always in the car reference frame), to the point that, if they are not wearing seat belts, they may end up hitting the dashboard, or the seat in front of them.

Again, nobody has pushed on them, and in fact what we can see in this case, from outside the car, is nothing but the law of inertia at work: the passengers were just keeping their initial velocity, when the car suddenly slowed down under and around them. So there is nothing wrong with the law of inertia, but *there is a problem with the reference frame*: if I want to describe the motion of objects in a reference frame like a plane being shaken up or a car that is speeding up or slowing down, I need to allow for the fact that objects may move—always relative to that frame—in an *apparent* violation of the law of inertia.

The way we deal with this in physics is by introducing the very important concept of an *inertial reference frame*, by which we mean a reference frame in which all objects will, at all times, be observed to move (or not move) in a way fully consistent with the law of inertia. In other words, the law of inertia has to hold *when we use that frame's own coordinates to calculate the objects' velocities*. This, of course, is what we always do instinctively: when I am on a plane I locate the various objects around me relative to the plane frame itself, not relative to the distant ground.

To ascertain whether a frame is inertial or not, we start by checking to see if the description of motion using that frame's coordinates obeys the law of inertia: does an object left at rest on the counter in the laboratory stay at rest? If set in motion, does it move with constant velocity on a straight line? The Earth's surface, as it turns out, is *not* quite a perfect inertial reference frame, but it *is* good enough that it made it possible for us to discover the law of inertia in the first place!

What spoils the inertial-ness of an Earth-bound reference frame is the Earth's rotation, which, as we shall see later, is an example of *accelerated motion*. In fact, if you think about the grossly non-inertial frames I have introduced above—the bouncy plane, the braking car—they all have this in common: that their velocities are changing; they are *not* moving with constant speed on a straight line.

So, once you have found an inertial reference frame, to decide whether another one is inertial or not is simple: if it is moving with constant velocity (relative to the first, inertial frame), then it is itself inertial; if not, it is not. I will show you how this works, formally, in a little bit (section 2.2.4, below), after I (finally!) get around to properly introducing the concept of acceleration.

It is a fundamental principle of physics that *the laws of physics take the same form in all inertial reference frames*. The law of inertia is, of course, an example of such a law. Since all inertial frames are moving with constant velocity relative to each other, this is another way to say that absolute motion is undetectable, and all motion is ultimately relative. Accordingly, this principle is known as **the principle of relativity**.

## 2.2 Acceleration

### 2.2.1 Average and instantaneous acceleration

Just as we defined average velocity in the previous chapter, using the concept of displacement (or change in position) over a time interval $\Delta t$, we define *average acceleration* over the time $\Delta t$ using the change in velocity:

$$a_{av} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i} \tag{2.1}$$

Here, $v_i$ and $v_f$ are the initial and final velocities, respectively, that is to say, the velocities at the beginning and the end of the time interval $\Delta t$. As was the case with the average velocity, though, the average acceleration is a concept of somewhat limited usefulness, so we might as well proceed straight away to the definition of the *instantaneous acceleration* (or just "the" acceleration, without modifiers), through the same sort of limiting process by which we defined the instantaneous velocity:

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \tag{2.2}$$

Everything that we said in the previous chapter about the relationship between velocity and position can now be said about the relationship between acceleration and velocity. For instance (if you know calculus), the acceleration as a function of time is the derivative of the velocity as a function of time, which makes it the second derivative of the position function:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{2.3}$$

(and if you do *not* know calculus yet, do not worry about the superscripts "2" on that last expression! It is just a weird notation that you will learn someday.)

Similarly, we can "read off" the instantaneous acceleration from a *velocity* versus time graph, by looking at the slope of the line tangent to the curve at any point. However, if what we are given is a *position* versus time graph, the connection to the acceleration is more indirect. Figure 2.1 (next page) provides you with such an example. See if you can guess at what points along this curve the acceleration is positive, negative, or zero.

The way to do this "from scratch," as it were, is to try to figure out what the velocity is doing, first, and infer the acceleration from that. Here is how that would go:

Starting at $t = 0$, and keeping an eye on the slope of the $x$-vs-$t$ curve, we can see that the velocity starts at zero or near zero and increases steadily for a while, until $t$ is a little bit more than 2 s (let us say, $t = 2.2$ s for definiteness). That would correspond to a period of positive acceleration, since $\Delta v$ would be positive for every $\Delta t$ in that range.

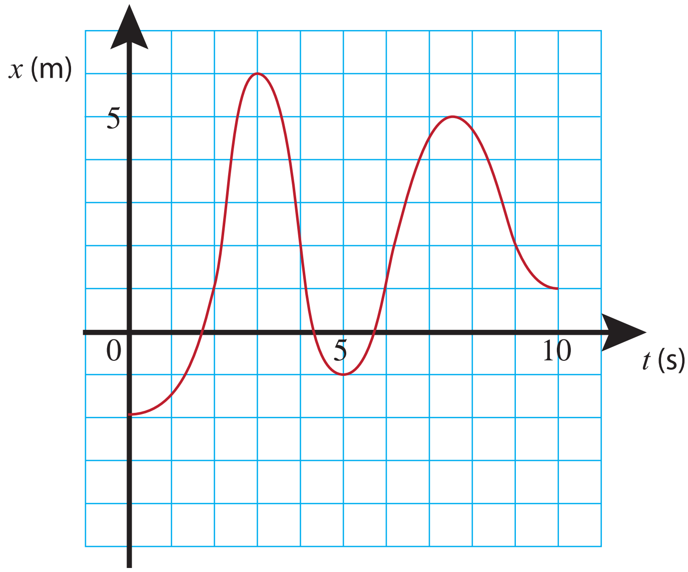



**Figure 2.1:** A possible position vs. time graph for an object whose acceleration changes with time.

Between $t = 2.2$ s and $t = 2.5$ s, as the object moves from $x = 2$ m to $x = 4$ m, the velocity does not appear to change very much, and the acceleration would correspondingly be zero or near zero. Then, around $t = 2.5$ s, the velocity starts to decrease noticeably, becoming (instantaneously) zero at $t = 3$ s ($x = 6$ m). That would correspond to a negative acceleration. Note, however, that the velocity afterwards continues to decrease, becoming more and more negative until around $t = 4$ s. This also corresponds to a negative acceleration: even though the object is speeding up, it is speeding up in the negative direction, so $\Delta v$, and hence $a$, is negative for every time interval there. We conclude that $a < 0$ for all times between $t = 2.5$ s and $t = 4$ s.

Next, as we just look past $t = 4$ s, something else interesting happens: the object is still going in the negative direction (negative velocity), but now it is slowing down. Mathematically, that corresponds to a *positive* acceleration, since the algebraic value of the velocity is in fact increasing (a number like $-3$ is larger than a number like $-5$). Another way to think about it is that, if we have less and less of a negative thing, our overall trend is positive. So the acceleration is positive all the way from $t = 4$ s through $t = 5$ s (where the velocity is instantaneously zero as the object's direction of motion reverses), and beyond, until about $t = 6$ s, since between $t = 5$ s and $t = 6$ s the velocity is positive and growing.

You can probably figure out on your own now what happens after $t = 6$ s, reasoning as I did above, but you may also have noticed a pattern that makes this kind of analysis a lot easier. The acceleration (as those with a knowledge of calculus may have understood already), being proportional to the second derivative of the function $x(t)$ with respect to $t$, is directly related to the *curvature* of the $x$-vs-$t$ graph. As figure 2.2 below shows, if the graph is *concave* (sometimes called "concave upwards"), the acceleration is positive, whereas it is negative whenever the graph is *convex* (or "concave downwards"). It is (instantly) zero at those points where the curvature changes (which you may know as *inflection points*), as well as over stretches of time when the $x$-vs-$t$ graph is a straight line (motion with constant velocity).

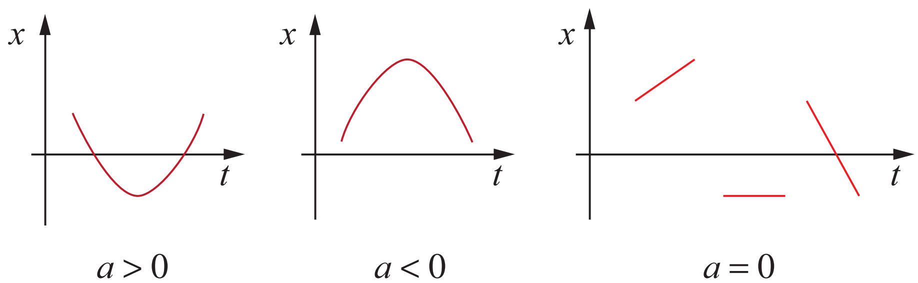



**Figure 2.2:** What the $x$-vs-$t$ curves look like for the different possible signs of the acceleration.

Figure 2.3 (in the next page) shows position, velocity, and acceleration versus time for a hypothetical motion case. Please study it carefully until every feature of every graph makes sense, relative to the other two! You will see many other examples of this in the homework and the lab.

Notice that, in all these figures, the sign of $x$ or $v$ at any given time has nothing to do with the sign of $a$ at that same time. It is true that, for instance, a negative $a$, if sustained for a sufficiently long time, will *eventually* result in a negative $v$ (as happens, for instance, in Fig. 2.3 over the interval from $t = 1$ to $t = 4\,\mathrm{s}$) but this may take a long time, depending on the size of $a$ and the initial value of $v$. The graphical clues to follow, instead, are: the acceleration is given by the slope of the tangent to the $v$-vs-$t$ curve, or the curvature of the $x$-vs-$t$ curve, as explained in Fig. 2.2; and the velocity is given by the slope of the tangent to the $x$-vs-$t$ curve.

(Note: To make the interpretation of Figure 2.3 simpler, I have chosen the acceleration to be "piecewise constant," that is to say, constant over extended time intervals and changing in value discontinuously from one interval to the next. This is physically unrealistic: in any real-life situation, the acceleration would be expected to change more or less smoothly from instant to instant. We will see examples of that later on, when we start looking at realistic models of collisions.)

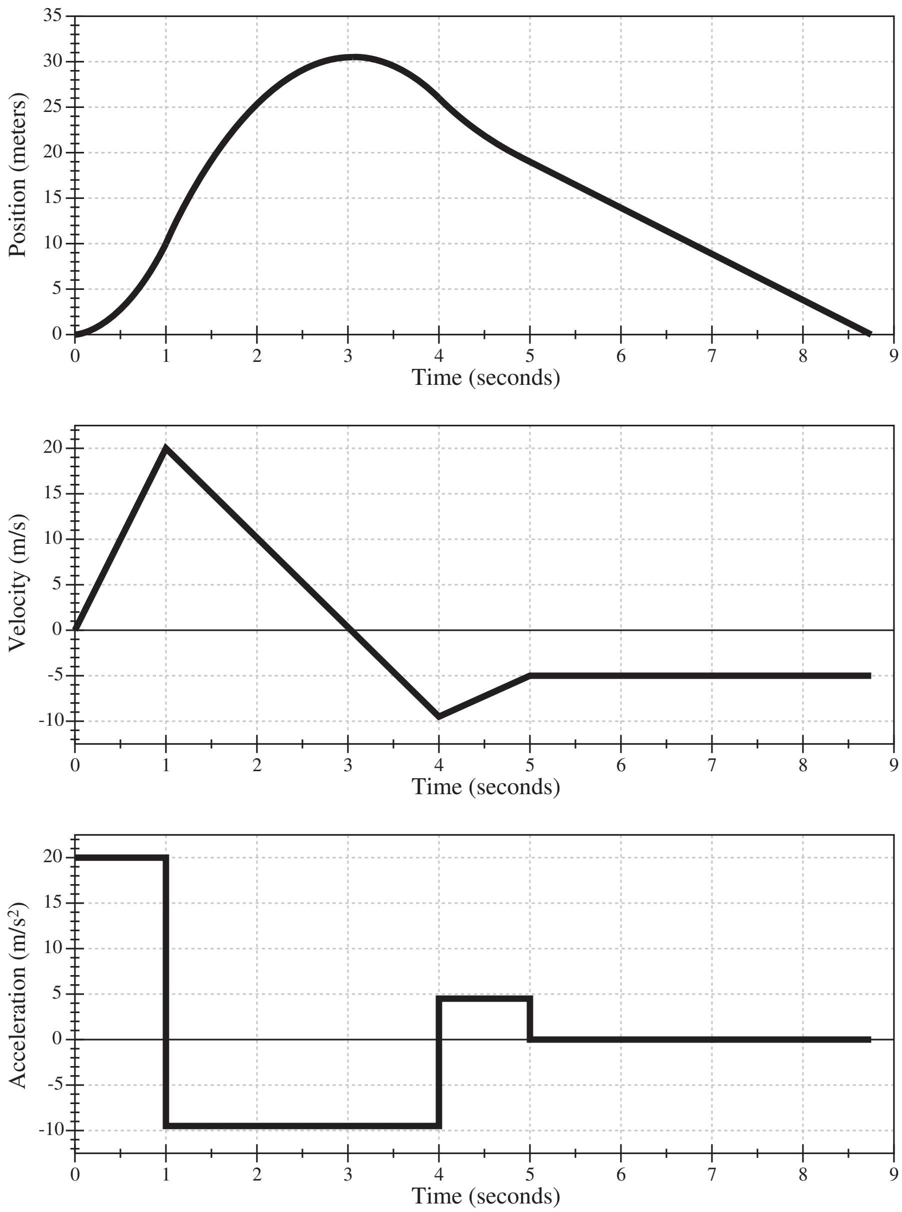



**Figure 2.3:** Sample position, velocity and acceleration vs. time graphs for motion with piecewise-constant acceleration.

### 2.2.2 Motion with constant acceleration

A particular kind of motion that is both relatively simple and very important in practice is motion with constant acceleration (see Figure 2.3 again for examples). If $a$ is constant, it means that the velocity changes with time at a constant rate, by a fixed number of m/s each second. (These are, incidentally, the units of acceleration: meters per second per second, or m/s$^2$.) The change in velocity over a time interval $\Delta t$ is then given by

$$\Delta v = a\,\Delta t \tag{2.4}$$

which can also be written

$$v = v_i + a\,(t - t_i) \tag{2.5}$$

Equation (2.5) is the form of the velocity function ($v$ as a function of $t$) for motion with constant acceleration. This, in turn, has to be the derivative with respect to time of the corresponding position function. If you know simple derivatives, then, you can verify that the appropriate form of the position function must be

$$x = x_i + v_i(t - t_i) + \frac{1}{2}\,a\,(t - t_i)^2 \tag{2.6}$$

or in terms of intervals,

$$\Delta x = v_i \Delta t + \frac{1}{2}\,a\,(\Delta t)^2 \tag{2.7}$$

Most often Eq. (2.6) is written with the implicit assumption that the initial value of $t$ is zero:

$$x = x_i + v_i t + \frac{1}{2}\,at^2 \tag{2.8}$$

This is simpler, but not as general as Eq. (2.6). Always make sure that you know what conditions apply for any equation you decide to use!

As you can see from Eq. (2.5), for intervals during which the acceleration is constant, the velocity vs. time curve should be a straight line. Figure 2.3 (previous page) illustrates this. Equation (2.6), on the other hand, shows that for those same intervals the position vs. time curve should be a (portion of a) parabola, and again this can be seen in Figure 2.3 (sometimes, if the acceleration is small, the curvature of the graph may be hard to see; this happens in Figure 2.3 for the interval between $t = 4\,\text{s}$ and $t = 5\,\text{s}$).

The observation that $v$-vs-$t$ is a straight line when the acceleration is constant provides us with a simple way to derive Eq. (2.7), when combined with the result (from the end of the previous chapter) that the displacement over a time interval $\Delta t$ equals the area under the $v$-vs-$t$ curve for that time interval. Indeed, consider the situation shown in Figure 2.4. The total area under the segment shown is equal to the area of a rectangle of base $\Delta t$ and height $v_i$, plus the area of a triangle of base $\Delta t$ and height $v_f - v_i$. Since $v_f - v_i = a\Delta t$, simple geometry immediately yields Eq. (2.7), or its equivalent (2.6).

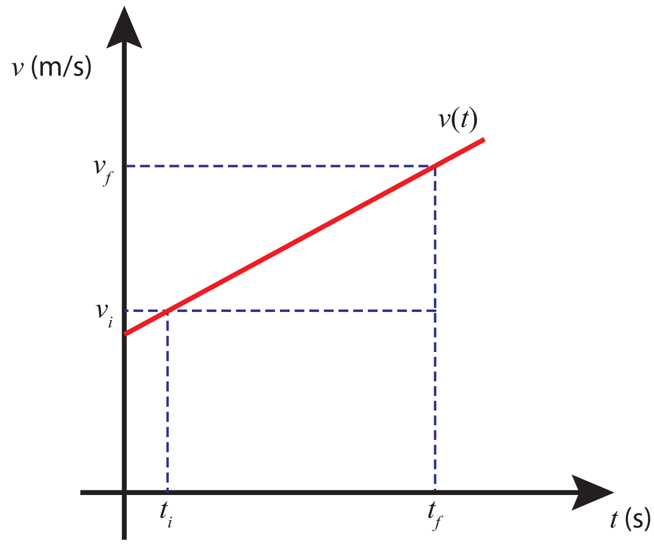



**Figure 2.4:** Graphical way to find the displacement for motion with constant acceleration.

Lastly, consider what happens if we solve Eq. (2.4) for $\Delta t$ and substitute the result in (2.7). We get

$$\Delta x = \frac{v_i \Delta v}{a} + \frac{(\Delta v)^2}{2a} \tag{2.9}$$

Letting $\Delta v = v_f - v_i$, a little algebra yields

$$v_f^2 - v_i^2 = 2a\Delta x \tag{2.10}$$

This is a handy little result that can also be seen to follow, more directly, from the work-energy theorems to be introduced in Chapter 7[1].

[1]In fact, equation (2.10) turns out to be *so* handy that you will probably find yourself using it over and over this semester, and you may even be tempted to use it for problems involving motion in two dimensions. However, unless you really know what you are doing, you should resist the temptation, since it is very easy to use (2.10) incorrectly when the acceleration and the displacement do not lie along the same line. You should use the appropriate form of a work-energy theorem instead.

### 2.2.3 Acceleration as a vector

In two (or more) dimensions we introduce the average acceleration vector

$$\vec{a}_{av} = \frac{\Delta \vec{v}}{\Delta t} = \frac{1}{\Delta t}\left(\vec{v}_f - \vec{v}_i\right) \tag{2.11}$$

whose components are $a_{av,x} = \Delta v_x/\Delta t$, etc.. The instantaneous acceleration is then the vector given by the limit of Eq. (2.11) as $\Delta t \to 0$, and its components are, therefore, $a_x = dv_x/dt, a_y = dv_y/dt, \ldots$

Note that, since $\vec{v}_i$ and $\vec{v}_f$ in Eq. (2.11) are vectors, and have to be subtracted as such, the acceleration vector will be nonzero whenever $\vec{v}_i$ and $\vec{v}_f$ are different, even if, for instance, their magnitudes (which are equal to the object's speed) are the same. In other words, you have accelerated motion whenever the *direction* of motion changes, even if the speed does not.

As long as we are working in one dimension, I will follow the same convention for the acceleration as the one I introduced for the velocity in Chapter 1: namely, I will use the symbol $a$, without a subscript, to refer to the relevant component of the acceleration $(a_x, a_y, \ldots)$, and *not* to the magnitude of the vector $\vec{a}$.

### 2.2.4 Acceleration in different reference frames

In Chapter 1 you saw that the following relation (Eq. (1.19)) holds between the velocities of a particle P measured in two different reference frames, A and B:

$$\vec{v}_{AP} = \vec{v}_{AB} + \vec{v}_{BP} \tag{2.12}$$

What about the acceleration? An equation like (2.12) will hold for the initial and final velocities, and subtracting them we will get

$$\Delta\vec{v}_{AP} = \Delta\vec{v}_{AB} + \Delta\vec{v}_{BP} \tag{2.13}$$

Now suppose that reference frame B moves with *constant velocity* relative to frame A. In that case, $\vec{v}_{AB,f} = \vec{v}_{AB,i}$, so $\Delta\vec{v}_{AB} = 0$, and then, dividing Eq. (2.13) by $\Delta t$, and taking the limit $\Delta t \to 0$, we get

$$\vec{a}_{AP} = \vec{a}_{BP} \qquad \text{(for constant } \vec{v}_{AB}) \tag{2.14}$$

So, if two reference frames are moving at constant velocity relative to each other, observers in both frames measure the *same* acceleration for any object they might both be tracking.

The result (2.14) means, in particular, that if we have an inertial frame then any frame moving at constant velocity relative to it will be inertial too, since the respective observers' measurements will agree that an object's velocity does not change (otherwise put, its acceleration is zero) when no forces act on it. Conversely, an accelerated frame will *not* be an inertial frame, because Eq. (2.14) will not hold. This is consistent with the examples I mentioned in Section 2.1 (the bouncing plane, the car coming to a stop). Another example of a non-inertial frame would be a car going around a curve, even if it is going at constant speed, since, as I just pointed out above, this is also an accelerated system. This is confirmed by the fact that objects in such a car tend to move—relative to the car—towards the outside of the curve, even though no actual force is acting on them.

## 2.3 Free fall

An important example of motion with (approximately) constant acceleration is provided by *free fall* near the surface of the Earth. We say that an object is in "free fall" when the only force acting on it is the force of gravity (the word "fall" here may be a bit misleading, since the object could actually be moving upwards some of the time, if it has been thrown straight up, for instance). The space station is in free fall, but because it is nowhere near the surface of the earth its direction of motion (and hence its acceleration, regarded as a two-dimensional vector) is constantly changing. Right next to the surface of the earth, on the other hand, the planet's curvature is pretty much negligible and gravity provides an approximately constant, vertical acceleration, which, in the absence of other forces, turns out to be *the same for every object*, regardless of its size, shape, or weight.

The above result—that, in the absence of other forces, all objects should fall to the earth at the same rate, regardless of how big or heavy they are—is so contrary to our common experience that it took many centuries to discover it. The key, of course, as with the law of inertia, is to realize that, under normal circumstances, frictional forces are, in fact, acting all the time, so an object falling through the atmosphere is never *really* in "free" fall: there is always, at a minimum, and in addition to the force of gravity, an air drag force that opposes its motion. The magnitude of this force does depend on the object's size and shape (basically, on how "aerodynamic" the object is); and thus a golf ball, for instance, falls much faster than a flat sheet of paper. Yet, if you crumple up the sheet of paper till it has the same size and shape as the golf ball, you can see for yourself that they do fall at approximately the same rate! The equality can never be exact, however, unless you get rid completely of air drag, either by doing the experiment in an evacuated tube, or (in a somewhat extreme way), by doing it on the surface of the moon, as the Apollo 15 astronauts did with a hammer and a feather back in 1971[2].

This still leaves us with something of a mystery, however: the force of gravity is the only force known to have the property that it imparts all objects the *same* acceleration, regardless of their mass or constitution. A way to put this technically is that the force of gravity on an object is

[2]The video of this is available online: https://www.youtube.com/watch?v=oYEgdZ3iEKA. It is, however, pretty low resolution and hard to see. A very impressive modern-day demonstration involving feathers and a bowling ball in a completely evacuated (airless) room is available here: https://www.youtube.com/watch?v=E43-CfukEgs.

proportional to that object's *inertial mass*, a quantity that we will introduce properly in the next chapter. For the time being, we will simply record here that this acceleration, near the surface of the earth, has a magnitude of approximately $9.8\,\mathrm{m/s^2}$, a quantity that is denoted by the symbol $g$. Thus, if we take the upwards direction as positive (as is usually done), we get for the acceleration of an object in free fall $a = -g$, and the equations of motion become

$$\Delta v = -g\,\Delta t \tag{2.15}$$

$$\Delta y = v_i \Delta t - \frac{1}{2}\,g\,(\Delta t)^2 \tag{2.16}$$

where I have used $y$ instead of $x$ for the position coordinate, since that is a more common choice for a vertical axis. Note that we could as well have chosen the downward direction as positive, and that may be a more natural choice in some problems. Regardless, the quantity $g$ is always defined to be positive: $g = 9.8\,\mathrm{m/s^2}$. The acceleration, then, is $g$ or $-g$, depending on which direction we take to be positive.

In practice, the value of $g$ changes a little from place to place around the earth, for various reasons (it is somewhat sensitive to the density of the ground below you, and it decreases as you climb higher away from the center of the earth). In a later chapter we will see how to calculate the value of $g$ from the mass and radius of the earth, and also how to calculate the equivalent quantity for other planets.

In the meantime, we can use equations like (2.15) and (2.16) (as well as (2.10), with the appropriate substitutions) to answer a number of interesting questions about objects thrown or dropped straight up or down (always, of course, assuming that air drag is negligible). For instance, back at the beginning of this chapter I mentioned that if I dropped an object it might take about half a second to hit the ground. If you use Eq. (2.16) with $v_i = 0$ (since I am dropping the object, not throwing it down, its initial velocity is zero), and substitute $\Delta t = 0.5\,\mathrm{s}$, you get $\Delta y = 1.23\,\mathrm{m}$ (about 4 feet). This is a reasonable height from which to drop something.

On the other hand, you may note that half a second is not a very long time in which to make accurate observations (especially if you do not have modern electronic equipment), and as a result of that there was considerable confusion for many centuries as to the precise way in which objects fell. Some people believed that the speed did increase in some way as the object fell, while others appear to have believed that an object dropped would "instantaneously" (that is, at soon as it left your hand) acquire some speed and keep that unchanged all the way down. In reality, in the presence of air drag, what happens is a combination of both: initially the speed increases at an approximately constant rate (free, or nearly free fall), but the drag force increases with the speed as well, until eventually it balances out the force of gravity, and from that point on the speed does not increase anymore: we say that the object has reached "terminal velocity." Some objects reach terminal velocity almost instantly, whereas others (the more "aerodynamic" ones) may take a long time to do so. This accounts for the confusion that prevailed before Galileo's experiments in the early 1600's.

Galileo's main insight, on the theoretical side, was the realization that it was necessary to separate clearly the effect of gravity and the effect of the drag force. Experimentally, his big idea was to use an inclined plane to slow down the "fall" of an object, so as to make accurate measurements possible (and also, incidentally, reduce the air drag force!). These "inclined planes" were just basically ramps down which he sent small balls (like marbles) rolling. By changing the steepness of the ramp he could control how slowly the balls moved. He reasoned that, ultimately, the force that made the balls go down was essentially the same force of gravity, only not the whole force, but just a fraction of it. Today we know that, in fact, an object sliding (*not* rolling!) up or down on a *frictionless* incline will experience an acceleration directed downwards along the incline and with a magnitude equal to $g \sin\theta$, where $\theta$ is the angle that the slope makes with the horizontal:

$$a = g \sin\theta \quad \text{(inclined plane, taking } \textit{downwards} \text{ to be positive)} \tag{2.17}$$

(for some reason, it seems more natural, when dealing with inclined planes, to take the downward direction as positive!). Equation (2.17) makes sense in the two extreme cases in which the plane is completely vertical ($\theta = 90^\circ$, $a = g$) and completely horizontal ($\theta = 0^\circ$, $a = 0$). For intermediate values, you will carry out experiments in the lab to verify this result.

We will show, in a later chapter, how Eq. (2.17) comes about from a careful consideration of all the forces acting on the object; we will also see, later on, how it needs to be modified for the case of a rolling, rather than a sliding, object. This modification does not affect Galileo's main conclusion, which was, basically, that the natural falling motion in the absence of friction or drag forces is motion with *constant acceleration* (at least, near the surface of the earth, where $g$ is constant to a very good approximation).

## 2.4 In summary

1. The *law of inertia* states that, if no external influences (forces) are acting on an object, then, if the object is initially at rest it will stay at rest, and if it is initially moving it will continue to move with constant velocity (unchanging speed and direction).

2. Reference frames in which the law of inertia is seen to hold (when the velocities of objects are calculated from their coordinates in that frame) are called *inertial*. A reference frame that is moving at constant velocity relative to an inertial frame is also an inertial frame. Conversely, accelerated reference frames are non-inertial.

3. Motion with constant velocity is fundamentally indistinguishable from no motion at all (i.e., rest). As long as the velocity (of the objects involved) does not change, only *relative* motion can be detected. This is known as the **principle of relativity**. Another way to state it is that the laws of physics must take the same form in all inertial reference frames (so you cannot single out one as being in "absolute rest" or "absolute motion").

