Common Denominator Method

Think / Pair / Share

Solve each of the following fraction division problems using the "groups of equal size" method:

$$\frac{6}{4} \div \frac{3}{4}, \quad \frac{6}{10} \div \frac{3}{10}, \quad \frac{8}{9} \div \frac{4}{9}, \quad \frac{6}{33} \div \frac{2}{33},$$

$$\frac{5}{4} \div \frac{2}{4}, \quad \frac{5}{2} \div \frac{2}{2}, \quad \frac{5}{10} \div \frac{2}{10}.$$

What do you notice?

This leads to our first fraction division method:

Common denominator method

If two fractions have the same denominator, then when you divide them, you can just divide the numerators. In symbols,

$$\frac{a}{d} \div \frac{b}{d} = \frac{a}{b}.$$

Think / Pair / Share

- Use the common denominator method to find these quotients:

$$\frac{1}{3} \div \frac{2}{3}, \quad \frac{5}{8}, \div \frac{3}{8}, \quad \frac{3}{8} \div \frac{5}{8}, \quad \frac{15}{33} \div \frac{1}{33}, \quad \frac{1}{2} \div \frac{1}{2}.$$

- What if the fractions do not have a common denominator? Is the method useless, or can you find a way to make it work? Can you solve these problems?

$$\frac{3}{5} \div \frac{3}{4}, \quad \frac{3}{4} \div \frac{8}{7}, \quad \frac{2}{3} \div \frac{1}{2}, \quad \frac{5}{8} \div \frac{1}{4}.$$

Missing Factor Method

We know that we can always turn a division problem into a "missing factor" multiplication problem. Can that help us compute fraction division? Sometimes!

Think / Pair / Share

For each division problem, rewrite it as a missing factor multiplication question. Then find the quotient using what you know about multiplying fractions.

$$\frac{9}{10} \div \frac{3}{5}, \qquad \frac{7}{8} \div \frac{1}{4}, \qquad \frac{6}{7} \div \frac{3}{7}, \qquad \frac{10}{9} \div \frac{2}{3}, \qquad \frac{25}{12} \div \frac{5}{6}.$$

Unfortunately, the missing factor method doesn't always work out so nicely. For example,

$$\frac{3}{4} \div \frac{1}{3} = \underline{\hspace{2em}}$$

can be rewritten as

$$\frac{1}{3} \cdot \underline{\hspace{2em}} = \frac{3}{4}.$$

There isn't a nice ratio of whole numbers that obviously fills in the blank, but we'll come back to this idea and resolve it soon.

Dividing Fractions: Invert and Multiply

The missing factor method is a particularly nice way to understand fraction division. It builds on what we know about multiplication and division, reinforcing that these operations have the same relationship whether the numbers are whole number, fractions, or anything else. It makes sense. But we've seen that it doesn't always work out nicely. For example,

$$\frac{3}{4} \div \frac{1}{3} = \underline{\quad}$$

can be rewritten as

$$\frac{1}{3} \cdot \underline{\quad} = \frac{3}{4}.$$

You want to ask:

- For the numerator: $1 \cdot \_\_ = 3$. We can fill in the blank with a 3.
- For the denominator: $3 \cdot \_\_ = 4$. We can fill in the blank with $\frac{4}{3}$. (Why does that work?)

So we have:

$$\frac{1}{3} \cdot \frac{3}{\frac{4}{3}} = \frac{3}{4}.$$

You learned about fractions like

$$\frac{3}{\frac{4}{3}}$$

back in the "What is a Fraction?" chapter. This means that each $\frac{4}{3}$ of a kid gets 3 pies. So how much does an individual kid (one whole kid) get? You could draw a picture to help you figure it out. But we can also use the key fraction rule to help us out.

$$\frac{3}{\frac{4}{3}} = \frac{3 \cdot 3}{3 \cdot \frac{4}{3}} = \frac{9}{4}.$$

This process is going to be key to understanding why the "invert and multiply" rule for fraction division actually makes sense.

Simplify an Ugly Fraction

Example

$7\frac{2}{3}$ pies are shared equally by $5\frac{1}{4}$ children. How much pie does each child get?

Technically, we could just write down the answer as

$$\frac{7\frac{2}{3}}{5\frac{1}{4}}$$

and be done! The answer is equivalent to this fraction, so why not?

Is there a way to make this look friendlier? Well, if we change those mixed numbers to "improper" fractions, it helps a little:

$$\frac{7\frac{2}{3}}{5\frac{1}{4}} = \frac{\frac{23}{3}}{\frac{21}{4}}.$$

That's a bit better, but it's still not clear how much pie each kid gets. Let's use the key fraction rule to make the fraction even friendlier. Let's multiply the numerator and denominator each by 3. (Why three?) Remember, this means we're multiplying the fraction by $\frac{3}{3}$, which is just a special form of 1, so we don't change its value.

$$\frac{3 \cdot \frac{23}{3}}{3 \cdot \frac{21}{4}} = \frac{23}{\frac{63}{4}}.$$

Now multiply numerator and denominator each by 4. (Why four?)

$$\frac{4 \cdot 23}{4 \cdot \frac{63}{4}} = \frac{92}{63}.$$

We now see that the answer is $\frac{92}{63}$. That means that sharing $7\frac{2}{3}$ pies among $5\frac{1}{4}$ children is the same as sharing 92 pies among 63 children. (In both situations, the individual child get exactly the same amount of pie.)

Example

Let's forget the context now and just focus on the calculations so that we can see what is going on more clearly. Try this one:

$$\frac{\frac{3}{5}}{\frac{2}{3}}.$$

Multiplying the numerator and denominator each by 5 (why did we choose 5?) gives

$$\frac{\frac{3}{5}}{\frac{2}{3}} = \frac{5 \cdot \frac{3}{5}}{5 \cdot \frac{2}{3}} = \frac{3}{\frac{10}{3}}.$$

Now multiply the numerator and denominator each by 3 (why did we choose 3?):

$$\frac{3 \cdot 3}{3 \cdot \frac{10}{3}} = \frac{9}{10}.$$

On Your Own

1. Each of the following is a perfectly nice fraction, but it could be written in a simpler form. So do that! Write each of them in a simpler form following the examples above.

$$\frac{\frac{2}{3}}{\frac{1}{3}}, \qquad \frac{2\frac{1}{5}}{2\frac{1}{4}}, \qquad \frac{\frac{5}{7}}{\frac{3}{5}}, \qquad \frac{\frac{3}{7}}{\frac{4}{5}}.$$

Think / Pair / Share

- Jessica calculated the second exercise above this way:

$$\frac{2\frac{1}{5}}{2\frac{1}{4}} = \frac{\cancel{2}\frac{1}{5}}{\cancel{2}\frac{1}{4}} = \frac{\frac{1}{5}}{\frac{1}{4}} = \frac{\frac{1}{5} \cdot 4}{\frac{1}{4} \cdot 4} = \frac{\frac{4}{5}}{1} = \frac{4}{5}.$$

Is her solution correct, or is she misunderstanding something? Carefully explain what is going on with her solution, and what you would do as Jessica's teacher.

- Isaac calculated the last exercise above this way:

$$\frac{\frac{3}{7}}{\frac{4}{5}} = \frac{\frac{3}{7}\cdot 7}{\frac{4}{5}\cdot 5} = \frac{3}{4}.$$

Is his solution correct, or is he misunderstanding something? Carefully explain what is going on with his solution, and what you would do as Isaac's teacher.

Perhaps without realizing it, you have just found another method to divide fractions.

Example: 3/5 ÷ 4/7

Consider $\frac{3}{5} \div \frac{4}{7}$. We know that a fraction is the answer to a division problem, meaning

$$\frac{3}{5} \div \frac{4}{7} = \frac{\frac{3}{5}}{\frac{4}{7}}.$$

And now we know how to simplify ugly fractions like this one! Multiply the numerator and denominator each by 5:

$$\frac{\left(\frac{3}{5}\right)\cdot 5}{\left(\frac{4}{7}\right)\cdot 5} = \frac{3}{\frac{20}{7}}.$$

Now multiply them each by 7:

$$\frac{(3)\cdot 7}{\left(\frac{20}{7}\right)\cdot 7} = \frac{21}{20}.$$

Done! So

$$\frac{3}{5} \div \frac{4}{7} = \frac{21}{20}.$$

Example: 5/9 ÷ 8/11

Let's do another! Consider $\frac{5}{9} \div \frac{8}{11}$:

$$\frac{5}{9} \div \frac{8}{11} = \frac{\frac{5}{9}}{\frac{8}{11}}.$$

Let's multiply numerator and denominator each by 9 and by 11 at the same time. (Why not?)

$$\frac{\frac{5}{9}}{\frac{8}{11}} = \frac{\left(\frac{5}{9}\right) \cdot 9 \cdot 11}{\left(\frac{8}{11}\right) \cdot 9 \cdot 11} = \frac{5 \cdot 11}{8 \cdot 9}.$$

(Do you see what happened here?)

So we have

$$\frac{\frac{5}{9}}{\frac{8}{11}} = \frac{5 \cdot 11}{8 \cdot 9} = \frac{55}{72}.$$

On Your Own

Compute each of the following, using the simplification technique in the examples above.

$$\frac{1}{2} \div \frac{1}{3}, \qquad \frac{4}{5} \div \frac{3}{7}, \qquad \frac{2}{3} \div \frac{1}{5}, \qquad \frac{45}{59} \div \frac{902}{902}, \qquad \frac{10}{13} \div \frac{2}{13}.$$

Invert and multiply

Consider the problem $\frac{5}{12} \div \frac{7}{11}$. Janine wrote:

$$\frac{\frac{5}{12}}{\frac{7}{11}} = \frac{\frac{5}{12} \cdot 12 \cdot 11}{\frac{7}{11} \cdot 12 \cdot 11} = \frac{5 \cdot 11}{7 \cdot 12} = \frac{5}{12} \cdot \frac{11}{7}.$$

She stopped before completing her final step and exclaimed: "Dividing one fraction by another is the same as multiplying the first fraction with the second fraction upside down!"

Think / Pair / Share

First check each step of Janine's work here and make sure that she is correct in what she did up to this point. Then answer these questions:

- Do you understand what Janine is saying? Explain it very clearly.

- Work out $\frac{\frac{3}{7}}{\frac{4}{13}}$ using the simplification method. Is the answer the same as $\frac{3}{7} \cdot \frac{13}{4}$?
- Work out $\frac{\frac{2}{5}}{\frac{3}{10}}$ using the simplification method. Is the answer the same as $\frac{2}{5} \cdot \frac{10}{3}$?
- Work out $\frac{\frac{a}{b}}{\frac{c}{d}}$ using the simplification method. Is the answer the same as $\frac{a}{b} \cdot \frac{d}{c}$?
- Is Janine right? Is dividing two fractions always the same as multiplying the two fractions with the second one turned upside down? What do you think? (Do not just think about examples. This is a question if something is *always true*.)

Summary

We now have several methods for solving problems that require dividing fractions:

Dividing fractions:

- Draw a picture using the rectangle method, and use that to solve the division problem.
- Find a common denominator and divide the numerators.
- Rewrite the division as a missing factor multiplication problem, and solve that problem.
- Simplify an ugly fraction.
- Invert the second fraction (the dividend) and then multiply.

Think / Pair / Share

Discuss your opinions about our four methods for solving fraction division problems with a partner:

- Which method for division of fractions is the easiest to *understand why it works* ?
- Which method for division of fractions is the easiest to *use in computations*?
- What are the benefits and drawbacks of each method? (Think both as a future teacher and as someone solving math problems here.)

Dividing Fractions: Problems

We've spent the last couple of chapters talking about dividing fractions: how to make sense of the operation, how to picture what's going on, and how to do the computations. But all of this kind of begs the question: When would you ever want to divide fractions, anyway? How does that even come up?

It's important that teachers are able to come up with situations and problems that model particular operations, which means you have to really understand what the operations mean and when they are used.

Think / Pair / Share

- Use one of our methods (draw a picture, rectangles, common denominator, missing factor) to compute $1\frac{3}{4} \div \frac{1}{2}$.
- Come up with a situation where you would want to compute $1\frac{3}{4} \div \frac{1}{2}$. (That is, write a word problem that would require you to do this computation to solve it.)

When to Multiply, When to Divide?

A common answer to

> Come up with a situation where you would want to compute $1\frac{3}{4} \div \frac{1}{2}$.

Is something like this:

> My recipe calls for $1\frac{3}{4}$ cups of flour, but I only want to make half a recipe. How much flour should I use?

But that problem doesn't ask you to divide fractions. It asks you to cut your recipe in half, which means dividing by 2 or *multiplying* by $\frac{1}{2}$.

Why is it so hard to come up with division problems that use fractions? Maybe it's because fractions are already the answer to a division problem, so you're dividing and then dividing some more. Maybe it's because they just make it look so complicated. In any case, it's worth spending some time thinking about division problems that involve fractions and how to recognize and solve them.

One handy trick: Write a problem that involves division of whole numbers, and then see if you can change the numbers to fractions in a sensible way.

Examples

Here are some division problems involving whole numbers:

- I have 10 feet of ribbon. How many 2-inch pieces can I cut from it?
- I have a fancy old clock that rings once every 15 minutes. How many times will it ring over the course of 2 hours (120 minutes)?
- My fish tank needs 6 gallons of water, and my bucket holds 3 gallons. How many times will I need to fill my bucket in order to fill the tank?
- A recipe calls for 6 cups of flour, and my largest scoop measures exactly 2 cups. How many times should I use it?
- I ran 12 miles and went around the the same route 3 times. How long was the route?

Here are some very similar problems, rewritten to use fractions instead:

- I have $1\frac{3}{4}$ feet of ribbon. How many 6-inch (that's $\frac{1}{2}$ a foot) pieces can I cut from it?
- My watch alarm goes off every half hour, and I don't know how to shut it off. How many times will it go off during the $1\frac{3}{4}$ hour movie?
- My fish tank needs $1\frac{3}{4}$ gallons of water, and my bucket holds $\frac{1}{2}$ gallon. How many times will I need to fill my bucket in order to fill the tank?
- I want to measure $1\frac{3}{4}$ cups of flour for a recipe, but I only have a $\frac{1}{2}$ cup measuring cup. How many times should I fill it?
- I ran $1\frac{3}{4}$ miles before I twisted my ankle. I only finished half the race. How long was the race course?

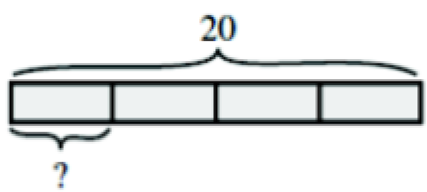
For each one of the fraction division questions, we can understand why it's a division problem:

- I have $1\frac{3}{4}$ feet of ribbon. How many 6-inch (that's $\frac{1}{2}$ a foot) pieces can I cut from it? This means making equal groups of $\frac{1}{2}$ foot each and asking how many groups. That's quotative division.
- My watch alarm goes off every half hour, and I don't know how to shut it off. How many times will it go off during the $1\frac{3}{4}$ hour movie? Again, we're making equal groups of $\frac{1}{2}$ hour each, and asking how many groups. Quotative division.
- My fish tank needs $1\frac{3}{4}$ gallons of water, and my bucket holds $\frac{1}{2}$ gallon. How many times will I need to fill

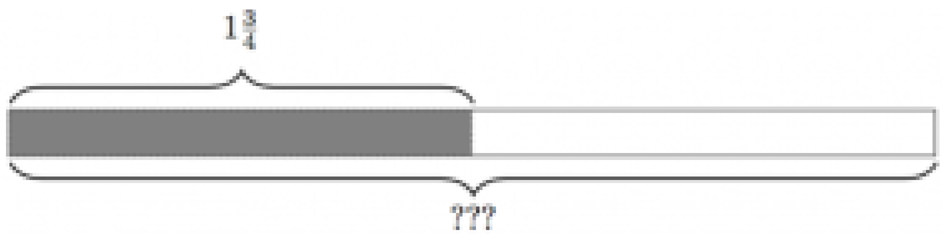
my bucket in order to fill the tank? Once again: we're making equal groups of $\frac{1}{2}$ gallon each, and asking how many groups (buckets).

- I want to measure $1\frac{3}{4}$ cups of flour for a recipe, but I only have a $\frac{1}{2}$ cup measuring cup. How many times should I fill it? This is making equal groups of $\frac{1}{2}$ cup and asking how many groups.
- I ran $1\frac{3}{4}$ miles before I twisted my ankle. I only finished half the race. How long was the race course? This one is a little different. This one is a little different. It's the fraction version of partitive division.

Recall what partitive division asks: For $20 \div 4$, we ask 20 is 4 groups of what size?



So for $1\frac{3}{4} \div \frac{1}{2}$, we ask: $1\frac{3}{4}$ is half a group of what size?



Think / Pair / Share

You try it.

- First write five different division word problems that use whole numbers. (Try to write at least a couple each of partitive and quotative division problems.)
- Then change the problems so that they are fraction division problems instead. You might need to rewrite the problem a bit so that it makes sense.
- Solve your problems!

Fractions involving zero

Zero in the Numerator

Think / Pair Share

Does the fraction $\frac{0}{11}$ make sense?

- Write a "pies per child" story for the fraction $\frac{0}{11}$. Does it make sense? How much pie does each individual child receive in your story?
- Think of $\frac{0}{11}$ as the answer to a division problem. What is that division problem? Can you solve it?

It seems pretty clear that zero pies among eleven kids gives zero pies per child:

$$\frac{0}{11} = 0.$$

The same reasoning would lead us to say:

$$\frac{0}{b} = 0 \quad \text{for any positive number } b.$$

The "Pies Per Child Model" offers one explanation: If there are no pies for us to share, no one gets any pie. It does not matter how many children there are. No pie is no pie is no pie.

We can also justify this claim by thinking about a missing factor multiplication problem:

$$\frac{0}{b} \quad \text{is asking us to fill in the blank:} \quad \_\_ \cdot b = 0.$$

The only way to fill that in and make a true statement is with 0, so $\frac{0}{b} = 0.$

Zero in the Denominator

What happens if things are flipped the other way round?

Think / Pair / Share

Does the fraction $\frac{11}{0}$ make sense?

- Write a "pies per child" story for the fraction $\frac{11}{0}$. Does it make sense? How much pie does each individual child receive in your story?
- Think of $\frac{11}{0}$ as the answer to a division problem. What is that division problem? Can you solve it?

Students often learn in school that "dividing by 0 is undefined." But they learn this as a rule, rather than thinking about why it makes sense or how it connects to other ideas in mathematics. In this case, the most natural connection is to a multiplication fact, the zero property for multiplication:

$$\text{any number} \cdot 0 = 0.$$

That says we can never find solutions to problems like

$$\_\_ \cdot 0 = 5, \qquad \_\_ \cdot 0 = 17, \qquad \_\_ \cdot 0 = 1.$$

Using the connection between fractions and division, and the connection between division and multiplication, that means there is no number $\frac{5}{0}$. There is no number $\frac{17}{0}$. And there is no number $\frac{1}{0}$. They are all "undefined" because they are not equal to any number at all.

Think / Pair / Share

Can we give meaning to $\frac{0}{0}$ at least? After all, a zero would appear on both sides of that equation!

- Cyril says that $\frac{0}{0} = 2$ since $0 \cdot 2 = 0$.
- Ethel says that $\frac{0}{0} = 17$ since $0 \cdot 17 = 0$.
- Wonhi says that $\frac{0}{0} = 887231243$ since $0 \cdot 887231243 = 0$.

Who is right? Can they all be correct? What do you think?

Cyril says that $\frac{0}{0} = 2$, and he believes he is correct because it passes the check: $0 \cdot 2 = 0$.

But 17 also passes the check, and so does 887231243. In fact, I can choose any number for x, and $0 \cdot x = 0$ will pass the check!

The trouble with the expression $\frac{a}{0}$ (with a not zero) is that there is *no meaningful value* to assign to it. The trouble with $\frac{0}{0}$ is different: There are *too many possible values* to give it!

Dividing by zero is simply too problematic to be done! It is best to avoid doing so and never will we allow zero as the denominator of a fraction. (But all is fine with 0 as a numerator.)

Problem Bank

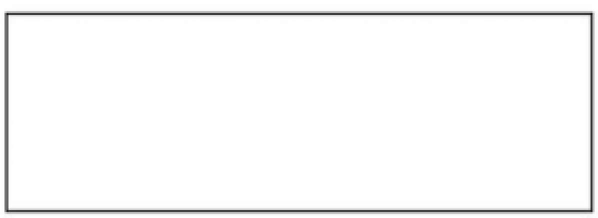
Problem 6

Harriet is with a group of five children who share four pies. Jeff is with a group of seven children who share four pies. Jean is in a group of seven children who share six pies.

1. Who gets more pie, Harriet or Jeff? Justify your answer!
2. Who gets more pie, Jeff or Jean? Justify your answer!
3. Who gets more pie, Harriet or Jean? Justify your answer!

Problem 7

Yesterday was Zoe's birthday, and she had a big rectangular cake. Today, $\frac{2}{5}$ of the cake is left. The **leftover cake** is shown here.



Draw a picture of the original (whole) cake and explain your work.

Problem 8

Use benchmarks and intuitive methods to arrange the fractions below in ascending order. Explain how you decided. (The point of this problem is to think more and compute less!):

$$\frac{2}{5}, \quad \frac{1}{3}, \quad \frac{5}{8}, \quad \frac{1}{4}, \quad \frac{2}{3}, \quad \frac{3}{4}, \quad \frac{4}{7}.$$

Problem 9

Which of these fractions has the larger value? Justify your choice.

$$\frac{10001}{10002} \quad \text{or} \quad \frac{10000001}{10000002}.$$

Problem 10

Solve each division problem. Look for a shortcut, and explain your work.

$$\frac{251 + 251 + 251 + 251}{4}$$

$$\frac{377 + 377 + 377 + 377 + 377}{5}$$

$$\frac{123123 + 123123 + 123123 + 123123 + 123123 + 123123}{3}$$

Problem 11

Yoko says

$$\frac{16}{64} = \frac{1}{4}$$

because she cancels the sixes:

$$\frac{1\not{6}}{\not{6}4} = \frac{1}{4}.$$

But note:

$$\frac{16}{64} = \frac{1 \cdot 16}{4 \cdot 16} = \frac{1 \cdot \not{16}}{4 \cdot \not{16}} = \frac{1}{4}.$$

So is Yoko right? Does her cancelation rule always work? If it does not always work, can you find any other example where it works? Can you find every example where it works?

Problem 12

Jimmy says that a fraction does not change in value if you add the same amount to the numerator and the denominator. Is he right? If you were Jimmy's teacher, how would you respond?

Problem 13

1. Shelly says that if $ab < cd$ then $\frac{a}{b} < \frac{c}{d}$. Is Shelly's claim always true, sometimes true, or never true? If you were Shelly's teacher, what would you say to her?
2. Rob says that if $ad < bc$ then $\frac{a}{b} < \frac{c}{d}$. Is Rob's claim always true, sometimes true, or never true? If you were Rob's teacher, what would you say to him?

Problem 14

Jill, her brother, and another partner own a pizza restaurant. If Jill owns $\frac{1}{3}$ of the restaurant and her brother owns $\frac{1}{4}$ of the restaurant, what fraction does the third partner own?

Problem 15

John spent a quarter of his life as a boy growing up, one-sixth of his life in college, and one-half of his life as a teacher. He spent his last six years in retirement. How old was he when he died?

Problem 16

Nana was planning to make a red, white, and blue quilt. One-third was to be red and two-fifths was to be white. If the area of the quilt was to be 30 square feet, how many square feet would be blue?[1]

Ku'u Hae Aloha (My Beloved Flag), Hawaiian cotton quilt from Waimea, before 1918, Honolulu Academy of Arts.

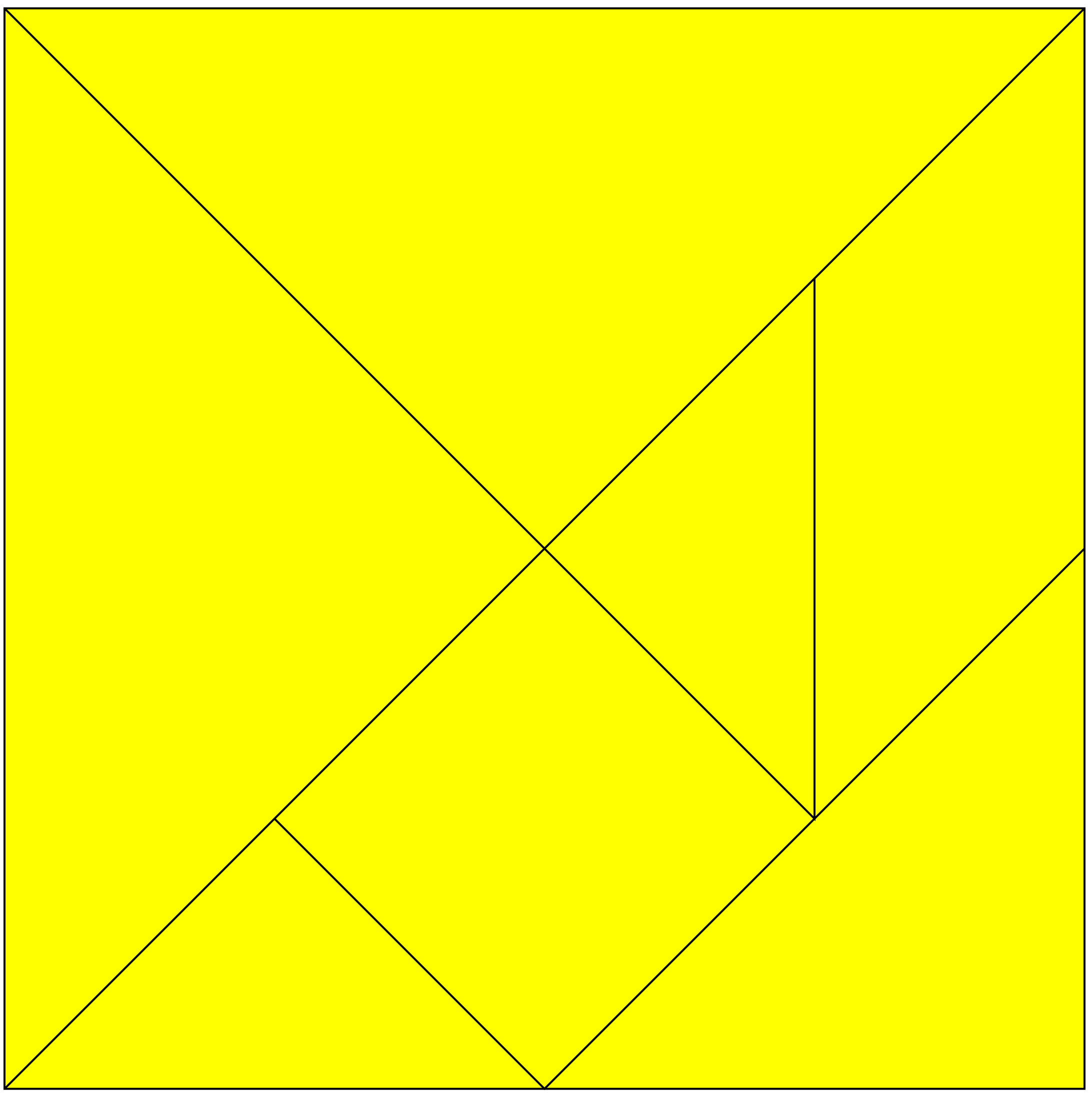
1. Image used under Creative Commons CC0 1.0 Universal Public Domain Dedication.

Problem 17

Rafael ate one-fourth of a pizza and Rocco ate one-third of it. What fraction of the pizza did they eat?

Problem 18

Problem 18 (Tangrams). Tangrams[2] are a seven-piece puzzle, and the seven pieces can be assembled into a big square.



1. If the large square shown above is one whole, assign a fraction value to each of the seven tangram

2. Tangram image from Wikimedia Commons, public domain.

pieces. Justify your answers.

2. The tangram puzzle contains a small square. If the small square (the single tangram piece) is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers.
3. The tangram set contains two large triangles. If a large triangle (the single tangram piece) is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers.
4. The tangram set contains one medium triangle. If the medium triangle (the single tangram piece) is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers.
5. The tangram set contains two small triangles. If a small triangle (the single tangram piece) is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers

Problem 19

Mikiko said her family made two square pizzas at home. One of the pizzas was 8 inches on each side, and the other was 12 inches on each side. Mikiko ate $\frac{1}{4}$ of the small pizza and $\frac{1}{12}$ of the large pizza. So she said that she ate

$$\frac{1}{4} + \frac{1}{12} = \frac{3}{12} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$$

of a pizza. Do you agree with Mikiko's calculation? Did she eat $\frac{1}{3}$ of a whole pizza? Carefully justify your answer. (This question is tricky. It's probably a good idea to draw a picture!)

Problem 20

Look at the triangle of numbers. There are lots of patterns here! Find as many as you can. In particular, try to answer these questions:

1. What pattern describes the first number in each row?
2. How is each fraction related to the two fractions below it?
3. Can you write down the next two rows of the triangle?

$$\frac{1}{1}$$

$$\frac{1}{2} \quad \frac{1}{2}$$

$$\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{3}$$

$$\frac{1}{4} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{4}$$

$$\frac{1}{5} \quad \frac{1}{20} \quad \frac{1}{30} \quad \frac{1}{20} \quad \frac{1}{5}$$

Problem 21

Marie made a sheet cake at home, but she saved some to bring to work and share with her co-workers the next day. Answer these questions about Marie's cake. (Draw a picture!)

1. Suppose Marie saved $\frac{1}{2}$ of the cake for her co-workers and the co-workers ate $\frac{3}{4}$ of this. What fraction of the entire cake did they eat?
2. What if Marie saved $\frac{1}{6}$ instead, and they ate $\frac{2}{3}$ of this?
3. What if she saved $\frac{5}{7}$ of the cake and they ate $\frac{1}{2}$ of this?

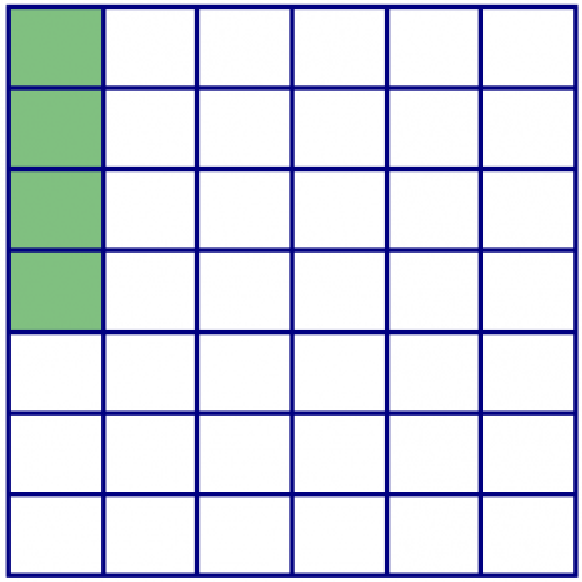
Problem 22

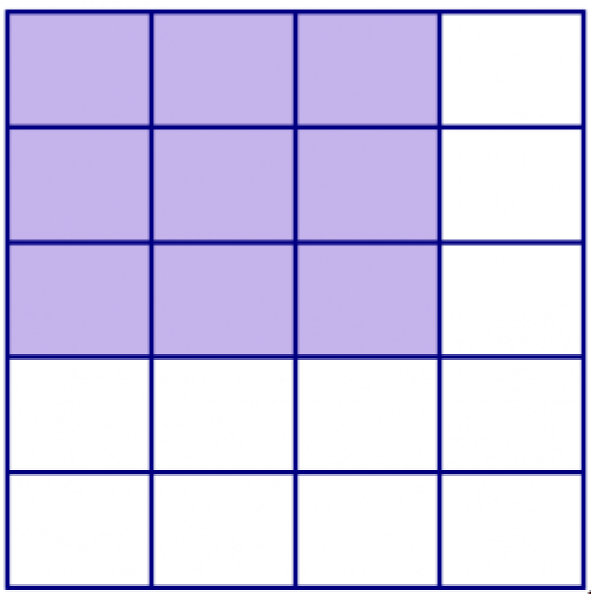
An elementary school held a "Family Math Night" event, and 405 students showed up. Two-thirds of the students who showed up won a door prize. How many students won prizes?

Problem 23

For each picture shown:

- What multiplication problem is represented?
- What is the product?





Problem 24

For each problem, use only the digits 0, 1, 2,. . . , 9 at most once each in place of the variables. Find the value closest to 1. Note that a can be a different value in each of the three problems. Justify your answer: How do you know it is the closest to 1?

1. $\dfrac{a}{b}$.
2. $\dfrac{a}{b} \cdot \dfrac{c}{d}$.
3. $\dfrac{a}{b} \cdot \dfrac{c}{d} \cdot \dfrac{e}{f}$.

Problem 25

A town plans to build a community garden that will cover $\frac{2}{3}$ of a square mile on an old farm. One side of the garden area will be along an existing fence that is $\frac{3}{4}$ of a mile long. If the garden is a rectangle, how long is the other side?

Problem 26

Nate used $90\frac{1}{2}$ pounds of seed to plant $1\frac{1}{4}$ acres of wheat. How many pounds of seed did he use per acre?

Problem 27

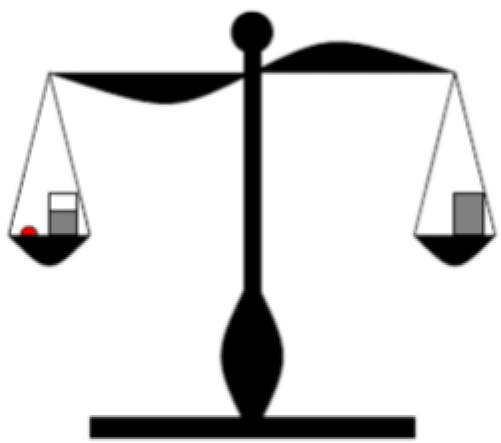
The family-sized box of laundry detergent contains 35 cups of detergent. Your family's machine requires $1\frac{1}{4}$ cup per load. How many loads of laundry can your family do with one box of detergent?

Problem 28

Jessica bikes to campus every day. When she is one-third of the way between her home and campus, she passes a grocery store. When she is halfway to school, she passes a Subway sandwich shop. This morning, Jessica passed the grocery store at 8:30am, and she passed Subway at 8:35am. What time did she get to campus?

Problem 29

If you place a full container of flour on a balance scale and place on the other side a $\frac{1}{3}$ pound weight plus a container of flour (the same size) that is $\frac{3}{4}$ full, then the scale balances. How much does the full container of flour weigh?



Problem 30

Geoff spent $\frac{1}{4}$ of his allowance on a movie. He spent $\frac{11}{18}$ of what was left on snacks at school. He also spent \$3 on a magazine, and that left him with $\frac{1}{24}$ of his total allowance, which he put into his savings account. How much money did Geoff save that week?

Problem 31

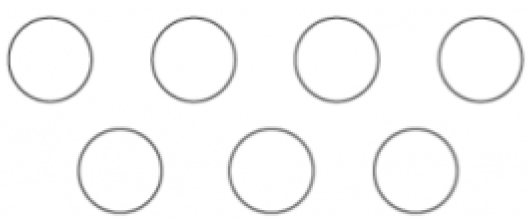
Lily was flying to San Francisco from Honolulu. Halfway there, she fell asleep. When she woke up, the distance remaining was half the distance traveled while she slept. For what fraction of the trip was Lily asleep?

Egyptian Fractions

Example: Egyptian fraction for 7/12

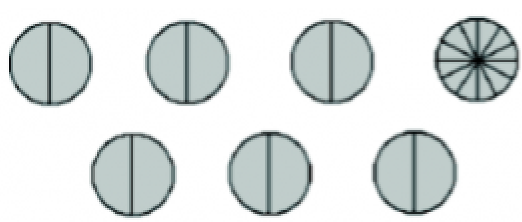
Consider the problem: Share 7 pies equally among 12 kids. Of course, given our model for fractions, each child is to receive the quantity "$\frac{7}{12}$" But this answer has little intuitive feel.

Suppose we took this task as a very practical problem. Here are the seven pies:



Is it possible to give each of the kids a whole pie? No.

How about the next best thing — can each child get half a pie? Yes! There are certainly 12 half pies to dole out. There is also one pie left over yet to be shared among the 12 kids. Divide this into twelfths and hand each kid an extra piece.



So each child gets $\frac{1}{2} + \frac{1}{12}$ of a pie, and it is indeed true that

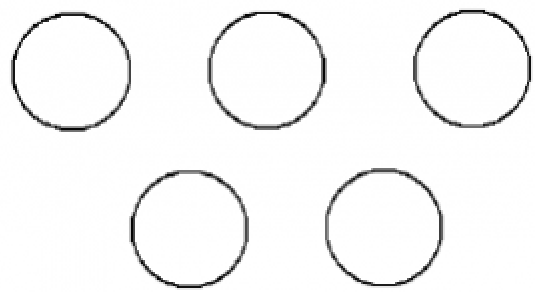
$$\frac{7}{12} = \frac{1}{2} + \frac{1}{12}.$$

(Check that calculation. . . don't just believe it!)

This seems quite reasonable. Instead of seven pieces each of size $\frac{1}{12}$, each kid gets a piece that is $\frac{1}{2}$ and a piece that is $\frac{1}{12}$. It's a lot less cutting, and a lot less messy!

Problem 32

1. Suppose you want to share five pies among six children, but you want each child to get a small number of (relatively) large pieces rather than five pieces of size $\frac{1}{6}$. Following the example above, how could you do it?



2. Using similar ideas, how could you share 4 pies among 7 kids?

History: Rhind Papyrus

The Egyptians (probably) were not particularly concerned with splitting up pies. But in fact, they did have a very strange (to us) way of expressing fractions. We know this by examining the Rhind Papyrus. This ancient document indicates that fractions were in use as many as four thousand years ago in Egypt, but the Egyptians seem to have worked primarily with **unit fractions**. They insisted on writing all of their fractions as sums of fractions with numerators equal to 1, and they insisted that the denominators of the fractions were all different.

> "Accurate reckoning for inquiring into things, and the knowledge of all things, mysteries…all secrets.

The Rhind Papyrus is an ancient account of Egyptian mathematics named after Alexander Henry Rhind. Rhind was a Scotsman who acquired the ancient papyrus in 1858 in Luxor, Egypt.

The papyrus dates back to around 1650 B.C. It was copied by a scribe named Ahmes (the earliest known contributor to the field of mathematics!) from a lost text written during the reign of king Amenehat III. The opening quote is taken from Ahmes introduction to the Rhind Papyrus[1]. The papyrus covers topics relating to fractions, volume, area, pyramids, and more.

1. Image of Rhind Papyrus from Wikimedia Commons, public domain.

Rhind Papyrus

Egyptian Fractions

To write a fraction as an Egyptian fraction, you must rewrite the fraction as:

- a **sum of unit fractions** (that means the numerator is 1), and
- the denominators must all be different.

Examples: Egyptian fractions for 3/10 and 5/7

The Egyptians would not write $\frac{3}{10}$, and they would not even write $\frac{1}{10}+\frac{1}{10}+\frac{1}{10}$. Instead, they wrote

$$\frac{1}{4}+\frac{1}{20}.$$

The Egyptians would not write $\frac{5}{7}$, and they would not even write $\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}$. Instead, they wrote

$$\frac{1}{2}+\frac{1}{5}+\frac{1}{70}.$$

(You should check that the sums above give the correct resulting fractions!)

Problem 33

Write the following as a sum of two *different* unit fractions. Be sure to check your answers.

$$\frac{2}{3}, \qquad \frac{2}{5}, \qquad \frac{2}{7}, \qquad \frac{2}{9}.$$

Can you find a general rule for how to write $\frac{2}{n}$ as an Egyptian fraction? (Assume n is an odd number.)

Problem 34

Write the following as a sum of distinct unit fractions. ("Distinct" means the fractions must have different denominators.) Note that you may need to use more than two unit fractions in some of the sums. Be sure to check your answers.

$$\frac{3}{4}, \qquad \frac{5}{6}, \qquad \frac{3}{5}, \qquad \frac{5}{9}.$$

Can you find a general process for fractions bigger than $\frac{1}{2}$?

Problem 35

Write the following fractions as Egyptian fractions.

$$\frac{17}{20}, \qquad \frac{3}{7}.$$

Can you find a general algorithm that will turn *any fraction at all* into an Egyptian fraction?

Algebra Connections

In an advanced algebra course students are often asked to work with complicated expressions like:

$$\frac{\frac{1}{x}+1}{\frac{3}{x}}$$

We can make it look friendlier by using the key fraction rule, exactly the same technique we used in the chapter on "Dividing Fractions: Invert and Multiply." In this example, let us multiply the numerator and denominator each by x. (Do you see why this is a good choice?) We obtain:

$$\frac{\left(\frac{1}{x}+1\right)\cdot x}{\left(\frac{1}{x}\right)\cdot x} = \frac{1+x}{3},$$

and $\frac{1+x}{3}$ is much less scary.

Notic that expressions like

$$\frac{1}{x}$$

cannot be rewritten as a decimal. Expressions like this arise in numerous applications, so it is important for math and science students to be able to work with fractions in fraction form, without always resorting to converting to decimals.

Example

As another example, given:

$$\frac{\frac{1}{a}-\frac{1}{b}}{ab},$$

one might find it helpful to multiply the numerator and the denominator each by a and then each by b:

$$\frac{\left(\frac{1}{a}-\frac{1}{b}\right)\cdot a\cdot b}{(ab)\cdot a\cdot b}=\frac{b-a}{a^2b^2}.$$

Examples

For

$$\frac{\frac{1}{(w+1)^2}-2}{\frac{1}{(w+1)^2}+5},$$

it might be good to multiply numerator and denominator each by $(w+1)^2$. (Why?)

$$\frac{\left(\frac{1}{(w+1)^2}-2\right)\cdot(w+1)^2}{\left(\frac{1}{(w+1)^2}+5\right)\cdot(w+1)^2}=\frac{1-2(w+1)^2}{1+5(w+1)^2}.$$

On Your Own

Can you make each of these expressions look less scary?

$$\frac{2-\frac{1}{x}}{1+\frac{1}{x}},\qquad \frac{\frac{1}{x+h}+3}{\frac{1}{x+h}},\qquad \frac{1}{\frac{1}{a}+\frac{1}{b}},\qquad \frac{\frac{1}{x+a}-\frac{1}{x}}{a}.$$

What is a Fraction? Part 3

So far, we have no single model that makes sense of fractions in all contexts. Sometimes a fraction is an action ("Cut this in half.") Sometimes it is a quantity ("We each get 2/3 of a pie!") And sometimes we want to treat fractions like numbers, like ticks on the number line in-between whole numbers.

We could say that a fraction is just a pair of numbers a and b, where we require that $b \neq 0$. We just happen to write the pair as $\frac{a}{b}$.

But again this is not quite right, since a whole infinite collection of pairs of numbers represent the same fraction! For example:

$$\frac{2}{3} \;=\; \frac{4}{6} \;=\; \frac{6}{9} \;=\; \frac{8}{12} \;=\; \cdots.$$

So a single fraction is actually a whole infinite class of pairs of numbers that we consider "equivalent."

How do mathematicians think about fractions? Well, in exactly this way. They think of pairs of numbers written as $\frac{a}{b}$, where we remember two important facts:

- $b \neq 0$, and
- $\frac{a}{b}$ is really shorthand for a whole infinite class of pairs that look like $\frac{xa}{xb}$ for all $x \neq 0$.

This is a hefty shift of thinking: The notion of a "number" has changed from being a specific combination of symbols to a whole class of combinations of symbols that are deemed equivalent.

Mathematicians then *define* the addition of fractions to be given by the daunting rule:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

This is obviously motivated by something like the "Pies Per Child Model." But if we just define things this way, we must worry about *proving* that choosing different representations for $\frac{a}{b}$ and $\frac{c}{d}$ lead to the same final answer.

For example, it is not immediately obvious that

$$\frac{2}{3} + \frac{4}{5} \quad \text{and} \quad \frac{4}{6} + \frac{40}{50}$$

give answers that are equivalent. (Check that they do!)

They also *define* the product of fractions as:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Again, if we start from here, we have to *prove* that you get equivalent answers for different choices of fractions equivalent to $\frac{a}{b}$ and $\frac{c}{d}$.

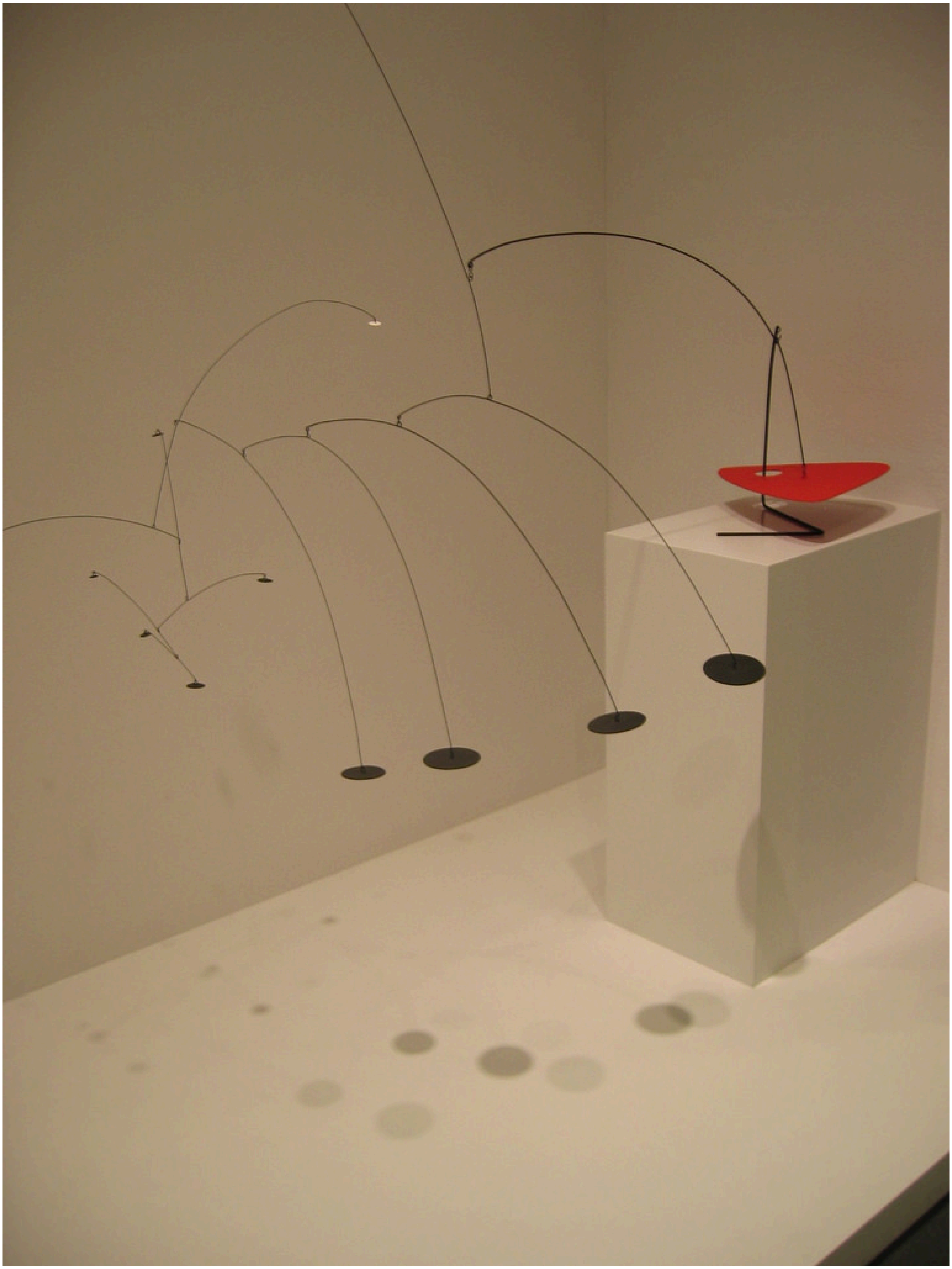
Then mathematicians establish that the axioms of an arithmetic system hold with these definitions and carry on from there! (That is, they check that addition and multiplication are both commutative and associative, that the distributive law holds, that all representations of 0 act like an additive identity, and so on.)

This is abstract, dry, and not at all the best first encounter to offer students on the topic of fractions. And, moreover, this approach completely avoids the question as to what a fraction really means in the "real world." But it is the best one can do if one is to be completely honest.

Think / Pair / Share

So… what is a fraction, really? How do you think about them? And what is the best way to talk about them with elementary school students?

Patterns and Algebraic Thinking



A mobile by the artist Alexander Calder.

[1]

> Doing mathematics should always mean finding patterns and crafting beautiful and meaningful explanations.
>
> -Paul Lockhart

1. Image used under the Creative Commons CC0 1.0 Universal Public Domain Dedication.

Introduction

Algebra skills are essential for your future students. Why? Here are just a few reasons:

- Mathematics, and especially algebra, is the language of science and modern technology. Thinking algebraically helps you to make sense of the world, to understand and interact with technology more productively, and to succeed in other fields.
- Algebra is a tool for solving problems. This may not be your experience so far, but it is true. If you are able to "algebratize" a problem, that often helps lead you to a solution.
- Algebra helps you to think abstractly. It is a tool for thinking about operations like addition, subtraction, multiplication, and division separate from doing calculations on particular numbers. Algebra helps you to *understand and explain why* the operations work the way they do, to *describe* their properties clearly, and to *manipulate* expressions to see the bigger picture.

You might wonder why future elementary teachers should master algebra, a topic usually studied (by that name, anyway) in 8th grade and beyond. But the Common Core Standards for School Mathematics has standards in "Operations and Algebraic Thinking" beginning in kindergarten!

Everyone who shows up to school has already learned a lot about abstraction and generalization — the fundamental ideas in algebra. They are all capable of learning to formalize these ideas. Your job as an elementary school teacher will be to provide your students with even more experiences in abstraction and generalization in a mathematical context, so that these ideas will seem quite natural when they get to a class with the name "Algebra."

