Technical Mathematics

Second Edition

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Morgan Chase

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TECHNICAL MATHEMATICS, 2ND EDITION

Morgan Chase

Bob Brown

Contents

Introduction

This developmental-level mathematics textbook is intended for career-technical students. It was made possible by grants from Open Oregon Educational Resources, which supports the development and implementation of high-quality materials at low or no cost to community college and university students.

Changes from the first edition

I won't list every detail, but here are the broad strokes.

- Added Modules 31, 32, 33, 34.
- Revised real-world data and examples from 2019 that felt outdated.
- Deleted or revised complicated explanations and examples.
- Redrew some low-resolution geometry figures.
- Cleaned up the formatting and made the online text easier to navigate.

To the student

This textbook is designed to bring a bit of fun to what can often be a boring or intimidating subject. Many of the examples are taken from everyday life rather than the trades, which helps make the material accessible to students in any field.

There are very few step-by-step examples, because—let's face it—does anyone ever read those? The general structure is a brief introduction of a topic followed by some practice exercises, a brief introduction of the next topic followed by more practice exercises, etc. Therefore, this book is not designed for self-study unless you already have some experience with the material and are simply looking for a refresher.

I have tried to indicate when a calculator would be appropriate, but your instructor has the final say on that. My personal philosophy is that a calculator is a useful tool, but the user needs to know enough to be able to estimate and judge whether a result is reasonable. I don't mention calculators until we get to formulas in Module 7 because I want students to be able to work with fractions, decimals, negatives, exponents, and all that stuff without a calculator if necessary.

In most careers, nobody is going to say you cheated if you use technology such as a calculator or spreadsheet; the important thing is that you use the available technology to get the correct answer. However, you're in school trying to obtain a degree or certificate, which means that you need to demonstrate certain knowledge and skills. Therefore, you need to show that you can (1) do the calculations without technology if necessary, and (2) determine whether an answer given by technology is reasonable or not.

One final thought about learning in general. After going through a particular topic, let's say fractions, you may feel that you understand everything, but there is a big difference between recognizing something when it's shown to you and actually doing it yourself. Think about learning to drive a car; your driving class has a lot of reading and videos and lectures to familiarize you with what you need to know, but it's impossible to learn to drive a car without actually driving the car yourself. Reading about it and watching other people do it is not enough; it is crucial that you actually *practice the techniques yourself* to be sure that you have mastered them. (And then you need to continue practicing the techniques to maintain that mastery, but that's another speech for another time.) Two ways to know that you've really and truly learned something are to either perform it yourself with no guidance or explain it to someone else. If you can follow someone else's explanation of how to add fractions, you're partway there. If you can explain how to add fractions to someone else, though? Now you're in business!

I wish you the best of luck, and I hope you enjoy pictures of cats!

To the instructor

To paraphrase the key points from above:

Many of the examples are taken from everyday life rather than specific trades. At my school, students in this course come from many different CTE programs—welding, machining, automotive, landscaping, horticulture, renewable energy—but not other programs such as nursing, electrical, or wastewater. I also have a few students in other non-CTE programs who don't need a course in the math of CNC machining. You should of course feel free to supplement the textbook with topics you'd like to emphasize… Or heck, this is an OER with a CC-BY-NC-SA 4.0 license; you can copy, revise, and remix it however you like! The only restrictions are that you must credit the author where you use his work and you may not charge money for your product, excluding a reasonable printing cost for a print edition. If you have a Pressbooks account, you can clone the book and then start revising as needed.

There are very few step-by-step worked out examples in the textbook. I believe that they add clutter without adding value. I provide my students with a playlist of YouTube videos that I ask them to watch before class, because watching and listening to someone work a problem—with the ability to pause, rewind, replay—is generally more effective than reading a static example printed on the page. Rather than reading examples that I've typed out step by step, students can actively try the exercises for themselves, with guidance from you.

If you use MyOpenMath for homework assignments, I have good news! I have written and/or collected over 600 questions and organized them into assignments for each module. You can copy course **#181716** and use it as you please.

I shared my personal philosophy on calculators above, but you're the expert on your situation and what your students need. Similarly, you (or your department) will decide how students will be assessed in your course. Will students be allowed to use the textbook or their notes during exams? Will you provide formulas and other reference information? Will a calculator be allowed for some or all of each exam? Will you be giving exams or assigning some kind of mastery worksheets instead? This paragraph is getting beyond the scope of a textbook introduction, but I would ask that you think about what it's important for your students to know how to do, with and without technology, and how they can best demonstrate that knowledge.

To everyone

Whether you are a student or instructor, I would love to hear your thoughts on the book and whether it works well for you. Feel free to let me know about any errors you find or suggestions for improvements. In fact, the trigonometry modules were written at the request of—and in collaboration with—instructors who reached out to let me know what they needed. I had intended to include a module on arc length but ran out of time, energy, and ideas.

Special thanks go out to Bob Brown at Kishwaukee College, who provided many of the diagrams and exercises in Modules 31 & 32.

Extra special thanks go out to my Math 50 students at CCC who have given me first-hand evidence of what worked well in the first edition and what did not. 😅

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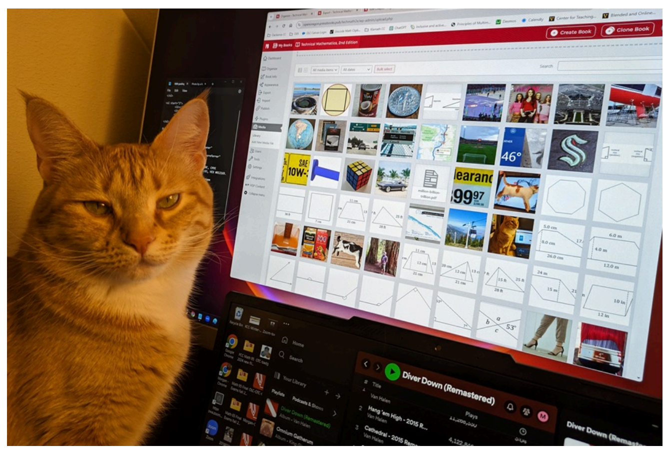
March, 2024



Pepper the cat oversees the revisions.

[1]

Order of Operations

We'll begin with a look at order of operations. In many situations in life, the order in which we perform certain actions is important. For example, if you are putting on your shoes and socks, you need to put a sock on your foot before you put a shoe on that foot. However, if you put on your left sock first, it doesn't really matter whether the next thing you put on is your left shoe or your right sock, as long as you don't try putting on your right shoe next. A multi-step math calculation can be the same way; you might have some steps that need to happen in a specific sequence, but there may be some steps that you can do in whatever order you prefer.

Photo by bastamanography on flickr.

Evaluating an Expression

To ***evaluate*** an expression means to simplify it and find its value.

Exercises

1. Evaluate by performing the addition first: $12 - 2 + 3$
2. Evaluate by performing the subtraction first: $12 - 2 + 3$

When we evaluate an expression, we want to have a single correct answer. It isn't very helpful for the answer to be "maybe 7, or maybe 13". Mathematicians have decided on an order of operations, which tells us which steps should be done before other steps. You can think of them as the rules of the road.

Order of Operations: PEMDAS

P: Work inside of **parentheses** or grouping symbols, following the order PEMDAS as necessary inside the grouping symbols.

E: Evaluate **exponents**.

MD: Perform **multiplications** and **divisions** from left to right.

AS: Perform **additions** and **subtractions** from left to right.

Exercises

Evaluate each expression.

3. $12 - (2 + 3)$
4. $12 - 2 + 3$

Based on Exercises 3 & 4, we can see that Exercise 1 told us to use the wrong order of operations. If there are no parentheses, we must evaluate $12 - 2 + 3$ by *first* performing the subtraction and *then* performing the addition.

Before we move on, you should be aware that there are a handful of ways to show multiplication. All of the following represent 3×4:

$$3 \cdot 4 \qquad 3 * 4 \qquad 3(4) \qquad (3)4 \qquad (3)(4)$$

In this textbook, you will most often see the dot, like $3 \cdot 4$, or parentheses directly next to a number, like $3(4)$. We tend to avoid using the 3×4 symbol because it can be mistaken for the letter x.

Exercises

Evaluate each expression.

5. $12 \div (3 \cdot 2)$

6. $12 \div 3 \cdot 2$

7. $5(1 + 3) - 2$

8. $5(1) + (3 - 2)$

Exponents

An exponent indicates repeated multiplication. For example, $6^2 = 6 \cdot 6 = 36$ and $4^3 = 4 \cdot 4 \cdot 4 = 64$. The exponent tells us how many factors of the base are being multiplied together.

Exercises

Evaluate each expression.

9. $3^2 + 4^2$

10. $(3 + 4)^2$

11. $(7 + 3)(7 - 5)^3$

12. $7 + 3(7 - 5)^3$

Grouping Symbols

In the next set of exercises, the only differences are the parentheses, but every exercise has a different answer.

Exercises

Evaluate each expression.

13. $39 - 7 \cdot 2 + 3$

14. $(39 - 7) \cdot 2 + 3$

15. $39 - (7 \cdot 2 + 3)$

16. $39 - 7 \cdot (2 + 3)$

17. $(39 - 7) \cdot (2 + 3)$

It is possible to have grouping symbols nested within grouping symbols; for example, $7 + (5^2 - (3(17 - 12 \div 4) + 2 \cdot 5) \div 4)$.

To make it somewhat easier to match up the pairs of left and right parentheses, we can use square brackets instead: $7 + (5^2 - [3(17 - 12 \div 4) + 2 \cdot 5] \div 4)$.

Exercises

Simplify the expression.

18. $7 + (5^2 - [3(17 - 12 \div 4) + 2 \cdot 5] \div 4)$

A fraction bar is another grouping symbol; it tells us to perform all of the steps on the top and separately perform all of the steps down below. The final step is to divide the top number by the bottom number.

Exercises

Evaluate each expression.

19. $\frac{15-1}{6+1}$

20. $\frac{(7+2)\cdot 4}{18\div(3+3)}$

21. $\frac{5\cdot 4^2}{2}$

22. $\frac{(5\cdot 4)^2}{2}$

23. $\frac{(5-1)^2}{2+6}$

24. $(5-1)^2 \div 2 + 6$

We will look at formulas in a later module, but let's finish by translating from words to a mathematical expression.

Exercises

25. You can find the approximate Fahrenheit temperature by doubling the Celsius temperature and adding 30. If the temperature is $9°$C, what is the approximate Fahrenheit temperature? Write an expression and evaluate it.

26. You can find the approximate Celsius temperature by subtracting 30 from the Fahrenheit temperature and then dividing by 2. If the temperature is $72°$F, what is the approximate Celsius temperature? Write an expression and evaluate it.

Exercise Answers

[2]

Negative Numbers

Negative numbers are a fact of life, from winter temperatures to our bank accounts. (And occasionally elevators, if they go underground.)

Before we start calculating with negative numbers, we'll take a look at ***absolute value***. This will make it easier for us to talk about what we're doing when we add, subtract, multiply, or divide signed numbers.

Absolute Value

The absolute value of a number is its distance from 0. You can think of it as the size of a number without identifying it as positive or negative. Numbers with the same absolute value but different signs, such as 3 and -3, are called ***opposites***. The absolute value of -3 is 3, and the absolute value of 3 is also 3, because both numbers are 3 units away from 0.

We use a pair of straight vertical bars to indicate absolute value; for example, $|-3| = 3$ and $|3| = 3$.

Exercises

Evaluate each expression.

1. $|-5|$

2. $|5|$

Adding Negative Numbers

To add two negative numbers, add their absolute values (i.e., ignore the negative signs) and make the final answer negative.

Exercises

Perform each addition.

3. $-8+(-7)$

4. $-13+(-9)$

To add a positive number and a negative number, we *subtract* the smaller absolute value from the larger. If the positive number has the larger absolute value, the final answer is positive. If the negative number has the larger absolute value, the final answer is negative.

Exercises

Perform each addition.

5. $7+(-3)$

6. $-7+3$

7. $14 + (-23)$

8. $-14 + 23$

9. The temperature at noon on a chilly Monday was $-7°$F. By the next day at noon, the temperature had risen $25°$F. What was the temperature at noon on Tuesday?

If an expression consists of only additions, we can break the rules for order of operations and add the numbers in whatever order we choose.

Exercises

Evaluate each expression using any shortcuts that you notice.

10. $-10 + 4 + (-4) + 3 + 10$

11. $-291 + 73 + (-9) + 27$

Subtracting Negative Numbers

The following image shows part of a paystub in which an $\$18$ payment needed to be made, but the payroll folks wanted to track the payment in the deductions category. Of course, a positive number in the deductions will subtract money away from the paycheck. Here, though, a deduction of negative 18 dollars has the effect of *adding* 18 dollars to the paycheck. Subtracting a negative amount is equivalent to adding a positive amount.

Deductions after Federal Tax	
Faculty Union Dues	$27.00
Stipend for Part Time Faculty	-$18.00
Workers Compensation Hourly Assessment	$0.13

To subtract two signed numbers, we *add* the first number to the *opposite* of the second number.

Exercises

Perform each subtraction.

12. $5 - 2$

13. $2 - 5$

14. $-2 - 5$

15. $-5 - 2$

16. $2 - (-5)$

17. $5 - (-2)$

18. $-2 - (-5)$

19. $-5 - (-2)$

Absolute Value, Revisited

Absolute value can be useful when we want to find the difference between two numbers but we want the result to be positive. For example, suppose that the temperature in Portland, Oregon is $43°$F, and the temperature in Portland, Maine is $-12°$F. What is the difference in temperature? The simplest way to find the difference is to do $43 - (-12) = 43 + 12 = 55$, and you would report that as a difference of "fifty-five degrees Fahrenheit". If you instead did $-12 - 43 = -55$, it would sound a bit strange to say the the difference is "negative fifty-five degrees Fahrenheit" and you would most likely ignore the negative sign when reporting the difference. To guarantee that the result of a subtraction is positive, we can put absolute value bars around the entire calculation. This is sometimes called the ***positive difference***.

Exercises

Evaluate each expression.

20. $|-12 - 43|$

21. $|43 - (-12)|$

22. The lowest point in Colorado is on the Arikaree River, with an elevation $3,317$ feet above sea level. The highest point in Colorado is the peak of Mount Elbert, with an elevation $14,440$ feet above sea level.[1] Find the positive difference between these elevations.

23. The lowest point in Louisiana is in New Orleans, with an elevation 8 feet below sea level. The highest point in Louisiana is the peak of Driskill Mountain, with an elevation 535 feet above sea level.[2] Find the positive difference between these elevations.

Multiplying Negative Numbers

Suppose you spend 3 dollars on a coffee every day. We could represent spending 3 dollars as a negative number, -3 dollars. Over the course of a 5-day work week, you would spend 15 dollars, which we could represent as -15 dollars. This shows that $-3 \cdot 5 = -15$, or $5 \cdot -3 = -15$.

If two numbers with *opposite* signs are multiplied, the product is negative.

Exercises

Find each product.

24. $-4 \cdot 3$

25. $5(-8)$

Going back to our coffee example, we saw that $5(-3) = -15$. Therefore, the *opposite* of $5(-3)$ must be positive 15. Because -5 is the opposite of 5, this implies that $-5(-3) = 15$.

If two numbers with the *same* sign are multiplied, the product is positive.

WARNING! These rules are different from the rules for addition; be careful not to mix them up.

Exercises

Find each product.

26. $-2(-9)$

27. $-3(-7)$

Recall that an exponent represents a repeated multiplication. Let's see what happens when we raise a negative number to an exponent.

Exercises

Evaluate each expression.

28. $(-2)^2$

29. $(-2)^3$

30. $(-2)^4$

31. $(-2)^5$

If a negative number is raised to an *even* power, the result is positive.
If a negative number is raised to an *odd* power, the result is negative.

Dividing Negative Numbers

Let's go back to the coffee example we saw earlier: $-3 \cdot 5 = -15$. We can rewrite this fact using division and see that $-15 \div 5 = -3$; a negative divided by a positive gives a negative result. Also, $-15 \div -3 = 5$; a negative divided by a negative gives a positive result. This means that the rules for division work exactly like the rules for multiplication.

If two numbers with *opposite* signs are divided, the quotient is negative.
If two numbers with the *same* sign are divided, the quotient is positive.

Exercises

Find each quotient.

32. $-42 \div 6$

33. $32 \div (-8)$

34. $-27 \div (-3)$

35. $0 \div 4$

36. $0 \div (-4)$

37. $4 \div 0$

Go ahead and check those last three exercises with a calculator. Any surprises?

0 divided by another number is 0.

A number divided by 0 is *undefined*, or *not a real number*.

Here's a quick explanation of why $4 \div 0$ can't be a real number. Suppose that there is a mystery number, which we'll call n, such that $4 \div 0 = n$. Then we can rewrite this division as a related multiplication, $n \cdot 0 = 4$. But because 0 times any number is 0, the left side of this equation is 0, and we get the result that $0 = 4$, which doesn't make sense. Therefore, there is no such number n, and $4 \div 0$ cannot be a real number.

Order of Operations with Negative Numbers

P: Work inside of **parentheses** or grouping symbols, following the order PEMDAS as necessary.

E: Evaluate **exponents**.

MD: Perform **multiplications** and **divisions** from left to right.

AS: Perform **additions** and **subtractions** from left to right.

Exercises

Evaluate each expression using the order of operations.

38. $(2-5)^2 \cdot 2+1$

39. $2-5^2 \cdot (2+1)$

40. $[7(-2)+16] \div 2$

41. $7(-2)+16 \div 2$

42. $\frac{1-3^4}{2(5)}$

43. $\frac{(1-3)^4}{2} \cdot 5$

Exercise Answers

Notes

1. Source: https://en.wikipedia.org/wiki/Colorado
2. Source: https://en.wikipedia.org/wiki/Louisiana

[3]

Decimals

Photo by Christian Collins on flickr.

Decimal notation is based on powers of 10: 0.1 is one tenth, 0.01 is one hundredth, 0.001 is one thousandth, and so on.

thousands	hundreds	tens	ones/units	.	tenths	hundredths	thousandths

Exercises

Write each number.