4. *Changes* in velocity *are* detectable, and, by (1) above, are evidence of unbalanced forces acting on an object.

5. The rate of change of an object's velocity is the object's *acceleration*: the average acceleration over a time interval $\Delta t$ is $a_{av} = \Delta v/\Delta t$, and the instantaneous acceleration at a time $t$ is the limit of the average acceleration calculated for successively shorter time intervals $\Delta t$, all with the same initial time $t_i = t$. Mathematically, this means the acceleration is the derivative of the velocity function, $a = dv/dt$.

6. In a velocity versus time graph, the acceleration can be read from the slope of the line tangent to the curve (just like the velocity in a position versus time graph).

7. In a position versus time graph, the regions with positive acceleration correspond to a concave curvature (like a parabola opening up), and those with negative acceleration correspond to a convex curvature (like a parabola opening down). Points of inflection (where the curvature changes) and straight lines correspond to points where the acceleration is zero.

8. The basic equations used to describe motion with constant acceleration are (2.4), (2.7) and (2.10) above. Alternative forms of these are also provided in the text.

9. In more than one dimension, a change in the *direction* of the velocity vector results in a nonzero acceleration, even if the object's speed does not change.

10. An object is said to be in *free fall* when the only force acting on it is gravity. All objects in free fall experience the same acceleration at the same point in their motion, regardless of their mass or composition. Near the surface of the earth, this acceleration is approximately constant and has a magnitude $g = 9.8\,\mathrm{m/s^2}$.

11. An object sliding on a frictionless inclined plane experiences (if air drag is negligible) an acceleration directed downward along the incline and with a magnitude $g \sin\theta$, where $\theta$ is the angle the incline makes with the horizontal.

## 2.5 Examples

### 2.5.1 Motion with piecewise constant acceleration

Construct the position vs. time, velocity vs. time, and acceleration vs. time graphs for the motion described below. For each of the intervals (a)–(d) you'll need to figure out the position (height) and velocity of the rocket at the beginning and the end of the interval, and the acceleration for the interval. In addition, for interval (b) you need to figure out the maximum height reached by the rocket and the time at which it occurs. For interval (d) you need to figure out its duration, that is to say, the time at which the rocket hits the ground.

(a) A rocket is shot upwards, accelerating from rest to a final velocity of 20 m/s in 1 s as it burns its fuel. (Treat the acceleration as constant during this interval.)
(b) From $t = 1\,\text{s}$ to $t = 4$ s, with the fuel exhausted, the rocket flies under the influence of gravity alone. At some point during this time interval (you need to figure out when!) it stops climbing and starts falling.
(c) At $t = 4\,\text{s}$ a parachute opens, suddenly causing an upwards acceleration (again, treat it as constant) lasting 1 s; at the end of this interval, the rocket's velocity is 5 m/s downwards.
(d) The last part of the motion, with the parachute deployed, is with constant velocity of 5 m/s downwards until the rocket hits the ground.

**Solution:**

(a) For this first interval (for which I will use a subscript "1" throughout) we have

$$\Delta y_1 = \frac{1}{2} a_1 (\Delta t_1)^2 \tag{2.18}$$

using Eq. (2.6) for motion with constant acceleration with zero initial velocity (I am using the variable $y$, instead of $x$, for the vertical coordinate; this is more or less customary, but, of course, I could have used $x$ just as well).

Since the acceleration is constant, it is equal to its average value:

$$a_1 = \frac{\Delta v}{\Delta t} = 20 \frac{\text{m}}{\text{s}^2}.$$

Substituting this into (2.18) we get the height at $t = 1\,\text{s}$ is 10 m. The velocity at that time, of course, is $v_{f1} = 20$ m/s, as we were told in the statement of the problem.

(b) This part is free fall with initial velocity $v_{i2} = 20$ m/s. To find how high the rocket climbs, use Eq. (2.15) in the form $v_{top} - v_{i2} = -g(t_{top} - t_{i2})$, with $v_{top} = 0$ (as the rocket climbs, its velocity decreases, and it stops climbing when its velocity is zero). This gives us $t_{top} = 3.04\,\text{s}$ as the time at which the rocket reaches the top of its trajectory, and then starts coming down. The corresponding displacement is, by Eq. (2.16),

$$\Delta y_{top} = v_{i2}(t_{top} - t_{i2}) - \frac{1}{2}g(t_{top} - t_{i2})^2 = 20.4\,\mathrm{m}$$

so the maximum height it reaches is 30.4 m.

At the end of the full 3-second interval, the rocket's displacement is

$$\Delta y_2 = v_{i2}\Delta t_2 - \frac{1}{2}g(\Delta t_2)^2 = 15.9\,\mathrm{m}$$

(so its height is 25.9 m above the ground), and the final velocity is

$$v_{f2} = v_{i2} - g\Delta t_2 = -9.43\,\frac{\mathrm{m}}{\mathrm{s}}.$$

(c) The acceleration for this part is $(v_{f3} - v_{i3})/\Delta t_3 = (-5 + 9.43)/1 = 4.43$ m/s$^2$. Note the positive sign. The displacement is

$$\Delta y_3 = -9.43 \times 1 + \frac{1}{2} \times 4.43 \times 1^2 = -7.22\,\mathrm{m}$$

so the final height is 25.9 − 7.21 = 18.7 m.

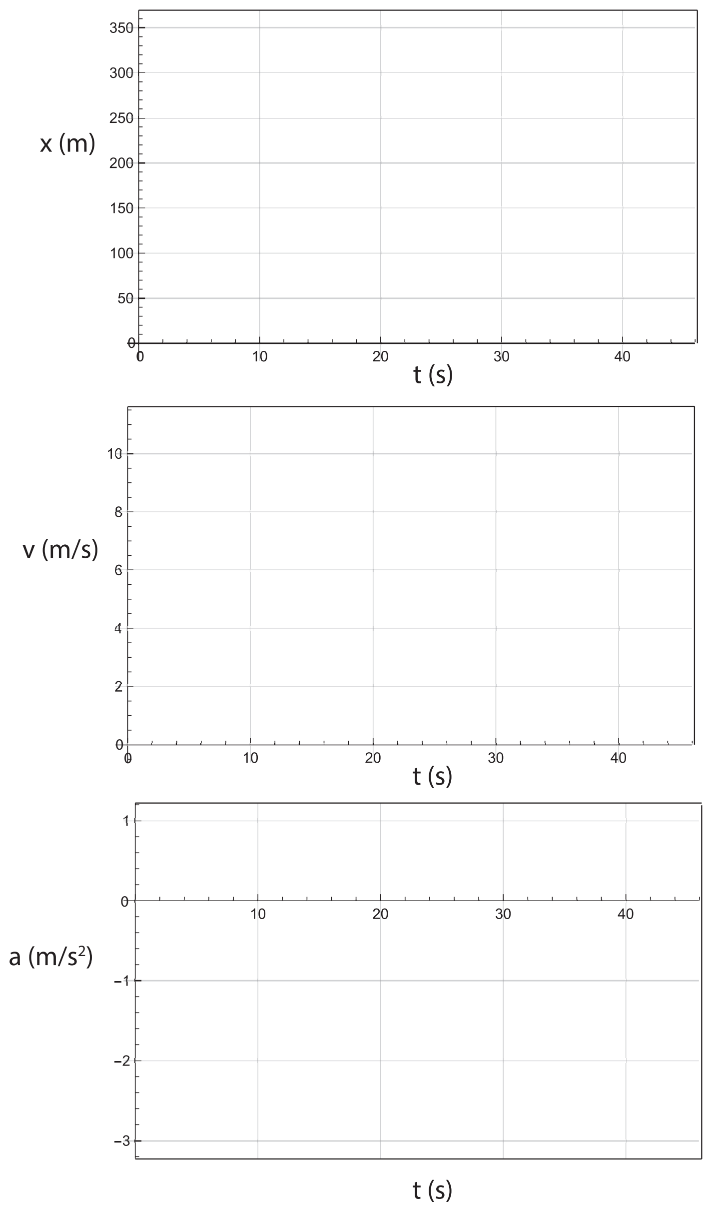
(d) This is just motion with constant speed to cover 18.7 m at 5 m/s. The time it takes is 3.74 s.

The graphs for this motion are shown earlier in the chapter, in Figure 2.3.

## 2.6 Problems

### Problem 1

You get on your bicycle and ride it with a constant acceleration of $0.5\,\mathrm{m/s^2}$ for $20\,\mathrm{s}$. After that, you continue riding at a constant velocity for a distance of $200\,\mathrm{m}$. Finally, you slow to a stop, with a constant acceleration, over a distance of $20\,\mathrm{m}$.



(a) How far did you travel while you were accelerating at $0.5\,\mathrm{m/s^2}$, and what was your velocity at the end of that interval?
(b) After that, how long did it take you to cover the next 200 m?
(c) What was your acceleration while you were slowing down to a stop, and how long did it take you to come to a stop?
(d) Considering the whole trip, what was your average velocity?
(e) Plot the position versus time, velocity versus time, and acceleration versus time graphs for the whole trip, in the grids provided above. Values at the beginning and end of each interval must be exact. Slopes and curvatures must be represented accurately. Do not draw any of the curves beyond the time the rider stops (or, if you do, make sure what you draw makes sense!).

**Problem 2**

You throw an object straight upwards and catch it again, when it comes down to the same initial height, 2 s later.

(a) How high did it rise above its initial height?
(b) With what initial speed did you throw it?
(Note: for this problem you should use the fact that, if air drag is negligible, the object will return to its initial height with the same speed it had initially.)

**Problem 3**

You are trying to catch up with a car that is in front of you on the highway. Initially you are both moving at 25 m/s, and the distance between you is 100 m. You step on the gas and sustain a constant acceleration for a time $\Delta t = 30\,\mathrm{s}$, at which point you have pulled even with the other car.

(a) What is 25 m/s, in miles per hour?
(b) What was your acceleration over the 30 s time interval?
(c) How fast were you going when you caught up with the other car?

**Problem 4**

Go back to Problem 4 of Chapter 1, and use the information in the figure to draw an accurate position vs. time graph for both runners.

**Problem 5**

A child on a sled slides (starting from rest) down an icy slope that makes an angle of 15° with the horizontal. After sliding 20 m down the slope, the child enters a flat, slushy region, where she slides for 2.0 s with a constant negative acceleration of $-1.5\,\mathrm{m/s^2}$ with respect to her direction of motion. She then slides up another icy slope that makes a 20° angle with the horizontal.

(a) How fast was the child going when she reached the bottom of the first slope? How long did it take her to get there?
(b) How long was the flat stretch at the bottom?
(c) How fast was the child going as she started up the second slope?
(d) How far up the second slope did she slide?

# Chapter 3

# Momentum and Inertia

## 3.1 Inertia

In everyday language, we speak of something or someone "having a large inertia" to mean, essentially, that they are very difficult to set in motion. This usage of the word "inertia" is consistent with the "law of inertia" we introduced in the previous chapter (which states, among other things, that an object at rest, if left to itself, will just remain at rest), but it goes a bit beyond that by trying to quantify just how hard it may be to get the object to move.

We do know, from experience, that lighter objects are easier to set in motion than heavier objects, but most of us probably have an intuition that gravity (the force that pulls an object towards the earth and hence determines its weight) is not involved in an essential way here. Imagine, for instance, the difference between slapping a volleyball and a bowling ball. It is not hard to believe that the latter would hurt as much if we did it while floating in free fall in the space station (in a state of effective "weightlessness") as if we did it right here on the surface of the earth. In other words, it is not (necessarily) how heavy something feels, but just how *massive* it is.

But just what is this "massiveness" quality that we associate intuitively with a large inertia? Is there a way (other than resorting to the weight again) to assign to it a numerical value?

### 3.1.1 Relative inertia and collisions

One possible way to determine the *relative* inertias of two objects, conceptually, at least, is to try to use one of them to set the other one in motion. Most of us are familiar with what happens when two identical objects (presumably, therefore, having the same inertia) collide: if the collision is head-on (so the motion, before and after, is confined to a straight line), they basically exchange velocities. For instance, a billiard ball hitting another one will stop dead and the second one will set off with the same speed as the first one. The toy sometimes called "Newton's balls" or "Newton's cradle" also shows this effect. Intuitively, we understand that what it takes to stop the first ball is exactly the same as it would take to set the second one in motion with the same velocity.

But what if the objects colliding have different inertias? We expect that the change in their velocities as a result of the collision will be different: the velocity of the object with the largest inertia will not change very much, and conversely, the change in the velocity of the object with the smallest inertia will be comparatively larger. A velocity vs. time graph for the two objects might look somewhat like the one sketched in Fig. 3.1.

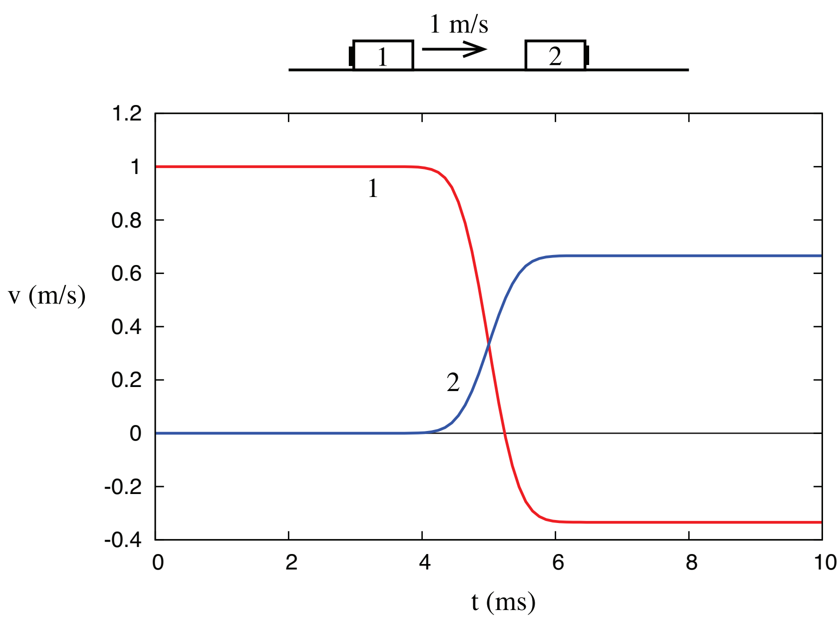



**Figure 3.1:** An example of a velocity vs. time graph for a collision of two objects with different inertias.

In this picture, object 1, initially moving with velocity $v_{1i} = 1\,\text{m/s}$, collides with object 2, initially at rest. After the collision, which here is assumed to take a millisecond or so, object 1 actually bounces back, so its final velocity is $v_{1f} = -1/3\,\text{m/s}$, whereas object 2 ends up moving to the right with velocity $v_{2f} = 2/3\,\text{m/s}$. So the change in the velocity of object 1 is $\Delta v_1 = v_{1f} - v_{1i} = -4/3\,\text{m/s}$, whereas for object 2 we have $\Delta v_2 = v_{2f} - v_{2i} = 2/3\,\text{m/s}$.

It is tempting to use this ratio, $\Delta v_1/\Delta v_2$, as a measure of the *relative inertia* of the two objects, only we'd want to use it upside down and with the opposite sign: that is, since $\Delta v_2/\Delta v_1 = -1/2$ we would say that object 2 has *twice* the inertia of object 1. But then we have to ask: is this a reliable, repeatable measure? Will it work for any kind of collision (within reason, of course: we clearly need to stay in one dimension, and eliminate external influences such as friction), and for any initial velocity?

To begin with, we have reason to expect that it does not matter whether we shoot object 1 towards object 2 or object 2 towards object 1, because we learned in the previous chapter that *only relative motion is detectable*, and the relative motion is the same in both cases. Consider, for instance, what the collision in Figure 3.1 appears like to a hypothetical observer moving along with object 1, at 1 m/s. To him, object 1 appears to be at rest, and it is object 2 that is coming towards him, with a velocity of $-1\,\text{m/s}$. To see what the outcome of the collision looks like to him, just add the same $-1\,\text{m/s}$ to the final velocities we obtained before: object 1 will end up moving at $v_{1f} = -4/3\,\text{m/s}$, and object 2 would move at $v_{2f} = -1/3\,\text{m/s}$, and we would have a situation like the one shown in Figure 3.2, where both curves have simply been shifted down by 1 m/s:

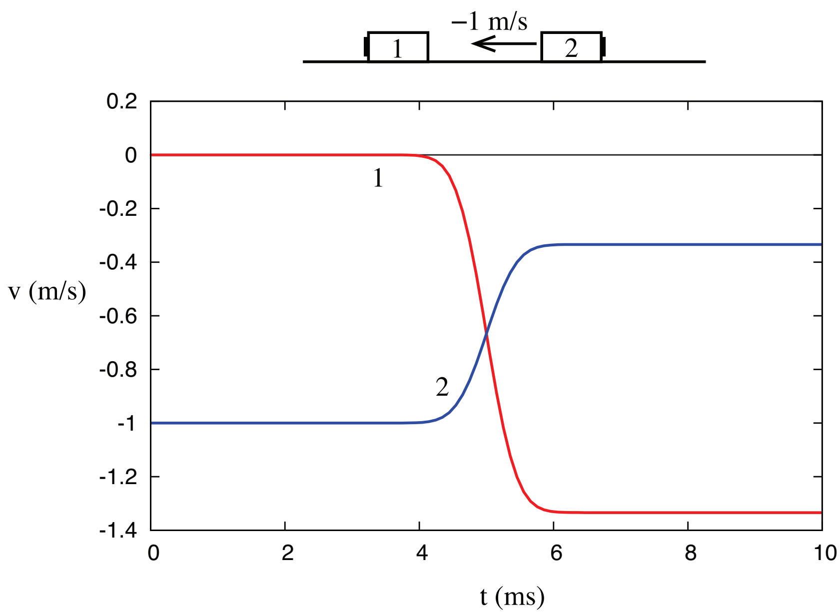



**Figure 3.2:** Another example (really the same collision as in Figure 1, only as seen by an observer initially moving to the right at 1 m/s).

But then, this is exactly what we should expect to find also in our laboratory if we actually did send the second object at 1 m/s towards the first one sitting at rest. All the individual velocities have changed relative to Figure 3.1, but the *velocity changes*, $\Delta v_1$ and $\Delta v_2$, are clearly still the same, and therefore so is our (tentative) measure of the objects' relative inertia.

Clearly, the same argument can be used to conclude that the same result will be obtained when both objects are initially moving towards each other, as long as their *relative velocity* is the same as in these examples, namely, 1 m/s. However, unless we do the experiments we cannot really predict what will happen if we increase (or decrease) their relative velocity. In fact, we could imagine smashing the two objects at very high speed, so they might even become seriously mangled in the process. Yet, experimentally (and this is not at all an obvious result!), we would still find the same value of $-1/2$ for the ratio $\Delta v_2/\Delta v_1$, at least as long as the collision is not so violent that the objects actually break up into pieces.

Perhaps the most surprising result of our experiments would be the following: imagine that the objects have a "sticky" side (for instance, the small black rectangles shown in the pictures could be strips of Velcro), and we turn them around so that when they collide they will end up stuck to each other. In this case (which, as we shall see later, is termed a **completely inelastic** collision), the $v$-vs-$t$ graph might look like Figure 3.3 below.

Now the two objects end up moving together to the right, fairly slowly: $v_{1f} = v_{2f} = 1/3\,\mathrm{m/s}$. The velocity changes are $\Delta v_1 = -2/3\,\mathrm{m/s}$ and $\Delta v_2 = 1/3\,\mathrm{m/s}$, both of which are different from what they were before, in Figs. 3.1 and 3.2: yet, the ratio $\Delta v_2/\Delta v_1$ is still equal to $-1/2$, just as in all the previous cases.

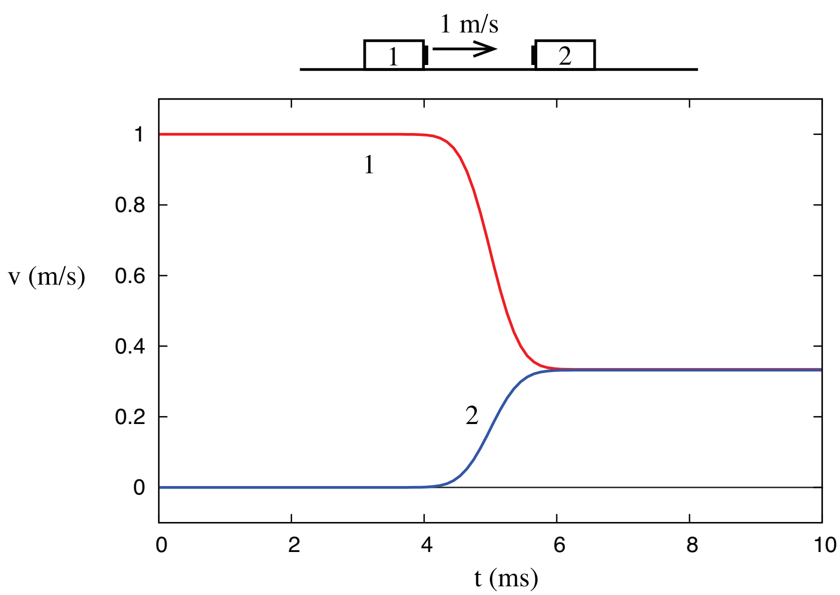



**Figure 3.3:** What would happen if the objects in Figure 1 became stuck together when they collided.

### 3.1.2 Inertial mass: definition and properties

At this point, it would seem reasonable to assume that this ratio, $\Delta v_2/\Delta v_1$, is, in fact, telling us something about an *intrinsic* property of the two objects, what we have called above their "relative inertia." It is easy, then, to see how one could assign a value to the inertia of any object (at least, conceptually): choose a "standard" object, and decide, arbitrarily, that its inertia will have the numerical value of 1, in whichever units you choose for it (these units will turn out, in fact, to be kilograms, as you will see in a minute). Then, to determine the inertia of another object, which we will label with the subscript 1, just arrange a one-dimensional collision between object 1 and the standard, under the right conditions (basically, no net external forces), measure the velocity changes $\Delta v_1$ and $\Delta v_s$, and take the quantity $-\Delta v_s/\Delta v_1$ as the numerical value of the ratio of the inertia of object 1 to the inertia of the standard object. In symbols, using the letter $m$ to represent an object's inertia,

$$\frac{m_1}{m_s} = -\frac{\Delta v_s}{\Delta v_1} \tag{3.1}$$

But, since $m_s = 1$ by definition, this gives us directly the numerical value of $m_1$.

The reason we use the letter $m$ is, as you must have guessed, because, in fact, the inertia defined in this way turns out to be identical to what we have traditionally called "mass." More precisely, the quantity defined this way is an object's *inertial mass*. The remarkable fact, mentioned earlier, that the force of gravity between two objects turns out to be proportional to their inertial masses, allows us to determine the inertial mass of an object by the more traditional procedure of simply weighing it, rather than elaborately staging a collision between it and the standard kilogram on an ice-hockey rink. But, in principle, we could conceive of the existence of two different quantities that should be called "inertial mass" and "gravitational mass," and the identity (or more precisely, the—so far as we know—exact proportionality) of the two is a rather mysterious experimental fact[1].

In any case, by the way we have constructed it, the inertial mass, defined as in Eq. (3.1), does capture, in a quantitative way, the concept that we were trying to express at the beginning of the chapter: namely, how difficult it may be to set an object in motion. In principle, however, other experiments would need to be conducted to make sure that it does have the properties we have traditionally associated with the concept of mass. For instance, suppose we join together two objects of mass $m$. Is the mass of the resulting object $2m$? Collision experiments would, indeed, show this to be the case with great accuracy in the macroscopic world (with which we are concerned this semester), but this is a good example of how you cannot take anything for granted: at the microscopic level, it is again a fact that the inertial mass of an atomic nucleus is a little *less* than the sum of the masses of all its constituent protons and neutrons[2].

Probably the last thing that would need to be checked is that *the ratio of inertias is independent*

[1]This fact, elevated to the category of a principle by Einstein (the *equivalence principle*) is the starting point of the general theory of relativity.

[2]And this is not just an unimportant bit of trivia: all of nuclear power depends on this small difference.

*of the standard*. Suppose that we have two objects, to which we have assigned masses $m_1$ and $m_2$ by arranging for each to collide with the "standard object" independently. If we now arrange for a collision between objects 1 and 2 directly, will we actually find that the ratio of their velocity changes is given by the ratio of the separately determined masses $m_1$ and $m_2$? We certainly would need that to be the case, in order for the concept of inertia to be truly useful; but again, we should not assume anything until we have tested it! Fortunately, the tests would indeed reveal that, in every case, the expected relationship holds[3]

$$-\frac{\Delta v_2}{\Delta v_1} = \frac{m_1}{m_2} \tag{3.2}$$

At this point, we are not just in possession of a useful definition of inertia, but also of a veritable *law of nature*, as I will explain next.

## 3.2 Momentum

For an object of (inertial) mass $m$ moving, in one dimension, with velocity $v$, we define its *momentum* as

$$p = mv \tag{3.3}$$

(the choice of the letter $p$ for momentum is apparently related to the Latin word "im*p*etus").

We can think of momentum as a sort of extension of the concept of inertia, from an object at rest to an object in motion. When we speak of an object's inertia, we typically think about what it may take to get it moving; when we speak of its momentum, we typically think of that it may take to stop it (or perhaps deflect it). So, both the inertial mass $m$ and the velocity $v$ are involved in the definition.

We may also observe that what looks like inertia in some reference frame may look like momentum in another. For instance, if you are driving in a car towing a trailer behind you, the trailer has only a large amount of inertia, but no momentum, relative to you, because its velocity relative to you is zero; however, the trailer definitely has a large amount of momentum (by virtue of both its inertial mass and its velocity) relative to somebody standing by the side of the road.

### 3.2.1 Conservation of momentum; isolated systems

For a system of objects, we treat the momentum as an *additive* quantity. So, if two colliding objects, of masses $m_1$ and $m_2$, have initial velocities $v_{1i}$ and $v_{2i}$, we say that the total initial momentum of

[3]Equation 3.2 actually is found to hold also at the microscopic (or *quantum*) level, although there we prefer to state the result by saying that conservation of momentum holds (see the following section).

the system is $p_i = m_1 v_{1i} + m_2 v_{2i}$, and similarly if the final velocities are $v_{1f}$ and $v_{2f}$, the total final momentum will be $p_f = m_1 v_{1f} + m_2 v_{2f}$.

We then assert that *the total momentum of the system is not changed by the collision.* Mathematically, this means

$$p_i = p_f \tag{3.4}$$

or

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \tag{3.5}$$

But this last equation, in fact, follows directly from Eq. (3.2): to see this, move all the quantities in Eq. (3.5) having to do with object 1 to one side of the equal sign, and those having to do with object 2 to the other side. You then get

$$\begin{aligned} m_1 (v_{1i} - v_{1f}) &= m_2 (v_{2f} - v_{2i}) \\ -m_1 \Delta v_1 &= m_2 \Delta v_2 \end{aligned} \tag{3.6}$$

which is just another way to write Equation (3.2). Hence, the result (3.2) ensures the conservation of the total momentum of a system of any two interacting objects ("particles"), regardless of the form the interaction takes, as long as there are no external forces acting on them.

Momentum conservation is one of the most important principles in all of physics, so let me take a little time to explain how we got here and elaborate on this result. First, as I just mentioned, we have been more or less implicitly assuming that the two interacting objects form an *isolated* system, by which we mean that, throughout, they interact with nothing other than each other. (Equivalently, there are no external forces acting on them.)

It is pretty much impossible to set up a system so that it is *really* isolated in this strict sense; instead, in practice, we settle for making sure that the external forces on the two objects *cancel out*. This is what happens on the air tracks with which you will be doing experiments this semester: gravity is acting on the carts, but that force is balanced out by the upwards push of the air from the track. A system on which there is no *net* external force is as good as isolated for practical purposes, and we will refer to it as such. (It is harder, of course, to completely eliminate friction and drag forces, so we just have to settle for approximately isolated systems in practice.)