Let's start with a problem:

Problem 1

I can use four 4's to make 0:

$$44 - 44 = 0.$$

I can also use four 4's to make the number 10:

$$(4 \times 4) - 4 - \sqrt{4} = 10.$$

Your challenge: Use four 4's to make all of the numbers between 0 and 20. (Try to find different solutions for 0 and 10 than the ones provided.) You can use any mathematical operations, but you can't use any digits other than the four 4's.

Think / Pair / Share

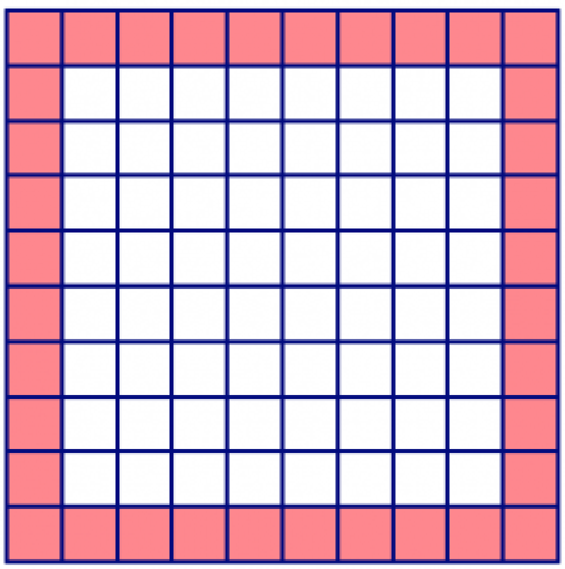
- What does "algebra" mean to you?
- What does Problem 1 have to do with "algebra"?
- What do you imagine when you think about using algebra to solve problems in school?
- Have you ever used algebra to solve problems outside of school?
- What is meant by "algebraic thinking," and what kinds of algebraic thinking can be done by elementary school students?

Borders on a Square

Here's another problem:

Problem 2

Here is a large square made up of 100 smaller unit squares. The unit squares along the border of the large square are colored red. Without counting one-by-one, can you figure out how many red squares there are in the picture?

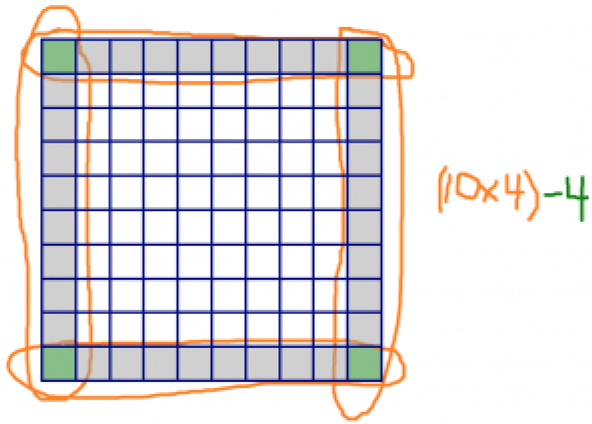


Clearly describe how you figured out the number of red squares, and how you know your answer is correct.

Justin calculated the number of squares as $(10 \times 4) - 4$. He justified his answer this way:

> Since the dimensions of the big square are 10×10, there are 10 squares along each of the four sides. So that gives me 40 red squares. But then each corner is part of two different sides. I've counted each of the corners twice. So I need to make up for that by subtracting 4 at the end.

Justin showed this picture to justify his work:



Think / Pair / Share

- What do you think about Justin's solution? Are you convinced? Could he have explained it more clearly?
- Was Justin's solution different from your solution or the same?
- Notice the color coding in Justin's picture. What do the colors represent? Why did he use the colors the way he did?

Problem 3

There are lots of different ways to calculate the number of colored squares along the border of a 10×10 square. Below are the calculations several other students did. For each calculation, write a justification and draw a picture to show why it calculates the number of squares correctly. Think about using color in your picture to make your work more clear.

1. Valerie calculated $10 + 10 + 8 + 8$.
2. Kayla calculated 4×9.
3. Linda calculated $(10 \times 10) - (8 \times 8)$.
4. Mark calculated $(4 \times 8) + 4$.
5. Allan calculated $(4 \times 8) + 4$.

Problem 4

Now suppose that you have a large 6×6 square with the unit squares along the border colored red. Adapt two of the techniques above to calculate the number of red unit squares.

For each technique you used, write an explanation and include a picture. Think about how to use colors or other methods to make your picture and explanation more clear.

Problem 5

Now suppose that you have a large 25×25 square with the unit squares along the border colored red. Adapt two of the techniques above to calculate the number of red unit squares.

For each technique you used, write an explanation and include a picture. Think about how to use colors or other methods to make your picture and explanation more clear.

Problem 6

1. Suppose that you have 64 red squares. Can you use all of those squares to make the border of a larger square in a picture like the one above? If yes, what are the dimensions of the larger square? If no, why not?
2. What if you have 30 red squares? Same questions.
3. What if you have 256 red squares? Same questions.

Think / Pair / Share

Describe some general rules:

- If you have a large $n \times n$ square with the border squares colored red, how many red squares will there be? Justify your answer with words and a picture.
- If you have k red squares, is there a quick test you can do to decide if you can use all of those

squares to make the border of a large square? Can you tell how big the square will be?

Careful Use of Language in Mathematics: =

The notion of equality is fundamental in mathematics, and especially in algebra and algebraic thinking. The symbol "="' expresses a *relationship*. It is *not* an operation in the way that + and are × operations. It should not be read left-to-right, and it definitely does not mean "… and the answer is …".

For your work to be clear and easily understood by others, it is essential that you use the symbol = appropriately. And for your future students to understand the meaning of the = symbol and use it correctly, it is essential that you are clear and precise in your use of it.

Let's start by working on some problems.

Problem 7

Akira went to visit his grandmother, and she gave him $1.50 to buy a treat.

He went to the store and bought a book for $3.20. After that, he had $2.30 left.

How much money did Akira have before he visited his grandmother?

Problem 8

Examine the following equations. Decide: Is the statement always true, sometimes true, or never true? Justify your answers.

$(a)\ 5+3=8. \qquad (b)\ \frac{2}{3}+\frac{1}{2}=\frac{3}{5}. \qquad (c)\ 5+3=y. \qquad (d)\ \frac{a}{5}=\frac{5}{a}.$

$(e)\ n+3=m. \qquad (f)\ 3x=2x+x. \qquad (g)\ 5k=5k+1.$

Problem 9

Consider the equation

$$18 - 7 = \underline{\qquad}.$$

1. Fill in the blank with something that makes the equation *always true*.
2. Fill in the blank with something that makes the equation *always false*.
3. Fill in the blank with something that makes the equation *sometimes true and sometimes false*.

Problem 10

If someone asked you to *solve* the equations in Problem 8, what would you do in each case and why?

Think / Pair / Share

Kim solved Problem 7 this way this way:

> Let's see:
>
> $$2.30 + 3.20 = 5.50 - 1.50 = 4,$$
>
> so the answer is 4.

What do you think about Kim's solution? Did she get the correct answer? Is her solution clear? How could it be better?

Although Kim found the correct numerical answer, her calculation really doesn't make any sense. It is true that

$$2.30 + 3.20 = 5.50.$$

But it is definitely *not* true that

$$2.30 + 3.20 = 5.50 - 1.40.$$

She is incorrectly using the symbol "=", and that makes her calculation hard to understand.

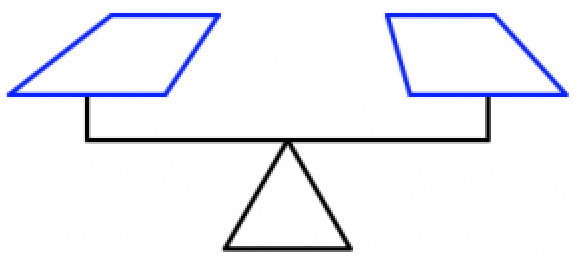
Think / Pair / Share

- Can you write a good *definition* of the symbol "="? What does it mean and what does it represent?
- Give some examples: When should the symbol "="' be used, and when should it *not* be used?
- Do these two equations express the same relationships or different relationships? Explain your answer.

$$x^2 - 1 = (x+1)(x-1).$$

$$(x+1)(x-1) = x^2 - 1.$$

This picture shows a (very simplistic) two-pan balance scale. Such a scale allows you to *compare* the weight of two objects. Place one object in each pan. If one side is lower than the other, then that side holds heavier objects. If the two sides are balanced, then the objects on each side weigh the same.

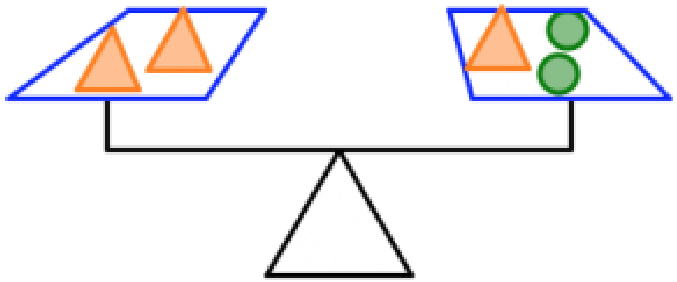


Think / Pair / Share

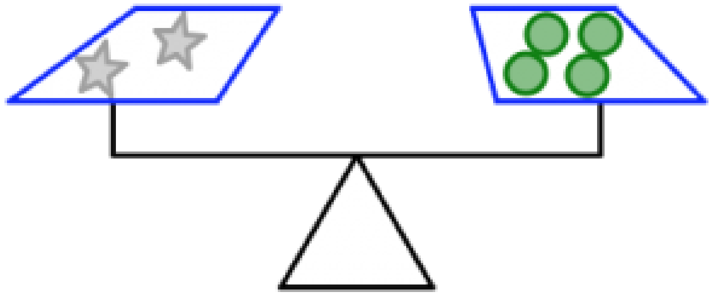
In the pictures below:

- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The silver stars all weigh the same.
- The scale is balanced.

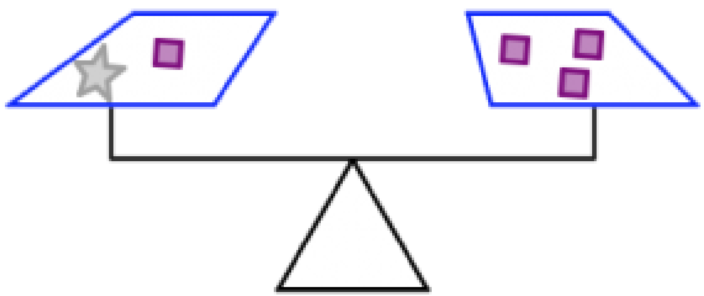
1. In the picture below, what do you know about the weights of the triangles and the circles? How do you know it?



2. In the picture below, what do you know about the weights of the circles and the stars? How do you know it?



3. In the picture below, what do you know about the weights of the stars and the squares? How do you know it?

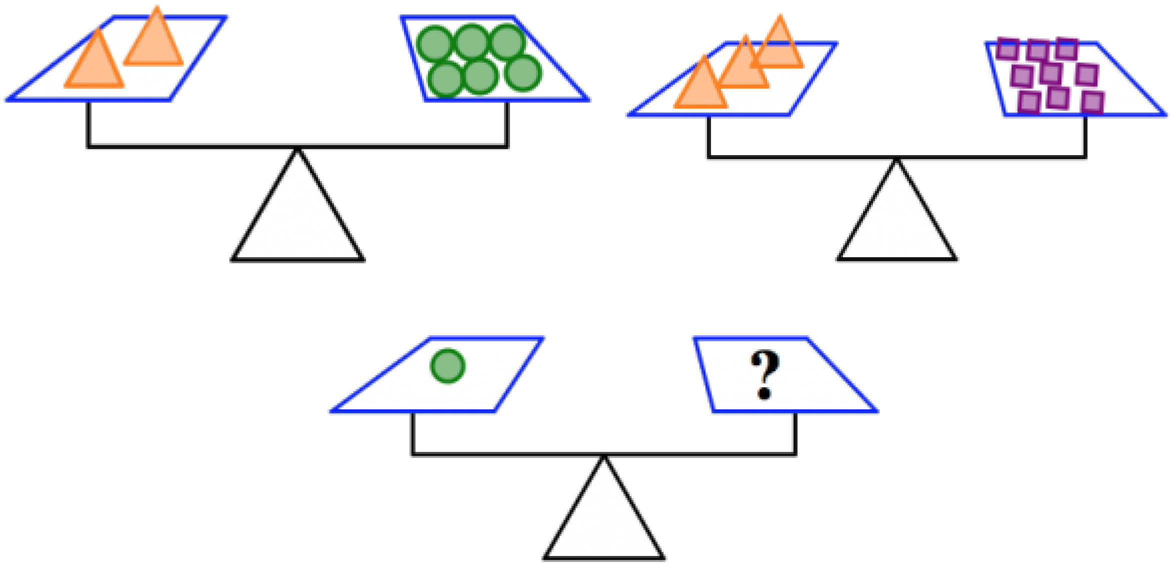


Problem 11

In the pictures below:

- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.

- The scale is balanced.

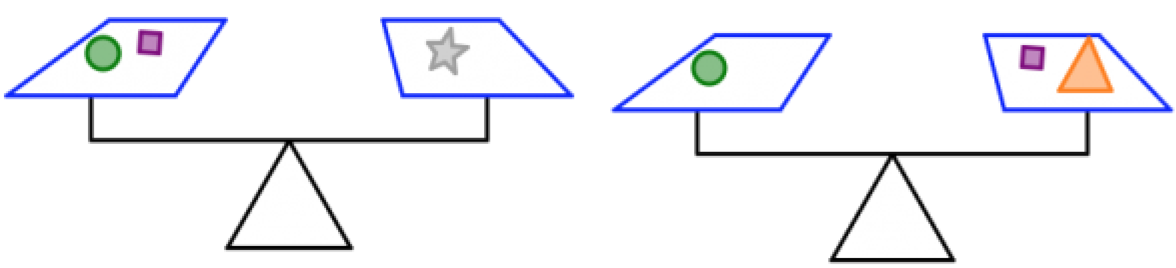


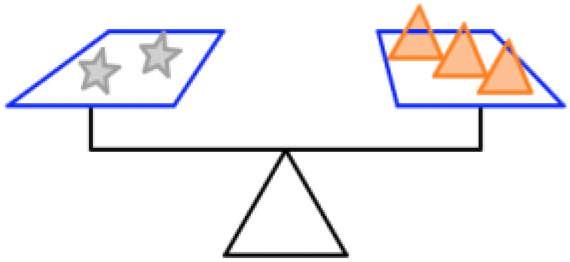
How many purple squares will balance with one circle? Justify your answer.

Problem 12

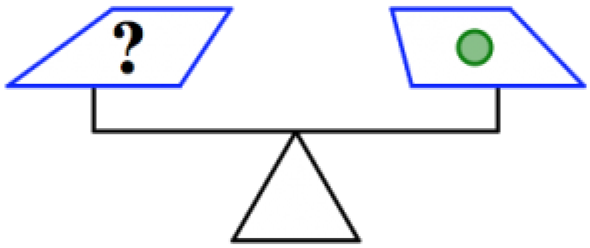
In the pictures below:

- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The silver stars all weigh the same.
- The scale is balanced.

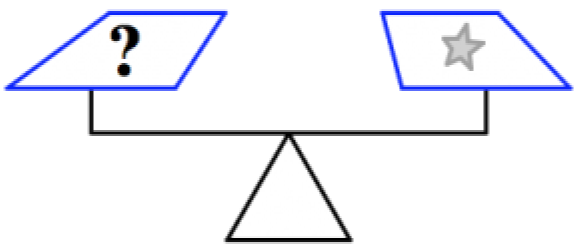




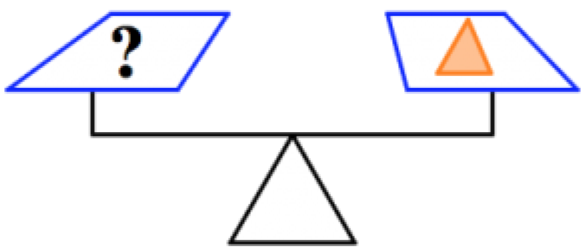
How many purple squares will balance the scale in each case? Justify your answers.



(a)



(b)

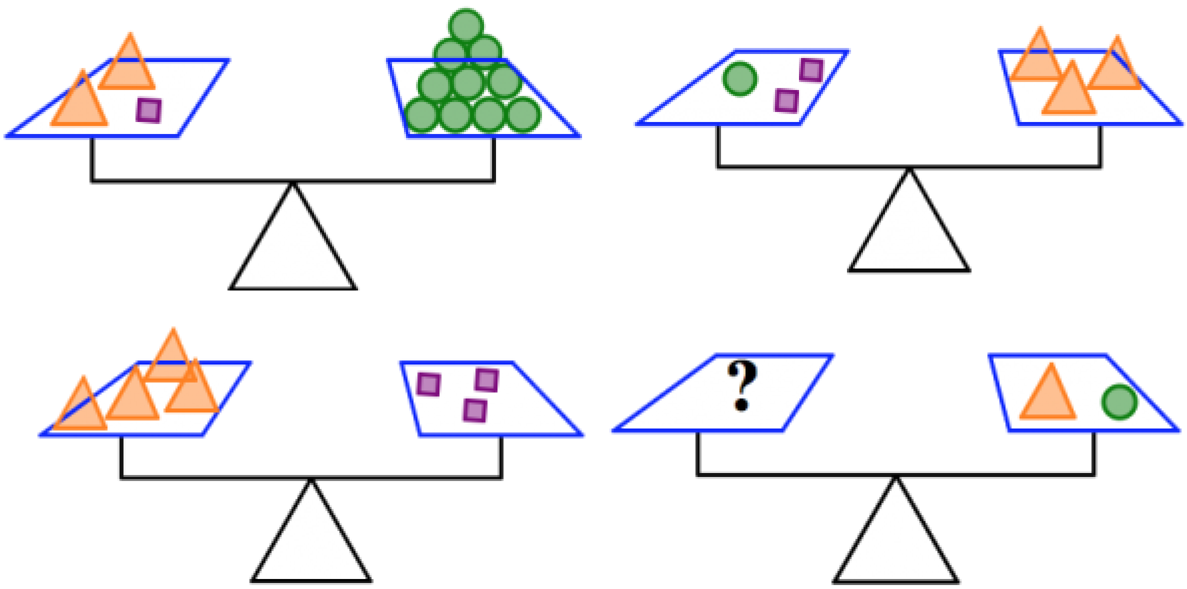


(c)

Problem 13

In the pictures below:

- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The scale is balanced.

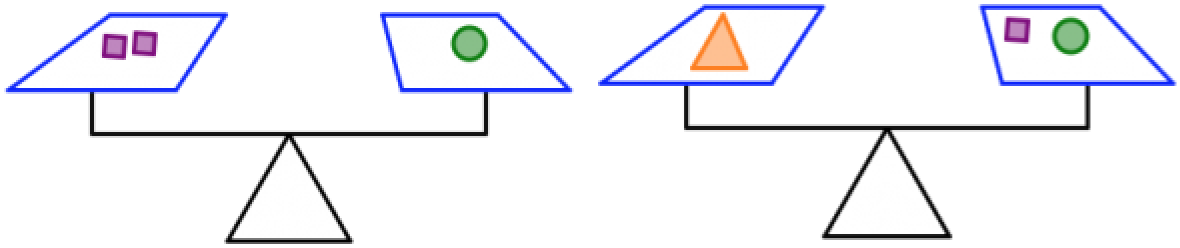


What will balance the last scale? Can you find more than one answer?

Problem 14

In the pictures below:

- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The scale is balanced.



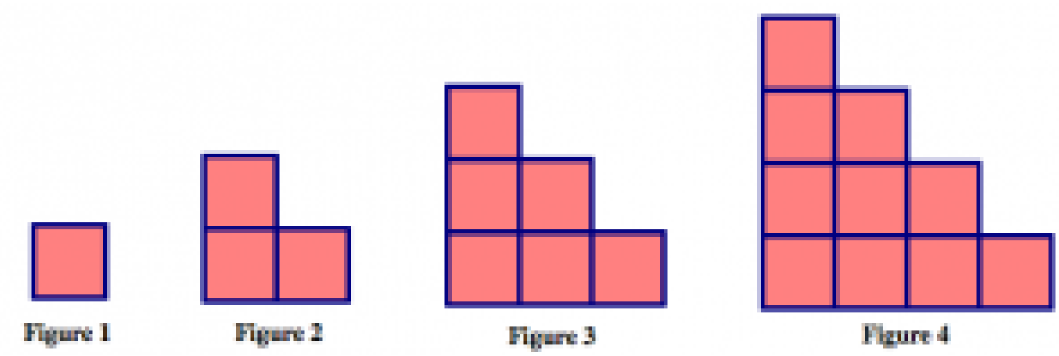
1. Which shape weighs the most: the square, the triangle, or the circle? Which shape weighs the least? Justify your answers.
2. Which of the two scales is holding the most total weight? How do you know you're right?

Think / Pair / Share

What do Problems 11–14 above have to do with the "=" symbol?

Growing Patterns

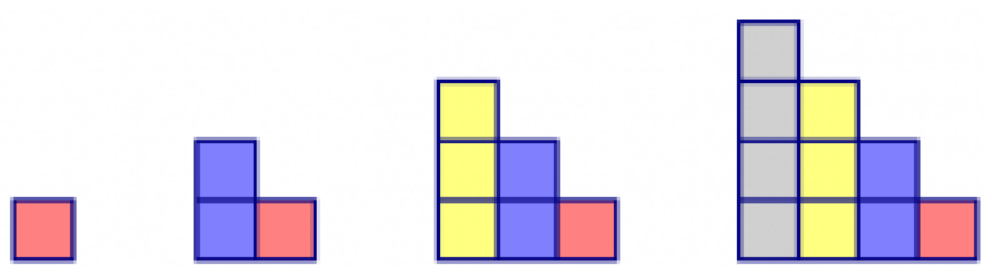
Here is a pattern made from square tiles.



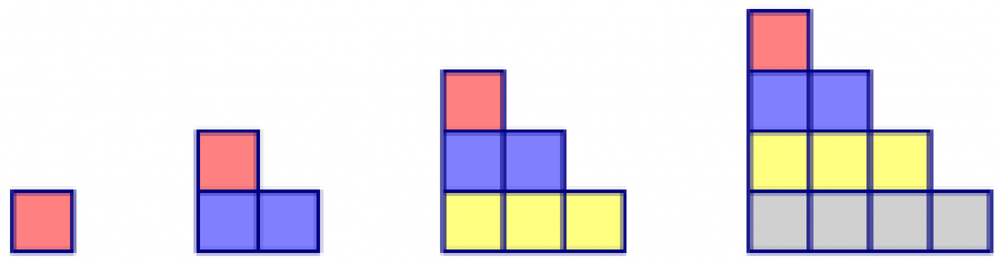
Think / Pair / Share

- Describe how you see this pattern growing. Be as specific as you can. Draw pictures and write an explanation to make your answer clear.
- Say as much as you can about this growing pattern. Can you draw pictures to extend the pattern?
- What mathematical questions can you ask about this pattern? Can you answer any of them?

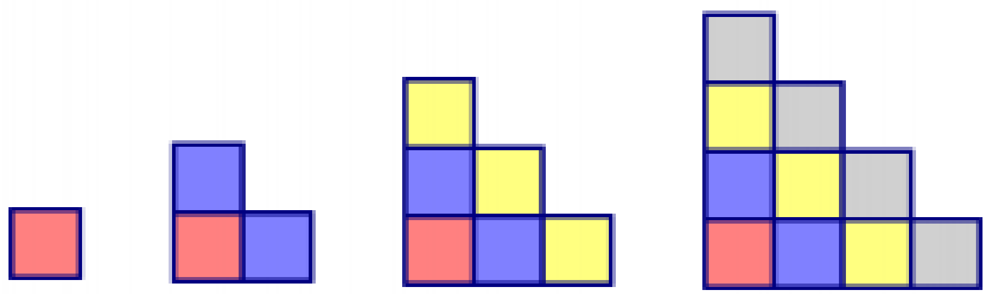
Here are some pictures that students drew to describe how the pattern was growing.



Ali’s picture



Michael’s picture



Kelli’s picture

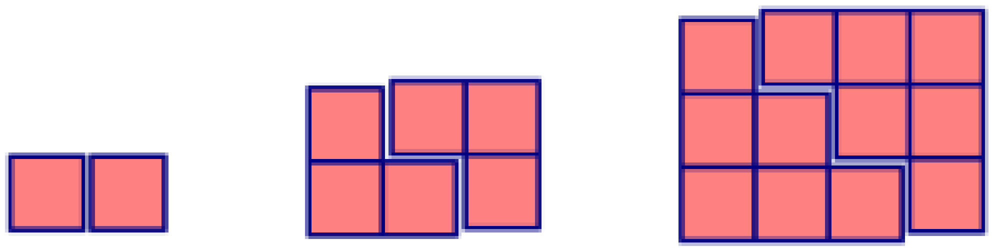
Think / Pair / Share

Describe in words how each student saw the pattern growing. Use the students’ pictures above (or your own method of seeing the growing pattern) to answer the following questions:

- How many tiles would you need to build the 5th figure in the pattern?
- How many tiles would you need to build the 10th figure in the pattern?
- How can you compute the number of tiles in any figure in the pattern?

Problem 15

Hy saw the pattern in a different way from everyone else in class. Here’s what he drew:



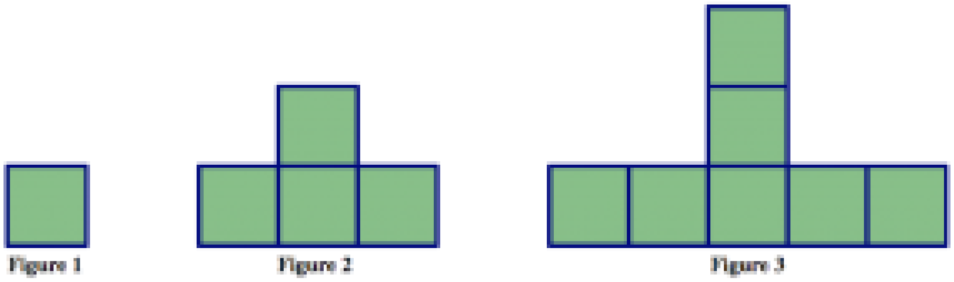
Hy's picture.

1. Describe in words how Hy saw the pattern grow.
2. How would Hy calculate the number of tiles needed to build the 10th figure in the pattern?
3. How would Hy calculate the number of tiles needed to build the 100th figure in the pattern?
4. How would Hy calculate the number of tiles needed to build any figure in the pattern?

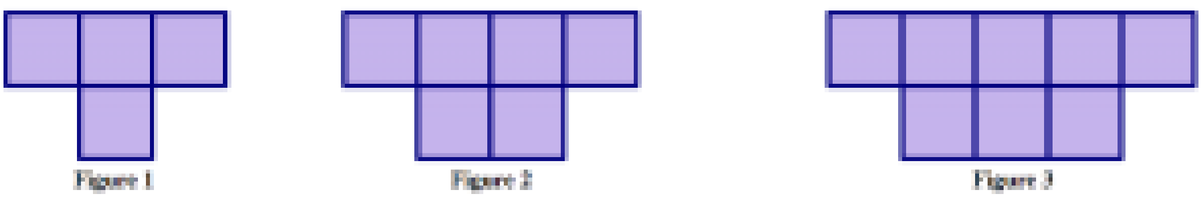
The next few problems present several growing patterns made with tiles. For each problem you work on, do the following:

1. Describe in words and pictures how you see the pattern growing.
2. Calculate the number of tiles you would need to build the 10th figure in the pattern. Justify your answer based on how the pattern grows.
3. Calculate the number of tiles you would need to build the 100th figure in the pattern.
4. Describe how you can figure out the number of tiles in any figure in the pattern. Be sure to justify your answer based on how the pattern grows.
5. Could you make one of the figures in the pattern using exactly 25 tiles? If yes, which figure? If no, why not? Justify your answer.
6. Could you make one of the figures in the pattern using exactly 100 tiles? If yes, which figure? If no, why not? Justify your answer.

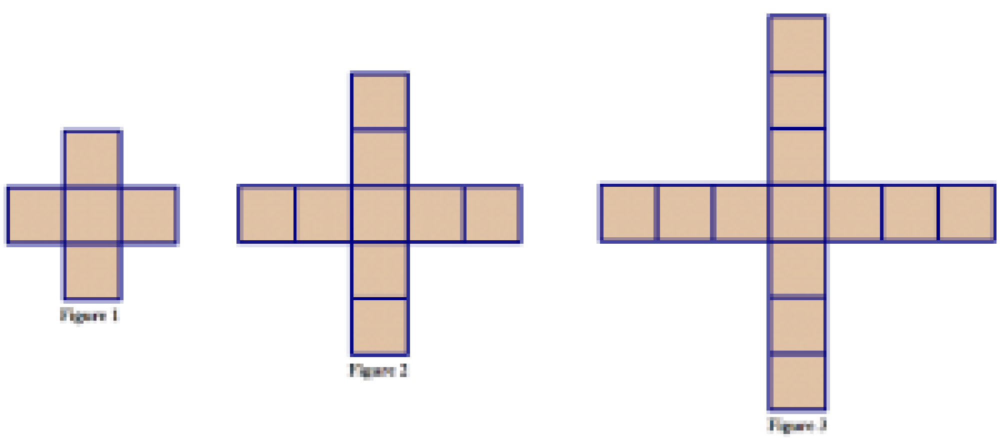
Problem 16



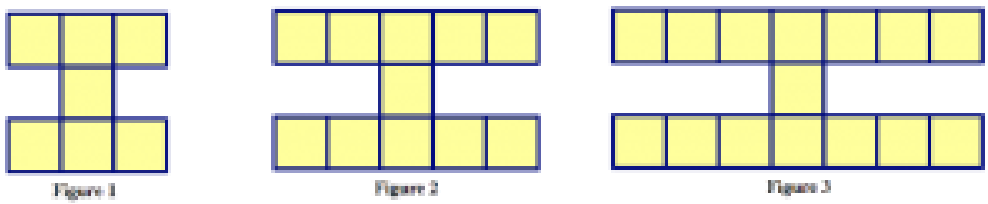
Problem 17



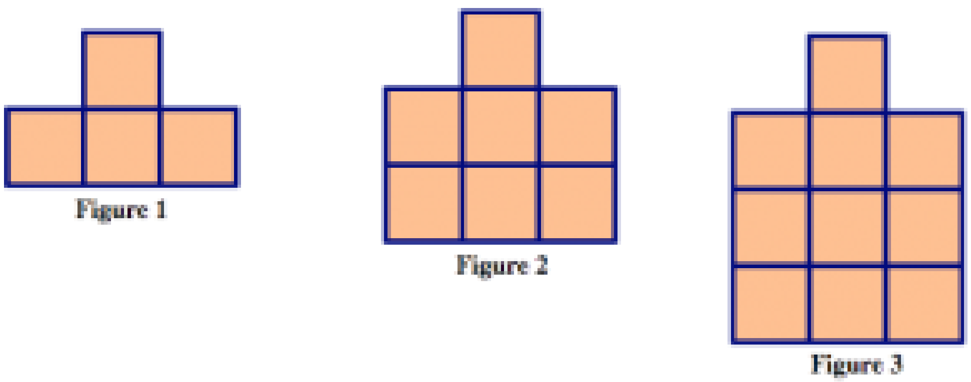
Problem 18



Problem 19



Problem 20



Matching Game

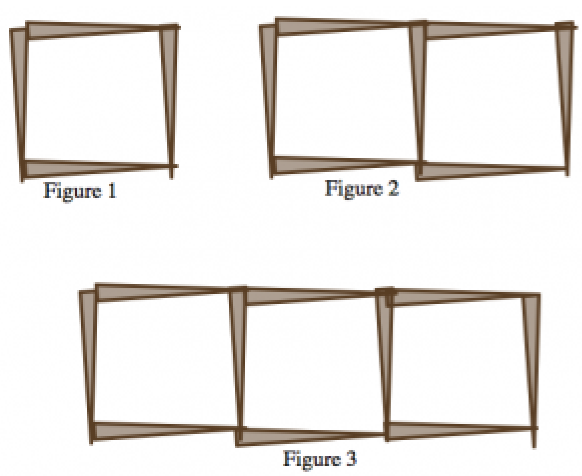
Below, you'll find patterns described in various ways: through visual representations, algebraic expressions, in tables of numbers, and in words. Your job is to match these up in a way that makes sense.

Note: there may be more than one algebraic expression to match a given pattern, or more than one pattern to match a given description. So be ready to justify your answers.

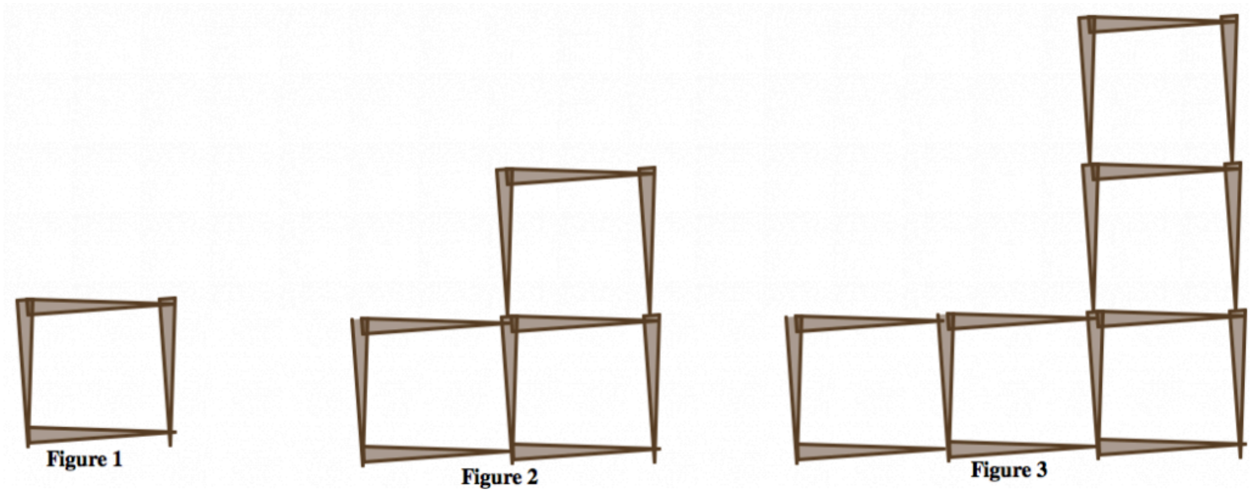
Algebraic Expressions

(a) t^2	(b) $2s+1$	(c) $2k+(k-1)+2k+(k-1)$
(d) $5n+5$	(e) $a+a$	(f) $3(\ell-1)+3(\ell-1)+4$
(g) $3b+1$	(h) $z+z+1$	(i) $m^2-(m-1)^2$
(j) $y \cdot y$	(k) $2x-1$	(l) $4e-(e-1)$
(m) $6f-2$	(n) $2c$	(o) $5(s+1)$

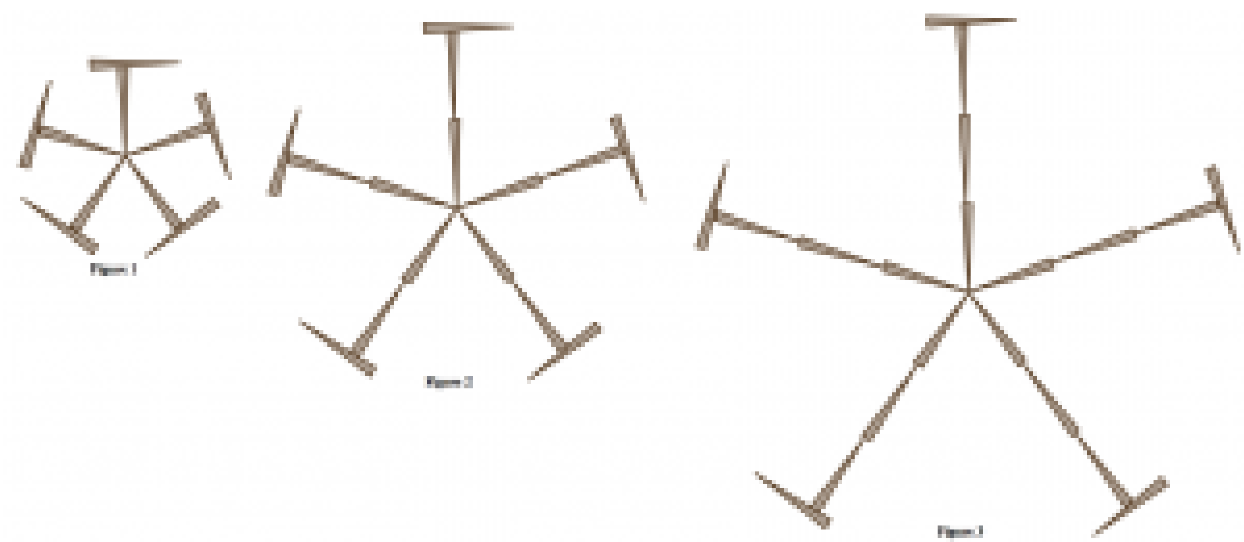
Visual Patterns



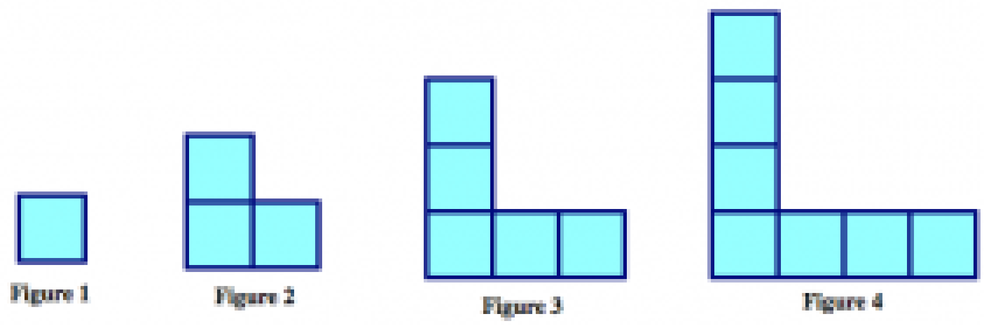
Pattern 1



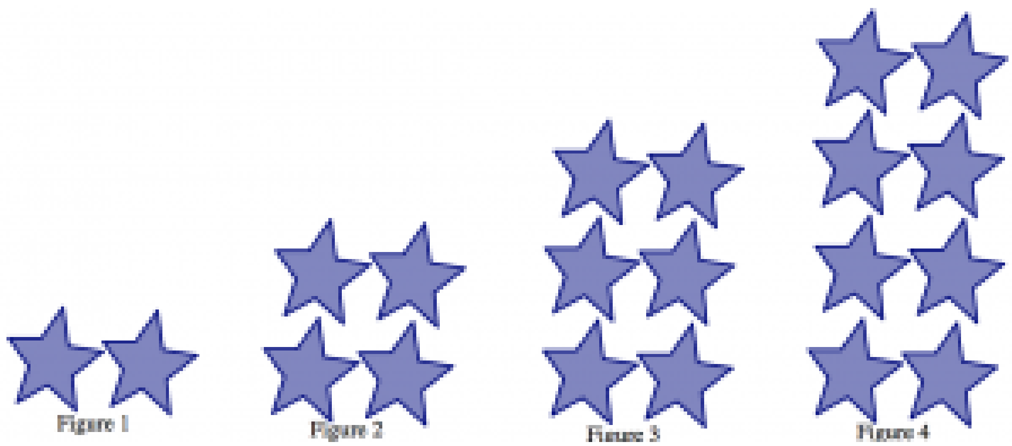
Pattern 2



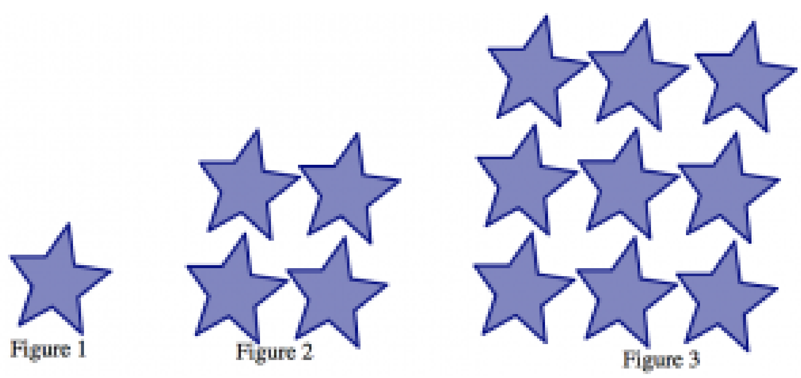
Pattern 3



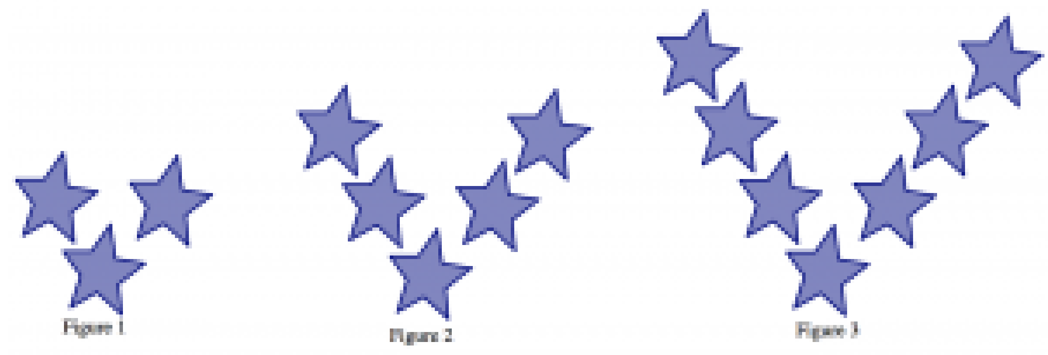
Pattern 4



Pattern 5



Pattern 6



Pattern 7

Tables of Numbers

Table A

Input	1	2	3	4
Output	1	4	9	16

Table B

Input	1	2	3	4
Output	10	15	20	25

Table C

Input	1	2	3	4
Output	1	3	5	7

Table D

Input	1	2	3	4
Output	3	5	7	9

Table E

Input	1	2	3	4
Output	4	7	10	13

Table F

Input	1	2	3	4
Output	4	10	16	22

Table G

Input	1	2	3	4
Output	2	4	6	8

Descriptions in Words

1. Count horizontal and vertical toothpicks separately. Horizontal: there are two rows of *n* toothpicks where *n* is the figure number. There are *n*-1 more of them on the vertical arm. The vertical toothpicks are just the same. There are two columns of *n* along the vertical arm, and then *n*-1 more of them on the horizontal arm.
2. To get a figure from the previous one, you add three toothpicks in a "C"' shape on the left side of the figure. The total number of toothpicks is three times the figure number, plus one extra to close off the square on the far right.
3. There are five spikes radiating out from the center. Each spike has the same number of toothpicks as the figure number. Each spike is capped off by one additional toothpick.
4. Each arm of the "L" shape has the same number of tiles as the figure number. But then we've counted the corner of the "L" twice, so we have to subtract one to get the total number of tiles needed.
5. The stars are in two equal rows, and each row has the same number of stars as the figure number.
6. To make the next figure, you always add five more toothpicks. Each arm has one more than the figure number of toothpicks, and there are five arms.
7. The stars are in a square, and the sides of the square have the same number of stars as the figure number.
8. Each arm of the "V" shape has the same number of stars as the figure number. Then we need to add one more star for the corner.
9. There are the same number of squares as the figure number, and each square uses four toothpicks. But then I've double-counted the toothpicks where the squares touch, so we have to subtract those out. There are one less of those than the figure number.
10. I can picture a square of tiles filled in. The side length of that square is the same as the figure number, so that's x^2. But then the square isn't really filled in. It's like I took away a square one size smaller from the top right, leaving just the border. What I took away was a square one size smaller, $(x-1)^2$.
11. Each time I go from one shape to the next, I add six new toothpicks. Three are added to the left in a "C" shape and three are added to the top in a rotated "C"' shape. So the total number will be six times the figure number plus or minus something. I can check to see that the right correction is to subtract 2.

Structural and Procedural Algebra

When most people think about algebra from school, they think about "solving for x." They imagine lots of equations with varying levels of complexity, but the goal is always the same: find the unknown quantity. This is a *procedural* view of algebra.

Even elementary students can be exposed to ideas in procedural algebra. This happens any time they think about unknown quantities and try to solve for them. For example, when first grade students learn to add and subtract numbers "within 10,'" they should frequently tackle problems like these:

- $3 + \_\_\_\_ = 7.$
- Find several pairs of numbers that add up to 10.

Although procedural algebra is important, it's not the most important skill, and it's certainly not the whole story.

You also need to foster thinking about *structural algebra* in your students: using symbols to express meaning in a situation. If there is an *x* on your page, you should be able to answer, "what does the *x* mean? What does it represent?"

Most of what you've done so far in this chapter is *structural algebra*. You've used letters and symbols not to represent a single unknown quantity, but a *varying* quantity. For example, in Section 4 you used letters to represent the "figure number'" or "case number" in a growing pattern. The letters could take on different values, and the expressions gave you information: how many tiles or toothpicks or stars you needed to build that particular figure in that particular pattern.

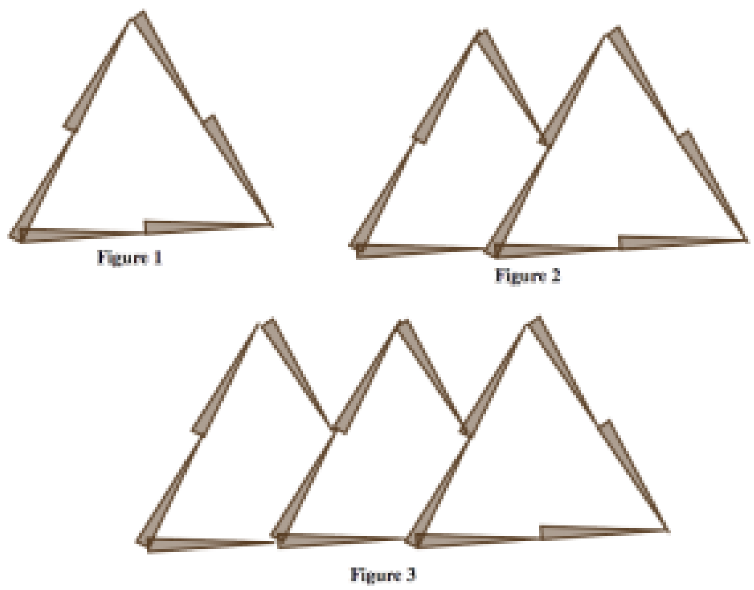
Think / Pair / Share

- Consider the expression $a + 3$. Give a real world situation that could be represented by this expression. Share your answer with your partner. Together, can you come up with even more ideas?
- Suppose the expression $3c + 2$ represents the number of tiles used at any stage of a growing pattern.
  - Evaluate the expression at $c = 1, 2, \text{ and } 3$. What do the values tell you about the pattern?

- Can you describe in words how the pattern is growing?
- Can you design a pattern with tiles that grows according to this rule?
- Where do you see the “3” in your pattern? Where do you see the “2”’? Where do you see the “c ”?

Problem 21

Krystal was looking at this pattern, which may be familiar to you from the Problem Bank:



She wrote down the equation

$$y = 4x + 2.$$

In Krystal’s equation, what does x represent? What does y represent? How do you know?

Problem 22

Candice was thinking about this problem:

> Today is Jennifer's birthday, and she's twice as old as her brother. When will she be twice as old as him again?

She wrote down the equation $2n = m$. In Candice's equation, what does n represent? What does m represent? How do you know?

Problem 23

Sarah and David collect old coins. Suppose the variable k stands for the number of coins Sarah has in her collection, and ℓ stands for the number of coins David has in his collection. What would each of these equations say about their coin collections?

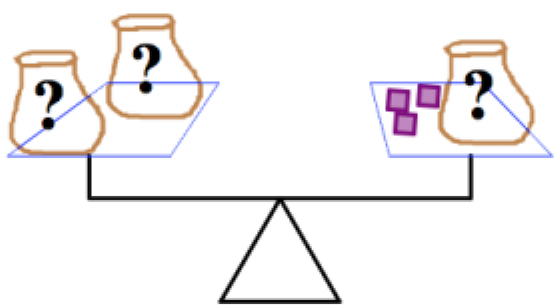
$(a)\ k = \ell + 1 \qquad (b)\ k = \ell \qquad (c)\ 3k = 2\ell \qquad (d)\ k = \ell - 11$

Problem 24

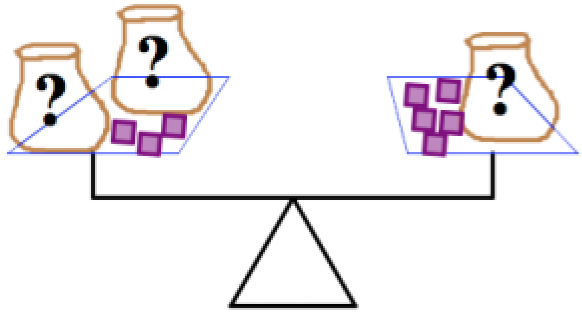
The pictures below show balance scales containing bags and blocks. The bags are marked with a "?"' because they contain some unknown number of blocks. In each picture:

- Each bag contains the same number of blocks.
- The scale is balanced.

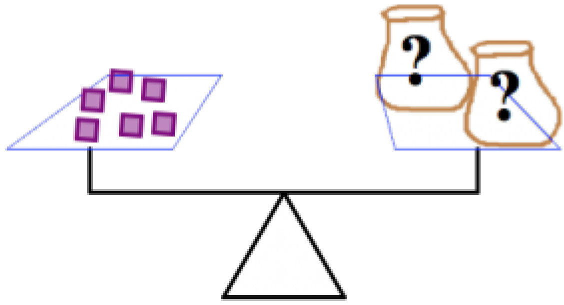
For each picture, determine how many blocks are in each bag. Justify your answers.