1. ninety and twenty-three hundredths
2. seven and fifty-six thousandths

Adding & Subtracting Decimals

Before you add or subtract decimals, you must line up the decimal points.

Exercises

Add each pair of numbers.

3. $3.75 + 12.8$

4. $71.085 + 112.93$

When subtracting, you may need to add zeros to the first number so you can borrow correctly.

Exercises

Subtract each pair of numbers.

5. $46.57 - 38.29$

6. $82.78 - 67.024$

Multiplying Decimals

To multiply decimal numbers:

1. Temporarily ignore the decimal points.
2. Multiply the numbers as though they are whole numbers.
3. Add the total number of decimal digits in the two numbers you multiplied. The result will have that number of digits to the right of the decimal point.

Note: You do NOT need to line up the decimal points when you are multiplying.

Exercises

Multiply each pair of numbers.

7. $143 \cdot 29$

8. $143 \cdot 2.9$

9. $14.3 \cdot 2.9$

10. $1.43 \cdot 2.9$

11. $375 \cdot 175$

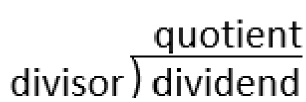
12. $375 \cdot 0.175$

13. $3.75 \cdot 1.75$

14. Evie worked 37.5 hours at a pay rate of $\$17.50$ per hour. How much did she earn in total?

Dividing Decimals

Let's review everyone's favorite topic, long division. The three parts of a division are named as follows: dividend ÷ divisor = quotient. When this is written with a long division symbol, the dividend is inside the symbol, the divisor is on the left, and the quotient is the answer we create on top.



To divide by a decimal:

1. Write in long division form.
2. Move the decimal point of the divisor until it is a whole number.
3. Move the decimal point of the dividend the same number of places to the right.
4. Place the decimal point in the quotient directly above the decimal point in the dividend.
5. Divide the numbers as though they are whole numbers.
6. If necessary, add zeros to the right of the last digit of the dividend to continue.

Exercises

Divide each pair of numbers.

15. $974 \div 4$
16. $974 \div 0.4$
17. $9,740 \div 0.04$
18. $0.0974 \div 0.004$

Rounding Numbers

It is often necessary to round a number to a specified place value. We will see more specific instructions in Modules 5 & 6, but let's review the basics of rounding a number.

Rounding a number:

1. Locate the **rounding digit** in the place to which you are rounding.
2. Look at the **test digit** directly to the right of the rounding digit.
3. If the test digit is 5 or greater, increase the rounding digit by 1 and drop all digits to its right.
4. If the test digit is less than 5, keep the rounding digit the same and drop all digits to its right.

Exercises

Round each number to the indicated place value.

19. $6,375$ (thousands)

20. $6,375$ (tens)

21. 0.7149 (hundredths)

22. 0.7149 (thousandths)

If a decimal answer goes on and on, it may be practical to round it off.

Exercises

23. A subscription to *The Chicago Manual of Style Online* costs $\$44.00$. Determine the monthly cost, rounded to the nearest cent.

24. In the summer of 1919, a military convoy (including Lt. Col. Dwight Eisenhower) drove from Washington, D.C. to San Francisco to assess the condition of the nation's developing highway system. The journal entry for August 1 says "Good dirt roads. Made 82 miles in 11 hrs."[1] What was the convoy's effective speed in miles per hour for that day? Round your result to the nearest tenth.

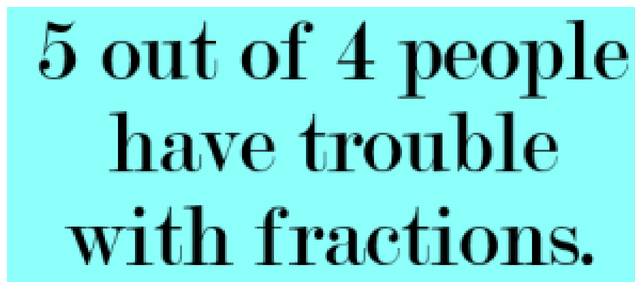
Exercise Answers

Notes

1. Source: https://after-ike.com/logbook-1919-transcontinental-military-convoy/. See https://www.nytimes.com/2019/07/07/opinion/the-most-important-road-trip-in-american-history.html if you're interested in the historical context.

[4]

Fractions



Working with fractions is one of the most hated/feared/avoided topics in lower-level mathematics. If you've always struggled with fractions, now is the time to face them. Don't avoid them and hope they'll go away. (They won't.) We'll start with the basics of what fractions are and proceed from there.

Writing Fractions

A fraction describes equal parts of a whole: $\frac{\text{part}}{\text{whole}}$

Using official math vocabulary: $\frac{\text{numerator}}{\text{denominator}}$

Exercises

The month of April had 11 rainy days and 19 days that were not rainy.

1. What fraction of the days were rainy?
2. What fraction of the days were not rainy?

Simplifying Fractions

Two fractions are ***equivalent*** if they represent the same number. (The same portion of a whole.) To build an equivalent fraction, multiply the numerator and denominator by the same number.

Exercises

3. Write $\frac{4}{5}$ as an equivalent fraction with a denominator of 15.

4. Write $\frac{2}{3}$ as an equivalent fraction with a denominator of 12.

Many fractions can be ***simplified***, or reduced. Here are four special cases.

Exercises

Simplify each fraction, if possible.

5. $\frac{7}{1}$

6. $\frac{7}{7}$

7. $\frac{0}{7}$

8. $\frac{7}{0}$

A fraction is ***completely reduced***, or in ***simplest form***, or in ***lowest terms***, when the numerator and denominator have no common factors other than 1. To reduce a fraction, divide the numerator and denominator by the same number.

Exercises

Reduce each fraction to simplest form.

9. $\frac{9}{12}$

10. $\frac{10}{6}$

Multiplying Fractions

To multiply fractions, multiply the numerators and multiply the denominators straight across. If possible, simplify your answer.

Exercises

Multiply each pair of numbers. Be sure that each answer is in simplest form.

11. $8 \cdot \frac{1}{4}$

12. $\frac{6}{7} \cdot \frac{7}{12}$

13. $\frac{5}{8} \cdot \frac{2}{3}$

14. $\frac{6}{5} \cdot \frac{10}{12}$

To find a fraction **of** a number, multiply.

Exercises

15. To pass his workplace training, Nathan must correctly answer at least $\frac{9}{10}$ of 50 questions. How many questions must he answer correctly to pass the training?

Dividing Fractions

To divide by a fraction, multiply by the reciprocal of the second number. (Flip the second fraction upside-down.)

Exercises

Divide. Be sure that each answer is in simplest form.

16. $12 \div \frac{3}{4}$

17. $\frac{3}{10} \div \frac{1}{2}$

18. Suppose you need to measure 2 cups of flour, but the only scoop you can find is $\frac{1}{3}$ cup. How many scoops of flour will you need?

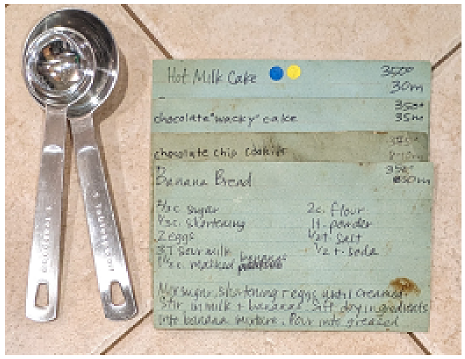
Comparing Fractions

If two fractions have the same denominator, we can simply compare their numerators.

If two fractions have different denominators, we can rewrite them with a common denominator and then compare their numerators.

Exercises

19. Banana bread recipe A requires $\frac{3}{4}$ cup of sugar, whereas banana bread recipe B requires $\frac{2}{3}$ cup of sugar. Which recipe requires more sugar?



Adding & Subtracting Fractions

To add or subtract two fractions with the same denominator, add or subtract the numerators and keep the common denominator.

Exercises

20. Jack ate $\frac{3}{8}$ of a pizza. Mack ate $\frac{1}{8}$ of the pizza. What fraction of the pizza did they eat together?

21. Tracy ate $\frac{5}{6}$ of a pizza. Stacy ate $\frac{1}{6}$ of the pizza. How much more of the pizza did Tracy eat?

To add or subtract two fractions with different denominators, first write them with a common denominator. Then add or subtract them.

Exercises

A $\frac{3}{8}$-inch thick sheet of plywood is going to be laid onto a $\frac{1}{4}$-inch thick sheet of plywood.

22. What is the combined thickness of the two sheets?

23. What is the difference in thickness of the two sheets of plywood?

Jacqueline budgets $\frac{1}{4}$ of her monthly income for food and $\frac{1}{3}$ of her monthly income for rent.

24. What fraction of her monthly income does she budget for these two expenses combined?

25. What fraction more of her monthly income does she budget for her rent than for her food?

Fractions and Decimals

To write a fraction as a decimal, divide the numerator by the denominator.

A decimal that ends (eventually has a remainder of 0) is called a terminating decimal. Fun fact: If the denominator of a fraction has no prime factors other than 2's and 5's, the decimal will terminate. Also, the fraction can be built up to have a denominator of 10, or 100, or $1,000$...

Exercises

Write each fraction as a decimal.

26. $\frac{11}{4}$

27. $\frac{7}{20}$

A decimal that continues a pattern of digits is called a repeating decimal. We can represent the repeating digits by using either an overbar or ellipsis (three dots)...

Exercises

Write each fraction as a decimal.

28. $\frac{5}{9}$

29. $\frac{18}{11}$

Improper Fractions & Mixed Numbers

A fraction which has a larger numerator than denominator is called an ***improper fraction***. Because an improper fraction is larger than 1, it can also be written as a ***mixed number***, with a whole number followed by a fractional part.

Keep in mind that a mixed number represents an addition, not a multiplication. For example, $6\frac{2}{3}$ means $6 + \frac{2}{3}$, not $6 \cdot \frac{2}{3}$.

To write an improper fraction as a mixed number:

1. Divide the numerator by the denominator to get the whole number part.
2. The remainder after dividing is the new numerator.
3. Keep the same denominator.

Exercises

Rewrite each improper fraction as a mixed number.

30. $\frac{23}{2}$

31. $\frac{14}{3}$

To write a mixed number as an improper fraction:

1. Multiply the whole number part by the denominator.
2. Add this result to the original numerator to get the new numerator.
3. Keep the same denominator.

Exercises

Rewrite each mixed number as an improper fraction.

32. $2\frac{1}{5}$

33. $6\frac{2}{3}$

Adding or subtracting mixed numbers can be fairly simple or more complicated, depending on the numbers. One approach is to work with the fractional parts separately from the whole numbers. For example, $5\frac{2}{3} - 3\frac{1}{2}$ can be rewritten as $5 + \frac{2}{3} + (-3) + (-\frac{1}{2})$ and rearranged to $[5 + (-3)] + [\frac{2}{3} + (-\frac{1}{2})]$. Then $5 + (-3) = 2$ and, with a little more work, $\frac{2}{3} + (-\frac{1}{2}) = \frac{1}{6}$, so the result is $2\frac{1}{6}$.

Exercises

34. $7\frac{5}{8} + 2\frac{3}{4}$

35. $7\frac{5}{8} - 2\frac{3}{4}$

Multiplying or dividing mixed numbers is more complicated than it may appear. Change any mixed numbers into improper fractions before doing the calculation, then change the answer back to a mixed number if possible.

Exercises

36. Multiply: $3\frac{1}{2} \cdot 2\frac{1}{3}$

37. $5\frac{1}{2}$ cups of water will be divided equally into 3 jars. How much water will go into each jar?

Exercise Answers

[5]

Accuracy and Significant Figures

Photo by Tudor Barker on flickr.

In the first few modules, we rarely concerned ourselves with rounding; we assumed that every number we were told was exact and we didn't have to worry about any measurement error. However, every measurement contains some error. A standard sheet of paper is 8.5 inches wide and 11 inches high, but it's possible that the actual measurements could be closer to 8.4999 and 11.0001 inches. Even if we measure something very carefully, with very sensitive instruments, we should assume that there could be some small measurement error.[1]

Exact Values and Approximations

A number is an ***exact value*** if it is the result of counting or a definition.

A number is an ***approximation*** if it is the result of a measurement or of rounding.

Exercises

Identify each number as an exact value or an approximation.

1. An inch is $\frac{1}{12}$ of a foot.
2. This board is 78 inches long.
3. There are 14 students in class.
4. A car's tachometer reads $3,000$ rpm.
5. A right angle measures $90°$.
6. The angle of elevation of a ramp is $4°$.

Accuracy and Significant Figures

African-American women were vital to NASA's success in the 1960s, as shown in the movie *Hidden Figures*. Photo by NASA/Kim Shiflett on flickr.

Because measurements are inexact, we need to consider how accurate they are. This requires us to think about ***significant figures***—often abbreviated "sig figs" in conversation—which are the digits in the measurement that we trust to be correct. The ***accuracy*** of a number is equal to the number of significant figures. (By the way, the terms "significant digits" and "significant figures" are used interchangeably.) The following rules aren't particularly difficult to understand but they can take time to absorb and internalize, so we'll include lots of examples and exercises.

Significant Figures

1. All nonzero digits are significant.
   Ex: $12,345$ has five sig figs, and 123.45 has five sig figs.
2. All zeros between other nonzero digits are significant.
   Ex: $10,045$ has five sig figs, and 100.45 has five sig figs.
3. Any zeros to the right of a decimal number are significant.
   Ex: 123 has three sig figs, but 123.00 has five sig figs.
4. Zeros on the left of a decimal number are NOT significant.
   Ex: 0.123 has three sig figs, and 0.00123 has three sig figs.
5. Zeros on the right of a whole number are NOT significant unless they are marked with an overbar.
   Ex: $12,300$ has three sig figs, but $12,30\overline{0}$ has five sig figs.

Another way to think about #4 and #5 above is that zeros that are merely showing the place value—where the decimal point belongs—are NOT significant.

Exercises

Determine the accuracy (i.e., the number of significant figures) of each number.

7. $63,400$
8. $63,040$
9. $63,004$
10. 0.085
11. 0.0805
12. 0.08050

In 1856, the first official measurement of the height of Mount Everest—called *Sagarmatha* in Nepal and *Chomolungma* in Tibet—was announced. The height was determined to be exactly $29,000$ feet, but there was concern that people would think this was only a rough estimate rounded to the nearest thousand feet. Therefore, the height was announced as $29,002$ feet, so that everyone seeing that number would believe that the measurement was correct to the nearest foot.[2] Yes, to demonstrate the correctness of the measurement, an incorrect measurement was announced.

Mt. Everest, Lohtse, and Nupse in the early morning. Photo by [Ralf Kayser]() on [flickr]().

Instead of fudging a number like $29,000$ to show that it is correct to the nearest foot, we can write it with an an ***overbar*** to indicate that the zeros are significant. Putting $29,00\overline{0}$ in a newspaper headline in 1856 would probably have confused people, but you can handle it because you're in a math class. Writing $29,00\overline{0}$ is our way of saying "Really, to the nearest foot, it's exactly $29,000$ feet!"

Exercises

Determine the accuracy (i.e., the number of significant figures) of each number.

13. $29,000$

14. $29,\overline{0}00$

15. $29,0\overline{0}0$

16. $29,00\overline{0}$

Two things to remember: we don't put an overbar over a nonzero digit, and we don't need an overbar for any zeros on the right of a decimal number because those are already understood to be significant.

Accuracy-Based Rounding

As we saw in Module 3, it is often necessary to round a number. We often round to a certain place value, such as the nearest hundredth, but there is another way to round. ***Accuracy-based rounding*** considers the number of significant figures rather than the place value.

Accuracy-based rounding:

1. Locate the **rounding digit** to which you are rounding by counting from the left until you have the correct number of significant figures.
2. Look at the **test digit** directly to the right of the rounding digit.
3. If the test digit is 5 or greater, increase the rounding digit by 1 and drop all digits to its right. If the test digit is less than 5, keep the rounding digit the same and drop all digits to its right.

Exercises

Round each number so that it has the indicated number of significant figures.

17. $51,837$ (three sig figs)

18. $51,837$ (four sig figs)

19. 4.2782 (two sig figs)

20. 4.2782 (three sig figs)

When the rounding digit of a whole number is a 9 that gets rounded up to a 0, we must write an overbar above that 0.

Similarly, when the rounding digit of a decimal number is a 9 that gets rounded up to a 0, we must include the 0 in that decimal place.

Exercises

Round each number so that it has the indicated number of significant figures. Be sure to include trailing zeros or an overbar if necessary.

21. $13,997$ (two sig figs)

22. $13,997$ (three sig figs)

23. 2.596 (two sig figs)

24. 2.596 (three sig figs)

The height of Mount Everest has changed over the years due to plate tectonics and earthquakes. In December 2020, it was jointly announced by Nepal and China that the summit of Mount Everest has an elevation of $29,031.69$ ft.[3]

25. Round $29,031.69$ ft to two sig figs.

26. Round $29,031.69$ ft to three sig figs.

27. Round $29,031.69$ ft to four sig figs.

28. Round $29,031.69$ ft to five sig figs.

29. Round $29,031.69$ ft to six sig figs.

Accuracy when Multiplying and Dividing

Suppose you needed to square the number $3\frac{1}{3}$. You could rewrite $3\frac{1}{3}$ as the improper fraction $\frac{10}{3}$ and then figure out that $(\frac{10}{3})^2 = \frac{100}{9}$, which equals the repeating decimal $11.111...$

Because most people prefer decimals to fractions, we might decide to round $3\frac{1}{3}$ to 3.33 and find that $3.33^2 = 11.0889$. The answer 11.0889 looks very accurate, but it is a false accuracy because there is round-off error involved. Only when we round to three sig figs do we get an accurate result: 11.0889 rounded to three sig figs is 11.1, which is accurate because $11.111...$ rounded to three sig figs is also 11.1. It turns out that because 3.33 has only three significant figures, our answer must be rounded to three significant figures.

When ***multiplying or dividing*** approximate numbers, the answer must be rounded to the same number of significant figures as the ***least*** accurate of the original numbers.

Don't round off the original numbers; do the necessary calculations first, then round the answer as your last step.

Exercises

Use a calculator to multiply or divide as indicated. Then round to the appropriate level of accuracy.