Secondly, we have assumed so far that the motion of the two objects is restricted to a straight line—one dimension. In fact, momentum is a *vector* quantity (just like velocity is), so in general we should write

$$\vec{p} = m\vec{v}$$

and conservation of momentum, in general, holds as a vector equation for any isolated system in three dimensions:

$$\vec{p}_i = \vec{p}_f \tag{3.7}$$

What this means, in turn, is that each separate component ($x$, $y$ and $z$) of the momentum will be separately conserved (so Eq. (3.7) is equivalent to three scalar equations, in three dimensions). When we get to study the vector nature of forces, we will see an interesting implication of this, namely, that it is possible for one component of the momentum vector to be conserved, but not another—depending on whether there is or there isn't a net external force in that direction or not. For example, anticipating things a bit, when you throw an object horizontally, as long as you can ignore air drag, there is no horizontal force acting on it, and so that component of the momentum vector is conserved, but the vertical component is changing all the time because of the (vertical) force of gravity.

Thirdly, although this may not be immediately obvious, for an isolated system of two colliding objects the momentum is truly conserved throughout the whole collision process. It is not just a matter of comparing the initial and final velocities: at any of the times shown in Figures 1 through 3, if we were to measure $v_1$ and $v_2$ and compute $m_1v_1 + m_2v_2$, we would obtain the same result. In other words, the total momentum of an isolated system is *constant*: it has the same value at all times.

Finally, all these examples have involved interactions between only two particles. Can we really generalize this to conclude that the total momentum of an isolated system of any number of particles is constant, even when all the particles may be interacting with each other simultaneously? Here, again, the experimental evidence is overwhelmingly in favor of this hypothesis[4], but much of our confidence on its validity comes in fact from a consideration of the nature of the internal interactions themselves. It is a mathematical fact that all of the interactions so far known to physics have the property of conserving momentum, whether acting individually or simultaneously. No experiments have ever suggested the existence of an interaction that does not have this property.

## 3.3 Extended systems and center of mass

Consider a collection of particles with masses $m_1, m_2, \ldots$, and located, at some given instant, at positions $x_1, x_2 \ldots$. (We are still, for the time being, considering only motion in one dimension, but all these results generalize easily to three dimensions.) The **center of mass** of such a system is a mathematical point whose position coordinate is given by

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + \ldots}{m_1 + m_2 + \ldots} \tag{3.8}$$

Clearly, the denominator of (3.8) is just the total mass of the system, which we may just denote by $M$. If all the particles have the same mass, the center of mass will be somehow "in the middle"

[4] For an important piece of indirect evidence, just consider that any extended object is in reality a collection of interacting particles, and the experiments establishing conservation of momentum almost always involve such extended objects. See the following section for further thoughts on this matter.

of all of them; otherwise, it will tend to be closer to the more massive particle(s). The "particles" in question could be spread apart, or they could literally be the "parts" into which we choose to subdivide, for computational purposes, a single extended object.

If the particles are in motion, the position of the center of mass, $x_{cm}$, will in general change with time. For a solid object, where all the parts are moving together, the displacement of the center of mass will just be the same as the displacement of any part of the object. In the most general case, we will have (by subtracting $x_{cmi}$ from $x_{cmf}$)

$$\Delta x_{cm} = \frac{1}{M}\left(m_1 \Delta x_1 + m_2 \Delta x_2 + \ldots\right) \tag{3.9}$$

Dividing Eq. (3.9) by $\Delta t$ and taking the limit as $\Delta t \to 0$, we get the instantaneous velocity of the center of mass:

$$v_{cm} = \frac{1}{M}\left(m_1 v_1 + m_2 v_2 + \ldots\right) \tag{3.10}$$

But this is just

$$v_{cm} = \frac{p_{sys}}{M} \tag{3.11}$$

where $p_{sys} = m_1 v_1 + m_2 v_2 + \ldots$ is the total momentum of the system.

### 3.3.1 Center of mass motion for an isolated system

Equation (3.11) is a very interesting result. Since the total momentum of an isolated system is constant, it tells us that the center of mass of an isolated system of particles moves at constant velocity, regardless of the relative motion of the particles themselves or their possible interactions with each other. As indicated above, this generalizes straightforwardly to more than one dimension, so we can write $\vec{v}_{cm} = \vec{p}_{sys}/M$. Thus, we can say that, for an isolated system in space, not only the speed, but also the direction of motion of its center of mass does not change with time.

Clearly this result is a sort of generalization of the law of inertia. For a solid, extended object, it does, in fact, provide us with the precise form that the law of inertia must take: in the absence of external forces, *the center of mass* will just move on a straight line with constant velocity, whereas the object itself may be moving in any way that does not affect the center of mass trajectory. Specifically, the most general motion of an isolated rigid body is a straight line motion of its center of mass at constant speed, combined with a possible rotation of the object as a whole around the center of mass.

For a system that consists of separate parts, on the other hand, the center of mass is generally just a point in space, which may or may not coincide at any time with the position of any of the parts, but which will nonetheless move at constant velocity as long as the system is isolated. This is illustrated by Figure 3.4, where the position vs. time curves have been drawn for the colliding objects of Figure 3.1. I have assumed that object 1 starts out at $x_{1i} = -5$ mm at $t = 0$, and object 2 starts at $x_{2i} = 0$ at $t = 0$. Because object 2 has twice the inertia of object 1, the position of the center of mass, as given by Eq. (3.8), will always be

$$x_{cm} = x_1/3 + 2x_2/3$$

that is to say, the center of mass will always be in between objects 1 and 2, and its distance from object 2 will always be half its distance to object 1:

$$|x_{cm} - x_1| = \frac{2}{3}|x_1 - x_2|$$

$$|x_{cm} - x_2| = \frac{1}{3}|x_1 - x_2|$$

Figure 4 shows that this simple prescription does result in motion with constant velocity for the center of mass (the green straight line), even though the $x$-vs-$t$ curves of the two objects themselves look relatively complicated. (Please check it out! Take a ruler to Fig. 3.4 and verify that the center of mass is, at every instant, where it is supposed to be.)

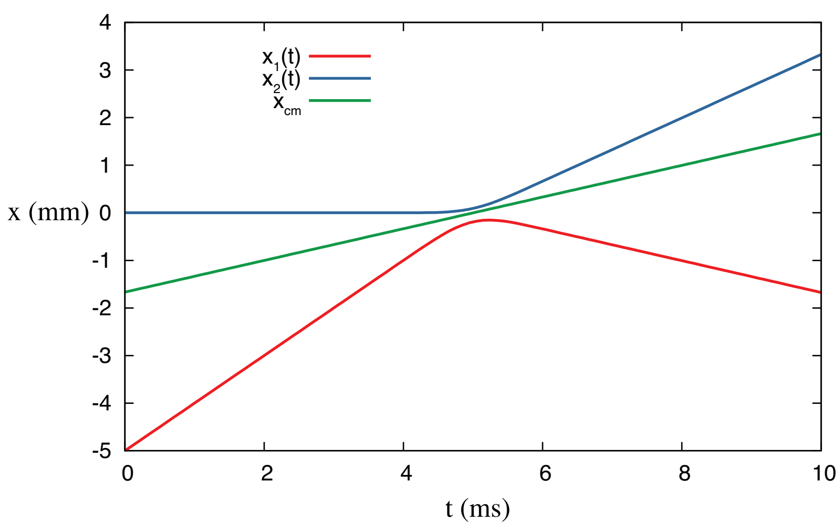



**Figure 3.4:** Position vs. time graph for the objects colliding in Figure 1. The green line shows the position of the center of mass as a function of time.

The concept of center of mass gives us an important way to simplify the description of the motion of potentially complicated systems. We will make good use of it in forthcoming chapters.

A very nice demonstration of the motion of the center of mass in two-body one-dimensional collisions can be found at
`https://phet.colorado.edu/sims/collision-lab/collision-lab_en.html`
(you need to check the “center of mass” box to see it).

### 3.3.2 Recoil and rocket propulsion

As we have just seen, you cannot alter the motion of your center of mass without relying on some external force—which is to say, some kind of external support. This is actually something you may have experienced when you are resting on a very slippery surface and you just cannot "get a grip" on it. There is, however, one way to circumvent this problem which, in fact, relies on conservation of momentum itself: if you are carrying something with you, and can throw it away from you at high speed, you will recoil as a result of that. If you can keep throwing things, you (with your store of as yet unthrown things) will speed up a little more every time. This is, in essence, the principle behind rocket propulsion.

Mathematically, consider two objects, of masses $m_1$ and $m_2$, initially at rest, so their total momentum is zero. If mass 1 is thrown away from mass 2 with a speed $v_{1f}$, then, by conservation of momentum (always assuming the system is isolated) we must have

$$0 = m_1 v_{1f} + m_2 v_{2f} \tag{3.12}$$

and therefore $v_{2f} = -m_1 v_{1f}/m_2$. This is how a rocket moves forward, by constantly expelling mass (the hot exhaust gas) backwards at a high velocity. Note that, even though both objects move, the center of mass of the whole system does *not* (in the absence of any external force), as discussed above.

## 3.4 In summary

1. The *inertia* of an object is a measure of its tendency to resist changes in its motion. It is quantified by the *inertial mass* (measured in kilograms).

2. A system of objects is called *isolated* (for practical purposes) when there are no *net* (or *unbalanced*) external forces acting on any of the objects (the objects may still interact with each other).

3. When two objects forming an isolated system collide in one dimension, the changes in their velocities are inversely proportional to their inertial masses:

$$\frac{\Delta v_1}{\Delta v_2} = -\frac{m_2}{m_1}$$

   This may be used, in principle, as a way to define the inertial mass operationally.

4. The inertial mass thus defined turns out to be exactly (as far as we know) proportional to the object's *gravitational mass*, which determines the gravitational force of attraction between it and any other object. For this reason, most often we measure an object's inertial mass simply by weighing it.

5. The *momentum* of an object of inertial mass $m$ moving with a velocity $\vec{v}$ is defined as $\vec{p} = m\vec{v}$. The total momentum of a system of objects is defined as the (vector) sum of all the individual momenta.

6. (**Conservation of momentum**) *The momentum of an isolated system remains always constant*, regardless of how the parts that make up the system may interact with one another.

7. In one dimension, the *center of mass* of a system of particles is a mathematical point whose $x$ coordinate is given by Equation (3.8) above. (In more dimensions, just change the $x$'s in Eq. (3.8) to $y$ and $z$ to get $y_{cm}$ and $z_{cm}$.)

8. The center of mass of a system always moves with a velocity

$$\vec{v}_{cm} = \frac{\vec{p}_{sys}}{M}$$

   where $\vec{p}_{sys}$ is the total momentum of the system, and $M$ its total mass.

9. It follows from 8 and 6 above that for an isolated system, the center of mass must always be at rest or moving with constant velocity. This result generalizes the law of inertia to extended objects, or systems of particles.

## 3.5 Examples

### 3.5.1 Reading a collision graph

The graph shows a collision between two carts (possibly equipped with magnets so that they repel each other before they actually touch) on an air track. The inertia (mass) of cart 1 is 1 kg. Note: this is a *position vs. time* graph!

(a) What are the initial velocities of the carts?
(b) What are the final velocities of the carts?
(c) What is the mass of the second cart?
(d) Does the air track appear to be level? Why? (Hint: does the graph show any evidence of acceleration, for either cart, outside of the collision region?)
(e) At the collision time, is the acceleration of the first cart positive or negative? How about the second cart? (Justify your answers.)
(f) For the system consisting of the two carts, what is its initial (total) momentum? What is its final momentum?
(g) Imagine now that one of the magnets is reversed, so when the carts collide they stick to each other. What would then be the final momentum of the system? What would be its final velocity?

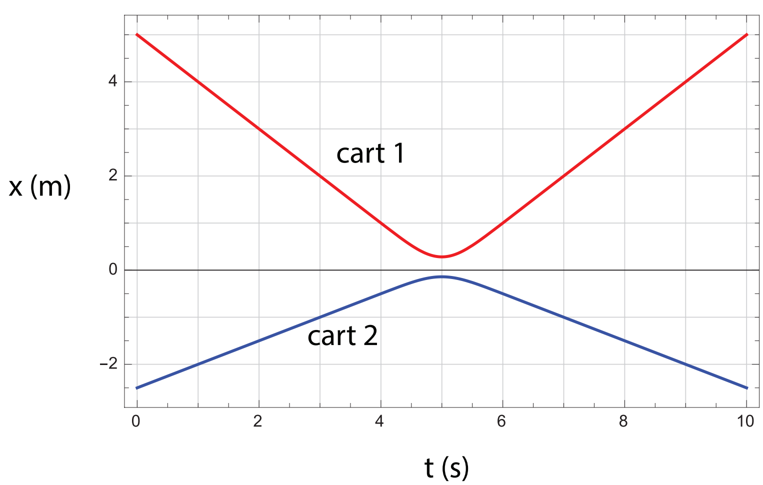



**Figure 3.5:** A collision between two carts.

### Solution

(a) All the velocities are to be calculated by picking an easy straight part of each curve and calculating

$$v = \frac{\Delta x}{\Delta t}$$

for suitable intervals. In this way one gets

$$v_{1i} = -1\,\frac{\mathrm{m}}{\mathrm{s}}$$
$$v_{2i} = 0.5\,\frac{\mathrm{m}}{\mathrm{s}}$$

(b) Similarly, one gets

$$v_{1f} = 1\,\frac{\mathrm{m}}{\mathrm{s}}$$
$$v_{2f} = -0.5\,\frac{\mathrm{m}}{\mathrm{s}}$$

(c) Use this equation, or equivalent (conservation of momentum is OK)

$$\frac{m_2}{m_1} = -\frac{\Delta v_1}{\Delta v_2}$$
$$\frac{m_2}{m_1} = -\frac{1-(-1)}{-0.5-0.5} = 2$$

so the mass of the second cart is 2 kg.

(d) Yes, the track appears to be level because the carts do not show any evidence of acceleration outside of the collision region (the position vs. time curves are straight lines outside of the region approximately given by $4.5\,\mathrm{s} < t < 5.5\,\mathrm{s}$).

(e) The acceleration of the first cart is positive. You can see this either graphically (the curve is like a parabola that opens upwards, i.e., concave), or algebraically (the cart's velocity increases, going from $-1\,\mathrm{m/s}$ to $1\,\mathrm{m/s}$)

Similarly, the acceleration of the second cart is negative. The curve is like a parabola that opens downwards, i.e., convex; or, algebraically, the cart's velocity decreases, going from $0.5\,\mathrm{m/s}$ to $-0.5\,\mathrm{m/s}$.

(f) The initial momentum of the system is

$$p_i = m_1 v_{1i} + m_2 v_{2i} = (1\,\mathrm{kg}) \times \left(-1\,\frac{\mathrm{m}}{\mathrm{s}}\right) + (2\,\mathrm{kg}) \times \left(0.5\,\frac{\mathrm{m}}{\mathrm{s}}\right) = 0$$

The final momentum is

$$p_f = m_1 v_{1f} + m_2 v_{2f} = (1\,\mathrm{kg}) \times \left(1\,\frac{\mathrm{m}}{\mathrm{s}}\right) + (2\,\mathrm{kg}) \times \left(-0.5\,\frac{\mathrm{m}}{\mathrm{s}}\right) = 0$$

You could also just say that the final momentum should be the same as the initial momentum, since the system appears to be isolated.

(g) The momentum should be conserved in this case as well, so $p_f = 0$. The velocity would be

$$v_f = \frac{p_f}{m_1 + m_2} = 0$$

### 3.5.2 Collision in different reference frames, center of mass, and recoil

An 80-kg hockey player (call him player 1), moving at 3 m/s to the right, collides with a 90-kg player (player 2) who was moving at 2 m/s to the left. For a brief moment, they are stuck sliding together as they grab at each other.
(a) What is their joint velocity as they slide together?
(b) What was the velocity of their center of mass before and after the collision?
(c) What does the collision look like to another player that was skating initially at 1.5 m/s to the right? Give all the initial and final velocities as seen by this player, and show explicitly that momentum is also conserved in this player's frame of reference.
(d) Eventually, the 90-kg player manages to push the other one back, in such a way that player 1 (the 80-kg player) ends up moving at 1 m/s to the left *relative to player 2*. What are their final velocities in the earth frame of reference?

**Solution**

(a) Call the initial velocities $v_{1i}$ and $v_{2i}$, the joint final velocity $v_f$, and assume the two players are an isolated system for practical purposes. Then conservation of momentum reads

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f \tag{3.13}$$

Solving for the final velocity, we get

$$v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} \tag{3.14}$$

Substituting the values given, we get

$$v_f = \frac{80 \times 3 - 90 \times 2}{170} = 0.353\,\frac{\mathrm{m}}{\mathrm{s}} \tag{3.15}$$

(b) According to Eq. (3.10), the velocity of the center of mass, $v_{cm}$, is just the same as what we just calculated (Eq. (3.14) above). This makes sense: after the collision, if the players are moving together, their system's center of mass has to be moving with them. Also, if the system is isolated, the center of mass velocity should be the same before and after the collision. So the answer is $v_{cm} = v_f = 0.353\,\mathrm{m/s}$

(c) Let me refer to this third player as "player 3," and introduce a subscript "3" to refer to the quantities as seen in his frame of reference. Let also the subscript "$E$" denote the original, "Earth" reference frame. From Eq. (1.19), we have then (for player 1, for instance)

$$v_{31} = v_{3E} + v_{E1} = v_{E1} - v_{E3} \tag{3.16}$$

because $v_{3E}$, the "velocity of the Earth in player 3's reference frame," is clearly equal to $-v_{E3}$, the negative of the velocity of player 3 relative to the Earth. Basically, what Eq. (3.16) is saying is that to convert all the Earth-frame velocities to the reference frame of player 3, we just need to subtract 1.5 m/s from them. This gives us

$$\begin{aligned} v_{31,i} &= 3\,\frac{\text{m}}{\text{s}} - 1.5\,\frac{\text{m}}{\text{s}} = 1.5\,\frac{\text{m}}{\text{s}} \\ v_{32,i} &= -2\,\frac{\text{m}}{\text{s}} - 1.5\,\frac{\text{m}}{\text{s}} = -3.5\,\frac{\text{m}}{\text{s}} \\ v_{31,f} = v_{32,f} &= 0.353\,\frac{\text{m}}{\text{s}} - 1.5\,\frac{\text{m}}{\text{s}} = -1.147\,\frac{\text{m}}{\text{s}} \end{aligned} \tag{3.17}$$

The total initial momentum in player's 3 reference frame is then

$$p_{sys,i} = m_1 v_{31,i} + m_2 v_{32,i} = 80 \times 1.5 + 90 \times (-3.5) = -195\,\frac{\text{kg}\cdot\text{m}}{\text{s}} \tag{3.18}$$

and the final momentum is

$$p_{sys,f} = (m_1 + m_2) v_{31,f} = 170 \times (-1.147) = -195\,\frac{\text{kg}\cdot\text{m}}{\text{s}} \tag{3.19}$$

So the total momentum is conserved in player 3's reference frame. The reason for this is that this is an inertial reference frame, because the velocity of player 3 does not change.

(d) For this part of the problem, we are back to the original reference frame (the Earth reference frame), and we can drop the "$E$" subscript. For this new process, the final velocities from part (a) become the initial velocities, so we have $v_{1i} = v_{2i} = 0.353$ m/s. [Note: alternatively, since the system is isolated throughout, it would be OK in this case to use the velocities *before* the collision to calculate its total momentum, which also needs to be conserved in this process.] We are also told that the final velocity of player 1 *relative to player 2* is $v_{21,f} = v_{1f} - v_{2f} = -1$ m/s. So we have two equations to solve:

$$\begin{aligned} (m_1 + m_2) \times \left(0.353\,\frac{\text{m}}{\text{s}}\right) &= m_1 v_{1f} + m_2 v_{2f} \qquad \text{(conservation of momentum)} \\ v_{1f} - v_{2f} &= -1\,\frac{\text{m}}{\text{s}} \qquad \text{(final relative velocity)} \end{aligned} \tag{3.20}$$

Leaving aside the units for the moment, to make the equations more readable (the final units will work out, if we make sure to use SI units all along), we have:

$$\begin{aligned} (80 + 90) \times 0.353 &= 80 v_{1f} + 90 v_{2f} \\ v_{1f} &= v_{2f} - 1 \end{aligned} \tag{3.21}$$

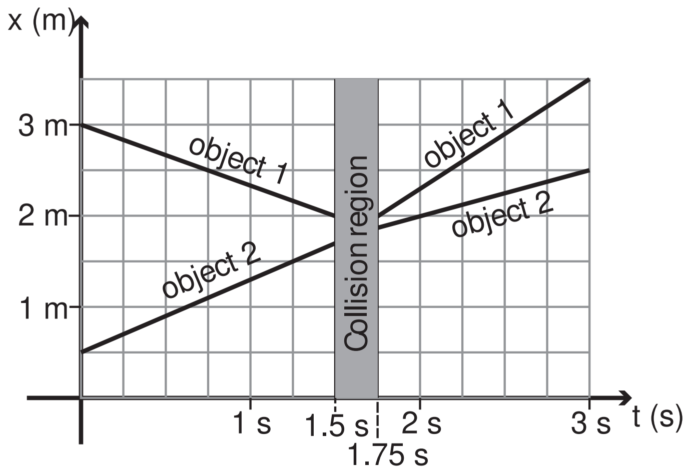
Now substitute the second equation, which I have "solved" already for $v_{1f}$, in the first equation and solve for $v_{2f}$. The result is $v_{2f} = 0.824$ m/s, which, when substituted back in the relative velocity equation, gives $v_{1f} = -0.176$ m/s.

## 3.6 Problems

**Problem 1**

This figure shows the position vs. time graph for two objects before and after they collide. Assume that they form an isolated system.

(a) What are the velocities of the two objects before and after the collision? (Hint: you will get a more accurate result if you choose the initial and final times where the lines go exactly through a point on the grid shown.)
(b) Given the result in (a), what is the ratio of the inertias of the two objects?




**Problem 2**

A car and a truck collide on a very slippery highway. The car, with a mass of 1600 kg, was initially moving at 50 mph. The truck, with a mass of 3000 kg, hit the car from behind at 65 mph. Assume the two vehicles form an isolated system in what follows.

(a) If, immediately after the collision, the vehicles separate and the truck's velocity is found to be 55 mph in the same direction it was going, how fast (in miles per hour) is the car moving?
(b) If instead the vehicles end up stuck together, what will be their common velocity immediately after the collision?

**Problem 3**

A 4-kg gun fires a 0.012-kg bullet at a 3-kg block of wood that is initially at rest. The bullet is embedded in the block, and they move together, immediately after the impact, with a velocity of 3.5 m/s.

(a) What was the velocity of the bullet just before impact?
(b) In order to shoot a bullet at this speed, what must have been the recoil speed of the gun?

**Problem 4**

A 2-kg object, moving at 1 m/s, collides with a 1-kg object that is initially at rest. After the collision, the two objects are found to move away from each other at 1 m/s. Assume they form an isolated system.

(a) What are their actual final velocities in the Earth reference frame?
(b) What is the velocity of the center of mass of this system? Does it change as a result of the collision?

**Problem 5**

Imagine you are stranded on a frozen lake (that means no friction—no traction!), with just a bow and a quiver of arrows. Each arrow has a mass of 0.02 kg, and with your bow can shoot them at a speed of 90 m/s (relative to you—but you might as well assume that this is the arrow's velocity relative to the earth, since, as you will see, your recoil velocity will end up being pretty small anyway). So you decide to use them to propel yourself back to shore.

(a) Suppose your mass (plus the bow and arrows) is 70 kg. When you shoot an arrow, starting from rest, with what speed do you recoil?
(b) Suppose you try to be really clever, and tie a string to the arrow, with the other end of the string tied around your waist. The idea is to get the arrow to pull you forward. Will this work? (Hint: remember part (a). What will happen when the string becomes taut?)

**Problem 6**

An object's position function is given by $x_1(t) = 5 + 10t$ (with $x_1$ in meters if $t$ is in seconds). A second object's position function is $x_2(t) = 5 - 6t$.

(a) If the first object's mass is 1/3 the mass of the second one, what is the position of the system's center of mass as a function of time?
(b) Under the same assumption, what is the velocity of the system's center of mass?

# Chapter 4

# Kinetic Energy

## 4.1 Kinetic Energy

For a long time in the development of classical mechanics, physicists were aware of the existence of two different quantities that one could define for an object of inertia $m$ and velocity $v$. One was the momentum, $mv$, and the other was something proportional to $mv^2$. Despite their obvious similarities, these two quantities exhibited different properties and seemed to be capturing different aspects of motion.

When things got finally sorted out, in the second half of the 19th century, the quantity $\frac{1}{2}mv^2$ came to be recognized as a form of *energy*—itself perhaps the most important concept in all of physics. *Kinetic energy*, as this quantity is called, may be the most obvious and intuitively understandable kind of energy, and so it is a good place to start our study of the subject.

We will use the letter $K$ to denote kinetic energy, and, since it is a form of energy, we will express it in the units especially named for this purpose, which is to say joules (J). 1 joule is 1 kg·m$^2$/s$^2$. In the definition

$$K = \frac{1}{2}mv^2 \tag{4.1}$$

the letter $v$ is meant to represent the *magnitude* of the velocity vector, that is to say, the *speed* of the particle. Hence, unlike momentum, *kinetic energy is not a vector, but a scalar*: there is no sense of direction associated with it. In three dimensions, one could write

$$K = \frac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right) \tag{4.2}$$

There is, therefore, some amount of kinetic energy associated with each component of the velocity vector, but in the end they are all added together in a lump sum.

For a system of particles, we will treat kinetic energy as an additive quantity, just like we did for momentum, so the total kinetic energy of a system will just be the sum of the kinetic energies of all the particles making up the system. Note that, unlike momentum, this is a scalar (not a vector) sum, and most importantly, that kinetic energy is, by definition, always positive, so there can be no question of a “cancellation” of one particle’s kinetic energy by another, again unlike what happened with momentum. Two objects of equal mass moving with equal speeds in opposite directions have a total momentum of zero, but their total kinetic energy is definitely nonzero. Basically, the kinetic energy of a system can never be zero as long as there is any kind of motion going on in the system.

### 4.1.1 Kinetic energy in collisions

To gain some further insights into the concept of kinetic energy, and the ways in which it is different from momentum, it is useful to look at it in the same setting in which we “discovered” momentum, namely, one-dimensional collisions in an isolated system. If we look again at the collision represented in Figure 1 of Chapter 3, reproduced below,

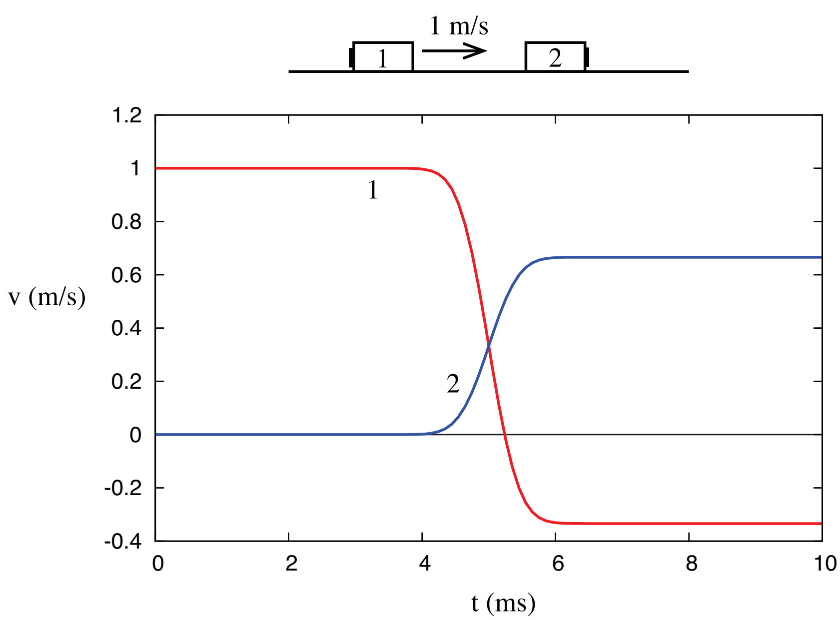



**Figure 4.1:** Elastic collision in an isolated system. (Figure 3.1.)

we can use the definition (4.1) to calculate the initial and final values of $K$ for each object, and for the system as a whole. Remember we found that, for this particular system, $m_2 = 2m_1$, so we can just set $m_1 = 1\,\text{kg}$ and $m_2 = 2\,\text{kg}$, for simplicity. The initial and final velocities are $v_{1i} = 1\,\text{m/s}$,

$v_{2i} = 0$, $v_{1f} = -1/3\,\mathrm{m/s}$, $v_{2f} = 2/3\,\mathrm{m/s}$, and so the kinetic energies are

$$K_{1i} = \frac{1}{2}\,\mathrm{J},\; K_{2i} = 0; \quad K_{1f} = \frac{1}{18}\,\mathrm{J},\; K_{2f} = \frac{4}{9}\,\mathrm{J}$$

Note that $1/18 + 4/9 = 9/18 = 1/2$, and so

$$K_{sys,i} = K_{1i} + K_{2i} = \frac{1}{2}\,\mathrm{J} = K_{1f} + K_{2f} = K_{sys,f}$$

In words, we find that, in this collision, the final value of the total kinetic energy is the same as its initial value, and so it looks like we have "discovered" *another* conserved quantity (besides momentum) for this system.