(a)



(b)



(c)

Problem 25

When he was working on Problem 24, Kyle wrote down these three equations.

(i) $2m = 6.$

(ii) $2x = x + 3.$

(iii) $z + 5 = 2z + 3.$

Match each equation to a picture, and justify your choices. Then solve the equations, and say (in a sentence) what the solution represents.

Problem 26

Draw a balance puzzle that represents the equation

$$2h + 3 = h + 8.$$

Now solve the balance puzzle. Where is the "h" in your puzzle? What does it represent?

Problem 27

Draw a balance puzzle that represents the equation

$$3b + 7 = 3b + 2.$$

Now solve the equation. Explain what happens.

Problem 28

Which equation below is most like the one in Problem 27 above? Justify your choice.

(a) $5{+}3 = 8$, (b) $\frac{2}{3}{+}\frac{1}{2} = \frac{3}{5}$, (c) $5{+}3 = y$, (d) $\frac{a}{5} = \frac{5}{a}$,

(e) $n + 3 = m$, (f) $3x = 2x + x$, (g) $5k = 5k + 1$.

Problem 29

Draw a balance puzzle that represents the equation

$$4\ell + 7 = 4\ell + 7.$$

Now solve the equation. Explain what happens.

Problem 30

Which equation below is most like the one in Problem 29 above? Justify your choice.

$(a)\ 5+3=8,$ $(b)\ \frac{2}{3}+\frac{1}{2}=\frac{3}{5},$ $(c)\ 5+3=y,$ $(d)\ \frac{a}{5}=\frac{5}{a},$

$(e)\ n+3=m,$ $(f)\ 3x=2x+x,$ $(g)\ 5k=5k+1.$

Problem 31

Create a balance puzzle where the solution is not a whole number of blocks. Can you solve it? Explain your answer.

Problem 32

There are three piles of rocks: pile A, pile B, and pile C. Pile B has two more rocks than pile A. Pile C has four times as many rocks as pile A. The total number of rocks in all three piles is 14.

1. Use *x* to represent the number of rocks in pile A, and write equations that describe the rules above. Then find the number of rocks in each pile.
2. Use *x* to represent the number of rocks in pile B, and write equations that describe the rules above. Then find the number of rocks in each pile.
3. Use *x* to represent the number of rocks in pile C, and write equations that describe the rules above. Then find the number of rocks in each pile.

Think / Pair / Share

Look back at Problems 21–32. Which of them felt like *structural algebraic thinking*? Which felt like *procedural algebraic thinking*? Did any of the problems feel like they involved both kinds of thinking?

Variables and Equations

You have seen that in algebra, letters and symbols can have different meanings depending on the context.

- A symbol could stand for some *unknown quantity*.
- A symbol could stand for some quantity that *varies*. (Hence the term "variable" to describe these symbols.)

In much the same way, *equations* can represent different things.

- They can represent a problem to be solved. This is the traditional procedural algebra type of question.
- They can represent a relationship between two or more quantities. For example, $A = s^2$ represents the relationship between the area of a square and its side length.
- They can represent *identities*: mathematical truths. For example,

  $$x^2 - 1 = (x+1)(x-1)$$

  is always true, for every value of x. There is nothing to solve for, and no relationship between varying quantities. (If you do try to "solve for x," you will get the equation $0 = 0$, much like you saw in Problem 29. Not very satisfying!)

Think / Pair / Share

Give an example of each type of equation. Be sure to say what the symbols in the equations represent.

Problem 33

Answer the following questions about the equation

$$x^2 - 1 = (x+1)(x-1).$$

1. Evaluate both sides of the above equation for $x = 1, 2, 3, 4, \text{ and } 5$. What happens?
2. Use the *distributive property* of multiplication over addition to expand the right side of the equation and simplify it.
3. Use the equation to compute 99^2 quickly, without using a calculator. Explain how you did it.

Problem Bank

Problems 34-36 ask you to solve problems about a strange veterinarian who created three mystifying machines.

Cat Machine: Place a cat in the input bin of this machine, press the button, and out jump two dogs and a mouse.

Dog Machine: This machine converts a dog into a cat and a mouse.

Mouse Machine: This machine can convert a mouse into a cat and three dogs.

Each machine can also operate in reverse. For example, if you have two dogs and a mouse, you can use the first machine to convert them into a cat.

Problem 34

The veterinarian hands you two cats, and asks you to convert them into exactly three dogs (no extra dogs and no other animals). Can you do it? If yes, say what process you would use. If no, say why not.

Problem 35

The veterinarian hands you one dog. He says he only wants cats, but he doesn't care how many. Can you help him? How?

Problem 36

The veterinarian hands you one cat. He says he only wants dogs, but he doesn't care how many. Can you help him? How?

Problems 37-40 present several growing patterns made with toothpicks. For each problem you work on, do the following:

1. Describe in words and pictures how you see the pattern growing.
2. Calculate the number of toothpicks you would need to build the 10th figure in the pattern. Justify your answer based on how the pattern grows.
3. Calculate the number of toothpicks you would need to build the 100th figure in the pattern.
4. Describe how you can figure out the number of toothpicks in any figure in the pattern. Be sure to justify your answer based on how the pattern grows.
5. Could you make one of the figures in the pattern using exactly 25 toothpicks? If yes, which figure? If no, why not? Justify your answer.
6. Could you make one of the figures in the pattern using exactly 100 toothpicks? If yes, which figure? If no, why not? Justify your answer.

Problem 37

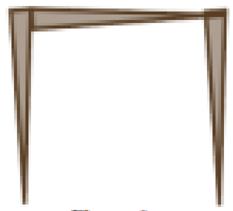


Figure 1

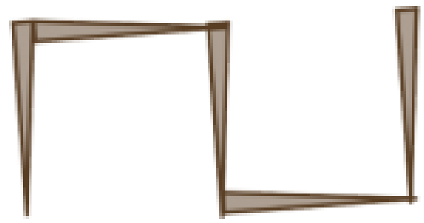


Figure 2

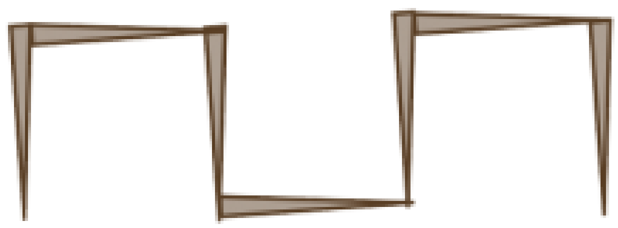
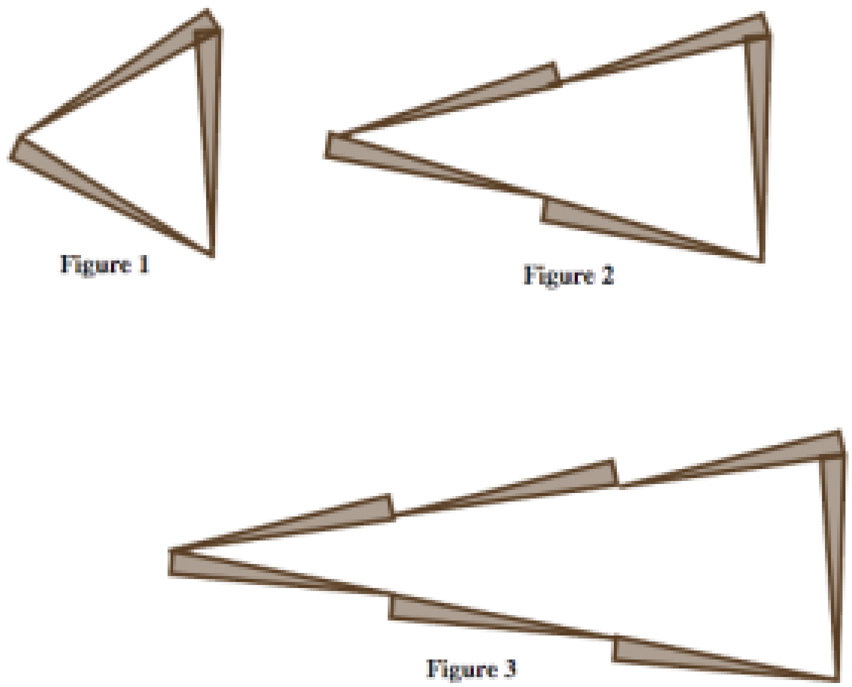
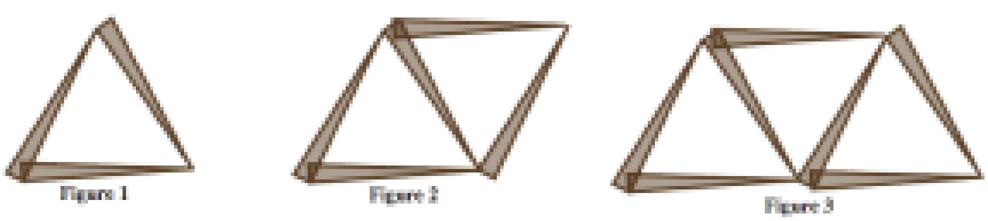


Figure 3

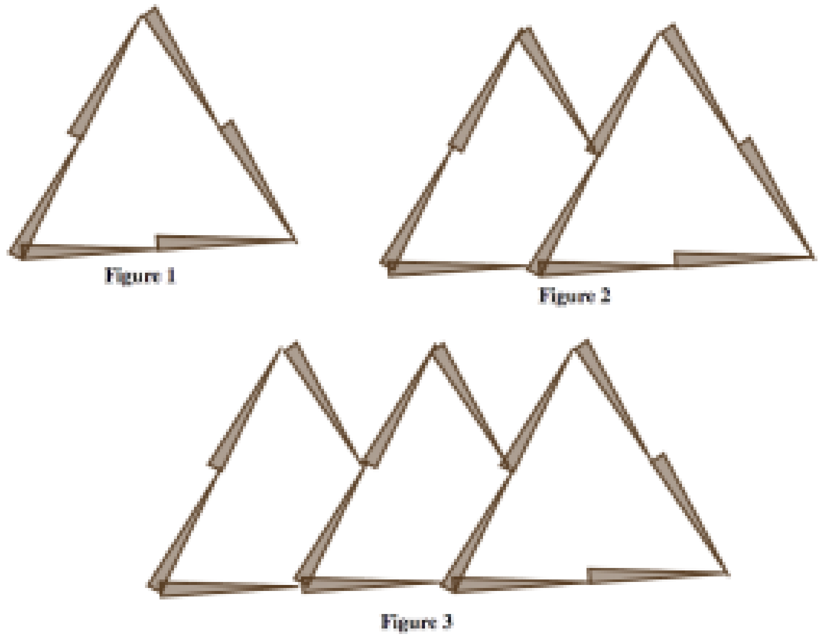
Problem 38



Problem 39



Problem 40



In a *mobile*, the arms must be perfectly balanced for it to hang properly. The artist Alexander Calder was famous for his artistic mobiles.You can view some of his amazing work here. Click “Explore Works.”

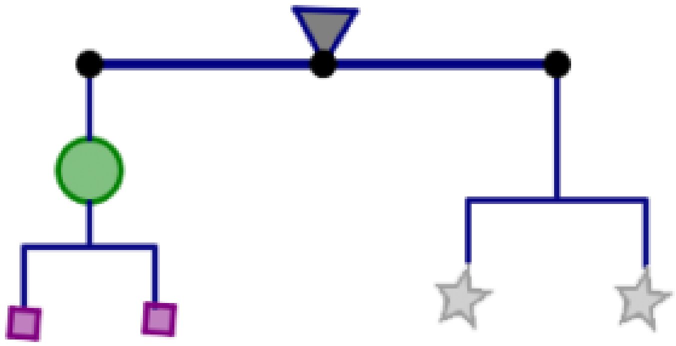
Problems 41-42 present you with mobile puzzles. In these puzzles:

- Objects that are the same shape have the same weight. (So all circles weigh the same, all squares weigh the same, etc.)
- Assume the strings and rods that hold the objects together don’t factor into the total weight.
- Each arm of the mobile must have exactly the same weight.

Problem 41

In this puzzle:

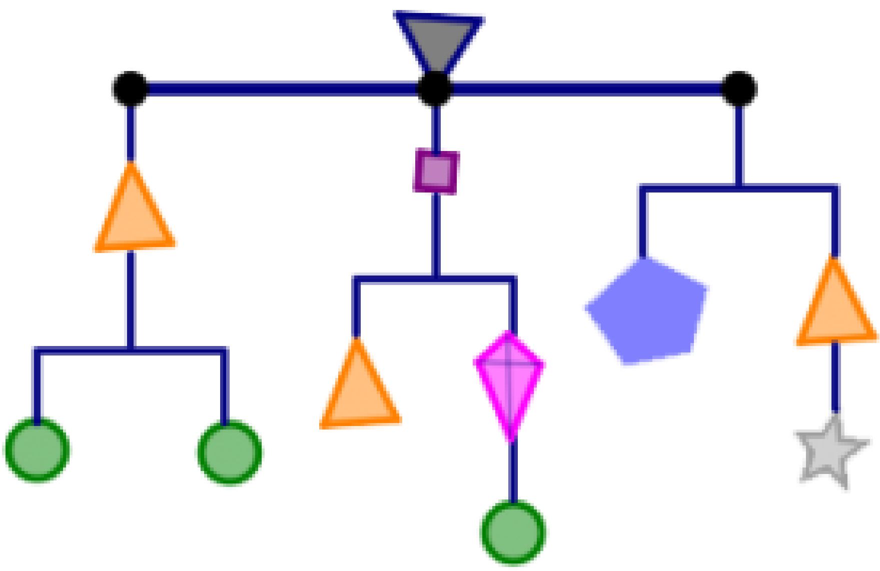
- The total weight is 36 grams.
- All shapes weigh less than 10 grams.
- All of the weights are whole numbers.
- One circle weighs more than one square.



Find the weight of each piece. Is there more than one answer? How do you know you are right?

Problem 42

In this puzzle, the total weight is 54 grams.



Find the weight of each piece. Is there more than one answer? How do you know you are right?

Place Value and Decimals

> I always say when you see that old black-and-white footage of the rocket on the launch pad and it falls over and explodes, that's because people had slide rules. Not having the decimal point is a real drawback. You want the decimal point, take it from me.
>
> -Bill Nye

The "Dots and Boxes" approach to place value used in this part (and throughout this book) and the "pies per child" approach to fractions comes from James Tanton, and are used with his permission. See his development of these and other ideas at http://gdaymath.com/.

Review of Dots & Boxes Model

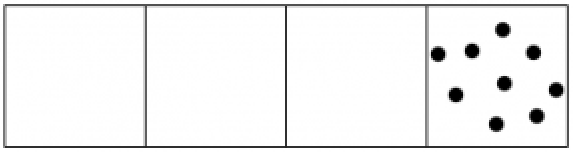
Let's start with a quick review of place value, different bases, and our "Dots & Boxes" model for thinking about these ideas.

The 1←2 Rule

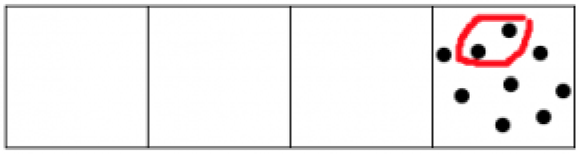
Whenever there are two dots in single box, they "explode," disappear, and become one dot in the box to the left.

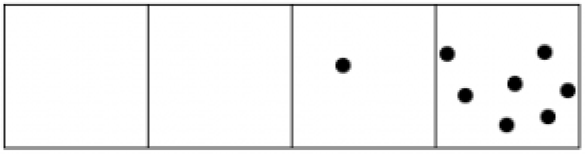
Example: Nine dots 1←2 in the system

We start by placing nine dots in the rightmost box.

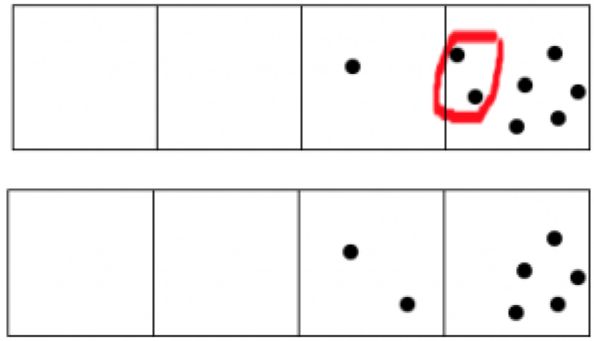


Two dots in that box explode and become one dot in the box to the left.

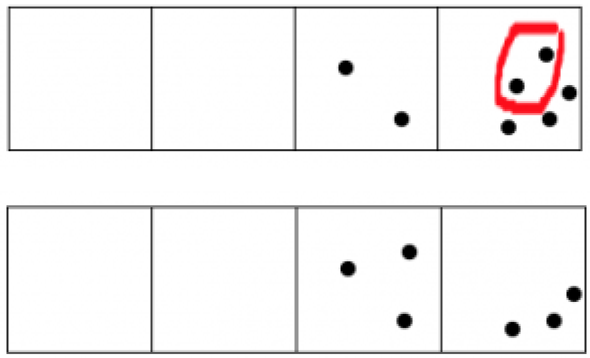




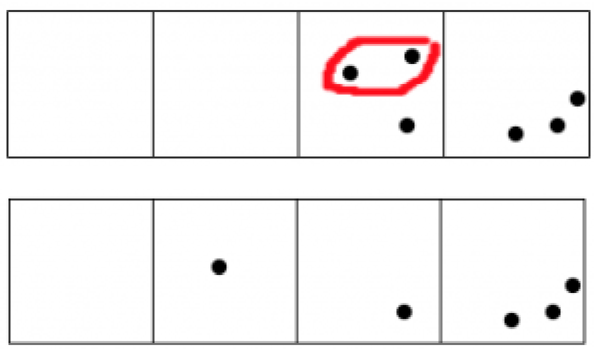
Once again, two dots in that box explode and become one dot in the box to the left.



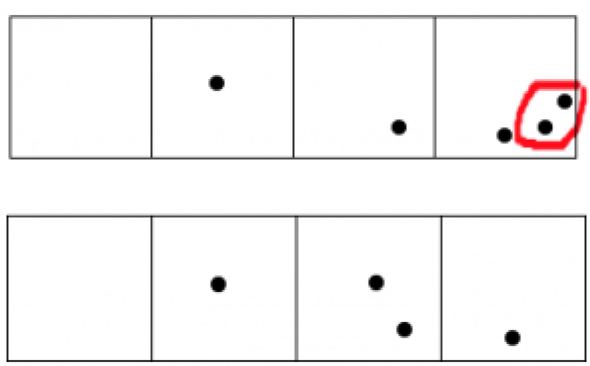
We do it again!



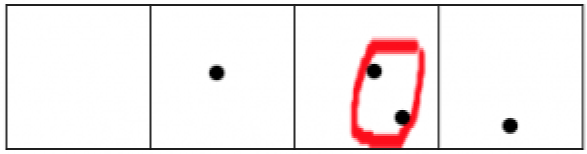
Hey, now we have more than two dots in the second box, so those can explode and move!

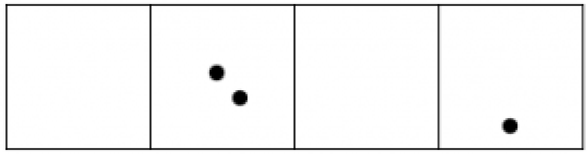


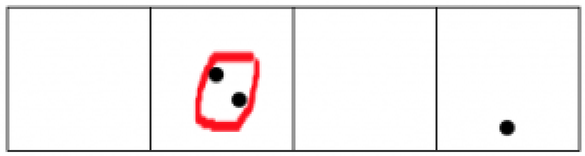
And the rightmost box still has more than two dots.

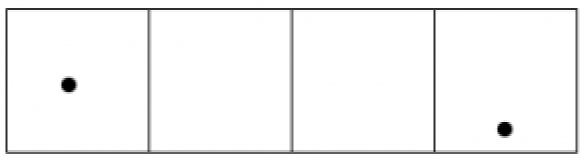


Keep going, until no box has two dots.









After all this, reading from left to right we are left with one dot, followed by zero dots, zero dots, and one final dot.

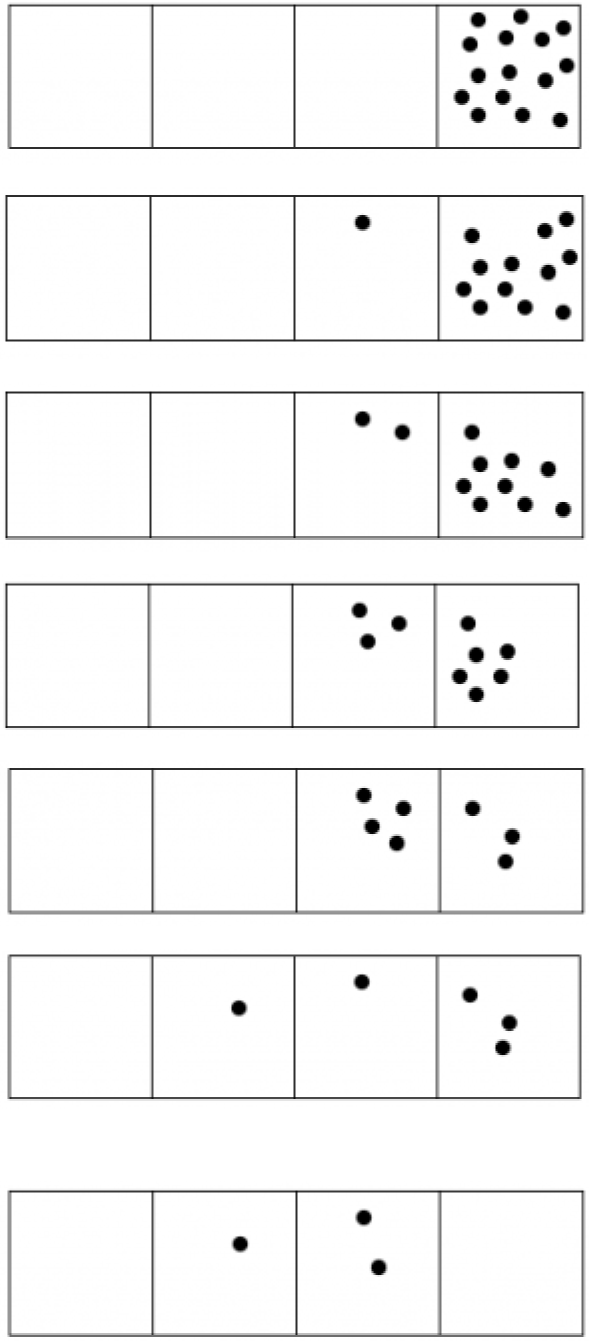
Solution: The $2 \leftarrow 1$ code for nine dots is: 1001.

The 1←3 Rule

Whenever there are three dots in single box, they "explode," disappear, and become one dot in the box to the left.

Example: Fifteen dots in the 1←3 system

Here's what happens with fifteen dots:



Solution: The $1 \leftarrow 3$ code for fifteen dots is: 120.

Definition

Recall that numbers written in the $1 \leftarrow 2$ system are called **binary** or **base two** numbers.

Numbers written in the $1 \leftarrow 3$ system are called **base three** numbers.

Numbers written in the $1 \leftarrow 4$ system are called **base four** numbers.

Numbers written in the $1 \leftarrow 10$ system are called **base ten** numbers.

In general, numbers written in the $1 \leftarrow b$ system are called **base *b*** numbers.

In a base b number system, each place represents a power of b, which means b^n for some whole number n. Remember this means b multiplied by itself n times:

- The right-most place is the units or ones place. (Why is this a power of b?)
- The second spot is the "b" place. (In base ten, it's the tens place.)
- The third spot is the "b^2" place. (In base ten, that's the hundreds place. Note that $10^2 = 100$.)
- The fourth spot is the "b^3" place. (In base ten, that's the thousands place, since $10^3 = 1000$.)
- And so on.

Notation

Whenever we're dealing with numbers written in different bases, we use a subscript to indicate the base so that there can be no confusion. So:

- 102_{three} is a base three number (read it as "one-zero-two base three"). This is the base three code for the number eleven.
- 222_{four} is a base four number (read it as "two-two-two base four"). This is the base four code for the number forty-two.
- 54321_{ten} is a base ten number. (It's ok to say "fifty-four thousand three hundred and twenty-one." Why?)

If the base is not written, we assume it's base ten.

Remember: when you see the subscript, you are seeing the **code** for some number of dots.

Think / Pair / Share

Work through the two examples above carefully to be sure you remember and understand how the "Dots & Boxes" model works. Then answer these questions:

- When we write 9 in base 2, why do we write 1001_{two} instead of just 11_{two}?
- When we write 15 in base 3, why do we write 120_{three} instead of just 12_{three}?
- How many different digits do you need in a base 7 system? In a base 12 system? In a base b system? How do you know?

On Your Own

Work on the following exercises on your own or with a partner.

1. In base 4, four dots in one box are worth one dot in the box one place to the left.
   - a. What is the value of each box?
   - b. How do you write 29_{ten} in base 4?
   - c. How do you write 132_{four} in base 10?
2. In our familiar base ten system, ten dots in one box are worth one dot in the box one place to the left.
   - a. What is the value of each box?
   - b. When we write the base ten number 7842:
     - i. What quantity does the "7" represent?
     - i. The "4" is four groups of what value?
     - i. The "8" is eight groups of what value?
     - i. The "2" is two groups of what value?
3. Write the following numbers of dots in base two, base three, base five, and base eight. Draw the "Dots & Boxes" model if it helps you remember how to do this! (Note: these numbers are all written in base ten. When we don't say otherwise, you should assume base ten.)

   $(a)\ 2 \qquad (b)\ 17 \qquad (c)\ 27 \qquad (d)\ 63.$

4. Convert these numbers to our more familiar base ten system. Draw out dots and boxes and "unexplode" the dots if it helps you remember.

   $(a)\ 1101_{\text{two}} \qquad (b)\ 102_{\text{three}} \qquad (c)\ 24_{\text{five}} \qquad (d)\ 24_{\text{nine}}.$

Think / Pair / Share

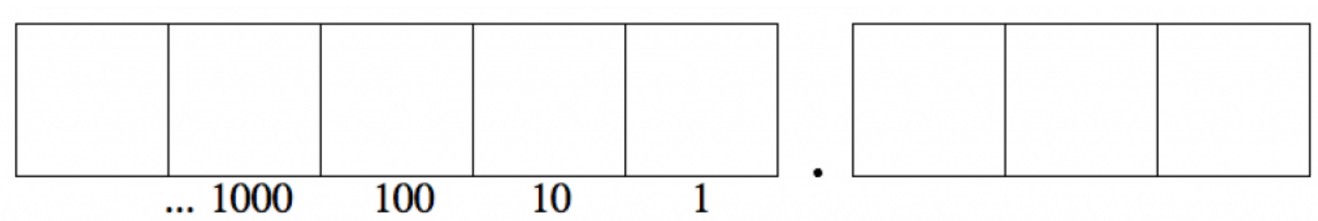
Quickly compute each of the following. Write your answer in the same base as the problem.

- 131_{ten} times ten.
- 263207_{eight} times eight.
- 563872_{nine} times nine.
- Use the $1 \leftarrow 10$ system to explain why multiplying a whole number in base ten by ten results in simply appending a zero to the right end of the number.
- Suppose you have a whole number written in base b. What is the effect of multiplying that number by b? Justify what you say.

Decimals

Up to now our "Dots & Boxes" model has consisted of a row of boxes extending infinitely far to the left. Why not have boxes extending to the right as well?

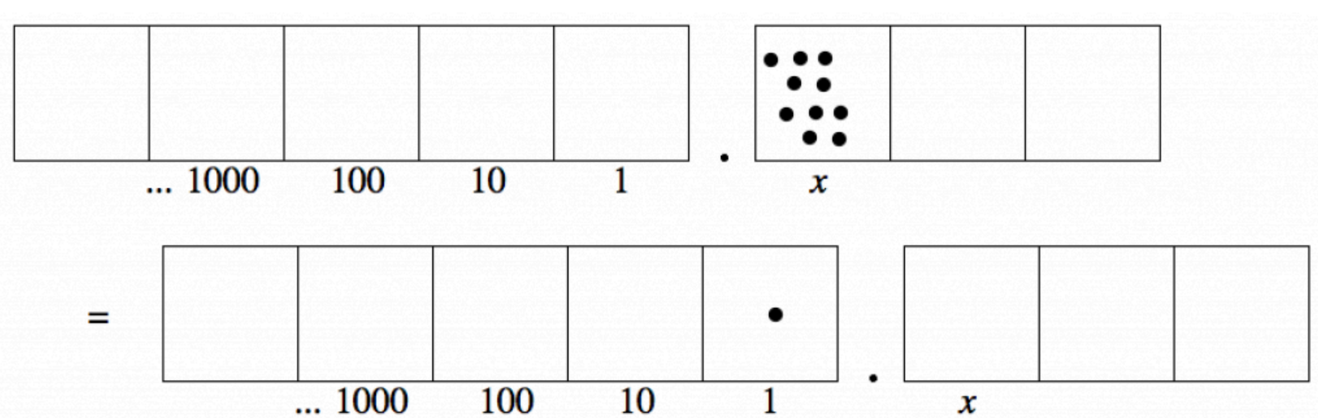
Let's work specifically with a 1←10 rule and see what boxes to the right could mean.



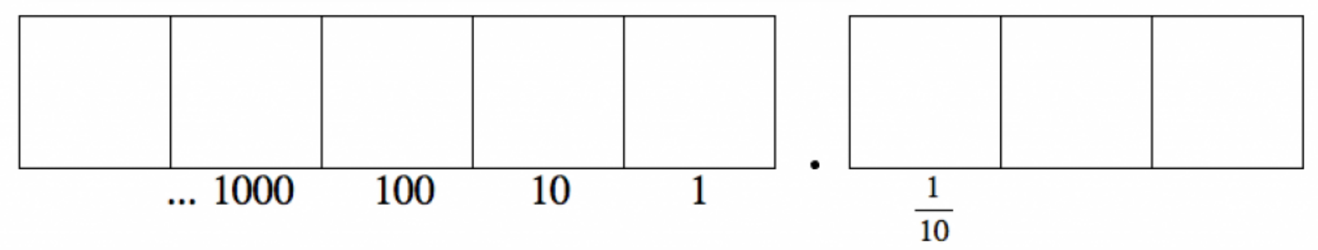
Notation

It has become convention to separate boxes to the right of the ones place with a decimal point. (At least, this is what the point is called in the base ten world… "dec" means "ten" after all!)

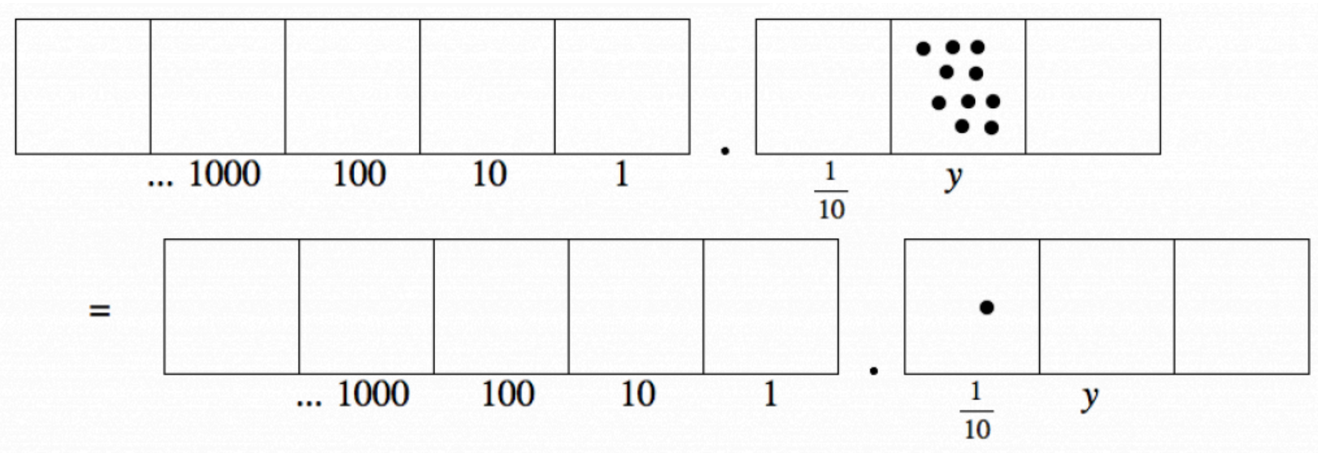
What is the value of the first box to the right of the decimal point? If we denote its value as x, we have that ten x 's is equivalent to 1. (Remember, we are using a 1 ← 10 rule.)



From $10x = 1$ we get that $x = \frac{1}{10}$.

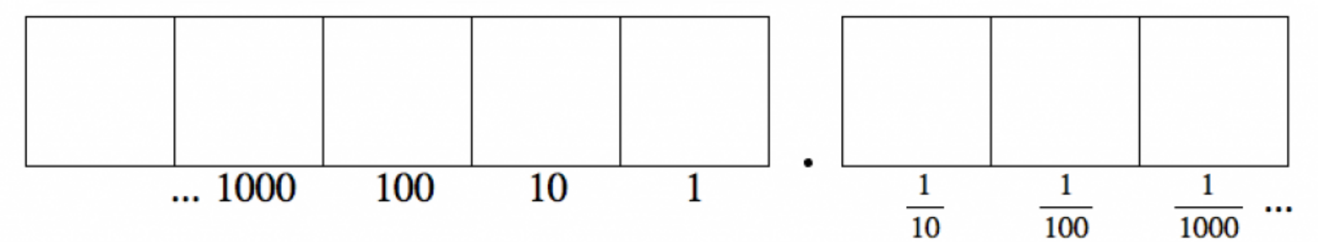


Call the value of the next box to the right y.



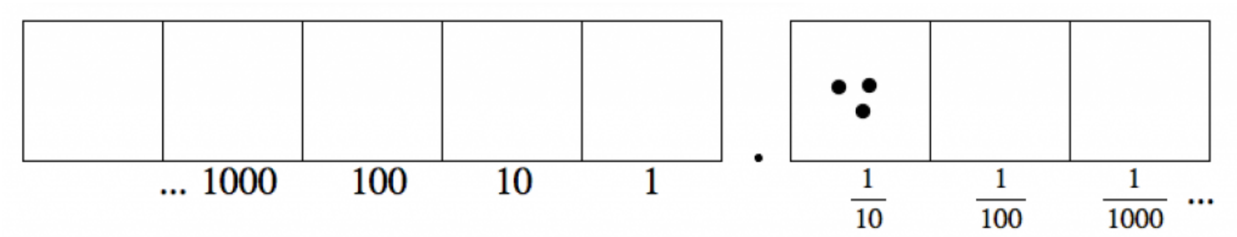
From $10y = \frac{1}{10}$ we get $y = \frac{1}{100}$.

If we keep doing this, we see that the boxes to the right of the decimal point represent the reciprocals of the powers of ten.



Example: 0.3

The decimal 0.3 is represented by the picture:

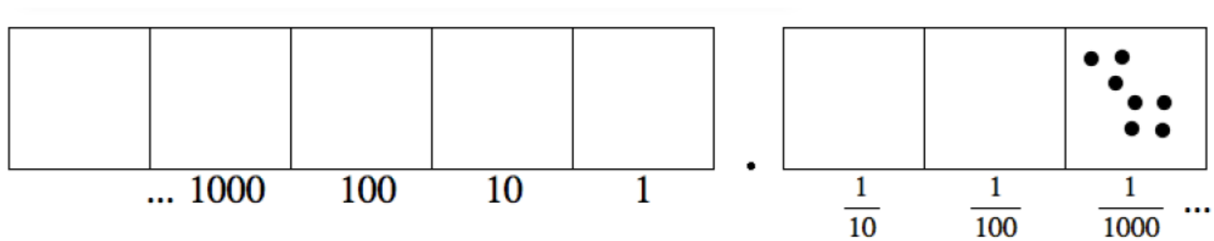


It represents three groups of $\frac{1}{10}$, that is:

$$0.3 = \frac{3}{10}.$$

Example: 0.007

The decimal 0.007 is represented by the picture:



It represents seven groups of $\frac{1}{1000}$.

Of course, some decimals represent fractions that can simplify further. For example:

$$0.5 = \frac{5}{10} = \frac{1}{2}.$$

Similarly, if a fraction can be rewritten to have a denominator that is a power of ten, then it is easy to convert it to a decimal. For example, $\frac{3}{5}$ is equivalent to $\frac{6}{10}$, and so we have:

$$\frac{3}{5} = \frac{6}{10} = 0.6.$$

Example: 12 3/4

Can you write $12\frac{3}{4}$ as a decimal? Well,

$$12\tfrac{3}{4} = 12 + \tfrac{3}{4}.$$

We can write the denominator as a power of ten using the key fraction rule:

$$\frac{3}{4} \cdot \frac{25}{25} = \frac{75}{100}.$$

So we see that:

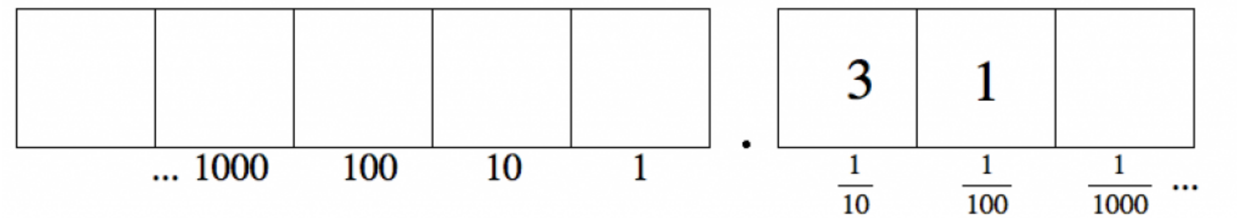
$$12 + \tfrac{3}{4} = 12 + \tfrac{75}{100} = 12.75.$$

Think / Pair / Share

- Draw a "Dots & Boxes" picture for each of the following decimals. Then say what fraction each decimal represents:
  $0.09, \quad 0.003, \quad 0.7, \quad 0.0000003.$
- Draw a "Dots & Boxes" picture for each of the following fractions. Then write the fraction as a decimal:
  $\frac{1}{1000}, \quad \frac{7}{100}, \quad \frac{9}{10}.$
- What fractions (in simplest terms) do the following decimals represent?
  $0.05, \quad 0.2, \quad 0.8, \quad 0.004.$
- Use the key fraction rule to write the following fractions as decimals. *Do not use a calculator!*
  $\frac{2}{5}, \quad \frac{1}{25}, \quad \frac{1}{20}, \quad \frac{1}{200}, \quad \frac{1}{1250}.$
- Some people read 0.6 out loud as "point six." Others read it out loud as "six tenths." Which is more helpful for understanding what the number really is? Why do you think so?

Example: 0.31

Here is a more interesting question: What fraction is represented by the decimal 0.31?



There are two ways to think about this.

Approach 1:

From the picture of the "Dots & Boxes" model we see:

$$0.31 = \frac{3}{10} + \frac{1}{100}.$$

We can add these fractions by finding a common denominator:

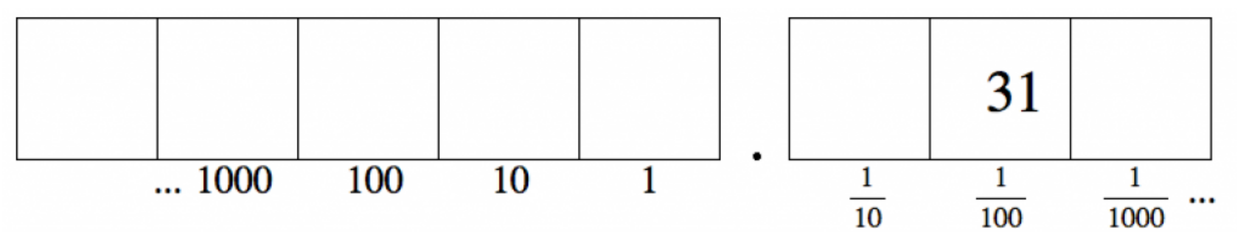
$$\frac{3}{10} + \frac{1}{100} = \frac{30}{100} + \frac{1}{100} = \frac{31}{100}.$$

So

$$0.31 = \frac{31}{100}.$$

Approach 2:

Let's unexplode the three dots in the $\frac{1}{10}$ position to produce an additional 30 dots in the $\frac{1}{100}$ position.



So we can see right away that

$$0.31 = \frac{31}{100}.$$

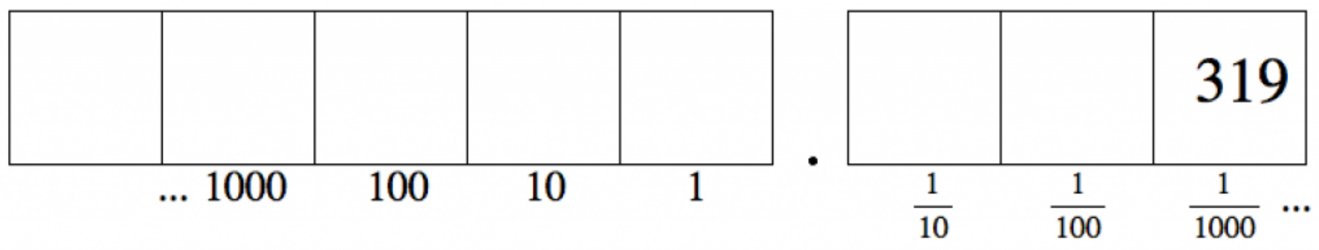
On Your Own

Work on the following exercises on your own or with a partner.

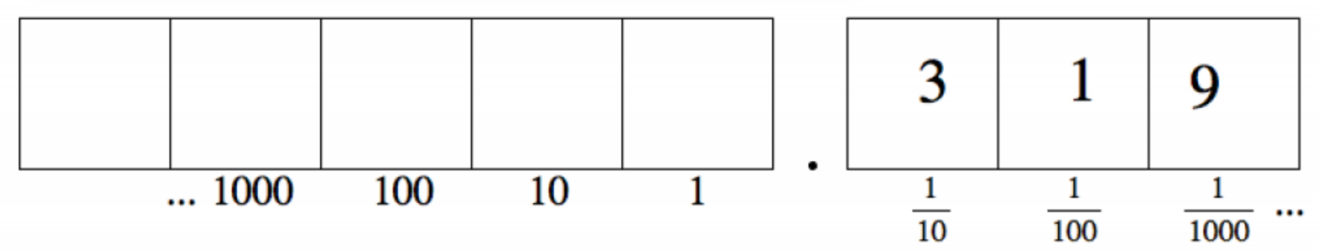
1. Brian is having difficulty seeing that 0.47 represents the fraction $\frac{47}{100}$. Describe the two approaches you could use to explain this to him.

2. A teacher asked his students to each draw a “Dots & Boxes” picture of the fraction $\frac{319}{1000}$.

Jin drew this:



Sonia drew this:



The teacher marked both students as correct.

- Are each of these solutions correct? Explain your thinking.
- Jin said he could get Sonia’s solution by performing some explosions. What did he mean by this? Is he right?

3. Choose the best answer and justify your choice. The decimal 0.23 equals:

$(a)\ \frac{23}{10}$ $\quad (b)\ \frac{23}{100}$

$(c)\ \frac{23}{1,000}$ $\quad (d)\ \frac{23}{10,000}$.

4. Choose the best answer and justify your choice. The decimal 0.0409 equals:

$(a)\ \frac{409}{100}$ $\quad (b)\ \frac{409}{1,000}$

$(c)\ \dfrac{409}{10,000}$ $(d)\ \dfrac{409}{100,000}.$

5. Choose the best answer and justify your choice. The decimal 0.050 equals:

$(a)\ \dfrac{50}{100}$ $(b)\ \dfrac{1}{20}$

$(c)\ \dfrac{1}{200}$ (d) None of these.

6. Choose the best answer and justify your choice. The decimal 0.000204 equals:

$(a)\ \dfrac{51}{250}$ $(b)\ \dfrac{51}{2500}$

$(c)\ \dfrac{51}{25000}$ $(d)\ \dfrac{51}{250000}.$

7. What fraction is represented by each of the following decimals?

$(a)\ 0.567$ $(b)\ 0.031$

$(c)\ 0.4077$ $(d)\ 0.101.$

8. Write each of the following fractions as decimals. Don't use a calculator!

$(a)\ \dfrac{73}{100}$ $(b)\ \dfrac{519}{1000}$

$(c)\ \dfrac{71}{1000}$ $(d)\ \dfrac{7001}{10000}.$

9. Write each of the following fractions as decimals. Don't use a calculator!

$(a)\ \dfrac{7}{20}$ $(b)\ \dfrac{16}{25}$

$(c)\ \dfrac{301}{500}$ $(d)\ \dfrac{17}{50}$ $(e)\ \dfrac{3}{4}.$

10. Write each of the following as a fraction (or mixed number).

$(a)\ 2.3$ $(b)\ 17.04$ $(c)\ 1003.1003$

11. Write each of the following numbers in decimal notation.

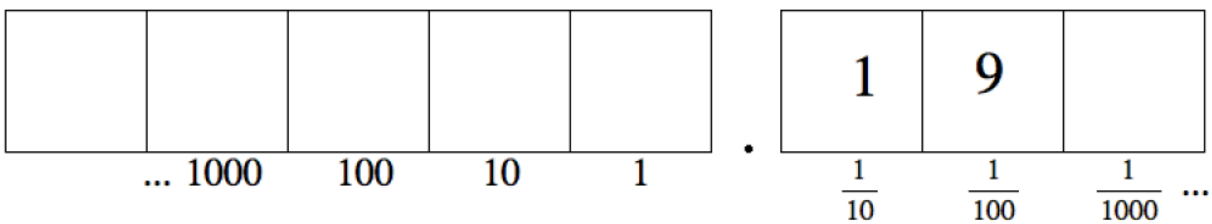
$(a)\ 5\frac{3}{10}$ $(b)\ 7\frac{1}{5}$ $(c)\ 13\frac{1}{2}$ $(d)\ 106\frac{3}{20}$

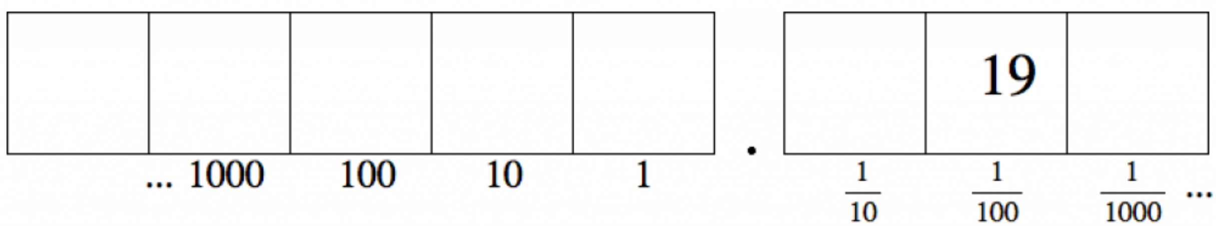
$(e)\ \frac{78}{25}$ $(f)\ \frac{9}{4}$ $(g)\ \frac{131}{40}$

Think Pair Share

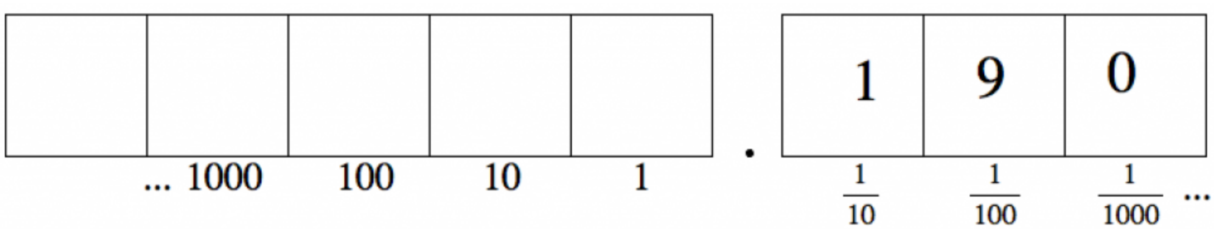
Do 0.19 and 0.190 represent the same number or different numbers?

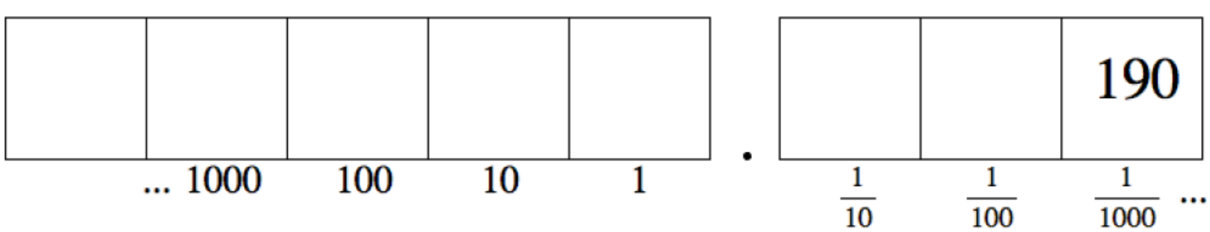
Here are two dots and boxes pictures for the decimal 0.19.





And here are two dots and boxes picture for the decimal 0.190.