30. $8.75 \cdot 12.25$

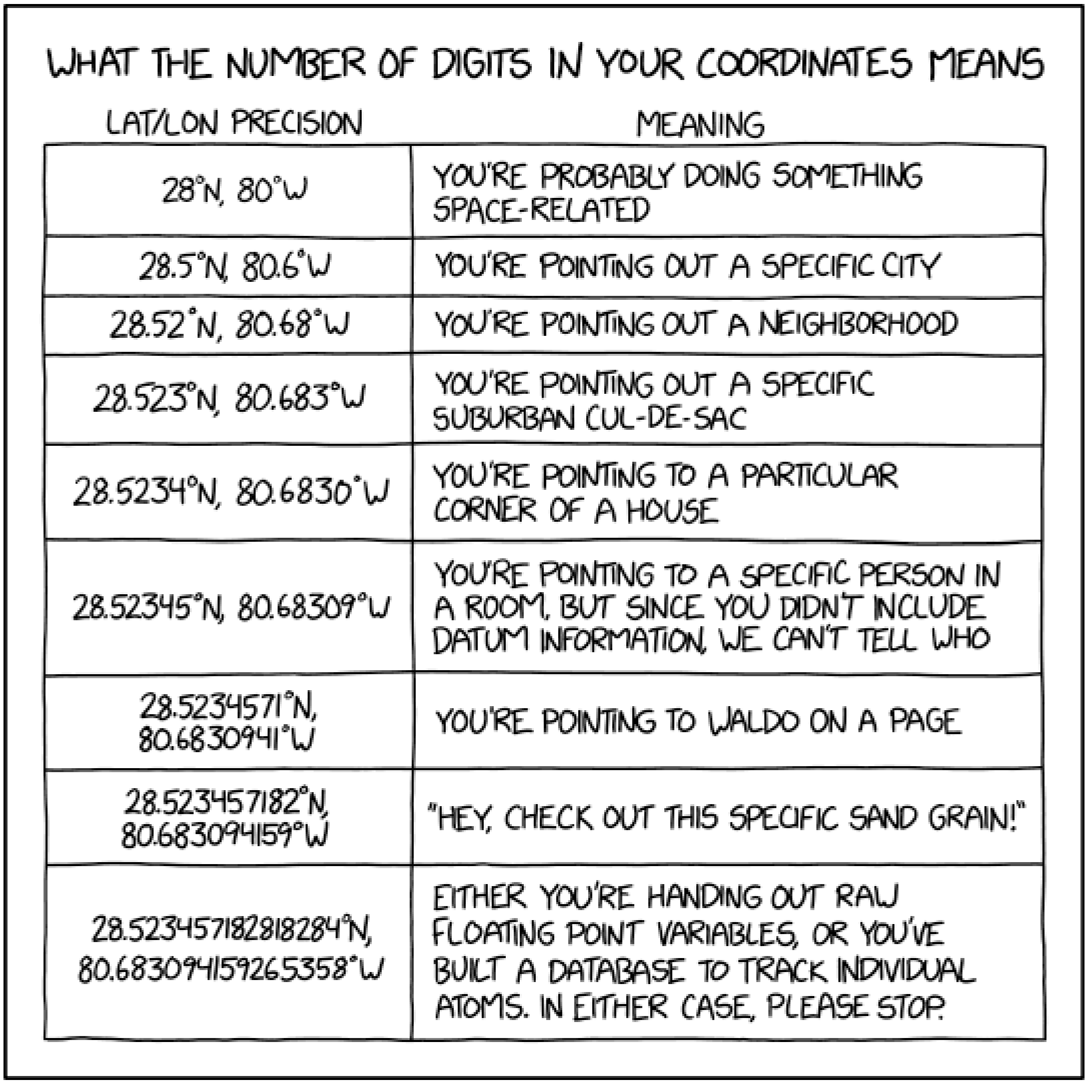
31. $355.12 \cdot 1.8$

32. $77.3 \div 5.375$

33. $53.2 \div 4.5$

34. Suppose you are filling a 5-gallon can of gasoline. The gasoline costs $\$4.579$ per gallon, and you estimate that you will buy 5.0 gallons. How much should you expect to spend?

Bonus material: Here is a comic strip from xkcd.com showing that including a lot of decimal digits can give a false sense of accuracy.



From the web comic xkcd.

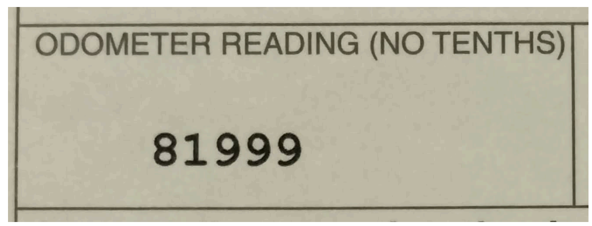
Exercise Answers

Notes

1. Here is an example of a speed camera with gigantic measurement error: https://youtube.com/shorts/cCYy29RuhV8
2. Source: https://en.wikipedia.org/wiki/Mount_Everest#cite_ref-tas1982_33-0
3. Source: https://www.washingtonpost.com/world/asia_pacific/mount-everest-height-nepal-china/2020/12/08/a7b3ad1e-389a-11eb-aad9-8959227280c4_story.html

[6]

Precision and GPE



When someone is selling a used car, its mileage might be listed as 80,000 miles or 82,000 miles because a buyer will want to know the approximate mileage but doesn't need an exact value. If you buy the car, however, you'll need to know the mileage to the nearest mile when you're completing the registration paperwork.

Precision

The ***precision*** of a number is the place value of the rightmost significant figure. For example, $82,000$ is precise to the thousands place, $81,999$ is precise to the ones place, and something like $81,999.2$ would be precise to the tenths place.

Exercises

Identify the precision (i.e., the place value of the rightmost significant figure) of each number.

1. $29,000$

2. $29,\bar{0}00$

3. $29,030$

4. 0.037
5. 0.0307
6. 0.03070

Precision-Based Rounding

In Module 3, we used ***precision-based rounding*** because we were rounding to a specified place value; for example, rounding to the nearest tenth. Let's practice this with overbars and trailing zeros.

Precision-based rounding:

1. Locate the **rounding digit** in the place to which you are rounding.
2. Look at the **test digit** directly to the right of the rounding digit.
3. If the test digit is 5 or greater, increase the rounding digit by 1 and drop all digits to its right. If the test digit is less than 5, keep the rounding digit the same and drop all digits to its right.

Remember, when the rounding digit of a whole number is a 9 that gets rounded up to a 0, we must write an overbar above that 0.

Also, when the rounding digit of a decimal number is a 9 that gets rounded up to a 0, we must include the 0 in that decimal place.

Exercises

Round each number to the indicated place value. Be sure to include an overbar or trailing zeros if necessary.

7. $81,999$ (thousands)

8. $81,999$ (hundreds)

9. $81,999$ (tens)

10. 0.5996 (tenths)

11. 0.5996 (hundredths)

12. 0.5996 (thousandths)

Precision when Adding and Subtracting

Suppose the attendance at a large event is estimated at $25,000$ people, but then you see 3 people leave. Is the new estimate $24,997$? No, because the original estimate was precise only to the nearest thousand. We can't start with an imprecise number and finish with a more precise number. If we estimated that $1,000$ people had left, then we could revise our attendance estimate to $24,000$ because this estimate maintains the same level of precision as our original estimate.

> When ***adding or subtracting*** numbers with different levels of precision, the answer must be rounded to the same precision as the ***least*** precise of the original numbers.

Don't round off the original numbers; do the necessary calculations first, then round the answer as your last step.

Exercises

Add or subtract as indicated. Round to the appropriate level of precision.

13. Find the combined weight of four packages with the following weights: 9.7 lb, 13.0 lb, 10.5 lb, 6.1 lb.

14. Find the combined weight of four packages with the following weights: 9.7 lb, 13 lb, 10.5 lb, 6.1 lb.

15. While purchasing renter's insurance, Chandra estimates the value of her insurable possessions at $\$10,200$. After selling some items valued at $\$375$, what would be the revised estimate?

16. Chandra knows that she has roughly $\$840$ in her checking account. After using her debit card to make two purchases of $\$25.95$ and $\$16.38$, how much would she have left in her account?

Before we move on, let's circle back to multiplication and division again. If you are multiplying by an exact number, you can consider this a repeated addition. For example, suppose you measure the weight of an object to be 4.37 ounces and you want to know the weight of three of these objects; multiplying 4.37 times 3 is the same as adding $4.37 + 4.37 + 4.37 = 13.11$ ounces. The answer is still precise to the hundredths place. When working with an exact number, we can assume that it has infinitely many significant figures. (Treat exact numbers like royalty; their accuracy is perfect and it would be an insult to even question it.)

Greatest Possible Measurement Error (GPE)

Suppose you are weighing a dog with a scale that displays the weight rounded to the nearest pound. If the scale says Sir Barks-A-Lot weighs 23 pounds, he could weigh anywhere from 22.5 pounds to almost 23.5 pounds. The true weight could be as much as 0.5 pounds above or below the measured weight, which we could write as 23 ± 0.5.

Now suppose you are weighing Sir Barks-A-Lot with a scale that displays the weight rounded to the nearest tenth of a pound. If the scale says Sir Barks-A-Lot weighs 23.0 pounds, we now know that he could weigh anywhere from 22.95 pounds to almost 23.05 pounds. The true weight could be as much as 0.05 pounds above or below the measured weight, which we could write as 23.0 ± 0.05.

As we increase the level of precision in our measurement, we decrease the ***greatest possible measurement error*** or ***GPE***. The GPE is always one half the precision; if the precision is to the nearest tenth, 0.1, the GPE is half of one tenth, or five hundredths, 0.05. The GPE will always be a 5 in the place to the right of the place value of the number's precision.

Another way to think about the GPE is that it gives the range of values that would round off to the number in question. 23 ± 0.5 tells us a lower value and an upper value. $23 - 0.5 = 22.5$ is the lowest weight that would round up to 23, and $23 + 0.5 = 23.5$ is the upper limit of the weights that would round down to 23. (Yes, 23.5 technically would round up to 24, but it is easier to just say 23.5 instead of 23.49.) Using inequalities, we could represent 23 ± 0.5 as the range of values $22.5 \leq \text{weight} < 23.5$.

Exercises

The attendance at a Portland Thorns match is estimated to be $14,000$ people.

17. What is the precision of this estimate?

18. What is the greatest possible error in this estimate?

A roll of plastic sheeting is 0.00031 inches thick.

19. What is the precision of this measurement?

20. What is the greatest possible error in this measurement?

Plastic sheeting 0.00031 inches thick is referred to as 0.31 mil.

21. What is the precision of this measurement, in mils?

22. What is the greatest possible measurement error, in mils?

Summary of Accuracy, Precision, GPE

Here is a summary of the important terms from Modules 5 & 6. It is easy to get them mixed up, but remembering that "precision" and "place value" both start with "p" can be helpful.

Summary of Terms

Significant figures: the digits in a number that we trust to be correct

Accuracy: the number of significant figures

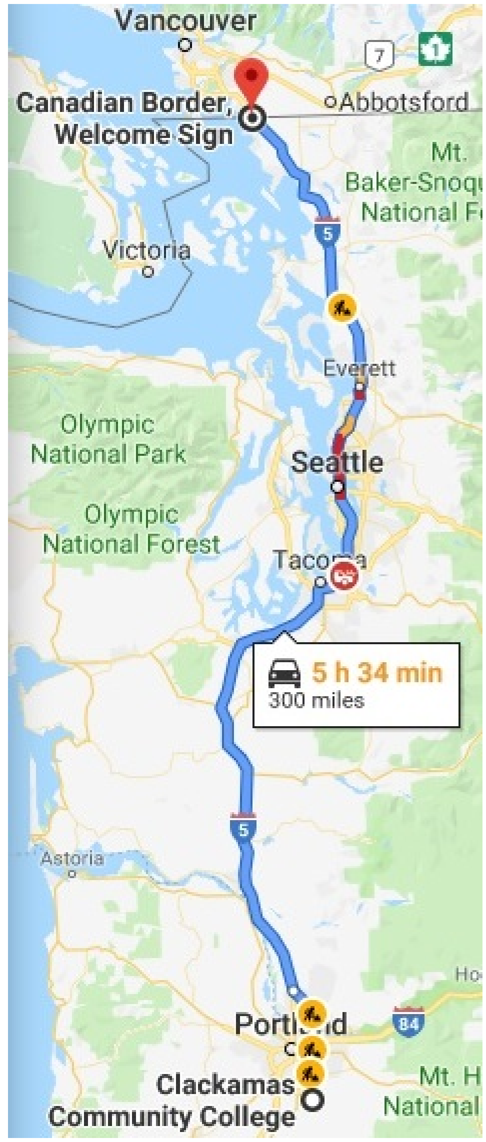
When multiplying or dividing, we use the accuracy to round the result.

Precision: the place value of the rightmost significant digit

Greatest possible measurement error (GPE): one half the precision

When adding or subtracting, we use the precision to round the result.

Exercises



Google Maps says that the driving distance from CCC's main campus to the Canadian border, rounded to the nearest mile, is 300 miles.

23. Write the distance with an overbar showing the correct precision.

24. What is the precision?

25. What is the GPE?

Google Maps says that the driving time, rounded to the nearest minute, is 5 hours and 34 minutes.

26. What is the precision of this estimate?

27. What is the GPE of this estimate?

Of course, the traffic conditions will change during the trip, making the estimate of 5 hours and 34 minutes unrealistically precise. Let's assume that the drive will take 5.5 hours.

28. What is the *accuracy* of this estimate?

29. What is the *accuracy* of the distance?

30. Calculate the average speed of the vehicle, rounded appropriately.

31. Why did we need to consider the accuracy instead of the precision for this calculation?

Exercise Answers

[7]

Formulas

You may use a calculator throughout this module if needed.

Photo by Rainier Ridao on Unsplash.

A formula is an equation or set of calculations that takes a number (or numbers) as input, and produces an output. The output is often a number, but it could also be a decision such as yes or no.

Each unknown number in a formula is called a ***variable*** because its value can vary. A variable is usually represented with a letter of the alphabet. To evaluate a formula, we substitute a number (or numbers) into the formula and then perform the steps using the order of operations.

Formulas with One Input

Many formulas will have just one input variable. Note: When a number is written directly next to a variable, it indicates multiplication. For example, $0.24w$ means $0.24 \cdot w$.

Exercises

The formula $C = 0.24w + 1.26$ gives the cost, in dollars, of mailing a large envelope weighing w ounces through the USPS.[1]

1. Find the cost of mailing a 6-ounce envelope.

2. Find the cost of mailing a 12-ounce envelope.

Radio Cab charges the following rates for a taxi ride: a fixed fee of $\$3.80$ to get in the taxi, plus a rate of $\$2.80$ per mile.[2] The total cost, in dollars, of a ride m miles long can be represented by the formula $C = 3.80 + 2.80m$.

3. Find the cost of a 5-mile ride.

4. Find the cost of a 7.5-mile ride.

5. Find the cost of getting in the taxi, then changing your mind and getting out without riding anywhere.

The number of members a state has in the U.S. House of Representatives can be approximated by the formula $R = P \div 0.76$, where P is the population in millions.[3] The 2020 populations of three states are as follows:[4]

Oregon	4.2 million
Washington	7.7 million
California	39.6 million

Round all answers to the nearest whole number.

6. How many U.S. Representatives does Oregon have?

7. How many U.S. Representatives does Washington have?

8. How many U.S. Representatives does California have?

The number of electoral votes a state has can be approximated by the formula $E = P \div 0.76 + 2$, where P is the population in millions.

9. How many electoral votes does Oregon have?

10. How many electoral votes does Washington have?

11. How many electoral votes does California have?

Temperature Conversions

You probably know that 32° Fahrenheit is the freezing point of water, and 212° Fahrenheit is the boiling point of water. The Celsius equivalents are 0°C and 100 $^\circ$C. The formulas shown below allow us to convert between the two temperature scales.

Temperature Formulas

$$F = \frac{9}{5}C + 32$$

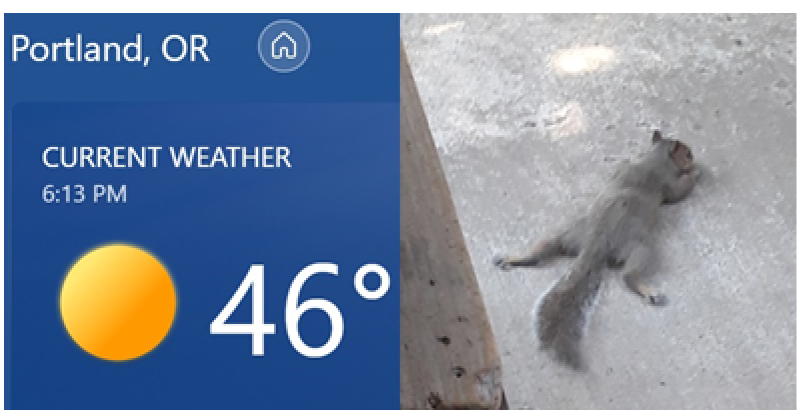
$$F = 1.8C + 32$$

$$C = \frac{5}{9}(F - 32)$$

$$C = (F - 32) \div 1.8$$

Exercises

12. The record high temperature in Portland, Oregon occurred during the “heat dome” event in June, 2021. As shown, it was 46°C at your author’s house and the squirrels were laying low. Convert this temperature to Fahrenheit.

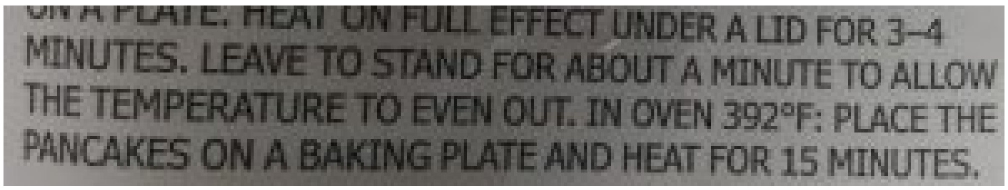


Based on the evidence, the melting point of squirrel is right around 46°C.