This belief may be reinforced if we look next at the collision depicted in Figure 2 of Chapter 3, again reproduced below. Recall I pointed out back then that we can think of this as being really the same collision as depicted in Figure 3.1, only looked at from another frame of reference (one moving initially to the right at $1\,\mathrm{m/s}$). We will have more to say about how to transform quantities from a frame of reference to another by the end of the chapter.

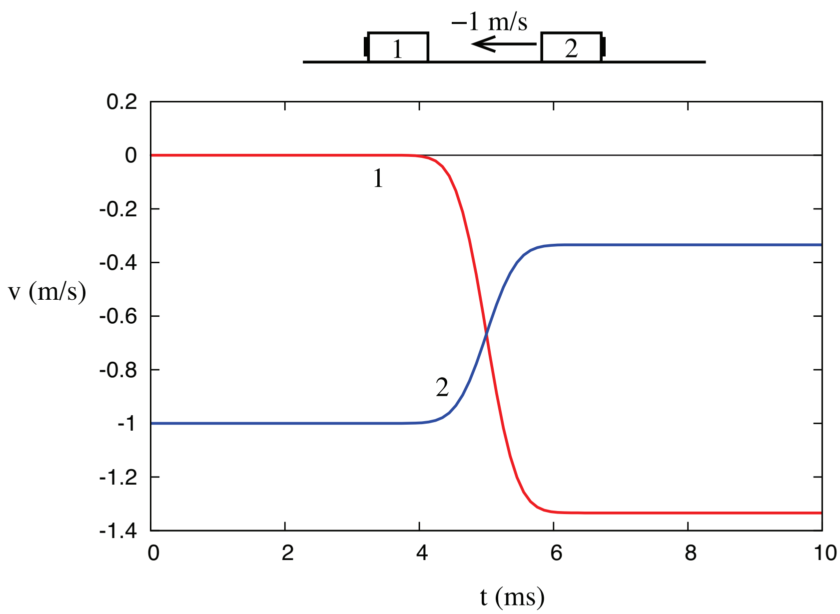



**Figure 4.2:** Another elastic collision, equivalent to the one in Figure 1 as seen from another reference frame. (Figure 3.2.)

In any case, as observed there, all we need to do is add $-1\,\mathrm{m/s}$ to all the velocities in the previous problem, so we have $v_{1i} = 0$, $v_{2i} = -1\,\mathrm{m/s}$, $v_{1f} = -4/3\,\mathrm{m/s}$, $v_{2f} = -1/3\,\mathrm{m/s}$. The corresponding kinetic energies are, accordingly, $K_{1i} = 0$, $K_{2i} = 1\,\mathrm{J}$, $K_{1f} = \frac{8}{9}\mathrm{J}$, $K_{2f} = \frac{1}{9}\mathrm{J}$. These are all different from the values we had in the previous example, but note that once again the total kinetic energy after the collision equals the total kinetic energy before—namely, 1 J in this case[1] .

Things are, however, very different when we consider the third collision example shown in Chapter 3, namely, the one where the two objects are stuck together after the collision.

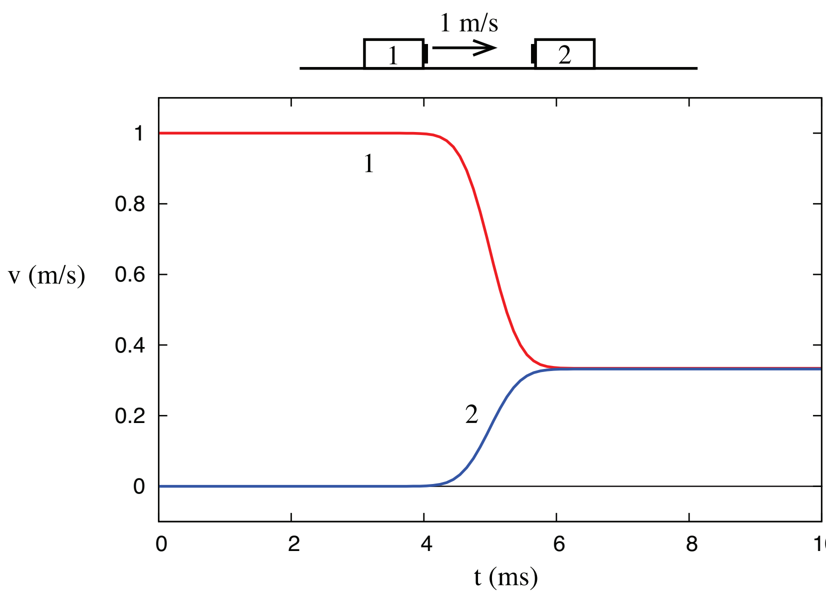



**Figure 4.3:** A totally inelastic collision. (Figure 3.3.)

Their joint final velocity, consistent with conservation of momentum, is $v_{1f} = v_{2f} = 1/3\,\mathrm{m/s}$. Since the system starts as in Figure 4.1, its kinetic energy is initially $K_{sys,i} = \frac{1}{2}$J, but after the collision we have only

$$K_{sys,f} = \frac{1}{2}(3\,\mathrm{kg})\left(\frac{1}{3}\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 = \frac{1}{6}\,\mathrm{J}$$

So kinetic energy is not conserved in this case at all.

What this shows, however, is that unlike the total momentum of a system, which is completely unaffected by internal interactions, the total kinetic energy does depend on the details of the interaction, and thus conveys some information about its nature. We can then refine our study of collisions to distinguish two kinds: the ones where the initial kinetic energy is recovered after the collision, which we will call **elastic**, and the ones where it is not, which we call **inelastic**. A

[1]This is, of course, consistent with the *principle of relativity* I told you about in Chapter 2: if the process in Fig. 4.2 is really the same as the one in Fig. 4.1, only viewed in a different inertial reference frame, then, if energy is seen to be conserved in one frame, it should also be seen to be conserved in the other. More on this below, in Section 4.2.1.

special case of inelastic collision is the one called *totally inelastic*, where the two objects end up stuck together, as in Figure 4.3. As we shall see later, the kinetic energy "deficit" is largest in that case.

I have said above that in an elastic collision the kinetic energy is "recovered," and I prefer this terminology to "conserved," because, in fact, unlike the total momentum, the total kinetic energy of a system does *not* remain constant throughout the interaction, not even during an elastic collision. The simplest example to show this would be an elastic, head-on collision between two objects of equal mass, moving at the same speed towards each other. In the course of the collision, both objects are brought momentarily to a halt before they reverse direction and bounce back, and at that instant, the total kinetic energy is zero.

You can also examine Figures 4.1 and 4.2 above, and calculate, from the graphs, the value of the total kinetic energy during the collision. You will see that it dips to a minimum, and then comes back to its initial value (see also Figure 4.5, later in this chapter). Conventionally, we may talk of kinetic energy as being "conserved" in elastic collisions, but it is important to realize that we are looking at a different kind of "conservation" than what we had with the total momentum, which was constant before, during, and after the interaction, as long as the system remained isolated.

Elastic collisions do suggest that, whatever the ultimate nature of this thing we call "energy" might be, it may be possible to *store* it in some form (in this case, during the course of the collision), and then recover it, as kinetic energy, eventually. This paves the way for the introduction of other kinds of "energy" besides kinetic energy, as we shall see in a later chapter, and the possibility of interconversion to take place among these kinds. For the moment, we shall simply say that in an elastic collision some amount of kinetic energy is temporarily stored as some kind of "internal energy," and after the collision this is converted back into kinetic energy; whereas, in an inelastic collision, some amount of kinetic energy gets irrevocably converted into some "internal energy," and we never get it back.

Since whatever ultimately happens depends on the details and the nature of the interaction, we will be led to distinguish between "conservative" interactions, where kinetic energy is *reversibly* stored as some other form of energy somewhere, and "dissipative" interactions, where the energy conversion is, at least in part, irreversible. Clearly, elastic collisions are associated with conservative interactions and inelastic collisions are associated with dissipative interactions. This preliminary classification of interactions will have to be reviewed a little more carefully, however, in the next chapter.

### 4.1.2 Relative velocity and coefficient of restitution

An interesting property of elastic collisions can be disclosed from a careful study of figures 4.1 and 4.2. In both cases, as you can see, the *relative velocity* of the two objects colliding has the same magnitude (but opposite sign) before and after the collision. In other words: *in an elastic collision, the objects end up moving apart at the same rate as they originally came together.*

Recall that, in Chapter 1, we defined the velocity of object 2 relative to object 1 as the quantity

$$v_{12} = v_2 - v_1 \tag{4.3}$$

(compare Eq. (1.21); and similarly the velocity of object 1 relative to object 2 is $v_{21} = v_1 - v_2$. With this definition you can check that, indeed, the collisions shown in Figs. 4.1 and 4.2 satisfy the equality

$$v_{12,i} = -v_{12,f} \tag{4.4}$$

(note that we could equally well have used $v_{21}$ instead of $v_{12}$). For instance, in Fig. 4.1, $v_{12,i} = v_{2i} - v_{1i} = -1\,\mathrm{m/s}$, whereas $v_{12,f} = 2/3 - (-1/3) = 1\,\mathrm{m/s}$. So the objects are initially moving towards each other at a rate of 1 m per second, and they end up moving apart just as fast, at 1 m per second. Visually, you should notice that the distance between the red and blue curves is the same before and after (but not during) the collision; the fact that they cross accounts for the difference in sign of the relative velocity, which in turns means simply that before the collision they were coming together, and afterwards they are moving apart.

It takes only a little algebra to show that Eq. (4.4) follows from the joint conditions of conservation of momentum and conservation of kinetic energy. The first one ($p_i = p_f$) clearly has the form

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \tag{4.5}$$

whereas the second one ($K_i = K_f$) can be written as

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \tag{4.6}$$

We can cancel out all the factors of 1/2 in Eq. (4.6)[2], then rearrange it so that quantities belonging to object 1 are on one side, and quantities belonging to object 2 are on the other. We get

$$\begin{aligned} m_1 \left(v_{1i}^2 - v_{1f}^2\right) &= -m_2 \left(v_{2i}^2 - v_{2f}^2\right) \\ m_1 \left(v_{1i} - v_{1f}\right)\left(v_{1i} + v_{1f}\right) &= -m_2 \left(v_{2i} - v_{2f}\right)\left(v_{2i} + v_{2f}\right) \end{aligned} \tag{4.7}$$

(using the fact that $a^2 - b^2 = (a+b)(a-b)$). Note, however, that Eq. (4.5) can also be rewritten as

$$m_1 \left(v_{1i} - v_{1f}\right) = -m_2 \left(v_{2i} - v_{2f}\right)$$

[2] You may be wondering, just why do we define kinetic energy with a factor 1/2 in front, anyway? There is no good answer at this point. Let's just say it will make the definition of "potential energy" simpler later, particularly as regards its relationship to *force*.

This immediately allows us to cancel out the corresponding factors in Eq (4.7), so we are left with $v_{1i} + v_{1f} = v_{2i} + v_{2f}$, which can be rewritten as

$$v_{1f} - v_{2f} = v_{2i} - v_{1i} \tag{4.8}$$

and this is equivalent to (4.4).

So, in an elastic collision the speed at which the two objects move apart is the same as the speed at which they came together, whereas, in what is clearly the opposite extreme, in a totally inelastic collision the final relative speed is *zero*—the objects do not move apart at all after they collide. This suggests that we can quantify how inelastic a collision is by the ratio of the final to the initial magnitude of the relative velocity. This ratio is denoted by $e$ and is called the *coefficient of restitution*. Formally,

$$e = -\frac{v_{12,f}}{v_{12,i}} = -\frac{v_{2f} - v_{1f}}{v_{2i} - v_{1i}} \tag{4.9}$$

For an elastic collision, $e = 1$, as required by Eq. (4.4). For a totally inelastic collision, like the one depicted in Fig. 3, $e = 0$. For a collision that is inelastic, but not totally inelastic, $e$ will have some value in between these two extremes. This knowledge can be used to "design" inelastic collisions (for homework problems, for instance!): just pick a value for $e$, between 0 and 1, in Eq. (4.9), and combine this equation with the conservation of momentum requirement (4.5). The two equations then allow you to calculate the final velocities for any values of $m_1$, $m_2$, and the initial velocities. Figure 4.4 below, for example, shows what the collision in Figure 4.1 would have been like, if the coefficient of restitution had been 0.6 instead of 1. You can check, by solving (4.5) and (4.9) together, and using the initial velocities, that $v_{1f} = -1/15\,\text{m/s} = -0.0667\,\text{m/s}$, and $v_{2f} = 8/15\,\text{m/s} = 0.533\,\text{m/s}$.

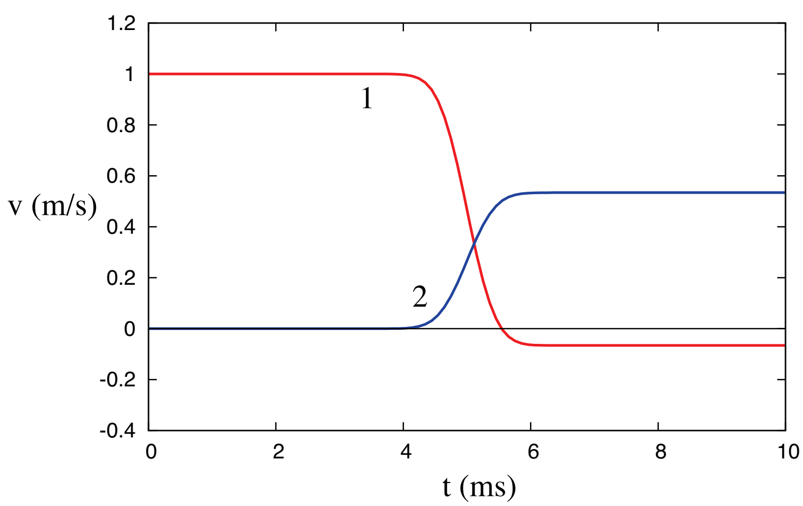



**Figure 4.4:** An $e = 0.6$ collision between objects with the same inertias and initial velocities as in Figure 1.

Although, as I just mentioned, for most "normal" collisions the coefficient of restitution will be a positive number between 1 and 0, there can be exceptions to this. If one of the objects passes through the other (like a bullet through a target, for instance), the value of $e$ will be negative (although still between 0 and 1 in magnitude). And $e$ can be greater than 1 for so-called "explosive collisions," where some amount of extra energy is released, and converted into kinetic energy, as the objects collide. (For instance, two hockey players colliding on the rink and pushing each other away.) In this case, the objects may well fly apart faster than they came together.

An extreme example of a situation with $e > 0$ is an *explosive separation*, which is when the two objects are initially moving together and then fly apart. In that case, the denominator of Eq. (4.9) is zero, and so $e$ is formally infinite. This suggests, what is in fact the case, namely, that although explosive processes are certainly important, describing them through the coefficient of restitution is rare, even when it would be formally possible. In practice, use of the coefficient of restitution is mostly limited to the elastic-to-completely inelastic range, that is, $0 \leq e \leq 1$.

## 4.2 "Convertible" and "translational" kinetic energy

Figure 4.5 shows how the total kinetic energy varies with time, for the two objects shown colliding in Figure 4.1 , depending on the details of the collision, namely, on the value of $e$. The three curves shown cover the elastic case, $e = 1$ (Figure 4.1), the totally inelastic case, $e = 0$ (Figure 4.3), and the inelastic case with $e = 0.6$ of Figure 4.4. Recall that the total momentum is conserved in all three cases.

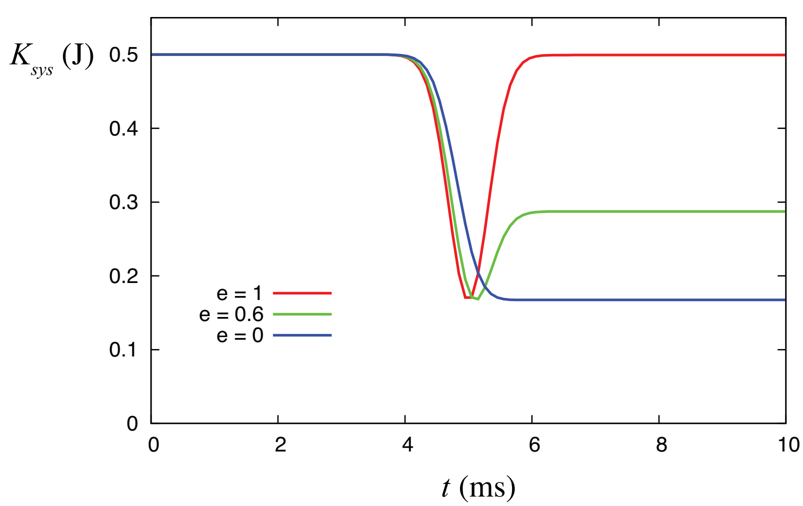



**Figure 4.5:** The total kinetic energy as a function of time for the collisions shown in Figures 1, 3 and 4, respectively.

Figure 4.5 shows that the greatest loss of kinetic energy happens for the totally inelastic collision, which, as we will see in a moment, is, in fact, a general result. That being the case, the figure also shows that it may not be always be possible to bring the total kinetic energy down to zero, even temporarily. The reason for this is that, if momentum is conserved, the velocity of the center of mass cannot change, so if the center of mass was moving before the collision, it must still be moving afterwards; and, as mentioned in this chapter's introduction, as long as there is motion in a system, its total kinetic energy cannot be zero.

All of this suggests that it should be possible to break up a system's total kinetic energy into two parts: one part associated with the motion of the center of mass, which cannot change in any momentum-conserving collision, and one part associated with the relative motion of the parts that make up the system. This second part would vanish irreversibly in a totally inelastic collision, whereas it would recover its original value in an elastic collision.

The way to see this mathematically, for a system of two objects with masses $m_1$ and $m_2$, is to introduce the center of mass velocity $v_{cm}$ [Eq. (3.10)]

$$v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

and the relative velocity $v_{12} = v_2 - v_1$ (Eq. (4.3) above), and observe that the velocities $v_1$ and $v_2$ can be written, respectively, as

$$\begin{aligned} v_1 &= v_{cm} - \frac{m_2}{m_1 + m_2} v_{12} \\ v_2 &= v_{cm} + \frac{m_1}{m_1 + m_2} v_{12} \end{aligned} \tag{4.10}$$

Substituting the equations (4.10) into the expression $K_{sys} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$, one finds that the cross-terms vanish, and all that is left is

$$K_{sys} = \frac{1}{2}(m_1 + m_2) v_{cm}^2 + \frac{1}{2} \frac{m_1 m_2^2 + m_2 m_1^2}{(m_1 + m_2)^2} v_{12}^2$$

A factor of $(m_1 + m_2)$ may be canceled in the last term, and the final expression takes the form

$$K_{sys} = K_{cm} + K_{conv} \tag{4.11}$$

where the center of mass kinetic energy (or *translational energy*) is just what one would have if the whole system was a single particle of mass $M = m_1 + m_2$ moving at the center of mass speed:

$$K_{cm} = \frac{1}{2} M v_{cm}^2 \tag{4.12}$$

and the *"convertible energy"* $K_{conv}$ is the part associated with the relative motion, which can be made to vanish entirely in an inelastic collision[3]:

$$K_{conv} = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2} v_{12}^2 = \frac{1}{2}\mu v_{12}^2 \tag{4.13}$$

The last equation implicitly defines a useful quantity that we call the *reduced mass* of a system of two particles, and denote by $\mu$:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{4.14}$$

Equation (4.11), with the definitions (4.12) and (4.13), pretty much explains everything that we see going on in Figure 4.5. The total kinetic energy is the sum of two terms, the first of which, $K_{cm}$, can never change: it is, in fact, as constant as the total momentum itself, since it involves the center of mass velocity, $v_{cm}$, which is proportional to the total momentum of the system (recall equation (3.11)). The term that can, and does change, is the second one, the convertible energy. In fact, in an ordinary collision in which the objects do not pass through each other, there must be at least an instant in time when $K_{conv} = 0$. This is because it involves the relative velocity, and since the relative velocity must change sign at some point (the objects are initially coming together, but end up moving apart), it must be zero at that time.

This explains why all the curves in Fig. 4.5 have the same minimum value (even though they may reach it at different times): that value is clearly $K_{cm}$ for the system (since $K_{conv}$ is zero at that time). It is the same for all the curves because all the systems considered have the same total mass and momentum (as determined by the initial velocities)—we just chose them that way.

Since $K_{cm}$ cannot change for an isolated system, the maximum kinetic energy that can be lost in a collision in such a system is the initial value of $K_{conv}$, which we would denote as $K_{conv,i}$. This is, in fact, completely lost in a totally inelastic collision, since in that case $v_{12,f} = 0$, and Eq. (4.13) then gives $K_{conv,f} = 0$. In fact, using Eq. (4.9), we can relate the final value of the convertible energy to its initial value via the coefficient of restitution:

$$K_{conv,f} = \frac{1}{2}\mu v_{12,f}^2 = \frac{1}{2}\mu e^2 v_{12,i}^2 = e^2 K_{conv,i} \tag{4.15}$$

Thus, for example, in a collision with $e = 0.6$, the final value of the convertible energy would be only 0.36 times its initial value: 64% of it would have been "lost." (This is not, however, the same as 64% of the *total* initial energy, since the latter still includes $K_{cm}$, which does not change.) We can also write Eq. (4.15) as

$$\Delta K_{sys} = \left(e^2 - 1\right) K_{conv,i} = \left(e^2 - 1\right) \frac{1}{2}\mu v_{12,i}^2 \tag{4.16}$$

since the only possible change in $K_{sys}$ must come from the convertible energy.

[3]Although the name "convertible energy" makes sense in this context, it is not, as far as I can tell, in general usage. I have borrowed it from Mazur's *The Principles and Practice of Physics*, but you should probably not expect to find it in other textbooks.

Although we have derived the decomposition (4.11) for the very restricted situation of two objects moving in one dimension, the basic result is quite general: first, everything in the derivation works if $v_1$ and $v_2$ are replaced by vectors $\vec{v}_1$ and $\vec{v}_2$, so the results holds in three dimensions as well. Second, for a system of any number of particles, one still can write $K_{sys}$ as $K_{cm}+$ another term that depends only on the relative motion of all the pairs of particles. This "generalized convertible energy," or *kinetic energy of relative motion* would have the form

$$K_{rel} = \frac{1}{2}\mu_{12}v_{12}^2 + \frac{1}{2}\mu_{13}v_{13}^2 + \ldots + \frac{1}{2}\mu_{23}v_{23}^2 + \ldots$$

(in this expression, something like $\mu_{23}$ means a reduced mass like the one in Eq. (4.14), only for masses $m_2$ and $m_3$, and so forth).

When we get to the study of rotational motion, for instance, we will see that the total kinetic energy of an extended rigid object can be written as $K_{cm} + K_{rot}$, where $K_{rot}$, the rotational kinetic energy, is just the same kind of thing as what we have called the "convertible energy" here.

All of the above still leaves unanswered the question of what happens to the convertible energy that is lost in an inelastic collision. Just what is it that it gets converted into? The answer to this question will be the subject of the following chapter.

### 4.2.1 Kinetic energy and momentum in different reference frames

I have pointed out repeatedly before that all motion is relative, and so, to some extent, kinetic energy and momentum must be somewhat relative as well. A car in a freight train has a lot of momentum relative to an observer on the ground, but its momentum relative to another car on the same train is zero, since they are not moving relative to each other. The same could be said about its kinetic energy.

In general, if you have a system with a total momentum $\vec{p}_{sys}$ and inertia $M$, its center of mass will have a velocity $\vec{v}_{cm} = \vec{p}_{sys}/M$. Then, if you were to move alongside the system with a velocity exactly equal to $\vec{v}_{cm}$, the total momentum of the system relative to you would be zero. If the system was a solid object, it would not "hit" you if you made contact; there would be no collision. It may help here to think, for instance, of aircraft refueling in flight: if the two planes' velocities are exactly matched, they can make contact without any damage, just as if they were at rest. A reference frame moving at a system's center of mass velocity is, for this reason, called a *zero-momentum frame* for the system in question.

Clearly, in such a reference frame, the translational kinetic energy of the system, $K_{cm} = \frac{1}{2}Mv_{cm}^2$, will also be zero (since, in that frame, the center of mass is not moving at all). However, the relative motion term, $K_{conv}$, would be *completely unaffected* by the change in reference frame. This is because, as you may have noticed by now, to convert velocities from one frame of reference to another we just add or subtract from all the velocities the relative velocity of the two frames. This operation, however, will not change any of the relative velocities of the parts of the system, since these are all differences to begin with. Mathematically,

$$(v_2 + v') - (v_1 + v') = v_2 - v_1$$

regardless of the value of $v'$.

So there something we might call *absolute* (as opposed to "relative") about the convertible kinetic energy: it is the same, it will have the same value, for any observer, regardless of how fast or in what direction that observer may be moving relative to the system as a whole. We may think of it as an *intrinsic* (meaning, observer-independent) property of the system.

## 4.3 In summary

1. The *kinetic energy* of a particle of mass $m$ moving with velocity $v$ is defined as $K = \frac{1}{2}mv^2$. It is a scalar quantity, and it is always positive. For a system of particles or an extended object, we define $K_{sys}$ as the sum of the kinetic energies of all the particles making up the system.

2. For any system, the total kinetic energy can be written as the sum of the *translational* (or *center of mass*) kinetic energy, $K_{cm}$, and another term that involves the motion of the parts of the system relative to each other. (See Eq. (4.11) above.) The translational kinetic energy is constant for an isolated system, and is always given by $K_{cm} = \frac{1}{2}Mv_{cm}^2$.

3. The kinetic energy of relative motion (which, in the context of collisions, is called the *convertible energy*) is given, for the special case of a system consisting of two particles (or two non-rotating extended objects), by $K_{conv} = \frac{1}{2}\mu v_{12}^2$, where $\mu = m_1m_2/(m_1+m_2)$ is the reduced mass, and $v_{12} = v_2 - v_1$ is the relative velocity of the two objects.

4. In a one-dimensional collision between two objects that do not pass through each other, the convertible energy always drops to zero at some point, as a result of the interaction; that is, it is converted entirely into some other form of energy. At the end of the interaction, all the convertible energy may be recovered (elastic collision), or only part of it (inelastic collision), or none of it (completely inelastic collision).

5. In terms of the *coefficient of restitution* $e$, defined as $e = -v_{12,f}/v_{12,i}$, elastic collisions have $e = 1$, totally inelastic collisions have $e = 0$, and inelastic collisions $0 < e < 1$. The total change in kinetic energy in the collision can be written as $\Delta K_{sys} = \Delta K_{conv} = (e^2 - 1)K_{conv,i}$.

6. Another way to say the above is that in an elastic collision in one dimension, the two objects move apart after the collision at the same rate (relative speed) at which they approached each other initially. In a totally inelastic collision, conversely, the two objects do not move apart at all after the collision—they become "stuck together."

7. Besides the cases considered above, one may have collisions where the objects pass through each other, giving $e < 0$, and "explosive collisions," where $e > 1$. In these latter collisions some internal source of energy is converted into additional kinetic energy when the objects interact. The extreme case of this is an *explosive separation*, which is the reverse of a totally inelastic collision—two objects initially moving together fly apart, with a net increase in the system's kinetic energy.

8. The translational kinetic energy of a system will, in general, have different values for observers moving with different velocities. The convertible kinetic energy, on the other hand, is seen by all observers to have the same value, regardless of their relative state of motion.

## 4.4 Examples

### 4.4.1 Collision graph revisited

Look again at the collision graph from example 3.5.1 from the point of view of the kinetic energy of the two carts.
(a) What is the initial kinetic energy of the system?
(b) How much of this is in the center of mass motion, and how much of is convertible?
(c) Does the convertible kinetic energy go to zero at some point during the collision? If so, when? Is it fully recovered after the collision is over?
(d) What kind of collision is this? (Elastic, inelastic, etc.) What is the coefficient of restitution?