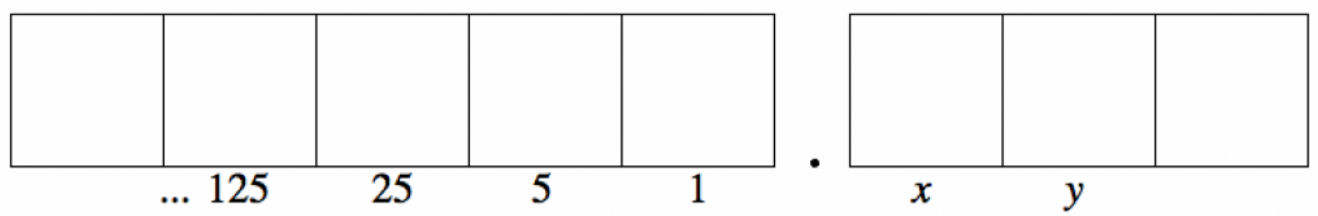




- Explain how one "unexplosion" establishes that the first picture of 0.19 is equivalent to the second picture of 0.19.
- Explain how several unexplosions establishes that the first picture of 0.190 is equivalent to the second picture of 0.190.
- Use explosions and unexplosions to show that all four pictures are equivalent to each other.
- So … does 0.19 represent the same number as 0.190?

x-mals

Just like in base 10, we can add boxes to the right of the decimal point in other bases, like base 5.

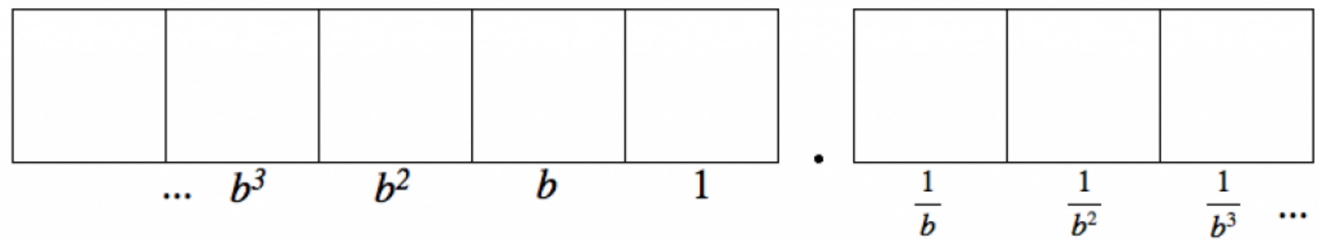


However, the prefix “dec” in “decimal point” means ten. So we really shouldn’t call it a decimal point anymore. Maybe a “pentimal point”? (In fact, the general term is **radix point**.)

Think / Pair / Share

- Use reasoning like you saw on page 6 for the base ten system to think about other number systems:
- Figure out the values of x and y in the picture of the base-5 system above. Be sure you can explain your reasoning.
- Draw a base-4 “Dots & Boxes” model, including a radix point and some boxes to the right. Label at least three boxes to the left of the ones place and three boxes to the right of the ones place.
- Draw a base-6 “Dots & Boxes” model, including a radix point and some boxes to the right. Label at least three boxes to the left of the ones place and three boxes to the right of the ones place

In general, in a base-b system, the boxes to the left of the ones place represent positive powers of the base b. Boxes to the right of the ones place represent reciprocals of those powers.



On Your Own

Work on the following exercises on your own or with a partner.

1. Draw a "Dots & Boxes" picture of each number.

   (a) 0.03_{five} (b) 0.22_{six}

   (c) 0.103_{four} (d) 0.002_{three}.

2. Find a familiar (base-10) fraction value for each number.

   (a) 0.04_{five} (b) 0.3_{six}

   (c) 0.02_{four} (d) 0.03_{nine}.

3. Find a familiar (base-10) fraction value for each number. (You might want to re-read the example of 0.31 in base ten from the previous chapter.)

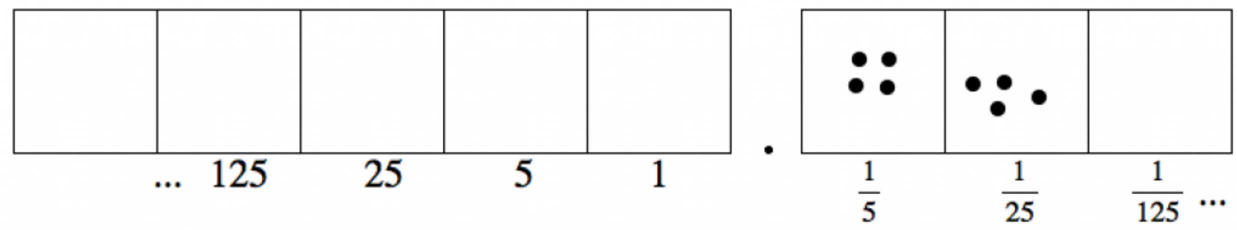
   (a) 0.13_{five} (b) 0.25_{six} (c) 0.101_{two}

   (d) 0.24_{seven} (e) 0.55_{eight} .

Think / Pair / Share

Tami and Courtney were working on converting 0.44_{five} to a familiar base-10 fraction. Courtney said this:

> The places in base five to the right of the point are like $\frac{1}{5}$ and then $\frac{1}{25}$. Since this has two places, the answer should be $\frac{44}{25}$.



Tami thought about what Courtney said and replied:

> I don't know what the right answer is, but I know that can't be right. The number 0.44_{five} is less than one, since there are no numbers in the ones place and no explosions that we can do. But the fraction $\frac{44}{25}$ is more than one. It's almost two. So they can't be the same number.

- Who makes the most sense, Courtney or Tami? Why do you think so?
- Find the right answer to the problem Courtney and Tami were working on.

Problem 1

Find the "decimal" representation of $\frac{1}{4}$ in each of the following bases. Be sure that you can justify your answer. (You might want to review the example of $12\frac{3}{4}$ in the previous chapter.)

base 2	base 4	base 6
base 8	base 10	base 12

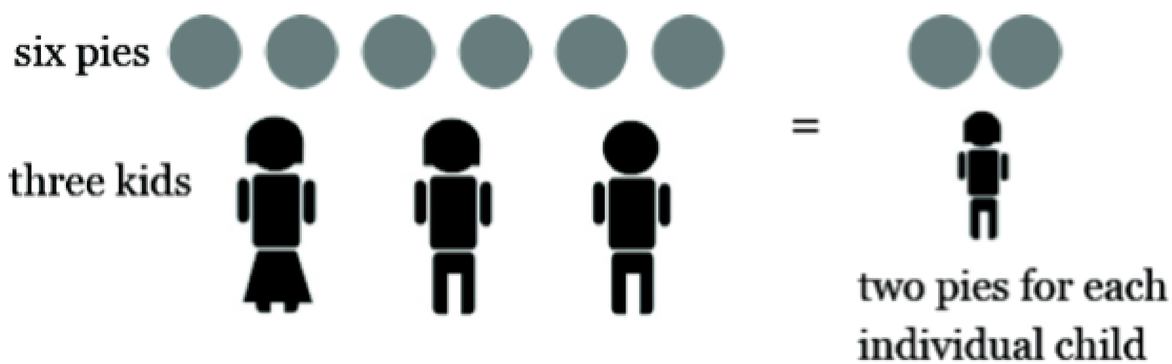
Division and Decimals

When you studied fractions, you had lots of different ways to think about them. But the first way, and the one we keep coming back to, is to think of a fraction as the answer to a division problem.

Example

Suppose 6 pies are to be shared equally among 3 children. This yields 2 pies per child. We write:

$$\frac{6}{3} = 2.$$



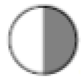
The fraction $\frac{6}{3}$ is equivalent to the answer to the division problem $6 \div 3 = 2$. It represents the number of pies one whole child receives.

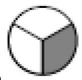
In the same way…

sharing 10 pies among 2 kids yields $\frac{10}{2} = 5$ pies per kid,

sharing 8 pies among 2 kids yields $\frac{8}{2} = 4$ pies per kid,

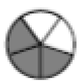
sharing 5 pies among 5 kids yields $\frac{5}{5} = 1$ pies per kid, and the answer to sharing 1 pies among 2 children is $\frac{1}{2}$, which we call “one-half.”

We associate the number “$\frac{1}{2}$” to the picture .

In the same way, the picture  represents “one third,” that is, $\frac{1}{3}$.

(This is the amount of pie an individual child would receive if one pie is shared among three children.)

The picture  is called “one fifth” and is indeed $\frac{1}{5}$, the amount of pie an individual child receives when one pie is shared by five kids.

And the picture  is called “three fifths” to represent $\frac{3}{5}$, the amount of pie an individual receives if three pies are shared by five kids.

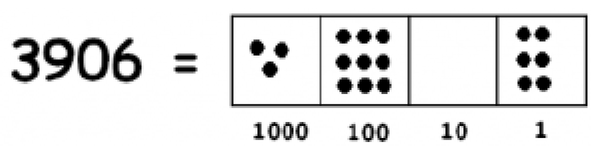
We know how to do division in our “Dots & Boxes” model.

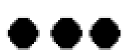
Example: 3906 ÷ 3

Suppose you are asked to compute $3906 \div 3$. One way to interpret this question (there are others) is:

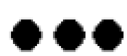
“How many groups of 3 fit into 3906?”

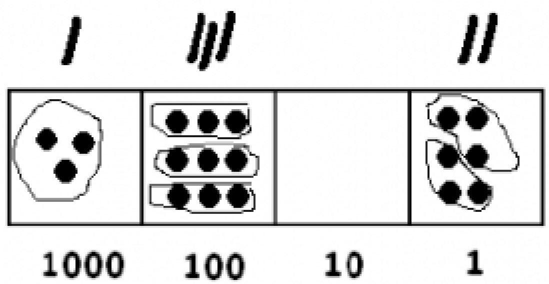
In our “Dots & Boxes” model, the dividend 3906 looks like this:



and three dots looks like this: 

So we are really asking:

“How many groups of  fit into the picture of 3906?”



Notice what we have in the picture:

- One group of 3 in the thousands box.
- Three groups of 3 in the hundreds box.
- Zero groups of 3 in the tens box.
- Two groups of 3 in the ones box.

This shows that 3 goes into 3906 one thousand, three hundreds and two ones times. That is, $3906 \div 3 = 1302$.

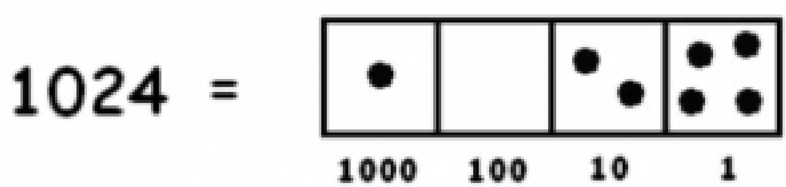
Of course, not every division problem works out evenly! Here's a different example.

Example: 1024 ÷ 3

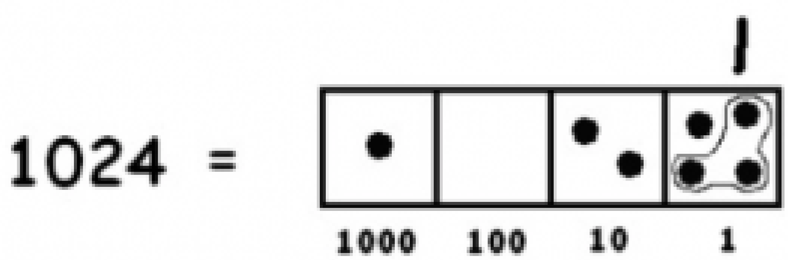
Suppose you are asked to compute $1024 \div 3$. One way to interpret this question is:

"How many groups of 3 fit into 1024?"

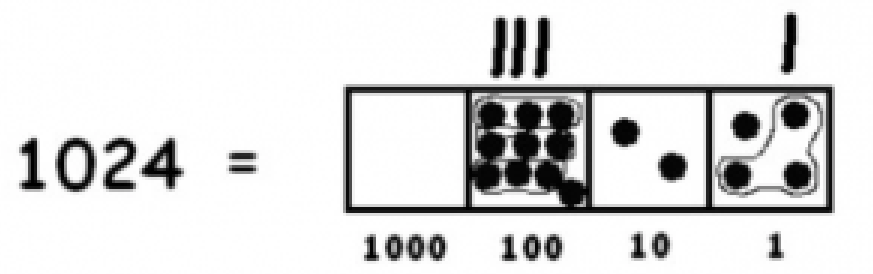
So we're looking for groups of three dots in this picture:



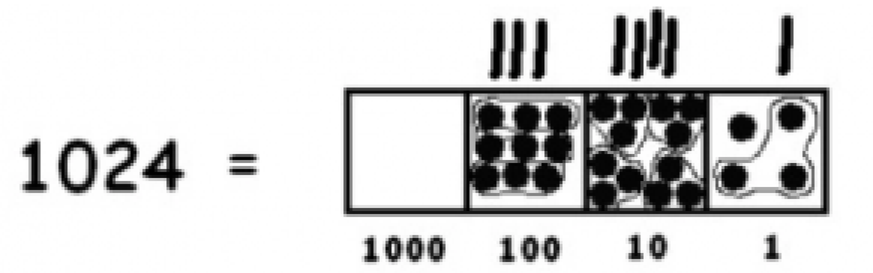
One group of three is easy to spot:



To find more groups of three dots, we must "unexplode" a dot:



We need to unexplode again:



This leaves one stubborn dot remaining in the ones box and no more group of three. So we conclude:

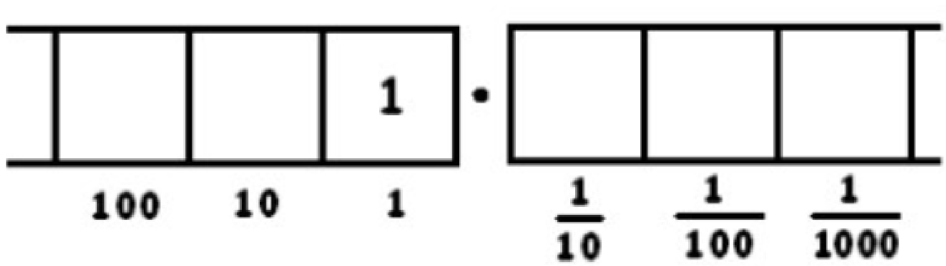
$$1024 \div 3 \;=\; 341 \;\; R1, \;\; \text{meaning } 1024 = 341 \cdot 3 + 1.$$

In words: 1024 gives 341 groups of 3, plus one extra dot.

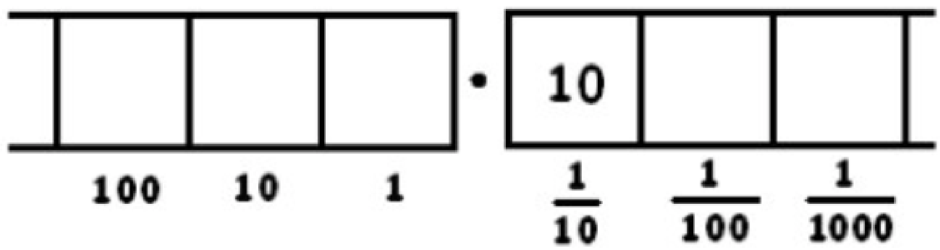
We can put these two ideas together — fractions as the answer to a division problem and what we know about division in the "Dots & Boxes" model — to help us think more about the connection between fractions and decimals.

Example: 1/8

The fraction $\frac{1}{8}$ is the result of dividing 1 by 8. Let's actually compute $1 \div 8$ in a "Dots & Boxes" model, making use of decimals. We want to find groups of eight in the following picture:

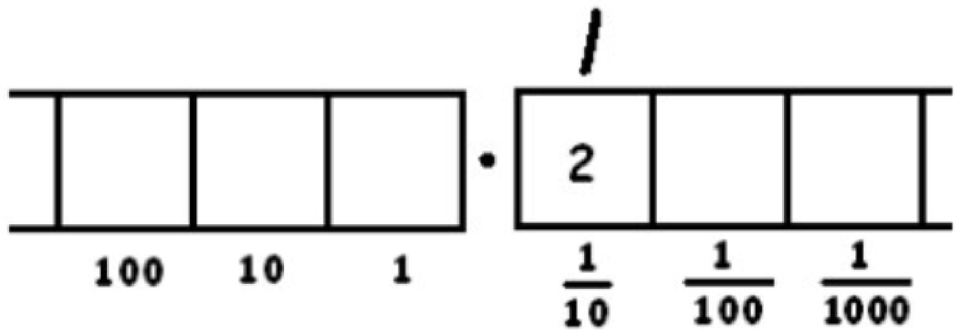


Clearly none are to be found, so let's unexplode:

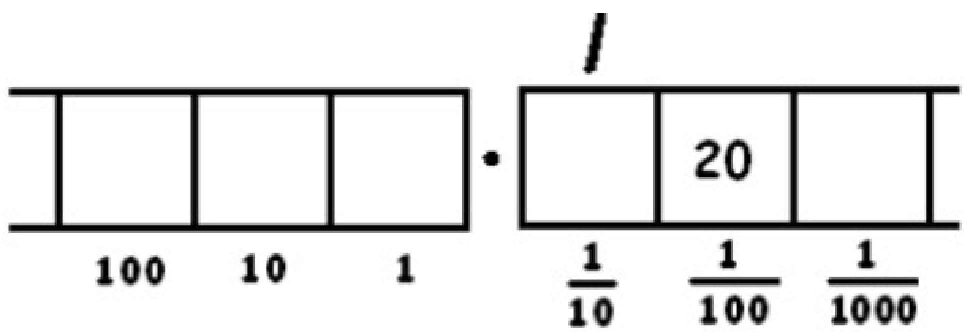


(We're being lazy and not drawing all the dots. As you follow along, you might want to draw the dots rather than the number of dots, if it helps you keep track.)

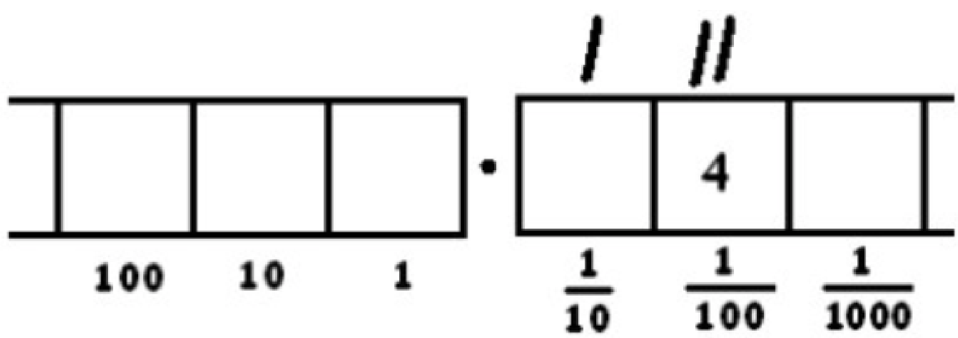
Now there is one group of 8, leaving two behind. We write a tick-mark on top, to keep track of the number of groups of 8, and leave two dots behind in the box.



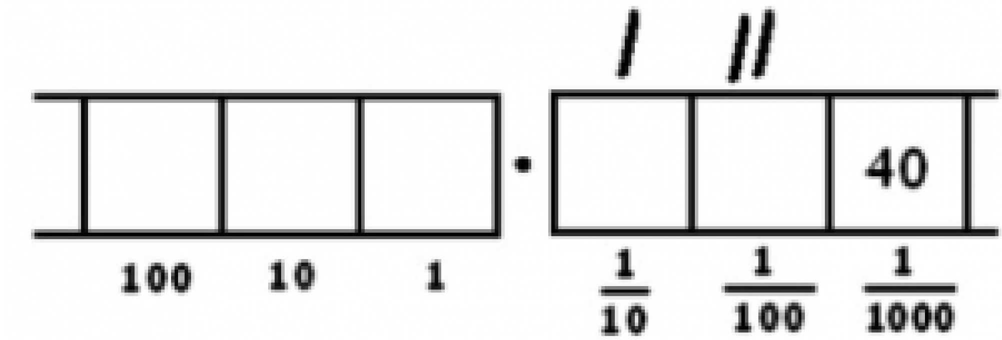
We can unexplode the two dots in the $\frac{1}{10}$ box:



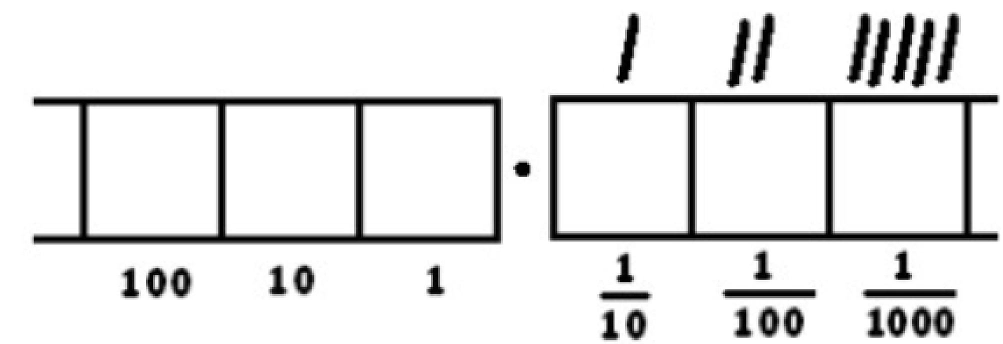
This gives two groups of 8 leaving four behind. Remember: the two tick marks represent two groups of 8. And there are four dots left in the $\frac{1}{100}$ box.



Unexploding those four remaining dots:



Now we have five groups of 8 and no remainder.



Remember: the tick marks kept track of how many groups of eight there were in each box. We have

- One group of 8 dots in the $\frac{1}{10}$ box
- Two groups of 8 dots in the $\frac{1}{100}$ box.
- Five groups of 8 dots in the $\frac{1}{1000}$ box.

So we conclude that:

$$\frac{1}{8} \;=\; 1 \div 8 \;=\; 0.125.$$

Of course, it's a good habit to check our answer:

$$0.125 \;=\; \frac{125}{1000} \;=\; \frac{\not{5}\cdot 25}{\not{5}\cdot 200} \;=\; \frac{\not{5}\cdot 5}{\not{5}\cdot 40} \;=\; \frac{\not{5}\cdot 1}{\not{5}\cdot 8} \;=\; \frac{1}{8}.$$

On Your Own

Work on the following exercises on your own or with a partner. Be sure to show your work.

1. Perform the division in a "Dots & Boxes" model to show that $\frac{1}{4}$, as a decimal, is 0.25.

2. Perform the division in a “Dots & Boxes” model to show that $\frac{1}{2}$, as a decimal, is 0.5.
3. Perform the division in a “Dots & Boxes” model to show that $\frac{3}{5}$, as a decimal, is 0.6.
4. Perform the division in a “Dots & Boxes” model to show that $\frac{3}{16}$, as a decimal, is 0.1875.
5. In simplest terms, what fraction is represented by each of these decimals?

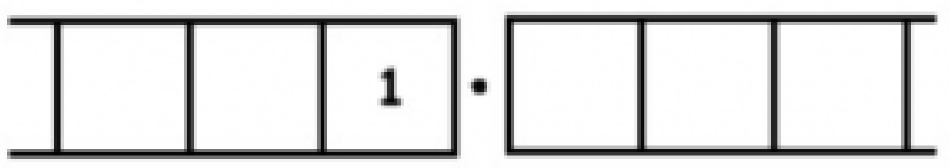
0.75, 0.625, 0.16, 0.85, 0.0625.

Repeating Decimals

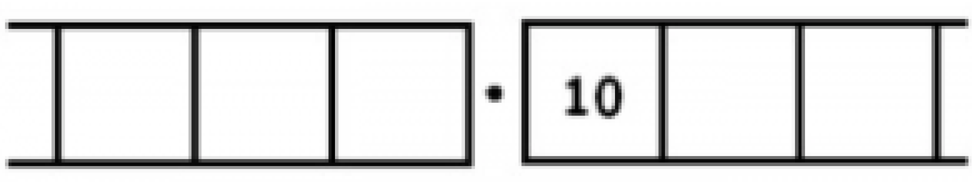
Not all fractions lead to simple decimal representations.

Example: 1/3

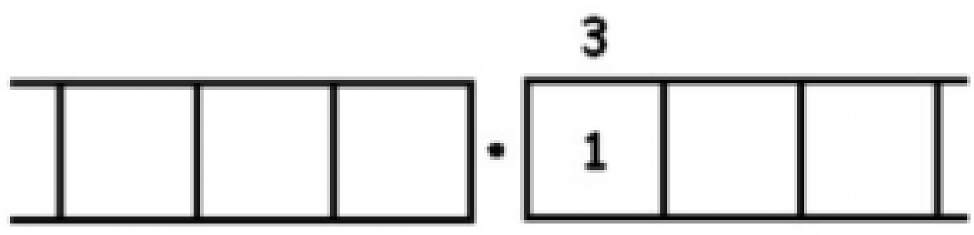
Consider the fraction $\frac{1}{3}$. We seek groups of three in the following picture:



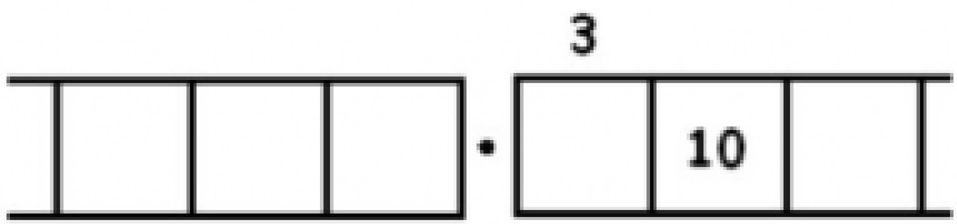
Unexploding requires us to look for groups of 3 in:



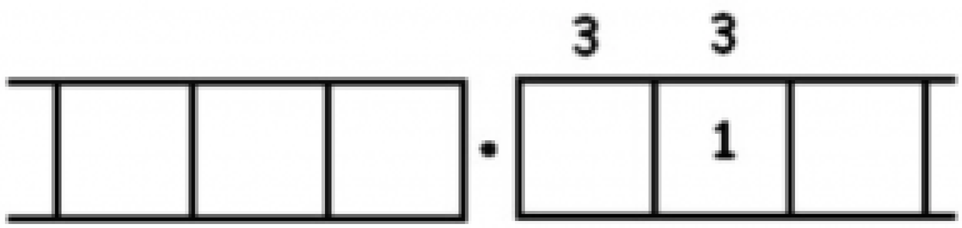
Here there are three groups of 3 leaving one behind:



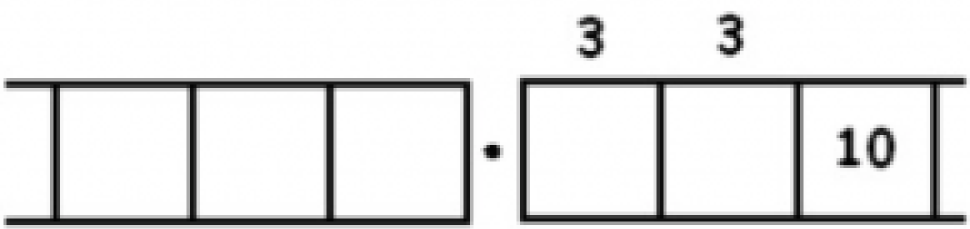
Unexploding gives:

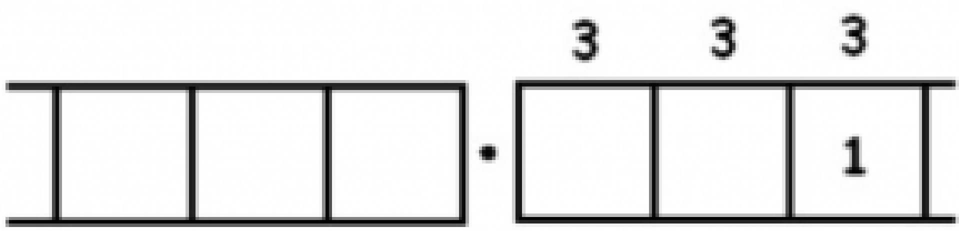


We find another three groups of 3 leaving one behind:



Unexploding gives:





And we seem to be caught in an infinitely repeating cycle.

We are now in a philosophically interesting position. As human beings, we cannot conduct this, or any, activity an infinite number of times. But it seems very tempting to write:

$$\frac{1}{3} = 0.33333\ldots,$$

with the ellipsis "..." meaning "keep going forever with this pattern." We can *imagine* what this means, but we cannot actually *write down* those infinitely many 3's represented by the ...

Notation

Many people make use of a **vinculum** (horizontal bar) to represent infinitely long repeating decimals. For example, $0.\overline{3}$ means "repeat the 3 forever":

$$0.\overline{3} \;=\; 0.33333\ldots,$$

and $0.296\overline{412}$ means "repeat the 412 forever":

$$0.296\overline{412} \;=\; 0.296412412412412\ldots$$

Now we're in a position to give a perhaps more satisfying answer to the question $1024 \div 3$. In the example above, we found the answer to be

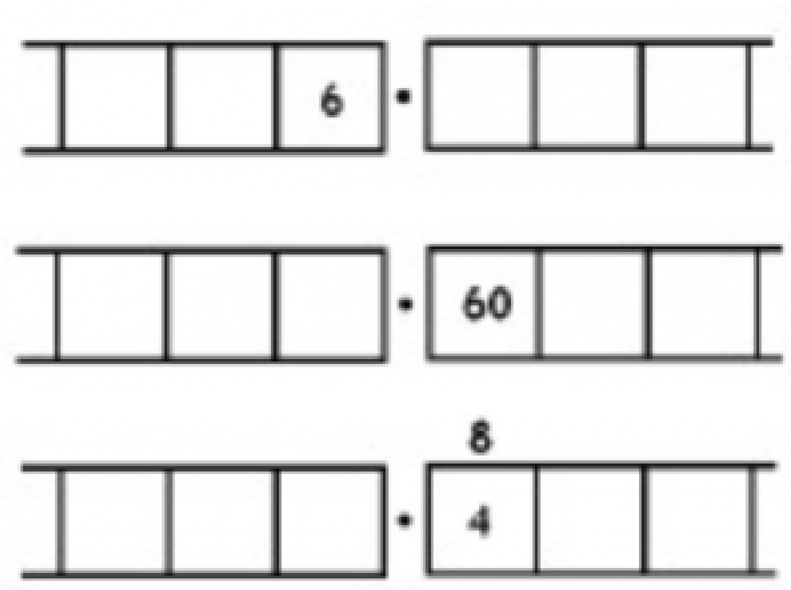
$$1024 \div 3 \;=\; 341\ R1.$$

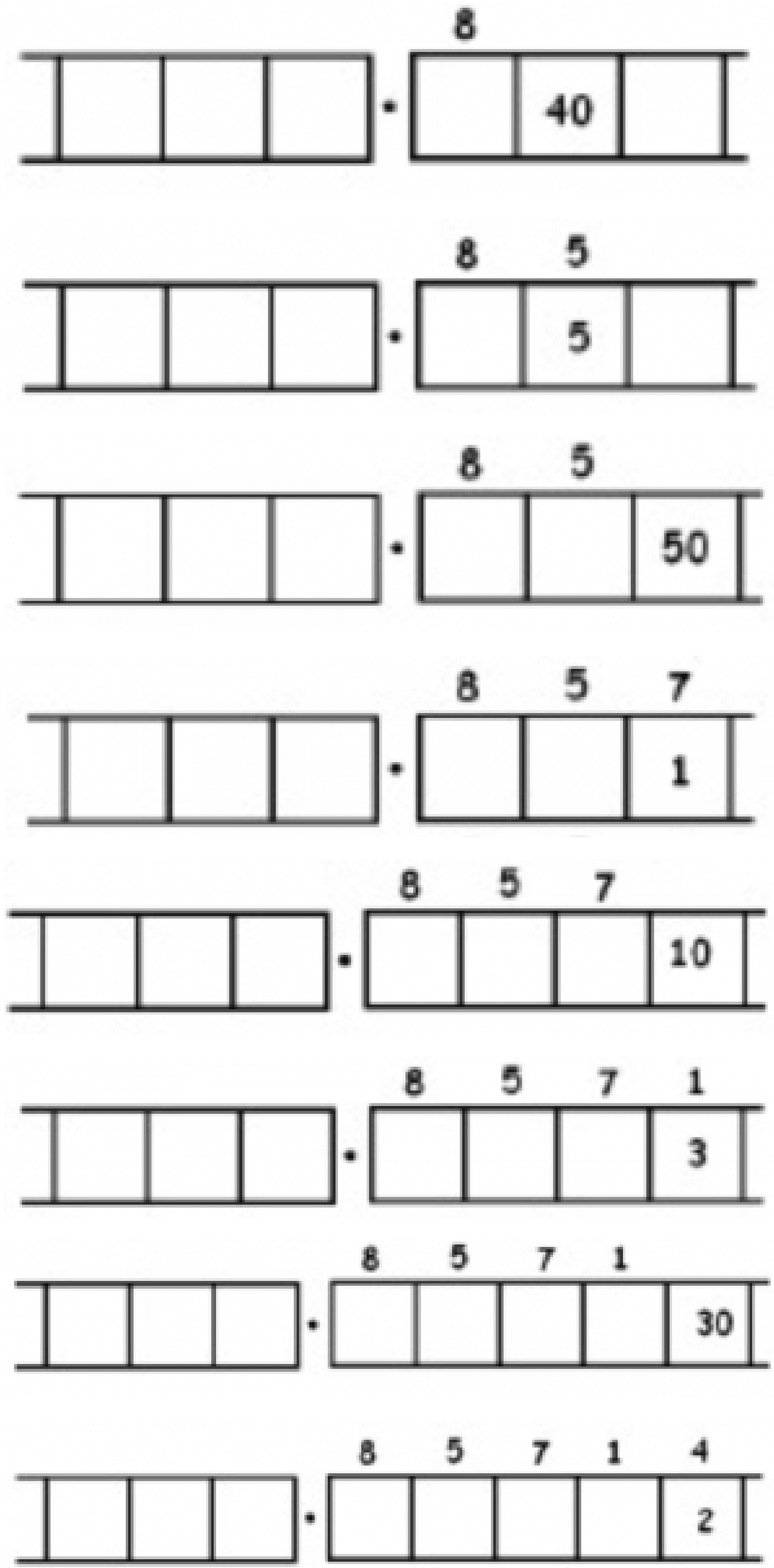
But now we know we can keep dividing that last stubborn dot by 3. Remember, that represents a single dot in the ones place, so if we keep dividing by three it really represents $\frac{1}{3}$. So we have:

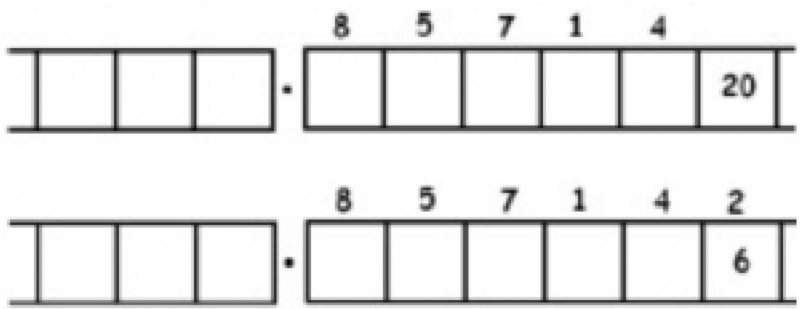
$$1024 \div 3 \;=\; 341\ R1 \;=\; 341\tfrac{1}{3} \;=\; 341.333333\ldots \;=\; 341.\overline{3}.$$

Example: 6/7

As another (more complicated) example, here is the work that converts the fraction $\frac{6}{7}$ to an infinitely long repeating decimal. Make sure to understand the steps one line to the next.




8
40
8 5
5
8 5
50
8 5 7
1
8 5 7
10
8 5 7 1
3
8 5 7 1
30
8 5 7 1 4
2



With this 6 in the final right-most box, we have returned to the very beginning of the problem. (Do you see why? Remember, we started with a six in the ones box!)

This means that we will simply repeat the work we have done and obtain the same sequence of answers: 857142. And then again, and then again, and then again. We have:

$$\frac{6}{7} = 0.857142857142857142857142\ldots$$

$$= 0.\overline{857142}.$$

On Your Own

Work on the following exercises on your own or with a partner. Be sure to show your work.

1. Compute $\frac{4}{7}$ as an infinitely long repeating decimal.
2. Compute $\frac{1}{9}$ as an infinitely long repeating decimal.
3. Use a “Dots & Boxes” model to compute $133 \div 6$. Write the answer as a decimal.
4. Use a “Dots & Boxes” model to compute $255 \div 11$. Write the answer as a decimal.

More x -mals

It should come as no surprise that we can use this reasoning about division in the “Dots & Boxes” model in other bases as well.

The following picture shows that working in base 5,

$$1432_{\text{five}} \div 13_{\text{five}} = 110_{\text{five}} \text{ R}2_{\text{five}}, \text{ meaning } 1432_{\text{five}} = 110_{\text{five}} \cdot 13_{\text{five}} + 2_{\text{five}}.$$

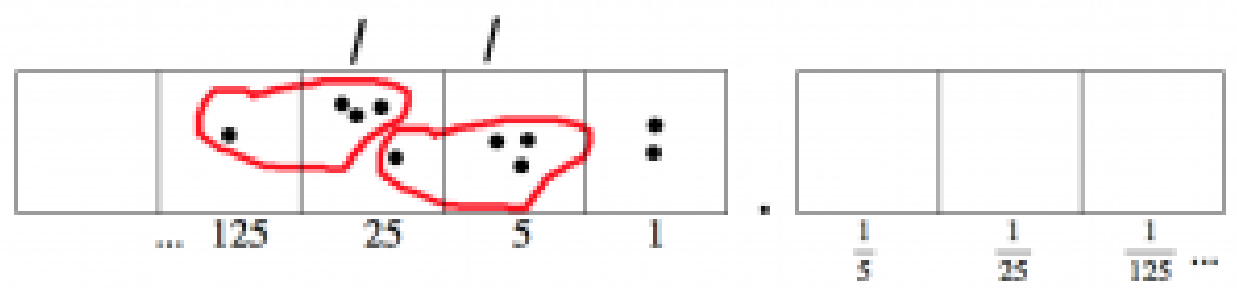
Think / Pair / Share

Carefully explain the connection between the picture and the equation shown above.

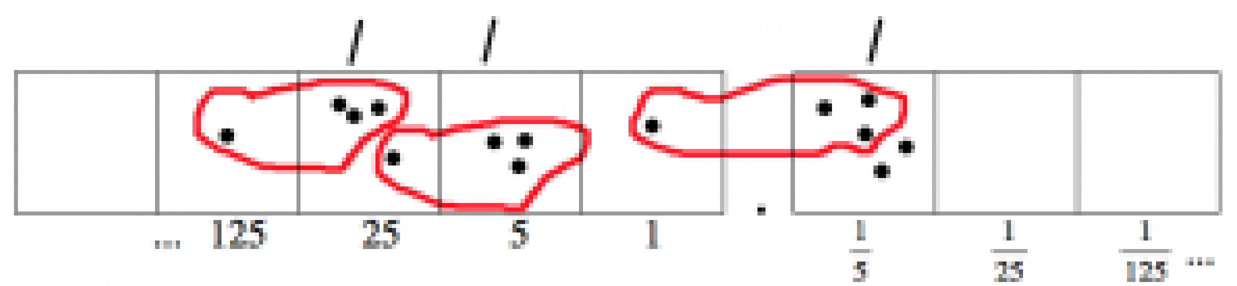
- Show in the picture where you see 1432_{five} from the equation.
- Where do you see 13_{five}?
- Where do you see 110_{five} and 2_{five}?

Example: $1432_{five} \div 13_{five}$

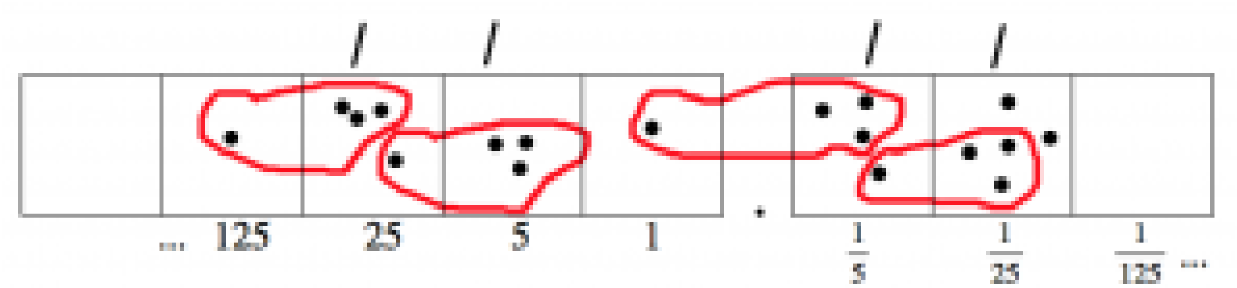
Here's where we left off the division, with a remainder of 2:



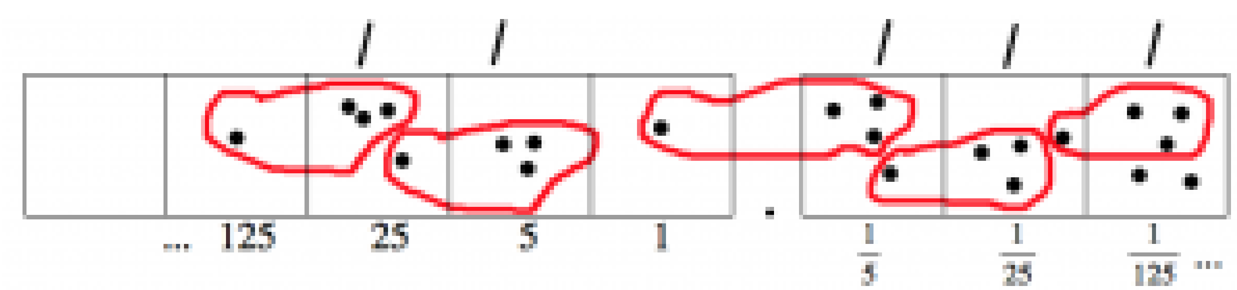
Now we can unexplode one of those two remaining dots. Then we're able to make another group of 13_{five}.



Once again, there are two dots left over, not in any group. So let's unexplode one of them.



And we still have two dots left over. Why not do it again?



It seems like we're going to be doing the same thing forever:

- Start with two dots in some box.

- Unexplode one one of the dots, so you have one dot in your original box and five in the box to the right.
- Form a group of 13_{five}. That uses the one dot in your original box and three dots in the box to the right.
- So you have two dots left in a box.
- Unexplode one of the dots, so you have one dot in your original box and five in the box to the right.
- This feels familiar…

We conclude:

$$1432_{\text{five}} \div 13_{\text{five}} = 110.111\ldots_{\text{five}} = 110.\overline{1}_{\text{five}}.$$

Think / Pair / Share

The equation

$$1432_{\text{five}} \div 13_{\text{five}} = 110.\overline{1}_{\text{five}}.$$

is a statement in base five. What is it saying in base ten?

"1432_{five}" is the number

$$1 \cdot 125 + 4 \cdot 25 + 3 \cdot 5 + 2 \cdot 1 = 242_{\text{ten}}.$$

- What is 13_{five} in base 10? Be sure to explain your answer.
- What is $110.\overline{1}_{\text{five}}$ in base 10? Explain how you got your answer.
- Translate the equation above to a statement in base ten and check that it is correct.

Problem 2

1. Draw pictures to compute $8 \div 3$ in a base ten system, and show the answer is $2.\overline{6}$.
2. Draw the pictures to compute $8_{\text{nine}} \div 3_{\text{nine}}$ in a base 9 system, and write the answer as a decimal. (Or is it a "nonimal"?)

Problem 3

1. Draw the pictures to compute $1 \div 11$ in a base ten system, and show the answer is $0.\overline{09}$.
2. Draw the base 3 pictures to compute $1_{\text{three}} \div 11_{\text{three}}$, and write the answer as a decimal ("trimal"?) number.
3. Draw the base four pictures to compute $1_{\text{four}} \div 11_{\text{four}}$, and write the answer as a decimal ("quadimal"?) number.
4. Draw the base six pictures to compute $1_{\text{six}} \div 11_{\text{six}}$, and write the answer as a decimal ("heximal"?) number.
5. Describe any patterns you notice in the computations above. Do you have a conjecture of a general rule? Can you prove your general rule is true?

Problem 4

Remember that the fraction $\frac{2}{5}$ represents the division problem $2 \div 5$. (This is all written in base ten.)

1. What is the decimal expansion (in base ten) of the fraction $\frac{2}{5}$?
2. Rewrite the base-ten fraction $\frac{2}{5}$ as a base four division problem. Then find the decimal expansion for that fraction in base four.
3. Rewrite the base-ten fraction $\frac{2}{5}$ as a base five division problem. Then find the decimal expansion for that fraction in base five.
4. Rewrite the base-ten fraction $\frac{2}{5}$ as a base seven division problem. Then find the decimal expansion for that fraction in base seven.
5. Barry said that in base fifteen, the division problem looks like

$$2_{\text{fifteen}} \div 5_{\text{fifteen}},$$

and the decimal representation would be 0.6_{fifteen}. Check Barry's answer. Is he right?

Problem 5

Expand each of the following as a "decimal" number in the base given. (The fraction is given in base ten.)

(a) $\frac{1}{9}$ in base 10

(b) $\frac{1}{2}$ in base 3

(c) $\frac{1}{3}$ in base 4

(d) $\frac{1}{4}$ in base 5

(e) $\frac{1}{5}$ in base 6

(f) $\frac{1}{6}$ in base 7

(g) $\frac{1}{7}$ in base 8

(h) $\frac{1}{8}$ in base 9.

Do you notice any patterns? Any conjectures?

Problem 6 (Challenge)

What fraction has decimal expansion $0.\overline{3}_{\text{seven}}$? How do you know you are right?

Terminating or Repeating?

You've seen that when you write a fraction as a decimal, sometimes the decimal *terminates*, like:

$$\frac{1}{2} = 0.5 \quad \text{and} \quad \frac{33}{1000} = 0.033.$$

However, some fractions have decimal representations that go on forever in a repeating pattern, like:

$$\frac{1}{3} = 0.33333\ldots \quad \text{and} \quad \frac{6}{7} = 0.857142857142857142857142\ldots.$$

It's not totally obvious, but it is true: Those are the only two things that can happen when you write a fraction as a decimal.

Of course, you can *imagine* (but never write down) a decimal that goes on forever but doesn't repeat itself, for example:

$$0.101001000100001000001\ldots \quad \text{and} \quad \pi = 3.14159265358979\ldots.$$

But these numbers can never be written as a nice fraction $\frac{a}{b}$ where a and b are whole numbers. They are called *irrational numbers*. The reason for this name: Fractions like $\frac{a}{b}$ are also called *ratios*. Irrational numbers cannot be expressed as a *ratio* of two whole numbers.

For now, we'll think about the question: Which fractions have decimal representations that terminate, and which fractions have decimal representations that repeat forever? We'll focus just on *unit fractions*.

Definition

A **unit fraction** is a fraction that has 1 in the numerator. It looks like $\frac{1}{n}$ for some whole number n.

Think / Pair / Share

- Which of the following fractions have infinitely long decimal representations and which do not?
$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7} \quad \frac{1}{8} \quad \frac{1}{9} \quad \frac{1}{10}.$$
- Try some more examples on your own. Do you have a conjecture?

A fraction $\frac{1}{b}$ *has an infinitely long decimal expansion if:*

________________________________.

Problem 7

Complete the table below which shows the decimal expansion of unit fractions where the denominator is a power of 2. (You may want to use a calculator to compute the decimal representations. The point is to look for and then explain a pattern, rather than to compute by hand.)

Try even more examples until you can make a conjecture: What is the decimal representation of the unit fraction $\frac{1}{2^n}$?

Fraction	Denominator	Decimal
$\frac{1}{2}$	2	0.5
$\frac{1}{4}$	2^2	0.25
$\frac{1}{8}$	2^3	0.125
$\frac{1}{16}$		
$\frac{1}{32}$		
$\frac{1}{64}$		
$\frac{1}{128}$		
$\frac{1}{256}$		

Problem 8

Complete the table below which shows the decimal expansion of unit fractions where the denominator is a power of 5. (You may want to use a calculator to compute the decimal representations. The point is to look for and then explain a pattern, rather than to compute by hand.)

Try even more examples until you can make a conjecture: What is the decimal representation of the unit fraction $\frac{1}{5^n}$?

Fraction	Denominator	Decimal
$\frac{1}{5}$	5	0.2
$\frac{1}{25}$	5^2	0.04
$\frac{1}{125}$	5^3	
$\frac{1}{625}$		
$\frac{1}{3125}$		
$\frac{1}{15625}$		

Marcus noticed a pattern in the table from Problem 7, but was having trouble explaining exactly what he noticed. Here's what he said to his group:

> *I remembered that when we wrote fractions as decimals before, we tried to make the denominator into a power of ten. So we can do this:*
>
> $$\frac{1}{2} = \frac{1}{2} \cdot \frac{5}{5} = \frac{5}{10} = 0.5.$$
>
> $$\frac{1}{4} = \frac{1}{4} \cdot \frac{25}{25} = \frac{25}{100} = 0.25.$$
>
> $$\frac{1}{8} = \frac{1}{8} \cdot \frac{125}{125} = \frac{125}{1000} = 0.125.$$
>
> *When we only have 2's, we can always turn them into 10's by adding enough 5's.*

Think / Pair / Share

- Write out several more examples of what Marcus discovered.
- If Marcus had the unit fraction $\frac{1}{2^n}$, what would be his first step to turn it into a decimal? What would the decimal expansion look like and why?
- Now think about unit fractions with powers of 5 in the denominator. If Marcus had the unit fraction $\frac{1}{5^n}$, what would be his first step to turn it into a decimal? What would the decimal expansion look like and why?

Marcus had a really good insight, but he didn't explain it very well. He doesn't really mean that we "turn 2's into 10's." And he's not doing any addition, so talking about "adding enough 5's" is pretty confusing.

Problem 9

1. Complete the statement below by filling in the numerator of the fraction.

> The unit fraction $\frac{1}{2^n}$ has a decimal representation that terminates. The representation will have n decimal digits, and will be equivalent to the fraction
>
> $$\frac{?}{10^n}.$$

2. Write a better version of Marcus's explanation to justify why this fact is true.

Problem 10

Write a statement about the decimal representations of unit fractions $\frac{1}{5^n}$ and justify that your statement is correct. (Use the statement in Problem 9 as a model.)