13. The traditionally accepted "normal" body temperature of a human[5] is $98.6°$F. What is this temperature in Celsius?

14. The FDA recommends that a freezer be set below $-18°$C. What is the Fahrenheit equivalent?

15. A package of frozen pancakes from IKEA calls for the oven to be set to 392 °F. IKEA is based in Sweden, and this temperature clearly was originally calculated in Celsius. What is the corresponding Celsius temperature?



Formulas with More than One Input

Some formulas have more than one input variable. Just pay attention to which number goes in for each variable.

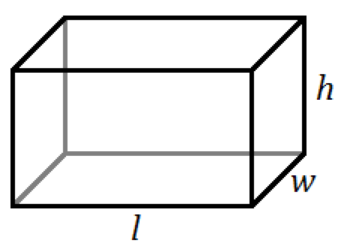
Exercises

When a patient's blood pressure is checked, they are usually told two numbers: the systolic blood pressure (SBP) and the diastolic blood pressure (DBP). The mean arterial pressure (MAP) can be estimated by the following formula: $MAP = \frac{SBP+2\cdot DBP}{3}$. (The units are mm Hg.) Calculate the mean arterial pressure for each patient.

16. SBP = 120, DBP = 75

17. SBP = 140, DBP = 90

UPS uses the following formula[6] to determine the "measurement" of a package with length l, width w, and height h: $m = l + 2w + 2h$. Determine the measurement of a package with the following dimensions.



18. length 18 inches, width 12 inches, height 14 inches

19. length 16 inches, width 14 inches, height 15 inches

Formulas with a Yes or No Answer

Some formulas give a yes or no answer: success or failure, approved or disapproved, etc. After calculating the result from the formula, we need to compare it to a given number to see whether the result is within a specified range.

Exercises

In Australia, a chicken egg is designated “large” if its mass, in grams, satisfies the following formula: $|m - 54.1| \leq 4.1$. Determine whether each egg qualifies as large.[7]

20. Egg 1’s mass is 57.8 grams.
21. Egg 2’s mass is 58.3 grams.
22. Egg 3’s mass is 49.8 grams.
23. Egg 4’s mass is 50.0 grams.

Exercise Answers

Notes

1. Source: https://pe.usps.com/text/dmm300/Notice123.htm#_c037
2. Source: https://www.radiocab.net/services-radio-cab/
3. The value 0.76 comes from dividing the total U.S. population in 2020, around 331 million people, by the 435 seats in the House of Representatives.
4. Source: https://www.census.gov/data/tables/2020/dec/2020-apportionment-data.html
5. See this article for a counterargument: https://www.nytimes.com/2023/10/12/well/live/fever-normal-body-temperature.html
6. Source: https://www.ups.com/us/en/help-center/packaging-and-supplies/prepare-overize.page
7. Source: https://en.wikipedia.org/wiki/Chicken_egg_sizes

[8]

Perimeter and Circumference

You may use a calculator throughout this module if needed.

Perimeter

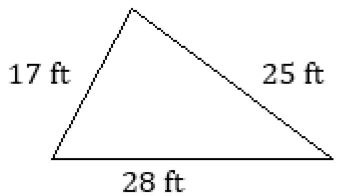
A ***polygon*** is a closed geometric figure with straight sides. Common polygons include triangles, squares, rectangles, parallelograms, trapezoids, pentagons, hexagons, octagons… Just as a perimeter fence runs along the outside edge of a region, the ***perimeter*** of a polygon is the total distance around the outside. In general, to find the perimeter of a polygon, you can add up the lengths of all of its sides.

Photo by Hermes Rivera on Unsplash

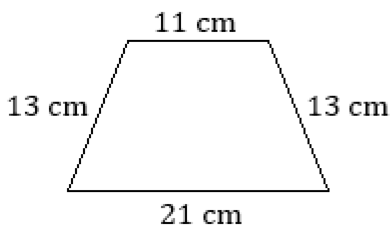
Also, if you haven't already, now is the time to get in the habit of including units in your answers.

Exercises

1. Find the perimeter of the triangle.



2. Find the perimeter of the trapezoid.



If we know that some of the sides of a polygon are equal, we can use a formula as an alternative to adding up all of the lengths individually. The first formula shown below uses the variable s for the side of a square. The rectangle formulas use l for length and w for width, or b for base and h for height; these terms are interchangeable.

Perimeter Formulas

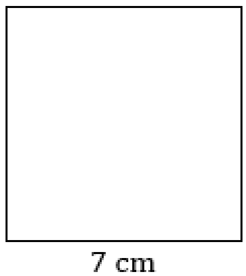
Square: $P = 4s$

Rectangle: $P = 2l + 2w$ or $P = 2b + 2h$

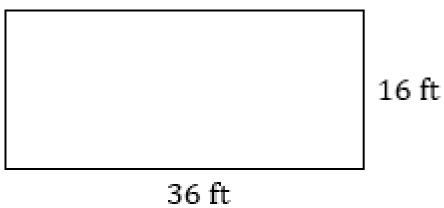
Rectangle: $P = 2(l + w)$ or $P = 2(b + h)$

Exercises

3. Find the perimeter of the square.



4. Find the perimeter of the rectangle.



5. A storage area, which is a rectangle that is 45 feet long and 20 feet wide, needs to be fenced around all four sides. How many feet of fencing is required? (To keep it simple, ignore any gates or other complications.)

6. Giancarlo is putting crown molding around the edge of the ceiling of his living room. If the room is a 12-foot by 16-foot rectangle, how much crown molding does he need?

The sides of a ***regular polygon*** are all equal in length. Therefore, multiplying the length of a side by the number of sides will give us the perimeter.

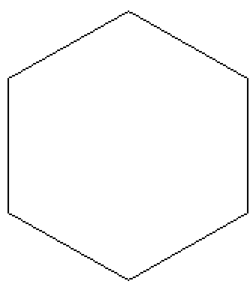
Perimeter Formula

Regular Polygon with n sides of length s: $P = n \cdot s$

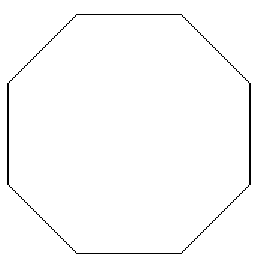
Exercises

Find the perimeter of each regular polygon.

7. Each side of the hexagon is 4 inches long.



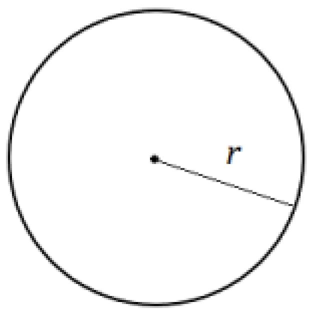
8. Each side of the octagon is 2.5 centimeters long.

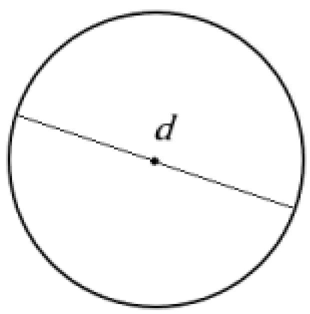


Circumference

The distance around the outside of a circle is called the ***circumference***, rather than the perimeter. Let's review some circle vocabulary before moving on.

Every point on a circle is the same distance from its center. This distance from the center to the edge of the circle is called the ***radius***. The distance from one edge to another, through the center of the circle, is called the ***diameter***. As you can see, the diameter is twice the length of the radius.





Throughout history, different civilizations have discovered that the circumference of a circle is slightly more than 3 times the length of its diameter. (By the year 2000 BCE, the Babylonians were using the value $3\frac{1}{8} = 3.125$ and the Egyptians were using the value $3\frac{13}{81} \approx 3.1605$.)[1] The value $3\frac{1}{7} \approx 3.1429$ is an even better approximation for the ratio of the circumference to the diameter. However, the actual value cannot be written as an exact fraction; it is the irrational number π, pronounced "pie", which is approximately 3.1416.

Circumference Formulas

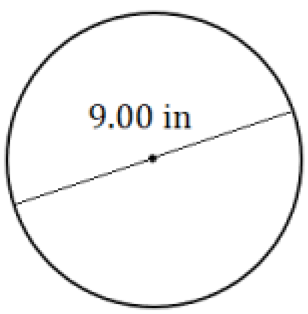
$C = \pi d$

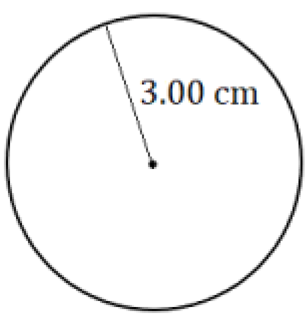
$C = 2\pi r$

Any scientific calculator will have a π key; using this will give you the most accurate result, although you should be sure to round your answer appropriately. (Remember from Module 5 that we need to pay attention to significant figures when multiplying or dividing.) Many people use 3.14 as an approximation for π, but this can lead to round-off error. If you must use an approximation for π, use 3.1416.

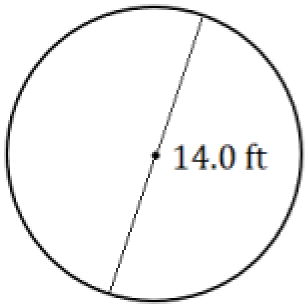
Exercises

Calculate the circumference of each circle. Round each answer to the appropriate level of accuracy.

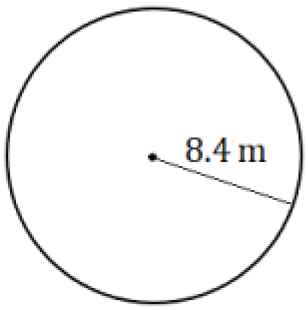
9. 

10. 

11.



12.



Exercise Answers

Notes

1. This information comes from Chapter 1 of the book *A History of Pi* by Petr Beckmann. It is a surprisingly interesting read.

[9]

Percents Part 1

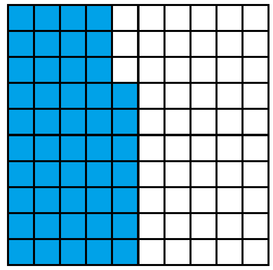
In this module, we will look at the basics of percents, and how percents are related to fractions and decimals. Then we will solve some straightforward percent problems.

We'll return to percents in modules 12 and 27 and solve more complicated problems. This will give you a chance to develop your skills gradually without getting confused.

Percent Basics

The word ***percent*** means "per one hundred". You can think of a percent as a fraction with a denominator of 100.

Exercises



1. What percent of the squares are shaded blue?
2. What percent of the squares are not shaded blue?

To write a percent as a fraction: drop the percent sign, write the number over 100, and simplify if possible. (Notice that if a percent is greater than 100%, the fraction will be greater than 1, and if a percent is less than 1%, the fraction will be less than $\frac{1}{100}$.)

Exercises

Write each percent as a fraction, and simplify if possible.

3. About 71% of Earth's surface is covered by water.[1]
4. About 1.3% of Earth's land surface is permanent cropland.[2]
5. About 0.04% of Earth's atmosphere is carbon dioxide.[3]

Photo by NASA on Unsplash

6. The number of HVACR technician jobs in the U.S. in 2032 is predicted to be 106% of the number of jobs in 2022.[4]

To write a percent as a decimal: drop the percent sign and move the decimal point two places to the left. (Notice that if the percent is not a whole number, the decimal will extend beyond the hundredths place.)

Exercises

Write each percent from Exercises 3 through 6 as a decimal.

7. 71%
8. 1.3%
9. 0.04%
10. 106%

To write a decimal as a percent: move the decimal point two places to the right and insert a percent sign.

Exercises

Write each decimal number as a percent.

11. 0.23
12. 0.07
13. 0.085
14. 2.5

To write a fraction as a percent, write the fraction as a decimal by dividing the numerator by the denominator, then move the decimal point two places to the right and insert a percent sign.

Alternate method: If the denominator of the fraction is a factor of 100, it can easily be built up to have a denominator of 100.

Exercises

15. 7 out of 25 students were tardy on Wednesday. Write $\frac{7}{25}$ as a percent.

16. A package of 24 m&m's contained 3 orange m&m's. Write $\frac{3}{24}$ as a percent.

Solving Percent Problems: Finding the Amount

You may use a calculator for the remainder of this module if needed.

We often use the words ***amount*** and ***base*** in a percent problem. The ***amount*** is the answer we get after finding the percent of the original number. The ***base*** is the original number, the number we find the percent of. (You may also think of the amount as the part, and the base as the whole.) We can call the percent the ***rate***.

$$\text{Amount} = \text{Rate} \cdot \text{Base}$$

$$A = R \cdot B$$

Be sure to change the percent to a decimal before multiplying.

Exercises

17. What is 9% of 350?
18. 30% of 75 is what number?
19. Find 13.5% of 500.
20. 125% of 80 is equal to what amount?
21. What number is 40% of 96.5?
22. Calculate 0.5% of 450.

Suppose you buy an electric drill with a retail price of $\$109.97$ in a city with 8.5% sales tax.

23. Find the amount of the tax. Round to the nearest cent, if necessary.
24. How much do you pay in total?

Exercise Answers

Notes

1. Source: https://en.wikipedia.org/wiki/Earth#Surface
2. Source: https://en.wikipedia.org/wiki/Earth#Surface
3. Source: https://en.wikipedia.org/wiki/Atmosphere_of_Earth#Composition
4. Source: Bureau of Labor Statistics, U.S. Department of Labor, Occupational Outlook Handbook, Heating, Air Conditioning, and Refrigeration Mechanics and Installers, at https://www.bls.gov/ooh/installation-maintenance-and-repair/heating-air-conditioning-and-refrigeration-mechanics-and-installers.htm

[10]

Ratios, Rates, Proportions

As we saw in the previous module, a percent is a fraction that compares a number to 100. This is an example of a ratio. In this module, we will focus on ratios and rates, which look like fractions, and then we'll finish up with proportions, which look like two fractions set equal to each other. Many of the examples in this module can be worked without a calculator, while others are best done with a calculator because you'll be multiplying big numbers or dividing messy decimals. You can ask your instructor for guidance if you aren't sure whether a calculator is appropriate.

Ratios & Rates

A ***ratio*** is the quotient of two numbers or the quotient of two quantities with the same units. (Pop quiz... Which operation gives you a quotient: addition, subtraction, multiplication, or division?)

When writing a ratio as a fraction, the first quantity is the numerator and the second quantity is the denominator. If the fraction can be simplified, reduce it to lowest terms.

Exercises

1. Find the ratio of 45 minutes to 2 hours. Simplify the fraction, if possible.

A ***rate*** is the quotient of two quantities with different units. You must include the units.

When writing a rate as a fraction, the first quantity is the numerator and the second quantity is the denominator. If the fraction can be simplified, reduce it to lowest terms.

Exercises

2. A car travels 105 miles in 2 hours. Write the rate as a fraction.

Unit Rates

For practical purposes, expressing a rate as a reduced fraction is not always useful. Expressing a rate as a single number with units such as miles per hour is often more meaningful. This is called a ***unit rate*** because it expresses the quantity in the numerator that corresponds to *one* unit of the denominator.

To find the unit rate, divide the numerator by the denominator and express the rate as a mixed number or decimal. The units can be expressed with the word "per": miles per hour, dollars per gallon, grams per deciliter, pounds per square inch, and so on.

Exercises

3. A car travels 105 miles in 2 hours. Write the car's average speed as a unit rate.

A ***unit price*** is a rate with the price in the numerator and a denominator equal to 1. The unit price tells the cost of one unit or one item. To find the unit price, divide the cost by the size or number of items.

You may use a calculator to calculate the unit price.

Exercises

4. A 15.8-ounce box of Carmella Creeper cereal costs \$5.29. Determine the unit price.

5. An 18.8-ounce box of Count Chocula cereal costs \$5.29. Determine the unit price.

6. The Monster Cereals are on sale, 2 boxes for \$9. If you buy a 16-ounce box of Franken Berry cereal and a 16-ounce box of Boo Berry cereal, what is the unit price per ounce?

Proportions

A ***proportion*** says that two ratios (or rates) are equal.

Exercises

Determine whether each proportion is true or false by simplifying each fraction.

7. $\frac{6}{8} = \frac{21}{28}$

8. $\frac{10}{15} = \frac{16}{20}$

A common method of determining whether a proportion is true or false is called ***cross-multiplying*** or finding the ***cross products***. We multiply diagonally across the equal sign. In a true proportion, the cross products are equal.

$$\frac{a}{b} = \frac{c}{d} \rightarrow \; a \cdot d = b \cdot c$$

Exercises

Determine whether each proportion is true or false by cross-multiplying.

9. $\frac{6}{8} = \frac{21}{28}$

10. $\frac{10}{15} = \frac{16}{20}$

11. $\frac{14}{4} = \frac{15}{5}$

12. $\frac{0.8}{4} = \frac{5}{25}$

As we saw in Module 7, we can use a ***variable*** to stand for a missing number. If a proportion has a missing number, we can use cross multiplication to solve for the missing number. This is as close to algebra as we get in this textbook.

To solve a proportion for a variable:

1. Set the cross products equal to form an equation of the form $a \cdot d = b \cdot c$.
2. Isolate the variable by rewriting the multiplication equation as a division equation.
3. Check the solution by substituting the answer into the original proportion and finding the cross products.

You may discover slightly different methods that you prefer. If you think "Hey, can't I do this a different way?", you're probably right.

Exercises

Solve for the variable.

13. $\frac{8}{10} = \frac{x}{15}$

14. $\frac{3}{2} = \frac{7.5}{n}$

15. $\frac{3}{k} = \frac{18}{24}$

16. $\frac{w}{6} = \frac{15}{9}$

17. $\frac{5}{4} = \frac{13}{x}$

18. $\frac{3.2}{7.2} = \frac{m}{4.5}$ (calculator recommended)

Problems that involve rates, ratios, scale models, etc. can be solved with proportions. When solving a real-world problem using a proportion, be consistent with the units.

You may use a calculator for the remainder of this module if needed.