#### Solution

(a) From the solution to example 3.5.1 we know that

$$v_{1i} = -1\,\frac{\text{m}}{\text{s}} \qquad v_{2i} = 0.5\,\frac{\text{m}}{\text{s}}$$
$$v_{1f} = 1\,\frac{\text{m}}{\text{s}} \qquad v_{2f} = -0.5\,\frac{\text{m}}{\text{s}}$$

and $m_1 = 1\,\text{kg}$ and $m_2 = 2\,\text{kg}$. So the initial kinetic energy is

$$K_{sys,i} = \frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = 0.5\,\text{J} + 0.25\,\text{J} = 0.75\,\text{J} \tag{4.17}$$

(b) To calculate $K_{cm} = \frac{1}{2}(m_1 + m_2)v_{cm}^2$, we need $v_{cm}$, which in this case is equal to

$$v_{cm} = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} = \frac{-1 + 2 \times 0.5}{3} = 0$$

so $K_{cm} = 0$, which means all the kinetic energy is convertible. We can also calculate that directly:

$$K_{conv,i} = \frac{1}{2}\mu v_{12,i}^2 = \frac{1}{2}\left(\frac{1 \times 2}{1+2}\,\text{kg}\right) \times \left(0.5\,\frac{\text{m}}{\text{s}} - (-1)\,\frac{\text{m}}{\text{s}}\right)^2 = \frac{1.5^2}{3}\,\text{J} = 0.75\,\text{J} \tag{4.18}$$

(c) If we look at figure 3.5, we can see that the carts do not pass through each other, so their relative velocity must be zero at some point, and with that, the convertible energy. In fact, the figure makes it quite clear that *both* $v_1$ and $v_2$ are zero at $t = 5\,\text{s}$, so at that point also $v_{12} = 0$, and the convertible energy $K_{conv} = 0$. (And so is the total $K_{sys} = 0$ at that time, since $K_{cm} = 0$ throughout.)

On the other hand, it is also clear that $K_{conv}$ is fully recovered after the collision is over, since the relative velocity just changes sign:

$$\begin{aligned} v_{12,i} &= v_{2i} - v_{1i} = 0.5\,\frac{\text{m}}{\text{s}} - (-1)\,\frac{\text{m}}{\text{s}} = 1.5\,\frac{\text{m}}{\text{s}} \\ v_{12,f} &= v_{2f} - v_{1f} = -0.5\,\frac{\text{m}}{\text{s}} - 1\,\frac{\text{m}}{\text{s}} = -1.5\,\frac{\text{m}}{\text{s}} \end{aligned} \tag{4.19}$$

Therefore

$$K_{conv,f} = \frac{1}{2}\mu v_{12,f}^2 = \frac{1}{2}\mu v_{12,i}^2 = K_{conv,i}$$

(d) Since the total kinetic energy (which in this case is only convertible energy) is fully recovered when the collision is over, the collision is elastic. Using equation (4.19), we can see that the coefficient of restitution is

$$e = -\frac{v_{12,f}}{v_{12,i}} = -\frac{-1.5}{1.5} = 1$$

as it should be.

### 4.4.2 Inelastic collision and explosive separation

Analyze example 3.5.2 from the point of view of the system's kinetic energy. In particular, answer the following questions:

(a) What is the total kinetic energy of the system (*i*) before the players collide, (*ii*) right after the collision, when they are holding to one another, and (*iii*) after they separate. How much of this energy is translational (that is, center-of-mass kinetic energy), and how much is convertible?
(b) Answer the same questions from the point of view of the player who is skating at a constant 1.5 m/s to the right (player 3)
(To avoid needless repetition, you may use already established results, such as conservation of momentum.)

**Solution** (a) Before the players collide, we have

$$K_{sys,i} = \frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = \frac{1}{2}\,(80\,\text{kg}) \times \left(3\,\frac{\text{m}}{\text{s}}\right)^2 + \frac{1}{2}\,(90\,\text{kg}) \times \left(-2\,\frac{\text{m}}{\text{s}}\right)^2 = 540\,\text{J} \tag{4.20}$$

While they are still holding to each other, we know from the solution to example 3.5.2 that their joint velocity is 0.353 , and that this has to be also the velocity of their center of mass, which is unchanged by the collision. So, we have

$$K_{cm} = \frac{1}{2}(m_1 + m_2)v_{cm}^2 = \frac{1}{2}\,(170\,\text{kg})\left(0.353\,\frac{\text{m}}{\text{s}}\right)^2 = 10.6\,\text{J} \tag{4.21}$$

This is $K_{cm}$ throughout, as well as $K_{sys}$ right after the collision, since the collision is totally inelastic and that means that $K_{conv}$ drops to zero. Also, subtracting this from (4.20) will give us the initial value of the convertible energy, without the need for a separate calculation, so

$$K_{conv,i} = K_{sys,i} - K_{cm} = 540\,\text{J} - 10.6\,\text{J} = 529.4\,\text{J} \simeq 529\,\text{J} \tag{4.22}$$

After the separation, the new total kinetic energy (for which I will use the subscript $f$) is

$$K_{sys,i} = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2 = \frac{1}{2}(80\,\text{kg}) \times \left(-0.176\,\frac{\text{m}}{\text{s}}\right)^2 + \frac{1}{2}(90\,\text{kg}) \times \left(0.824\,\frac{\text{m}}{\text{s}}\right)^2 = 31.8\,\text{J} \tag{4.23}$$

where I have gotten the values for $v_{1f}$ and $v_{2f}$ from the solution to part (d) of Example 3.5.2. Subtracting $K_{cm}$ from this will give us the final value of the convertible energy:

$$K_{conv,f} = K_{sys,f} - K_{cm} = 31.8\,\text{J} - 10.6\,\text{J} = 21.2\,\text{J} \tag{4.24}$$

To summarize, then, we have:

- Before the collision:

$$K_{sys,i} = 540\,\text{J}, \qquad K_{cm} = 10.6\,\text{J}, \qquad K_{conv,i} = 529.4\,\text{J}$$

- Right after the collision (players still holding to each other):

$$K_{sys} = K_{cm} = 10.6\,\text{J}, \qquad K_{conv} = 0$$

- After the (explosive) separation:

$$K_{sys,f} = 31.8\,\text{J}, \qquad K_{cm} = 10.6\,\text{J}, \qquad K_{conv,i} = 21.2\,\text{J}$$

So, in the collision, approximately 529 J of kinetic energy "disappeared" from the system (or, we could say, were "converted into some form of internal energy"), whereas the players' pushing on each other managed to put about 21 J of kinetic energy back into the system; we will explore these kinds of processes in more detail in the following chapter!

(b) We need to repeat all the above calculations with all the velocities shifted down by 1.5 m/s, to bring them to the reference frame of player 3. Instead of putting a subscript "3" on all the quantities, since we already have tons of subscripts to worry about, I'm going to follow an alternative convention and use a "prime" superscript ($'$) to denote all the quantities in this frame of reference. In brief, we have

$$K'_{sys,i} = \frac{1}{2}m_1 (v'_{1i})^2 + \frac{1}{2}m_2 (v'_{2i})^2 = \frac{1}{2}(80\,\text{kg}) \times \left(1.5\,\frac{\text{m}}{\text{s}}\right)^2 + \frac{1}{2}(90\,\text{kg}) \times \left(-3.5\,\frac{\text{m}}{\text{s}}\right)^2 = 641.3\,\text{J} \tag{4.25}$$

$$K'_{cm} = \frac{1}{2}(m_1 + m_2)(v'_{cm})^2 = \frac{1}{2}\,(170\,\text{kg})\left(0.353\,\frac{\text{m}}{\text{s}} - 1.5\,\frac{\text{m}}{\text{s}}\right)^2 = 111.8\,\text{J} \tag{4.26}$$

$$K'_{conv,i} = K'_{sys,i} - K'_{cm} = 641.3\,\text{J} - 111.8\,\text{J} = 529.5\,\text{J} \simeq 529\,\text{J} \tag{4.27}$$

This shows explicitly that the convertible energy, as I pointed out earlier in this chapter, is the same in every reference frame! (The equality is exact, if you keep enough decimals in the calculation.)

Knowing this, we can simplify the calculation of the final kinetic energy, after the explosive separation: the convertible energy, $K'_{conv,f}$, will be the same as in the earth reference frame, that is to say, 21.2 J, and the total kinetic energy will be $K'_{sys,f} = K'_{cm} + K'_{conv,f} = 111.8\,\text{J} + 21.2\,\text{J} = 133\,\text{J}$.

So, in this frame of reference, we have (to three significant figures):

$$\begin{aligned} K'_{sys,i} = 641\,\text{J}, \quad K'_{cm} = 112\,\text{J}, \quad K'_{conv,i} &= 529\,\text{J} \quad \text{(before the collision)} \\ K'_{sys} = K'_{cm} = 112\,\text{J}, \quad K'_{conv} &= 0 \quad \text{(right after the collision)} \\ K'_{sys,f} = 133\,\text{J}, \quad K'_{cm} = 112\,\text{J}, \quad K'_{conv,i} &= 21.2\,\text{J} \quad \text{(after the separation)} \end{aligned}$$

So, even though the total kinetic energy is different in the two reference frames, all the (inertial) observers will agree as to the amount of kinetic energy "lost" in the collision, as well as the amount of kinetic energy put back into the system by the players' pushing on each other.

## 4.5 Problems

**Problem 1**

A 71-kg man can throw a 1-kg ball with a maximum speed of 6 m/s relative to himself. Imagine that one day he decides to try to do that on roller skates. Starting from rest, he throws the ball as hard as he can, so it ends up moving at 6 m/s relative to him, but he himself is recoiling as a result of the throw.

(a) Assuming conservation of momentum, find the velocities of the man and the ball relative to the ground.
(b) What is the kinetic energy of the system right after the throw? (By the system here we mean the man and the ball throughout.) Where did this kinetic energy come from?
(c) Is the man's reference frame inertial throughout this process? Why or why not?
(d) Does the center of mass of the system move at all throughout this process?

**Problem 2**

Analyze Problem 1 from Chapter 3 from the point of view of the system's kinetic energy. In particular, answer the following questions:

(a) What is the total kinetic energy of the system before and after the collision? How much of this energy is translational (that is, center-of-mass kinetic energy), and how much is convertible?
(b) What kind of collision is this? (Elastic, inelastic, etc.) What is the coefficient of restitution?

**Problem 3**

Analyze Problem 2 from Chapter 3 from the point of view of the system's kinetic energy. In particular, answer the following questions:

(a) What is the coefficient of restitution for the collision described in part (a) of the problem, and how much kinetic energy is "lost" in that collision?
(b) What is the coefficient of restitution for the collision described in part (b) of the problem, and how much kinetic energy is "lost" in that collision?

**Problem 4**

A 0.012-kg bullet, traveling at 850 m/s, hits a 2-kg block of wood that is initially at rest, and goes straight through it. Assume that the final velocity of the bullet *relative to the block* is 400 m/s, and that the system is isolated.

(a) What is the coefficient of restitution for this collision?
(b) How much kinetic energy is "lost" in the collision?
(c) What is the final velocity of the block?

**Problem 5**

A 2-kg object, moving at 1 m/s, collides with a 1-kg object that is initially at rest. Assume they form an isolated system.

(a) What is the initial kinetic energy of the system? How much of this is center of mass energy, and how much is convertible?
(b) What is the maximum amount of kinetic energy that could be "lost" (converted to other forms of energy) in this collision?
(c) If 60% of the amount you calculated in part (b) is in fact converted into other forms of energy in the collision, what are the final velocities of the two objects?

# Chapter 5

# Interactions and energy

## 5.1 Conservative interactions

Let me summarize the physical concepts and principles we have encountered so far in our study of classical mechanics. We have "discovered" one important quantity, the inertia or inertial mass of an object, and introduced two different quantities based on that concept, the momentum $m\vec{v}$ and the kinetic energy $\frac{1}{2}mv^2$. We found that these quantities have different but equally intriguing properties. The total momentum of a system is insensitive to the interactions between the parts that make up the system, and therefore it stays constant in the absence of external influences (a more general statement of the law of inertia, the first important principle we encountered). The total kinetic energy, on the other hand, changes while any sort of interaction is taking place, but in some cases it may actually return to its original value afterwards.

In this chapter, we will continue to explore this intriguing behavior of the kinetic energy, and use it to gain some important insights into the kinds of interactions we encounter in classical physics. In the next chapter, on the other hand, we will return to the momentum perspective and use it to formally introduce the concept of force. Hence, we can say that this chapter deals with interactions from an energy point of view, whereas next chapter will deal with them from a force point of view.

In the previous chapter I suggested that what was going on in an elastic collision could be interpreted, or described (perhaps in a figurative way) more or less as follows: as the objects come together, the total kinetic energy goes down, but it is as if it was being temporarily stored away somewhere, and as the objects separate, that "stored energy" is fully recovered as kinetic energy. Whether this does happen or not in any particular collision (that is, whether the collision is elastic or not) depends, as we have seen, on the kind of interaction ("bouncy" or "sticky," for instance) that takes place between the objects.

We are going to take the above description literally, and use the name *conservative interaction* for any interaction that can "store and restore" kinetic energy in this way. The "stored energy" itself—which is *not* actually kinetic energy while it remains stored, since it is not given by the value of $\frac{1}{2}mv^2$ at that time—we are going to call *potential energy*. Thus, conservative interactions will be those that have a "potential energy" associated with them, and vice-versa.

### 5.1.1 Potential energy

Perhaps the simplest and clearest example of the storage and recovery of kinetic energy is what happens when you throw an object straight upwards, as it rises and eventually falls back down. The object leaves your hand with some kinetic energy; as it rises it slows down, so its kinetic energy goes down, down… all the way down to zero, eventually, as it momentarily stops at the top of its rise. Then it comes down, and its kinetic energy starts to increase again, until eventually, as it comes back to your hand, it has very nearly the same kinetic energy it started out with (exactly the same, actually, if you neglect air resistance).

The interaction responsible for this change in the object's kinetic energy is, of course, the gravitational interaction between it and the Earth, so we are going to say that the "missing" kinetic energy is temporarily stored as *gravitational potential energy* of the system formed by the Earth and the object.

We even have a way to describe what is going on mathematically. Recall the equation $v_f^2 - v_i^2 = 2a\Delta x$ for motion under constant acceleration. Let us use $y$ instead of $x$, for the vertical motion; let $a = -g$, and let $v_f$ just be the generic velocity, $v$, at some arbitrary height $y$. We have

$$v^2 - v_i^2 = -2g(y - y_i)$$

Now multiply both sides of this equation by $\frac{1}{2}m$:

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_i^2 = -mg(y - y_i) \tag{5.1}$$

The left-hand side of (5.1) is just the change in kinetic energy (from its initial value when the object was launched). We will interpret the right-hand side as the negative of the change in gravitational potential energy. To make this clearer, rearrange Eq. (5.1) by moving all the "initial" quantities to one side:

$$\frac{1}{2}mv^2 + mgy = \frac{1}{2}mv_i^2 + mgy_i \tag{5.2}$$

We see, then, that the quantity $\frac{1}{2}mv^2 + mgy$ stays *constant* (always equal to its initial value) as the object goes up and down. Let us define the *gravitational potential energy* of a system formed by the Earth and an object a height $y$ above the Earth's surface as the following simple function of $y$:

$$U^G(y) = mgy \tag{5.3}$$

Then we see from Eq. (5.2) that

$$K + U^G = \text{constant} \tag{5.4}$$

This is a statement of conservation of energy under the gravitational interaction. For any interaction that has a potential energy associated with it, the quantity $K + U$ is called the (total) *mechanical energy.*

Figure 5.1 shows how the kinetic and potential energies of an object thrown straight up change with time. To calculate $K$ I have used the equation $v = v_i - gt$ (taking $t_i = 0$); to calculate $U^G = mgy$, I have used $y = y_i + v_i t - \frac{1}{2}gt^2$. I have arbitrarily assumed that the object has a mass of 1 kg and an initial velocity of 2 m/s, and it is thrown from an initial height of 0.5 m above the ground. Note how the change in potential energy exactly mirrors the change in kinetic energy (so $\Delta U^G = -\Delta K$, as indicated by Eq. (5.1)), and the total mechanical energy remains equal to its initial value of 6.9 J throughout.

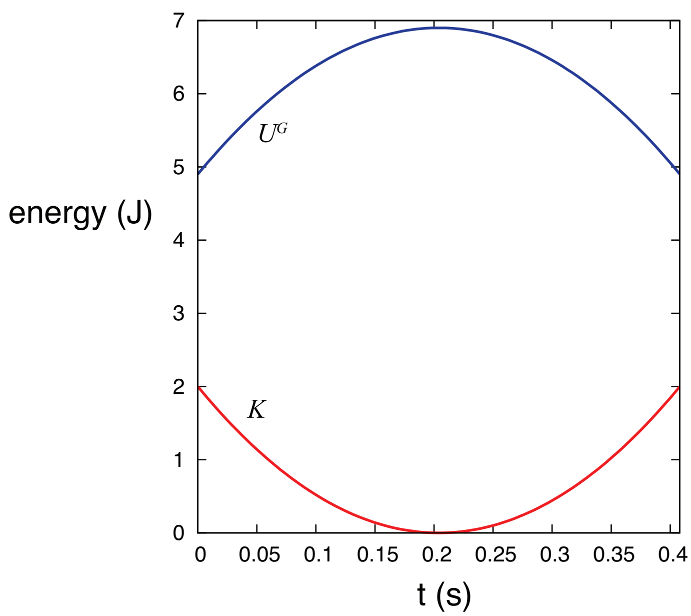



**Figure 5.1:** Potential and kinetic energy as a function of time for a system consisting of the earth and a 1-kg object sent upwards with $v_i = 2\,\text{m/s}$ from a height of 0.5 m.

There is something about potential energy that probably needs to be mentioned at this point. Because I have chosen to launch the object from 0.5 m above the ground, and I have chosen to measure $y$ from the ground, I started out with a potential energy of $mgy_i = 4.9$ J. This makes sense, in a way: it tells you that if you simply dropped the object from this height, it would have picked up an amount of kinetic energy equal to 4.9 J by the time it reached the ground. But, actually, where I choose the vertical origin of coordinates is arbitrary. I could start measuring $y$

from any height I wanted to—for instance, taking the initial height of my hand to correspond to $y = 0$. This would shift the blue curve in Fig. 5.1 down by 4.9 J, but it would not change any of the physics. The only important thing I really want the potential energy for is to calculate the kinetic energy the object will lose or gain *as it moves from one height to another*, and for that only *changes* in potential energy matter. I can always add or subtract any (constant) number[1] to or from $U$, and it will still be true that $\Delta K = -\Delta U$.

What about potential energy in the context in which we first encountered it, that of elastic collisions in one dimension? Imagine that we have two carts collide on an air track, and one of them, let us say cart 2, is fitted with a spring. As the carts come together, they compress the spring, and some of their kinetic energy is "stored" in it as elastic potential energy. In physics, we use the following expression for the potential energy stored in what we call an *ideal spring*[2]:

$$U^{spr}(x) = \frac{1}{2}k(x - x_0)^2 \tag{5.5}$$

where $k$ is something called the spring constant; $x_0$ is the "equilibrium length" of the spring (when it is neither compressed nor stretched); and $x$ its actual length, so $x > x_0$ means the spring is stretched, and $x < x_0$ means it is compressed. For the system of the two carts colliding, we can take the potential energy to be given by Eq. (5.5) if the distance between the carts is less than $x_0$, and 0 (corresponding to a relaxed spring) otherwise. If we put cart 1 on the left and cart 2 on the right, then the distance between them is $x_2 - x_1$, and so we can write, for the whole interaction

$$\begin{aligned} U(x_2 - x_1) =& \frac{1}{2}k(x_2 - x_1 - x_0)^2 && \text{if } x_2 - x_1 < x_0 \\ & 0 && \text{otherwise} \end{aligned} \tag{5.6}$$

This is enough to solve for the motion of the two carts, given the initial conditions. To see how, look in the "Examples" section at the end of this chapter. Here, I will just give you the result.

For the calculation, shown in Fig. 5.2 below, I have chosen cart 1 to have a mass of 1 kg, an initial position (at $t = 0$) of $x_{1i} = -5$ cm and an initial velocity of 1 m/s, whereas cart 2 has a mass of 2 kg and starts at rest at $x_{2i} = 0$. I have assumed the spring has a length of $x_0 = 2$ cm and a spring constant $k = 1000\ \text{J/m}^2$ (which sounds like a lot but isn't really). The collision begins at $t_c = (x_{2i} - x_0 - x_{1i})/v_{1i} = 0.03$ s, which is the time it takes cart 1 to travel the 3 cm separating it from the end of the spring. Prior to that point, the total kinetic energy $K_{sys} = 0.5$ J, and the total potential energy $U = 0$.

[1] Of course, some choices may result in the potential energy, and even the total energy, being *negative* sometimes! If this notion of a negative total energy bothers you a bit, wait until the chapter on gravity (Chapter 10), where we will try to make some sense out of it...

[2] An "ideal spring" is basically defined, mathematically, by this expression, or by the corresponding force equation (6.21) (which we will study in the next chapter, and which goes by the name of *Hooke's law*); usually, we also require that the spring be "massless" (by which we mean that its mass should be negligible compared to all the other masses involved in any given problem). Of course, for Eq. (5.5) to hold for $x < x_0$, it must be possible to compress the spring as well as stretch it, which is not always possible with some springs.

As a result of the collision, the spring compresses and undergoes "half a cycle" of oscillation with an "angular frequency" $\omega = \sqrt{k/\mu}$ (where $\mu$ is, as in previous chapters, the "reduced mass" of the system, $\mu = m_1 m_2/(m_1+m_2)$). That is, the spring is compressed and then pushes out until it gets back to its equilibrium length[3]. This lasts from $t = t_c$ until $t = t_c + \pi/\omega$, during which time the potential and kinetic energies of the system can be written as

$$U(t) = \frac{1}{2}\mu v_{12,i}^2 \sin^2[\omega(t-t_c)]$$
$$K(t) = K_{cm} + \frac{1}{2}\mu v_{12,i}^2 \cos^2[\omega(t-t_c)] \tag{5.7}$$

(don't worry, all this will make a lot more sense after we get to Chapter 11 on simple harmonic motion, I promise!). After $t = t_c + \pi/\omega$, the interaction is over, and $K$ and $U$ go back to their initial values.

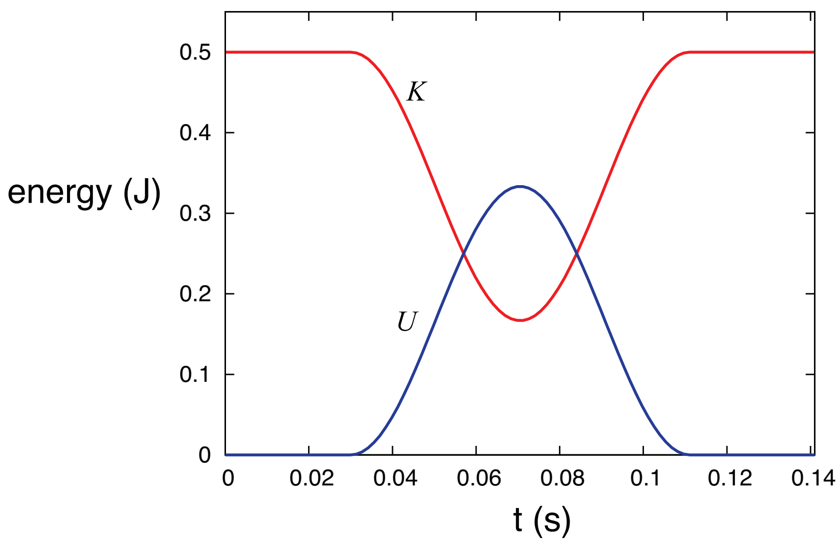



**Figure 5.2:** Potential and kinetic energy as a function of time for a system of two carts colliding and compressing a spring in the process.

If you compare Figure 5.2 with Figure 4.5 of Chapter 4, you'll see that the kinetic energy curve looks very similar, except for the time scale, which here is hundredths of a second and over there was taken to be milliseconds. The quantity that determines the time scale here is the "half period" of oscillation, $\pi/\omega = \pi\sqrt{\mu/k} = 0.081\,\mathrm{s}$ for the values of $k$ and $\mu$ assumed here. We could make this smaller by making the spring stiffer (increasing $k$), or the blocks lighter (reducing $\mu$), but there's not much point in trying, since the collisions in Chapters 3 and 4 were all just made up in any case.

[3]As noted earlier, we shall always assume our springs to be "massless," that is, that their inertia is negligible. In turn, negligible inertia means that the spring does not "keep going": it stops stretching as soon as it is back to its original length.

The main point is that this kind of physical setup (a cart fitted with a spring) would indeed give us an elastic collision, and a kinetic energy curve very much like the ones I used, for illustration purposes, in Chapter 4; only now we also have a potential energy curve to go with it, and to show where the energy is "hiding" while the collision lasts.

(You might wonder, anyway, what kind of potential energy function would actually produce the made-up elastic collision curves in Chapters 3 and 4? The (perhaps surprising) answer is, I do not really know, and I have no way to find out! If you are curious about why, again look at the "Examples" section at the end of the chapter.)

### 5.1.2 Potential energy functions and "energy landscapes"

The potential energy function of a system, as illustrated in the above examples, serves to let us know how much energy can be stored in, or extracted from, the system by changing its *configuration*, that is to say, the positions of its parts relative to each other. We have seen this in the case of the gravitational force (the "configuration" in this case being the distance between the object and the earth), and just now in the case of a spring (how stretched or compressed it is). In all these cases we should think of the potential energy as being a property of the system as a whole, not any individual part; it is, very loosely speaking, something akin to a "stress" in the system that can be turned into motion under the right conditions.

It is a consequence of the principle of conservation of momentum that, if the interaction between two particles can be described by a potential energy function, this should be a function only of their relative position, that is, the quantity $x_1 - x_2$ (or $x_2 - x_1$), and not of the individual coordinates, $x_1$ and $x_2$, separately[4]. The example of the spring in the previous section illustrates this, whereas the gravitational potential energy example shows how this can be simplified in an important case: in Eq. (5.3), the height $y$ of the object above the ground is really a measure of the distance between the object and the earth, something that we could write, in full generality, as $|\vec{r}_o - \vec{r}_E|$ (where $\vec{r}_o$ and $\vec{r}_E$ are the position vectors of the Earth and the object, respectively). However, since we do not expect the Earth to move very much as a result of the interaction, we can take its position to be constant, and only include the position of the object explicitly in our potential energy function, as we did above[5].

Generally speaking, then, we can identify a large class of problems where a "small" object or "particle" interacts with a much more massive one, and it is a good approximation to write the potential energy of the whole system as a function of only the position of the particle. In one

[4]We will see why in the next chapter! But, if you want to peek ahead, nothing's preventing you from reading sections 6.1 and 6.2 right now. Basically, to conserve momentum we need Eq. (6.6) to hold, and as you can see from Eq. (6.18), having the potential energy depend only on $x_1 - x_2$ ensures that.

[5]This will change in Chapter 10, when we get to study gravity over a planetary scale.

dimension, then, we have a situation where, once the initial conditions (the particle's initial position and velocity) are known, the motion of the particle can be completely determined from the function $U(x)$, where $x$ is the particle's position at any given time. This can be done, using calculus, essentially by the method illustrated in Example 5.6.3 at the end of this chapter (namely, let $v = \pm\sqrt{2m(E - U(x))}$ and solve the resulting differential equation); but it is also possible to get some pretty valuable insights into the particle's motion without using any calculus at all, through a mostly *graphical* approach that I would like to show you next.