Problem 11

Each of the fractions listed below has a terminating decimal representation. Explain how you could know this for sure, without actually calculating the decimal representation.

$$\frac{1}{10} \quad \frac{1}{20} \quad \frac{1}{50} \quad \frac{1}{200} \quad \frac{1}{500} \quad \frac{1}{4000}.$$

The Period of a Repeating Decimal

If the denominator of a fraction can be factored into just 2's and 5's, you can always form an equivalent fraction where the denominator is a power of ten.

For example, if we start with the fraction

$$\frac{1}{2^a 5^b},$$

we can form an equivalent fraction

$$\frac{1}{2^a 5^b} = \frac{1}{2^a 5^b} \cdot \frac{2^b 5^a}{2^b 5^a} = \frac{2^b 5^a}{2^{a+b} 5^{a+b}} = \frac{2^b 5^a}{10^{a+b}}.$$

The denominator of this fraction is a power of ten, so the decimal expansion is finite with (at most) $a + b$ places.

What about fractions where the denominator has other prime factors besides 2's and 5's? Certainly we *can't* turn the denominator into a power of 10, because powers of 10 have just 2's and 5's as their prime factors. So in this case the decimal expansion will go on forever. But why will it have a *repeating pattern*? And is there anything else interesting we can say in this case?

Definition

The **period** of a repeating decimal is the smallest number of digits that repeat.

For example, we saw that

$$\frac{1}{3} = 0.33333\cdots = 0.\overline{3}.$$

The repeating part is just the single digit 3, so the period of this repeating decimal is one.

Similarly, we know that

$$\frac{6}{7} = 0.857142857142857142857142\ldots = 0.\overline{857142}. \qquad .$$

The smallest repeating part is the digits 857142, so the period of this repeating decimal is 6.

You can think of it this way: the *period* is the length of the string of digits under the vinculum (the horizontal bar that indicates the repeating digits).

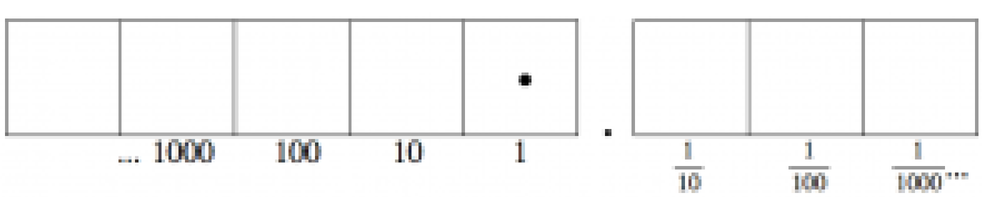
Problem 12

Complete the table below which shows the decimal expansion of unit fractions where the denominator has prime factors besides 2 and 5. (You may want to use a calculator to compute the decimal representations. The point is to look for and then explain a pattern, rather than to compute by hand.)

Try even more examples until you can make a conjecture: What can you say about the period of the fraction $\frac{1}{n}$ when n has prime factors besides 2 and 5?

Fraction	Decimal	Period
$\frac{1}{3}$	$0.\overline{3}$	1
$\frac{1}{6}$	$0.1\overline{6}$	1
$\frac{1}{7}$	$0.\overline{142857}$	6
$\frac{1}{9}$		
$\frac{1}{11}$		
$\frac{1}{12}$		
$\frac{1}{13}$		
$\frac{1}{14}$		

Imagine you are doing the "Dots & Boxes" division to compute the decimal representation of a unit fraction like $\frac{1}{6}$. You start with a single dot in the ones box:



To find the decimal expansion, you "unexplode" dots, form groups of six, see how many dots are left, and repeat.

Draw your own pictures to follow along this explanation:

Picture 1: When you unexplode the first dot, you get 10 dots in the $\frac{1}{10}$ box, which gives one group of six with remainder of 4.

Picture 2: When you unexplode those four dots, you get 40 dots in the $\frac{1}{100}$ box, which gives six group of six with remainder of 4.

Picture 3: Unexplode those 4 dots to get 40 in the next box to the right.

Picture 4: Make six groups of 6 dots with remainder 4.

Since the remainder repeated (we got a remainder of 4 again), we can see that the process will now repeat forever:

- unexplode 4 dots to get 40 in the next box to the right,
- make six groups of 6 dots with remainder 4,
- unexplode 4 dots to get 40 in the next box to the right,
- make six groups of 6 dots with remainder 4,
- and so on forever…

On Your Own

Work on the following exercises on your own or with a partner.

1. Use "Dots & Boxes" division to compute the decimal representation of $\frac{1}{11}$. Explain how you know for sure the process will repeat forever.
2. Use "Dots & Boxes" division to compute the decimal representation of $\frac{1}{12}$. Explain how you know for sure the process will repeat forever.
3. What are the possible *remainders* you can get when you use division to compute the fraction $\frac{1}{7}$? How can you be sure the process will eventually repeat?
4. What are the possible *remainders* you can get when you use division to compute the fraction $\frac{1}{9}$? How can you be sure the process will eventually repeat?

Problem 13

Suppose that n is a whole number, and it has some prime factors besides 2's and 5's. Write a convincing argument that:

1. The decimal representation of $\frac{1}{n}$ will go on forever (it will not terminate).
2. The decimal representation of $\frac{1}{n}$ will be an infinite *repeating* decimal.
3. The period of the decimal representation of $\frac{1}{n}$ will be less than n.

Problem 14

1. Find the "decimal" expansion for $\frac{1}{2}$ in the following bases. Be sure to show your work:
   $\text{two}, \quad \text{three}, \quad \text{four}, \quad \text{five}, \quad \text{six}, \quad \text{seven}.$
2. Make a conjecture: If I write the decimal expansion of $\frac{1}{2}$ in base b, when will that expansion be finite and when will it be an infinite repeating decimal expansion?
3. Can you prove your conjecture is true?

Matching Game

In this section, you'll find numbers described in various ways: as fractions, as points on a number line, as decimals, and in a picture. Your job is to match these up in a way that makes sense.

Note: there may be more than one fraction to match a given decimal, or more than one picture to match a given point on the number line. So be ready to justify your answers.

Fractions

(a) $\frac{1}{5}$	(b) $\frac{1}{3}$	(c) $\frac{2}{3}$
(d) $\frac{9}{8}$	(e) $\frac{15}{16}$	(f) $\frac{25}{100}$
(g) $\frac{3}{4}$	(h) $\frac{33}{100}$	(i) $\frac{3}{25}$
(j) $\frac{1}{4}$	(k) $\frac{6}{5}$	(ℓ) $\frac{2}{5}$
(m) $\frac{4}{100}$	(n) $\frac{2}{10}$	(o) $\frac{1}{2}$

Points on a number line

Point 1

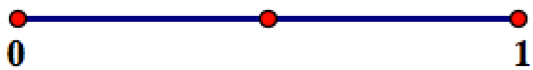
Point 2

Point 3

Point 4

Point 5

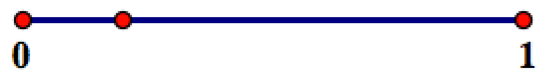
Point 6



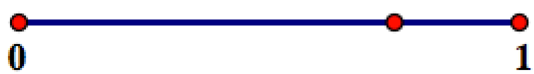
Point 7

Point 8

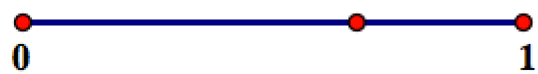
Point 9



Point 10



Point 11



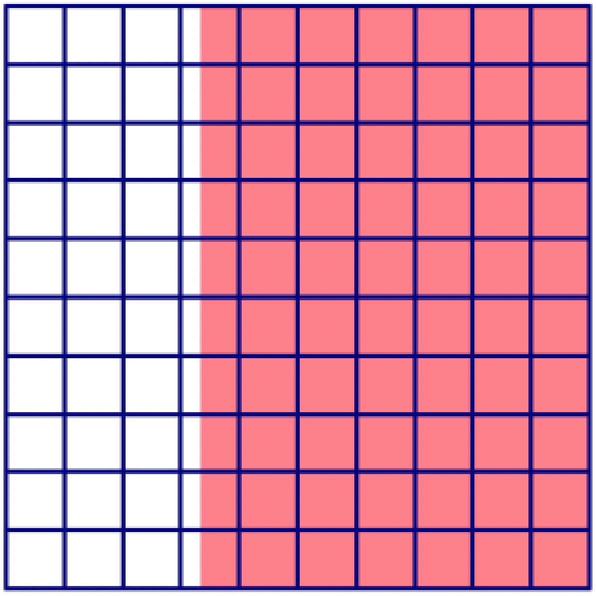
Point 12

Point 13

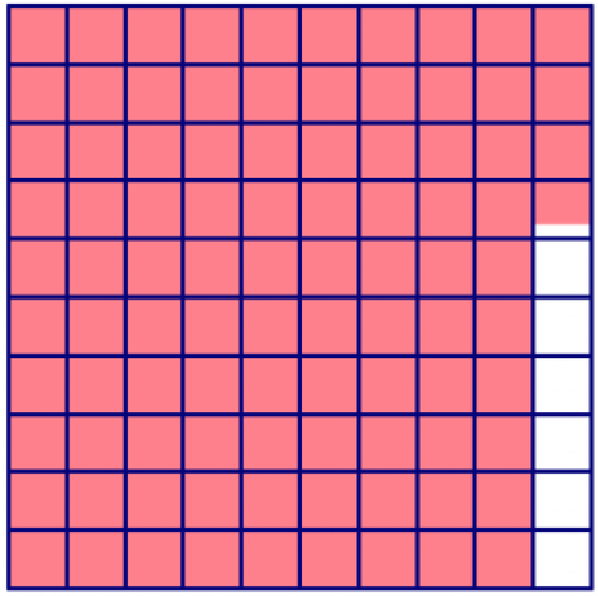
Decimals

(i) 1.20	(ii) $0.\overline{6}$	(iii) 0.33
(iv) 0.25	(v) 0.5	(vi) 0.200
(vii) 0.75	(viii) $0.\overline{3}$	(ix) 0.2
(x) 1.125	(xi) 0.12	(xii) 0.04
(xiii) 0.40	(xiv) 0.50	(xv) 0.9375

Pictures



Picture A



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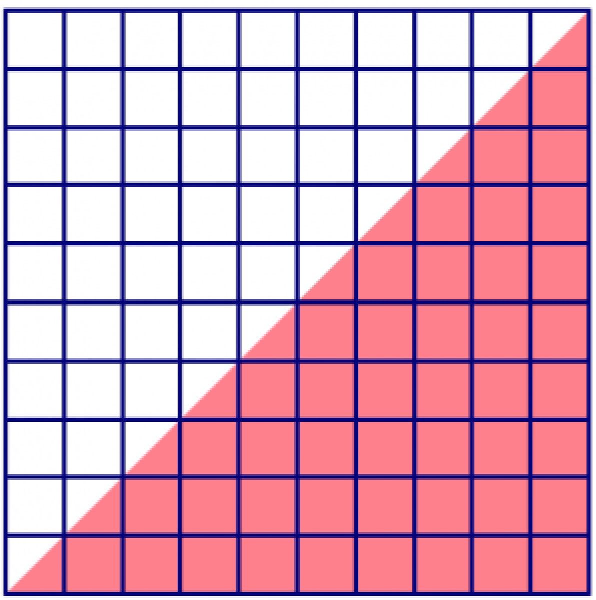
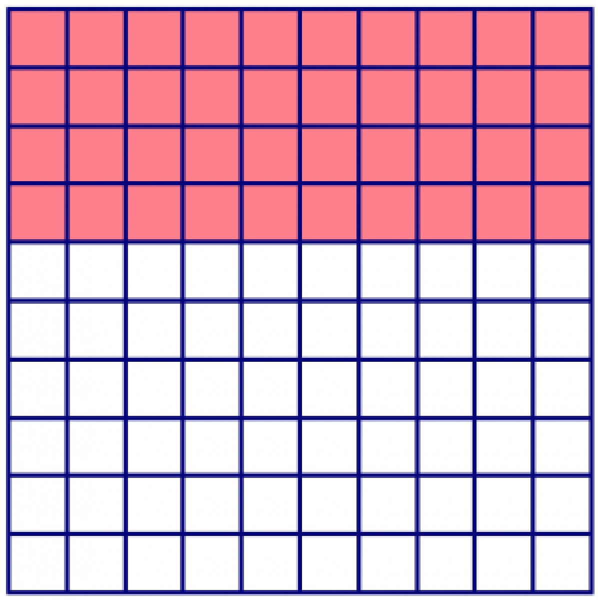
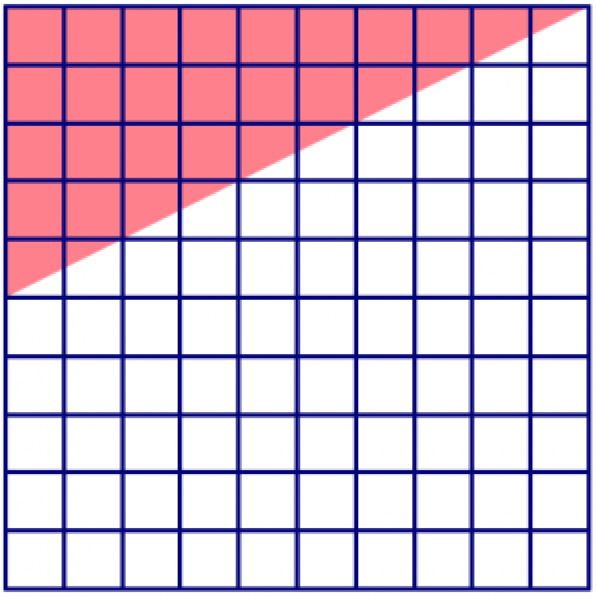


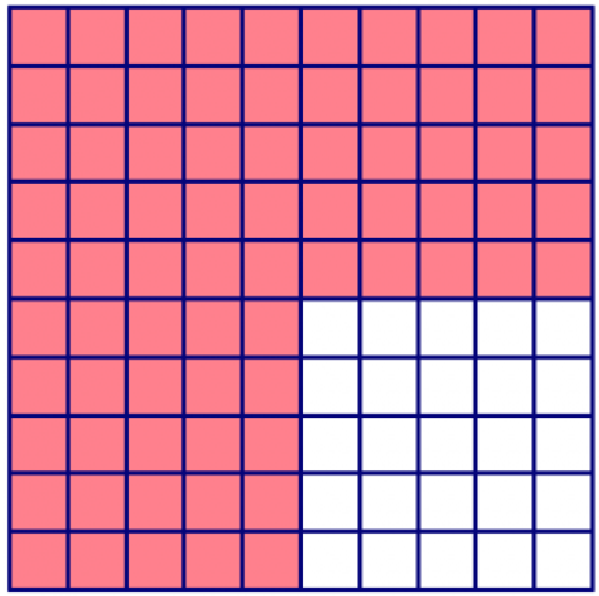
Picture C



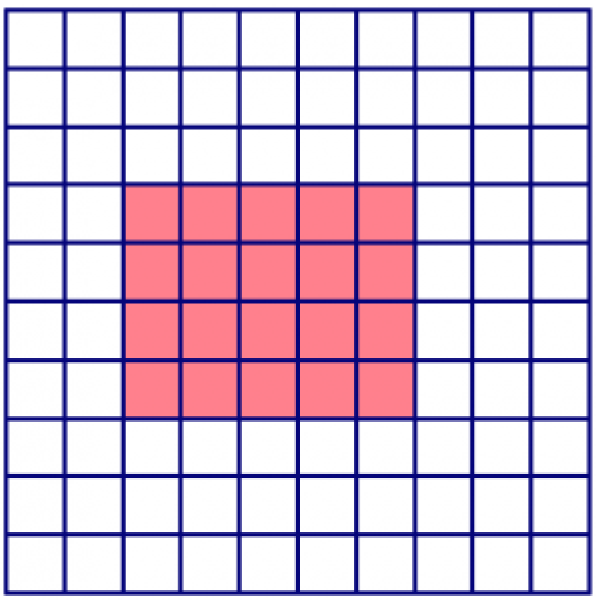
Picture D



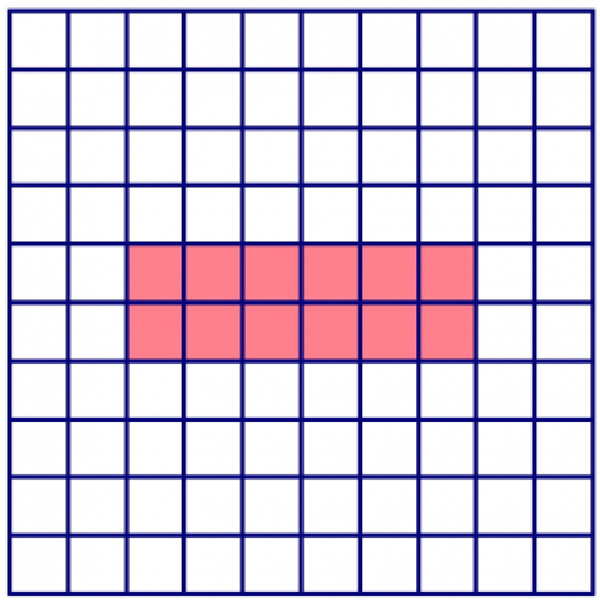
Picture E



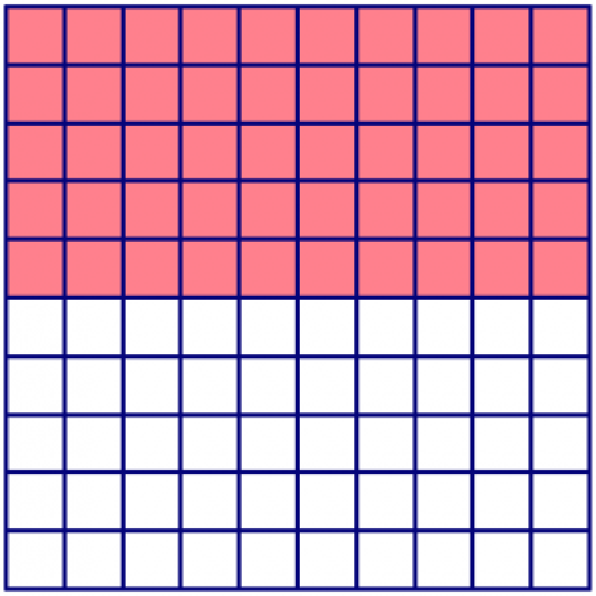
Picture F

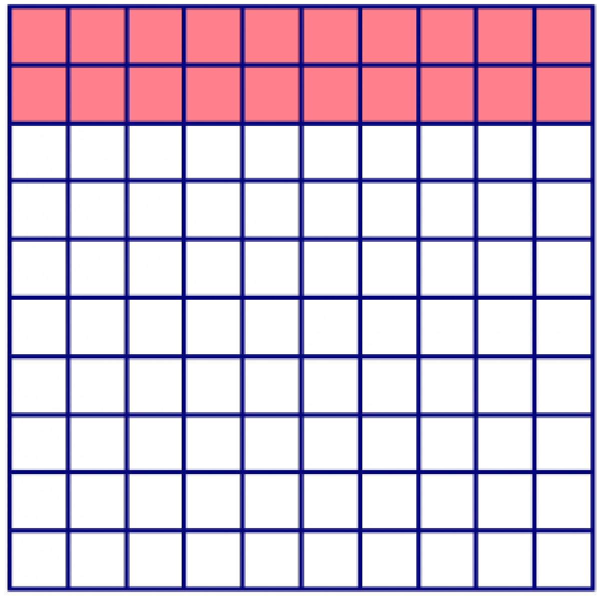


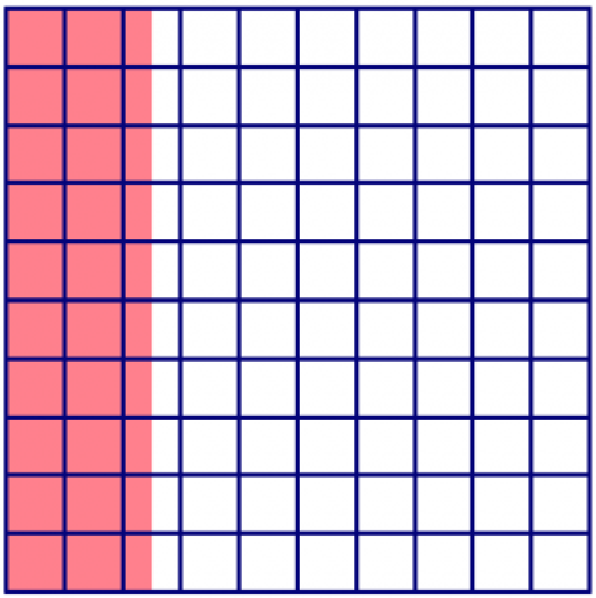
Picture G

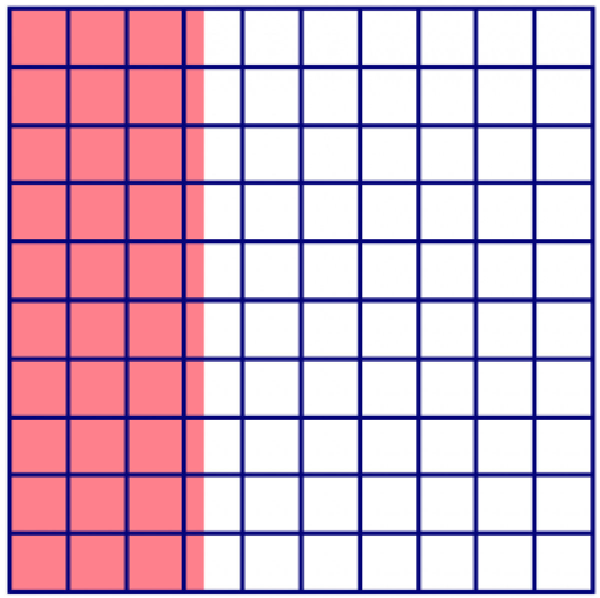


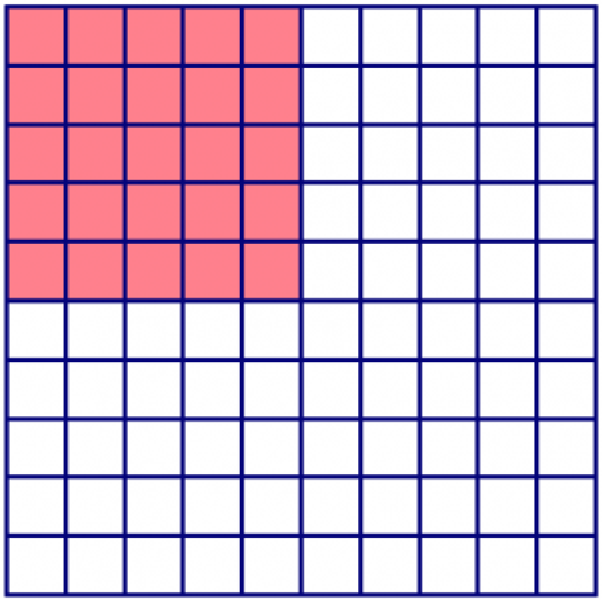
Picture H


Picture I

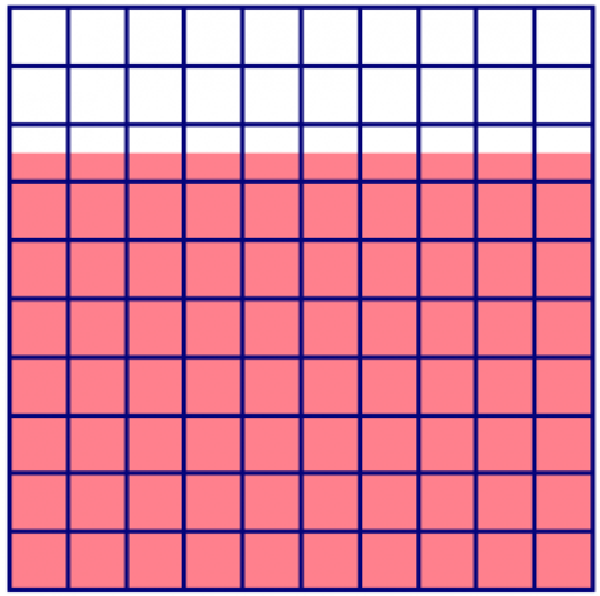

Picture J


Picture K

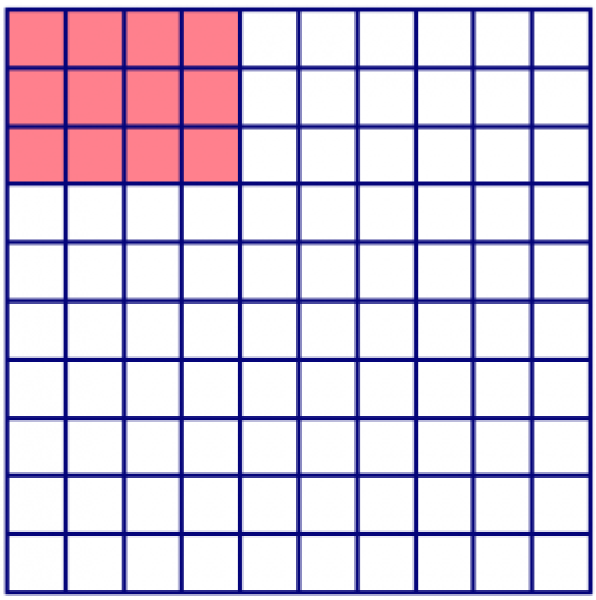

Picture L



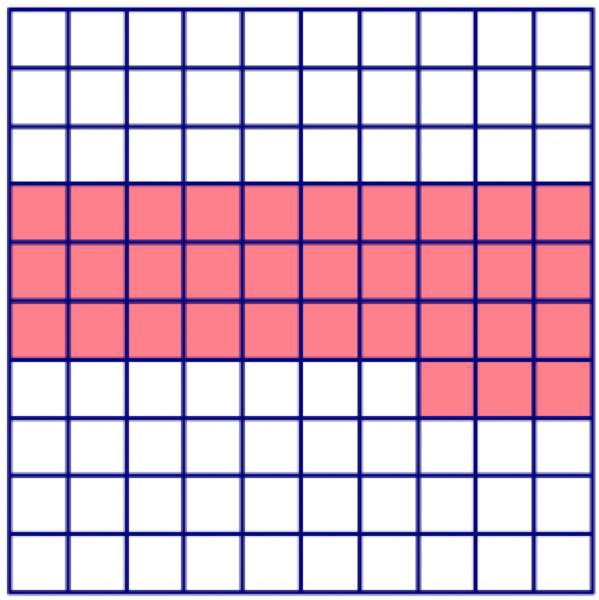
Picture M



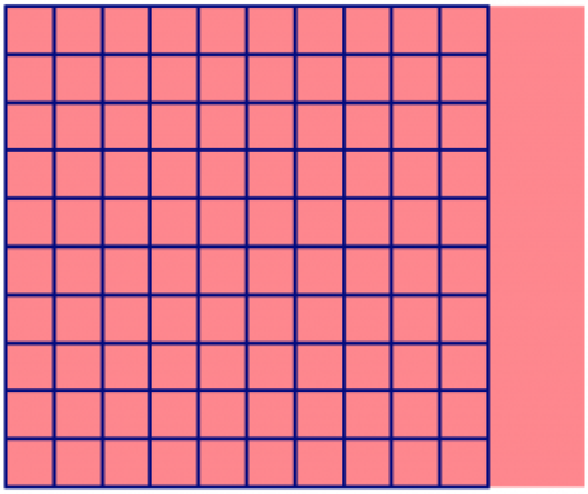
Picture N



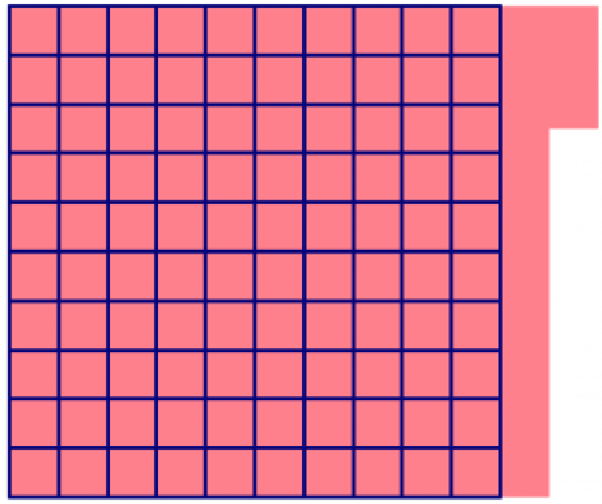
Picture O



Picture P



Picture Q



Picture R

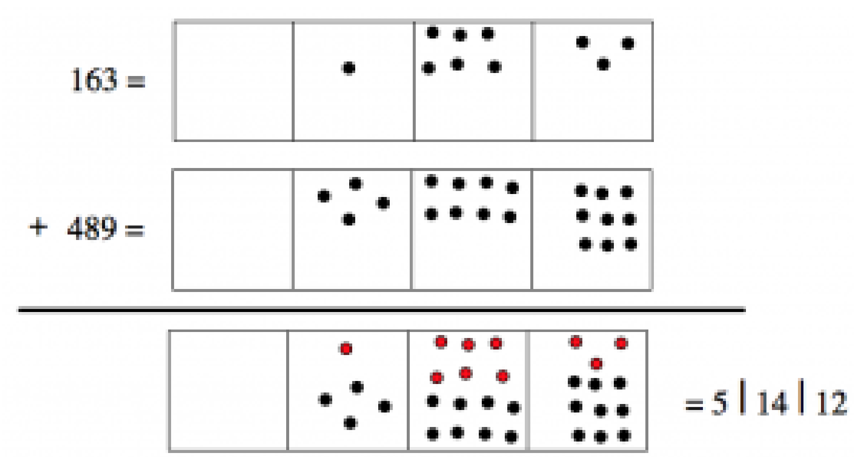
Operations on Decimals

Of course we can add, subtract, multiply, and divide decimal numbers by rewriting them as fractions and using the algorithms we know there. Of course, sometimes it is a lot more work to convert to fractions than it is to just work directly with the decimals (as long as you know what you're doing). So let's think about place value and computing with decimals.

Adding and Subtracting Decimals

Remember that when we used the "Dots & Boxes" model to add, it looked like this.

Example: 163 + 489



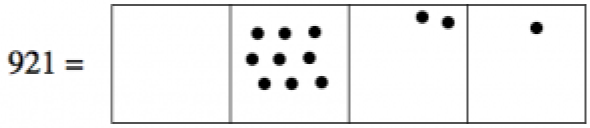
We then perform explosions until there are fewer than ten dots in each box, and we find that:

$163 + 489 = 652.$

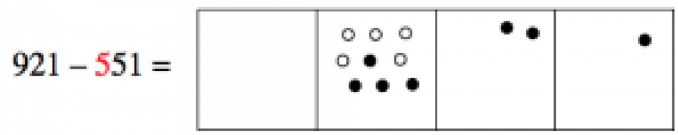
Subtraction was a little more complicated.

Example: 921 – 551

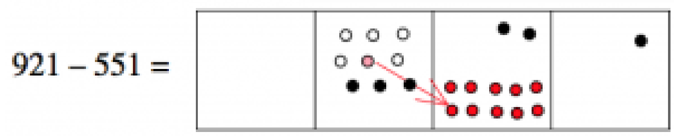
We start with the representation of 921:

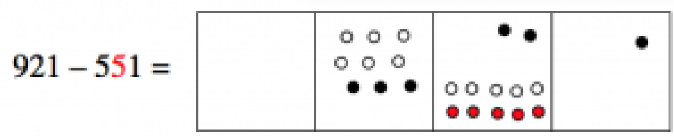


Since we want to "take away" 551, that means we take away five dots from the hundreds box, leaving four dots.



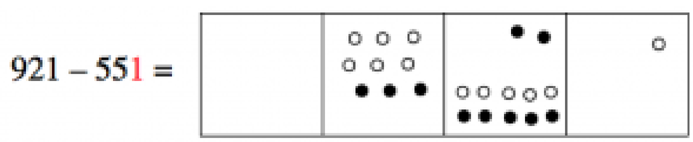
Now we want to take away five dots from the tens box, but we can't do it! There are only two dots there. What can we do? Well, we still have some hundreds, so we can "unexplode" a hundreds dot, and put ten dots in the tens box instead. Then we'll be able to take five of them away, leaving seven.





(Notice that we also have one less dot in the hundreds box; there's only three dots there now.)

Now we want to take one dot from the ones box, and that leaves no dots there.



We conclude that: $921 - 551 = 370.$

On Your Own

Work on the following exercises on your own or with a partner.

1. For each calculation, draw a "Dots & Boxes" model and use it to find the result of the calculation.

$$3.56 + 7.95 \qquad 1.452 + 32.27 \qquad 3.0205 + 409.2019$$

$$15.225 - 7.209 \qquad 14.793 - 8.95 \qquad 12.5 - 3.0002.$$

2. For each calculation below, add the decimals quickly, and say why your method is faster than converting to fractions and finding a common denominator.

$0.0066 + 0.9 \qquad 0.25 + 0.0088 \qquad 0.\overline{20} + 0.\overline{01}.$

Think / Pair / Share

- Chloe added 0.2 and 0.02 and got an answer of 0.4. What was Chloe's likely mistake? As her teacher, how could you help Chloe understand the operation of addition better?
- In elementary school, students are taught to add and subtract decimals by "lining up the decimal points." Use the "Dots & Boxes" model to explain why this shorthand makes sense.

Multiplying and Dividing: Powers of 10

Let's quickly review the "Dots & Boxes" model for multiplication of whole numbers before we get back to talking about decimals.

Example: 243192 × 4

If we want to compute 243192×4, it helps to remember what multiplication *means*. One interpretation is: I want to add 243192 to itself a total of four times. So there will be:

- 2×4 dots in the ones place,
- 9×4 dots in the tens place,
- 1×4 dots in the hundreds place,
- and so on.

Here's the start of the computation:

$$243192 \times 4 = 8 \mid 16 \mid 12 \mid 4 \mid 36 \mid 8.$$

To finish the computation, we need to do some explosions to write the result as a familiar base 10 number:

$$243192 \times 4 = 972768.$$

On Your Own

Work on the following exercises on your own or with a partner.

1. Do each computation, using reasoning like in the multiplication example above.

$(a)\ 2.3 \times 10 \qquad (b)\ 3.56 \times 10 \qquad (c)\ 1.452 \times 100.$

2. Do each computation, using reasoning like in the "Division and Decimals" examples.

$7.1 \div 10 \qquad 98.55 \div 10 \qquad 145.2 \div 100.$

You know that multiplying a base-ten whole number by 10 results in appending a zero to the right end of the number. Your work above should convince you that this does not work for decimals!

Think / Pair / Share

- Write a new rule that works for both whole numbers and decimals:

> If I multiply a whole number or a decimal by 10, a simple way to find the result is ________________________.

- Justify the claim you made above.
- One can go much further with this thinking. What is the effect of *dividing* a number written in decimal notation by ten? By one-hundred? Justify what you say.

Multiplying Decimals

You probably know an algorithm for multiplying decimal numbers by hand. But if you think carefully about the algorithm, it should **make sense** based on what the decimal numbers represent and what it means to multiply. Let's start by using number sense to think about multiplying whole numbers by decimals.

Think / Pair / Share

Consider the expression

$16 \times \square.$

Fill in the box with a whole number or decimal so that the product is:

- Greater than 100.
- Greater than 64 but less than 100.
- At least 17, but less than 32.
- Equal to 16.
- Greater than 8 but less than 16.
- Less than 8, but greater than 0.

Be sure to justify your answers. You should use your number sense rather than computing by hand or with a calculator!

Earlier in this chapter, you multiplied decimal numbers by converting them to fractions and then using what you know about multiplying fractions. There are other ways to think about multiplying that focus on number sense and place value rather than on the mechanics of computation.

Example: 321 × 0.4

Suppose a student wanted to compute 321×0.4, but he didn't already know the standard algorithm. What might she do? Here is one idea:

> I know that $321 \times 4 = 1284$. Since I want to multiply by 0.4 and not by 4, my answer should be $\frac{1}{10}$ of this one. So
> $$321 \times 0.4 = 128.4.$$

You should notice that the student is using the **associative property** of multiplication:

$$321 \times 0.4 \;=\; 321 \times \frac{4}{10} \;=\; 321 \times \left(4 \times \frac{1}{10}\right) \;=\; (321 \times 4) \times \frac{1}{10}.$$

Problem 15

For each computation below, the result of the computation is shown correctly, but the decimal point is missing. Use number sense and reasoning to correctly place the decimal point, and briefly justify how you know you're right.

(Don't use a calculator, don't work out the multiplication by hand, and don't use the trick of "counting the number of decimal places." Use your number sense!)

(a) $855 \times 1.7 = 14535$ (b) $549 \times 0.33 = 18117$

(c) $2.03 \times 1028 = 208684$ (d) $999 \times 0.53 = 52947$

(e) $30.02 \times 472 = 1416944$ (f) $173 \times 0.09 = 1557.$

On Your Own

Work on the following exercises on your own or with a partner.

1. Write each number given as a fraction. (Write them as "improper fractions," not "mixed numbers.")

$(a)\ 15.2$ $(b)\ 3.43$

$(c)\ 0.0021$ $(d)\ 13.02026.$

2. In exercise (1) above, how does the number of digits to the right of the decimal point compare to the number of zeros in the denominator? Use what you know about place value to explain why your answer is always true (not just for the examples above).
3. Find each product.

$(a)\ 10 \times 10000$ $(b)\ 100 \times 1000$

$(c)\ 100000 \times 1000$ $(d)\ 10^m \times 10^n.$

4. In exercise (3) above, how is the number of zeros in the product related to the number of zeros in the two factors? Use what you know about place value to explain why your answer is always true (not just for the examples above).
   a. If you write 0.037 as a fraction, how many zeros would be in the denominator?
   b. What if you write 0.59 as a fraction,how many zeros would be in the denominator?
   c. How many zeros would be in the denominator of the product of 0.037 and 0.59? (Don't compute the product to answer this question!)
5. Use the fact that $37 \times 59 = 2183$ and your answers to the exercises above to find 0.037×0.59. Explain how you got your answer.

Standard multiplication algorithm

The standard algorithm for multiplying decimal numbers can be described this way:

Step 1

Compute the product as if the two factors were whole numbers. (Ignore the decimal points.)

Step 2

Count the number of digits to the right of the decimal point in each factor, and add those numbers together. Call the result n.

Step 3

The sum n that you found in Step 2 will be the number of digits to the right of the decimal point in the product. So place the decimal point according by counting the appropriate number of places from the right.

Think / Pair / Share

- Write down two examples of multiplying decimal numbers using the standard algorithm above.
- Use what you know about place value, fractions, and multiplication to **carefully explain why** the standard algorithm described above makes sense.

Dividing Decimals

As you might expect, dividing decimals is more complicated to explain than any of the other operations. It's hard to adapt our "Dots & Boxes" model for division. Suppose we want to compute $15.37 \div 0.013$. We can certainly draw the picture for 15.37, but how could we make groups of 0.013 dots?

Think / Pair / Share

Let's start by sharing what you already know. Perform this computation (by hand, not with a calculator), showing all of your work. Explain your method to a partner, and see if your partner computed the same way.

$$0.0351 \div 0.074.$$

On Your Own

Work on the following exercises on your own or with a partner.

1. Explain why these two fractions are equivalent.

$$\frac{12.33}{44.1} \quad \text{and} \quad \frac{123.3}{441}.$$

2. Explain why these two division computations give the same result.

$$12.33 \div 44.1 \quad \text{and} \quad 123.3 \div 441.$$

3. Explain why these three fractions are equivalent.

$$\frac{325.5}{75.133}, \quad \frac{3255}{751.33}, \quad \text{and} \quad \frac{32550}{7513.3}.$$

4. Explain why these three division computations give the same result.

$$325.5 \div 75.133, \quad 3255 \div 751.33, \quad \text{and} \quad 32550 \div 7513.3.$$

5. Fill in the box to make the equation true. Be sure to justify your answer.

$$\frac{325.5}{75.133} = \frac{\square}{75133}.$$

Standard division algorithm

The standard algorithm for dividing numbers represented by finite decimal expansions is something like this:

Step 1

Move the decimal point of the divisor to the end of the number.

Step 2

Move the decimal point of the dividend the same number of positions (the same distance and direction).

Step 3

Divide the new decimal dividend (from Step 2) by the new whole number divisor (from Step 1). Since we're dividing by a whole number, our standard methods make sense.

This is a pretty mechanical description, and doesn't give a lot of insight into **why** this algorithm works.

Think / Pair / Share

Write down at least two examples of computing with the algorithm described above. (Make up your own numbers to test. Be sure to show every step clearly.) You can do the division by drawing a "Dots & Boxes" picture or by another method (but don't use a calculator). Then answer these more general questions.

- Suppose you want to compute $a \div b$ where a and b are decimal numbers. Carefully explain why $(10 \cdot a) \div (10 \cdot b)$ will give the same result.
- Suppose you want to compute $a \div b$ where a and b are decimal numbers. Carefully explain why $(100 \cdot a) \div (100 \cdot b)$ will give the same result.
- Suppose you want to compute $a \div b$ where a and b are decimal numbers. Carefully explain why $(10^k \cdot a) \div (10^k \cdot b)$ will give the same result.
- Suppose b has a finite decimal expansion. Carefully explain why you can find a power of 10 so that $10^k \cdot b$ is a whole number.

Problem 16

Carefully explain **why** the algorithm described above in three steps works for computing division of decimal numbers. You need to explain what is going on when you "move the decimal point" in Steps 1 and 2, and why the result you compute in Step 3 is the same as the original problem.

Orders of Magnitude

Problem 17

How old were you when you were one million seconds old? (That's $1,000,000$.)

- Before you figure it out, write down a guess. What's your gut instinct? About a day? A week? A month? A year? Have you already reached that age? Or maybe you won't live that long?
- Now figure it out! When was / will be your million-second birthday?

Problem 18

How old were you when you were one *billion* seconds old? (That's $1,000,000,000$.)

- Again, before you figure it out, write down a guess.
- Now figure it out! When was / will be your billion-second birthday?

Were you surprised by the answers? People (most people, anyway) tend to have a very good sense for small, everyday numbers, but have very bad instincts about big numbers. One problem is that we tend to think *additively*, as if one billion is about a million plus a million more (give or take). But we need to think *mulitplicatively* in situations like this. One billion is $1,000 \times \text{ a million.}$

So you could have just taken your answer to Problem 17 and multiplied it by $1,000$ to get your answer to Problem 18. Of course, you would probably still need to do some calculations to make sense of the answer.

Think / Pair / Share

When is your one trillion second birthday? What will you do to celebrate?

Think / Pair / Share

The US debt is total amount the government has borrowed. (This borrowing covers the *deficit* — the difference between what the government spends and what it collects in taxes.) In summer of 2013, the US debt was *on the order of* 10 trillion dollars. (That means more than 10 trillion but less than 100 trillion. If you were to write out the dots-and-boxes picture, the dots would be as far left as the $10,000,000,000$ place.)

- If the US pays back one penny every second, will the national debt be paid off in your lifetime? Explain your answer.
- A headline from April 2013 said, "US to Pay Down $35 billion in Quarter 2." Suppose the US pays down $35 billion dollars *every* quarter (so four times per year). About how many years would it take to pay of the total national debt?

Here are some big-number problems to think about. Can you solve them?

Problem 19

1. Suppose you have a million jelly beans, and you tile the floor with them. How big of an area will they cover? The classroom? A football field? Something bigger? What if it was a billion jelly beans?
2. Suppose you have a million jelly beans and you stack them up. How tall would it be? As tall as you? As a tree? As a skyscraper? What if it was a billion jelly beans? About how many jelly beans (what *order of magnitude*) would you need to stack up to reach the moon? Explain your answers.

Fermi Problems

James Boswell wrote,

> Knowledge is of two kinds. We know a subject ourselves, or we know where we can find information upon it.

But math proves this wrong. There is actually a third kind of knowledge: Knowledge that you *figure out for yourself*. In fact, this is what scientists and mathematicians do for a living: they create new knowledge! Starting with what is already known, they ask "what if…" questions. And eventually, they figure out something new, something no one ever knew before!

Even for knowledge that you *could* look up (or ask someone), you can often figure out the answer (or a close approximation to the answer) on your own. You need to use a little knowledge, and a little ingenuity.

Fermi problems, named for the physicist Enrico Fermi, involve using your knowledge, making educated guesses, and doing reasonable calculations to come up with an answer that might at first seem unanswerable.

Example

Here's a classic Fermi problem: How many elementary school teachers are there in the state of Hawaii?

You might think: How could I possibly answer that? Why not just google it? (But some Fermi problems we meet will have — gasp! — non-googleable answers.)

First let's define our terms. We'll say that we care about classroom teachers (not administrators, supervisors, or other school personnel) who have a permanent position (not a sub, an aide, a resource room teacher, or a student teacher) in a grade K–5 classroom.

But let's stop and think. Do you know the population of Hawaii? It's about $1,000,000$ people. (That's not exact, of course. But this is an exercise is estimation. We're trying to get at the *order of magnitude* of the answer.)

How many of those people are elementary school students? Well, what do you know about the population of Hawaii? Or what do you *suspect* is true? A reasonable guess would be that the population is evenly distributed across all age groups. That would give a population that looks something like this:

age range	# of people
0 – 9	125,000
10 – 19	125,000
20 – 29	125,000
30 – 39	125,000
40 – 49	125,000
50 – 59	125,000
60 – 69	125,000
70 – 79	125,000

We'll assume people don't live past 80. Of course some people do! But we're all about making simplifying assumptions right now. That gives us eight age categories, with about 125,000 people in each category.

An even better guess (since we have a large university that draws lots of students) is that there's a "bump" around college age. And some people live past 80, but there are probably fewer people in the older age brackets. Maybe the breakdown is something like this? (If you have better guesses, use them!)

age range	# of people
0 – 9	125,000
10 – 19	130,000
20 – 29	140,000
30 – 39	125,000
40 – 49	125,000
50 – 59	125,000
60 – 69	120,000
> 70	105,000

So, how many K–5 students are in Hawaii? That covers about six years of the 0–9 range. If we are still going with about the same number of people at each age, there should be about 12,500 in each grade for a total of $12,500 \times 6 = 75,000$ K–5 students.

OK, but we really wanted to know about K–5 *teachers*. One nice thing about elementary school: there tends to be just one teacher per class. So we need an estimate of how many classes, and that will tell us how many teachers.

So, how many students in each class? It probably varies a bit, with smaller kindergarten classes (since they are more rambunctious and need more attention), and larger fifth grade classes. There are also smaller classes in private schools and charter schools, but larger classes in public schools. A reasonable average might be 25 students per class across all grades K–5 and all schools.

So that makes $75,000 \div 25 = 3,000$ K–5 classrooms in Hawaii. And that should be the same as the number of K–5 teachers.

How good is this estimate? Can you think of a way to check and find out for sure?

So now you see the process for tackling a Fermi problem:

- Define your terms.
- Write down what you know.
- Make some reasonable guesses / estimates.
- Do some simple calculations.

Try your hand at some of these:

Problem 20

How much money does your university earn in parking revenue each year?

Problem 21

How many tourists visit Waikiki in a year?

Problem 22

How much gas would be saved in Hawaii if one out of every ten people switched to a carpool?

Problem 23

How high can a climber go up a mountain on the energy in one chocolate bar?

Problem 24

How much pizza is consumed each month by students at your university?

Problem 25

How much would it cost to provide free day care to every four-year-old in the US?

Problem 26

How many books are in your university's main library?

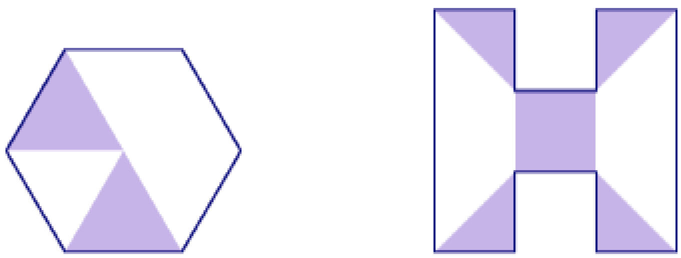
Problem 27

Make up your own Fermi problem… what would you be interested in calculating? Then try to solve it!

Problem Bank

Problem 28

Express the shaded portion of each figure as both a fraction and as a decimal. Justify your answers.



(a) (b)

Problem 29

Which number is bigger: 0.135 or 0.14? Justify your answer.

Problem 30

Arrange the digits 1, 2, 3, and 4 in the boxes to create the smallest possible sum. Use each digit exactly once. Justify that your answer is as small as possible.

$$\frac{\square}{\square} + \frac{\square}{\square}$$

Problem 31

Arrange the digits 1, 2, 3, and 4 in the boxes to create the smallest possible (positive) difference. Use each digit exactly once. Justify that your answer is as small as possible.

$$\frac{\square}{\square} - \frac{\square}{\square}$$

Problem 32

Use the "Dots & Boxes" model to show that $\frac{1}{9} = 0.\overline{1}$. Then use this fact to answer these questions and justify your answers.

1. What fraction is given by $0.\overline{2}$?
2. What fraction is given by $0.\overline{5}$?
3. What fraction is given by $0.\overline{6}$?
4. What fraction is given by $0.\overline{8}$?
5. What fraction is given by $0.\overline{9}$?

Problem 33

In this problem, you will focus on the calculation

$$170 \times \square.$$

Your goal is to get a product that is close to 200.

1. Will you multiply 170 by a number greater or less than 1? Greater or less than 2? Justify your

answers.

2. Suppose you can use only one decimal place. Fill in the box with a number that gets as close to 200 as possible.
3. Suppose you can use only two decimal places. Fill in the box with a number that gets as close to 200 as possible.
4. Suppose you can use only three decimal places. Fill in the box with a number that gets as close to 200 as possible.

Problem 34

Do each computation below without using a calculator. Explain your thinking.