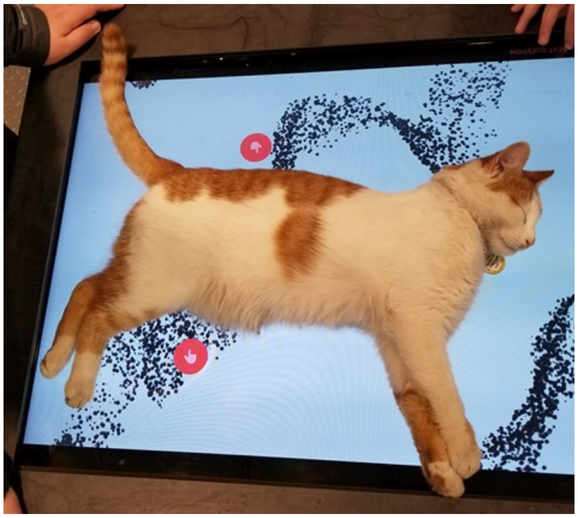
Exercises

19. Tonisha drove her car 320 miles and used 12.5 gallons of gas. At this rate, how far could she drive using 10 gallons of gas?

20. Marcus worked 14 hours and earned $\$210$. At the same rate of pay, how long would he have to work to earn $\$300$?

21. Taylor is trying to figure out the straight-line distance from Portland to Mt. Hood. On their map, $\frac{3}{4}$ inch represents 5 miles. The distance between Portland and Mt. Hood on the map is pretty close to 7 inches. What is the actual distance?

22. Púki the cat lives in a Reykjavík bookstore and likes to sleep on a warm tabletop video screen. Suppose you have a picture of Púki that is 285 pixels wide and 255 pixels high but you need to reduce it in size so that it is 170 pixels high. If the height and width are kept proportional, what is the width of the picture after it has been reduced?



That's pronounced "pookee", not "pyookee".

Exercise Answers

[11]

Scientific Notation

Powers of Ten

Decimal notation is based on powers of 10: 0.1 is $\frac{1}{10^1}$, 0.01 is $\frac{1}{10^2}$, 0.001 is $\frac{1}{10^3}$, and so on.

We represent these powers with negative exponents: $\frac{1}{10^1} = 10^{-1}$, $\frac{1}{10^2} = 10^{-2}$, $\frac{1}{10^3} = 10^{-3}$, etc.

Negative exponents: $\frac{1}{10^n} = 10^{-n}$

Note: This is true for any base, not just 10, but we will focus only on 10 in this course.

With our base 10 number system, any power of 10 can be written as a 1 in a certain decimal place.

10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$10,000$	$1,000$	100	10	1	0.1	0.01	0.001	0.0001

If you haven't watched the video "Powers of Ten" from 1977 on YouTube, take ten minutes right now and check it out. Your mind will never be the same again.

Scientific Notation

Let's consider how we could rewrite some different numbers using these powers of 10.

Let's take 50,000 as an example. 50,000 is equal to $5 \times 10,000$ or 5×10^4.

Looking in the other direction, a decimal such as 0.0007 is equal to 7×0.0001 or 7×10^{-4}.

The idea behind scientific notation is that we can represent very large or very small numbers in a more compact format: a number between 1 and 10, multiplied by a power of 10.

> A number is written in scientific notation if it is written in the form $a \times 10^n$, where n is an integer and a is any real number such that $1 \leq a < 10$.

Note: An **integer** is a number with no fraction or decimal part: … -3, -2, -1, 0, 1, 2, 3 …

Although we generally try to avoid using the "x" shaped multiplication symbol, it is frequently used with scientific notation.

Exercises

1. Earth and Mars are two of the smaller planets. Earth has a mass of approximately 5,970,000,000,000,000,000,000,000 kilograms, and Mars has a mass of approximately 639,000,000,000,000,000,000,000 kilograms. Can you determine which mass is larger?

Clearly, it is difficult to keep track of all those zeros. Let's rewrite those huge numbers using scientific notation.

Exercises

2. Earth has a mass of approximately 5.97×10^{24} kilograms, and Mars has a mass of approximately 6.39×10^{23} kilograms. Can you determine which mass is larger?

It is much easier to compare the powers of 10 and determine that the mass of the Earth is larger because it has a larger power of 10. You may be familiar with the term ***order of magnitude***; this simply refers to the difference in the powers of 10 of the two numbers. Earth's mass is one order of magnitude larger because 24 is 1 more than 23.

Exercises

Suppose someone tells you their salary is "six figures".

3. To the nearest dollar, what is their minimum possible salary? Write the answer in standard notation and in scientific notation.

4. To the nearest dollar, what is their maximum possible salary? Write the answer in standard notation and in scientific notation.

We can apply scientific notation to small decimals as well.

Exercises

5. The radius of a hydrogen atom is approximately 0.000000000053 meters. The radius of a chlorine atom is approximately 0.00000000018 meters. Can you determine which radius is larger?

Again, keeping track of all those zeros is a chore. Let's rewrite those decimal numbers using scientific notation.

Exercises

6. The radius of a hydrogen atom is approximately 5.3×10^{-11} meters. The radius of a chlorine atom is approximately 1.8×10^{-10} meters. Can you determine which radius is larger?

The radius of the chlorine atom is larger because it has a larger power of 10; the digits 1 and 8 for chlorine begin in the tenth decimal place, but the digits 5 and 3 for hydrogen begin in the eleventh decimal place.

Scientific notation is very helpful for really large numbers, like the mass of a planet, or really small numbers, like the radius of an atom. It allows us to do calculations or compare numbers without going cross-eyed counting all those zeros.

Exercises

Write each of the following numbers in scientific notation.

7. $1,234$

8. $10,200,000$

9. 0.000870

10. 0.0732

Convert the following numbers from scientific notation to standard decimal notation.

11. 3.5×10^4

12. 9.012×10^7

13. 8.25×10^{-3}

14. 1.4×10^{-5}

Multiplying & Dividing with Scientific Notation

You may be familiar with a shortcut for multiplying numbers with zeros on the end; for example, to multiply $300 \times 4,000$, we can multiply the significant digits $3 \times 4 = 12$ and count up the total number of zeros, which is five, and write five zeros on the back end of the 12: $1,200,000$. This shortcut can be applied to numbers in scientific notation.

To multiply powers of 10, add the exponents: $10^m \cdot 10^n = 10^{m+n}$

Exercises

Multiply each of the following and write the answer in scientific notation.

15. $(2 \times 10^3)(4 \times 10^4)$

16. $(5 \times 10^4)(7 \times 10^8)$

17. $(3 \times 10^{-2})(2 \times 10^{-3})$

18. $(8 \times 10^{-5})(6 \times 10^9)$

When the numbers get messy, it's probably a good idea to use a calculator. If you are dividing numbers in scientific notation with a calculator, you may need to use parentheses carefully. The following rule is true, but you may just want to use a calculator instead.

To divide powers of 10, subtract the exponents: $10^m \div 10^n = 10^{m-n}$

Exercises

19. New Jersey has the highest population density of the 50 states.[1] Its population is 9.29×10^6 people and its land area is 7.35×10^3 square miles. Divide these numbers to find the population density in people per square mile.

20. California has the highest population density of all states west of the Mississippi River.[2] Its population is 3.90×10^7 people and its land area is 1.56×10^5 square miles. Divide these numbers to find the population density in people per square mile.

21. The mass of a proton is 1.67×10^{-27} kg. The mass of an electron is 9.11×10^{-31} kg. Divide these numbers to determine approximately how many times greater the mass of a proton is than the mass of an electron.

Engineering Notation

Closely related to scientific notation is ***engineering notation***, which uses powers of $1,000$. This is the way large numbers are often reported in the news; if roughly $37,000$ people live in Oregon City, we say "thirty-seven thousand" and we might see it written as "37 thousand"; it would be unusual to think of it as $3.7 \times 10,000$ and report the number as "three point seven ten thousands".

One thousand = 10^3, one million = 10^6, one billion = 10^9, one trillion = 10^{12}, and so on.

In engineering notation, the power of 10 is always a multiple of 3, and the other part of the number must be between 1 and $1,000$.

A number is written in engineering notation if it is written in the form $a \times 10^n$, where n is a multiple of 3 and a is any real number such that $1 \leq a < 1,000$.

Note: Prefixes for large numbers such as kilo, mega, giga, and tera are essentially engineering notation, as are prefixes for small numbers such as micro, nano, and pico. We'll see these in *Module 16*.

Exercises

Write each number in engineering notation, then in scientific notation.

22. The U.S. population is around 335.9 million people.[3]

23. The world population is around 8.020 billion people.[4]

24. The U.S. national debt is around 33.9 trillion dollars.[5]

---

25. Divide the U.S. national debt by the U.S. population to determine the amount of debt per person.

For a visualization of the relative sizes of a million, a billion, and a trillion, see this graphic made by Chris Kane, graphic artist and my friend from high school (PDF file).

Exercise Answers

Notes

1. Source: https://en.wikipedia.org/wiki/New_Jersey
2. Source: https://en.wikipedia.org/wiki/List_of_states_and_territories_of_the_United_States_by_population_density
3. Retrieved from *https://www.census.gov/popclock/*, January 1, 2024

4. Retrieved from *https://www.census.gov/popclock/*, January 1, 2024

5. Retrieved from *https://fiscaldata.treasury.gov/datasets/debt-to-the-penny/*, January 1, 2024

[12]

Percents Part 2 and Error Analysis

You may use a calculator throughout this module.

How could anyone pass up that 0.005% discount?

Recall: The ***amount*** is the answer we get after finding the percent of the original number. The ***base*** is the original number, the number we find the percent of. We can call the percent the ***rate***.

When we looked at percents in Module 9, we focused on finding the amount. In this module, we will practice finding the percentage rate and the base.

$$\text{Amount} = \text{Rate} \cdot \text{Base}$$

$$A = R \cdot B$$

We can translate from words into algebra.

- "is" means equals
- "of" means multiply
- "what" means a variable

Solving Percent Problems: Finding the Rate

Suppose you earned 56 points on a 60-point quiz. To figure out your grade as a percent, you need to answer the question "56 is what percent of 60?" We can translate this sentence into the equation $56 = R \cdot 60$.

Exercises

1. 56 is what percent of 60?
2. What percent of 120 is 45?

Be aware that this method gives us the answer in decimal form and we must move the decimal point to convert the answer to a percent.

Also, if the instructions don't explicitly tell you how to round your answer, use your best judgment: to the nearest whole percent or nearest tenth of a percent, to two or three significant figures, etc.

Solving Percent Problems: Finding the Base

Suppose you earn 2% cash rewards for the amount you charge on your credit card. If you want to earn $\$50$ in cash rewards, how much do you need to charge on your card? To figure this out, you need to answer the question "50 is 2% of what number?" We can translate this into the equation $50 = 0.02 \cdot B$.

Exercises

3. $\$50$ is 2% of what dollar value?
4. 5% of what number is 36?

Solving Percent Problems: Using Proportions

Recall that a percent is a ratio, a fraction out of 100. Instead of translating word for word as we just were, we can set up a proportion with the percentage rate over 100. Because the base is the original amount, it corresponds to 100%.

$$\frac{\text{amount}}{\text{base}} = \frac{\text{percent}}{100}$$

Let's try Exercises 1 through 4 again, using proportions.

Exercises

5. 56 is what percent of 60?
6. What percent of 120 is 45?
7. $\$50$ is 2% of what dollar value?
8. 5% of what number is 36?

Now that we have looked at both methods, you are free to use whichever method you prefer: percent equations or proportions.

Exercises

9. The University of Oregon women's basketball team made 13 of the 29 three-points shots they attempted during a game against UNC. What percent of their three-point shots did the team make?
10. A bottle of Dr. Pepper contains 65 grams of added sugars, which is 129% of the recommended daily intake.[1] What is the recommended daily intake?

Solving Percent Problems: Percent Increase

When a quantity changes, it is often useful to know by what percent it changed. If the price of a candy bar is increased by 50 cents, you might be annoyed because it's it's a relatively large percentage of the original price. If the price of a car is increased by 50 cents, though, you wouldn't care because it's such a small percentage of the original price.

To find the percent of increase:

1. Subtract the two numbers to find the amount of increase.
2. Using this result as the amount and the **original** number as the base, divide and find the unknown percent.

Notice that we always use the ***original*** number for the base, the number that occurred earlier in time. In the case of a percent increase, this is the smaller of the two numbers.

Exercises

11. The price of a candy bar increased from $0.89 to $1.39. By what percent did the price increase?
12. Your author bought an overpriced t-shirt at a Seattle Kraken hockey game with a retail price of $40.00. Including sales tax, the actual cost was $44.04. What was the sales tax rate?

Solving Percent Problems: Percent Decrease

Finding the percent decrease in a number is very similar.

To find the percent of decrease:

1. Subtract the two numbers to find the amount of decrease.
2. Using this result as the amount and the **original** number as the base, divide and find the unknown percent.

Again, we always use the ***original*** number for the base, the number that occurred earlier in time. For a percent decrease, this is the larger of the two numbers.

Exercises

13. During a sale, the price of a candy bar was reduced from $\$1.39$ to $\$0.89$. By what percent did the price decrease?

14. The estimated population of Portland, Oregon in April 2020 was $652,500$. The estimated population in July 2022 was $635,100$. Find the percent of decrease in the population, to the nearest tenth of a percent.[2]

To summarize, we can determine the percent change using the following formula, which works whether we're finding a percent increase or a percent decrease.

$$\text{percent change} = \frac{|\text{new} - \text{original}|}{\text{original}} \cdot 100$$

Relative Error

In Module 5, we said that a measurement will always include some error, no matter how carefully we measure. It can be helpful to consider the size of the error relative to the size of what is being measured. As we saw in the examples above, a difference of 50 cents is important when we're pricing candy bars but insignificant when we're pricing cars. In the same way, an error of an eighth of an inch could be a deal-breaker when you're trying to fit a screen into a window frame, but an eighth of an inch is insignificant when you're measuring the length of your garage.

The ***expected outcome*** is what the number would be in a perfect world. If a window screen is supposed to be exactly 25 inches wide, we call this the expected outcome, and we treat it as though it has infinitely many significant digits. In theory, the expected outcome is $25.000000...$

To find the ***absolute error***, we subtract the measurement and the expected outcome. Because we always treat the expected outcome as though it has unlimited significant figures, the absolute error should have the same precision (place value) as the *measurement*, not the expected outcome.

To find the ***relative error***, we divide the absolute error by the expected outcome. We usually express the relative error as a percent. In fact, the procedure for finding the relative error is identical to the procedures for finding a percent increase or percent decrease!

To find the relative error:

1. Subtract the two numbers to find the absolute error.
2. Using the **absolute error** as the amount and the **expected outcome** as the base, divide and find the unknown percent.

Exercises

15. A window screen is measured to be $25\frac{3}{16}$ inches wide instead of the advertised 25 inches. Determine the relative error, rounded to the nearest tenth of a percent.

16. The contents of a box of cereal are supposed to weigh 10.8 ounces, but they are measured at 10.67 ounces. Determine the relative error, rounded to the nearest tenth of a percent.

The following formula has the same structure as the percent change formula we saw earlier.

$$\text{percent error} = \frac{|\text{measured} - \text{expected}|}{\text{expected}} \cdot 100$$

Tolerance

The ***tolerance*** is the maximum amount that a measurement is allowed to differ from the expected outcome. For example, the U.S. Mint needs its coins to have a consistent size and weight so that they will work in vending machines. A dime (10 cents) weighs 2.268 grams, with a tolerance of ± 0.091 grams.[3] This tells us that the minimum acceptable weight is $2.268 - 0.091 = 2.177$ grams, and the maximum acceptable weight is $2.268 + 0.091 = 2.359$ grams. A dime with a weight outside of the range $2.177 \leq \text{weight} \leq 2.359$ would be unacceptable.

Exercises

A U.S. nickel (5 cents) weighs 5.000 grams with a tolerance of ± 0.194 grams.

17. Determine the lowest acceptable weight and highest acceptable weight of a nickel.
18. Determine the relative error of the weight of a nickel with an absolute error of 0.194 grams.

A U.S. quarter (25 cents) weighs 5.670 grams with a tolerance of ± 0.227 grams.

19. Determine the lowest acceptable weight and highest acceptable weight of a quarter.
20. Determine the relative error of the weight of a quarter with an absolute error of 0.227 grams.

Exercise Answers

Notes

1. Source (PDF file): https://www.fda.gov/media/135299/download
2. Source: https://www.census.gov/quickfacts/fact/table/portlandcityoregon/PST045222
3. Sources: https://www.usmint.gov/learn/coin-and-medal-programs/coin-specifications and https://www.thesprucecrafts.com/how-much-do-coins-weigh-4171330

[13]

The US Measurement System

You may use a calculator throughout this module if needed.

Robin the cat has the spirit of '76.

The U.S. customary system of measurement developed from the system used in England centuries ago.[1] To convert from one unit to another, we often have to perform messy calculations like dividing by 16 or multiplying by $5,280$.

We could solve these unit conversions using proportions, but there is another method that is more versatile, especially when a conversion requires more than one step. This method goes by various names, such as *dimensional analysis* or the *factor label method*. The basic idea is to begin with the measurement you know, then multiply it by a conversion ratio that will cancel the units you don't want and replace it with the units you do want.

It's okay if you don't have the conversion ratios memorized; just be sure to have them available. If you discover other conversion ratios that aren't provided here, go ahead and write them down!

U.S. System: Measurements of Length

1 foot = 12 inches

1 yard = 3 feet

1 mile = $5,280$ feet

Let's walk through two examples to demonstrate the process.

Suppose you're a fan of Eminem[2] or the Byrds[3] and you're curious about how many feet are in 8 miles. We can start by writing 8 mi as a fraction over 1 and then use the conversion ratio $1 \text{ mi} = 5,280 \text{ ft}$ to cancel the units.

$$\frac{8 \text{ mi}}{1} \cdot \frac{5280 \text{ ft}}{1 \text{ mi}} = 8 \cdot 5280 \text{ ft} = 42,240 \text{ ft}$$

Now suppose that you want to convert a measurement from feet to miles. (Maybe you're watching *The Twilight Zone* episode "Nightmare at 20,000 Feet"[4] and wondering how many miles high that is.) We'll start by writing 20000 ft as a fraction over 1 and then use the conversion ratio $1 \text{ mi} = 5,280 \text{ ft}$ to cancel the units.

$$\frac{20000 \text{ ft}}{1} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} = \frac{20000}{5280} \text{ mi} \approx 3.8 \text{ mi}$$

As it happens, the first situation became a multiplication problem but the second situation became a division problem. Rather than trying to memorize rules about when you'll multiply versus when you'll divide, just set up the conversion ratio so the units will cancel out and then the locations of the numbers will tell you whether you need to multiply or divide them.