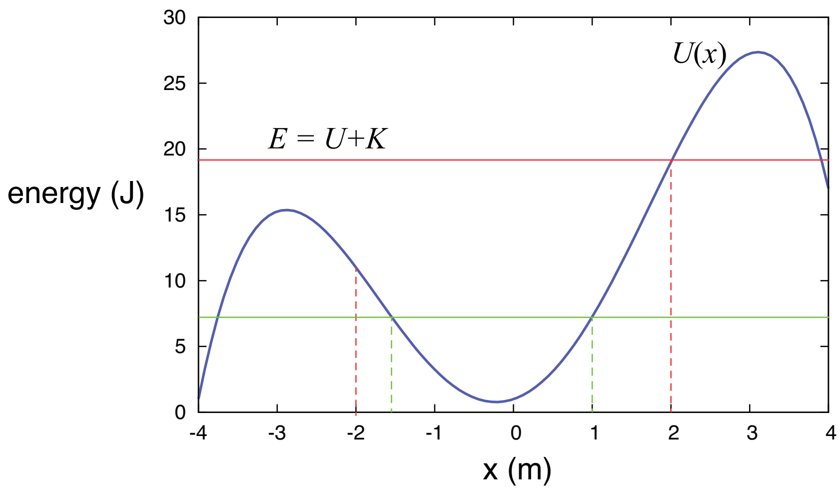



**Figure 5.3:** A hypothetical potential energy curve for a particle in one dimension. The horizontal red line shows the total mechanical energy under the assumption that the particle starts out at $x = -2$ m with $K_i = 8$ J. The green line assumes the particle starts instead from rest at $x = 1$ m.

In Figure 5.3 above I have assumed, as an example, that the potential energy of the system, as a function of the position of the particle, is given by the function $U(x) = -x^4/4 + 9x^2/2 + 2x + 1$ (in joules, if $x$ is given in meters). Consider then what happens if the particle has a mass $m = 4$ kg and is found initially at $x_i = -2$ m, with a velocity $v_i = 2$ m/s. (This scenario goes with the red lines in Fig. 5.3, so please ignore the green lines for the time being.) Its kinetic energy will then be $K_i = 8$ J, whereas the potential energy will be $U(-2) = 11$ J. The total mechanical energy is then $E = 19$ J, as indicated by the red horizontal line.

Now, as the particle moves, the total energy remains constant, so as it moves to the right, its potential energy goes down at first, and consequently its kinetic energy goes up—that is, it accelerates. At some point, however (around $x = -0.22$ m) the potential energy starts to go up, and so the particle starts to slow down, although it keeps going, because $K = E - U$ is still nonzero. However, when the particle eventually reaches the point $x = 2$ m, the potential energy $U(2) = 19$ J, and the kinetic energy becomes zero.

At that point, the particle stops and turns around, just like an object thrown vertically upwards. As it moves "down the potential energy hill," it recovers the kinetic energy it used to have, so that when it again reaches the starting point $x = -2\,\text{m}$, its speed is again $2\,\text{m/s}$, but now it is moving in the opposite direction, so it just passes through and over the next "hill" (since it has enough total energy to do so), and eventually moves outside the region shown in the figure.

As another example, consider what would have happened if the particle had been released at, say, $x = 1\,\text{m}$, but with zero velocity. (This is illustrated by the green lines in Fig. 5.3.) Then the total energy would be just the potential energy $U(1) = 7.25\,\text{J}$. The particle could not possibly move to the right, since that would require the total energy to go up. It can only move to the left, since in that direction $U(x)$ decreases (initially, at first), and that means $K$ can increase (recall $K$ is always positive as long as the particle is in motion). So the particle speeds up to the left until, past the point $x = -0.22\,\text{m}$, $U(x)$ starts to increase again and $K$ has to go down. Eventually, as the figure shows, we reach a point (which we can calculate to be $x = -1.548\,\text{m}$) where $U(x)$ is once again equal to $7.25\,\text{J}$. This leaves no room for any kinetic energy, so the particle has to stop and turn back. The resulting motion consists of the particle oscillating back and forth forever between $x = -1.548\,\text{m}$ and $x = 1\,\text{m}$.

At this point, you may have noticed that the motion I have described as following from the $U(x)$ function in Figure 5.3 resembles very much the motion of a car on a roller-coaster having the shape shown, or maybe a ball rolling up and down hills like the ones shown in the picture. In fact, the correspondence can be made *exact*—if we substitute sliding for rolling, since rolling motion has complications of its own. Given an arbitrary potential energy function $U(x)$ for a particle of mass $m$, imagine that you build a "landscape" of hills and valleys whose height $y$ above the horizontal, for a given value of the horizontal coordinate $x$, is given by the function $y(x) = U(x)/mg$. (Note that $mg$ is just a constant scaling factor that does not change the shape of the curve.) Then, for an object of mass $m$ sliding without friction over that landscape, under the influence of gravity, the gravitational potential energy at any point $x$ would be $U^G(x) = mgy = U(x)$, and therefore its speed at any point will be precisely the same as that of the original particle, if it starts at the same point with the same velocity.

This notion of an "energy landscape" can be extended to more than one dimension (although they are hard to visualize in three!), or generalized to deal with configuration parameters other than a single particle's position. It can be very useful in a number of disciplines (not just physics), to predict the ways in which the configuration of a system may be likely to change.

## 5.2 Dissipation of energy and thermal energy

From all the foregoing, it is clear that when an interaction can be completely described by a potential energy function we can define a quantity, which we have called the total mechanical energy of the system, $E_{mech} = K + U$, that is constant throughout the interaction. However, we already know from our study of inelastic collisions that this is rarely the case. Essential to the concept of potential energy is the idea of "storage and retrieval" of the kinetic energy of the system during the interaction process. When kinetic energy simply disappears from the system and does not come back, a full description of the process in terms of a potential energy is not possible.

Processes in which some amount of mechanical energy disappears (that is, it cannot be found anywhere anymore as either macroscopic kinetic or potential energy) are called *dissipative*. Mysterious as they may appear at first sight, there is actually a simple, intuitive explanation for them. All macroscopic systems consist of a great number of small parts that enjoy, at the microscopic level, some degree of independence from each other and from the body to which they belong. Macroscopic motion of an object requires all these parts to move together as a whole, at least on average; however, a collision with another object may very well "rattle" all these parts and leave them in a more or less disorganized state. If the total energy is conserved, then after the collision the object's atoms or molecules may be, on average, vibrating faster or banging against each other more often than before, but they will do so in random directions, so this increased "agitation" will not be perceived as macroscopic motion of the object as a whole.

This kind of random agitation at the microscopic level that I have just introduced is what we know today as *thermal energy*, and it is by far the most common "sink" or reservoir where macroscopic mechanical energy is "dissipated." In our example of an inelastic collision, the energy the objects had is not gone from the universe, in fact it is still right there inside the objects themselves; it is just in a disorganized or incoherent state from which, as you can imagine, it would be pretty much impossible to retrieve it, since we would have to somehow get all the randomly-moving parts to get back to moving in the same direction again.

We will have a lot more to say about thermal energy in a later chapter, but for the moment you may want to think of it as essentially *noise*: it is what is left (the residual motional or configurational energy, at the microscopic level) after you remove the average, macroscopically-observable kinetic or potential energy. So, for example, for a solid object moving with a velocity $v_{cm}$, the kinetic part of its thermal energy would be the sum of the kinetic energies of all its microscopic parts, calculated *in its center of mass* (or zero-momentum) *reference frame*; that way you remove from every molecule's velocity the quantity $v_{cm}$, which they all must have in common—on average (since the body as a whole is moving with that velocity)[6].

In order to establish conservation of energy as a fact (which was one of the greatest scientific triumphs of the 19th century) it was clearly necessary to show experimentally that a certain amount

[6]Note that thermal energy is not necessarily just kinetic; it may have a configurational component to it as well. For example, imagine a collection of vibrating diatomic molecules. You may think of each one as two atoms connected by a spring. The length of the "spring" at rest determines the molecule's nominal *chemical energy*; thermal vibrations cause this length to change, resulting in a net increase in energy that—as for two masses connected by a spring—has both a kinetic and a configurational (or "potential") component.

of mechanical energy lost always resulted in the same predictable increase in the system's thermal energy. Thermal energy is largely "invisible" at the macroscopic level, but we detect it indirectly through an object's *temperature*. The crucial experiments to establish what at the time was called the "mechanical equivalent of heat" were carried out by James Prescott Joule in the 1850's, and required exceedingly precise measurements of temperature (in fact, getting the experiments done was only half the struggle; the other half was getting the scientific establishment to believe that he could measure changes in temperature so accurately!)

## 5.3 Fundamental interactions, and other forms of energy

At the most fundamental (microscopic) level, physicists today believe that there are only four (or three, depending on your perspective) basic interactions: gravity, electromagnetism, the strong nuclear interaction (responsible for holding atomic nuclei together), and the weak nuclear interaction (responsible for certain nuclear processes, such as the transmutation of a proton into a neutron[7] and vice-versa). In a technical sense, at the quantum level, electromagnetism and the weak nuclear interactions can be regarded as separate manifestations of a single, consistent quantum field theory, so they are sometimes referred to as "the electroweak interaction."

All of these interactions are conservative, in the sense that for all of them one can define the equivalent of a "potential energy function" (generalized, as necessary, to conform to the requirements of quantum mechanics and relativity), so that for a system of elementary particles interacting via any one of these interactions the total kinetic plus potential energy is a constant of the motion. For gravity (which we do not really know how to "quantize" anyway!), this function immediately carries over to the macroscopic domain without any changes, as we shall see in a later chapter, and the gravitational potential energy function I introduced earlier in this chapter is an approximation to it valid near the surface of the earth (gravity is such a weak force that the gravitational interaction between any two earth-bound objects is virtually negligible, so we only have to worry about gravitational energy when one of the objects involved is the earth itself).

As for the strong and weak nuclear interactions, they are only appreciable over the scale of an atomic nucleus, so there is no question of them directly affecting any macroscopic mechanical processes. They are responsible, however, for various nuclear reactions in the course of which *nuclear energy* is, most commonly, transformed into electromagnetic energy (X- or gamma rays) and thermal energy.

All the other forms of energy one encounters at the microscopic, and even the macroscopic, level have their origin in electromagnetism. Some of them, like the electrostatic energy in a capacitor or the magnetic interaction between two permanent magnets, are straightforward enough scale-ups of their microscopic counterparts, and may allow for a potential energy description at the macroscopic

[7] Plus a positron and a neutrino.

level (and you will learn more about them next semester!). Many others, however, are more subtle and involve quantum mechanical effects (such as the exclusion principle) in a fundamental way.

Among the most important of these is *chemical energy*, which is an extremely important source of energy for all kinds of macroscopic processes: combustion (and explosions!), the production of electrical energy in batteries, and all the biochemical processes that power our own bodies. However, the conversion of chemical energy into macroscopic mechanical energy is almost always a dissipative process (that is, one in which some of the initial chemical energy ends up irreversibly converted into thermal energy), so it is generally impossible to describe them using a (macroscopic) potential energy function (except, possibly, for electrochemical processes, with which we will not be concerned here).

For instance, consider a chemical reaction in which some amount of chemical energy is converted into kinetic energy of the molecules forming the reaction products. Even when care is taken to "channel" the motion of the reaction products in a particular direction (for example, to push a cylinder in a combustion engine), a lot of the individual molecules will end up flying in the "wrong" direction, striking the sides of the container, etc. In other words, we end up with a lot of the chemical energy being converted into *disorganized microscopic agitation*—which is to say, *thermal energy*.

Electrostatic and quantum effects are also responsible for the elastic properties of materials, which *can* sometimes be described by macroscopic potential energy functions, at least to a first approximation (like the spring we studied earlier in the chapter). They are also responsible for the adhesive forces between surfaces that play an important role in friction, and various other kinds of what might be called "structural energies," most of which play only a relatively small part in the energy balance where macroscopic objects are involved.

## 5.4 Conservation of energy

Today, physics is pretty much founded on the belief that the energy of a closed system (defined as one that does not exchange energy with its surroundings—more on this in a minute) is always *conserved*: that is, internal processes and interactions will only cause energy to be "converted" from one form into another, but the total, after all the forms of energy available to the system have been carefully accounted for, will not change. This belief is based on countless experiments, on the one hand, and, on the other, on the fact that all the fundamental interactions that we are aware of do conserve a system's total energy.

Of course, recognizing whether a system is "closed" or not depends on having first a complete catalogue of all the ways in which energy can be stored and exchanged—to make sure that there is, in fact, no exchange of energy going on with the surroundings. Note, incidentally, that a "closed" system is not necessarily the same thing as an "isolated" system: the former relates to the total energy, the latter to the total momentum. A parked car getting hotter in the sun is not a closed system (it is absorbing energy all the time) but, as far as its total momentum is concerned, it is certainly fair to call it "isolated." (And as you keep this in mind, make sure you also do not mistake "isolated" for "insulated"!) Hopefully all these concepts will be further clarified when we introduce the additional auxiliary concepts of force, work, and heat (although the latter will not come until the end of the semester).

For a closed system, we can state the principle of conservation of energy (somewhat symbolically) in the form

$$K + U + E_{source} + E_{diss} = \text{constant} \tag{5.8}$$

where $K$ is the total, macroscopic, kinetic energy; $U$ the sum of all the applicable potential energies associated with the system's *internal* interactions; $E_{source}$ is any kind of internal energy (such as chemical energy) that is *not* described by a potential energy function, but can increase the system's mechanical energy; and $E_{diss}$ stands for the contents of the "dissipated energy reservoir"—typically thermal energy. As with the potential energy $U$, the absolute value of $E_{source}$ and $E_{diss}$ does not (usually) really matter: all we are interested in is how much they change in the course of the process under consideration.

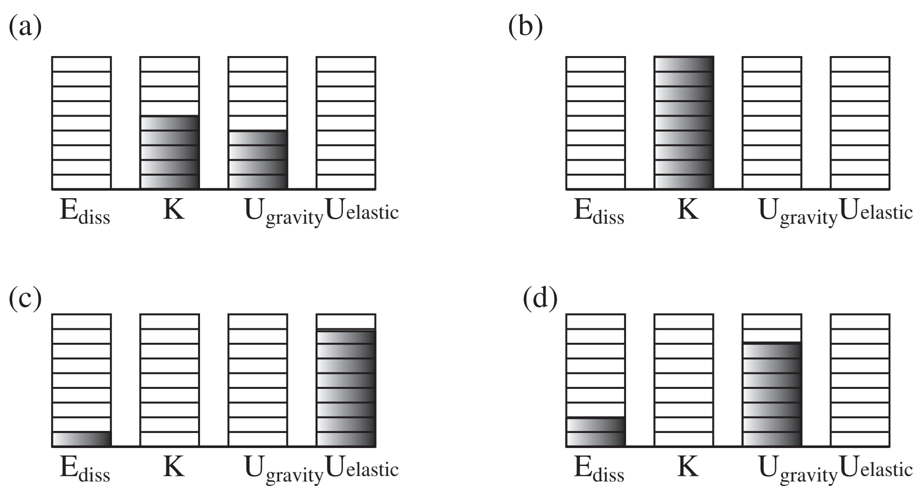



**Figure 5.4:** Energy bar diagrams for a system formed by the earth and a ball thrown downwards. (a) As the ball leaves the hand. (b) Just before it hits the ground. (c) During the collision, at the time of maximum compression. (d) At the top of the first bounce. The total number of energy "units" is the same in all the diagrams, as required by the principle of conservation of energy. From the diagrams you can tell that the coefficient of restitution $e = \sqrt{7/9}$.

Figure 5.4 above is an example of this kind of "energy accounting" for a ball bouncing on the ground. If the ball is thrown down, the system formed by the ball and the earth initially has both gravitational potential energy, and kinetic energy (diagram (a)). Note that we could write the total kinetic energy as $K_{cm} + K_{conv}$, as we did in the previous chapter, but because of the large mass of the earth, the center of mass of the system is essentially the center of the earth, which, in our earth-bound coordinate system, does not move at all, so $K_{cm}$ is, to an excellent approximation, zero. Then, the reduced mass of the system, $\mu = m_b M_E/(m_b + M_E)$ is, also to an excellent approximation, just equal to the mass of the ball, so $K_{conv} = \frac{1}{2}\mu v_{12}^2 = \frac{1}{2}m_b(v_b - v_e)^2 = \frac{1}{2}m_b v_b^2$ (again, because the earth does not move). So all the kinetic energy that we have is the kinetic energy of the ball, *and* it is all, in principle, convertible (as you can see if you replace the ball, for instance, with a bean bag).

As the ball falls, gravitational potential energy is being converted into kinetic energy, and the ball speeds up. As it is about to hit the ground (diagram (b)), the potential energy is zero and the kinetic energy is maximum. During the collision with the ground, all the kinetic energy is temporarily converted into other forms of energy, which are essentially elastic energy of deformation (like the energy in a spring) and some thermal energy (diagram (c)). When it bounces back, its kinetic energy will only be a fraction $e^2$ of what it had before the collision (where $e$ is the coefficient of restitution). This kinetic energy is all converted into gravitational potential energy as the ball reaches the top of its bounce (diagram (d)). Note there is more dissipated energy in diagram (d) than in (c); this is because I have assumed that dissipation of energy takes place both during the compression and the subsequent expansion of the ball.

## 5.5 In summary

1. For *conservative interactions* one can define a potential energy $U$, such that that in the course of the interaction the total mechanical energy $E = U + K$ of the system remains constant, even as $K$ and $U$ separately change. The function $U$ is a measure of the energy stored in the *configuration* of the system, that is, the relative position of all its parts.

2. The potential energy function for a system of two particles must be a function of their *relative* position only: $U(x_1 - x_2)$. However, if one of the objects is very massive, so it does not move during the interaction, its position may be taken to be the origin of coordinates, and $U$ written as a function of the lighter object's coordinate alone.

3. For a system formed by the earth and an object of mass $m$ at a height $y$ above the ground, the gravitational potential energy can be written as $U^G = mgy$ (approximately, as long as $y$ is much smaller than the radius of the earth).

4. The elastic potential energy stored in an ideal spring of spring constant $k$ and relaxed length $x_0$, when stretched or compressed to an actual length $x$, is $U^{spr} = \frac{1}{2}k(x - x_0)^2$.

5. For an object in one dimension, with position coordinate $x$, which is part of a system with potential energy $U(x)$, the motion can be predicted from the "energy landscape" formed by

the graph of the function $U(x)$. The idea, elaborated in Section 5.1.2 above, is to imagine the equivalent motion of an object sliding without friction over the same landscape, under the influence of gravity.

6. The fundamental interactions currently known in physics are gravity, the strong nuclear interaction and the electroweak interaction (which includes all electromagnetic phenomena). These are all conservative.

7. At a macroscopic level, one finds a number of interactions and associated energies that are derived from electromagnetism and quantum mechanics. Two important examples are chemical energy, and elastic energy (which is energy associated with the elasticity or "springiness" of a body). Elastic energy can often be described approximately by a potential energy function, and as such be included in calculations of the total mechanical energy of a system.

8. Interactions between macroscopic objects almost always involve the conversion of some type of energy into another. Typically, some of the total mechanical energy is always lost in the conversion process, because it is impossible to keep at least some of the energy from spreading itself randomly among the microscopic parts that make up the interacting objects. This is an intrinsically irreversible process known as *dissipation of energy.*

9. Most of the time the *dissipated energy* ends up as *thermal energy*, which is energy associated with a random agitation at the atomic or molecular level.

10. A *closed system* is one that does not exchange energy with its surroundings. This is not necessarily the same thing as an isolated system (which is one that does not exchange momentum with its surroundings). For a closed system, the sum of its macroscopic mechanical energy (kinetic + potential) and all its other "internal" energies (chemical, thermal), must be a constant.

## 5.6 Examples

### 5.6.1 Inelastic collision in the middle of a swing

Tarzan swings on a vine to rescue a helpless explorer (as usual) from some attacking animal or another. He begins his swing from a branch a height of 15 m above the ground, grabs the explorer at the bottom of his swing, and continues the swing, upwards this time, until they both land safely on another branch. Suppose that Tarzan weighs 90 kg and the explorer weighs 70 , and that Tarzan doesn't just drop from the branch, but pushes himself off so that he starts the swing with a speed of 5 m/s. How high a branch can he and the explorer reach?

**Solution**

Let us break this down into parts. The first part of the swing involves the conversion of some amount of initial gravitational potential energy into kinetic energy. Then comes the collision with the explorer, which is completely inelastic and we can analyze using conservation of momentum (assuming Tarzan and the explorer form an isolated system for the brief time the collision lasts). After that, the second half of their swing involves the complete conversion of their kinetic energy into gravitational potential energy.

Let $m_1$ be Tarzan's mass, $m_2$ the explorer's mass, $h_i$ the initial height, and $h_f$ the final height. We also have three velocities to worry about (or, more properly in this case, speeds, since their direction is of no concern, as long as they all point the way they are supposed to): Tarzan's initial velocity at the beginning of the swing, which we may call $v_{top}$; his velocity at the bottom of the swing, just before he grabs the explorer, which we may call $v_{bot1}$, and his velocity just after he grabs the explorer, which we may call $v_{bot2}$. (If you find those subscripts confusing, I am sorry, they are the best I could do; please feel free to make up your own.)

• *First part: the downswing.* We apply conservation of energy, in the form (5.8), to the first part of the swing. The system we consider consists of Tarzan and the earth, and it has kinetic energy as well as gravitational potential energy. We ignore the source energy and the dissipated energy terms, and consider the system closed despite the fact that Tarzan is holding onto a vine (as we shall see in a couple of chapters, the vine does no "work" on Tarzan—meaning, it does not change his energy, only his direction of motion—because the force it exerts on Tarzan is always perpendicular to his displacement):

$$K_{top} + U^G_{top} = K_{bot1} + U^G_{bot} \tag{5.9}$$

In terms of the quantities I introduced above, this equation becomes:

$$\frac{1}{2} m_1 v_{top}^2 + m_1 g h_i = \frac{1}{2} m_1 v_{bot1}^2 + 0$$

which can be solved to give

$$v_{bot1}^2 = v_{top}^2 + 2 g h_i \tag{5.10}$$

(note that this is just the familiar result (2.10) for free fall! This is because, as I pointed out above, the vine does no work on the system.). Substituting, we get

$$v_{bot1} = \sqrt{\left(5\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 + 2\left(9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)\times(15\,\mathrm{m})} = 17.9\,\frac{\mathrm{m}}{\mathrm{s}}$$

• *Second part: the completely inelastic collision.* The explorer is initially at rest (we assume he has not seen the wild beast ready to pounce yet, or he has seen it and he is paralyzed by fear!). After Tarzan grabs him they are moving together with a speed $v_{bot2}$. Conservation of momentum gives

$$m_1 v_{bot1} = (m_1 + m_2) v_{bot2} \tag{5.11}$$

which we can solve to get

$$v_{bot2} = \frac{m_1 v_{bot1}}{m_1 + m_2} = \frac{(90\,\mathrm{kg})\times(17.9\,\mathrm{m/s})}{160\,\mathrm{kg}} = 10\,\frac{\mathrm{m}}{\mathrm{s}}$$

• *Third part: the upswing.* Here we use again conservation of energy in the form

$$K_{bot2} + U^G_{bot} = K_f + U^G_f \tag{5.12}$$

where the subscript $f$ refers to the very end of the swing, when they both safely reach their new branch, and all their kinetic energy has been converted to gravitational potential energy, so $K_f = 0$ (which means that is as high as they can go, unless they start climbing the vine!). This equation can be rewritten as

$$\frac{1}{2}(m_1 + m_2)v^2_{bot2} + 0 = 0 + (m_1 + m_2) g h_f$$

and solving for $h_f$ we get

$$h_f = \frac{v^2_{bot2}}{2g} = \frac{(10\,\mathrm{m/s})^2}{2\times 9.8\,\mathrm{m/s}^2} = 5.15\,\mathrm{m}$$

### 5.6.2 Kinetic energy to spring potential energy: block collides with spring

A block of mass $m$ is sliding on a frictionless, horizontal surface, with a velocity $v_i$. It hits an ideal spring, of spring constant $k$, which is attached to the wall. The spring compresses until the block momentarily stops, and then starts expanding again, so the block ultimately bounces off.

(a) In the absence of dissipation, what is the block's final speed?

(b) By how much is the spring compressed?

**Solution**

This is a simpler version of the problem considered in Section 5.1.1, and in the next example. The problem involves the conversion of kinetic energy into elastic potential energy, and back. In the absence of dissipation, Eq. (5.8), specialized to this system (the spring and the block) reads:

$$K + U^{spr} = \text{constant} \tag{5.13}$$

For part (a), we consider the whole process where the spring starts relaxed and ends relaxed, so $U_i^{spr} = U_f^{spr} = 0$. Therefore, we must also have $K_f = K_i$, which means the block's final speed is the same as its initial speed. As explained in the chapter, this is characteristic of a conservative interaction.

For part (b), we take the final state to be the instant where the spring is maximally compressed and the block is momentarily at rest, so all the energy in the system is spring (which is to say, elastic) potential energy. If the spring is compressed a distance $d$ (that is, $x - x_0 = -d$ in Eq. (5.5)), this potential energy is $\frac{1}{2}kd^2$, so setting that equal to the system's initial energy we get:

$$K_i + 0 = 0 + \frac{1}{2}kd^2 \tag{5.14}$$

or

$$\frac{1}{2}mv_i^2 = \frac{1}{2}kd^2$$

which can be solved to get

$$d = \sqrt{\frac{m}{k}}\, v_i$$

## 5.7 Advanced Topics

### 5.7.1 Two carts colliding and compressing a spring

Unlike the example 5.6.2, which considered a stationary spring and asked only questions about initial and final states, this example is intended to show you how one can use "energy methods" to solve for the actual motion of a relatively complicated system as a function of time. The system is the two carts colliding, one of them fitted with a spring, considered in Section 5.1.1. Although all the physics involved is straightforward, the math is at a higher level than you will be using this semester, so I'm presenting this here as a "curiosity" only.

First, recall the total kinetic energy for a collision problem can be written as $K = K_{cm} + K_{conv}$, where (if the system is isolated) $K_{cm}$ remains constant throughout. Then, the total mechanical energy $E = K + U = K_{cm} + K_{conv} + U$. This is also constant, and before the interaction happens, when $U = 0$, we have $E = K_{cm} + K_{conv,i}$, so setting these two things equal and canceling out $K_{cm}$ we get

$$K_{conv} = K_{conv,i} - U \tag{5.15}$$

where the potential energy function is given by Eq. (5.6). Introducing the relative coordinate $x_{12} = x_2 - x_1$, and the relative velocity $v_{12}$, Eq. (5.15) becomes

$$\frac{1}{2}\mu v_{12}^2 = \frac{1}{2}\mu v_{12,i}^2 - \frac{1}{2}k(x_{12} - x_0)^2 \tag{5.16}$$

an equation that must hold while the interaction is going on. We can solve this for $v_{12}$, and then notice that both $x_{12}$ and $v_{12}$ are functions of time, and the latter is the derivative with respect to time of the former, so

$$v_{12} = \pm\sqrt{v_{12,i}^2 - (k/\mu)\,(x_{12} - x_0)^2} \tag{5.17}$$

$$\frac{dx_{12}}{dt} = \pm\sqrt{v_{12,i}^2 - (k/\mu)\,(x_{12}(t) - x_0)^2} \tag{5.18}$$

(The "±" sign means that the quantity on the right-hand side has to be negative at first, when the carts are coming together, and positive later, when they are coming apart. This is because I have assumed cart 1 starts to the left of cart 2, so going in cart 2, as seen from cart 1, appears to be moving to the left.)

Equation (5.18) is what is known, in calculus, as a differential equation. The problem is to find a function of $t$, $x_{12}(t)$, such that when you take its derivative you get the expression on the right-hand side. If you know how to calculate derivatives, you can check that the solution is in fact

$$x_{12}(t) = x_0 - \frac{v_{12,i}}{\omega}\sin[\omega(t - t_c)] \qquad \text{for } t_c \le t \le t_c + \pi/\omega \tag{5.19}$$

where the quantity $\omega = \sqrt{k/\mu}$, and the time $t_c$ is the time cart 1 first makes contact with the spring: $t_c = (x_{2i} - x_0 - x_{1i})/v_{1i}$. The solution (5.19) is valid for as long as the spring is compressed, which is to say, for as long as $x_{12}(t) < x_0$, or $\sin[\omega(t - t_c)] > 0$, which translates to the condition on $t$ shown above.

Having a solution for $x_{12}$, we could now obtain explicit results for $x_1(t)$ and $x_2(t)$ separately, using the fact that $x_1 = x_{cm} - m_2 x_{12}/(m_1 + m_2)$, and $x_2 = x_{cm} + m_1 x_{12}/(m_1 + m_2)$ (compare Eqs. (4.10), in chapter 4), and finding the position of the center of mass as a function of time is a trivial problem, since it just moves with constant velocity.