1. $(23 \times 0.1) + (0.001 \times 55)$.
2. $18.45 \div (0.63 \div 0.7)$.
3. $22.65 - (0.03 \cdot 10)$.

Problem 35

Without actually calculating anything (just use your number sense!), order x, y, and z from smallest to largest. Explain your ordering.

$x = 0.07 + 0.000001$

$y = 0.07 \times 0.000001$

$z = 0.07 \div 0.000001$

Problem 36

For each question below, choose the correct calculation and explain your choice. Then estimate the answer (don't calculate it exactly) and explain why your estimate is a good one.

1. A large pizza has eight slices and costs $15.95. How much does each slice of pizza cost? Should you calculate 15.95×8 or $15.95 \div 8$?
2. There are 2.54 centimeters in an inch. A standard sheet of notebook paper is $8\frac{1}{2}$ inches wide and 11 inches long. How many centimeters wide is the page? Should you calculate 8.5×2.54 or 11×2.54 or $8.5 \div 2.54$ or $11 \div 2.54$?
3. In a model train set, 1.38 inches represents one foot in real life. The height of One World Trade Center in New York City is 1776 feet. How tall would a scale model of the building be? Should you calculate 1776×1.38 or $1776 \div 1.38$?
4. Eight-tenths of a jumprope is 1.75 meters long. How long is the whole rope? Should you calculate 0.8×1.75 or $0.8 \div 1.75$ or $1.75 \div 0.8$?

Problem 37

Kaimi had no money at all when he cashed his paycheck. As he left the bank, he bought a piece of candy for a nickel from a machine. Later, he realized that the money in his pocket was equal to twice his paycheck. After a quick calculation, he figured out what happened: the teller accidentally switched the dollars and cents. How much was Kaimi supposed to be paid, and what did the teller give him? Justify your answer.

Problem 38

Here are the rules to a card game. Read the rules carefully and then answer the questions below.

- Each player starts with 10 points. The goal is to score as close to 100 points as possible without going over.
- On your turn: draw two cards, which will each have a decimal number on them. Using estimation (no computation), you can choose to multiply or divide your current score by one of the decimal numbers.
- After you decide, compute your new score exactly using a calculator. If your new score is over 100, you lose. If not, the other player takes a turn.
- At the end of your turn, you can decide to end the game. If you do, the other player gets one more turn. Then, the player with the score that is closest to 100 without going over wins the game.

Here are the questions:

1. On your turn, your score is 50. You draw the cards 0.2 and 1.75. Remember that your choices are:

   $\text{divide by } \ 0.2 \qquad\qquad \text{multiply by } \ 0.2$

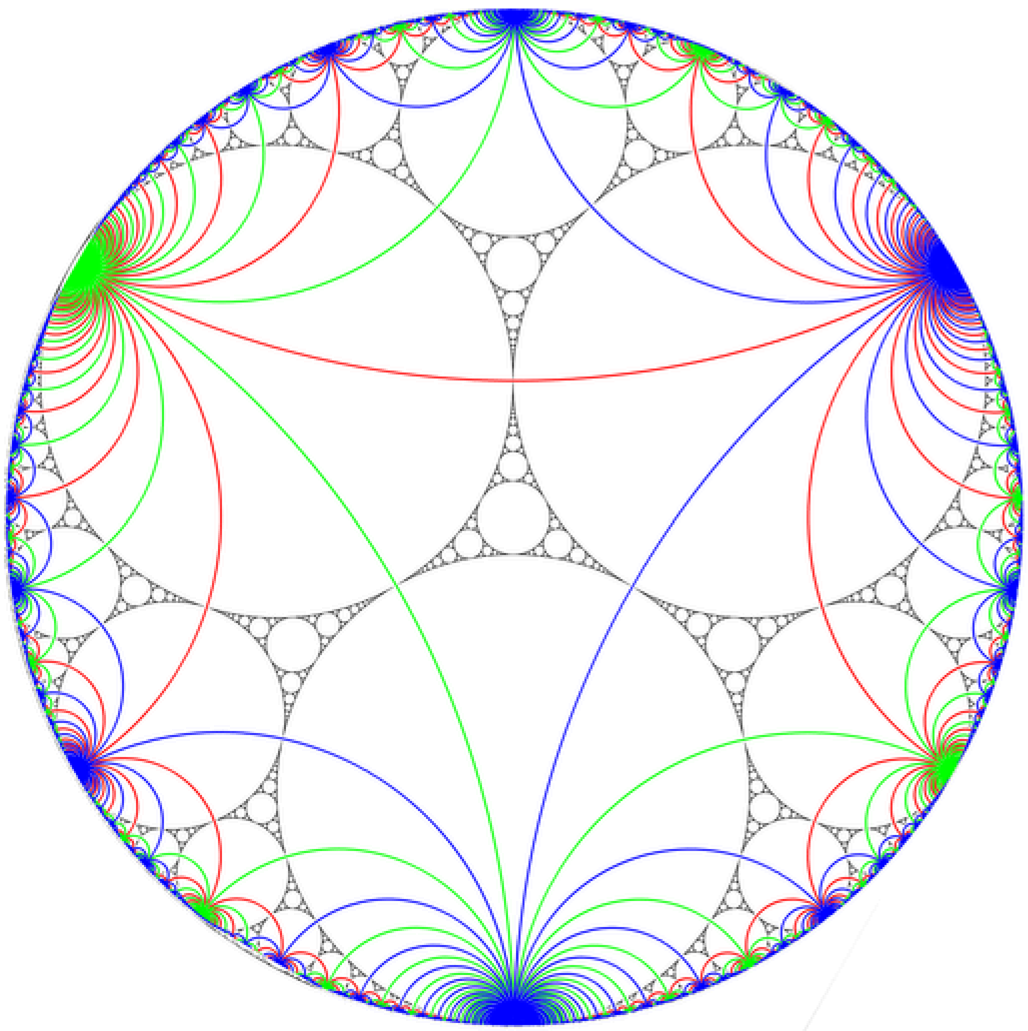
   $\text{divide by } \ 1.75 \qquad\qquad \text{multiply by } \ 1.75.$

   What is your best move and why?

2. On your turn, your score is 88. You draw 1.3 and 0.6. What is your best move and why?

3. Your partner has a score of 57, and your score is 89. On her turn, your partner draws 0.8 and 1.8. She says she wants to end the game. On your final turn, you draw 0.7 and 1.2. If you both make the best possible move, who will win the game? Justify your answer.

Geometry


[1]

Geometry is the art of good reasoning from bad drawings.

– Henri Poincaré

1. Image by Tomruen (Own work) [CC BY-SA 3.0 (https://creativecommons.org/licenses/by-sa/3.0)], via Wikimedia Commons.

Introduction

The word "geometry" comes from the ancient Greek words "geo" meaning Earth and "metron" meaning measurement. It is probably the oldest field of mathematics, because of its usefulness in calculating lengths, areas, and volumes of everyday objects.

The study of geometry has evolved a great deal during the last 3,000 years or so. Like all of mathematics, what's really important in geometry is reasoning, making sense of problems, and justifying your solutions.

The mathematician Henri Poincaré said that

> *Geometry is the art of good reasoning from bad drawings.*

This insight should guide your study in this chapter. You should never trust a drawing. You might find that one line segment appears to be longer than another, or an angle looks like it might be 90 degrees. But "appears to be" and "looks like" are simply not good enough. You have to reason through the situation and figure out what you know for sure and why you know it.

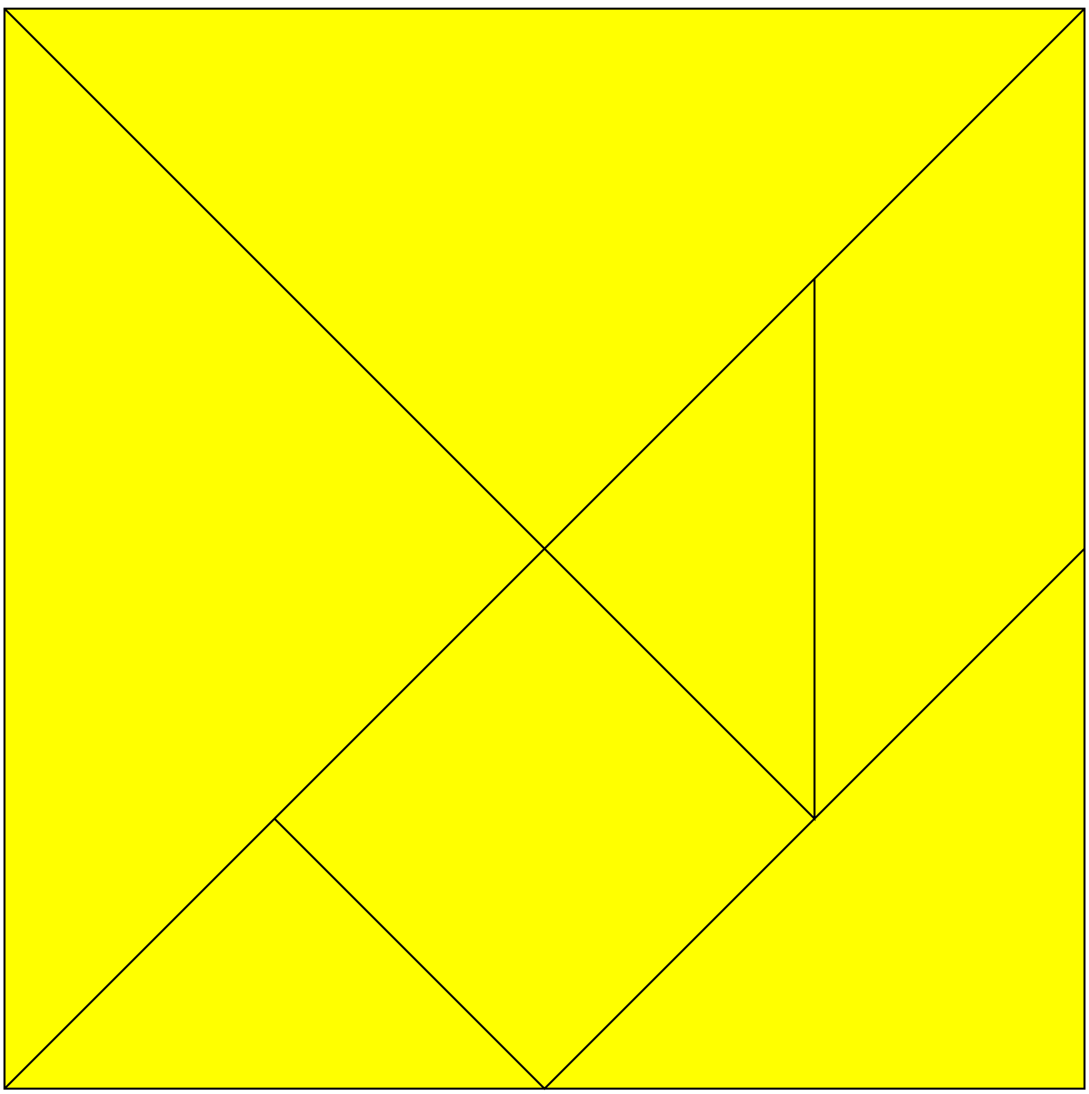
Think / Pair / Share

Reflect on your learning of geometry in the past. What is geometry really about? Also think about these questions:

- What is a point?
- What is a line? A segment? A ray?
- What is a plane?
- What is a circle?
- What is an angle?
- Which of these basic objects can be measured? How are they measured? What kinds of tools are useful?

Tangrams

Tangrams are a seven-piece geometric puzzle that dates back at least to the Song Dynasty in China (about 1100 AD). Below[1] you will find the seven puzzle pieces. Make a careful copy (a photocopy or printout is best), cut out the puzzle pieces, and then use them to solve the problems in this section.



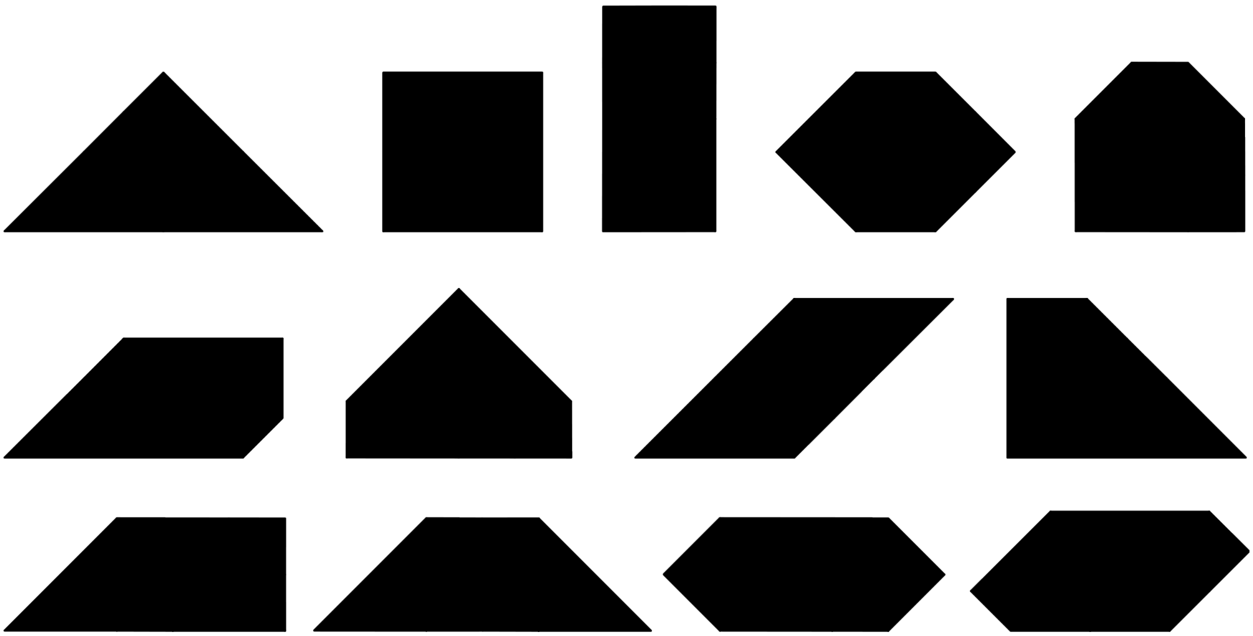
1. Image of tangram puzzle from Wikimedia Commons, public domain.

Whenever you solve a tangram puzzle, your job is to **use all seven pieces to form the shape**. They should fit together like puzzle pieces, sitting flat on the table; **no overlapping** of the pieces is allowed.

You can trace around your solutions to remember what you have done and to have a record of your work.

Problem 1

Use your tangram pieces to build the following designs[2]. How many can you make?

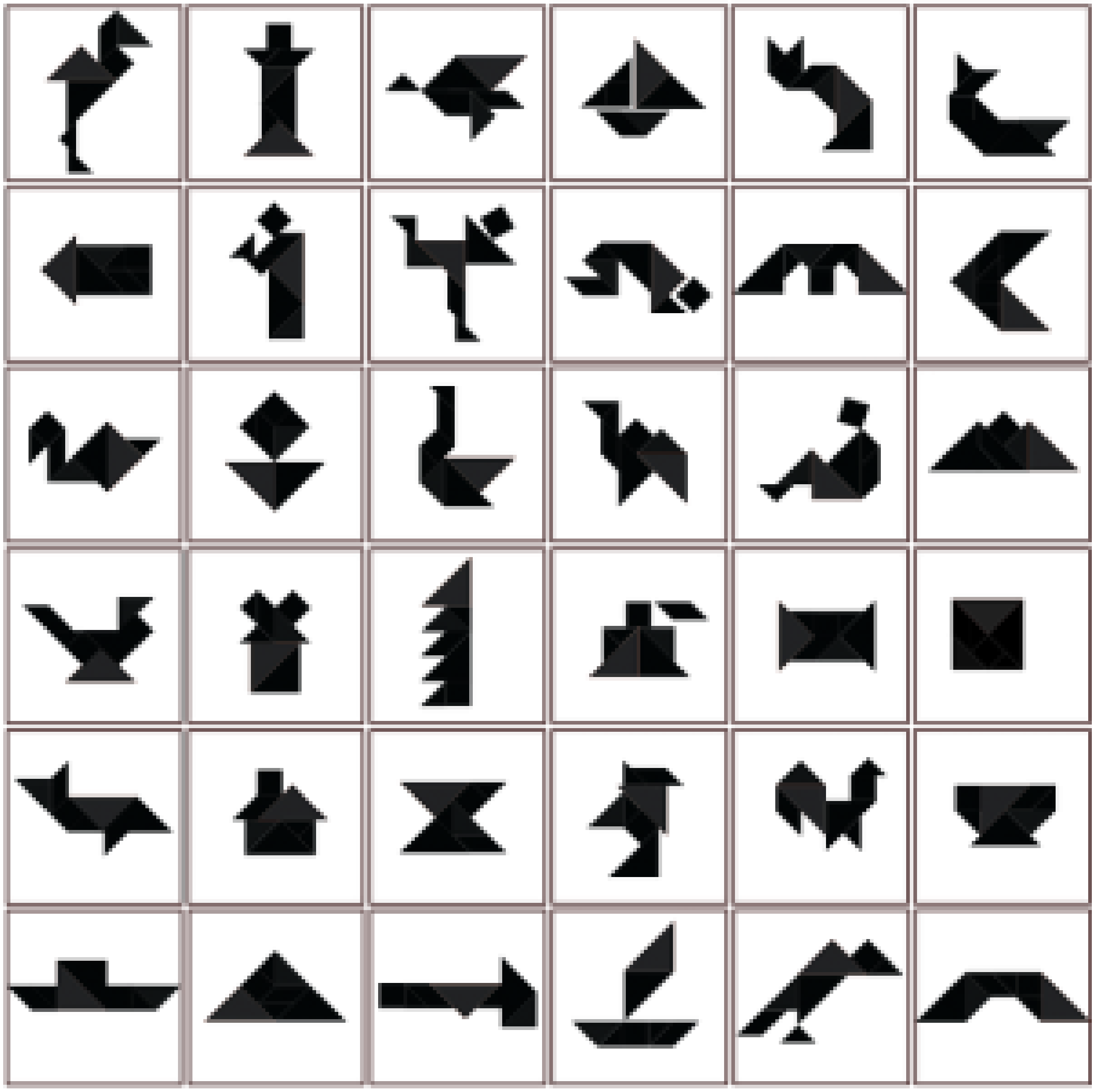


(These are all separate challenges. Each one requires all seven pieces. Once you solve one, trace your solution. Then try to solve another one.)

Problem 2

Use your tangram pieces to build the following designs[3]. How many can you make?

2. Tangram puzzles from Wikimedia Commons, public domain.
3. Tangram puzzles from pixababy.com,CC0 Creative Commons.



(These are all separate challenges. Each one requires all seven pieces. Once you solve one, trace your solution. Then try to solve another one.)

Think / Pair / Share

- Which tangram problems were easier and which were harder: making "real life" objects like cats and people, or purely mathematical objects like the rectangle?
- What do you think made one kind of problem easier or harder?

Triangles and Quadrilaterals

Think / Pair / Share

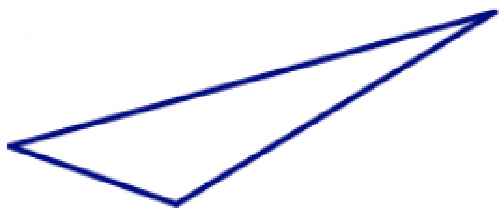
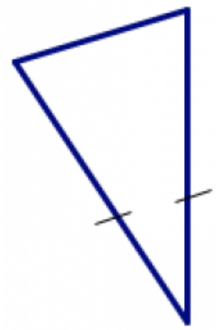
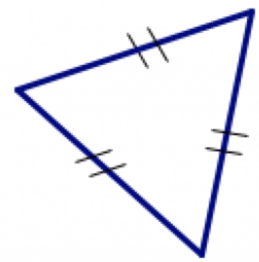
Follow these directions on your own:

- Draw any triangle on your paper.
- Draw a second triangle that is different in some way from your first one. Write down a sentence or two to say how it is different.
- Draw a third triangle that is different from both of your other two. Describe how it is different.
- Draw two more triangles, different from all the ones that came before.

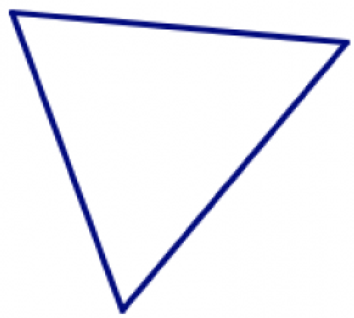
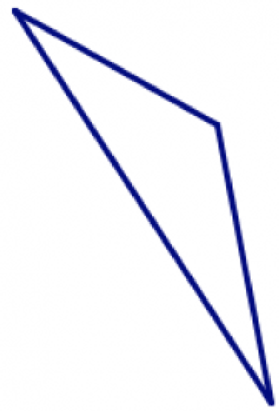
Compare your triangles and descriptions with a partner. To make "different" triangles, you have to change some feature of the triangle. Make a list of the features that you or your partner changed.

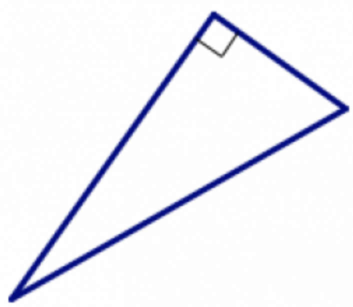
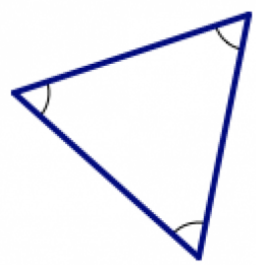
Triangles are classified according to different properties. The point of learning geometry is not to learn a lot of vocabulary, but it's useful to use the correct terms for objects, so that we can communicate clearly. Here's a quick dictionary of some types of triangles.

Classification by sides

scalene	isosceles	equilateral
		
all sides have different lengths	*two sides have the same length*	*all three sides have the same length*

Classification by angles

acute	obtuse
	
all interior angles measure less than 90°	*one interior angle measures more than 90°*

right	equiangular
	
one interior angle measures exactly 90°	*all interior angles have the same measure*

Remember that “geometry is the art of good reasoning from bad drawings.” That means you can’t always trust your eyes. If you look at a picture of a triangle and one side looks like it’s longer than another, that may just mean the drawing was done a bit sloppily.

Notation: Tick marks

Mathematicians either write down measurements or use tick marks to indicate when sides and angles are supposed to be equal.

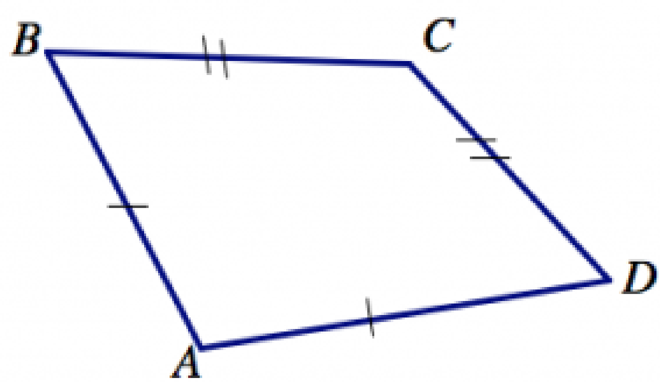
If two sides have the same measurement or the same number of tick marks, you must believe they are equal and work out the problem accordingly, *even if it doesn’t look that way to your eyes*.

You can see examples of these in some of the pictures above. Another example is the little square used to indicate a right angle in the picture of the right triangle.

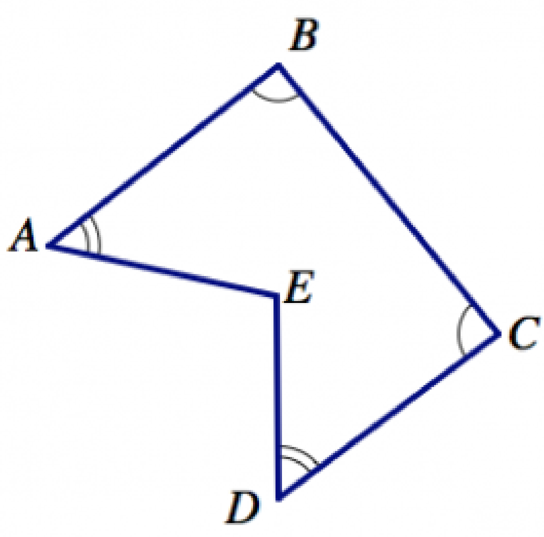
On Your Own

Work on the following exercises on your own or with a partner.

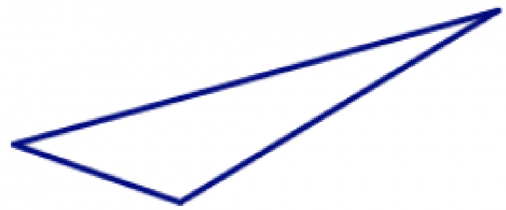
1. In the picture below, which sides are understood to have the same length (even if it doesn't look that way in the drawing)?



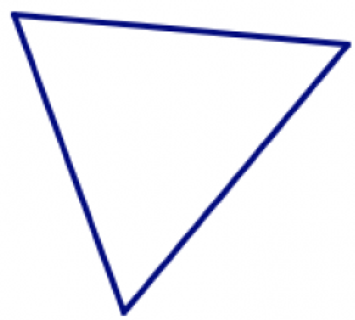
2. In the picture below, which angles are understood to have the same measure (even if if doesn't look that way in the drawing)?



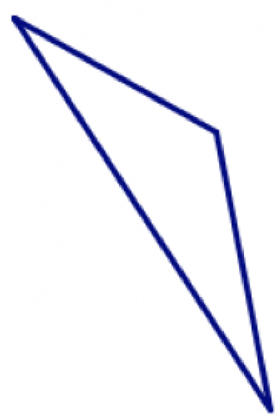
3. Here is a scalene triangle. Sketch two more scalene triangles, each of which is different from the one shown here in some way.



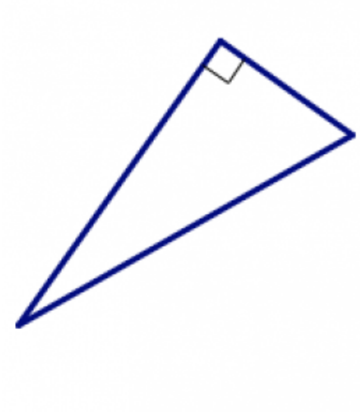
4. Here is an acute triangle. Sketch two more acute triangles, each of which is different from the one shown here in some way.



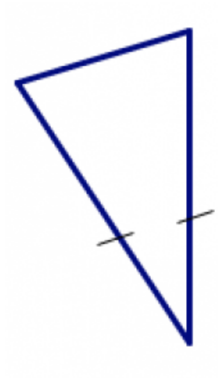
5. Here is an obtuse triangle. Sketch two more obtuse triangles, each of which is different from the one shown here in some way.



6. Here is a right triangle. Sketch two more right triangles, each of which is different from the one shown here in some way. Be sure to indicate which angle is 90°.



7. Here is an isosceles triangle. Sketch two more isosceles triangles, each of which is different from the one shown here in some way. Use tick marks to indicate which sides are equal.

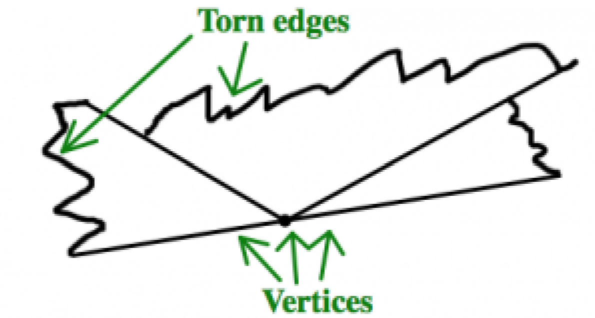


Angle Sum

Think / Pair / Share

By now, you have drawn several different triangles on your paper. Choose one of your triangles, and follow these directions:

- Using scissors, cut the triangle out.
- Tear (do not cut) off the corners, and place the three vertices together. Your should have something that looks a bit like this picture:



What do you notice? What does this suggest about the angles in a triangle?

You may remember learning that the sum of the angles in any triangle is 180°. In your class, you now have lots of examples of triangles where the sum of the angles seems to be 180°. But remember, our drawings are not exact. How can we be sure that our eyes are not deceiving us? How can we be sure that the sum of the angles in a triangle isn't 181° or 178°, but is really 180° on the nose in every case?

Think / Pair / Share

What would convince you beyond all doubt that the sum of the angles in any triangle is 180°? Would testing lots of cases be enough? How many is enough? Could you ever test every possible triangle?

History: Euclid's axioms

Often high school geometry teachers prove the sum of the angles in a triangle is 180°, usually using some facts about parallel lines. But (maybe surprisingly?) it's just as good to take this as an *axiom*, as a given fact about how geometry works, and go from there. Perhaps this is less satisfying than proving it from some other statement, and if you're curious you can certainly find a proof or your instructor can share one with you.

In about 300BC, Euclid[1]Creative Commons Attribution 4.0 International license. was the first mathematician (as far as we know) who tried to write down careful axioms and then build from those axioms rigorous proofs of mathematical truths.

1. Portrait of Euclid from Wikimedia Commons, licensed under the

Euclid

Euclid had five axioms for geometry, the first four of which seemed pretty obvious to mathematicians. People felt they were reasonable assumptions from which to build up geometric truths:

1. Given two points, you can connect them with a straight line segment.

2. Given a line segment, you can extend it as far as you like in either direction, making a line.

3. Given a line segment, you can draw a circle having that segment as a radius.

4. All right angles are congruent.

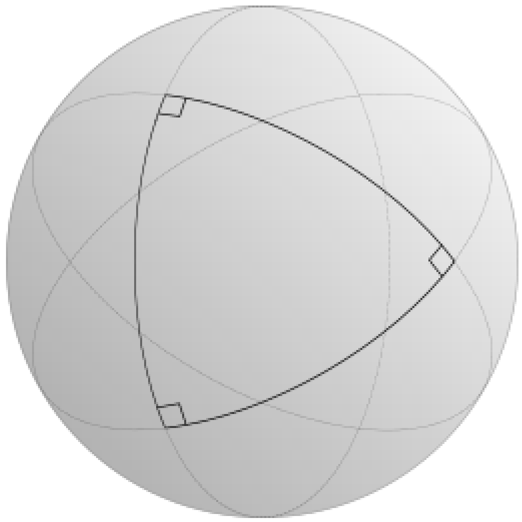
The fifth postulate bothered people a bit more. It was originally stated in more flowery language, but it was equivalent to this statement:

5. The sum of the angles in a triangle is 180°.

It’s easy to see why this fifth axiom caused such a ruckus in mathematics. It seemed much less obvious than the other four, and mathematicians felt like they were somehow cheating if they just assumed it rather than proving it had to be true. Many mathematicians spent many, many years trying to prove this fifth axiom from the other axioms, but they couldn’t do it. And with good reason: There are other kinds of geometries where the first four axioms are true, but the fifth one is not!

For example, if you do geometry on a sphere — like a basketball or more importantly on the surface of the Earth

— rather than on a flat plane, the first four axioms are true. But triangles are a little strange on the surface of the earth. Every triangle you can draw on the surface of the earth has an angle sum strictly *greater than* 180°. In fact, you can draw a triangle on the Earth that has three right angles[2]Creative Commons Attribution-Share Alike 3.0 Unported., making an angle sum of 270°.



Triangle with three right angles on a sphere.

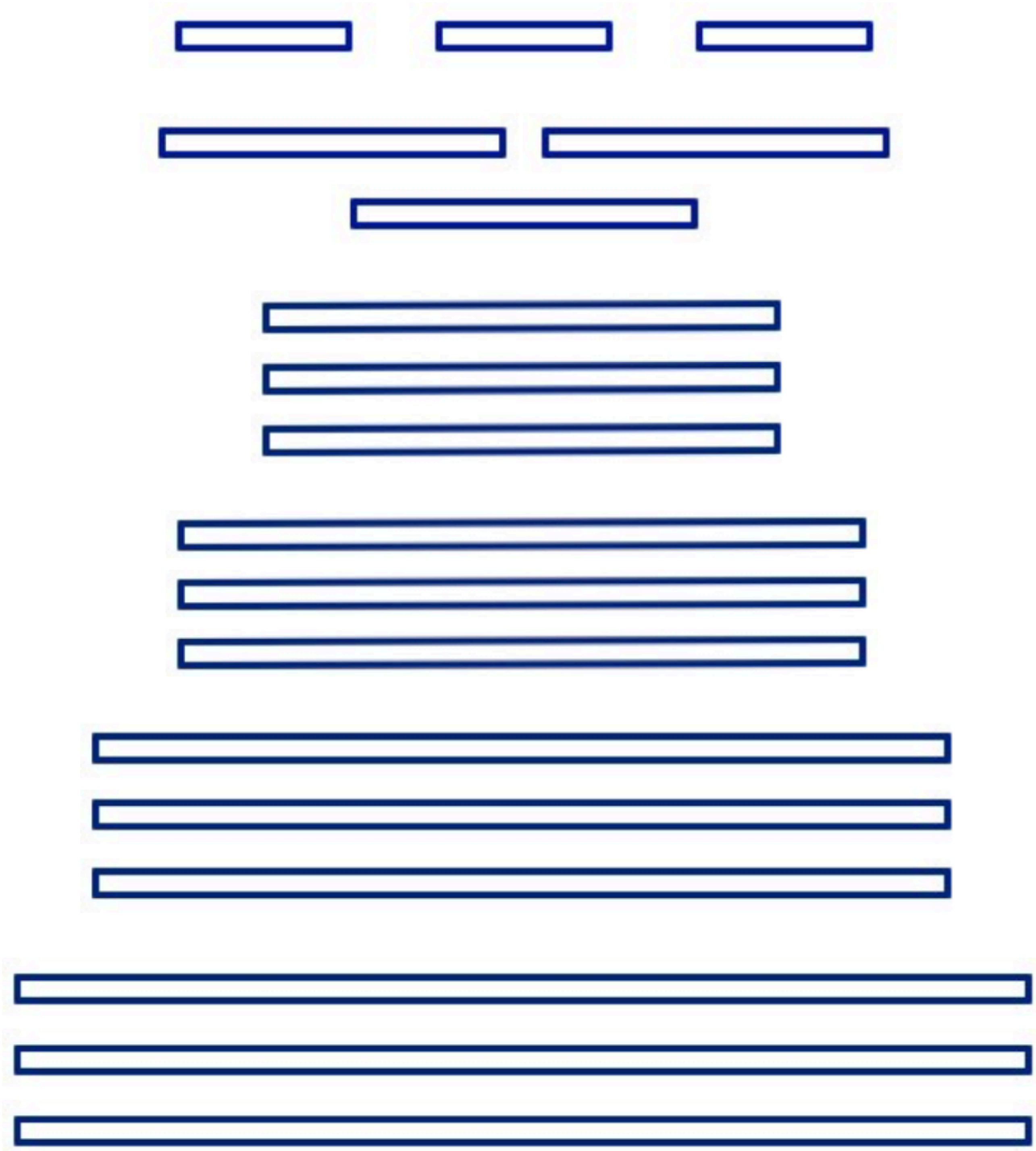
On a sphere like the Earth, the angle sum is not constant among all triangles. Bigger triangles have bigger angle sums, and smaller triangles have smaller angle sums, but even tiny triangles have angle sums that are greater than 180°.

The geometry you study in school is called *Euclidean geometry*; it is the geometry of a flat plane, of a flat world. It's a pretty good approximation for the little piece of the Earth that we see at any given time, but it's not the only geometry out there!

Triangle Inequality

Make a copy of these strips of paper and cut them out. They have lengths from 1 unit to 6 units. You may want to color the strips, write numbers on them, or do something that makes it easy to keep track of the different lengths.

2. Image by Coyau / Wikimedia Commons, via Wikimedia Commons, licensed under



Problem 3

Repeat the following process several times (at least 10) and keep track of the results (a table has been started for you).

- Pick three strips of paper. (The lengths do not have to be all different from each other; that's why you have multiple copies of each length.)
- Try to make a triangle with those three strips, and decide if you think it is possible or not. (Don't overlap the strips, cut them, or fold them. The length of the strips should be the length of the sides of the triangle.)

Length 1	Length 2	Length 3	Triangle?
4	3	2	yes
4	2	1	no
4	2	2	??

Your goal is to come up with a **rule that describes when three lengths will make a triangle** and when they will not. Write down the rule in your own words.

Think / Pair / Share

Compare your rule with other students. Then use your rule to answer the following questions. Keep in mind the goal is not to try to build the triangle, but to **predict the outcome** based on your rule.

- Suppose you were asked to make a triangle with sides 40 units, 40 units, and 100 units long. Do you think you could do it? Explain your answer.
- Suppose you were asked to make a triangle with sides 2.5 units, 2.6 units, and 5 units long. Do you think you could do it? Explain your answer.

You probably came up with some version of this statement:

Theorem: Triangle Inequality

The sum of the lengths of two sides in a triangle is greater than the length of the third side.

Of course, we know that in geometry we should not believe our eyes. You need to look for an *explanation*. Why does your statement make sense?

Remember that "geometry is the art of good reasoning from bad drawings." Our materials weren't very precise, so how can we be sure this rule we've come up with is is correct?

Well in this case, the rule is really just the same as the saying "the shortest distance between two points is a straight line." In fact, this is exactly what we mean by the words *straight line* in geometry.

SSS Congruence

We say that two triangles (or any two geometric objects) are *congruent* if they are exactly the same shape and the same size. That means that if you could pick one of them up and move it to put down on the other, they would exactly overlap.

Problem 4

Repeat the following process several times and keep track of the results.

- Pick three strips of paper that will definitely form a triangle.
- Try to make two *different* (non-congruent) triangles with the same three strips of paper. Record if you were able to do so.

Problem 5

Repeat the following process several times and keep track of the results.

- Pick four strips of paper and form a quadrilateral with them. (If your four strips do not form a quadrilateral, pick another four strips.)
- Try to make two *different* (non-congruent) quadrilaterals with the same four strips of paper. Record if you were able to do so.

Think / Pair / Share

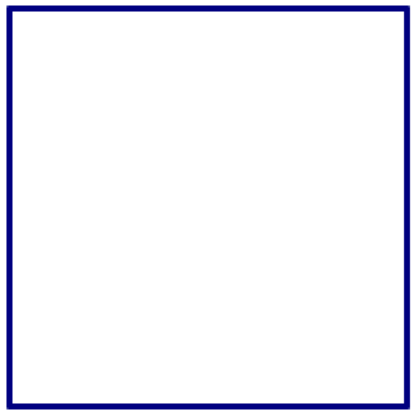
What do you notice from Problems 4 and 5? Can you make a general statement to describe what's going on? Can you explain why your statement makes sense?

You probably came up with some version of this statement:

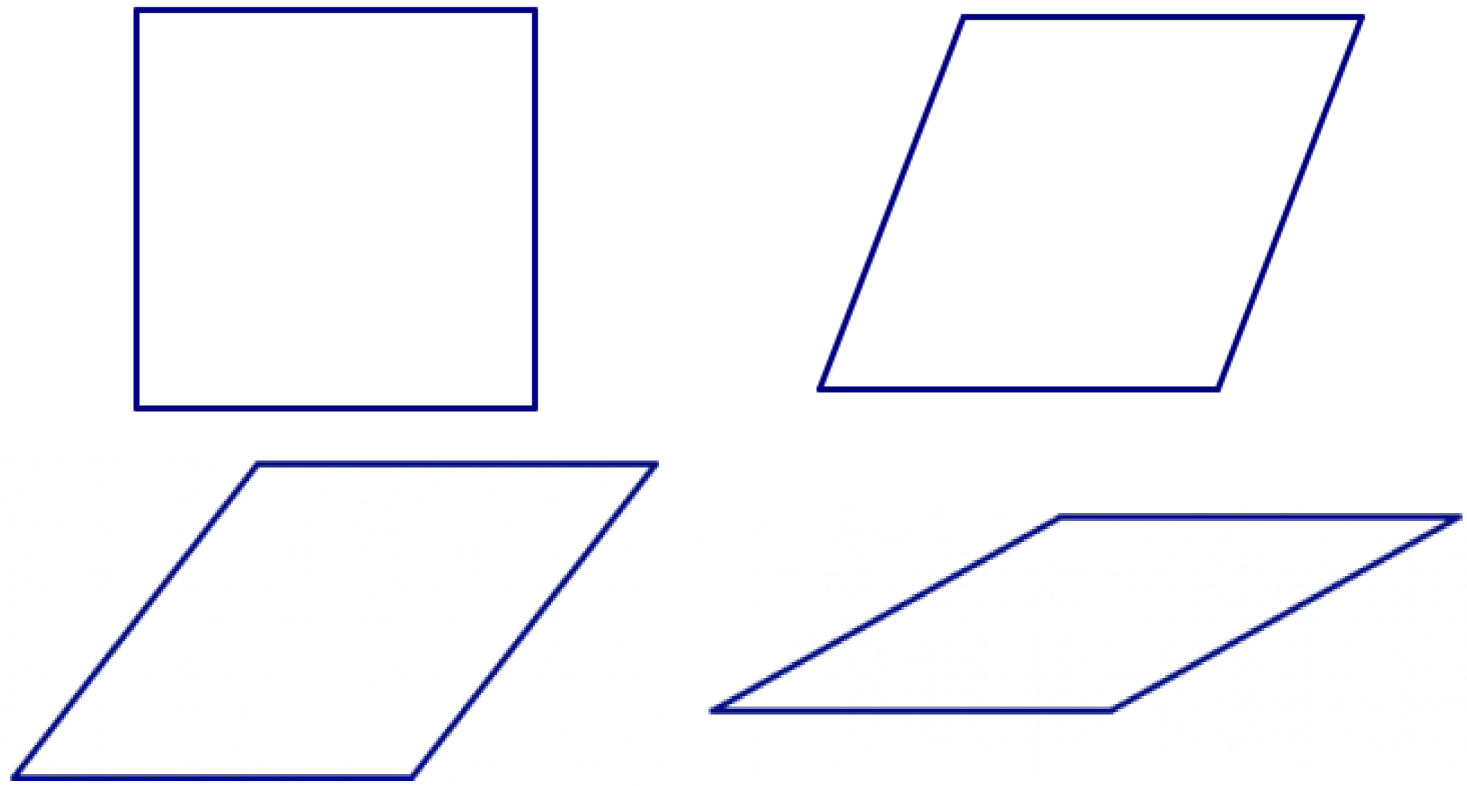
Theorem: SSS (side-side-side) Congruence

If two triangles have the same side lengths, then the triangles are congruent.

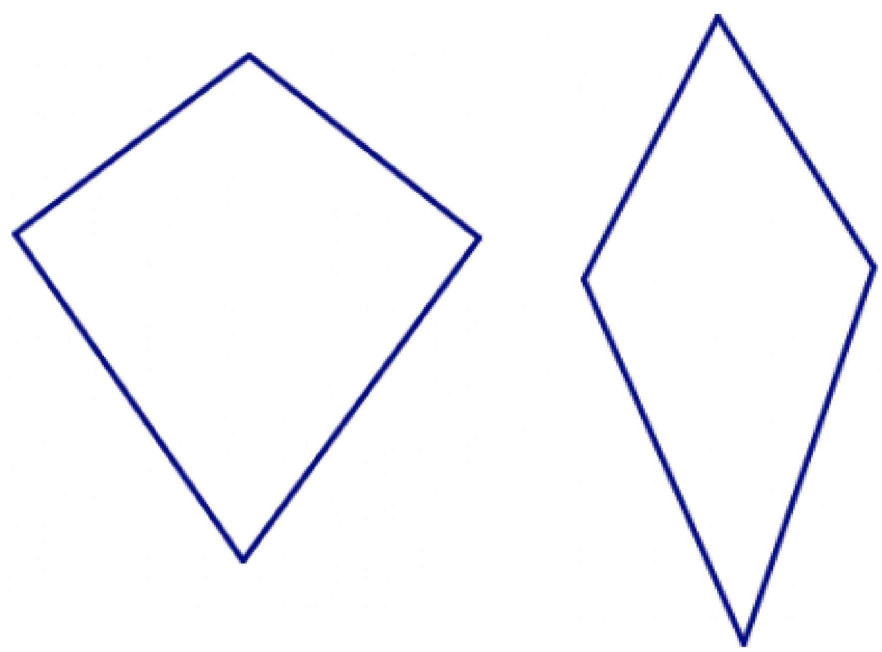
This most certainly is not true for quadrilaterals. For example, if you choose four strips that are all the same length, you can make a square:



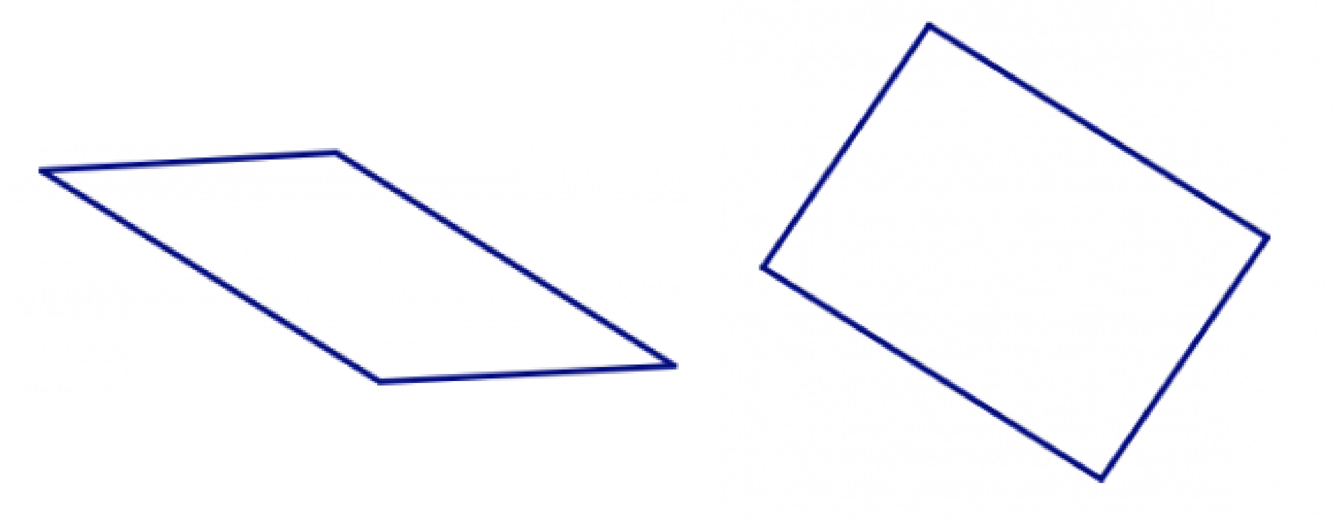
But you can also squish that square into a non-square rhombus. (Try it!)



If you don't choose four lengths that are all the same, in addition to "squishing" the shape, you can rearrange the sides to make different (non-congruent) shapes. (Try it!)



These two quadrilaterals have the same four side lengths in the same order.



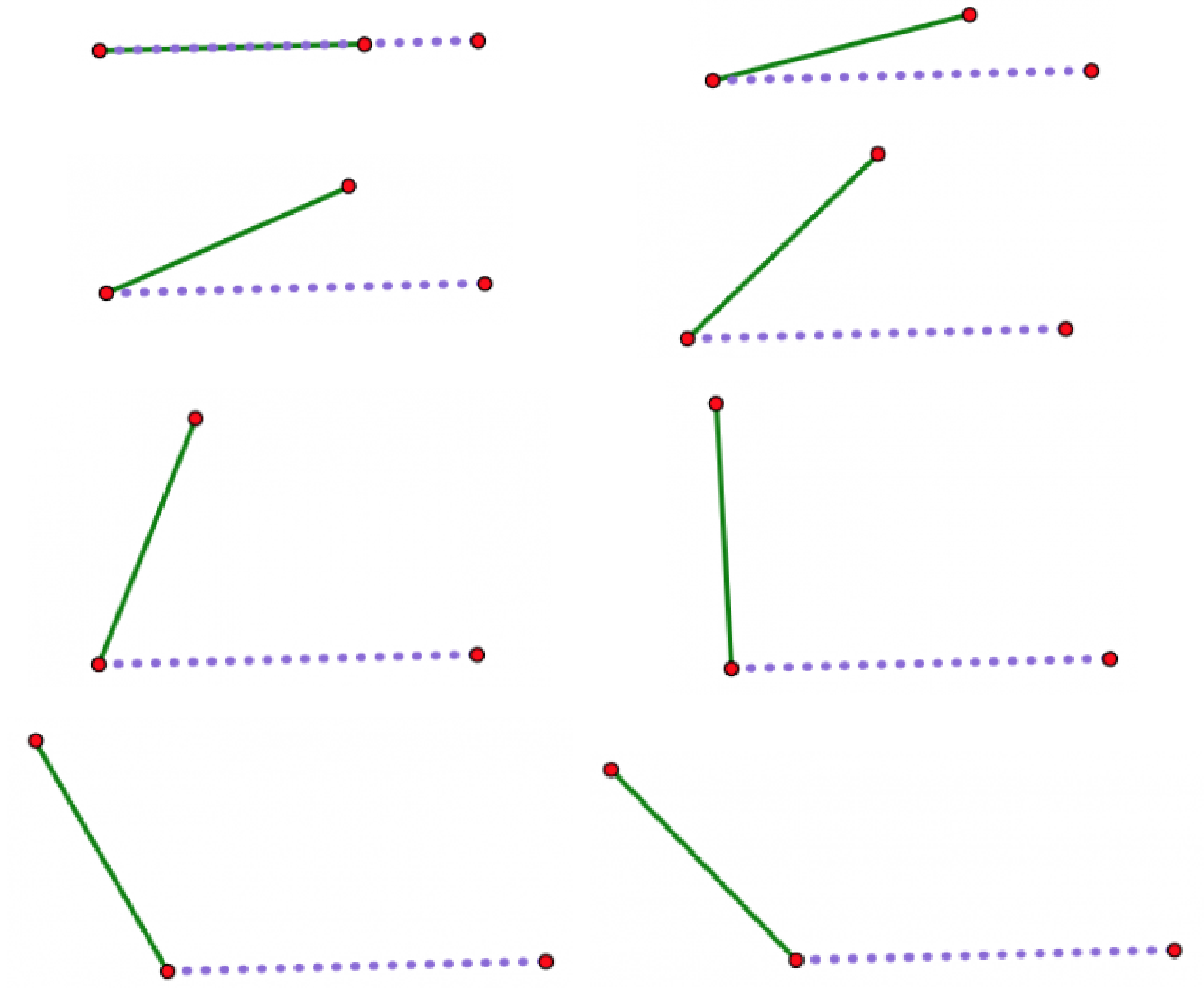
These two quadrilaterals have the same four side lengths as the two above, but the sides are in a different order.