Exercises

1. How many inches are in 4.5 feet?
2. How many feet make up 18 yards?
3. 1 yard is equal to how many inches?

4. 1 mile is equivalent to how many yards?
5. How many feet is 176 inches?
6. 45 feet is what length in yards?
7. Convert $10,560$ feet into miles.
8. How many yards are the same as $1,080$ inches?

Notice that Exercises 3 & 4 give us two more conversion ratios that we could add to our list.

U.S. System: Measurements of Weight or Mass

1 pound = 16 ounces

1 ton = $2,000$ pounds

The procedure is the same; start with the measurement you know and write it as a fraction over 1. Then write the conversion factor so that the units you don't want will cancel out.

Exercises

9. How many ounces are in 2.5 pounds?
10. How many pounds are equivalent to 1.2 tons?
11. Convert 300 ounces to pounds.
12. 1 ton is equivalent to what number of ounces?

U.S. System: Measurements of Volume or Capacity

1 cup = 8 fluid ounces

1 pint = 2 cups

1 quart = 2 pints

1 gallon = 4 quarts

There are plenty of other conversions that could be provided, such as the number of fluid ounces in a gallon, but let's keep the list relatively short.

Exercises

13. How many fluid ounces are in 6 cups?

14. How many pints are in 3.5 quarts?

15. 1 gallon is equal to how many pints?

16. How many cups equal 1.25 quarts?

17. Convert 20 cups into gallons.

18. How many fluid ounces are in one half gallon?

U.S. System: Using Mixed Units of Measurement

Measurements are frequently given with mixed units, such as a person's height being given as 5 ft 7 in instead of 67 in, or a newborn baby's weight being given as 8 lb 3 oz instead of 131 oz. This can sometimes make the calculations more complicated, but if you can convert between improper fractions and mixed numbers, you can handle this.

Exercises

19. A bag of apples weighs 55 ounces. What is its weight in pounds and ounces?

20. A carton of orange juice contains 59 fluid ounces. Determine its volume in cups and fluid ounces.

21. A hallway is 182 inches long. Give its length in feet and inches.

22. The maximum loaded weight of a Ford F-150 pickup truck is $8,500$ lb. Convert this weight into tons and pounds.

We'll finish up this module by adding and subtracting with mixed units. Again, it may help to think of them as mixed numbers, with a whole number part and a fractional part.

Exercises

Comet weighs 8 lb 7 oz and Fred weighs 11 lb 9 oz.

23. Comet and Fred are being put into a cat carrier together. What is their combined weight?

24. How much heavier is Fred than Comet?

Two tables are 5 ft 3 in long and 3 ft 10 in long.

25. If the two tables are placed end to end, what is their combined length?

26. What is the difference in length between the two tables?

Exercise Answers

Notes

1. Want some Wikipedia rabbit holes? Visit https://en.wikipedia.org/wiki/United_States_customary_units and https://en.wikipedia.org/wiki/English_units.
2. https://en.wikipedia.org/wiki/8_Mile_(film)
3. https://en.wikipedia.org/wiki/Eight_Miles_High
4. https://en.wikipedia.org/wiki/Nightmare_at_20,000_Feet

[14]

The Metric System

You will NOT need a calculator for this module.

The metric system was first implemented following the French Revolution; if we're overthrowing the monarchy, why should we use a unit of a "foot" that is based on the length of a king's foot?

The metric system was designed to be based on the natural world, and different units are related to each other by powers of 10 instead of weird numbers like 3, 12, 16, and $5,280$... This makes converting between metric units incredibly easy because all we need to do is move the decimal point.

The table below shows the most common metric prefixes. The prefixes are arranged in order so that we can convert between them simply by moving the decimal point the same number of places shown in the table.

kilo- (k)	hecta- (h)	deka- (da)	**[base unit]**	deci- (d)	centi- (c)	milli- (m)
$1,000$	100	10	1	0.1	0.01	0.001
$1,000$	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1{,}000}$
10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}

Because deka- and deci- both start with d, the abbreviation for deka- is da.

Metric System: Measurements of Length

The base unit of length is the ***meter***, which is a bit longer than a yard (three feet). Because the prefix ***kilo-*** means one thousand, 1 kilometer is $1,000$ meters. (One kilometer is around six tenths of a mile.) Similarly, because the prefix ***centi-*** means one hundredth, 1 centimeter is $\frac{1}{100}$ of a meter, or 1 meter is 100 centimeters. (One centimeter is roughly the thickness of a pen.) And because the prefix ***milli-*** means one thousandth, 1 millimeter is $\frac{1}{1{,}000}$ of a meter, or 1 meter is $1,000$ millimeters. (One millimeter is roughly the thickness of a credit card.)

Exercises

From each of the four choices, choose the most reasonable measure.

1. The length of a car:
   5 kilometers, 5 meters, 5 centimeters, 5 millimeters
2. The height of a notebook:
   28 kilometers, 28 meters, 28 centimeters, 28 millimeters
3. The distance to the next town:
   3.8 kilometers, 3.8 meters, 3.8 centimeters, 3.8 millimeters
4. An adult woman's height:
   1.6 kilometers, 1.6 meters, 1.6 centimeters, 1.6 millimeters
5. An adult woman's height:
   160 kilometers, 160 meters, 160 centimeters, 160 millimeters
6. The thickness of a pane of glass:
   3 kilometers, 3 meters, 3 centimeters, 3 millimeters

kilo- (km)	hecta- (hm)	deka- (dam)	**meter (m)**	deci- (dm)	centi- (cm)	milli- (mm)
$1,000$	100	10	1	0.1	0.01	0.001
$1,000$	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1{,}000}$
10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}

To convert metric units, you can simply move the decimal point left or right the number of places indicated in the table above. No calculator required!

Exercises

A 2024 Chevrolet Silverado 1500 pickup truck is 5.36 meters long.[1]

7. Convert 5.36 meters to centimeters.

8. Convert 5.36 meters to millimeters.

One mile is approximately 1.609 kilometers.

9. Convert 1.609 kilometers to meters.

10. Convert 1.609 kilometers to centimeters.

A sheet of A4 paper[2] is 297 millimeters long.

11. Convert 297 millimeters to meters.

12. Convert 297 millimeters to centimeters.

The Burj Khalifa in Dubai is the world's tallest building, with a height of 828 meters.[3]

13. Convert 828 meters to kilometers.

14. Convert 828 meters to dekameters.

Metric System: Measurements of Weight or Mass

The base unit for mass is the ***gram***, which is about the mass of a paper clip. A ***kilogram*** is $1,000$ grams; as we'll see in the next module, this is around 2.2 pounds. The active ingredients in medicines may be measured using the ***milligram***.

Exercises

From each of the three choices, choose the most reasonable measure.

15. The mass of an apple:
100 kilograms, 100 grams, 100 milligrams

16. The mass of an adult man:
80 kilograms, 80 grams, 80 milligrams

17. The amount of active ingredient in a pain relief pill:
500 kilograms, 500 grams, 500 milligrams

18. The base vehicle weight of a Chevrolet Silverado 1500 pickup truck:
$2,000$ kilograms, $2,000$ grams, $2,000$ milligrams

kilo- (kg)	hecta- (hg)	deka- (dag)	**gram (g)**	deci- (dg)	centi- (cg)	milli- (mg)
$1,000$	100	10	1	0.1	0.01	0.001
$1,000$	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1{,}000}$
10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}

This table is identical to the previous table; the only difference is that the base unit “meter” has been replaced by “gram”. This means that converting metric units of mass is exactly the same process as converting metric units of length.

Exercises

A five-pound bag of flour weighs about 2.27 kilograms.

19. Convert 2.27 kilograms to grams.

20. Convert 2.27 kilograms to milligrams.

A 20-ounce bottle of Dr. Pepper contains 65 grams of sugars.

21. Convert 65 grams to centigrams.

22. Convert 65 grams to milligrams.

23. Convert 65 grams to kilograms.

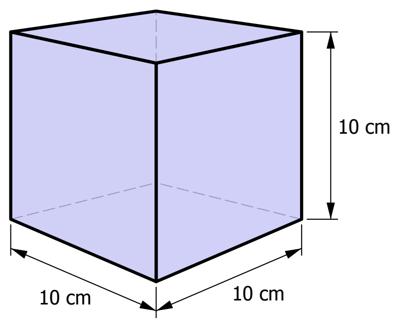
A 20-ounce bottle of Dr. Pepper contains 95 milligrams of sodium.

24. Convert 95 milligrams to centigrams.

25. Convert 95 milligrams to grams.

Metric System: Measurements of Volume or Capacity

The base unit of volume is the ***liter***, which is slightly larger than one quart. The ***milliliter*** is also commonly used; of course, there are $1,000$ milliliters in 1 liter.



1 liter is equivalent to a cube with sides of 10 centimeters. Image adapted by Cristianrodenas on Wikimedia Commons.

In case you were wondering, the units of volume, length, and mass are all connected; one cubic centimeter (a cube with each side equal to 1 cm) has the same volume as one milliliter, and one milliliter of water has a mass of one gram.

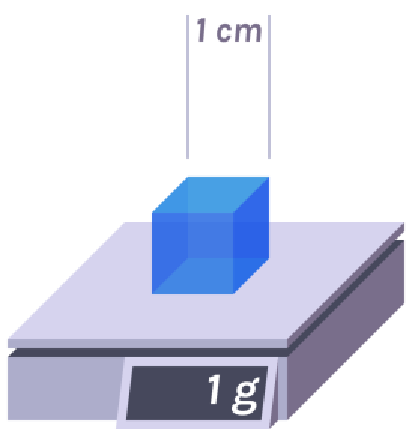


Image by nclm on Wikimedia Commons.

Exercises

From each of the two choices, choose the more reasonable measure.

26. The capacity of a car's gas tank: 50 liters, 50 milliliters

27. A dosage of liquid cough medicine: 30 liters, 30 milliliters

kilo- (kL)	hecta- (hL)	deka- (daL)	**liter (L)**	deci- (dL)	centi- (cL)	milli- (mL)
$1,000$	100	10	1	0.1	0.01	0.001
$1,000$	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1,000}$
10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}

Again, this table is identical to the previous tables; just move the decimal point left or right to convert the units.

Exercises

28. A bottle of sparkling water is labeled 50 cl. Convert 50 centiliters to liters.

29. A "1-liter" bag of saline solution for intravenous use actually contains about 1.05 liters of solution.[4] How many deciliters is this?

30. A carton of orange juice has a volume of 1.75 liters. Convert this into mL.

31. One cup (8 fluid ounces) is approximately 250 milliliters. Convert 250 milliliters into liters.

32. While being served drinks on IcelandAir, you notice that one mini bottle is labeled 50 mL, but another mini bottle is labeled 5 cL. How do the two bottles compare in size?

33. The engine displacement of a Yamaha Majesty scooter is 125 cc (cubic centimeters), and the engine displacement of a Chevrolet Spark automobile is 1.4 L (liters). What is the approximate ratio of these engine displacements?

34. How many 500-milliliter bottles of Coke are equivalent to one 2-liter bottle?

Exercise Answers

Notes

1. Source: https://www.caranddriver.com/chevrolet/silverado-1500/specs
2. A4 is a bit narrower and a bit longer than standard letter paper that we use in the U.S. and Canada.
3. Source: https://en.wikipedia.org/wiki/Burj_Khalifa
4. Source: https://pubmed.ncbi.nlm.nih.gov/11096388/

[15]

Converting Between Systems

You may use a calculator throughout this module.

Converting between the U.S. system and metric system is important in today's global economy; like it or not, the metric system is infiltrating our lives.

The numbers in these conversion ratios are usually difficult to work with, so we will use a calculator whenever necessary and pay attention to rounding. If you discover other conversion ratios that aren't provided here, write them down!

Converting Measurements of Length

Some of these conversion ratios are exact, because a yard is defined to be exactly 0.9144 meters, which means that a foot is exactly 0.3048 meters and an inch is exactly 2.54 centimeters.[1] The conversions that are not exact are rounded to four significant figures.

1 in = 2.54 cm

1 ft = 30.48 cm

1 ft = 0.3048 m or 1 m $\approx$ 3.281 ft

1 yd = 0.9144 m or 1 m $\approx$ 1.094 yd

1 mi $\approx$ 1.609 km or 1 km $\approx$ 0.6214 mi

Let's verify that 1.98 m is actually $6\text{ ft }6\text{ in}$. We can start by writing 1.98 m as a fraction over 1 and then use the conversion ratio $1\text{ ft} = 0.3048\text{ m}$ to cancel the units.

$$\frac{1.98\text{ m}}{1} \cdot \frac{1\text{ ft}}{0.3048\text{ m}} = \frac{1.98}{0.3048}\text{ ft} \approx 6.49606\text{ ft}$$

Rounding to three sig figs,[2] that's 6.50 ft, which of course is $6\text{ ft }6\text{ in}$.

Let's verify that again using the approximate conversion ratio $1\text{ m} \approx 3.281\text{ ft}$.

$$\frac{1.98\text{ m}}{1} \cdot \frac{3.281\text{ ft}}{1\text{ m}} = 1.98 \cdot 3.281\text{ ft} \approx 6.49638\text{ ft}$$

Okay, everything looks good; both conversions give us a result that rounds to 6.50 ft. As long as we don't try to keep more than four sig figs in our result, we can use either conversion ratio and get the same result.

Exercises

1. In Canada, if a road in a city does not have a speed limit posted, the default speed limit is 50 km per hour.[3] Convert 50 km into miles.
2. A *Star Wars* clone trooper is 6.00 feet tall. Convert this height to cm.
3. An Olympic-size swimming pool is 50.000 meters in length.[4] How many feet is this?
4. A 4-inch paintbrush is labeled 101.6 millimeters. Verify the accuracy of this conversion.

5. An electric fan has an advertised diameter of 20 inches, or 50.0 centimeters. Verify the accuracy of this conversion.

6. Is 21 kilometers equivalent to 13 miles? If not, what is the percent error?

Converting Measurements of Weight or Mass

These conversions are approximate. (Technically, one pound is exactly 0.45359237 kilograms[5], but we'll stick to four significant figures.)

$1 \text{ oz} \approx 28.35 \text{ g}$

$1 \text{ kg} \approx 2.205 \text{ lb}$ or $1 \text{ lb} \approx 0.4536 \text{ kg}$

Exercises

As shown in the picture, a shelving system is rated to hold a total weight of $3,250$ pounds, or 650 pounds on each of its five shelves. The metric equivalents printed on the box are $1,474.1$ kilograms and 294.8 kilograms.

7. Convert 650 pounds into kilograms. Does your answer agree with the number printed on the box?

8. Convert $3,250$ pounds into kilograms. Does your answer agree with the number printed on the box?

---

9. A hand weight weighs 5 kilograms. Convert 5 kilograms to pounds.

10. How many grams is a half pound of ground beef?

11. A gravy recipe calls for 4 ounces of flour. Convert 4 ounces into grams.

12. A smoothie recipe has 50 grams of protein. Convert 50 grams to ounces.

13. In around 2010, the National Collector's Mint (not affiliated with the U.S. Mint) ran a TV commercial selling an imitation $\$50$ gold coin modeled after the U.S. "buffalo" nickel. The commercial made the following claims. *This replica coin is coated in* 31 *milligrams of pure gold! And the price of gold keeps going up; gold is worth about* $\$1,000$ *per ounce! But you can order these fake coins for only* $\$19.95$ *apiece!* What was the approximate dollar value of the gold in one of these coins?

14. In 2023, the National Collector's Mint was still selling the imitation $\$50$ gold "buffalo" nickel, but had cut the price to $\$9.95$. This version of the coin was coated in 14 milligrams of pure gold.[6] The price of gold in 2023 was $\$2,000$ per ounce. What was the approximate dollar value of the gold in one of these coins?

Converting Measurements of Volume or Capacity

These conversion ratios are approximations rounded to four significant figures.

$1 \text{ fl oz} \approx 29.57 \text{ mL}$

$1 \text{ L} \approx 33.81 \text{ fl oz}$

$1 \text{ L} \approx 1.057 \text{ qt}$

$1 \text{ gal} \approx 3.785 \text{ L}$

Exercises

15. Stephanie lives in Vermont and buys her home renovation supplies at *Réno-Dépôt* in Quebec. She buys a toilet that uses 4.8 L of water per flush. How many gallons is this?

16. How many milliliters of drink are in a 12-ounce can?

17. TSA airline regulations limit liquids in carry-on luggage to 100 milliliters.

How many fluid ounces is this? (Round your answer to the nearest tenth.)

18. If you visit an oil change shop, you may notice large boxes of motor oil. (If you teach math, you may even take a picture of one so you can use it in your course materials.) Verify that 6 gallons of motor oil is equivalent to 22.7 liters.

Converting Measurements: Extensions

Let's finish up with some rates that require conversions.

Exercises

Maxine is driving across Canada. Her car has a 14.2-gallon gas tank and gets an average of 26 miles per gallon.

19. Approximately how many kilometers—actually, the Canadian spelling is kilometres. Approximately how many kilometres can she travel on a full tank of gas?

20. Of course, Canada measures gas in liters. Actually, litres. Convert Maxine's mileage rate, 26 miles per gallon, to kilometres per litre.

[Exercise Answers](#)

Notes

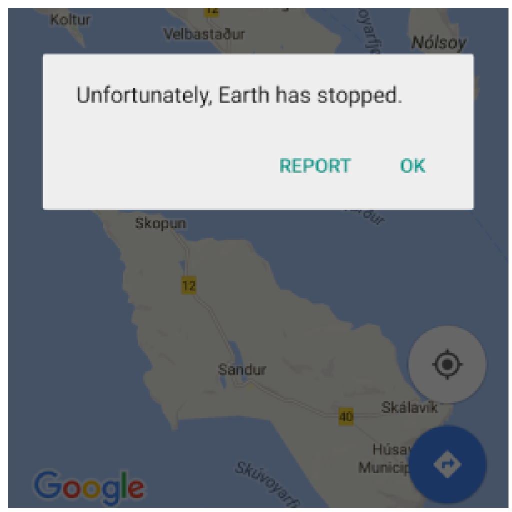
1. Source: https://en.wikipedia.org/wiki/International_yard_and_pound
2. We round to three sig figs because the measurement 1.98 meters has only three sig figs.
3. Source: https://niagarafalls.ca/city-hall/transportation-services/traffic/speed-limits.aspx
4. Source: https://swimswam.com/how-big-is-an-olympic-sized-swimming-pool/
5. Source: https://en.wikipedia.org/wiki/International_yard_and_pound
6. Source: https://ncmint.com/2023-buffalo-tribute-proof/

[16]

Other Conversions

You may use a calculator in this module as needed.