We do not, however, need to do any of this in order to generate the plots of the kinetic and potential energy shown in Fig. 5.2: the potential energy depends only on $x_2 - x_1$, which is given explicitly by Eq. (5.19), and the kinetic energy is equal to $K_{cm} + K_{conv}$, where $K_{cm}$ is constant and $K_{conv}$ is given by Eq. (5.16), which can also be easily rewritten in terms of Eq. (5.19). The results are Eqs. (5.7) in the text.

### 5.7.2 Getting the potential energy function from collision data

Consider the collision illustrated in Figure 3.4 (back in Chapter 3). Can we tell what the potential energy function is for the interaction between the two carts?

At first sight, it may seem that all the information necessary to "reconstruct" the function $U(x_1 - x_2)$ is available already, at least in graphical form: From Figure 3.4 you could get the value of $x_2 - x_1$ at any time $t$; then from Figure 4.5 you can get the value of $K$ (in the elastic-collision scenario) for the same value of $t$; and then you could plot $U = E - K$ (where $E$ is the total energy) as a function of $x_2 - x_1$.

But there is a catch: Figure 3.4 shows that the colliding objects never get any closer than $x_2 - x_1 \simeq 0.28\,\text{mm}$, so we have no way to tell what $U(x_2 - x_1)$ is for smaller values of $x_2 - x_1$. This is essentially the problem faced by particle physicists when they use collisions (which they do regularly) to try to determine the precise nature of the interactions between the particles they study!

You can check this for yourself. The functions I used for $x_1(t)$ and $x_2(t)$ in figure 3.4 are

$$
\begin{aligned}
x_1(t) &= \frac{1}{3}\left((2t - 10)\,\mathrm{erf}(10 - 2t) + 10\,\mathrm{erf}(10) + t - \frac{e^{-4(t-5)^2}}{\sqrt{\pi}}\right) - 5 \\
x_2(t) &= \frac{1}{3}\left((5 - t)\,\mathrm{erf}(10 - 2t) - 5\,\mathrm{erf}(10) + t + \frac{e^{-4(t-5)^2}}{2\sqrt{\pi}}\right)
\end{aligned}
\tag{5.20}
$$

Here, "erf" is the so-called "error function," which you can find in any decent library of mathematical functions. This looks complicated, but it just gives you the shapes you want for the velocity curves. The derivative of the above is

$$
\begin{aligned}
v_1(t) &= \frac{1}{3}\left(1 + 2\,\mathrm{erf}(10 - 2t)\right) \\
v_2(t) &= \frac{1}{3}\left(1 - \mathrm{erf}(10 - 2t)\right)
\end{aligned}
\tag{5.21}
$$

and you may want to try plotting these for yourself; the result should be Figure 3.1.

Now, assume (as I did for figure 4.5) that $m_1 = 1\,\mathrm{kg}$, and $m_2 = 2\,\mathrm{kg}$, and use these values and the results (5.21) (assumed to be in m/s) to calculate $K_{sys}$ as a function of $t$. Then $U = E_{sys} - K_{sys}$, with $E_{sys} = 1/2\,\mathrm{J}$:

$$
U = \frac{1}{2} - \frac{1}{2}m_1 v_1^2(t) - \frac{1}{2}m_2 v_2^2(t) = \frac{1}{3}\left(1 - \mathrm{erf}^2(10 - 2t)\right) \tag{5.22}
$$

and now do a parametric plot of $U$ versus $x_2 - x_1$, using $t$ as a parameter. You will end up with a figure like the one below:

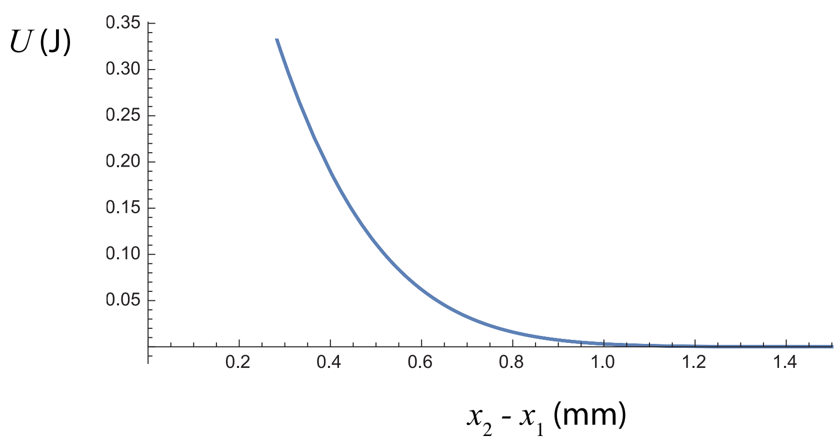



**Figure 5.5:** The potential energy function reconstructed from the information available for the collision shown in Figs. 3.1, 3.4, 4.5. No information can be gathered from those figures (nor from the explicit expressions (5.20) and (5.21) above) on the values of $U$ for $x_2 - x_1 < 0.28\,\mathrm{mm}$, the distance of closest approach of the two carts.

## 5.8 Problems

**Problem 1**

A particle is in a region where the potential energy has the form $U = 5/x$ (in joules, if $x$ is in meters).

(a) Sketch this potential energy function for $x > 0$.
(b) Assuming the particle starts at rest at $x = 0.5\,\mathrm{m}$, which way will it go if released? Why?
(c) Under the assumption in part (b), what will be the particle's kinetic energy after it has moved $0.1\,\mathrm{m}$ from its original position?
(d) Now assume that initially the particle is at $x = 1\,\mathrm{m}$, moving towards the left with an initial velocity $v_i = 2\,\mathrm{m/s}$. If the mass of the particle is $1\,\mathrm{kg}$, how close to the origin can it get before it stops?

**Problem 2**

A "ballistic pendulum" is a device (now largely obsolete, but very useful in its day) to measure the speed of a bullet as it hits a target. Let the target be a block of wood suspended from a string, as in the figure below. When the bullet hits, it is embedded in the wood, and together they swing, like a pendulum, to some maximum height $h$. The question is, how do you find the initial speed of the bullet ($v_i$) if you know the mass of the bullet ($m_1$), the mass of the block ($m_2$), and the height $h$?

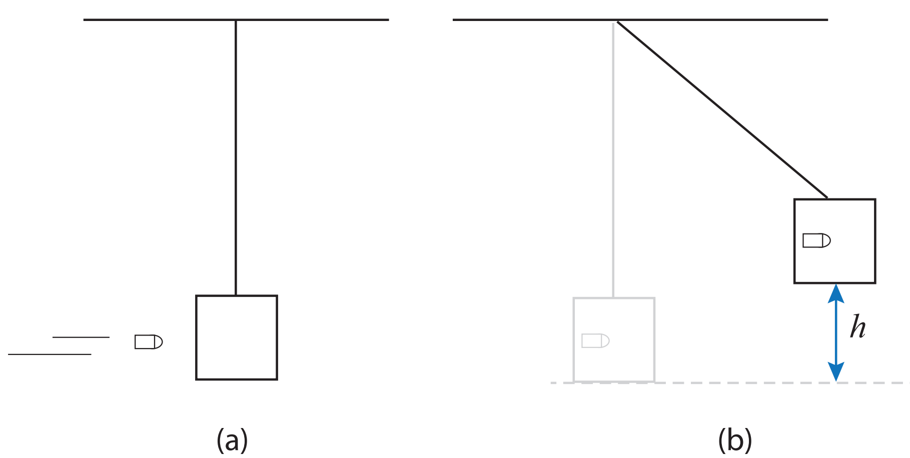
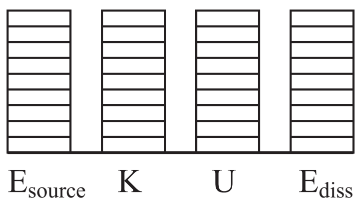



**Figure 5.6:** Ballistic pendulum. (a) Before the bullet hits. (b) After the bullet hits and is embedded in the block, at the maximum height of the swing.

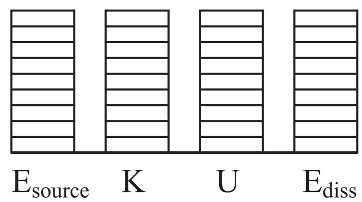
**Problem 3**

You drop a 0.5 kg ball from a height of 2 m, and it bounces back to a height of $1.5\,\mathrm{m}$. Consider the system formed by the ball and the Earth, so we can speak properly of its gravitational potential energy.

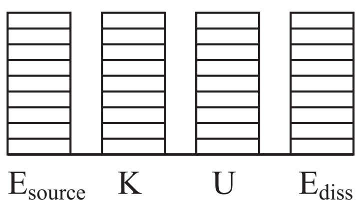
(a) What is the kinetic energy of the ball just before it hits the ground?
(b) What is the kinetic energy of the ball just after it bounces up?
(c) What is the coefficient of restitution for this collision?
(d) What kind of collision is this (elastic, inelastic, etc.)? Why?
(e) If the coefficient of restitution does not change, how high would the ball rise on a *second* bounce?
(f) On the graphs below, draw the energy bar diagrams for the system: (1) as the ball leaves your hand; (2) just before it hits the ground (assume $h = 0$ for practical purposes); (3) just after it leaves the ground on its way up ($h = 0$ still), and (4) at the top of its (first) bounce. Make sure to do this to scale, consistent with the values for the energies you have calculated above.



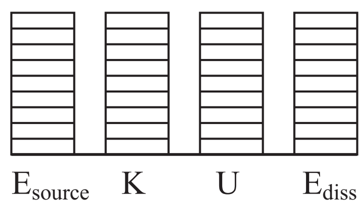
(1) As the ball leaves your hand



(2) Just before it hits the ground



(3) Just after it leaves the ground on its way up



(4) At the top of its bounce

**Problem 4**

A 60-kg skydiver jumps from an airplane 4000 m above the earth. After falling 450 m, he reaches a terminal speed of 55 m/s (about 120 mph). This means that after this time his speed does not increase any more.

(a) At the moment of the jump, what is the initial (gravitational) potential energy of the system formed by the earth and the skydiver? (Take $U^G = 0$ at ground level.)
(b) After the skydiver has fallen 450 m, what is the (gravitational) potential energy of the system? (Call this the "final" potential energy.)
(c) What is the final kinetic energy of the diver at that time?
(d) Assume the initial kinetic energy of the skydiver is zero. Is $\Delta K = -\Delta U$ for this system? If not, explain what happened to the "missing" energy.
(e) Can the skydiver and the earth below (*excluding the atmosphere!*) be considered a closed system

here? Explain.
(f) After the skydiver reaches terminal speed (and before he opens his parachute), he falls for a while at constant speed. What kind of energy conversion is taking place during this time? (Consider the system to be the earth, the skydiver, and the air around him).

**Problem 5**

You shoot a 1-kg projectile straight up from a spring toy gun, and find that it reaches a height of 5 m. (How do you figure out the height? From the time of flight, of course! See problem 2 from Chapter 2.) You also measure that when you load the gun, the spring compresses a distance 10 cm. What is the value of the spring constant?

# Chapter 6

# Interactions, part 2: Forces

## 6.1 Force

As we saw in the previous chapter, when an interaction can be described by a potential energy function, it is possible to use this to get a full solution for the motion of the objects involved, at least in one dimension. In fact, energy-based methods (known as the Lagrangian and Hamiltonian methods) can be also generalized to deal with problems in three dimensions, and they also provide the most direct pathway to quantum mechanics and quantum field theory. It might be possible to write an advanced textbook on classical mechanics without mentioning the concept of force at all.

On the other hand, as you may have also gathered from the example I worked out at the end of the previous chapter (section 5.6.3), solving for the equation of motion using energy-based methods may involve somewhat advanced math, even in just one dimension, and it only gets more complicated in higher dimensions. There is also the question of how to deal with interactions that are not conservative (at the macroscopic level) and therefore cannot be described by a potential energy function of the macroscopic coordinates. And, finally, there are specialized problem areas (such as the entire field of statics) where you actually *want* to know the forces acting on the various objects involved. For all these reasons, the concept of force will be introduced here, and the next few chapters will illustrate how it may be used to solve a variety of elementary problems in classical mechanics. This does not mean, however, that we are going to forget about energy from now on: as we will see, energy methods will continue to provide useful shortcuts in a variety of situations as well.

We start, as usual, by considering two objects that form an isolated system, so they interact with each other and with nothing else. As we have seen, under these circumstances their individual momenta change, but the total momentum remains constant. We are going to take the *rate of* *change* of each object's momentum as a measure of the *force* exerted on it by the other object. Mathematically, this means we will write for the *average force exerted by 1 on 2* over the time interval $\Delta t$ the expression

$$(F_{12})_{av} = \frac{\Delta p_2}{\Delta t} \tag{6.1}$$

Please observe the notation we are going to use: the subscripts on the symbol $F$ are in the order "by,on", as in "force exerted by" (object identified by first subscript) "on" (object identified by second subscript). (The comma is more or less optional.)

You can also see from Eq. (6.1) that the SI units of force are kg·m/s$^2$. This combination of units has the special name "newton," and it's abbreviated by an uppercase N.

In the same way as above, we can write the average force exerted by object 2 on object 1:

$$(F_{21})_{av} = \frac{\Delta p_1}{\Delta t} \tag{6.2}$$

and we know, by conservation of momentum, that we must have $\Delta p_1 = -\Delta p_2$, so we get our first important result,

$$(F_{12})_{av} = -(F_{21})_{av} \tag{6.3}$$

That is, *whenever two objects interact, they always exert equal* (in magnitude) *and opposite* (in direction) *forces on each other*. This is most often called **Newton's third law** of motion, or informally "the law of action and reaction."

We might as well now proceed along familiar lines and take the limit of Eqs. (6.1) and (6.2) above, as $\Delta t$ goes to zero, in order to introduce the more general concept of the instantaneous force (or just the "force," without any further qualifiers). We then get

$$\begin{aligned} F_{12} &= \frac{dp_2}{dt} \\ F_{21} &= \frac{dp_1}{dt} \end{aligned} \tag{6.4}$$

and, since Eq. (6.3) should hold for a time interval of any size,

$$F_{12} = -F_{21} \tag{6.5}$$

Now, under most circumstances the mass of, say, object 2 will not change during the interaction, so we can write

$$F_{12} = \frac{d}{dt}(m_2 v_2) = m_2 \frac{dv_2}{dt} = m_2 a_2 \tag{6.6}$$

This is the result that we often refer to as "$F = ma$", also known as **Newton's second law** of motion: *the (net) force acting on an object is equal to the product of its inertial mass and its acceleration*. The formulation in terms of the rate of change of momentum, as in Eqs. (6.4), is, however, somewhat more general, so it is technically preferred, even though this semester we will directly use $F = ma$ throughout.

If you want an example of a physical situation where $F = dp/dt$ is *not* equivalent to $F = ma$, consider a system where object 1 is a rocket, including its fuel, and "object" 2 are the gases ejected by the rocket. In this case, the mass of both "objects" is constantly changing, as the fuel is burned and more gases are ejected, and so the more general form $F = dp/dt$ needs to be used to calculate the force on the rocket (the thrust) at any given time.

At this point you may be wondering just what is Newton's *first* law? It is just the law of inertia: an object on which no force acts will stay at rest if it is initially at rest, or will move with constant velocity.

### 6.1.1 Forces and systems of particles

What if you had, say, three objects (let us make them "particles," for simplicity), all interacting with one another? In physics we find that all our interactions are pairwise additive, that is, we can write the total potential energy of the system as the sum of the potential energies associated with each pair of particles separately. As we will see in a moment, this means that the corresponding forces are additive too, so that, for instance, the total force on particle 1 could be written as

$$F_{all,1} = F_{21} + F_{31} = \frac{dp_1}{dt} \tag{6.7}$$

Consider now the most general case of a system that has an arbitrary number of particles, and is *not* isolated; that is, there are other objects, outside the system, that exert forces on some or all of the particles that make up the system. We will call these *external forces*. The sum of all the forces (both internal and external) acting on all the particles will take a form like this:

$$F_{total} = F_{ext,1} + F_{21} + F_{31} + \ldots + F_{ext,2} + F_{12} + F_{32} + \ldots + \ldots = \frac{dp_1}{dt} + \frac{dp_2}{dt} + \ldots \tag{6.8}$$

where $F_{ext,1}$ is the sum of all the external forces acting on particle 1, and so on. But now, observe that because of Newton's third law, Eq. (6.5), for every term of the form $F_{ij}$ appearing in the sum (6.8), there is a corresponding term $F_{ji} = -F_{ij}$ (you can see this explicitly already in Eq. (6.8) with $F_{12}$ and $F_{21}$), so all those terms (which represent all the internal forces) are going to cancel out, and we will be left only with the sum of the external forces:

$$F_{ext,1} + F_{ext,2} + \ldots = \frac{dp_1}{dt} + \frac{dp_2}{dt} + \ldots \tag{6.9}$$

The left-hand side of this equation is the sum of all the external forces; the right-hand side is the rate of change of the total momentum of the system. But the total momentum of the system is just equal to $Mv_{cm}$ (compare Eq. (3.11), in the "Momentum" chapter). So we have

$$F_{ext,all} = \frac{dp_{sys}}{dt} = \frac{d}{dt}(Mv_{cm}) \tag{6.10}$$

This extends a previous result. We already knew that in the absence of external forces, the momentum of a system remained constant. Now we see that the system's momentum responds to the net external force as if the whole system was a single particle of mass equal to the total mass $M$ and moving at the center of mass velocity $v_{cm}$. In fact, assuming that $M$ does not change we can rewrite Eq. (6.10) in the form

$$F_{ext,all} = Ma_{cm} \tag{6.11}$$

where $a_{cm}$ is the acceleration of the center of mass. This is the key result that allows us to treat extended objects as if they were particles: as far as the motion of the center of mass is concerned, all the internal forces cancel out (as we already saw in our study of collisions), and the point representing the center of mass responds to the sum of the external forces as if it were just a particle of mass $M$ subject to Newton's second law, $F = ma$. The result (6.11) applies equally well to an extended solid object that we choose to mentally break up into a collection of particles, as to an actual collection of separate particles, or even to a collection of separate extended objects; in the latter case, we would just have each object's motion represented by the motion of its own center of mass.

Finally, note that all the results above generalize to more than one dimension. In fact, forces are *vectors* (just like velocity, acceleration and momentum), and all of the above equations, in 3 dimensions, apply separately to each vector component. In one dimension, we just need to be aware of the sign of the forces, whenever we add several of them together.

## 6.2 Forces and potential energy

In the last chapter I mentioned a special case that we encounter often, in which a lighter object is interacting with a much more massive one, so that the massive one essentially does not move at all as a result of the interaction. Note that this does not contradict Newton's 3rd law, Eq. (6.5): the forces the two objects exert on each other *are* the same in magnitude, but the acceleration of each object is inversely proportional to its mass, so $F_{12} = -F_{21}$ implies

$$m_2 a_2 = -m_1 a_1 \tag{6.12}$$

and so if, for instance, $m_2 \gg m_1$, we get $|a_2| = |a_1| m_1/m_2 \ll |a_1|$. In words, the more massive object is less responsive than the less massive one to a force of the *same* magnitude. This is just how we came up with the concept of inertial mass in the first place!

Anyway, you'll recall that in this situation I could just write the potential energy function of the whole system as a function of only the lighter object's coordinate, $U(x)$. I am going to use this simplified setup to show you a very interesting relationship between potential energies and forces. Suppose this is a closed system in which no dissipation of energy is taking place. Then the total mechanical energy is a constant:

$$E_{mech} = \frac{1}{2}mv^2 + U(x) = \text{constant} \tag{6.13}$$

(Here, $m$ is the mass of the lighter object, and $v$ its velocity; the more massive object does not contribute to the total kinetic energy, since it does not move!)

As the lighter object moves, both $x$ and $v$ in Eq. (6.13) change with time (recall, for instance, our study of "energy landscapes" in the previous chapter, section 5.1.2). So I can take the derivative of Eq. (6.13) with respect to time, using the chain rule, and noting that, since the whole thing is a constant, the total value of the derivative must be zero:

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{1}{2}m\big(v(t)\big)^2 + U\big(x(t)\big)\right) \\ &= mv(t)\,\frac{dv}{dt} + \frac{dU}{dx}\,\frac{dx}{dt} \end{aligned} \tag{6.14}$$

But note that $dx/dt$ is just the same as $v(t)$. So I can cancel that on both terms, and then I am left with

$$m\,\frac{dv}{dt} = -\frac{dU}{dx} \tag{6.15}$$

But $dv/dt$ is just the acceleration $a$, and $F = ma$. So this tells me that

$$F = -\frac{dU}{dx} \tag{6.16}$$

and this is how you can always get the force from a potential energy function.

Let us check it right away for the force of gravity: we know that $U^G = mgy$, so

$$F^G = -\frac{dU^G}{dy} = -\frac{d}{dy}(mgy) = -mg \tag{6.17}$$

Is this right? It seems to be! Recall all objects fall with the same acceleration, $-g$ (assuming the upwards direction to be positive), so if $F = ma$, we must have $F^G = -mg$. So the gravitational force exerted by the earth on any object (which I would denote in full by $F^G_{E,o}$) is proportional to the inertial mass of the object—in fact, it is what we call the object's *weight*—but since to get the acceleration you have to divide the force by the inertial mass, that cancels out, and $a$ ends up being the same for all objects, regardless of how heavy they are.

Now that we have this result under our belt, we can move on to the slightly more challenging case of two objects of comparable masses interacting through a potential energy function that must be, as I pointed out in the previous chapter, a function of just the relative coordinate $x_{12} = x_2 - x_!$.

I claim that in that case you can again get the force on object 1, $F_{21}$, by taking the derivative of $U(x_2 - x_1)$ with respect to $x_1$ (leaving $x_2$ alone), and reciprocally, you get $F_{12}$ by taking the derivative of $U(x_2 - x_1)$ with respect to $x_2$. Here is how it works, again using the chain rule:

$$
\begin{aligned}
F_{21} &= -\frac{d}{dx_1}U(x_{12}) = -\frac{dU}{dx_{12}}\frac{d}{dx_1}(x_2 - x_1) = \frac{dU}{dx_{12}} \\
F_{12} &= -\frac{d}{dx_2}U(x_{12}) = -\frac{dU}{dx_{12}}\frac{d}{dx_2}(x_2 - x_1) = -\frac{dU}{dx_{12}}
\end{aligned}
\tag{6.18}
$$

and you can see that this automatically ensures that $F_{21} = -F_{12}$. In fact, it was in order to ensure this that I required that $U$ should depend only on the *difference* of $x_1$ and $x_2$, rather than on each one separately. Since we got the condition $F_{21} = -F_{12}$ originally from conservation of momentum, you can see now how the two things are related[1].

The only example we have seen so far of this kind of potential energy function was in last chapter's Section 5.1.1, for two carts interacting through an "ideal" spring. I told you there that the potential energy of the system could be written as $\frac{1}{2}k(x_2 - x_1 - x_0)^2$, where $k$ was the "spring constant" and $x_0$ the relaxed length of the spring. If you apply Eqs. (6.18) to this function, you will find that the force exerted (through the spring) by cart 2 on cart 1 is

$$F_{21} = k(x_2 - x_1 - x_0) \tag{6.19}$$

Note that this force will be negative under the assumptions we made last chapter, namely, that cart 2 is on the right, cart 1 on the left, and the spring is compressed, so that $x_2 - x_1 < x_0$. Similarly,

$$F_{12} = -k(x_2 - x_1 - x_0) \tag{6.20}$$

and this one, as it should, is positive.

The results (6.19) and (6.20) basically tell you what we mean by an "ideal spring" in physics: it is a spring that pulls (if stretched) or pushes (if compressed) with a force that is proportional to the change from its equilibrium length. Thus, if you fasten one end of the spring at $x = 0$, and stretch it or compress it so that the other end is at $x$, the spring will respond by exerting a force

$$F^{spr} = -k(x - x_0) \tag{6.21}$$

As you can see, this is negative if $x > x_0 > 0$ (spring stretched, pulling force) and positive if $x < x_0$ (spring compressed, pushing force). In fact, the spring exerts an equal (in magnitude) and opposite (in direction) force at the other end (the one attached to the wall), so Eq. (6.21) only gives the correct sign of the force at the end that is denoted by the coordinate value $x$. Equations (6.19) and (6.20) are a bit clearer in this respect: Eq. (6.19) gives the correct sign of the force at point $x_1$, and Eq. (6.20) the correct sign at point $x_2$.

[1] The result (6.18) generalizes to more dimensions, but to do it properly you need vectors and partial derivative notation, and I'm already bending the notational rules a little bit here...

Figure 6.1 shows, in black, all the forces exerted by a spring with one fixed end, according as to whether it is relaxed, compressed, or stretched. I have assumed that it is pushed or pulled by a hand (not shown) at the "free" end, hence the subscript "$h$", whereas the subscript "$w$" stands for "wall." Note that the wall and the hand, in turn, exert equal and opposite forces on the spring, shown in red in the figure.

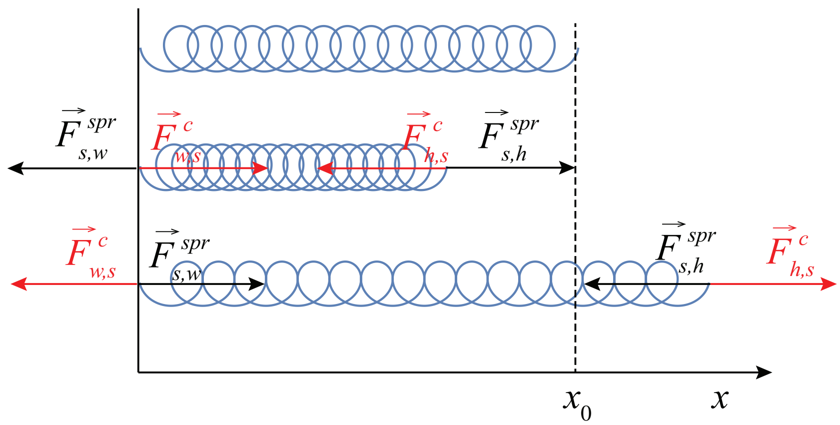



**Figure 6.1:** Forces (in black) exerted *by* a spring with one end attached to a wall and the other pushed or pulled by a hand (not shown). In every case the force is proportional to the change in the length of the spring from its equilibrium, or relaxed, value, shown here as $x_0$. For this figure I have set the proportionality constant $k = 1$. The forces exerted *on* the spring, by the wall and by the hand, are shown in red.

Equation (6.21) is generally referred to as *Hooke's law*, after the British scientist Robert Hooke (a contemporary of Newton). Of course, it is not a "law" at all, merely a useful approximation to the way most springs behave as long as you do not stretch them or compress them too much[2].

A note on the way the forces have been labeled in Figure 6.1. I have used the generic symbol "$c$", which stands for "contact," to indicate the type of force exerted by the wall and the hand on the spring. In fact, since each pair of forces (by the hand on the spring and by the spring on the hand, for instance), at the point of contact, arises from one and the same interaction, I should have used the same "type" notation for both, but it is widespread practice to use a superscript like "spr" to denote a force whose origin is, ultimately, a spring's elasticity. This does not change the fact that the spring force, at the point where it is applied, is indeed a contact force.

So, next, a word on "contact" forces. Basically, what we mean by that is forces that arise where objects "touch," and we mean this by opposition to what are called instead "field" forces (such as gravity, or magnetic or electrostatic forces) which "act at a distance." The distinction is actually

[2]Assuming that you *can* compress them! Some springs, such as slinkies, actually cannot be compressed because their coils are already in contact when they are relaxed. Nevertheless, Eq. (6.21) will still apply approximately to such a spring when it is stretched, that is, when $x > x_0$.

only meaningful at the macroscopic level, since at the microscopic level objects never *really* touch, and all forces are field forces, it is just that some are “long range” and some are “short range.” For our purposes, really, the word “contact” will just be a convenient, catch-all sort of moniker that we will use to label the force vectors when nothing more specific will do.

## 6.3 Forces not derived from a potential energy

As we have seen in the previous section, for interactions that are associated with a potential energy, we are always able to determine the forces from the potential energy by simple differentiation. This means that we do not have to rely exclusively on an equation of the type $F = ma$, like (6.4) or (6.6), to *infer* the value of a force from the observed acceleration; rather, we can work in reverse, and *predict* the value of the acceleration (and from it all the subsequent motion) from our knowledge of the force.