But this can't happen with triangles. Why not? Well, certainly you can't rearrange the three sides. That would be just the same as rotating the triangle or flipping it over, but not making a new shape.

Why can't the triangles "squish" the way a quadrilateral (and other shapes) can? Here's one way to understand it. Imagine you pick two of your three lengths and lay them on top of each other, hinged at one corner.

This shows a longer purple dashed segment and a shorter green segment. The two segments are hinged at the red dot on the left.

Now imagine opening up the hinge a little at a time.



As the hinge opens up, the two non-hinged endpoints get farther and farther apart. Whatever your third length is (assuming you are actually able to make a triangle with your three lengths), there is **exactly one position** of the hinge where it will just exactly fit to close off the triangle. No other position will work.

Polygons

It can seem like the study of geometry in elementary school is nothing more than learning a bunch of definitions and then classifying objects. In this part, you'll explore some problem solving and reasoning activities that are based in geometry. But definitions are still important! So let's start with this one.

Definition

A **polygon** is:

1. a plane figure
2. that is bounded by a finite number of straight line segments
3. in which each segment meets exactly two others, one at each of its endpoints.

Think / Pair / Share

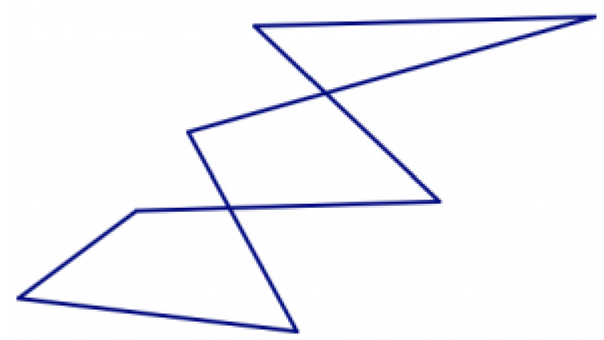
Just as the first step in problem solving is to understand the problem, the first step in reading a mathematical definition is to understand the definition.

- Use the definition above to draw several examples of figures that are definitely polygons. (You should be able to say why your example fits the definition.)
- Draw several non-examples as well: shapes that are definitely not polygons. (You should be able to say which part of the definition fails for your non-examples.)

A few comments about polygons:

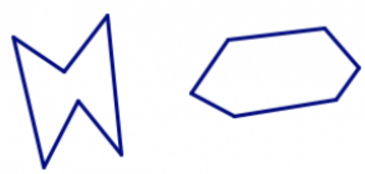
- The line segments that make up a polygon are called its **edges** and the points where they meet are called its **vertices** (singular: **vertex**).

- Because of properties (2) and (3) in the definition, the boundaries of polygons are not self-intersecting.



Not a polygon.

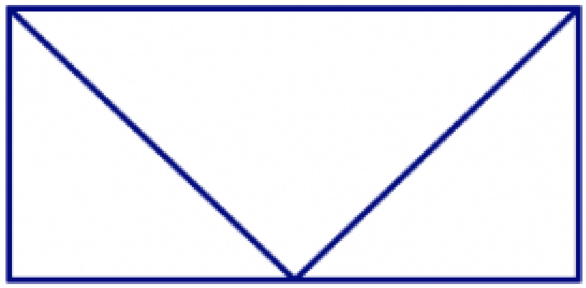
- Polygons are named based on the number of sides they have.

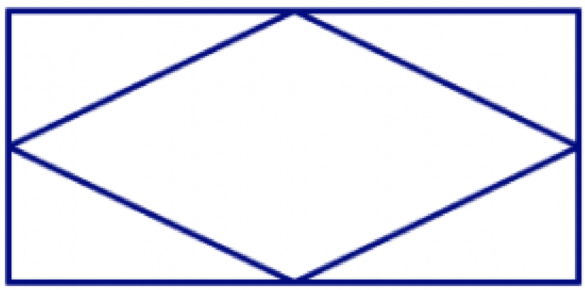
name	# of sides	examples
triangle	3	
quadrilateral	4	
pentagon	5	
hexagon	6	
heptagon	7	
octagon	8	
nonagon	9	
decagon	10	

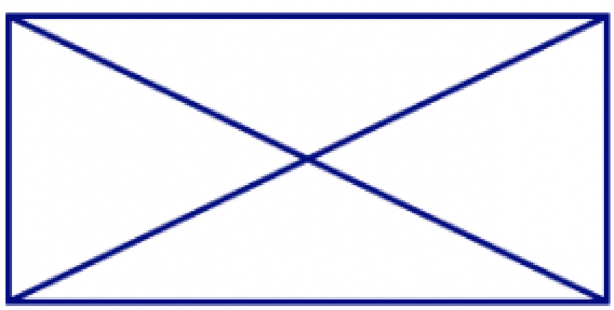
- In general, we call a polygon with *n* sides an *n-gon*.

Problem 6

In the pictures below, there are polygons hidden in the design. In each design, find all of the triangles, quadrilaterals, pentagons, and hexagons. How can you be sure you've found them all and haven't counted any twice?







Angle Sum

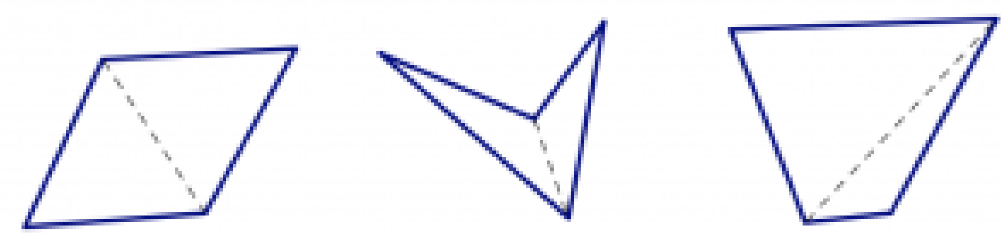
You know that the sum of the interior angles in any triangle is 180°. Can you say anything about the angles in other polygons?

You probably know that rectangles have four 90° angles. So *if if all quadrilaterals have the same interior angle sum*, it must be 360° (since 4 × 90° = 360°).

But notice: We don't necessarily have any reason to believe this constant sum would be true. Remember that SSS congruence is true for triangles, but not for any other polygons. Triangles are special, and we shouldn't assume that true statements about triangles will hold true for other shapes.

Think / Pair / Share

Any quadrilateral can be split into two triangles, where the vertices of the triangles all coincide with the vertices of the quadrilateral:



Use the pictures above to carefully explain why all quadrilaterals do, indeed, have an angle sum of 360°.

On Your Own

Work on the following exercises on your own or with a partner.

1. Draw several different pentagons on your paper. Show that each of them can be split into exactly three triangles in such a way that the vertices of the triangles all coincide with the vertices of the pentagon.
2. Use the fact that every pentagon can be split into three triangles in this way to find the sum of the angles in any pentagon.
3. Draw several different hexagons on your paper. Show that each of them can be split into exactly four triangles so that the vertices of the triangles all coincide with the vertices of the hexagon.
4. Use the fact that every hexagon can be split into four triangles in this way to find the sum of the angles in any hexagon.

Problem 7

Use your work on the exercises above to complete this general statement:

Angle Sum in Polygons

The sum of the interior angles in an n-gon (a polygon with n sides) is

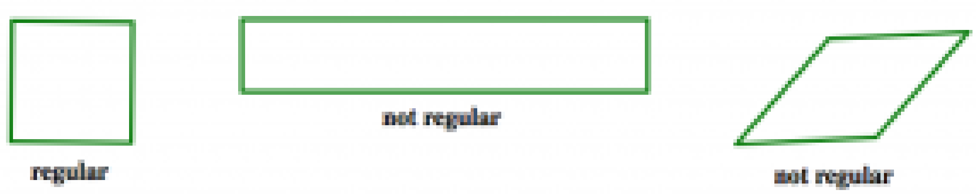
____________________________.

Explain how you know your statement is true.

Definition

A **regular polygon** has all sides the same length and all angles the same measure.

For example, squares are regular quadrilaterals — all four sides are the same length, and all four angles measure 90°. But a non-square rectangle is *not regular*. Even though all of the angles are 90°, the sides are not all the same length. Similarly, a non-square rhombus is *not regular*. Even though the sides of a rhombus are all the same length, the angles can be different.



Problem 8

Since a square is a regular quadrilateral, you know that every angle in a regular quadrilateral measure 90°. What about angles in other regular polygons?

1. What is the measure of each angle in a regular triangle? Explain how you know you are right.
2. What is the measure of each angle in a regular pentagon? Explain how you know you are right.
3. What is the measure of each angle in a regular hexagon? Explain how you know you are right.
4. What is the measure of each angle in a regular *n-gon*? Explain how you know you are right.

Platonic Solids

Of course, we live in a three-dimensional world (at least!), so only studying flat geometry doesn't make a lot of sense. Why not think about some three-dimensional objects as well?

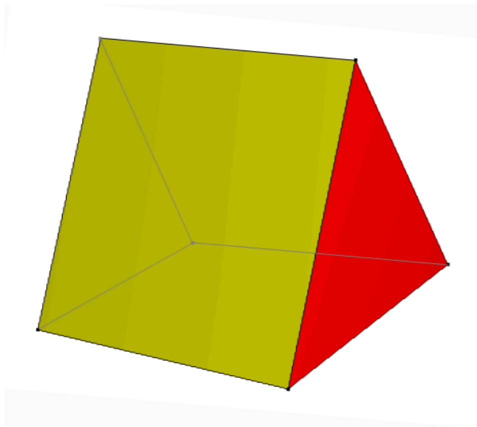
Definition

A **polyhedron** is a solid (3-dimensional) figure bounded by polygons. A polyhedron has **faces** that are flat polygons, straight **edges** where the faces meet in pairs, and **vertices** where three or more edges meet.

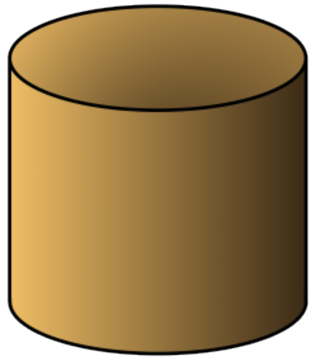
The plural of polyhedron is **polyhedra.**

Think / Pair / Share

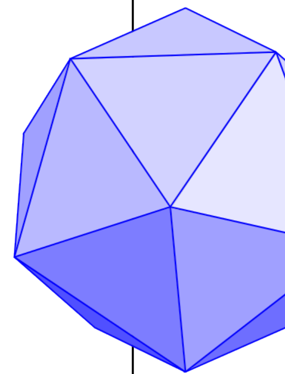
Look at the pictures of solids below, and decide which are polyhedra and which are not. You should be able to say why each figure does or does not fit the definition.



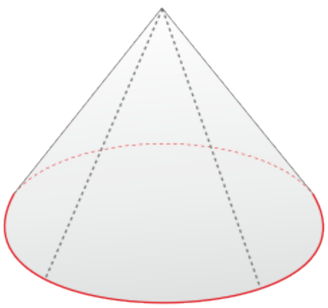
(a)[1]



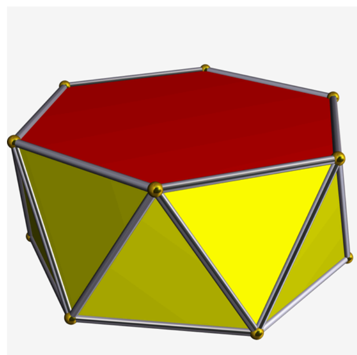
(b)[2]



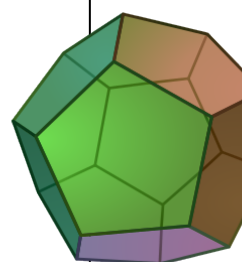
(c)[3]



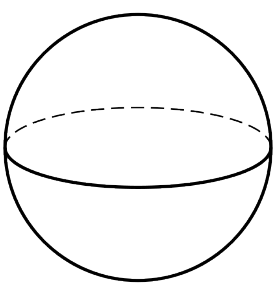
(d)[4]



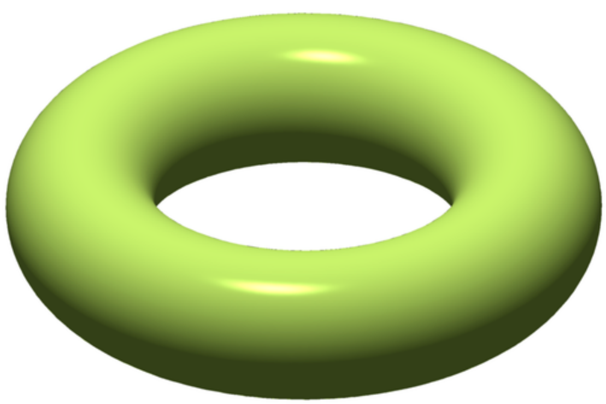
(e)[5]



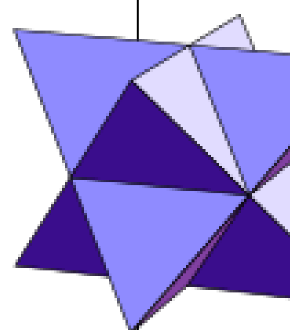
(f)[6]



(g)[7]



(h)[8]



(i)[9]

1. Image by Tom Ruen [Public domain], via Wikimedia Commons
2. Image via pixababy.com, CC0 Creative Commons license.
3. Image by Aldoaldoz (Own work) [CC BY-SA 3.0, via Wikimedia Commons.
4. Image by By Thinkingarena (Own work) [CC BY-SA 4.0], via Wikimedia Commons
5. Image by Robert Webb's Stella software: http://www.software3d.com/Stella.php, via Wikimedia Commons.
6. Image DTR CC-BY-SA-3.0], via Wikimedia Commons
7. Imgae by Stephen.G.McAteer (Own work) [CC BY-SA 3.0], via Wikimedia Commons.
8. Imgae via Wikimedia Commons [Public domain].
9. Image by self [CC BY-SA 3.0], via Wikimedia Commons.

Remember that a *regular polygon* has all sides the same length and all angles the same measure. There is a similar (if slightly more complicated) notion of *regular* for solid figures.

Definition

A r**egular polyhedron** has faces that are all *identical (congruent) regular polygons*. All vertices are also identical (the same number of faces meet at each vertex).

Regular polyhedra are also called **Platonic solids** (named for Plato).

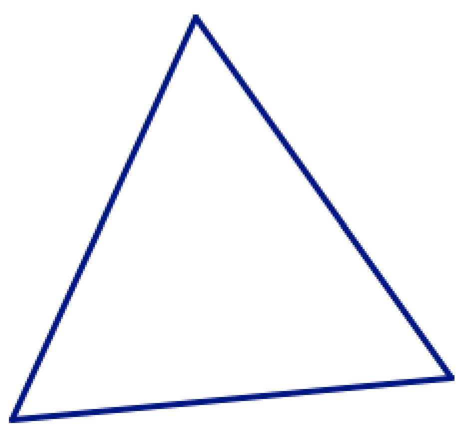
If you fix the number of sides and their length, there is one and only one regular polygon with that number of sides. That is, every regular quadrilateral is a square, but there can be different sized squares. Every regular octagon looks like a stop sign, but it may be scaled up or down. Your job in this section is to figure out what we can say about regular polyhedra.

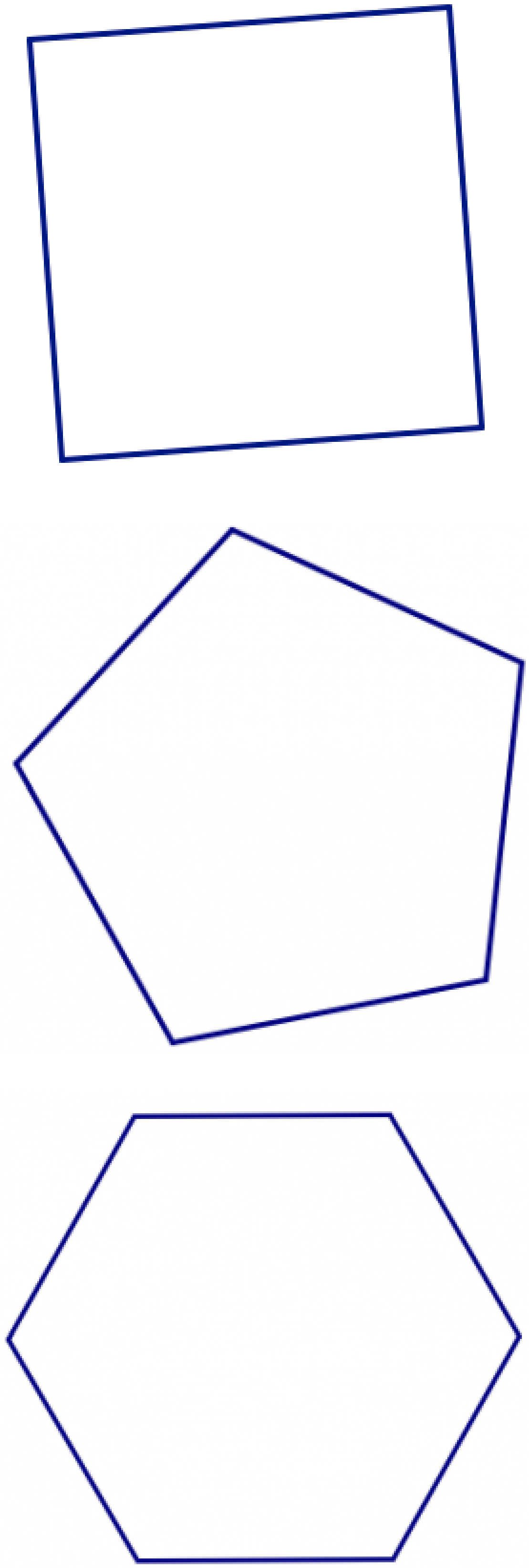
On Your Own

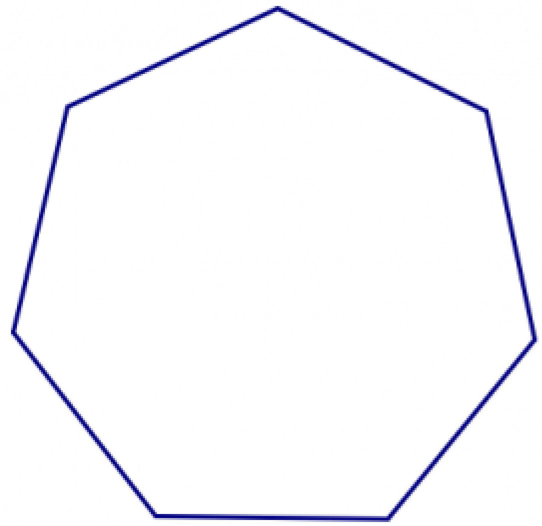
Work on the following exercises on your own or with a partner. You will need to make lots of copies of the regular polygons below. Copy and cut out at least:

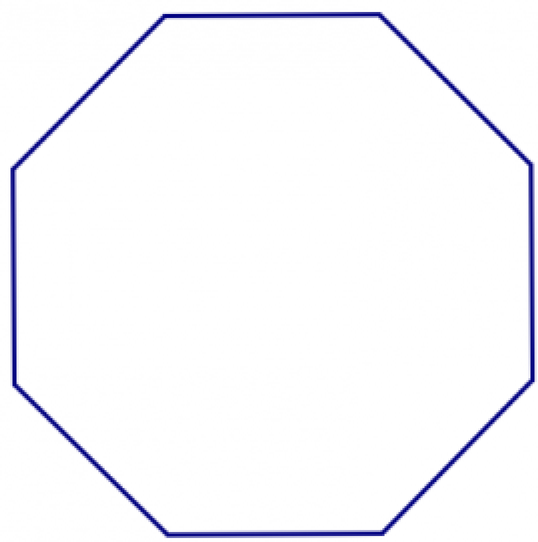
- 40 copies of the equilateral triangle,
- 15 copies of the square,
- 20 copies of the regular pentagon, and
- 10 copies each of the hexagon, heptagon, and octagon.

You will also need some tape.









1. In any polyhedron, at least three polygons meet at each vertex. Start with the equilateral triangles: Put **three** of them together meeting at a vertex and tape them together. Then close them up so they form a solid shape. Can you complete this shape into a platonic solid? Be sure to check that at every vertex you have **exactly three** triangles meeting.
2. Now repeat this process, but start with **four** equilateral triangles around a single vertex. Then close them up so they form a solid shape. Can you complete this into a platonic solid? Be sure to check that at every vertex you have **exactly four** triangles meeting.
3. Repeat this process with **five** equilateral triangles, then six, then seven, and so on. Keep going until you are convinced you understand what's happening with Platonic solids that have triangular faces.
4. When you are done with triangular faces, move on to square faces. Work systematically: Try to build a Platonic solid with three squares at each vertex, then four, then five, etc. Keep going until you can make a definitive statement about Platonic solids with square faces.

5. Repeat this process with the other regular polygons you cut out: pentagons, hexagons, heptagons, and octagons.

You must have noticed that the situation for Platonic solids is quite different from the situation for regular polygons. There are *infinitely many* regular polygons (even if you don't account for size). There is a regular polygon with n sides for every value of n bigger than 2. But for solids, we have the following (perhaps surprising) result.

Theorem

There are exactly five Platonic solids.

The key fact is that for a three-dimensional solid to close up and form a polyhedron, there must be less than 360° around each vertex. Otherwise, it either lies flat (if there is exactly 360°) or folds over on itself (if there is more than 360°).

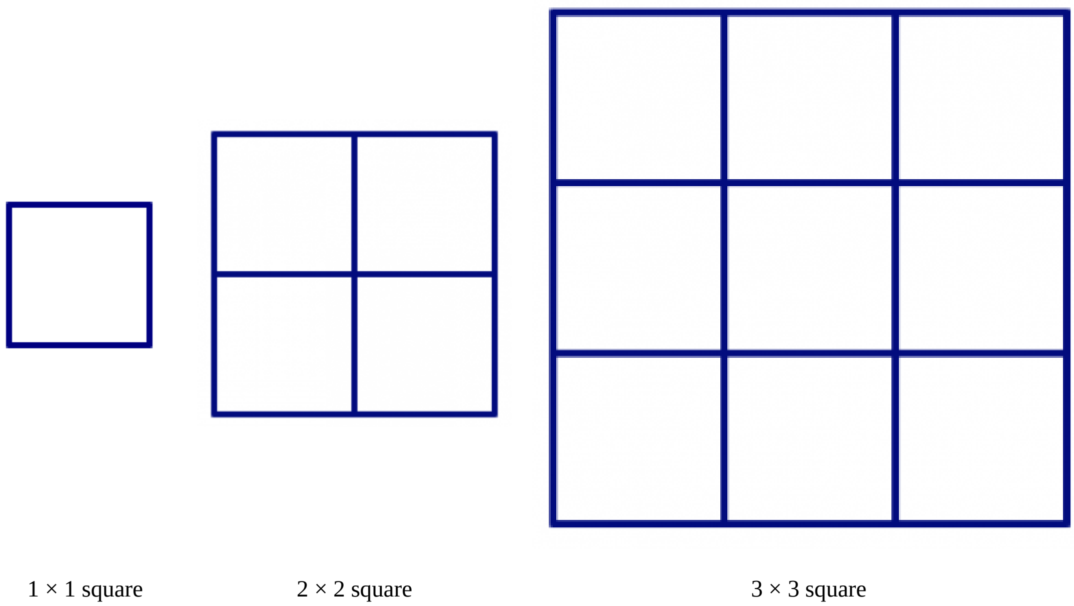
Problem 9

Based on your work in the exercises, you should be able to write a convincing justification of the Theorem above. Here's a sketch, and you should fill in the explanations.

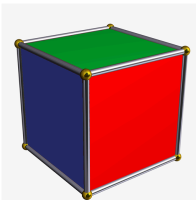
1. If a Platonic solid has faces that are equilateral triangles, then fewer than 6 faces must meet at each vertex. Why?
2. If a Platonic solid has square faces, then three faces can meet at each vertex, but not more than that. Why?
3. If a Platonic solid has faces that are regular pentagons, then three faces can meet at each vertex, but not more than that. Why?
4. Regular hexagons cannot be used as the faces for a Platonic solid. Why?
5. Similarly, regular n-gons for n bigger than 6 cannot be used as the faces for a Platonic solid. Why?

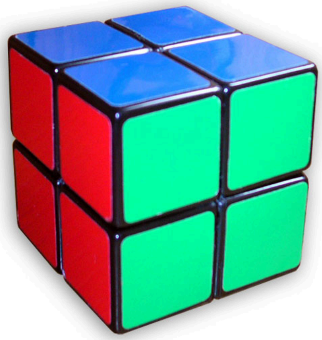
Painted Cubes

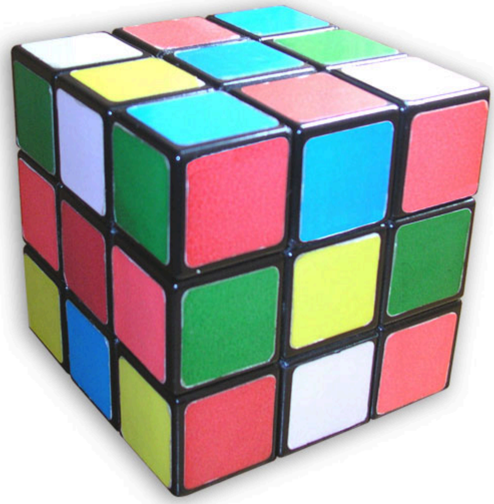
You can build up squares from smaller squares:



In a similar way, you can build up cubes from smaller cubes:







$1 \times 1 \times 1$ cube[1] $2 \times 2 \times 2$ cube[2] $3 \times 3 \times 3$ cube[3]

Think / Pair / Share

We call a $1 \times 1 \times 1$ cube a **unit cube**.

- How many unit cubes are in a $2 \times 2 \times 2$ cube?
- How many unit cubes are in a $3 \times 3 \times 3$ cube?
- How many unit cubes are in a $n \times n \times n$ cube?

Explain your answers.

Problem 10

Imagine you build a $3 \times 3 \times 3$ cube from 27 small white unit cubes. Then you take your cube and dip it into a bucket of bright blue paint. After the cube dries, you take it apart, separating the small unit cubes.

1. After you take the cube apart, some of the unit cubes are still all white (no blue paint). How many? How do you know you are right?
2. After you take the cube apart, some of the unit cubes have blue paint on just one face. How many? How do you know you are right?

1. Image by Robert Webb's Stella software: http://www.software3d.com/Stella.php, via Wikimedia Commons.
2. Image by Mike Gonzalez (TheCoffee) (Work by Mike Gonzalez (TheCoffee)) [CC BY-SA 3.0], via Wikimedia Commons.
3. Image by Mike Gonzalez (TheCoffee) (Work by Mike Gonzalez (TheCoffee)) [CC BY-SA 3.0], via Wikimedia Commons.

3. After you take the cube apart, some of the unit cubes have blue paint on two faces. How many? How do you know you are right?
4. After you take the cube apart, some of the unit cubes have blue paint on three faces. How many? How do you know you are right?
5. After you take the cube apart, do any of the unit cubes have blue paint on more than three faces? How many? How do you know you are right?

Problem 11

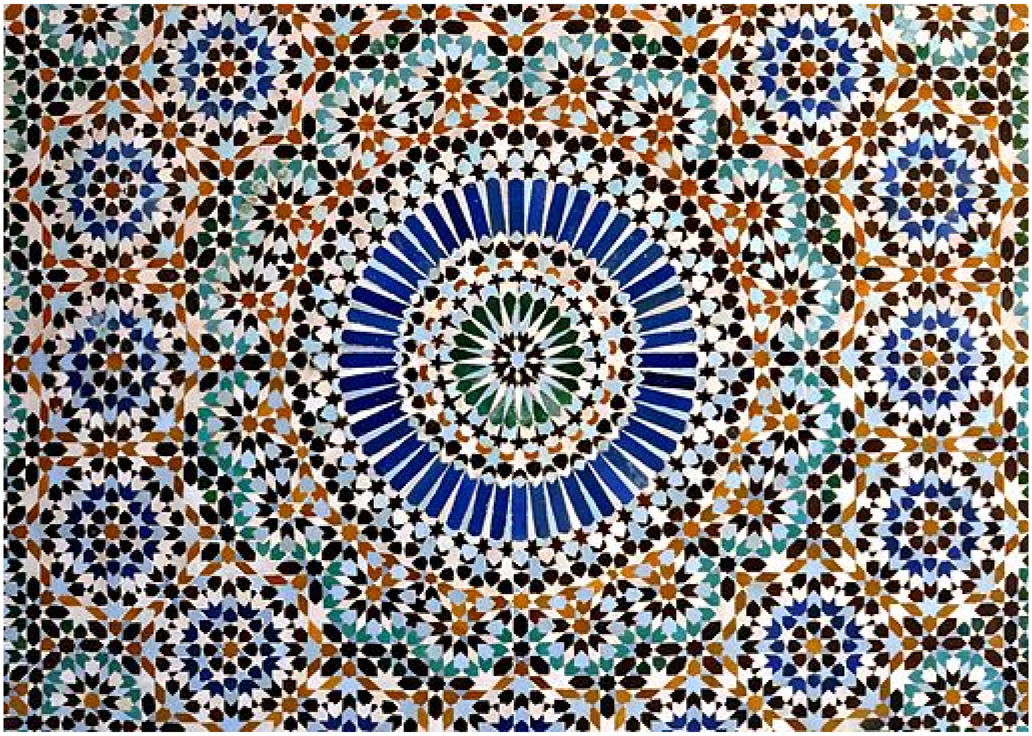
Generalize your work on Problem 10. What if you started with a $2 \times 2 \times 2$ cube? Answer the same questions. What about a $4 \times 4 \times 4$ cube? How about an $n \times n \times n$ cube? Be sure to justify what you say.

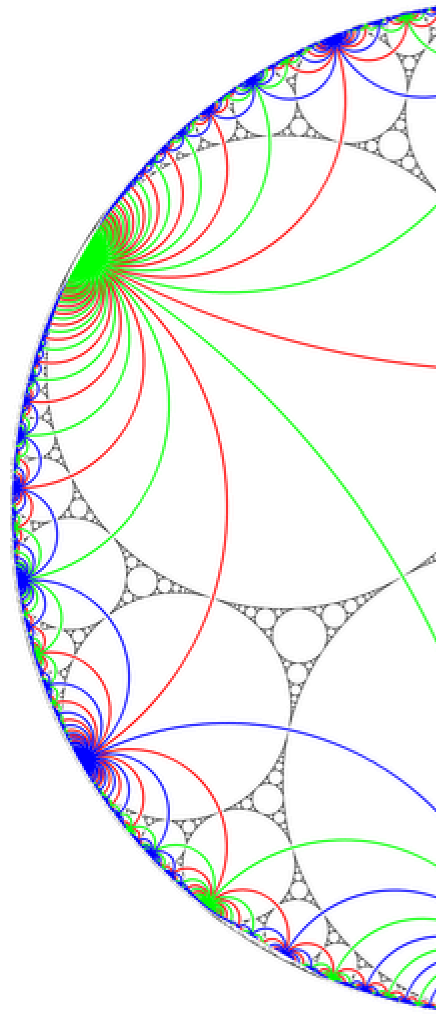
Symmetry

Mathematicians use symmetry in all kinds of situations. There can be symmetry in calculations, for example. But the most recognizable kinds of symmetry are those in geometric designs.

Geometric and real-world objects can have different kinds of symmetries[1].

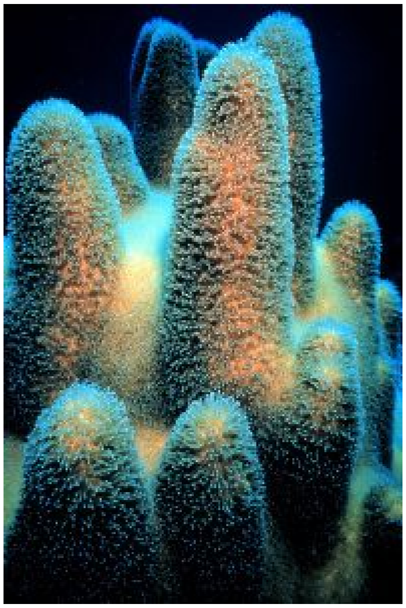
1. Mosaic image by MarcCooperUK (Flickr: Paris central mosque) [CC BY 2.0], via Wikimedia Commons. Apollonian Circle Packing by Tomruen (Own work) [CC BY-SA 3.0], via Wikimedia Commons. Butterfly by Bernard DUPONT from FRANCE (Swallowtail Butterfly (Papilio oribazus)) [CC BY-SA 2.0], via Wikimedia Commons. Starfish by Paul Shaffner [CC BY 2.0], via Wikimedia Commons. Normal distribution from Wikimedia Commons [Public domain]. Water drop from pixababy.com [CC0 Creative Commons].

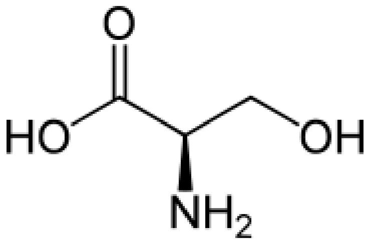




Or they might have no symmetry[2] at all.

2. Pillar coral, wave, and molecule from Wikimedia commons [Public domain]. Head of a woman by Pablo Picasso, image from Gandalf's Gallery on flickr [CC-BY-NC-SA 2.0]



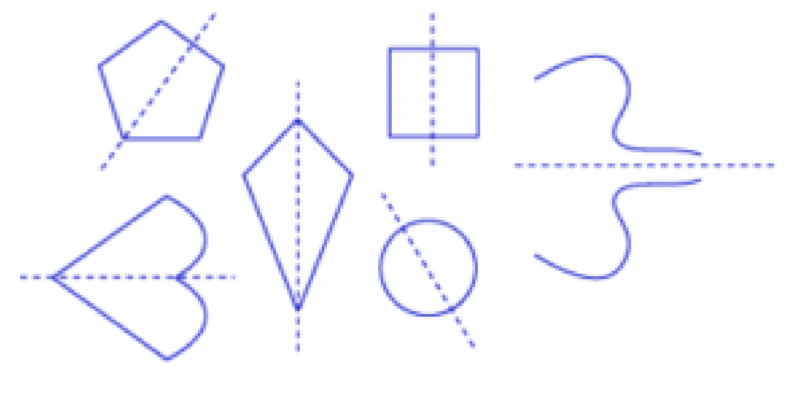


Think / Pair / Share

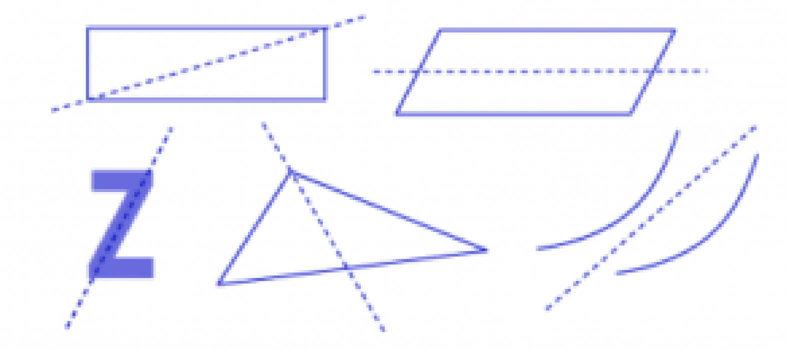
- What do you already know about the idea of *symmetry*? What does it mean to say a design is *symmetric*?
- Do you know about different types of symmetry? What types?
- Can you give examples of real-world objects that are symmetric? What about objects that are not symmetric?

Line Symmetry

If you can flip a figure over a line — this is called *reflecting* the figure — and then it appears unchanged, then the figure has **reflection symmetry** or **line symmetry**. A **line of symmetry** divides an object into two mirror-image halves. The dashed lines below are lines of symmetry:



Compare with the dashed lines below. Though they do cut the figures in half, they don't create mirror-image halves. These are **not** lines of symmetry:



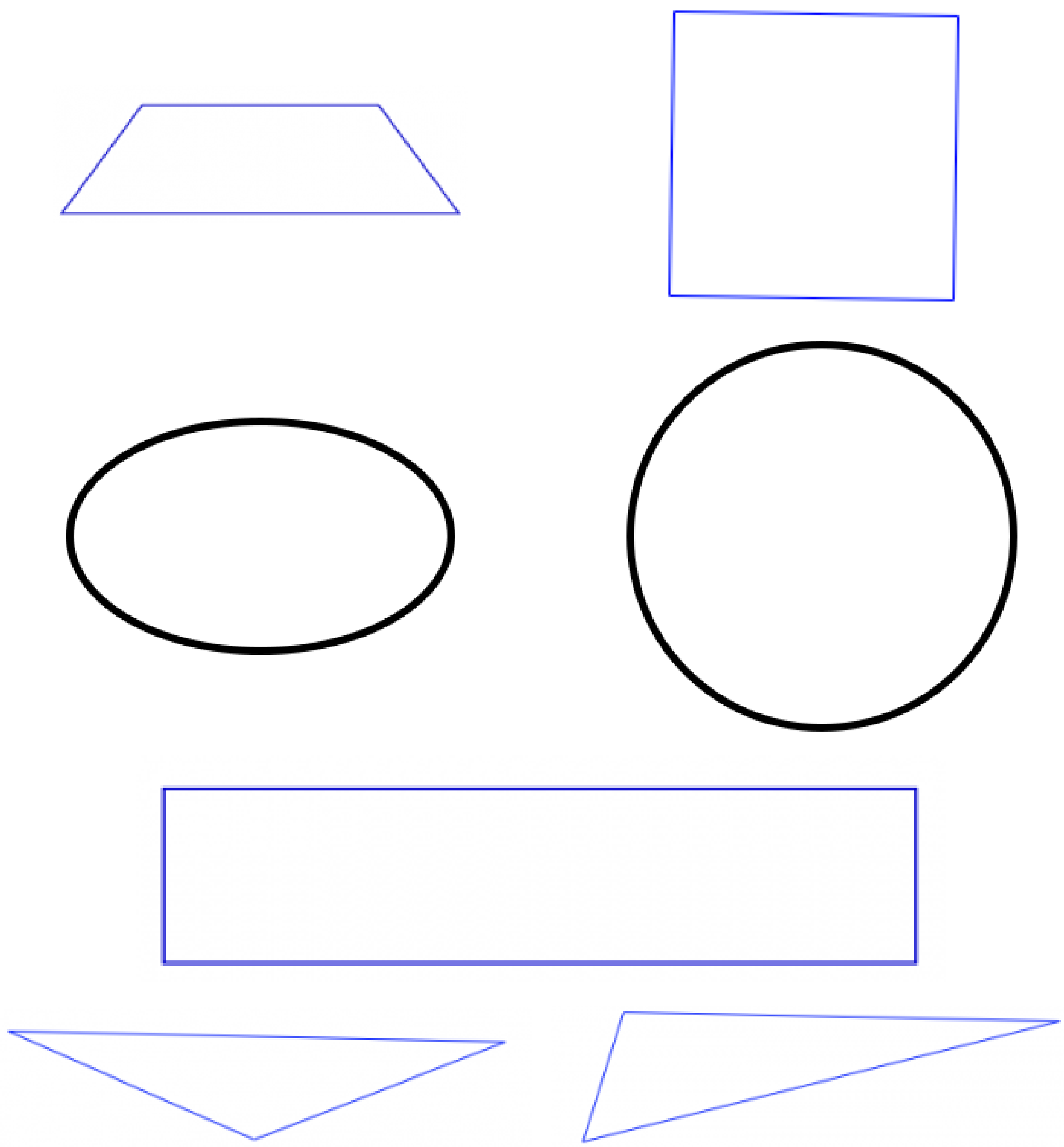
Think / Pair / Share

Look at the first set of pictures at the start of this chapter. Do any of them have lines of symmetry? How can you tell?

Problem 12

For each of the figures[3] below:

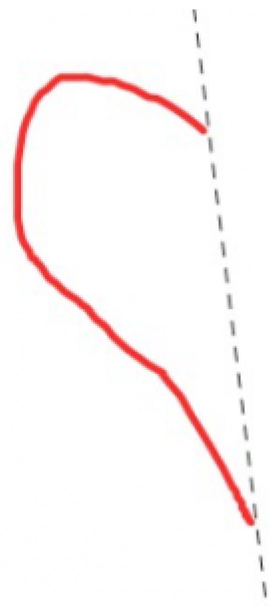
1. Decide if it has any lines of symmetry. If not, how do you know?
2. If it does have one or more lines of symmetry, find / describe all of them. Explain how you did it.

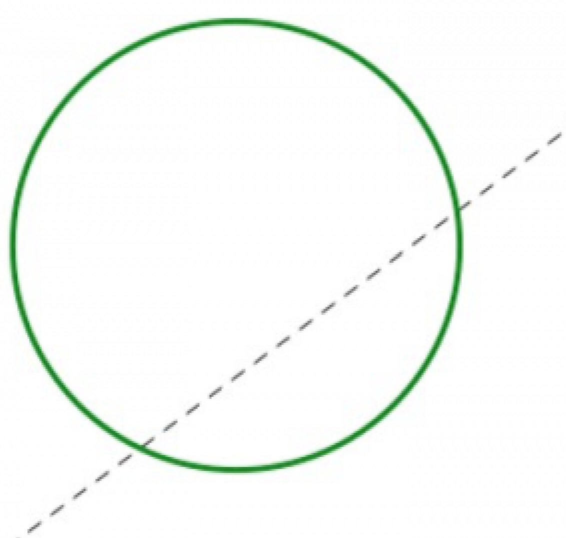


3. Circle and ellipse by Paris 16 (Own work) [CC BY-SA 4.0], via Wikimedia Commons

Problem 13

Each picture below shows **half** of a design with line symmetry. The line of symmetry (dashed) is shown. Can you complete the design? Explain how you did it.

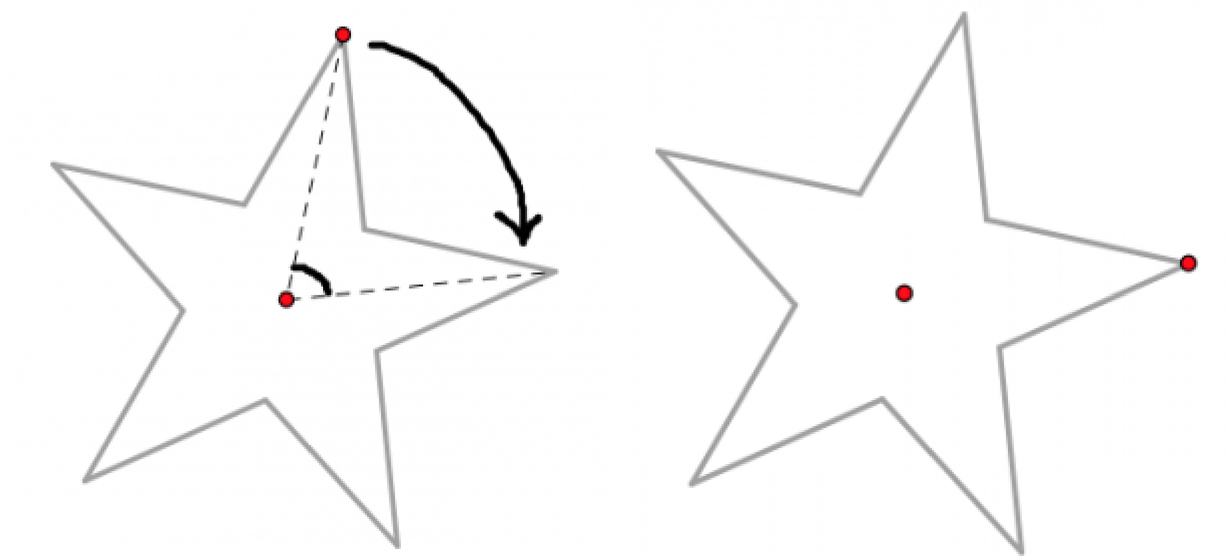




Rotational Symmetry

If you can turn a figure around a center point less than a full circle — this is called a *rotation* — and the figure appears unchanged, then the figure has **rotational symmetry**. The point around which you rotate is called the center of rotation, and the smallest angle you need to turn is called the angle of rotation.

This star has rotational symmetry of 72°, and the center of rotation is the center of the star. One point is marked to help you visualize the rotation.



Think / Pair / Share

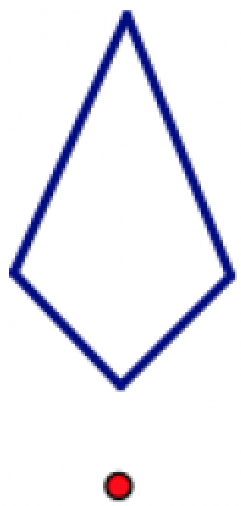
- How can you be certain that the angle of rotation for the star is exactly 72°?
- Look at the first set of pictures at the start of this chapter. Do any of them have rotational symmetry? How can you tell?

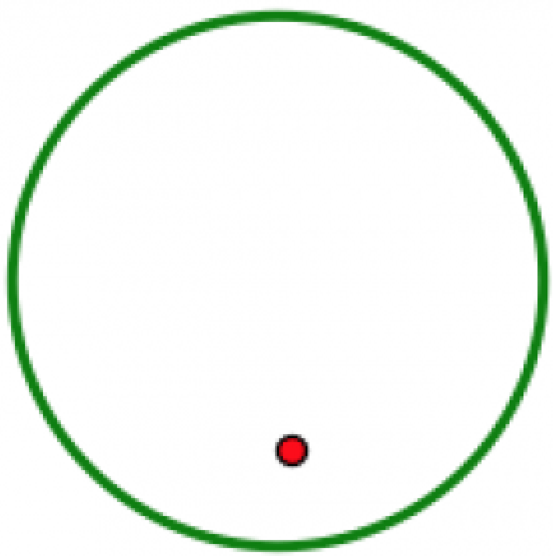
Problem 14

Each of the figures below has rotational symmetry. Find the center of rotation and the angle of rotation. Explain your thinking.

Problem 15

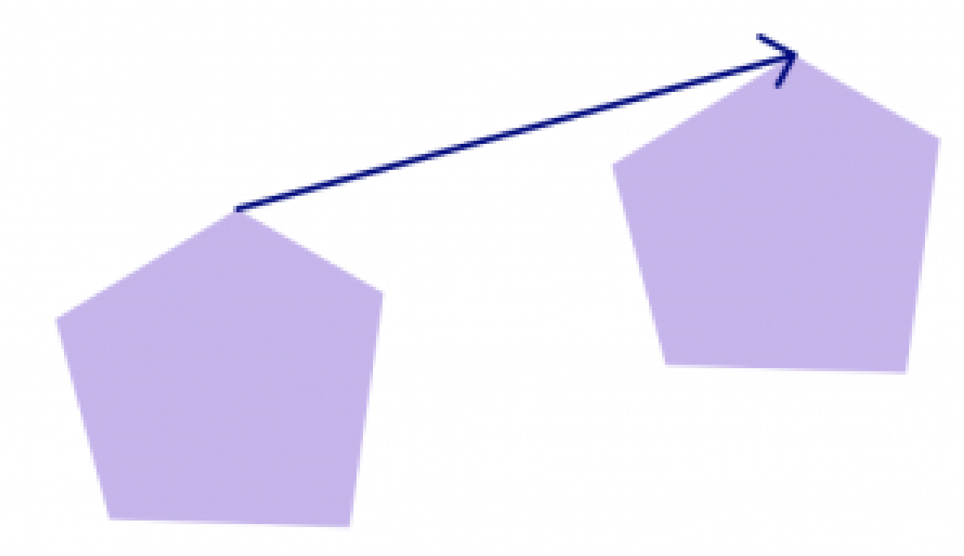
Each picture below shows part of a design with a marked center of rotation and an angle of rotation given. Can you complete the design so that it has the correct rotational symmetry? Explain how you did it.