Converting Measurements of Time



You probably know all of the necessary conversions for time: 60 seconds in a minute, 24 hours in a day, etc.

When we get to units of time larger than weeks, however, we encounter problems because not all months have the same number of days, a year is not exactly 52 weeks, and the time it takes for the Earth to orbit the Sun is not exactly 365 days. Therefore, it doesn't make sense to expect an exact answer to a question like "how many minutes are in one month?" We will have to use our best judgment in situations such as these.

1 min = 60 sec

1 hr = 60 min

1 day (dy) = 24 hr

1 week (wk) = 7 dy

1 year (yr) = 365 dy

Exercises

1. How many minutes is one standard 365-day year?
2. Have you been alive for one billion seconds? Is this even possible?

Converting Rates

Previously when we were converting units, we began with units in the numerator only. If we need to convert a rate, however, we'll begin with units in both the numerator and denominator.

Exercises

Usain Bolt holds the world record time for the 100-meter dash, 9.58 seconds.

3. What was his average speed in kilometers per hour?
4. What was his average speed in miles per hour?

The more information we know, the more things we can figure out.

Exercises

An F-15 fighter jet can reach a sustained top speed of roughly Mach 2.3; this is 2.3 times the speed of sound, which is 770 miles per hour.[1]

5. What is the jet's top speed in miles per hour?
6. At this speed, how many miles would the jet travel in one minute?
7. The jet's range at this speed before it runs out of fuel is around 600 miles. If the jet flies 600 miles at top speed, for how many minutes will it fly?
8. The jet's maximum fuel capacity is $3,475$ gallons. If the jet flies 600 miles and burns $3,475$ gallons of fuel, find the jet's fuel efficiency, in miles per gallon.
9. Rewrite the jet's fuel efficiency, in gallons per mile.
10. How many gallons of fuel does the jet consume in one minute?

Measurement Prefixes: Larger

Now let's turn our attention to converting units based on their prefixes. We'll start with large units of measure.

tera- (T)	giga- (G)	mega- (M)	kilo- (k)	**[base unit]**
trillion	billion	million	thousand	one
$1,000,000,000,000$	$1,000,000,000$	$1,000,000$	$1,000$	1
10^{12}	10^9	10^6	10^3	10^0

Notice that the powers of these units are multiples of 3, just as with the engineering notation we saw at the end of Module 11. Each prefix is $1,000$ times the next smaller prefix, so moving one place in the chart is equivalent to moving the decimal point *three* places. Also notice that capitalization is important; megagram (which is also called a metric ton) is Mg with a capital M, but milligram is mg with a lowercase m.

Using computer memory as an example:

1 kilobyte = $1,000$ bytes

1 megabyte = $1,000$ kilobytes = $1,000,000$ bytes

1 gigabyte = $1,000$ megabytes = $1,000,000$ kilobytes, etc.

1 terabyte = $1,000$ gigabytes = $1,000,000$ megabytes, etc.

Note: There can be inconsistencies with different people's understanding of these prefixes with regards to computer memory, which is counted in powers of 2, not 10. Computer engineers originally defined 1 kilobyte as $1,024$ bytes because $2^{10} = 1,024$, which is very close to $1,000$. However, we will consider these prefixes to be powers of $1,000$, not $1,024$. There is an explanation at https://physics.nist.gov/cuu/Units/binary.html.

Exercises

11. A $5\frac{1}{4}$ inch floppy disk from the 1980s could store about 100 kB; a $3\frac{1}{2}$ inch floppy disk from the 1990s could store about 1.44 MB. By what factor was the storage capacity increased?

12. How many times greater is the storage capacity of a 2 TB hard drive than a 500 GB hard drive?

13. In an article describing small nuclear reactors that are designed to be retrofitted into coal plants, Dr. Jose Reyes of Oregon State University says "One module will produce 60 megawatts of electricity. That's enough for about 50 thousand homes."[2] How much electricity per home is this?

14. In the same article, Dr. Reyes says "a 60 megawatt module could produce about 60 million gallons of clean water per day using existing technologies in reverse osmosis." What is the rate of watts per gallon?

15. The destructive power of nuclear weapons is measured in kilotons (the equivalent of $1,000$ tons of TNT) or megatons (the equivalent of $1,000,000$ tons of TNT). The first nuclear device ever tested, the US's *Trinity*, was measured at roughly 20 kilotons on July 16, 1945. The largest thermonuclear weapon ever detonated, at 50 megatons, was the USSR's *Tsar Bomba*, on October 31, 1961.[3] (Video of *Tsar Bomba* was declassified in 2020.) How many times more powerful was *Tsar Bomba* than *Trinity*?

Measurement Prefixes: Smaller

Now we'll go in the other direction and look at small units of measure.

[base unit]	milli- (m)	micro- (µ or mc)	nano- (n)	pico (p)
one	thousandth	millionth	billionth	trillionth
1	0.001	0.000001	0.000000001	0.000000000001
10^0	10^{-3}	10^{-6}	10^{-9}	10^{-12}

The symbol for micro- is the Greek letter µ (pronounced "myoo"). Because this character can be difficult to replicate, you may see the letter "u" standing in for "µ" in web-based or plaintext technical articles… or you may see the prefix "mc" instead.

Again, the powers are multiples of 3; each prefix gets smaller by a factor of $\frac{1}{1000}$.

The negative exponents can sometime be complicated to work with, and it may help to think about things in reverse.

1 meter = 10^3 millimeters = 10^6 micrometers = 10^9 nanometers = 10^{12} picometers

1 second = 10^3 milliseconds = 10^6 microseconds = 10^9 nanoseconds = 10^{12} picoseconds

…and so on.

See https://physics.nist.gov/cuu/Units/prefixes.html for a list of more prefixes.

Exercises

16. An article about network latency compares the following latency times: “So a 10 Mbps link adds 0.4 milliseconds to the RTT, a 100 Mbps link 0.04 ms and a 1 Gbps link just 4 microseconds.”[4] Rewrite these times so that they are all in terms of milliseconds, then rewrite them in terms of microseconds.

17. The wavelength of red light is around 700 nm. Infrared radiation has a wavelength of approximately 10 µm.[5] Find the ratio of these wavelengths.

18. Nuclear radiation is measured in units called Sieverts, but because this unit is too large to be practical when discussing people’s exposure to radiation, milliSieverts and microSieverts are more commonly used. In 1986, workers cleaning up the Chernobyl disaster were exposed to an estimated dose of 250 mSv.[6] A typical chest x-ray exposes a person to 100 µSv.[7] How many chest x-rays’ worth of radiation were the workers exposed to?

Exercise Answers

Notes

1. My sources for the following set of questions are a combination of former students in the Air National Guard and people who sound like they know what they're talking about on the internet, particularly in this Quora discussion.
2. Source: https://www.kgw.com/article/news/local/oregon-company-get-approval-to-build-nuclear-power-plants/283-7b26b8cd-12d5-4116-928a-065731f7a0f6
3. Source: https://en.wikipedia.org/wiki/Nuclear_weapon_yield
4. Source: https://www.noction.com/blog/network-latency-effect-on-application-performance
5. Source: http://labman.phys.utk.edu/phys222core/modules/m6/The%20EM%20spectrum.html
6. Source: https://en.wikipedia.org/wiki/Chernobyl_disaster
7. Source: https://www.cancer.org/treatment/understanding-your-diagnosis/tests/understanding-radiation-risk-from-imaging-tests.html

[17]

Angles

You will need a calculator near the end of this module.

Angle Measure

Angle measurement is important in construction, surveying, physical therapy, and many other fields. We can visualize an angle as the figure formed when two line segments share a common endpoint.

We can also think about an angle as a measure of rotation. One full rotation or a full circle is $360°$, so a half rotation or U-turn is $180°$, and a quarter turn is $90°$. We often classify angles by their size relative to these $90°$ and $180°$ benchmarks.

Photo by Rangga Cahya Nugraha on Unsplash

Acute Angle: between $0°$ and $90°$

Right Angle: exactly $90°$

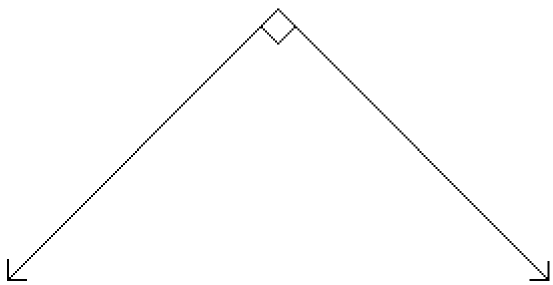
Obtuse Angle: between $90°$ and $180°$

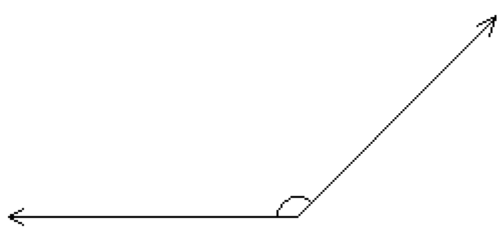
Straight Angle: exactly $180°$

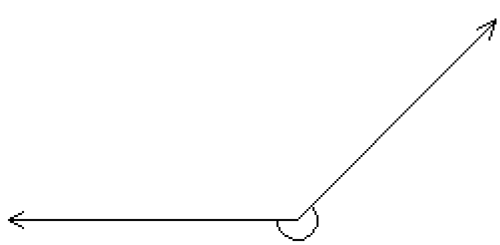
Reflexive Angle: between $180°$ and $360°$

Exercises

Identify each angle shown below as acute, right, obtuse, straight, or reflexive.

1. 

2. 

3. 

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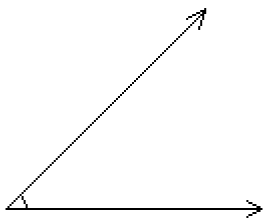


5.

Lines that form a $90°$ angle are called ***perpendicular***. As shown below, the needle should be perpendicular to the body surface for an intramuscular injection.

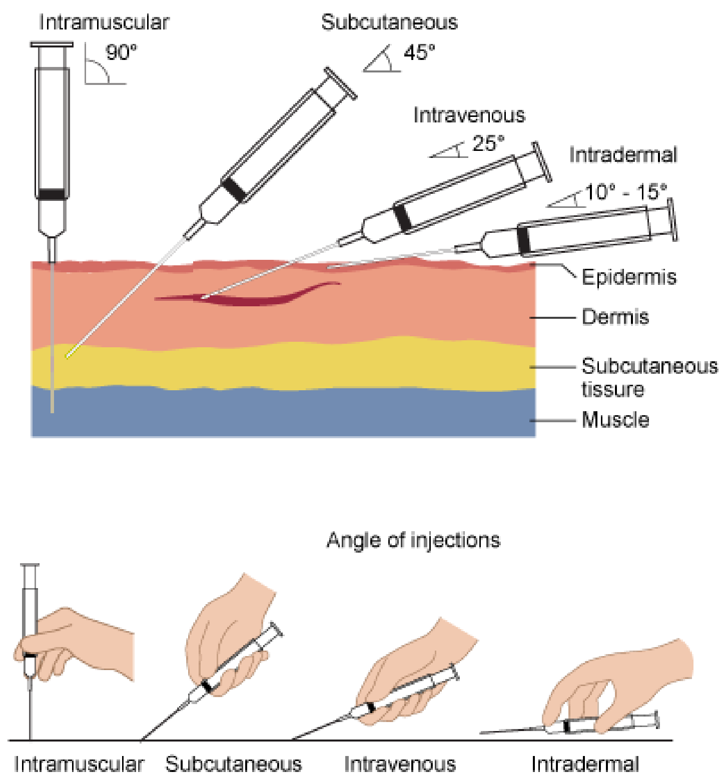
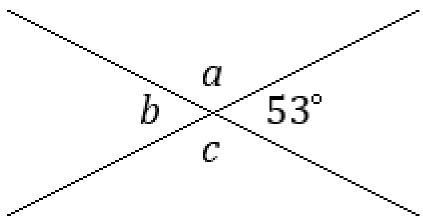


Image by British Columbia Institute of Technology (BCIT) on [Wikimedia Commons](https://commons.wikimedia.org)

When lines cross, they form angles. No surprises there. If we know the measure of one angle, we may be able to determine the measures of the remaining angles using a little logic.

Exercises

Find the measure of each unknown angle.

6. 

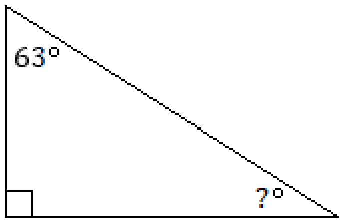
Angles in Triangles

If you need to find the measures of the angles in a triangle, there are a few rules that can help.

> The sum of the angles of every triangle is $180°$.
>
> If any sides of a triangle have equal lengths, then the angles opposite those sides will have equal measures.

Exercises

Find the measures of the unknown angles in each triangle.



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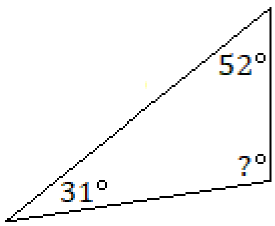

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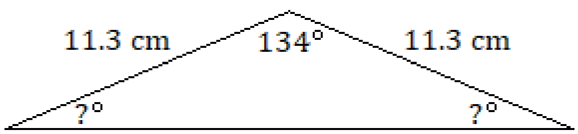

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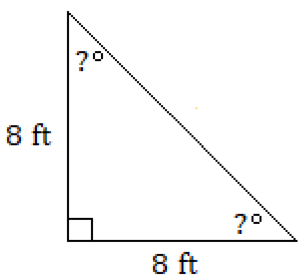

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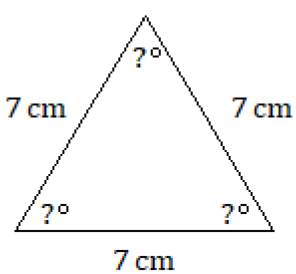
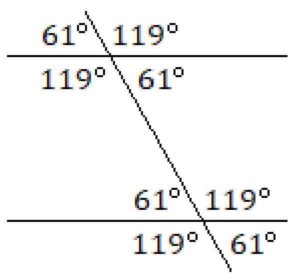


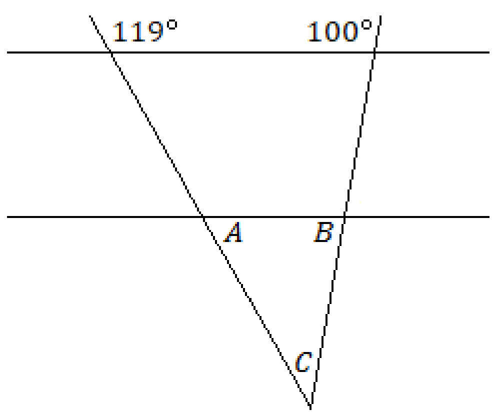
11.

Angles and Parallel Lines

Two lines that point in the exact same direction and will never cross are called ***parallel*** lines. If two parallel lines are crossed by a third line, sets of equally-sized angles will be formed, as shown in the following diagram. All four acute angles will be equal in measure, all four obtuse angles will be equal in measure, and any acute angle and obtuse angle will have a combined measure of $180°$.



Exercises



12. Find the measures of angles A, B, and C.

Degrees, Minutes, Seconds

It is possible to have angle measures that are not a whole number of degrees. It is common to use decimals in these situations, but the older method—which is called the ***degrees-minutes-seconds*** or ***DMS*** system—divides a degree using fractions out of 60: a minute is $\frac{1}{60}$ of a degree, and a second is $\frac{1}{60}$ of a minute, which means a second is $\frac{1}{3{,}600}$ of a degree. (Fortunately, these units work exactly like time; think of 1 degree as 1 hour.) For example, $2.5° = 2°30'$.

We will look at the procedure for converting between systems, but there are online calculators such as the one at https://www.fcc.gov/media/radio/dms-decimal which will do the conversions for you.

If you have latitude and longitude in DMS, like N $18°54'40''$ W $155°40'51''$, and need to convert it to decimal degrees, the process is fairly simple with a calculator.

Converting from DMS to Decimal Degrees

Enter $\text{degrees} + \text{minutes} \div 60 + \text{seconds} \div 3600$ in your calculator. Round the result to the fourth decimal place, if necessary.[1]

Exercises

Convert each angle measurement from degrees-minutes-seconds into decimal form. Round to the nearest ten-thousandth, if necessary.

13. $18°54'40''$

14. $155°40'51''$

15. $34°11'32.5''$

Going from decimal degrees to DMS is a more complicated process.

Converting from Decimal Degrees to DMS

1. The whole-number part of the angle measurement gives the number of degrees.
2. Multiply the decimal part by 60. The whole number part of this result is the number of minutes.
3. Multiply the decimal part *of the minutes* by 60. This gives the number of seconds (including any decimal part of seconds).

For example, let's convert $15.3740°$.

1. The *degrees* part of our answer will be 15.
2. The decimal part times 60 is $0.3740 \cdot 60 = 22.44$ minutes. The *minutes* part of our answer will be 22.
3. The decimal part times 60 is $0.44 \cdot 60 = 26.4$ seconds. The *seconds* part of our answer will be 26.4.

So $15.3740° = 15°22'26.4''$.

Exercises

Convert each angle measurement from decimal into degrees-minutes-seconds form.

16. $29.9750°$
17. $31.1375°$
18. $76.3467°$

Exercise Answers

Notes

1. We round to four decimal places because 1 second of angle is $\frac{1}{3,600}$ of a degree. This is a smaller fraction than $\frac{1}{1,000}$ so our precision is slightly better than the thousandths place.