I have said before that, on a microscopic level, all the interactions can be derived from potential energies, yet at the macroscopic level this is not generally true: we have many kinds of interactions for which the associated “stored” or converted energy cannot, in general, be written as a function of the macroscopic position variables for the objects making up the system (by which I mean, typically, the positions of their centers of mass). So what do we do in those cases?

The forces of this type with which we shall deal this semester actually fall into two different categories: the ones that do not dissipate energy, and that we *could*, in fact, associate with a potential energy if we wanted to[3], and the ones that definitely dissipate energy and need special handling. The former category includes the normal force, tension, and the static friction force; the second category includes the force of kinetic (or sliding) friction, and air resistance. A brief description of all these forces, and the methods to deal with them, follows.

### 6.3.1 Tensions

*Tension* is the force exerted by a stretched spring, and, similarly, by objects such as cables, ropes, and strings in response to a stretching force (or load) applied to them. It is ultimately an elastic force, so, as I said above, we could in principle describe it by a potential energy, but in practice cables, strings and the like are so stiff that it is often all right to neglect their change in length altogether and assume that *no* potential energy is, in fact, stored in them. The price we pay for this simplification (and it *is* a simplification) is that we are left without an independent way to determine the value of the tension in any specific case; we just have to infer it from the acceleration

[3] If we wanted to complicate our life, that is...

of the object on which it acts (since it is a reaction force, it can assume any value as required to adjust to any circumstance—up to the point where the rope snaps, anyway).

Thus, for instance, in the picture below, which shows two blocks connected by a rope over a pulley, the tension force exerted by the rope on block 1 must equal $m_1 a_1$, where $a_1$ is the acceleration of that block, provided there are no other horizontal forces (such as friction) acting on it. For the hanging block, on the other hand, the net force is the sum of the tension on the other end of the rope (pulling up) and gravity, pulling down. If we choose the upward direction as positive, we can write Newton's second law for the second block as

$$F^t_{r,2} - m_2 g = m_2 a_2 \tag{6.22}$$

Two things need to be realized now. First, if the rope is inextensible, both blocks travel the same distance in the same time, so their speeds are always the same, and hence the *magnitude* of their accelerations will always be the same as well; only the sign may be different depending on which direction we choose as positive. If we take to the right to be positive for the horizontal motion, we will have $a_2 = -a_1$. I'm just going to call $a_1 = a$, so then $a_2 = -a$.

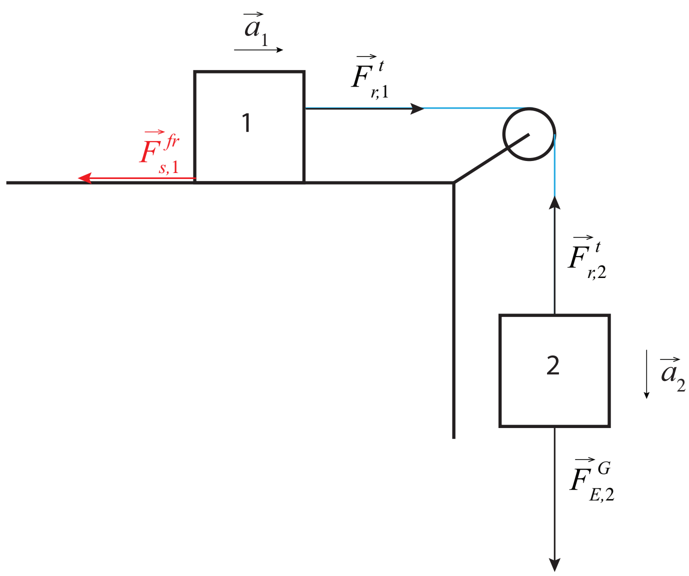



**Figure 6.2:** Two blocks joined by a massless, inextensible strength threaded over a massless pulley. An optional friction force (in red, where *fr* could be either $s$ or $k$) is shown for use later, in the discussion in subsection 3.3. In this subsection, however, it is assumed to be zero.

The second thing to note is that, if the rope's mass is negligible, it will, like an ideal spring, pull with a force with the same *magnitude* on both ends. With our specific choices (up and to the right is positive), we then have $F^t_{r,2} = F^t_{r,1}$, and I'm just going to call this quantity $F^t$. All this yields, then, the following two equations:

$$\begin{aligned} F^t &= m_1 a \\ F^t - m_2 g &= -m_2 a \end{aligned} \tag{6.23}$$

The system (6.23) can be easily solved to get

$$\begin{aligned} a &= \frac{m_2 g}{m_1 + m_2} \\ F^t &= \frac{m_1 m_2 g}{m_1 + m_2} \end{aligned} \tag{6.24}$$

### 6.3.2 Normal forces

Normal force is the reaction force with which a surface pushes back when it is being pushed on. Again, this works very much like an extremely stiff spring, this time under compression instead of tension. And, again, we will eschew the potential energy treatment by assuming that the surface's actual displacement is entirely negligible, and we will just calculate the value of $F^n$ as whatever is needed in order to make Newton's second law work. Note that this force will always be perpendicular to the surface, by definition (the word "normal" means "perpendicular" here); the task of dealing with a sideways push on the surface will be delegated to the static friction force, to be covered next.

If I am just standing on the floor and not falling through it, the net vertical force acting on me must be zero. The force of gravity on me is $mg$ downwards, and so the upwards normal force must match this value, so for this situation $F^n = mg$. But don't get too attached to the notion that the normal force must always be equal to $mg$, since this will often not be the case. Imagine, for instance, a person standing inside an elevator at the time it is accelerating upwards. With the upwards direction as positive, Newton's second law for the person reads

$$F^n - mg = ma \tag{6.25}$$

and therefore for this situation

$$F^n = mg + ma \tag{6.26}$$

If you were weighing yourself on a bathroom scale in the elevator, this is the upwards force that the bathroom scale would have to exert on you, and it would do that by compressing a spring inside, and it would record the "extra" compression (beyond that required by your actual weight, $mg$) as extra weight. Conversely, if the elevator were accelerating downward, the scale would record you as being lighter. In the extreme case in which the cable of the elevator broke and you, the elevator and the scale ended up (briefly, before the emergency brake caught on) in free fall, you would all be falling with the same acceleration, you would not be pushing down on the scale at all, and its normal force as well as your recorded weight would be zero. This is ultimately the reason for the apparent weightlessness experienced by the astronauts in the space station, where the force of gravity is, in fact, not very much smaller than on the surface of the earth. (We will return to this effect after we have a good grip on two-dimensional, and in particular circular, motion.)

### 6.3.3 Static and kinetic friction forces

The *static friction* force is a force that prevents two surfaces in contact from slipping relative to each other. It is an extremely useful force, since we would not be able to drive a car, or ride a bicycle, or even walk, without it—as we know from experience, if we have ever tried to do any of those things on a low-friction surface (such as a sheet of ice).

The science behind friction (known technically as *tribology*) is actually not very simple at all, and it is of great current interest for many reasons—whether the ultimate goal is to develop ways to reduce friction or to increase it. On an elementary level, we are all aware of the fact that even a surface that looks smooth on a macroscopic scale will actually exhibit irregularities, such as ridges and valleys, under a microscope. It makes sense, then, that when two such surfaces are pressed together, the bumps on one of them will hit, and be held in place by, the bumps on the other one, and that will prevent sliding until and unless a sufficient force is applied to temporarily "flatten" the bumps enough to allow the thing to move[4].

As long as this does not happen, that is, as long as the surfaces do *not* slide relative to each other, we say we are dealing with the *static* friction force, which is, at least approximately, an elastic force that does not dissipate energy: the small distortion of the "bumps" on the surfaces that takes place when you push on them typically happens slowly enough, and is small enough, to be reversible, so that when you stop pushing the two surfaces just go back to their initial state. This is no longer the case once the surfaces start sliding relative to each other. At that point the character of the friction force changes, and we have to deal with the *sliding*, or *kinetic* friction force, as I will explain below.

The static friction force is also, like tension and the normal force, a reaction force that will adjust itself, within limits, to take any value required to prevent slippage in a given circumstance. Hence, its actual value in a particular situation cannot really be ascertained until the other relevant forces—the other forces pushing or pulling on the object—are known.

For instance, for the system in Figure 6.2, imagine there is a force of static friction between block 1 and the surface on which it rests, sufficiently large to keep it from sliding altogether. How large

[4]This picture based, essentially, on classical physics, leaves out an atomic-scale effect that may be important in some cases, which is the formation of weak bonds between the atoms of both surfaces, resulting in an actual "adhesive" force. This is, for instance, how geckos can run up vertical walls. For our purposes, however, the classical picture (of small ridges and valleys bumping into each other) will suffice to qualitatively understand all the examples we will cover this semester.

does this have to be? If there is no acceleration ($a = 0$), the equivalent of system (6.23) will be

$$\begin{aligned} F^s_{s,1} + F^t &= 0 \\ F^t - m_2 g &= 0 \end{aligned} \tag{6.27}$$

where $F^s_{s,1}$ is the force of static friction exerted by the surface on block 1, and we are going to let the math tell us what sign it is supposed to have. Solving the system (6.27) we just get the condition

$$F^s_{s,1} = -m_2 g \tag{6.28}$$

so this is how large $F^s_{s,1}$ has to be in order to keep the whole system from moving in this case.

There is an empirical formula that tells us approximately how large the force of static friction *can* get in a given situation. The idea behind it is that, microscopically, the surfaces are in contact only near the top of their respective ridges. If you press them together harder, some of the ridges get flattened and the effective contact area increases; this in turn makes the surfaces more resistant to slippage. A direct measure of how strongly the two surfaces press against each other is, actually, just the normal force they exert on each other. So, in general, we expect the maximum force that static friction will be able to exert to be proportional to the *normal* force between the surfaces:

$$\left|F^s_{s1,s2}\right|_{max} = \mu_s \left|F^n_{s1,s2}\right| \tag{6.29}$$

where $s1$ and $s2$ just mean "surface 1" and "surface 2," respectively, and the number $\mu_s$ is known as the *coefficient of static friction*: it is a tabulated quantity that is determined experimentally, by testing the slippage of different surfaces against each other under different loads.

In our example, the normal force exerted by the surface on block 1 has to be equal to $m_1 g$, since there is no vertical acceleration for that block, and so the maximum value that $F^s$ may have in this case is $\mu_s m_1 g$, whatever $\mu_s$ might happen to be. In fact, this setup would give us a way to determine $\mu_s$ for these two surfaces: start with a small value of $m_2$, and gradually increase it until the system starts moving. At that point we will know that $m_2 g$ has just exceeded the maximum possible value of $|F^s_{12}|$, namely, $\mu_s m_1 g$, and so $\mu_s = (m_2)_{max}/m_1$, where $(m_2)_{max}$ is the largest mass we can hang before the system starts moving.

By contrast with all of the above, the *kinetic friction* force, which always acts so as to oppose the relative motion of the two surfaces when they *are* actually slipping, is not elastic, it is definitely dissipative, and, most interestingly, it is also *not* much of a reactive force, meaning that its value can be approximately predicted for any given circumstance, and does not depend much on things such as how fast the surfaces are actually moving relative to each other. It *does* depend on how hard the surfaces are pressing against each other, as quantified by the normal force, and on another tabulated quantity known as the *coefficient of kinetic friction*:

$$\left|F^k_{s1,s2}\right| = \mu_k \left|F^n_{s1,s2}\right| \tag{6.30}$$

Note that, unlike for static friction, this is *not* the maximum possible value of $|F^k|$, but its *actual* value; so if we know $F^n$ (and $\mu_k$) we know $F^k$ without having to solve any other equations (its sign does depend on the direction of motion, of course). The coefficient $\mu_k$ is typically a little smaller than $\mu_s$, reflecting the fact that once you get something you have been pushing on to move, keeping it in motion with constant velocity usually does not require the same amount of force.

To finish off with our example in Figure 2, suppose the system *is* moving, and there is a kinetic friction force $F^k_{s,1}$ between block 1 and the surface. The equations (6.23) then have to be changed to

$$\begin{aligned} F^t - \mu_k m_1 g &= m_1 a \\ F^t - m_2 g &= -m_2 a \end{aligned} \tag{6.31}$$

and the solution now is

$$\begin{aligned} a &= \frac{m_2 - \mu_k m_1}{m_1 + m_2}\, g \\ F^t &= \frac{m_1 m_2 (1+\mu_k)}{m_1 + m_2}\, g \end{aligned} \tag{6.32}$$

You may ask, why does kinetic friction dissipate energy? A qualitative answer is that, as the surfaces slide past each other, their small (sometimes microscopic) ridges are constantly “bumping” into each other; so you have lots of microscopic collisions happening all the time, and they cannot all be perfectly elastic. So mechanical energy is being “lost.” In fact, it is primarily being converted to thermal energy, as you can verify experimentally: this is why you rub your hands together to get warm, for instance. More dramatically, this is how some people (those who really know what they are doing!) can actually start a fire by rubbing sticks together.

### 6.3.4 Air resistance

Air resistance is an instance of fluid resistance or *drag*, a force that opposes the motion of an object through a fluid. Microscopically, you can think of it as being due to the constant collisions of the object with the air molecules, as it cleaves its way through the air. As a result of these collisions, some of its momentum is transferred to the air, as well as some of its kinetic energy, which ends up as thermal energy (as in the case of kinetic friction discussed above). The very high temperatures that air resistance can generate can be seen, in a particularly dramatic way, on the re-entry of spacecraft into the atmosphere.

Unlike kinetic friction between solid surfaces, the fluid drag force does depend on the velocity of the object (relative to the fluid), as well as on a number of other factors having to do with the object's shape and the fluid's density and viscosity. Very roughly speaking, for low velocities the drag force is proportional to the object's speed, whereas for high velocities it is proportional to the square of the speed.

In principle, one can use the appropriate drag formula together with Newton's second law to calculate the effect of air resistance on a simple object thrown or dropped; in practice, this requires a somewhat more advanced math than we will be using this course, and the formulas themselves are complicated, so I will not introduce them here.

One aspect of air resistance that deserves to be mentioned is what is known as "terminal velocity" (which I already introduced briefly in Section 2.3). Since air resistance increases with speed, if you drop an object from a sufficiently great height, the upwards drag force on it will increase as it accelerates, until at some point it will become as large as the downward force of gravity. At that point, the net force on the object is zero, so it stops accelerating, and from that point on it continues to fall with constant velocity. When the Greek philosopher Aristotle was trying to figure out the motion of falling bodies, he reasoned that, since air was just another fluid, he could slow down the fall (in order to study it better) without changing the physics by dropping objects in liquids instead of air. The problem with this approach, though, is that terminal velocity is reached much faster in a liquid than in air, so Aristotle missed entirely the early stage of approximately constant acceleration, and concluded (wrongly) that the natural way all objects fell was with constant velocity. It took almost two thousand years until Galileo disproved that notion by coming up with a better method to slow down the falling motion—namely, by using inclined planes.

## 6.4 Free-body diagrams

As Figure 6.1 shows, trying to draw every single force acting on every single object can very quickly become pretty messy. And anyway, this is not usually what we need: what we need is to separate cleanly all the forces acting on any given object, one object at a time, so we can apply Newton's second law, $F_{net} = ma$, to each object individually.

In order to accomplish this, we use what are known as *free-body diagrams*. In a free-body diagram, a potentially very complicated object is replaced symbolically by a dot or a small circle, and all the forces acting on the object are drawn (approximately to scale and properly labeled) as acting on the dot. Regardless of whether a force is a pulling or pushing force, the convention is to always draw it *as a vector that originates at the dot*. If the system is accelerating, it is also a good idea to indicate the acceleration's direction also somewhere on the diagram.

The figure below (next page) shows, as an example, a free-body diagram for block 1 in Figure 6.2, in the presence of both a nonzero acceleration and a kinetic friction force. The diagram includes all the forces, even gravity and the normal force, which were left out of the picture in Figure 6.2.

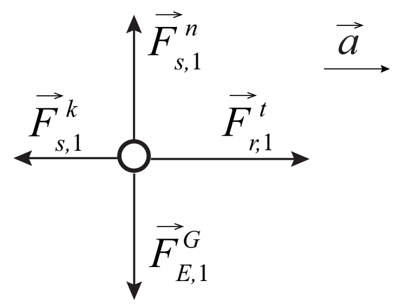



**Figure 6.3:** Free-body diagram for block 1 in Figure 6.2, with the friction force adjusted so as to be compatible with a nonzero acceleration to the right.

Note that I have drawn $F^n$ and the force of gravity $F^G_{E,1}$ as having the same magnitude, since there is no vertical acceleration for that block. If I know the value of $\mu_k$, I should also try to draw $F^k = \mu_k F^n$ approximately to scale with the other two forces. Then, since I know that there is an acceleration to the right, I need to draw $F^t$ greater than $F^k$, since the net force on the block must be to the right as well. And, if I were drawing a free-body diagram for block 2, I would have to make sure that I drew its weight, $F^G_{E,2}$, as being greater in magnitude than $F^t$, since the net force on that block needs to be downwards.

## 6.5 In summary

1. Whenever two objects interact, they exert *forces* on each other that are equal in magnitude and opposite in direction (*Newton's 3rd law*).

2. Forces are vectors, and they are additive. The total (or net) force on an object or system is equal to the rate of change of its total momentum (*Newton's 2nd law*). If the system's mass is constant, this can be written as $F_{ext,all} = Ma_{cm}$, where $M$ is the system's total mass and $a_{cm}$ is the acceleration of its center of mass. Only the *external* forces contribute to this equation; the internal forces cancel out because of point 1 above.

3. For any interaction that can be derived from a potential energy function $U(x_1 - x_2)$, the force exerted by object 2 on object 1 is equal to $-dU/dx_1$ (where the derivative is calculated treating $x_2$ as a constant), and vice-versa.

4. The force of gravity on an object near the surface of the earth is known as the object's *weight*, and it is equal (in magnitude) to $mg$, where $m$ is the object's inertial mass.

5. An ideal spring whose relaxed length is $x_0$, when stretched or compressed to a length $x$, exerts a pulling or pushing force, respectively, at both ends, with magnitude $k|x - x_0|$, where $k$ is called the spring constant.

6. When dealing with macroscopic objects we introduce several "constraint" forces whose values need to be determined from the accelerations through Newton's second law: the *tension* $F^t$ in ropes, strings or cables; the *normal force* $F^n$ exerted by a surface in response to applied pressure; and the static friction force $F^s$ that prevents surfaces from slipping past each other.

7. The maximum possible value of the static friction force is $\mu_s|F^n|$, where $\mu_s$ is the coefficient of static friction.

8. The force of sliding or kinetic friction, $F^k$, appears when two surfaces are sliding past each other. Its magnitude is $\mu_k|F^n|$ ($\mu_k$ is the coefficient of static friction), and its sign is such as to oppose the sliding motion. Unlike the forces in 6 above, it is a dissipative force.

9. A *free-body diagram* is a way to depict *all* (and *only*) the forces *acting on* an object. The object should be represented as a small circle or dot. The forces should all be drawn as vectors originating on the dot, with their directions correctly shown and their lengths approximately to scale. The acceleration of the object should also be indicated elsewhere in the picture. The forces should be labeled like this: $F_{by,on}^{type}$.

## 6.6 Examples

### 6.6.1 Dropping an object on a weighing scale

(Short version) Suppose you drop a 5-kg object on a spring scale from a height of 1 m. If the spring constant is $k = 20,000\,\mathrm{N/m}$, what will the scale read?

(Long version) OK, let's break that up into parts. Suppose that a spring scale is just a platform (of negligible mass) sitting on top of a spring. If you put an object of mass $m$ on top of it, the spring compresses so that (in equilibrium) it exerts an upwards force that matches that of gravity.

(a) If the spring constant is $k$ and the object's mass is $m$ and the whole system is at rest, what distance is the spring compressed?
(b) If you drop the object from a height $h$, what is the (instantaneous) *maximum* compression of the spring as the object is brought to a momentary rest? (This part is an *energy* problem! Assume that $h$ is much greater than the actual compression of the spring, so you can neglect that when calculating the change in gravitational potential energy.)
(c) What mass would give you that same compression, if you were to place it gently on the scale, and wait until all the oscillations died down?
(d) OK, now answer the question at the top!

**Solution**

(a) The forces acting on the object sitting at rest on the platform are the force of gravity, $F^G_{E,o} = -mg$, and the normal force due to the platform, $F^n_{p,o}$. This last force is equal, in magnitude, to the force exerted on the platform by the spring (it has to be, because the platform itself is being pushed down by a force $F^n_{o,p} = -F^n_{p,o}$, and this has to be balanced by the spring force). This means we can, for practical purposes, pretend the platform is not there and just set the upwards force on the object equal to the spring force, $F^{spr}_{s,p} = -k(x - x_0)$. So, Newton's second law gives

$$F_{net} = F^G_{E,o} + F^{spr}_{s,p} = ma = 0 \tag{6.33}$$

For a compressed spring, $x - x_0$ is negative, and we can just let $d = x_0 - x$ be the distance the spring is compressed. Then Eq. (6.33) gives

$$-mg + kd = 0$$

so

$$d = mg/k \tag{6.34}$$

when you just set an object on the scale and let it come to rest.

(b) This part, as the problem says, is a conservation of energy situation. The system formed by the spring, the object and the earth starts out with some gravitational potential energy, and ends up, with the object momentarily at rest, with only spring potential energy:

$$\begin{aligned} U_i^G + U_i^{spr} &= U_f^G + U_f^{spr} \\ mgy_i + 0 &= mgy_f + \frac{1}{2}kd_{max}^2 \end{aligned} \tag{6.35}$$

where I have used the subscript "max" on the compression distance to distinguish it from what I calculated in part (a) (this kind of makes sense also because the scale is going to swing up and down, and we want only the maximum compression, which will give us the largest reading). The problem said to ignore the compression when calculating the change in $U^G$, meaning that, if we measure height from the top of the scale, $y_i = h$ and $y_f = 0$. Then, solving Eq. (6.35) for $d_{max}$, we get

$$d_{max} = \sqrt{\frac{2mgh}{k}} \tag{6.36}$$

(c) For this part, let us rewrite Eq. (6.34) as $m_{eq} = kd_{max}/g$, where $m_{eq}$ is the "equivalent" mass that you would have to place on the scale (gently) to get the same reading as in part (b). Using then Eq. (6.36),

$$m_{eq} = \frac{k}{g}\sqrt{\frac{2mgh}{k}} = \sqrt{\frac{2mkh}{g}} \tag{6.37}$$

(d) Now we can substitute the values given: $m = 5\,\text{kg}$, $h = 1\,\text{m}$, $k = 20,000\,\text{N/m}$. The result is $m_{eq} = 143\,\text{kg}$.

(Note: if you found the purely algebraic treatment above confusing, try substituting numerical values in Eqs. (6.34) and (6.36). The first equation tells you that if you just place the 5-kg mass on the scale it will compress a distance $d = 2.45\,\text{mm}$. The second tells you that if you drop it it will compress the spring a distance $d_{max} = 70\,\text{mm}$, about 28.6 times more, which corresponds to an "equivalent mass" 28.6 times greater than 5 kg, which is to say, 143 kg. Note also that 143 kg is an equivalent weight of 309 pounds, so if you want to try this on a bathroom scale I'd advise you to use smaller weights and drop them from a much smaller height!)

### 6.6.2 Speeding up and slowing down

(a) A 1400-kg car, starting from rest, accelerates to a speed of 30 mph in 10 s. What is the force on the car (assumed constant) over this period of time?
(b) Where does this force comes from? That is, what is the (external) object that exerts this force on the car, and what is the nature of this force?
(c) Draw a free-body diagram for the car. Indicate the direction of motion, and the direction of the acceleration. (d) Now assume that the driver, traveling at 30 mph, sees a red light ahead and

pushes on the brake pedal. Assume that the coefficient of static friction between the tires and the road is $\mu_s = 0.7$, and that the wheels don't "lock": that is to say, they continue rolling without slipping on the road as they slow down. What is the car's minimum stopping distance?
(e) Draw a free-body diagram of the car for the situation in (d). Again indicate the direction of motion, and the direction of the acceleration. (f) Now assume that the driver again wants to stop as in part (c), but he presses on the brakes too hard, so the wheels lock, and, moreover, the road is wet, and the coefficient of kinetic friction is only $\mu_k = 0.2$. What is the distance the car travels now before coming to a stop?

**Solution**

(a) First, let us convert 30 mph to meters per second. There are 1,609 meters to a mile, and 3,600 seconds to an hour, so 30 mph $= 10 \times 1609/3600\,\mathrm{m/s} = 13.4\,\mathrm{m/s}$.

Next, for constant acceleration, we can use Eq. (2.4): $\Delta v = a\Delta t$. Solving for $a$,

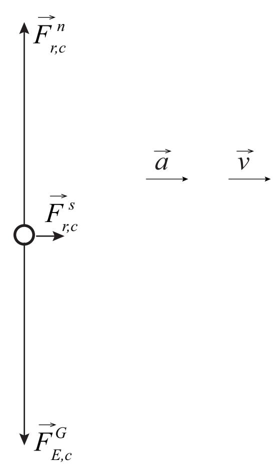
$$a = \frac{\Delta v}{\Delta t} = \frac{13.4\,\mathrm{m/s}}{10\,\mathrm{s}} = 1.34\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

Finally, since $F = ma$, we have

$$F = ma = 1400\,\mathrm{kg} \times 1.34\,\frac{\mathrm{m}}{\mathrm{s}^2} = 1880\,\mathrm{N}$$

(b) The force must be provided by the road, which is the only thing external to the car that is in contact with it. The force is, in fact, the force of *static* friction between the car and the tires. As explained in the chapter, this is a reaction force (the tires push on the road, and the road pushes back). It is *static* friction because the tires are not slipping relative to the road. In fact, we will see in Chapter 9 that the point of the tire in contact with the road has an instantaneous velocity of zero (see Figure 9.8).

(c) This is the free-body diagram. Note the force of static friction pointing *forward*, in the direction of the acceleration. The forces have been drawn to scale.



(d) This is the opposite of part (a): the driver now relies on the force of static friction to *slow down* the car. The shortest stopping distance will correspond to the largest (in magnitude) acceleration, as per our old friend, Eq. (2.10):

$$v_f^2 - v_i^2 = 2a\Delta x \tag{6.38}$$

In turn, the largest acceleration will correspond to the largest force. As explained in the chapter, the static friction force cannot exceed $\mu_s F^n$ (Eq. (6.29)). So, we have

$$F^s_{max} = \mu_s F^n = \mu_s mg$$

since, in this case, we expect the normal force to be equal to the force of gravity. Then

$$|a_{max}| = \frac{F^s_{max}}{m} = \frac{\mu_s mg}{m} = \mu_s g$$

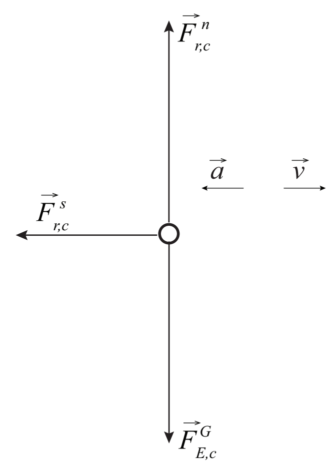
We can substitute this into Eq. (6.38) with a negative sign, since the acceleration acts in the opposite direction to the motion (and we are implicitly taking the direction of motion to be positive). Also note that the final velocity we want is zero, $v_f = 0$. We get

$$-v_i^2 = 2a\Delta x = -2\mu_s g\Delta x$$

From here, we can solve for $\Delta x$:

$$\Delta x = \frac{v_i^2}{2\mu_s g} = \frac{(13.4\,\text{m/s})^2}{2 \times 0.7 \times 9.81\,\text{m/s}^2} = 13.1\,\text{m}$$

(e) Here is the free-body diagram. The interesting feature is that the force of static friction has reversed direction relative to parts (a)–(c). It is also much larger than before. (The forces are again to scale.)




(f) The math for this part is basically identical to that in part (d). The difference, physically, is that now you are dealing with the force of *kinetic* (or "sliding") friction, and that is always given by $F^k = \mu_k F^n$ (this is not an upper limit, it's just what $F^k$ is). So we have $a = -F^k/m = -\mu_k g$, and, just as before (but with $\mu_k$ replacing $\mu_s$),

$$\Delta x = \frac{v_i^2}{2\mu_k g} = \frac{(13.4\,\text{m/s})^2}{2 \times 0.2 \times 9.81\,\text{m/s}^2} = 45.8\,\text{m}$$