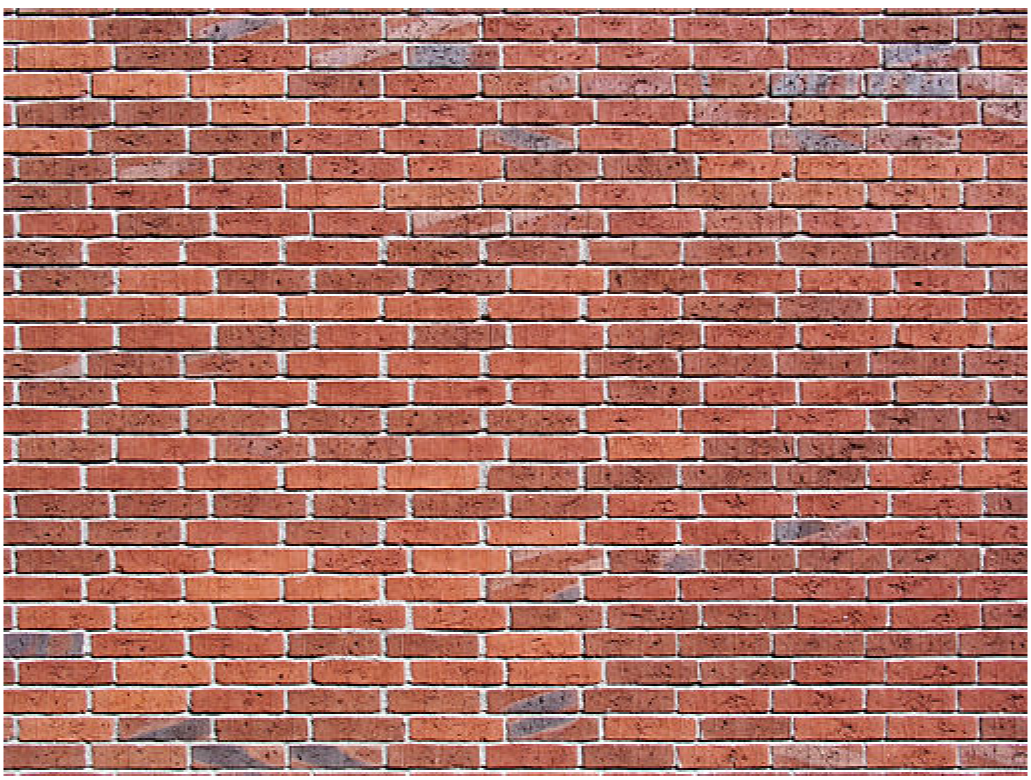


Translational Symmetry

A **translation** (also called a slide) involves moving a figure in a specific direction for a specific distance. A **vector** (a line segment with an arrow on one end) can be used to describe a translation, because the vector communicates both a distance (the length of the segment) and a direction (the direction the arrow points).

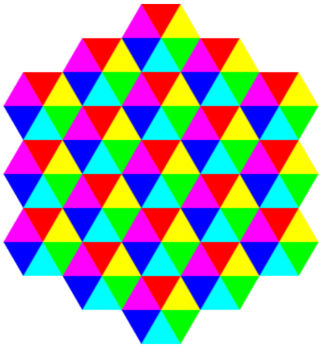
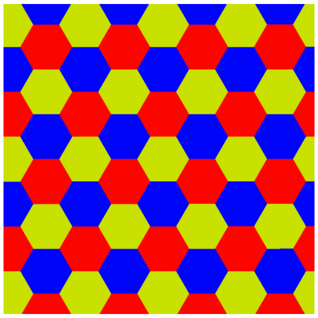
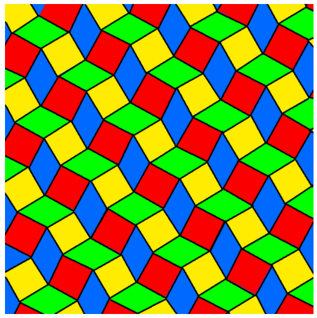


A design has **translational symmetry** if you can perform a translation on it and the figure appears unchanged. A brick wall[4] has translational symmetry in lots of directions!



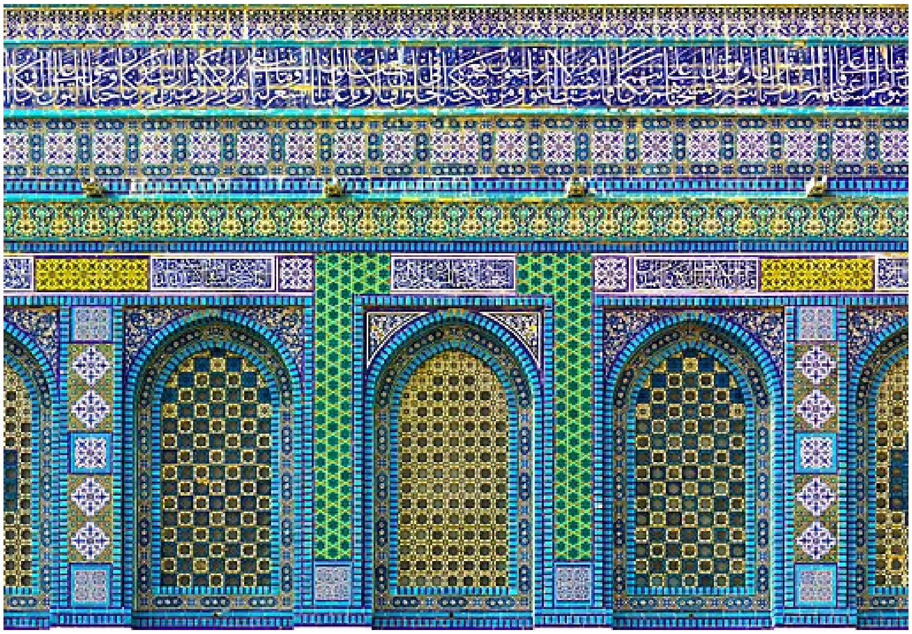
The brick wall is one example of a *tessellation*[5], which you'll learn more about in the next chapter.

4. Image by I, Xauxa [CC-BY-SA-3.0], via Wikimedia Commons
5. Triangular tessellation from pixababy [CC0]. Hexagonal and rhombic tessellations from Wikimedia Commons [Public domain].

You can see translation symmetry in lots of places. It's in architecture and design[6].

6. Tile at Jerusalem temple by Andrew Shiva / Wikipedia, via Wikimedia Commons [CC BY-SA 4.0]. Mosque by Hisham Binsuwaif via flickr [CC BY-SA 2.0]. British Museum great court by Andrew Dunn, http://www.andrewdunnphoto.com/ (Own work) [CC BY-SA 2.0], via Wikimedia Commons



It's in art, most famously that by M.C. Escher. (You might want to visit http://www.mcescher.com/gallery/symmetry/ and browse the "Symmetry" gallery.)

And it appears in traditional Hawaiian and other Polynesian tattoo[7] designs.

7. Royal Hawaiian officer via Wikimedia Commons [Public domain]. Shoulder and arm tattoos by Micael Faccio on flicker [CC BY-2.0].

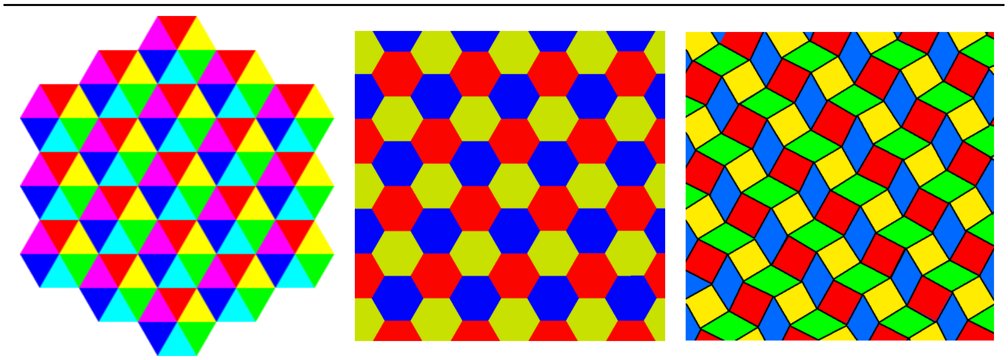
Think / Pair / Share

- On each of the pictures with translational symmetry above, sketch a vector to indicate the direction and distance of the translational symmetry.
- Create your own design with translational symmetry. Explain how you did it.

Geometry in Art and Science

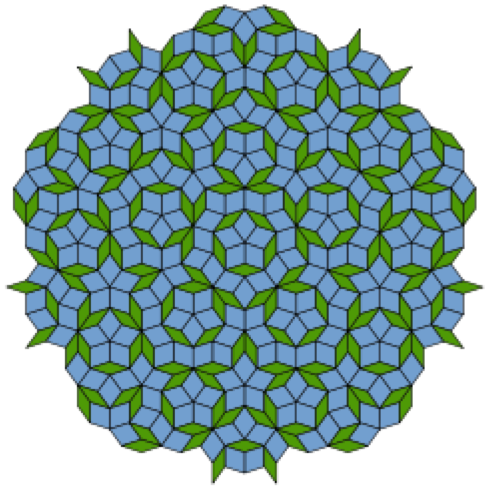
Tessellations

A *tessellation*[1] is a design using one ore more geometric shapes with no overlaps and no gaps. The idea is that the design could be continued infinitely far to cover the whole plane (though of course we can only draw a small portion of it).



Many tessellations have translational symmetry, but it's not strictly necessary. The *Penrose tiling* shown below[2] does not have any translational symmetry.

1. *Triangular tessellation from pixababy [CC0]. Hexagonal and rhombic tessellations from Wikimedia Commons [Public domain].*
2. Image via Wikimedia Commons [Public donmain].

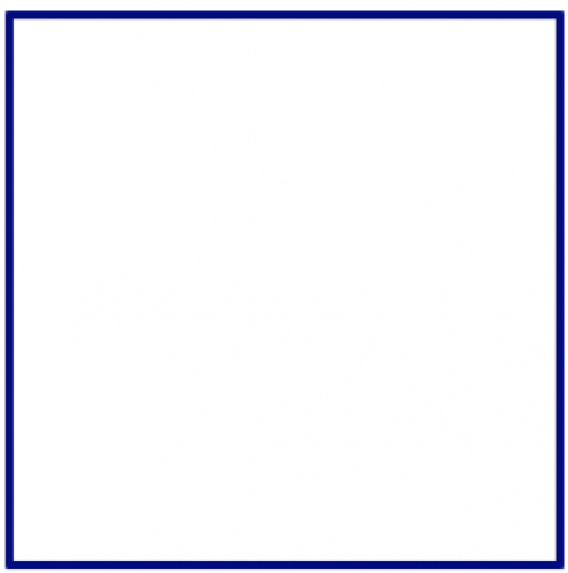


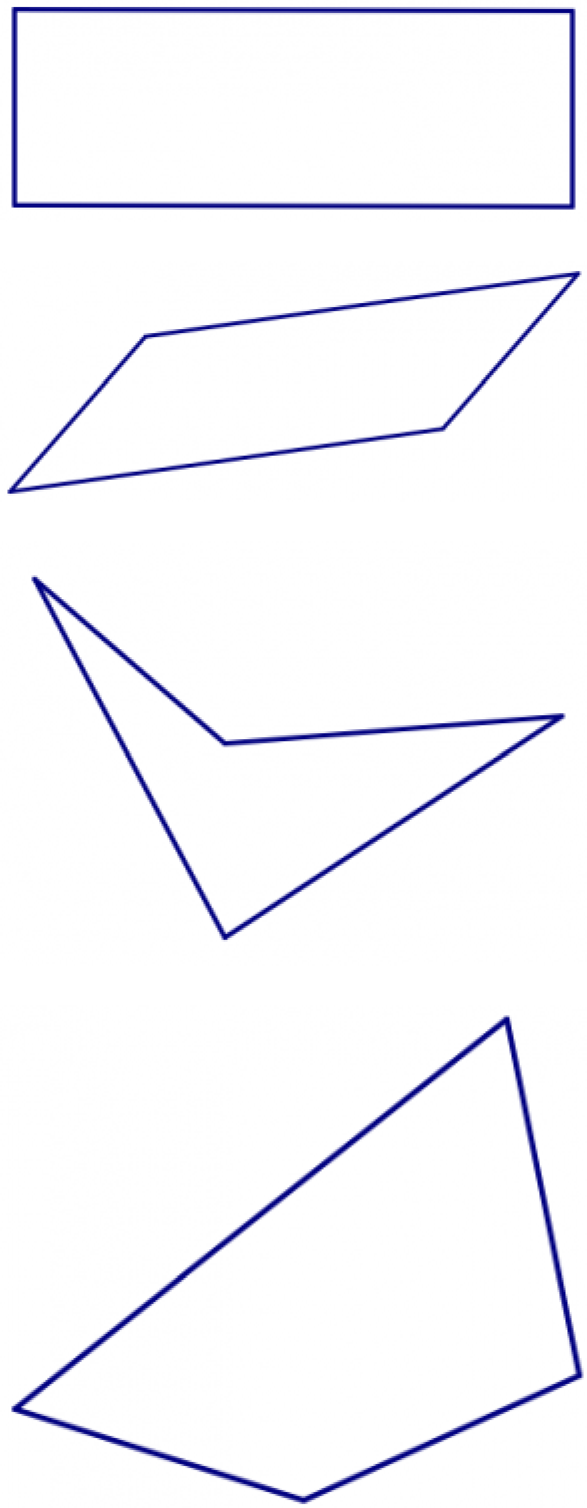
It's actually much harder to come up with these "aperiodic" tessellations than to come up with ones that have translational symmetry. So we'll focus on how to make symmetric tessellations.

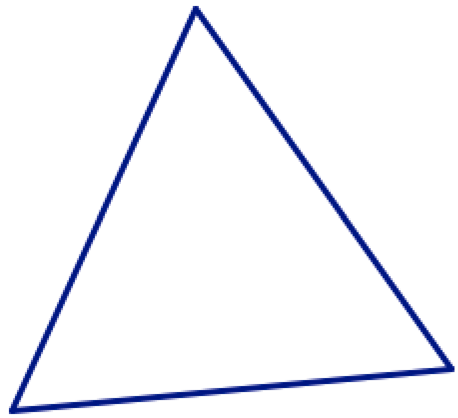
The first two tessellations above were made with a single geometric shape (called a *tile)* designed so that they can fit together without gaps or overlaps. The third design uses two basic tiles. Tessellations are often called *tilings*, and that's what you should think about: If I had tiles made in this shape, could I use them to tile my kitchen floor? Or would it be impossible?

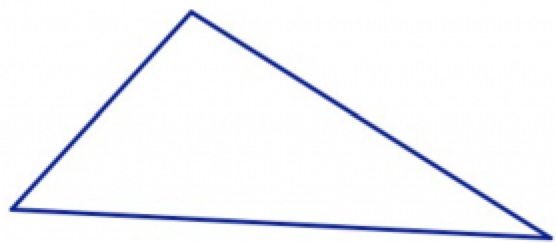
On Your Own

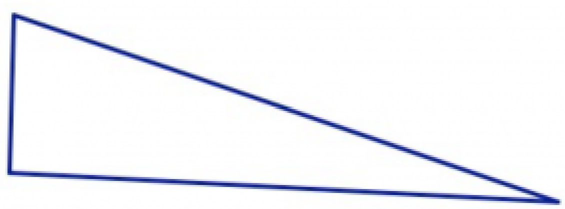
Work on these exercises on your own or with a partner. You will need lots of copies (maybe 10–15 each) of each shape below. In each problem, focus on just a single tile for making your tessellation.

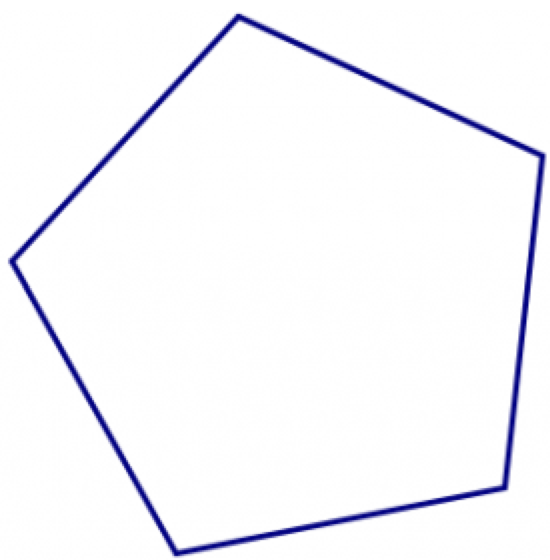


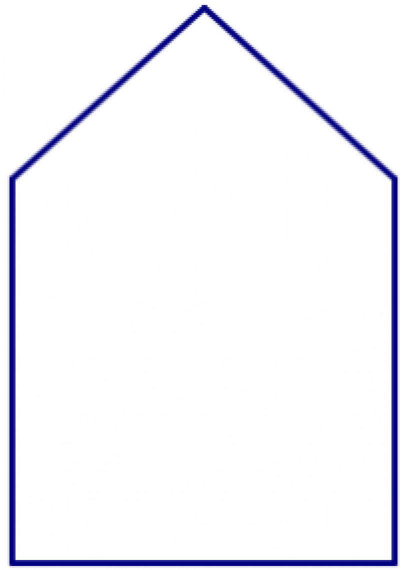


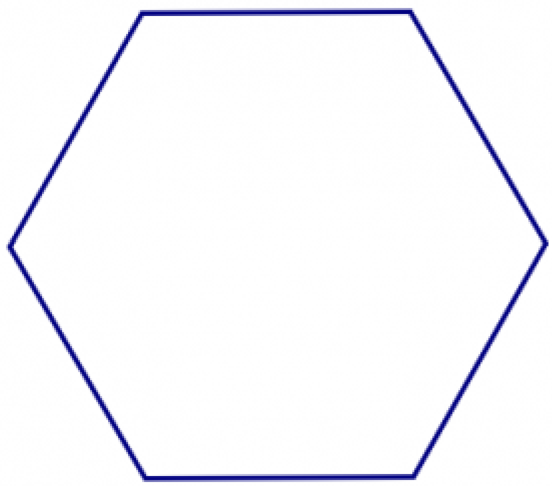


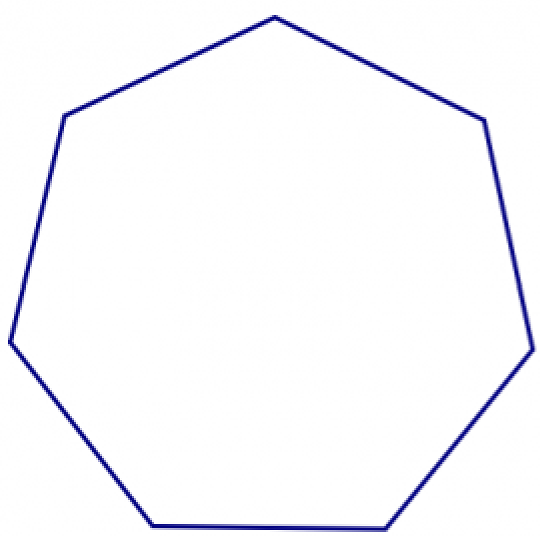


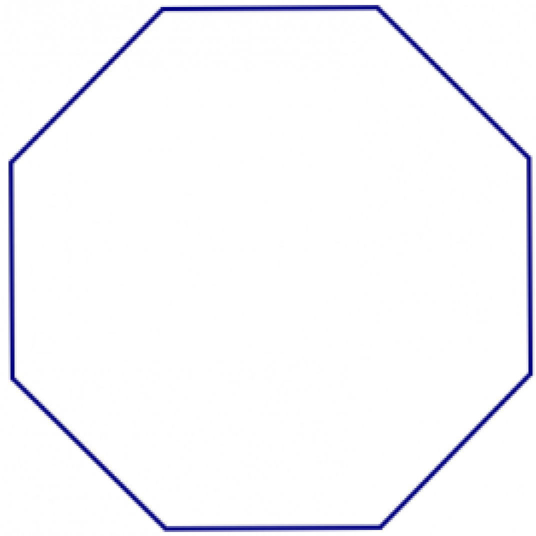












1. Start with the square tile. Can you fit the squares together in a pattern that could be continued forever, with no gaps and no overlaps? Can you do it in more than one way?
2. Now try one of the triangular tiles. Can you use many copies of a single triangle to tessellate the plane?
3. Repeat this process with each of the other tiles. Keep track of your findings.

Think / Pair / Share

- Which of the tiles given above tessellate, and which do not?
- Do you have any conjectures based on this experience, about which shapes will tessellate and which will not?

Escher Drawings

The artist M.C. Escher created many works of art inspired by mathematics, including some very beautiful tessellations. Below you will see some images[3] inspired by his work. You can view the real thing at http://www.mcescher.com/ in the “Symmetry” gallery.

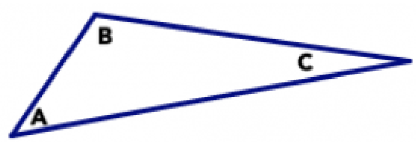
3. Images from flickr [CC BY-NC-SA 2.0]. Birds by Sharon Drummond. Lizard tiles by Ben Lawson.

You can make your own Escher-like drawings using some facts that you learned while studying tessellations.

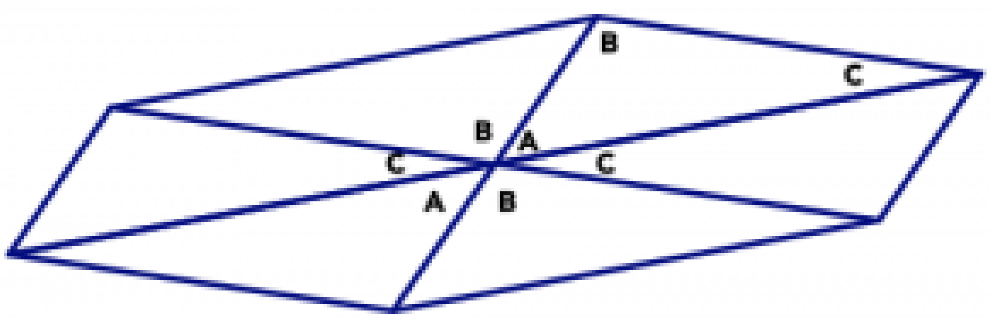
Theorem: Tessellations

Any triangle will tessellate. So will any quadrilateral.

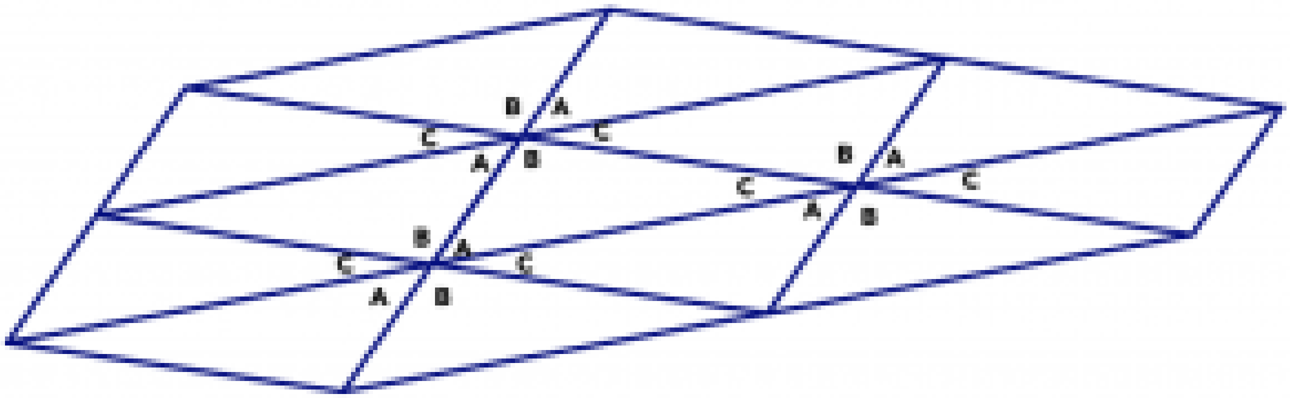
The explanation for this comes down to what you know about the sums of angles. The sum of the angles in a triangle is 180°.



So if you make six copies of a single triangle and put them together at a point so that each angle appears twice, there will be a total of 360° around the point, meaning the triangles fit together perfectly with no gaps and no overlaps.



You can then repeat this at every vertex, using more and more copies of the same triangles.



Think / Pair / Share

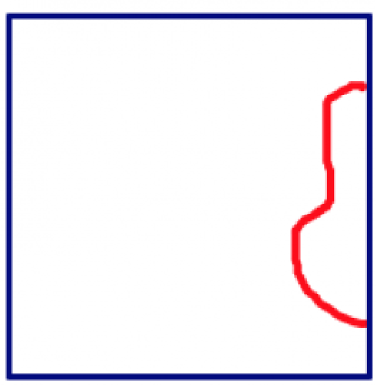
- Use the fact that the sum of the angles in any quadrilateral is 360° to explain why every quadrilateral will tessellate.
- Use angles to explain why regular hexagons will tessellate.
- Explain why regular pentagons will not tessellate.

On Your Own

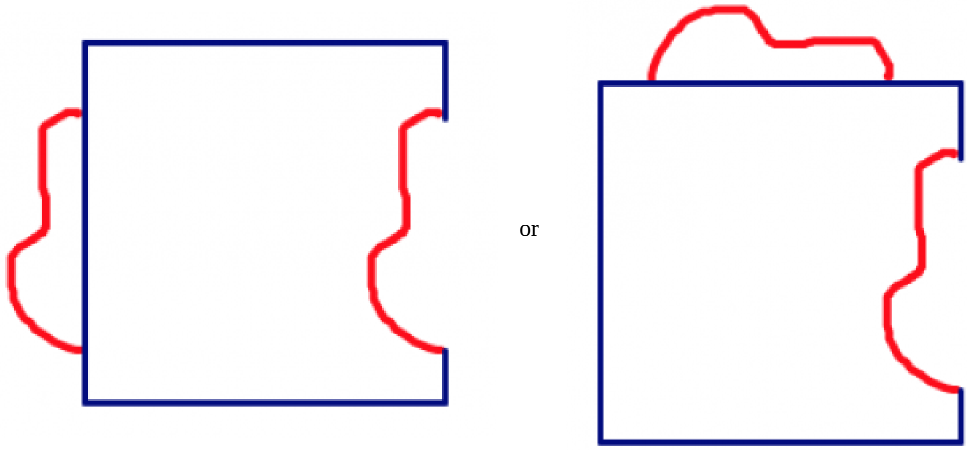
Work on the following exercises on your own or with a partner. Here's how you can create your own Escher-like drawings.

1. Select your basic tile. The first time you do this, it's easiest to start with a simple shape that you know will tessellate, like an equilateral triangle, a square, or a regular hexagon.

2. Draw a "squiggle" on one side of your basic tile.

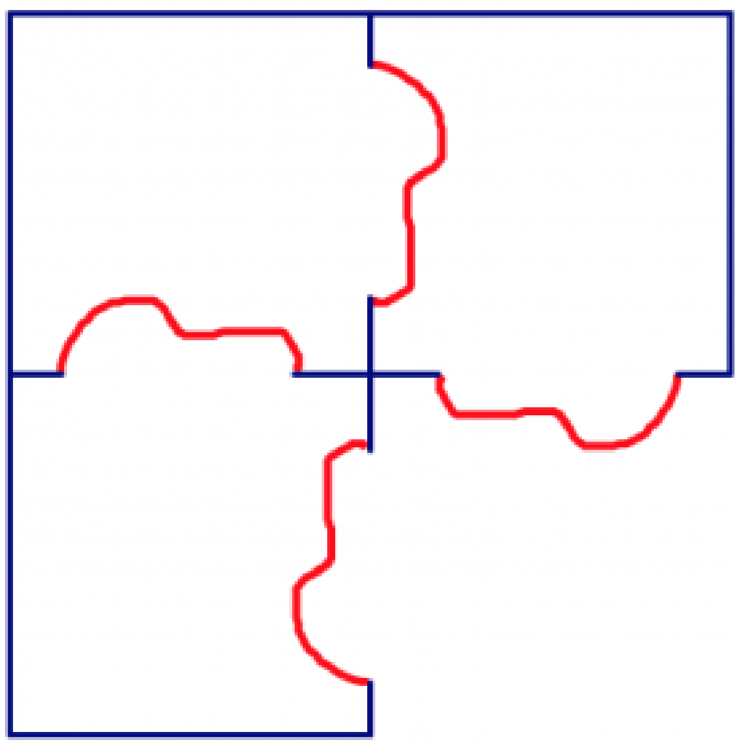


3. Cut out the squiggle, and move it to another side of your shape. You can either translate it straight across or rotate it.



4. It’s important that the cut-out lines up along the new edge in the same place that it appeared on its original edge.

5. Tape the squiggle into its new location. This is your basic tile. On a large piece of paper, trace around your tile. Then move it the same way you moved the squiggle (translate or rotate) so that the squiggle fits in exactly where you cut it out.



6. The shape will still tessellate, so go ahead and fill up your paper.

7. Now get creative. Color in your basic shape to look like something — an animal? a flower? a colorful blob? Add color and design throughout the tessellation to transform it into your own Escher-like drawing.

8. If you want to try a more complicated version, cut two different squiggles out of two different sides, and move them both.

Building Towers

For this activity, you will need some construction materials:

- You'll need lots of toothpicks.
- You'll also need something to connect the toothpicks together. The best material for this is mini marshmallows; you can stick the ends of the toothpicks into the marshmallows to connect them. You can also use pieces of clay, bits of gummy candies, or other similar (sticky) material.

Problem 16

Try this as a warm-up activity. Grab exactly six toothpicks. Your job is to make four triangles using all six toothpicks. You cannot break any of the toothpicks or add any other materials besides the marshmallow connectors.

Problem 17

Now comes the main challenge. You have ten minutes to build the *tallest free-standing structure* that you can make. "Free-standing" means that it will stand up on its own. You can't hold it up or lean it against something. When the ten minutes are up, back away from your tower and measure its height.

Think / Pair / Share

Look at your own tower and at other students' towers. Talk about these questions:

- What design choices led to taller free-stranding structures? Why do you think that is?
- If you had another ten minutes to try this activity again, what would you do differently and why?

Problem Bank

Problem 18

In the Tangrams chapter, you first saw all 7 tangram pieces arranged into a square.

1. If the large square you made with all seven pieces is one whole, assign a (fractional) value to each of the seven tangram pieces. Justify your answers.
2. The tangram puzzle contains a small square. If the small square (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.
3. The tangram set contains two large triangles. If a large triangle (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.
4. The tangram set contains one medium triangle. If the medium triangle (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.
5. The tangram set contains two small triangles. If a small triangle (the single tangram piece) is one whole, assign a value to each of the seven tangram pieces. Justify your answers.

Problem 19

If possible sketch an example of the following triangles. If it is not possible, explain why not.

1. A right triangle that is scalene.
2. A right triangle that is isosceles.
3. A right triangle that is equilateral.

Problem 20

If possible sketch an example of the following triangles. If it is not possible, explain why not.

1. An acute triangle that is scalene.
2. An acute triangle that is isosceles.
3. An acute triangle that is equilateral.

Problem 21

If possible sketch an example of the following triangles. If it is not possible, explain why not.

1. An obtuse triangle that is scalene.
2. An obtuse triangle that is isosceles.
3. An obtuse triangle that is equilateral.

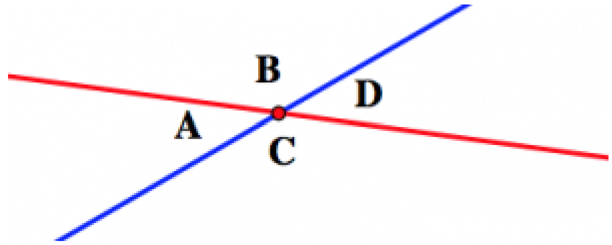
Problem 22

If possible sketch an example of the following triangles. If it is not possible, explain why not.

1. An equiangular triangle that is scalene.
2. An equiangular triangle that is isosceles.
3. An equiangular triangle that is equilateral.

Problem 23

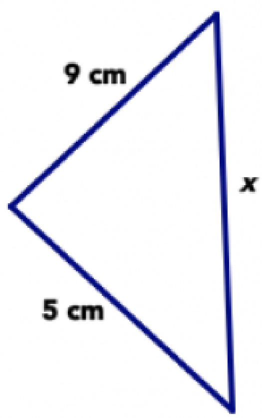
Look at the picture below, which shows two lines intersecting. Angles A and D are called “vertical angles,” and so are angles B and C.



Use this drawing to explain why vertical angles must have the same measure. (Hint: what is the sum of the measures of angle A angle B? How do you know?)

Problem 24

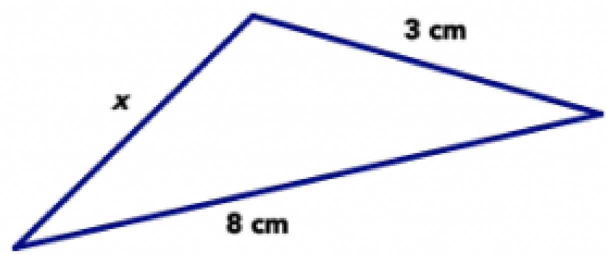
Answer the following questions about the triangle below. Be sure to focus on what you *know for sure* and not what the picture *looks like.*



1. Could it be true that $x = 4$ cm? Explain your answer.
2. Could it be true that $x = 20$ cm? Explain your answer.
3. Give three possible values of x, based on the information in the picture.

Problem 25

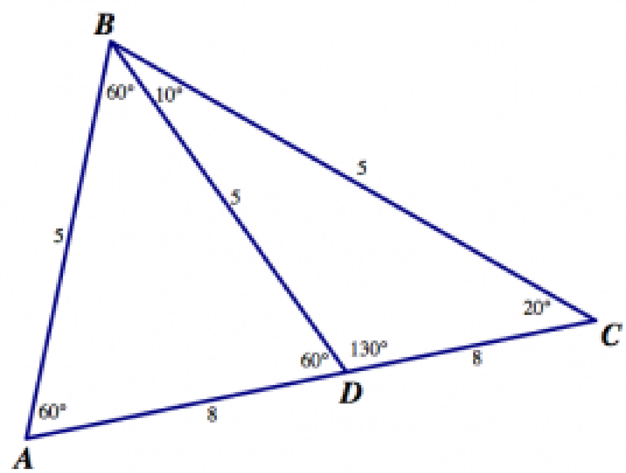
Answer the following questions about the triangle below. Be sure to focus on what you know for sure and not what the picture looks like.



1. If x = 3 cm, the triangle is isosceles. Is this possible? Explain your answer.
2. If x = 8 cm, the triangle is isosceles. Is this possible? Explain your answer.
3. Give three *impossible* values of x, based on the information in the picture.

Problem 26

Prof. Faber drew this picture on the board, saying it showed three triangles: △ABC, △ABD, and △CBD. Side lengths and angle measurements are shown for each of the triangles.



There are **lots of mistakes** in this picture. Use what you know about side lengths and angles in triangles to find all the mistakes you can. For each mistake, say what is wrong with the picture, and why it's a mistake. Explain your thinking as clearly as you can.

Problem 27

Because of SSS congruence, triangles are exceptionally sturdy. This means they are used frequently in architecture and design to provide supports for buildings, bridges, and other man-made objects. Take your camera with you, and find several places in your neighborhood or near your campus that use triangular supports. Snap a picture, and describe what the structure is and where you see the triangles.

Problem 28

It is possible to create designs that have multiple symmetries. See if you can find images (or create your own!) that have both:

1. reflection symmetry and rotational symmetry,
2. reflection symmetry and translational symmetry, or
3. rotational symmetry and translational symmetry.

Voyaging on Hōkūle`a

> We sail for peace, for the love of our planet and with the desire to leave the children of the world a hopeful and healthy future.
>
> -Hōkūle`a crew

Unless otherwise noted, photos and drawings in this Part come from the "Press Room at Outreach Tools" at http://hokulea.com and are used here non-commercially in accordance with their agreement.

Introduction

In the 1950's and 1960's, historians couldn't agree on how the Polynesian islands — including the Hawaiian islands — were settled. Some historians insisted that Pacific Islanders sailed deliberately around the Pacific Ocean, relocating as necessary, and settling the islands with purpose and planning. Others insisted that such a navigational and voyaging feat was impossible thousands of years ago, before European sailors would leave the sight of land and sail into the open ocean. These historians believed that the Polynesian canoes were caught up in storms, tossed and turned, and eventually washed up on the shores of faraway isles.

Think / Pair / Share

- How could such a debate ever be settled one way or the other, given that we can't go back in time to find out what happened?
- What kinds of evidence would support the idea of "intentional voyages"? What kinds of evidence would support the idea of "accidental drift"?
- What do you already know about how this debate was eventually settled?

Hōkūle`a

The Polynesian Voyaging Society (PVS) was founded in 1973 for scientific inquiry into the history and heritage of Hawai`i: How did the Polynesians discover and settle these islands? How did they navigate without instruments, guiding themselves across ocean distances of 2500 miles or more?

In 1973–1975, PVS built a replica of an ancient double-hulled voyaging canoe to conduct an experimental voyage from Hawai`i to Tahiti. The canoe was designed by founder Herb Kawainui Kāne and named Hōkūle`a ("Star of Gladness").

On March 8th, 1975, Hōkūle`a was launched. Mau Piailug, a master navigator from the island of Satawal in Micronesia, navigated her to Tahiti using traditional navigation techniques (no modern instruments at all).

Think / Pair / Share

- What are some mathematical questions you can ask about voyaging on Hōkūle`a?
- What kinds of problems (especially mathematics problems) did the crew have to solve before setting off on the voyage to Tahiti?
- What are you curious about, with respect to voyaging on Hōkūle`a?

When you teach elementary school, you will mostly likely be teaching all subjects to your students. One thing you should think about as a teacher: How can you connect the different subjects together? Specifically, how can you see mathematics in other fields of study, and how can you draw out that mathematical content?

In this chapter, you'll explore just a tiny bit of the mathematics involved in voyaging on a traditional canoe. You will apply your knowledge of geometry to create scale drawings and make a star compass. And you'll use your knowledge of operations and algebraic thinking to plan the supplies for the voyage. The focus here is on applying your mathematical knowledge to a new situation.

One of the first things to know about Hōkūle`a[1] is what she looks like. You can find more pictures at http://hokulea.com.

1. Hokulea homecoming picture by Michelle Manes.

Problem 1

Here's some information about the dimensions of Hōkūle`a. Your job is to draw a good scale model of the canoe, like a floor plan.

- Hōkūle`a is 62 feet 4 inches long. (This is "LOA" or "length overall" in navigation terms. It means the maximum length measured parallel to the waterline.)
- Hōkūle`a is 17 feet 6 inches wide. (This is "at beam" meaning at the widest point.)
- You can see from the picture that Hōkūle`a has two hulls, connected by a rectangular deck. The deck is about 40 feet long and 10 feet wide.

Imagine you are above the canoe looking down at it. Draw a scale model of the hulls and the deck. Do not include the sails or any details; you are aiming to convey the overall *shape* in a scale drawing.

You will use this scale drawing several times in the rest of this unit, so be sure to do a good job and keep it somewhere that you can find it later.

Note: You don't have all the information you need! So you either need to find out the missing information or make some reasonable estimates based on what you do know.

Problem 2

Crew for a voyage is usually 12–16 people. During meal times, the whole crew is on the deck together. About how much space does each person get when they're all together on the deck?

Worldwide Voyage

To Prepare for next activity:

1. Read this description of the daily life on Hōkūle`a: http://pvs.kcc.hawaii.edu/ike/canoe_living/daily_life.html.
2. Watch the video about the Worldwide Voyage:

```
<object id="flash-object" name="flash-object"
type="application/x-shockwave-flash" width="100%"
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name="flashvars"
value="clip_id=51118047&amp;api=1&amp;moogaloop_type
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name="allowscriptaccess" value="always"> <param
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value="high"> <param name="scalemode"
value="noscale"> <div class="fallback"><iframe
src="/video/51118047/fallback?noscript" frameborder="0"
title="Player Fallback Message"></iframe></div>
</object>
```

From the webpage above, you learned:

> *The quartermaster is responsible for provisioning the canoe — loading food, water and all needed supplies, and for maintaining Hōkūle`a's inventory. While this is not an on board job, it is critical to the safe and efficient sailing of the canoe.*

Problem 3

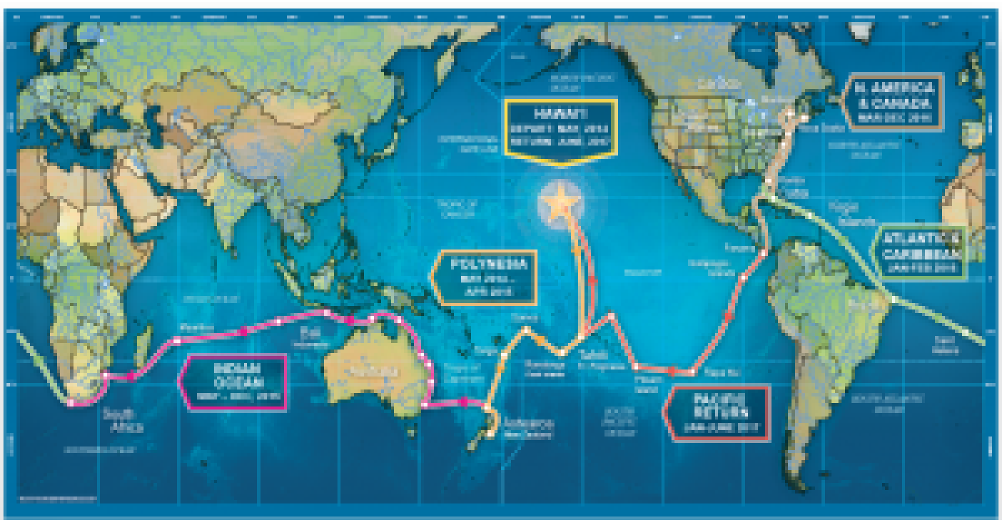
Imagine that you are part of the crew for the Worldwide Voyage, and you are going to help the quartermaster and the captain with provisioning the canoe for one leg of the voyage. You need to write a preliminary report for the quartermaster, documenting:

1. Which leg of the trip are you focused on? (See the map below.)
2. How long will that leg of the trip take? Explain how you figured that out.
3. How much food and water will you need for the voyage? Explain how you figured that out.

The rest of this section contains pointers to information that may or may not be helpful to you as you make your plans and create your report. Your job is to do the relevant research and then write your report. You should include enough detail about how you came to your conclusions that the quartermaster can understand your reasoning.

Pick a leg of the route:

Here's a picture of the route planned for the Worldwide Voyage, which you can find at the Worldwide Voyage website: http://www.hokulea.com/worldwide-voyage/ and a full-sized map here: https://tinyurl.com/WWVmap. On the map, the different colors correspond to different years of the voyage. A "leg" means a dot-to-dot route on the map.



After you pick a leg of the voyage, you'll need to figure out the total distance of that leg. This tool might help (or you can find another way): http://www.acscdg.com/.

Here is some relevant information to help you figure out how long it will take Hōkūle`a to complete your chosen leg:

- The first trip from Hawai`i to Tahiti in 1976 took a total of 34 days. (You probably want to use the tool above to compute the number of nautical miles.)

Plan the provisions:

Here is some information about provisions.

- Hōkūle`a can carry about 11,000 pounds, including the weight of the crew, provisions, supplies, and personal gear.
- The supplies (sails, cooking equipment, safety equipment, communications equipment, etc.) account for about 3,500 pounds.
- The crew eats three meals per day and each crew member gets 0.8 gallons of water per day.
- For a trip that is expected to take 30 days, the quartermaster plans for 40 days' worth of supplies, in case of bad weather and other delays.

Navigation

The following is from http://pvs.kcc.hawaii.edu/ike/hookele/modern_wayfinding.html.

> *A voyage undertaken using modern wayfinding has three components:*
>
> *Design a course strategy, which includes a reference course for reaching the vicinity of one's destination, hopefully upwind, so that the canoe can sail downwind to the destination rather than having to tack into the wind to get there. (Tacking involves sailing back and forth as closely as possible into the wind to make progress against the wind; its very arduous and time-consuming, something to be avoided if at all possible, particularly at the end of a long, difficult voyage.)*
>
> *During the voyage, holding as closely as possible to the reference course while keeping track of (1) distance and direction traveled; (2) one's position north and south and east and west of the reference course and (3) the distance and direction to the destination.*
>
> *Finding land after entering the vicinity of the destination, called a target screen or 'the box'.*

So how is the navigation done — especially component (2) — through thousands of miles of open ocean? You can't see land. How can you hold closely to the reference course? How can you keep track of distance and direction traveled? How can you even know if you're going in the right direction if all you can see is blue ocean and blue sky?

By day, the navigators use their deep knowledge of the oceans. Which way do the winds blow? Which way do the prevailing currents move? Clouds in the sky, flotsam in the water, and animal behaviors give them great insight into where land might be, and where they are in relation to it.

By night, they use the stars. In this section, you'll learn just a tiny fraction of what these master navigators know about the stars.

Think / Pair / Share

Here is a time-lapse picture[1] of the stars in the night sky:

1. Image from pixababy [CC0 Creative Commons].

- Describe what you see happening in this picture.
- What can you conclude about how the stars move through the night sky?
- How might that help a navigator find his way?

Star Compass

A fundamental tool for navigators on Hōkūle`a and other voyaging canoes is a star compass. Here's a picture of Mau Piailug[2] and a star compass[3] he used in his teaching.

2. Picture by Maiden Voyage Productions [CC BY-SA 3.0], via Wikimedia Commons
3. Image by Newportm (Own work) [CC BY-SA 3.0], via Wikimedia Commons.

The object in the center of the circle represents the canoe. The shells along the outside represent directional points. The idea is to imagine the stars rising up from the horizon in the east, traveling through the night sky, and setting past the horizon in the west. They move like they're on a sphere surrounding the Earth (it's called the celestial sphere).

Problem 4

Nainoa Thompson developed a star compass with 32 equidistant points around a circle. (Note this is more points than in Mau's star compass pictured above.) You will first try to make a rough sketch of Nainoa's star compass based on this information.

- Place 32 points around the circle so they are equally spaced.
- The arcs between these equidistant points are called "houses." You will label each house with its Hawaiian name. Start with the four cardinal points:

`Ākau: North.

Hema: South.

Hikina: East.

Komohana: West.

- The four quadrants also get names. (These cover all of the houses in the quadrant, so label them in the appropriate place inside the compass.)

Ko`olau: northeast.

Malani: southeast.

Kona: southwest.

Ho`olua: northwest.

- Moving from `Ākau to Hikina (clockwise), there are seven houses. They are labeled in order as you move away from `Ākau:

Haka: “empty,” describing the skies in this house.

Nā Leo: “the voices” of the stars speaking to the navigator.

Nālani: “the heavens.”

Manu: “bird,” the Polynesian metaphor for a canoe.

Noio: the Hawaiian tern (a bird).

`Āina: “land.”

Lā: “sun,” which stays in this house most of the year.

- The compass has a vertical line of symmetry, so there are the same seven houses in the same order as you move from `Ākau to Komohana (counterclockwise).
- The compass also has a horizontal line of symmetry. Use that fact to label the houses from Hema to Hikina (counterclockwise) and from Hema to Komohana (clockwise).

How is the star compass used in navigation? There are lots of ways. Here’s a (very!) quick overview:

- The canoe is pictured in the middle of the star compass, with all of the houses around.
- Winds and ocean swells move directly across the star compass from north to south or vice versa.
    - If the swells are coming from `Āina Ko`olau, they will be heading in the direction `Āina Kona. (Look at your star compass and trace out this path.)
    - If the wind is coming from Nālani Malani, it will be heading towards Nālani Ho`olua. (Look at your star compass and trace out this path.)
- Stars stay in their houses, but also in their hemisphere. They do not move across the center of the circle.
- Just like the sun, they rise in the east and set in the west.
    - `A`ā (Sirius) rises in Lā Malanai and sets in Lā Kona. (Look at your star compass and trace out this path.)
    - Hōkūle`a rises in `Āina Ko`olau and sets in `Āina Ho`olua. (Look at your star compass and trace out this path.)

A navigator memorizes the houses of over 200 stars. At sunrise and sunset (when the sun or the stars are rising), the navigator can use the star compass to memorize which way the wind is moving and which way the currents are moving. The navigator can then use that information throughout the day or night to ensure the canoe stays on course.

Think / Pair / Share

Look again at the time-lapse picture of the stars:

- Describe how this shows that stars "stay in their houses" and in their hemisphere as they move through the night sky.
- The star Ke ali`i o kona i ka lewa (Canopus), rises in Nālani Malanai. Where does it set?

When teaching navigation while sitting on land, it's perfectly fine to have a rough sketch or model of the star compass. But if you really have to do the navigation, you need to make a very, very precise star compass.

Imagine Nainoa Thompson, who navigated Hōkūle`a on the final leg of her journey from Hawai`i to Rapa Nui, an island even smaller and lower than Ni`ihau. You have to be within 30 miles of Rapa Nui to see it. But a mistake of even one degree would have led to Hōkūle`a being 60 miles off course. And if you end up drifting in the open ocean and supplies run out? Well…

Nainoa Thompson

Problem 5

Now that you have a rough sketch of the star compass and know what it should look like, your job is to draw one that's as perfect as possible. That means you want to draw:

1. A perfect circle (well, as perfect as possible). What tools can you use to do that? What tools would ancient Polynesian navigators have had to use?
2. Thirty-two points around the circle that are exactly evenly spaced apart. (What tools would help you? What tools would ancient Polynesian navigators have had to use?)
3. When you have finished, label your perfectly drawn star compass with the houses.

Of course, a star compass on a piece of paper isn't so useful when you're out on a canoe. How do you position it properly? And how do you keep it from getting lost, damaged, or soaking wet? You paint it on the rails of the canoe, permanently!

Look back at the drawing of Hōkūle`a. Find the "kilo" (navigator's seat) in the rear (aft) of the canoe. There is actually one navigator's seat on either side of the deck.

Problem 6

Go back to the scale drawing of Hōkūle`a that you made in Problem 1. Add the navigator's seats to your drawing. You will then add the star compass to the rails as follows:

1. Start with the kilo (seat) on the left (port) side of the canoe. That will be the center of your star compass. Imagine looking to the right. You want to see the star compass markings on the rails when you look to the right. Of course, Hōkūle`a is not a circular canoe, and the navigator doesn't sit at the center. So how can you make the markings in the right places?
2. Now repeat that process, using the seat on the right (starboard) side of the canoe.

The kilo onHōkūle`a[4].

Compass markings on the rails (to be painted more visibly before voyaging)[5].

Nainoa Thompson has said:

> *Initially, I depended on geometry and analytic mathematics to help me in my quest to navigate the ancient way. However as my ocean time and my time with Mau have grown, I have internalized this knowledge. I rely less on mathematics and come closer and closer to navigating the way the ancients did.*

Really he is still doing a lot of mathematics; it's just mathematics that he has internalized and that is now second nature to him. The ancient navigators may not have spoken of their navigation techniques in the same modern language we've been using — compass points and perfect circles and degrees. But their mathematical understanding was truly astonishing.

4. Photo by Michelle Manes.
5. Photo by Michelle Manes.