[18]

Triangles

Triangular supports hold a boat in the air while it undergoes repairs at the Front Street Shipyard in Belfast, Maine.

Classifying Triangles

We can classify triangles into three categories based on the lengths of their sides.

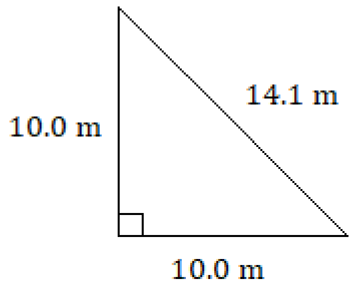
- Equilateral triangle: all three sides have the same length
- Isosceles triangle: exactly two sides have the same length
- Scalene triangle: all three sides have different lengths

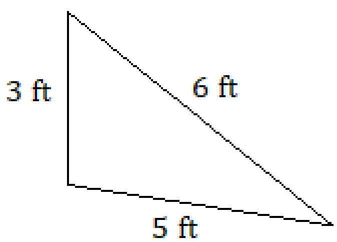
We can also classify triangles into three categories based on the measures of their angles.

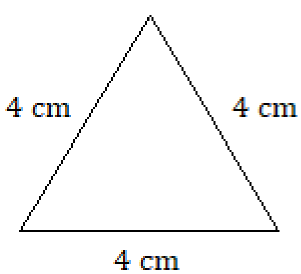
- Obtuse triangle: one of the angles is an obtuse angle
- Right triangle: one of the angles is a right angle
- Acute triangle: all three of the angles are acute

Exercises

Classify each triangle by angle and side. For example, "acute scalene".)

1. 

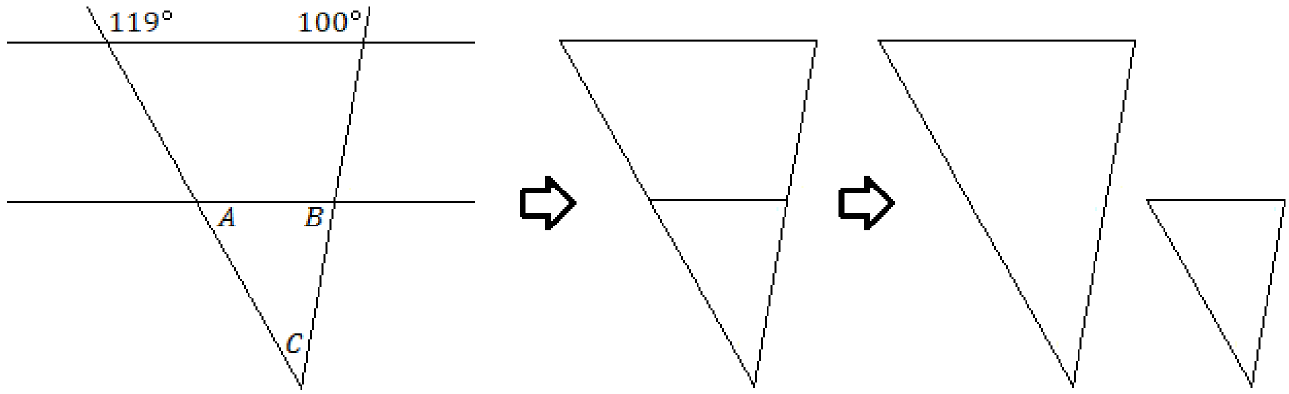
2. 

3. 

You may use a calculator for the remainder of this module.

Similar Triangles

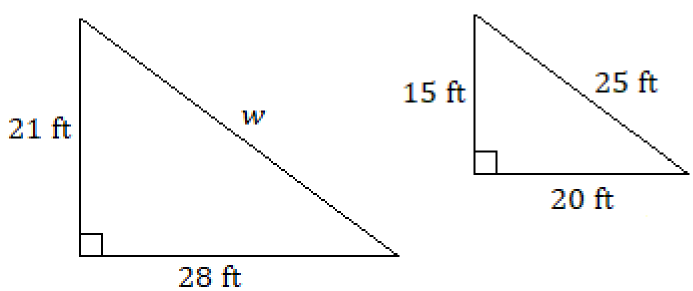
In one of the diagrams in Module 17, the parallel lines included two similar triangles, although they may be hard to see.

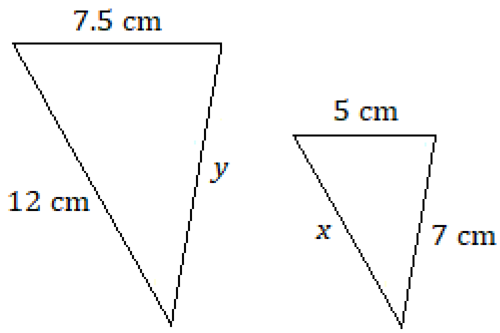


Two triangles are ***similar*** if the three angles of one triangle have the same measure as the three angles of the second triangle. The lengths of the sides of similar triangles will be in the same proportion. The triangles will have the same shape but the lengths will be scaled up or down.

Exercises

Assume that each pair of triangles are similar. Use a proportion to find each unknown length.

4. 

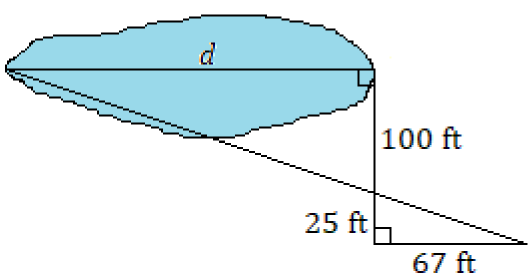


5.

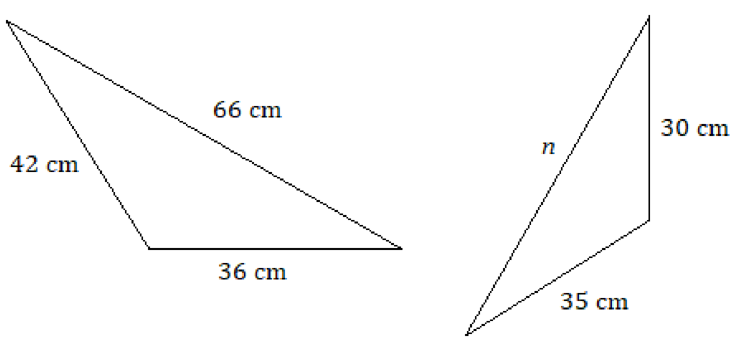
Recognizing corresponding sides can be more difficult when the figures are oriented differently.

Exercises

Assume that each pair of triangles are similar. Use a proportion to find each unknown length.



6.

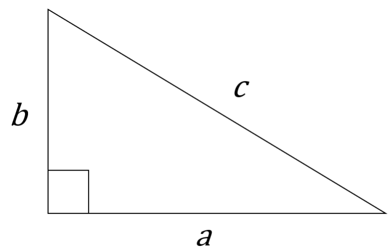


7.

The Pythagorean Theorem

In a right triangle, the two sides that form the right angle are called the ***legs***. The side opposite the right angle, which will always be the longest side, is called the ***hypotenuse***.

The ***Pythagorean theorem*** says that the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.



The Pythagorean Theorem

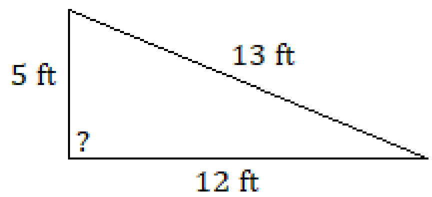
In a right triangle with legs a and b and hypotenuse c,

$$a^2 + b^2 = c^2$$

If you know the lengths of all three sides of a triangle, you can use the Pythagorean theorem to verify whether the triangle is a right triangle or not. The ancient Egyptians used this method for surveying when they needed to redraw boundaries after the yearly flooding of the Nile washed away their previous markings.[1]

Exercises

Use the Pythagorean theorem to determine whether either of the following triangles is a right triangle.



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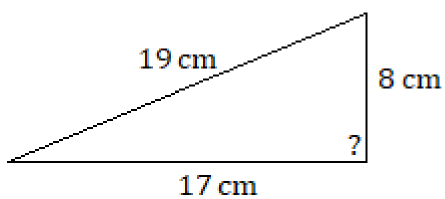


9.

Before we continue, we need to briefly discuss square roots. Calculating a square root is the opposite of squaring a number. For example, $\sqrt{49} = 7$ because $7^2 = 49$. If the number under the square root symbol is not a perfect square like 49, then the square root will be an irrational decimal that we will round off as necessary.

Exercises

Use a calculator to find the value of each square root. Round to the hundredths place.

10. $\sqrt{50}$

11. $\sqrt{296}$

12. $\sqrt{943}$

We most often use the Pythagorean theorem to calculate the length of a missing side of a right triangle. Here are three different versions of the Pythagorean theorem arranged to find a missing side, so you don't have to use algebra with $a^2 + b^2 = c^2$.

The Pythagorean Theorem, three other versions

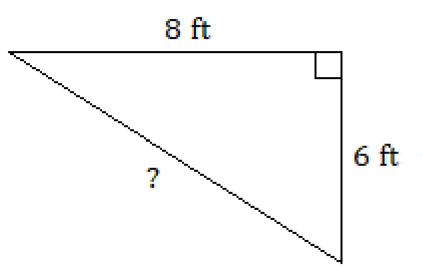
$c = \sqrt{a^2 + b^2}$

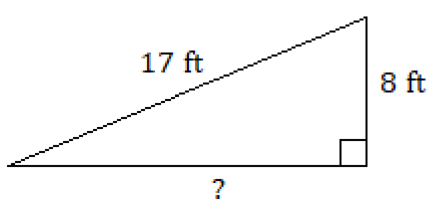
$b = \sqrt{c^2 - a^2}$

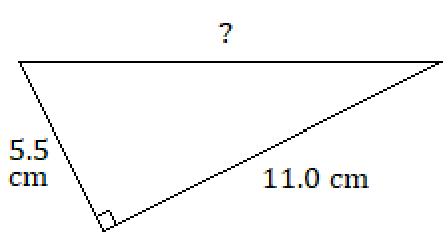
$a = \sqrt{c^2 - b^2}$

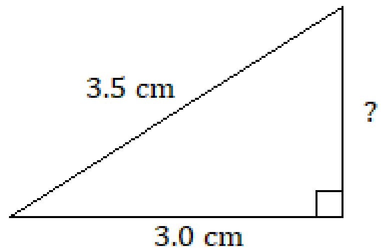
Exercises

Find the length of the missing side for each of these right triangles. Round to the nearest tenth, if necessary.

13. 

14. 

15. 



16.

[Exercise Answers](#)

Notes

1. The surveyors were called "rope-stretchers" because they used a loop of rope 12 units long with 12 equally-spaced knots. Three rope-stretchers each held a knot, forming a triangle with lengths 3, 4, and 5 units. When the rope was stretched tight, they knew that the angle between the 3-unit and 4-unit sides was a right angle because $3^2 + 4^2 = 5^2$. From *Discovering Geometry: an Inductive Approach* by Michael Serra, Key Curriculum Press, 1997.

[19]

Area of Polygons and Circles

You may use a calculator in this module as needed.

We have seen that the ***perimeter*** of a polygon is the distance around the outside. Perimeter is a length, which is one-dimensional, and so it is measured in linear units (feet, centimeters, miles, etc.). The ***area*** of a polygon is the amount of two-dimensional space inside the polygon, and it is measured in square units: square feet, square centimeters, square miles, etc.

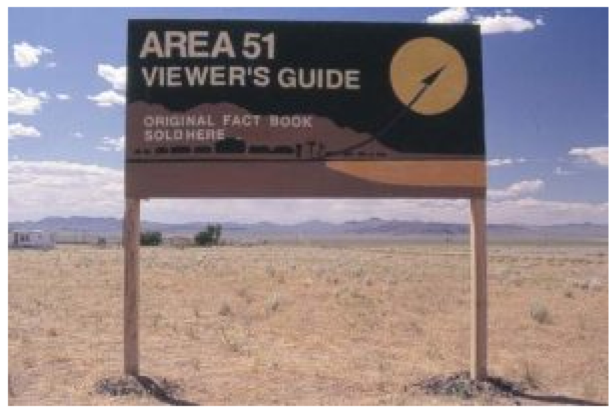


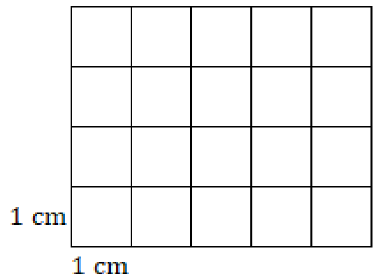
Photo by Paolo Macorig on flickr.

You can always think of area as the number of squares required to completely fill in the shape.

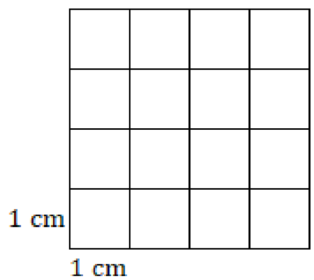
Area: Rectangles and Squares

Exercises

1. Find the area of this rectangle.



2. Find the area of this square.



There are of course formulas for finding the areas of rectangles and squares; we don't have to count little squares.

Area of a Rectangle

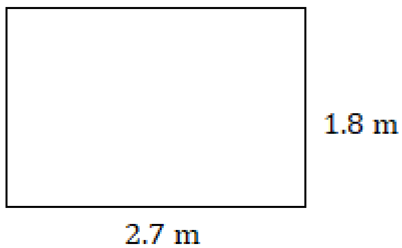
$A = lw$[1] or $A = bh$

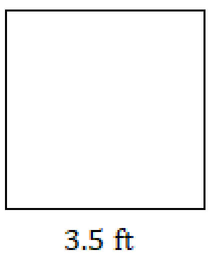
Area of a Square

$A = s^2$

Exercises

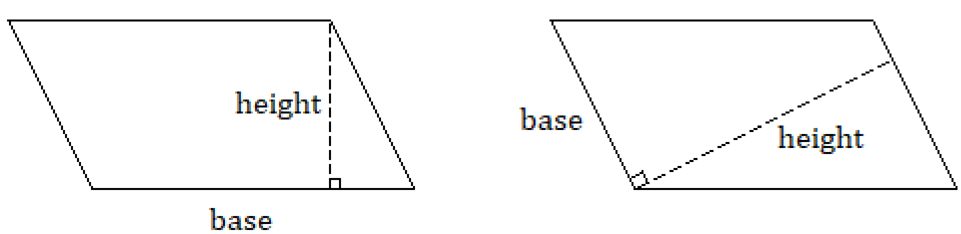
Find the area of each figure.

3. 

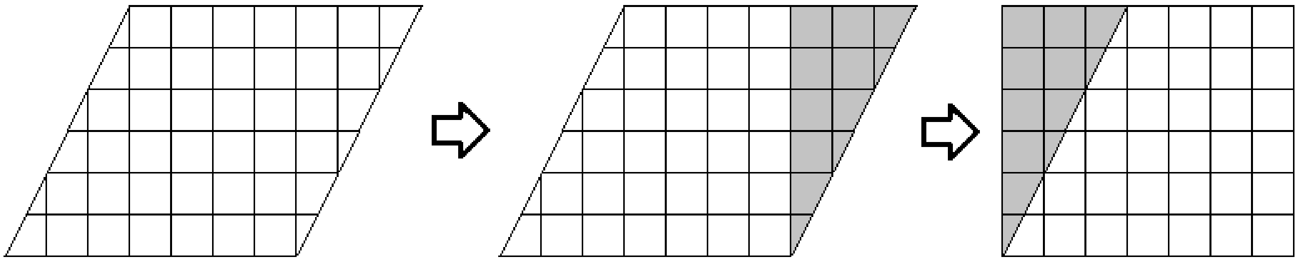
4. 

Area: Parallelograms

Another common polygon is the ***parallelogram***, which looks like a tilted rectangle. As the name implies, the pairs of opposite sides are parallel and have the same length. Notice that, if we label one side as a base of the parallelogram, we have a perpendicular height which is *not* the length of the other sides.



The following set of diagrams shows that we can cut off part of a parallelogram and rearrange the pieces into a rectangle with the same base and height as the original parallelogram. A parallelogram with a base of 7 units and a vertical height of 6 units is transformed into a 7 by 6 rectangle, with an area of 42 square units.



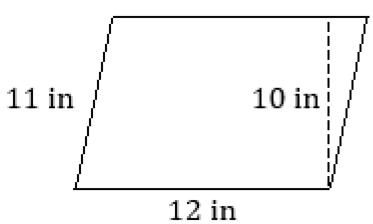
Therefore, the formula for the area of a parallelogram is identical to the formula for the area of a rectangle, provided that we are careful to use the base and the height, which must be perpendicular.

Area of a Parallelogram

$A = bh$

Exercises

Find the area of each parallelogram.



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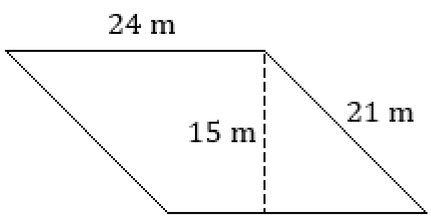
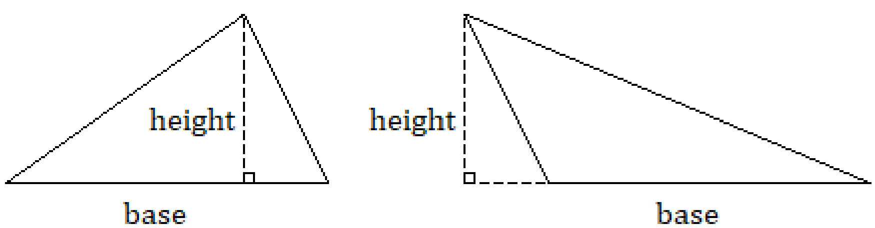


6.

Area: Triangles

When we are finding the area of a triangle, we need to identify a base and a height that is perpendicular to that base. If the triangle is obtuse, you may have to imagine the height outside of the triangle and extend the base line to meet it.



As shown below, any triangle can be doubled to form a parallelogram. Therefore, the area of a triangle is one half the area of a parallelogram with the same base and height.